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Non-linear Modifications of Black-Scholes Pricing Model with Diminishing Marginal Transaction Cost

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Abstract

In the field of quantitative financial analysis, the Black-Scholes Model has exerted significant influence on the booming of options trading strategies. Publishing in their Nobel Prize Work in 1973, the model was generated by Black and Scholes. Using Ito's Lemma and portfolio management methodology, they employed partial differential equation to provide a theoretical estimate of the price of European-style options.

This paper is interested in deriving non-linear modifications of the Black-Scholes model with diminishing marginal transaction cost.

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Chapter 1

Introduction

1.1 Background

With the growing economy and service globalization, the financial market played an important role in helping individuals making efficient decisions. New types of financial assets and new forms of trading applications have emerged in the market to satisfy the urging needs. Besides qualitative perspective and personal experience, facing a dynamic market, it is crucial for the companies and investors to have a systematic and numerical method to evaluate trading investment for financial products, rationalize behaviours, and better control uncertainties. Since asset pricing is the inspiration for more effective trading strategy, it has long been the focus of many studies, especially on options.

An option is an agreement that gives the holder a right, not obligation, to buy from, or sell to, the seller, or the buyer of option certain amounts of underlying assets at a specified price (strike price) at a future time (expiration date). In 1973, Black and Scholes[1] proposed the Black-Scholes Pricing model aiming to better hedge the option investment by buying and selling a correct amount of the underlying asset and eliminating risk. This method is frequently called as the "continuously revised delta hedging" method which laid the foundation of more complicated investment strategies in recent decades.

Though the Black-Scholes Model is fundational, many of its assumptions are too theoretical to be applied in the real world and many studies in the decades had been dedicated to further improve it. First, it assumes there is no dividend pay out, ignoring the impact of dividend on changing the price of the option. Published in May 1978, Robert Geske [2] derived the improved model in discrete time when the underlying stock has a stochastic dividend yield. The results indicate that whether there is dividend yield results in misestimation of the variance of the underlying stock. This estimation could diminish a bias of the Black-Scholes Model.

Second, the model also assumes the risk free return and volatility are constant, which is not applicable in the real market. There are various studies focused on the volatility improvement. For example, Gong, Hanlu and Thavaneswaran [3] developed a new method for pricing derivatives under the Black-Scholes Model with the volatility following a GARCH process (Generalized Autoregressive Conditional Heteroskedasticity). They view the call price as an expected value of a truncated normal distribution. The results demonstrate that the their model outperforms other GARCH pricing models and the pricing errors are very small.

Besides dividend and volatility assumptions, the Black-Scholes Model did not consider the issue of transaction cost. Most of the literature focusing on this track are mostly based on the idea of the improve model in Hoggard, Whalley and Wilmott [6] where they assumed the transaction cost is proportional to the value of underlying assets traded and add this consideration into the original Black-Scholes Model. Detailed description is later presented in section 2.

In this section, we first analyze the construction method of Black-Scholes Model. It is established based on the Ito's Lemma, Geometric Brownian Motion, and the delta hedging strategy, which will be illustrated in the following subsections.

1.2 Ito's Lemma

The Ito's Lemma, named after Kiyosi Ito, is an identity in calculus to find the differential of a time-dependent function of a stochastic process. It can be derived by forming the Taylor series expansion of the function up to its second derivatives and retaining terms up to first order in the time increment and second order in the Wiener process increment.

In its simplest form, an Ito drift-diffusion process has the formula:

$$dS(t) = \alpha(t)dt + \sigma(t)dW(t). \tag{1.1}$$

For any twice differentiable scalar function $f(t, S)$, it has the formula when it is applied to a stochastic process $S(t)$:

$$df(t, S(t)) = \left(\frac{df}{dt} + \alpha(t)\frac{df}{dS} + \frac{\sigma(t)^2}{2}\frac{d^2 f}{dS^2}\right)dt + \sigma(t)\frac{df}{dS}dW(t), \tag{1.2}$$

where $\alpha(t)$ is the drift, $\sigma(t)$ is the variance, $W(t)$ is a Wiener process, which is often used to measure continuous-time stochastic process. It has the following properties:

1. $W(0) = 0$;

2. W has independent increments: for every $t > 0$, the future increments $W(t+u) - W(t)$, $u \geq 0$, are independent of the past value;

3. W has Gaussian increments: $W(t+u) - W(t)$ is normally distributed with mean 0 and variance u;

4. W has continuous paths: $W(t)$ is continuous in t.

1.3 Geometric Brownian Motion

A stochastic process $S(t)$ is said to follow a Geometric Brownian Motion, if it satisfies the following stochastic differential equation:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t). \tag{1.3}$$

Here $W(t)$ is a Wiener process or Brownian motion, and α ('the percentage drift') and σ ('the percentage volatility') are constants.

1.4 Classic Black-Scholes Model

The classic Black-Scholes-Merton Model is established based on the employment of Ito's Lemma and Geometric Brownian Motion mentioned in the previous sections.

Before evaluating the asset, it proposes several assumptions. For assets, first, the rate of return on the riskless asset is constant. Second, the stock price follows a log-normal random walk, which is a geometric Brownian motion, and its drift and volatility are constant. Third, the stock does not pay a dividend.

For the financial market, there is no arbitrage opportunity. Second, it is possible to borrow and lend any amount of cash at the riskless rate. Moreover it is possible to buy and sell any amount of the stock, including short selling. Last but not least, the above transactions do not incur any fees or costs. The market is frictionless.

We them demonstrate the Black Scholes Model using the method and idea from Shreve [8]. Consider an agent who at each time t has a portfolio valued at $X(t)$. This portfolio invests in a money market account paying a constant rate of interest r and in stock modeled by the geometric Brownian motion:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t), \tag{1.4}$$

where α is the drift coefficient for S, the stock price, σ presents the volatility of stock price, and $W(t)$ is the geometric Brownian motion.

Suppose at each time t, the investor holds $\Delta(t)$ shares of stock. The position $\Delta(t)$ can be random but must be adapted to the filtration associated with the Brownian motion $w(t), t \geq 0$.

The value of the portfolio is due to two factors: the capital gain and the money market.

$$X(t) = \Delta(t)S(t) + X(t) - \Delta(t)S(t), \tag{1.5}$$

The differential $dX(t)$ for the investor's portfolio value at each time t is due to two factors, the capital gain $\Delta(t)dS(t)$ on the stock investment and the interest earnings in the money market $r(X(t)dt - \Delta(t))S(t))dt$. Specifically,

$$\begin{aligned} dX(t) =& \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt \\ =& rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t). \end{aligned} \tag{1.6}$$

We shall often consider the discounted stock price and the discounted portfolio value. For the differential of the discounted stock price, according to the Ito-Doeblin formula with $f(t,x) = e^{-rt}x$, it is

$$\begin{aligned} d(e^{-rt}S(t)) =& f_t(t, S(t))dt + f_x(t, S(t))dS(t) + \frac{1}{2}f_{xx}(t, S(t))dS(t)dS(t) \\ =& (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t). \end{aligned} \tag{1.7}$$

Hence the differential of the discounted portfolio value is

$$\begin{aligned} d(e^{-rt}X(t)) =& f_t(t, X(t))dt + f_x(t, X(t))dS(t) + \frac{1}{2}f_{xx}(t, X(t))dX(t)dX(t) \\ =& \Delta(t)d(e^{-rt}S(t)). \end{aligned} \tag{1.8}$$

Consider a European call option which gives the investor the right, but not the obligation, to buy the underlying assets for a certain price (strike price) at the expiration date. This option pays $\max\{S(T) - K, 0\}$ at time T. Here $S(t)$ is the price of the underlying asset; t is the current time, α is the drift of S; σ is the volatility of S; K is the strike price; T is the expiration data; r is the risk-free rate. Black, Scholes, and Merton assumed that the value of the call option is a function of various parameters in the contract $C(S, t, \alpha, \sigma, K, T, r)$.

Our goal is to determine the value of the function $C(t, S)$. We first compute the differential of it based on the Ito-Doeblin formula and equation (1.2):

$$
\begin{aligned}
dC(t,S(t)) =& C_t(t,S(t))dt + C_S(t,S(t))dS(t) + \frac{1}{2}C_{SS}(t,S(t))dS(t)dS(t) \\
=& C_t(t,S(t))dt + C_S(t,S(t))(\alpha S(t)dt + \sigma S(t)dW(t)) + \frac{1}{2}C_{SS}(t,S(t))\sigma^2 S^2(t)dt \\
=& [C_t(t,S(t)) + \alpha S(t)C_S(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)C_{SS}(t,S(t))]dt \\
&+ \sigma S(t)C_S(t,S(t))dW(t).
\end{aligned}
\tag{1.9}
$$

Taking the differential, we have

$$
\begin{aligned}
d(e^{-rt}C(t,S(t))) =& df(t,C(t,S(t))) \\
=& f_t(t,C(t,S(t)))dt + f_x(t,C(t,S(t))))dC(t,S(t)) \\
&+ \frac{1}{2}f_{xx}(t,C(t,S(t)))dC(t,S(t))dC(t,S(t)) \\
=& e^{-rt}[-rC(t,S(t)) + C_t(t,S(t)) + \alpha S(t)C_x(t,S(t)) \\
&+ \frac{1}{2}\sigma^2 S^2(t)C_{xx}(t,S(t))]dt + e^{-rt}\sigma S(t)C_x(t,S(t))dW(t).
\end{aligned}
\tag{1.10}
$$

A (short option) hedging portfolio starts with some initial capital $X(0)$ and invest in the stock and money market so that the portfolio value $X(t)$ at each time $t \in [0,T]$ agrees with $C(t,S(t))$. Such condition happens if and only if:

$$
e^{-rt}X(t) = e^{-rt}C(t,S(t)) \tag{1.11}
$$

which is to ensure:

$$
d(e^{-rt}X(t)) = d(e^{-rt}C(t,S(t))), \quad t \in [0,T), \tag{1.12}
$$

and $X(0) = C(0,S(0))$.

Comparing equation (1.8) and (1.10), we see that equation (1.12) holds if and only if

$$
\begin{aligned}
&\Delta(\alpha - r)S(t)dt + \Delta\sigma S(t)dW(t) \\
=& \left[-rC(t,S(t)) + C_t(t,S(t)) + \alpha S(t)C_S(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)C_{SS}(t,S(t))\right]dt \\
&+ \sigma S(t)C_S(t,S(t))dW(t).
\end{aligned}
\tag{1.13}
$$

We examine what is required in order for (1.13) to hold. We first equate the $dW(t)$ terms in (1.13), which gives:

$$\Delta(t) = C_S(t, S(t)), \quad t \in [0, T]. \tag{1.14}$$

This is the delta-hedging rule: at each time t prior to expiration, the number of shares held by the hedge of the short option position is the partial derivative with respect to the stock price of the option value at that time. $C_S(t, S(t))$ is the delta of the option.

Then, we equate the dt terms in (1.13) and (1.14):

$$\begin{aligned} &(\alpha - r)S(t)C_S(t, S(t)) \\ =& - rC(t, S(t)) + C_t(t, S(t)) + \alpha S(t)C_S(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)C_{SS}(t, S(t)). \end{aligned} \tag{1.15}$$

With simplification, we have:

$$rC(t, S(t)) = C_t(t, S(t)) + rS(t)C_S(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)C_{SS}(t, S(t)). \tag{1.16}$$

In conclusion, we have the following Black-Scholes-Merton partial differential equation that satisfies the call option terminal condition:

$$\begin{cases} C_t(t, S) + rxC_S(t, S) + \frac{1}{2}\sigma^2 S^2 C_{SS}(t, S) = rC(t, S), & S > 0,\ 0 < t < T, \\ C(T, S) = \max\{S - K, 0\}, & S > 0. \end{cases} \tag{1.17}$$

Solving (1.17), we obtain the the Black-Scholes formula for option pricing:

$$C(t, S) = N(d_1)S - N(d_2)Ke^{-r(T-t)}, \tag{1.18}$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left(\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T - t)\right), \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy$$

is the standard normal cumulative distribution function.

Beyond the call option, other options' value can also be presented and analyzed through the Black-Scholes equation with their specific end-condition at the end of period T, which is illustrated below. The put option is a contract giving the owner the right, but not the obligation, to sell a specified amount of an underlying security at a pre-determined price within a specified time frame. A binary call option pays off the corresponding amount if at maturity the underlying asset price is above the strike price and zero otherwise. Similarly, the binary put option pays off that amount if the underlying asset price is less than the strike price and zero otherwise. That is

1. $C(S,T) = \max\{S-K, 0\}$ for call option;
2. $P(S,T) = \max\{K-S, 0\}$ for put option;
3. $BC(S,T) = H(S-K)$ for binary call option;
4. $BP(S,T) = H(K-S)$ for binary put option.

Here function $H(x)$ is the Heaviside function satisfying $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$.

The solutions of Black-Scholes equation with these different end-conditions are

Option	Value Formula
Call	$N(d_1)S - N(d_2)Ke^{-r(T-t)}$
Put	$N(-d_2)Ke^{-r(T-t)} - N(-d_1)S$
Binary Call	$N(d_2)e^{-r(T-t)}$
Binary put	$(1-N(d_2))e^{-r(T-t)}$

Table 1.1: Pricing formulas for different options.

The diagrams below illustrate the pricing formula of various options. Given parameters as $K = 50, r = 0.01, T = 10, \sigma = 0.2$, Figures 1.1-1.4 show the option value at various time before expiration.

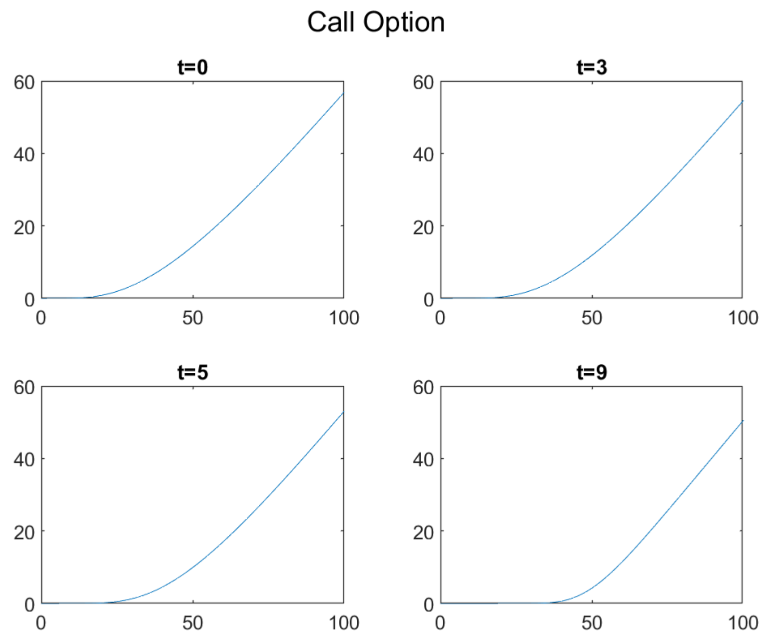

Figure 1.1: Call option

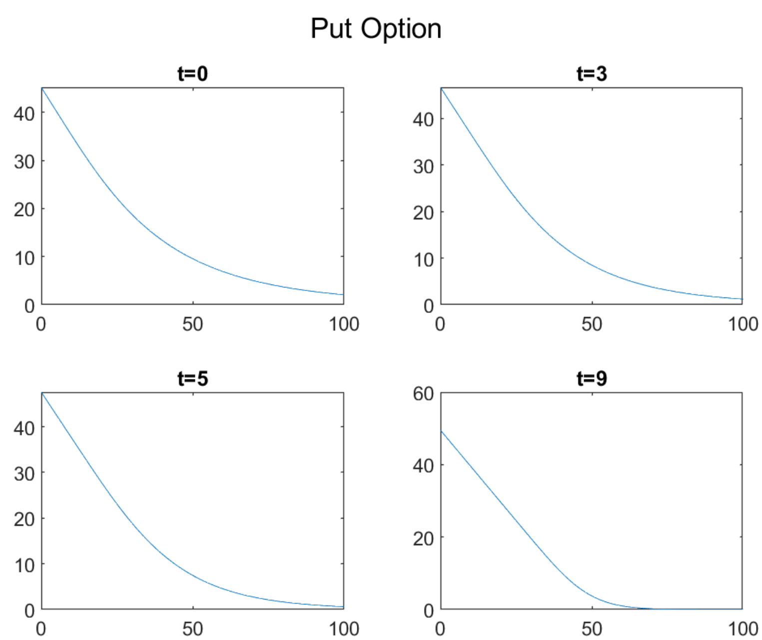

Figure 1.2: Put option

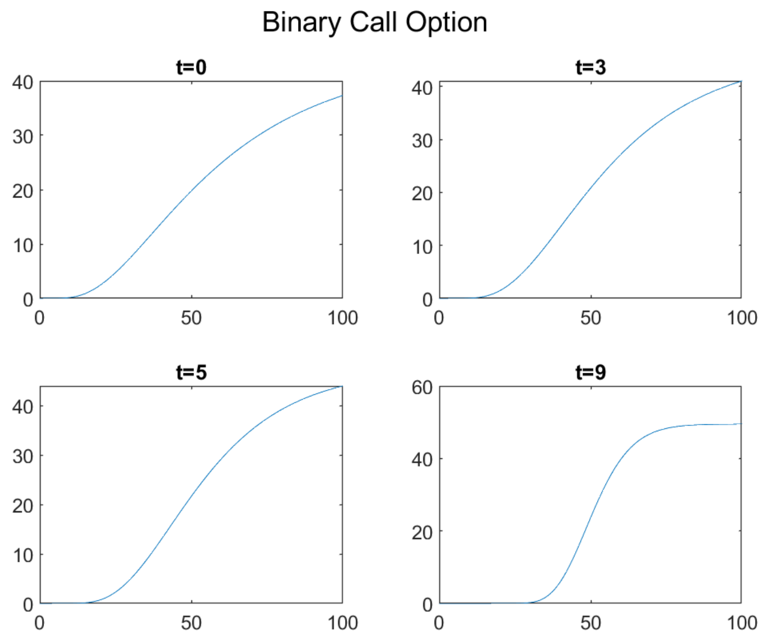

Figure 1.3: Binary Call option

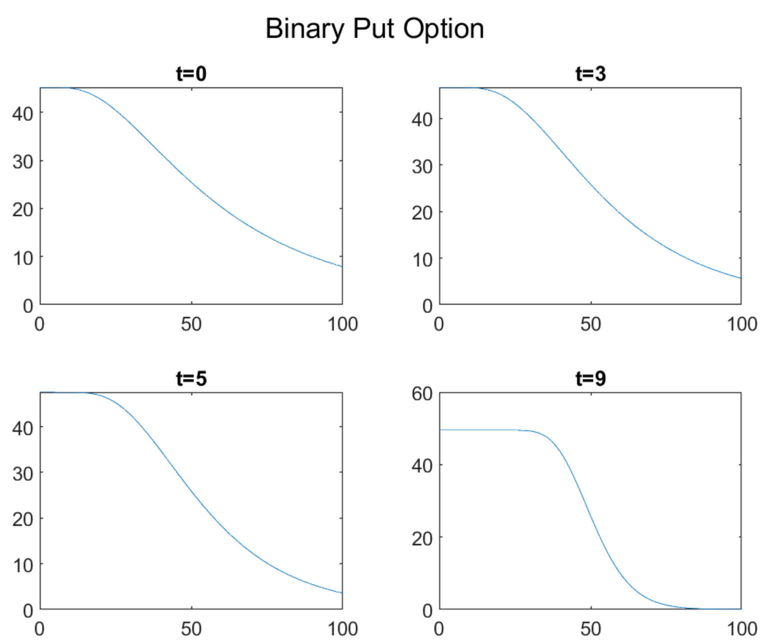

Figure 1.4: Binary Put option

The derivative of the function $C(t, S)$ of (1.18) with respect to various variables are called the Greeks. For Black-Scholes Model, various measure with Greek letter are essential for constructing option strategies. Using call options as an example:

Delta Δ is defined as:

$$\Delta = C_S(t, S).$$

It measures the sensitivity of the option or portfolio to the underlying asset. Call deltas are positive and put deltas are negative, reflecting the fact that the call option price is positively correlated to the underlying asset price while the put option price and the underlying asset price are inversely related. The table below demonstrates the delta for various option. (Here $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$)

Option	Delta
Call	$N(d_1)$
Put	$N(d_1) - 1$
Binary Call	$\frac{e^{-r(T-t)} N'(d_2)}{\sigma S \sqrt{T-t}}$
Binary put	$\frac{e^{-r(T-t)} N'(d_2)}{\sigma S \sqrt{T-t}}$

Table 1.2: Delta of Options

Gamma Γ is defined as:

$$\Gamma = C_{SS}(t, S).$$

It measures how fast the delta changes for small change in the underlying stock price. It shows by how much or how often a position should be re-hedged in order to keep a delta neutral position. For hedging a portfolio with the delta-hedge strategy, then we want to keep gamma as small as possible, since the smaller it is the less often we will have to adjust the hedge to maintain a delta neutral position. It is always positive for call options while negative for put options.

Theta Θ is defined as:

$$\Theta = C_t(t, S).$$

It measures the sensitivity of the value of the option to the change of time.

Vega V measures the sensitivity of the option price to the volatility of the underlying asset.

$$Vega = C_\sigma$$

Practically, it is expressed as the amount that the option's value will gain or lose as volatility rises or falls.

Rho ρ measures the rate of change of the option with respect to the interest rate.

$$\rho = C_r$$

.

Chapter 2

Model Modifications

2.1 Hoggard-Whalley-Wilmott Model

Transaction costs are the costs appearing in the buying and selling of the underlying asset. The Black-Scholes model requires the continuous rebalancing of a hedged portfolio and assumes no transaction costs in buying and selling. Neverthless, in reality, transaction costs do exist. Depending on the underlying asset and the market condition, transaction costs may or may not be important. For example, transaction costs in emerging markets are more expensive and therefore it is not desirable to rehedge frequently. However, in a more liquid market, transaction costs can be very low and a portfolio can be easily rehedged to keep a neutral position. Thus, the classic Black-Scholes model should be generalized to incorporate the effects of transaction costs in option pricing.

Based on Leland's model [5], Hoggard assumed that the transaction cost is proportional to the value of the underlying assets traded and the rate of proportion is a positive constant κ. Therefore, for buying (+) or selling (−) of $|\nu|$ shares at the price S, the transaction cost is $\kappa|\nu|S$. It is possible to generalize the model by considering the components of transaction costs as a fixed cost for each transaction or a cost proportional to the number of shares of traded assets. Whalley and Wilmott [6] discussed the general models in greater detail. For completeness of model derivation, we quickly make a sketch of the Hoggard-Whalley-Wilmott model, which is based on Leland's assumption of transaction cost.

We first demonstrate the model by considering the case of the call option. Since the transaction can only happen in discrete time step, we need to approximate the stochastic process (1.4) for the underlying asset by

$$dS(t) = \alpha S(t)dt + \sigma S\Phi\sqrt{dt}, \tag{2.1}$$

where Φ is a standardized normal random variable. According to the equation (1.5), a change in the value of the portfolio is given as:

$$\begin{aligned} \Delta X(t) =& X(t+\Delta t) - X(t) \\ =& rX(t)\Delta(t) + (\alpha - r)C_S(t, S(t))S(t)\Delta t \\ &+ \sigma C_S(t, S(t))S(t)\Phi\sqrt{\Delta t} - k|\nu|S(t). \end{aligned} \tag{2.2}$$

The change in the number of shares $|\nu|$ is:

$$\begin{aligned} |\nu| =& C_S(t+\Delta t, S(t) + \Delta S(t)) - C_S(t, S(t)) \\ =& [C_S(t+\Delta t, S(t) + \Delta S(t)) - C_S(t, S(t) + \Delta S(t))] \\ &+ [C_S(t, S(t) + \Delta S(t) - C_S(t, S(t))] \\ =& C_{SS}(t, S(t))\sigma S(t)\Phi\sqrt{\Delta t} + O(\Delta t). \end{aligned} \tag{2.3}$$

Thus, by employing Taylor expansion, we obtain

$$\nu = C_{SS}\sigma S\phi\sqrt{\Delta t} + O(\Delta t). \tag{2.4}$$

Thus, with the changes, we now have the equation:

$$C_t(t, S) + \frac{1}{2}\sigma^2 S^2 C_{SS}(t, S) - k\sigma S^2\sqrt{\frac{2}{\pi dt}}|C_{SS}(t, S)| + rSC_S(t, S) - rC(t, S) = 0, \tag{2.5}$$

which is the Hoggard-Whalley-Wilmott model.

2.2 Modifications

Though the nonlinear partial different model proposed by Hoggard, Whalley, and Wilmott presents an important sight on the improvement of the Black-Scholes Model, further modification corresponding to real financial market transaction can be implemented. In this section, we modify the Hoggard-Whalley-Wilmott model through considering a diminishing marginal transaction cost.

Specifically, Hoggard-Whalley-Wilmott model assumes the transaction cost is proportional to the value of underlying asset. Nevertheless, in some scenarios in real-world, the more underlying asset is invested, the less transaction cost for an additional unit of spending. With a diminishing rate:

$$\text{Transaction Cost} = \kappa \frac{|\nu|}{|\nu| + c} S(t), \tag{2.6}$$

where κ is the rate of proportion, c is a constant, ν is the number of shares buying for selling, and S is the strike price.

Thus, a change in the value of the portfolio is given as:

$$\begin{aligned} \Delta X(t) =& X(t + \Delta t) - X(t) \\ =& rX(t)\Delta(t) + (\alpha - r)C_S(t, S(t))S(t)\Delta t \\ &+ \sigma C_S(t, S(t))S(t)\Phi\sqrt{\Delta t} - \kappa \frac{|\nu|}{|\nu| + c} S(t). \end{aligned} \tag{2.7}$$

2.3 Case 1: European Call Option

For the call option, with equation (2.5), the nonlinear modified equation we have is:

$$\begin{aligned} &C_t(t, S)\Delta t + \frac{1}{2}\sigma^2 S^2 C_{SS}(t, S)\Delta t - \kappa S \frac{C_{SS}\sigma S\Phi\sqrt{\Delta t}}{C_{SS}\sigma S\Phi\sqrt{\Delta t} + c} \\ &+ rSC_S(t, S)\Delta t - rC(t, S)\Delta t = 0. \end{aligned} \tag{2.8}$$

In order to analyze the behavior of the model (2.8) with respect to the Black-Scholes model, we study the equation with a simpler term in the transaction cost. Since the function $h(y) = \dfrac{y}{y + c}$ satisfies

$$0 \leq \frac{y}{y+c} \leq 1, \quad \lim_{y \to 0} \frac{y}{y+c} = 0, \quad \lim_{y \to \infty} \frac{y}{y+c} = 1, \tag{2.9}$$

The solution of (2.8) is bounded by the one of

$$C_t(t,S) + \frac{1}{2}\sigma^2 S^2 C_{SS}(t,S) - \kappa S \frac{1}{\Delta t} + rSC_S(t,S) - rC(t,S) = 0, \tag{2.10}$$

and the original Black-Scholes equation:

$$C_t(t,S) + \frac{1}{2}\sigma^2 S^2 C_{SS}(t,S) + rSC_S(t,S) - rC(t,S) = 0. \tag{2.11}$$

To study the model (2.8) when $0 < |\nu| < \infty$, we first solve the equation (2.10). Let

$$\begin{cases} \tau = \dfrac{\sigma^2}{2}(T-t) \\ x = \ln\left(\dfrac{S}{K}\right) \\ C(t,S) = Ku(\tau, x). \end{cases} \tag{2.12}$$

Then by the multivariate chain rule, we have

$$C_s = K(u_x X_S + u_\tau \tau_s) = K u_x x_S = u_x e^{-x}. \tag{2.13}$$

$$\begin{aligned} C_{ss} =& K(u_x X_S + u_\tau \tau_S)_S \\ =& K(u_{xx}(x_S)^2 + u_x x_{SS} + u_{\tau\tau}(\tau_s)^2 + u_\tau \tau_{SS} + 2u_{x\tau} x_S \tau_S). \end{aligned} \tag{2.14}$$

$$C_t = K u_\tau \tau_t = -\frac{1}{2} K u_\tau \sigma^2. \tag{2.15}$$

Thus, substituting the equation (2.9), we obtain:

$$-\frac{1}{2}Ku_\tau\sigma^2 + \frac{1}{2}\sigma^2 S^2 K(u_{xx}\frac{1}{S^2} - u_x\frac{1}{S^2}) - ms + rKu_x - rKu = 0, \tag{2.16}$$

where $m = \dfrac{\kappa}{\Delta t}$.

Simplifying the equation, we have:

$$-\frac{1}{2}u_\tau\sigma^2 + \frac{1}{2}\sigma^2(u_{xx} - u_x) - me^x + ru_x - ru = 0. \tag{2.17}$$

Let $\lambda = \dfrac{2r}{\sigma^2}$, (2.17) becomes

$$u_\tau = u_{xx} + (\lambda - 1)u_x - \lambda u - \frac{m\lambda}{r}e^x. \tag{2.18}$$

The equation above is still not in the form of diffusion equation. To further transform to diffusion equation, we will let $W(\tau, x) = e^{ax+b\tau}u(\tau, x)$. Thus, we have

$$\begin{cases} W_x = e^{ax+b\tau}(au + u_x), \\ W_{xx} = e^{ax+b\tau}(a^2u + 2au_x + u_{xx}), \\ W_\tau = e^{ax+b\tau}(bu + u_\tau). \end{cases} \tag{2.19}$$

Let $a = \dfrac{\lambda - 1}{2}$ and $b = \dfrac{(\lambda + 1)^2}{4}$, the equation above can be simplified into

$$\begin{aligned} W_\tau =& W_{xx} - \frac{m\lambda}{r}e^x e^{ax+b\tau} \\ =& W_{xx} - \frac{m}{r}(2a+1)e^{x(a+1)+b\tau}. \end{aligned} \tag{2.20}$$

Performing the substitutions on the boundary conditions with a call option, we obtain:

$$\begin{cases} W_\tau = W_{xx} - \frac{m}{r}(2a+1)e^{x(a+1)+b\tau} \\ W(0, x) = \max\{e^{\frac{x(\lambda+1)}{2}} - e^{\frac{x(\lambda-1)}{2}}K, 0\} = e^{\frac{(\lambda-1)x}{2}}\max\{e^x - K, 0\}. \end{cases} \tag{2.21}$$

The solution to the initial value problem (2.21) can be written as a convolution integral of the initial data with the heat equation's fundamental solution:

$$\begin{aligned} W(x, \tau) =& \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} \exp\frac{-(x-\eta)^2}{4\tau} g(\eta)d\eta \\ &- \frac{m\lambda}{r}\int_0^\tau\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(\tau - S)}} \exp\frac{-(x-\eta)^2}{4\tau - S} \exp\frac{(\lambda+1)\eta}{2} \exp\frac{(\lambda+1)^2 S}{4} d\eta dS. \end{aligned} \tag{2.22}$$

For the second term in the equation above, let $\frac{\lambda+1}{2}=\beta$. We have:

$$
\begin{aligned}
&-\frac{m\lambda}{r}\int_0^{\tau}\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi(\tau-S)}}\exp(\frac{-(x-\eta)^2}{4\tau-S})\exp(\frac{(\lambda+1)\eta}{2})\exp(\frac{(\lambda+1)^2S}{4})d\eta dS\\
=&-\frac{m\lambda}{r}\int_0^{\tau}\frac{e^{\beta^2S}}{2\sqrt{\pi(\tau-S)}}\int_{-\infty}^{\infty}\exp(\frac{-(x-\eta)^2}{4\tau-S})\exp(\beta\eta)d\eta dS\\
=&-\frac{m\lambda}{r}\int_0^{\tau}\frac{e^{\beta^2S}}{2\sqrt{\pi(\tau-S)}}\exp(-\frac{x^2-(x+2(\tau-S)\beta)^2}{4(\tau-S)})\int_{-\infty}^{\infty}\exp(-(\frac{\eta-(x+2(\tau-S)\beta)}{2\sqrt{\tau-S}})^2)d\eta dS
\end{aligned}
\tag{2.23}
$$

With substitution $\frac{\eta-(x+2(\tau-S)\beta)}{2\sqrt{\tau-S}}=y$, we have:

$$
\begin{aligned}
&-\frac{m\lambda}{r}\int_0^{\tau}\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi(\tau-S)}}\exp(\frac{-(x-\eta)^2}{4\tau-S})\exp(\frac{(\lambda+1)\eta}{2})\exp(\frac{(\lambda+1)^2S}{4})d\eta dS\\
=&-\frac{m\lambda}{r}\int_0^{\tau}\frac{e^{\beta^2S}}{2\sqrt{\pi(\tau-S)}}\exp(-\frac{x^2-(x+2(\tau-S)\beta)^2}{4(\tau-S)})\int_{-\infty}^{\infty}e^{-y^2}2\sqrt{\tau-S}dydS\\
=&-\frac{m\lambda}{r}\int_0^{\tau}\frac{e^{\beta^2S}}{2\sqrt{\pi(\tau-S)}}\exp(-\frac{x^2-(x+2(\tau-S)\beta)^2}{4(\tau-S)})2\sqrt{\tau-S}\sqrt{\pi}dS\\
=&-\frac{m\lambda}{r}\int_0^{\tau}e^{\beta^2S}\exp(-\frac{x^2-(x+2(\tau-S)\beta)^2}{4(\tau-S)})dS\\
=&-\frac{m\lambda}{r}\int_0^{\tau}e^{\beta^2S}e^{\beta x}e^{(\tau-S)\beta^2}dS\\
=&-\frac{m\lambda}{r}e^{\beta x}e^{\tau\beta^2}\int_0^{\tau}1dS\\
=&-\frac{m\lambda}{r}\tau e^{\beta x+\beta^2\tau}
\end{aligned}
\tag{2.24}
$$

So equation (2.22) becomes:

$$
\begin{aligned}
W(x,\tau)=&\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\exp(\frac{-(x-\eta)^2}{4\tau})g(\eta)d\eta-\frac{m\lambda}{r}\tau e^{\beta x+\beta^2\tau}\\
=&\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\exp(\frac{-(x-\eta)^2}{4\tau})\exp(\frac{(\lambda-1)\eta}{2})(e^{\eta}-K)d\eta-\frac{m\lambda}{r}\tau e^{\beta x+\beta^2\tau}\\
=&\frac{1}{2\sqrt{\pi\tau}}\int_{\ln K}^{\infty}\exp(\frac{-(x-\eta)^2}{4\tau}+\frac{(\lambda-1)\eta}{2})(e^{\eta}-K)d\eta-\frac{m\lambda}{r}\tau e^{\beta x+\beta^2\tau}.
\end{aligned}
\tag{2.25}
$$

The first term of the integral can be evaluated by completing the square inside the exponential, producing:

$$\begin{aligned} W(x,\tau) = \frac{1}{2}\left[\exp(\frac{(\lambda+1)^2\tau}{4} + \frac{(\lambda+1)x}{2})\text{erfc}(\frac{\ln(K)-(\lambda+1)\tau-\eta}{2\sqrt{\tau}})\right] \\ -\frac{1}{2}\left[K\exp(\frac{(\lambda-1)^2\tau}{4} + \frac{(\lambda-1)x}{2})\text{erfc}(\frac{\ln K-(\lambda+1)\tau-x}{2\sqrt{\tau}})\right] - \frac{m\lambda}{r}\tau e^{\beta x+\beta^2\tau}, \end{aligned} \tag{2.26}$$

where the error function: $\text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-x^2}dz = 1-\text{erf}(x)$.

Since $W(\tau,x) = e^{ax+b\tau}u(\tau,x)$ and $C(t,S) = Ku(\tau,x)$,

$$u(\tau,x) = \exp(-\frac{(\lambda+1)^2\tau}{4} - \frac{(\lambda-1)x}{2})W(\tau,x) \tag{2.27}$$

Finally we have the solution of (2.10) to be

$$\begin{aligned} C_2(t,S) =& \frac{1}{2}\Big[S\cdot\text{erfc}(-\frac{(r+\frac{\sigma^2}{2})(T-t)+\ln(\frac{S}{K})}{\sqrt{2\sigma^2(T-t)}}) \\ & -Ke^{-r(T-t)}\text{erfc}(-\frac{(r-\frac{\sigma^2}{2})(T-t)+\ln(\frac{S}{K})}{\sqrt{2\sigma^2(T-t)}})\Big] - \frac{\kappa}{\Delta t K}(T-t). \end{aligned} \tag{2.28}$$

Since the solution of the modified equation with diminishing marginal transaction cost is bounded by both (2.10) which we already solved, and (2.11) which is the original Black-Scholes Equation. We have the solution of (2.11) to be

$$\begin{aligned} C_1(t,S) =& \frac{1}{2}\Big[S\cdot\text{erfc}(-\frac{(r+\frac{\sigma^2}{2})(T-t)+\ln(\frac{S}{K})}{\sqrt{2\sigma^2(T-t)}}) \\ & -Ke^{-r(T-t)}\text{erfc}(-\frac{(r-\frac{\sigma^2}{2})(T-t)+\ln(\frac{S}{K})}{\sqrt{2\sigma^2(T-t)}})\Big]. \end{aligned} \tag{2.29}$$

Finally we conclude that although we cannot explicitly solve (2.10), but the solution $C(t,S)$ of (2.10) satisfies

$$C_2(t,S) \le C(t,S) \le C_1(t,S),$$

where $C_1(t,S)$ and $C_2(t,S)$ are given in (2.29) and (2.28) respectively.

To better visualize the value of the call option with a diminishing marginal transaction cost, numerical analysis is conducted. A graph of the solutions of the two bounding equations is shown in Figure 2.1.

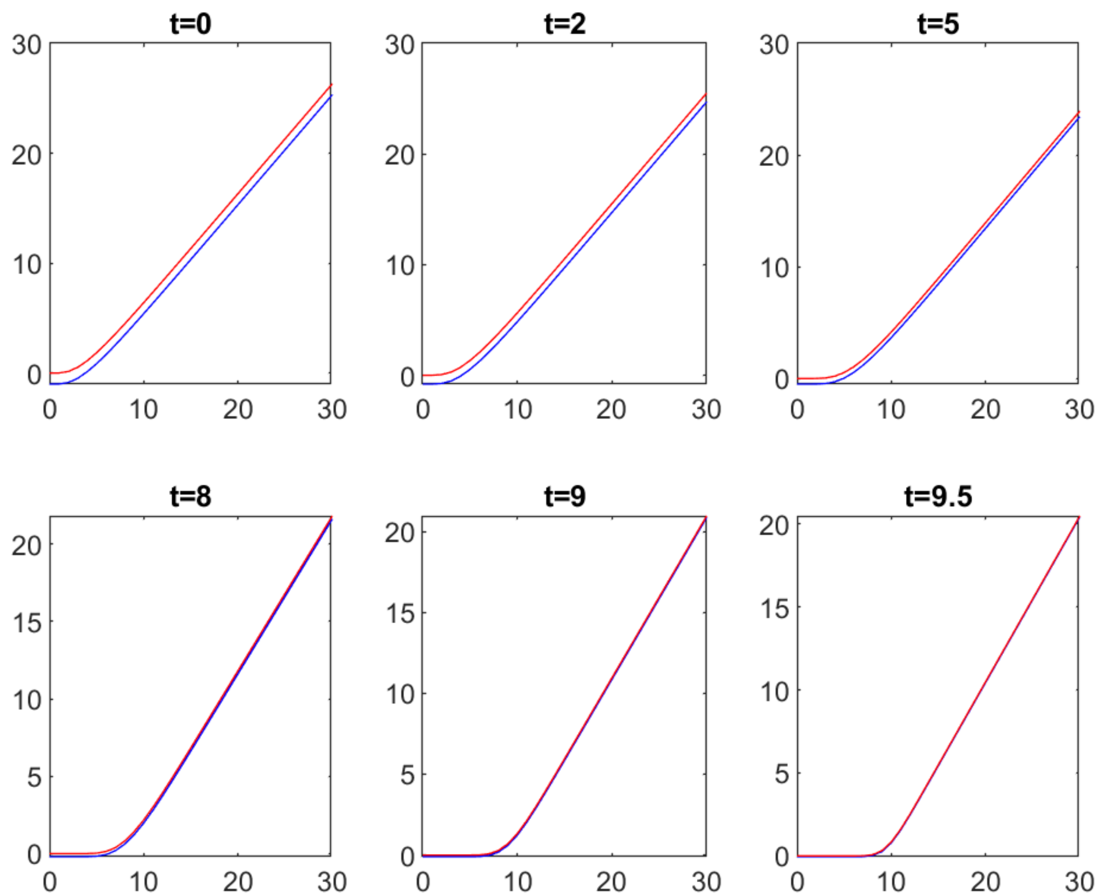

Figure 2.1: Modified Call option values. Red curve is $C_1(t,S)$ and the blue one is $C_2(t,S)$. Here $T=10,\ K=10,\ r=0.1, \sigma=0.2, \kappa=0.2, \Delta t=0.2$.

From Figure 2.1, as the time approaches to the expiration date, the two boundary functions have more similar trends, which implies the value of the call option is more well-defined with less range of fluctuation. While there is a general increasing trend, as time increases, there is one turning point where the slope of the functions changes: at 10, the strike price. Specifically, as time is closer to the expiration time, when the price is higher than 10, the value of the option increases more rapidly.

Since the call option gives the investor to buy the asset during the time before expiration, the higher the spot price of the option, the higher the profit when exercised. Thus, from Figure 2.1, investor can start to exercise the option when the price is larger than the strike price. What's more, the modified model implies that value of the call option in the real market is normally smaller than the value concluded from the original Black-Scholes model (given by red curve), whereas greater than the value concluded from the modified Model when time approaches infinity (given by the blue curve). So, the value of the option might reach the turning point earlier with a value smaller than 10, the strike price.

2.4 Case 2: European Put Option

In this section, we consider the case of the put option. Similar to the previous case, the nonlinear equation for the put option is:

$$\begin{aligned}&P_t(t,S)\Delta t+\frac{1}{2}\sigma^2S^2P_{SS}(t,S)\Delta t-\kappa S\frac{P_{SS}\sigma S\Phi\sqrt{\Delta t}}{P_{SS}\sigma S\Phi\sqrt{\Delta t}+c}\\&+rSP_S(t,S)\Delta t-rP(t,S)\Delta t=0.\end{aligned}\tag{2.30}$$

With same transformation, we arrive at equation (2.10). Now, we perform the substitutions on the boundary conditions with a put option. We obtain:

$$\begin{cases}W_\tau=W_{xx}-\frac{m}{r}(2a+1)e^{x(a+1)+b\tau},\\W(0,x)=max(e^{\frac{x(\lambda-1)}{2}}K-e^{\frac{x(\lambda+1)}{2}},0)=e^{\frac{(\lambda-1)x}{2}}max(K-e^x,0),\\W(x,\tau)\to 0,\quad \tau\in(0,\frac{\sigma^2}{2}T).\end{cases}\tag{2.31}$$

We thus arrive at

$$
\begin{aligned}
W(x,\tau) =& \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\exp(\frac{-(x-\eta)^2}{4\tau})g(\eta)d\eta - \frac{m\lambda}{r}\tau e^{\beta x+\beta^2\tau} \\
=& \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\exp(\frac{-(x-\eta)^2}{4\tau})\exp(\frac{(\lambda-1)\eta}{2})(K-e^{\eta})d\eta - \frac{m\lambda}{r}\tau e^{\beta x+\beta^2\tau} \\
=& \frac{1}{2\sqrt{\pi\tau}}\int_{0}^{\ln K}\exp(\frac{-(x-\eta)^2}{4\tau}+\frac{(\lambda-1)\eta}{2})(K-e^{\eta})d\eta - \frac{m\lambda}{r}\tau e^{\beta x+\beta^2\tau}
\end{aligned} \tag{2.32}
$$

By completing the square inside the exponential, we have

$$
\begin{aligned}
W(x,\tau) &= \frac{1}{2}\left[\exp(\frac{(\lambda+1)^2\tau}{4}+\frac{(\lambda+1)x}{2})\text{erfc}(\frac{-\ln(K)+(\lambda+1)\tau+x}{2\sqrt{\tau}})\right] \\
-\frac{1}{2}&\left[K\exp(\frac{(\lambda-1)^2\tau}{4}+\frac{(\lambda-1)x}{2})\text{erfc}(\frac{-\ln K+(\lambda+1)\tau+x}{2\sqrt{\tau}})\right] - \frac{m\lambda}{r}\tau e^{\beta x+\beta^2\tau}.
\end{aligned} \tag{2.33}
$$

By solving the equation, we have

$$
\begin{aligned}
P_2(t,S) &= \frac{1}{2}Ke^{-r(T-t)}\text{erfc}(\frac{(r-\frac{\sigma^2}{2})(T-t)+\ln(\frac{S}{K})}{\sqrt{2\sigma^2(T-t)}}) \\
-\frac{S}{2}&\cdot\text{erfc}(\frac{(r+\frac{\sigma^2}{2})(T-t)+\ln(\frac{S}{K})}{\sqrt{2\sigma^2(T-t)}})) - \frac{\kappa}{\Delta tK}(T-t),
\end{aligned} \tag{2.34}
$$

whereas the value of the put option from the original Black-Scholes Model is

$$
\begin{aligned}
P_1(t,S) &= \frac{1}{2}(Ke^{-r(T-t)}\text{erfc}(\frac{(r-\frac{\sigma^2}{2})(T-t)+\ln(\frac{S}{K})}{\sqrt{2\sigma^2(T-t)}}) \\
-\frac{S}{2}&\cdot\text{erfc}(\frac{(r+\frac{\sigma^2}{2})(T-t)+\ln(\frac{S}{K})}{\sqrt{2\sigma^2(T-t)}}))
\end{aligned} \tag{2.35}
$$

From the numerical analysis below, the range of the value of the put option from the modified model can be illustrated.

Similar to the trend of the call option, the value of the put option is also bounded by the two boundary functions of equation (2.34) and (2.35). While there is a general decreasing trend, as time increases, there is a turning point at 10 as illustrated by the original Black-Scholes model as an upper bound. Specifically, when the price is smaller than 10, the

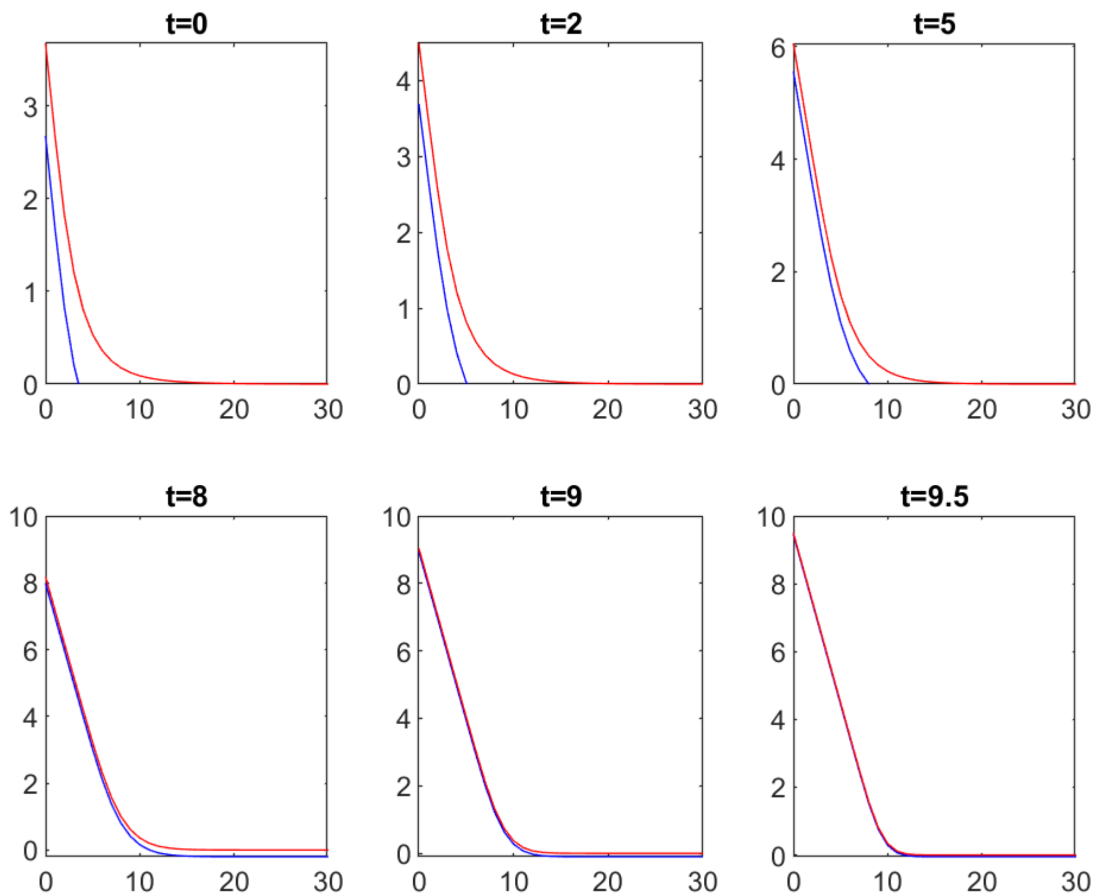

Figure 2.2: Modified Put option values. Red curve is $P_1(t, S)$ and the blue one is $P_2(t, S)$. Here $T = 10,\ K = 10,\ r = 0.1, \sigma = 0.2, \kappa = 0.2, \Delta t = 0.2$.

value of the option decreases more rapidly. Though the lower bound provided by the blue line function follow the general same trend, because the investor will not exercise the option when the spot price is greater than the strike price, the minimum value of the option is 0 and the negative values are not plotted.

Since the put option gives the investor to sell the asset during the time before expiration, the higher the spot price of the option, the less the profit when exercised. Thus, from the graph, investor can exercise the option when the spot price is smaller than the strike price.

One thing to be noticed is that the modified model implies that value of the put option in the real market is normally smaller than the value concluded from the original Black-Scholes model (given by red curve), whereas greater than the value concluded from the modified model when the time approaches infinity (given by blue curve). In the beginning period, the original model states the value of the put option will nearly equal to 0 when the spot price is larger than 10. Nevertheless, from the modified model, the price of the put option will be close to zero more rapidly at some time before the spot price reaches 10.

2.5 Case 3: Binary Call Option

In this section, we consider the case of the binary call; option. Similar to the previous case, the nonlinear equation for this type of option is:

$$\begin{aligned} &BC_t(t,S)\Delta t + \frac{1}{2}\sigma^2 S^2 BC_{SS}(t,S)\Delta t - \kappa S \frac{BC_{SS}\sigma S\Phi\sqrt{\Delta t}}{BC_{SS}\sigma S\Phi\sqrt{\Delta t} + c} \\ &+ rSBC_S(t,S)\Delta t - rBC(t,S)\Delta t = 0. \end{aligned} \tag{2.36}$$

With same transformation, since a cash-or-nothing call option pays out one unit of cash if the spot is above the strike at maturity, we perform the substitutions on the boundary conditions with a Heaviside function. We obtain:

$$\begin{cases} W_\tau = W_{xx} - \frac{m}{r}(2a+1)e^{x(a+1)+b\tau} \\ W(0,x) = \exp(\frac{(\lambda-1)x}{2}), S > K; 0, S < K, \\ W(x,\tau) \to 0, \quad \tau \in (0, \frac{\sigma^2}{2}T). \end{cases} \tag{2.37}$$

Thus

$$\begin{aligned} W(x,\tau) =& \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \exp(\frac{-(x-\eta)^2}{4\tau}) g(\eta) d\eta - \frac{m\lambda}{r}\tau e^{\beta x + \beta^2 \tau} \\ =& \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \exp(\frac{-(x-\eta)^2}{4\tau}) \exp(\frac{(\lambda-1)\eta}{2}) d\eta - \frac{m\lambda}{r}\tau e^{\beta x + \beta^2 \tau} \\ =& \frac{1}{2\sqrt{\pi\tau}} \int_{\ln K}^{\infty} \exp(\frac{-(x-\eta)^2}{4\tau} + \frac{(\lambda-1)\eta}{2}) d\eta - \frac{m\lambda}{r}\tau e^{\beta x + \beta^2 \tau} \\ =& \frac{1}{2}\left[\exp(\frac{(\lambda-1)^2\tau}{4} + \frac{(\lambda-1)x}{2}) \text{erfc}(\frac{\ln K - (\lambda+1)\tau - x}{2\sqrt{\tau}})\right]. \end{aligned} \tag{2.38}$$

So the price of the binary call option is between the original Black-Scholes given in the first section:

$$BC_1(t,S) = N(d_2)e^{-r(T-t)},$$

and the function:

$$BC_2(t,S) = \frac{1}{2}\left(e^{-r(T-t)}\text{erfc}(-\frac{(r-\frac{\sigma^2}{2})(T-t) + \ln(\frac{S}{K})}{\sqrt{2\sigma^2(T-t)}}\right) - \frac{\kappa}{\Delta t K}(T-t), \tag{2.39}$$

From Figure 2.3, as the time approaches to the expiration date, the two boundary functions have more similar trends, which implies the value of the binary call option are more well defined with less range of fluctuation. While there is a general increasing trend, as time increases, there are two turning point where the slope of the functions changes: one is at 10, the strike price, and the other is somewhere around 15. Specifically, as spot price increases, at any time, the value of the option first increases with a rapid speed then gradually slows down.

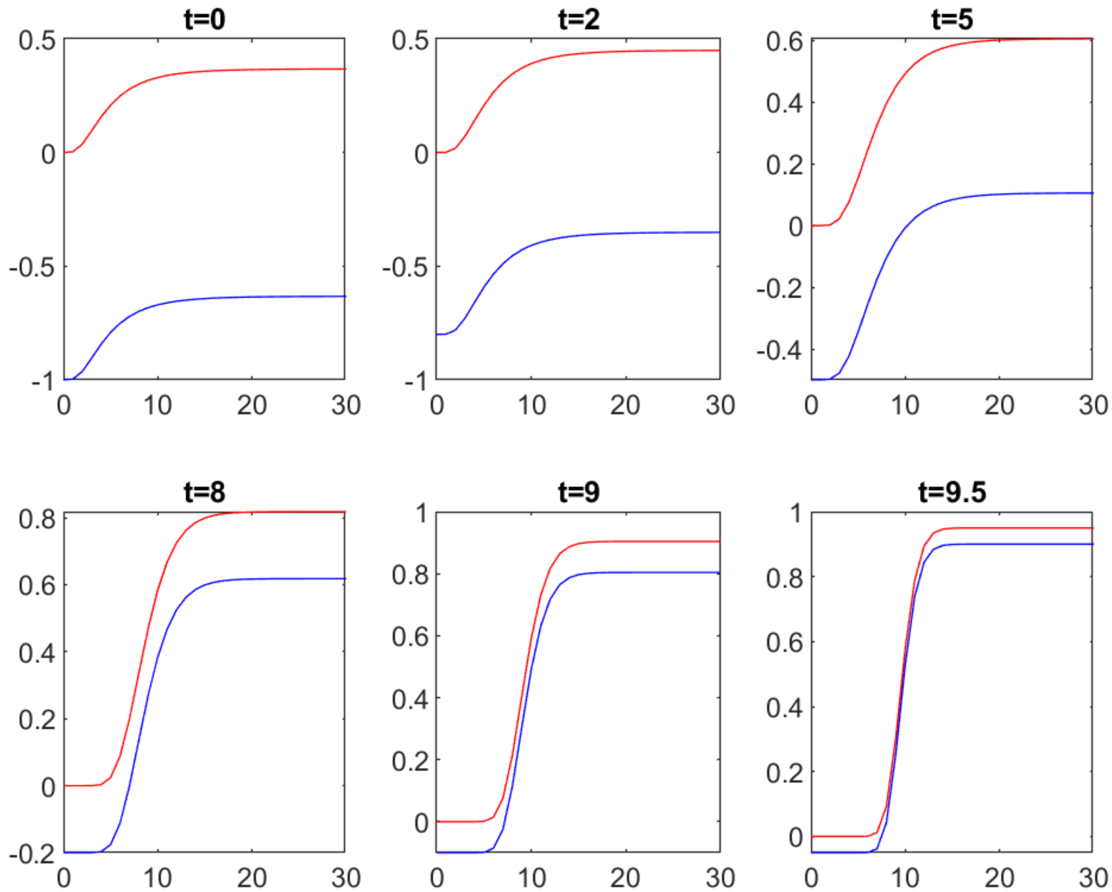

Figure 2.3: Modified Binary Call option values. Red curve is $BC_1(t, S)$ and the blue one is $BC_2(t, S)$. Here $T = 10,\ K = 10,\ r = 0.1, \sigma = 0.2, \kappa = 0.2, \Delta t = 0.2$.

Since the binary call option pays out one unit of cash if the spot is above the strike at maturity, from the graph, investor can maximize their profit if the spot price is greater than 10 at the exercise date.

Moreover, like other options, the modified model also implies that value of the binary call option in the real market is normally smaller than the value concluded from the original Black-Scholes model(given by red line), whereas greater than the value concluded from the modified Model when time approaches infinity (given by the blue line).Thus, while the original Black-Scholes model indicating at turning point at 10, the modified model states that this turning point may be earlier.

Nevertheless, there is limitation for this modified model. In the beginning period, the solution is bounded by a wider range of functions. The lower bound of the function has all negative values which does not correspond with the real-world situation where investors will choose to not exercise the option with a minimum profit and value of 0. However, since the European option can only be exercise at the expiration date, and as time closer to the time, boundary functions have smaller range with all positive values, the value of the option will be not significantly influenced.

2.6 Case 4: Binary Put Option

In this section, we consider the case of the binary put option. Similar to the previous case, the nonlinear equation for this type of option is:

$$\begin{aligned} &BP_t(t,S)\Delta t + \frac{1}{2}\sigma^2 S^2 BP_{SS}(t,S)\Delta t - \kappa S \frac{BP_{SS}\sigma S\Phi\sqrt{\Delta t}}{BP_{SS}\sigma S\Phi\sqrt{\Delta t} + c} \\ &+ rSBP_S(t,S)\Delta t - rBP(t,S)\Delta t = 0. \end{aligned} \tag{2.40}$$

With same transformation, since a cash-or-nothing put option pays out one unit of cash if the spot is below the strike at maturity, we perform the substitutions on the boundary conditions with a Heaviside function. We obtain:

$$\begin{cases} W_\tau = W_{xx} - \dfrac{m}{r}(2a+1)e^{x(a+1)+b\tau} \\ W(0,x) = \exp(\frac{(\lambda-1)x}{2}), S < K; 0, S > K, \\ W(x,\tau) \to 0, \quad \tau \in (0, \frac{\sigma^2}{2}T). \end{cases} \tag{2.41}$$

Thus

$$
\begin{aligned}
W(x,\tau) =& \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \exp(\frac{-(x-\eta)^2}{4\tau}) g(\eta) d\eta - \frac{m\lambda}{r}\tau e^{\beta x+\beta^2\tau} \\
=& \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \exp(\frac{-(x-\eta)^2}{4\tau}) \exp(\frac{(\lambda-1)\eta}{2}) d\eta - \frac{m\lambda}{r}\tau e^{\beta x+\beta^2\tau} \\
=& \frac{1}{2\sqrt{\pi\tau}} \int_{0}^{\ln K} \exp(\frac{-(x-\eta)^2}{4\tau} + \frac{(\lambda-1)\eta}{2}) d\eta - \frac{m\lambda}{r}\tau e^{\beta x+\beta^2\tau} \\
=& \frac{1}{2}\left[\exp(\frac{(\lambda-1)^2\tau}{4} + \frac{(\lambda-1)x}{2}) \operatorname{erfc}(\frac{-\ln K + (\lambda+1)\tau + x}{2\sqrt{\tau}})\right].
\end{aligned}
\tag{2.42}
$$

So, the price of the binary put option is between the original Black-Scholes equation given in the first section:

$$BP_1(t,S) = (1 - N(d_2))e^{-r(T-t)},$$

and the equation:

$$BP_2(t,S) = \frac{1}{2}\left(e^{-r(T-t)} \operatorname{erfc}(\frac{(r-\frac{\sigma^2}{2})(T-t) + \ln(\frac{S}{K})}{\sqrt{2\sigma^2(T-t)}}\right) - \frac{\kappa}{\Delta t K}(T-t). \tag{2.43}$$

From Figure 2.4, as the time approaches to the expiration date, the two boundary functions have more similar trends, which implies the value of the binary put option are more well defined with less range of fluctuation. While there is a general increasing trend, as time increases, there are two turning point where the slope of the functions changes: one is at 10, the strike price, and the other is somewhere around 15. Specifically, as spot price increases, at any time, the value of the option first decreases with a rapid speed then gradually slows down.

Since the binary put option pays out one unit of cash if the spot is below the strike at maturity, from the graph, investor can exercise their options if the spot price is approximately smaller than 10 at the exercise date.

What's more, like other options, the modified model also implies that value of the binary call option in the real market is normally smaller than the value concluded from the original Black-Scholes model (given by red curve), whereas greater than the value

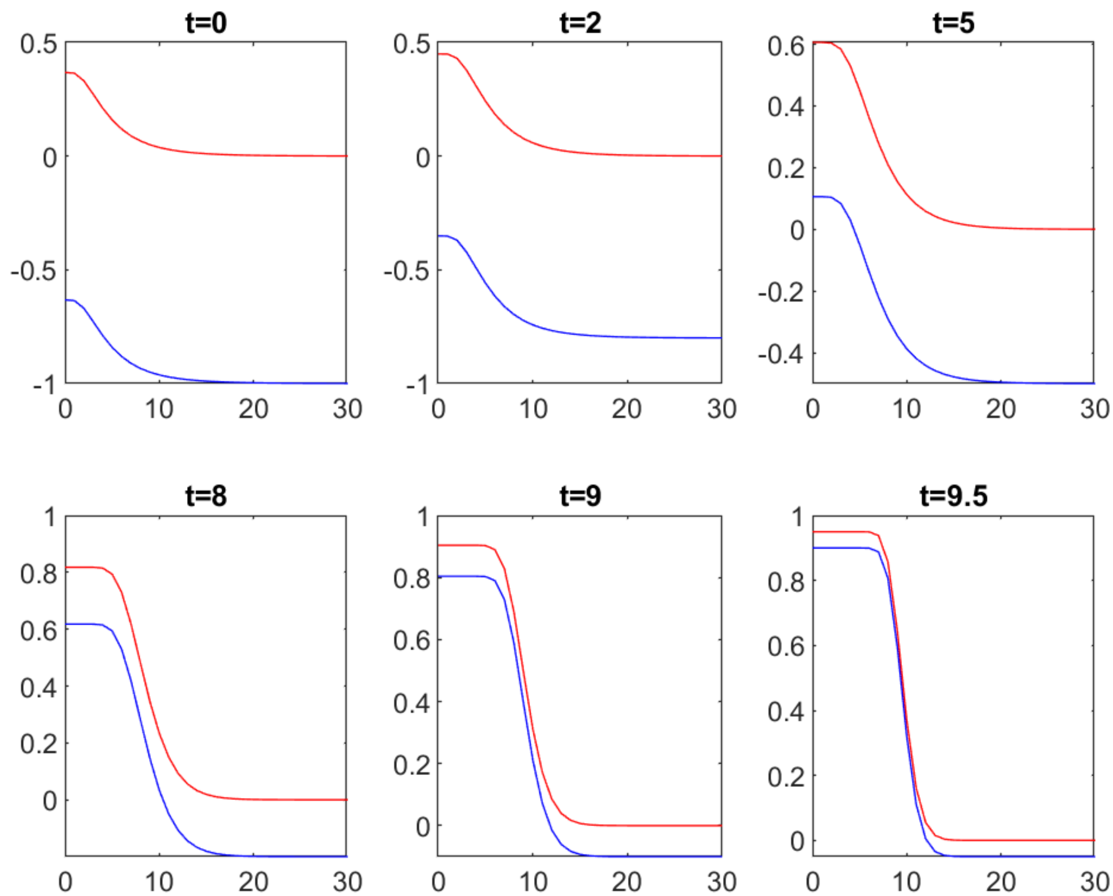

Figure 2.4: Modified Put option values. Red curve is $BP_1(t, S)$ and the blue one is $BP_2(t, S)$. Here $T = 10,\ K = 10,\ r = 0.1, \sigma = 0.2, \kappa = 0.2, \Delta t = 0.2$.

concluded from the modified Model when time approaches infinity (given by the blue line). Thus, while the original Black-Scholes model indicating a turning point at 10, the modified model states that this turning point may be earlier.

Nevertheless, similar to the case of binary call option, there is limitation for this modified model. In the beginning period, the solution is bounded by a wider range of functions. The lower bound of the function has all negative values which does not correspond with the real-world situation where investors will choose to not exercise the option with a minimum profit and value of 0. However, since the European option can only be exercise at the expiration date, and as time closer to the time, boundary functions have smaller range with all positive values, the value of the option will be not significantly influenced.

Chapter 3

Conclusion

We aim to construct an improved Black-Scholes Model to better study the pricing of European options. Modeling such outcomes is vital for not only financial corporations but also for individuals to better understand the pricing of financial assets and make more efficient decisions in future investment. With a more accurate description of the trend of the options in the market, it may provide more contribution to companies' asset management, derivatives' sales and trading, and new portfolio development.

Our modification is based on a well-known Hoggard-Whalley Model which assumes the transaction cost proportionally increases with the amount of trading assets. Nevertheless, we proposed that, in the real market, the more underlying assets that are traded, the less the transaction cost for an extra unit of the asset. We propose a function with decreasing marginals as the presentation of the transaction cost and add it into the original Black-Scholes Model for consideration.

Specifically, we classified the model into four different cases: call option, put option, binary call option, and binary put option. Here the binary options are cash-or-nothing binary options. Since each of them has a different pricing structure, they have different boundary conditions with the corresponding nonlinear partial differential equations. Because of the complexity of the model, it is extremely hard or even impossible to analytically solve for the solution of the system, so we have instead analyzed its solution based on the solution of its two boundary-condition functions: one for time approaches infinity, the other is as the time approaches zero which is the original Black-Scholes Model. We then perform some numerical simulations with Matlab.

We found that, as the time approaches to the expiration date, the two boundary functions have more similar trends, which implies the value of the option from the modified model is more well-defined with less range of fluctuation. For all four scenarios, the modified model implies that the value of the put option in the real market is normally smaller than the value concluded from the original Black-Scholes model, whereas greater than the value concluded from the modified model when the time approaches infinity. One thing to be noticed is that, while the original Black-Scholes Model indicates a turning point at the strike price when the behavior of the pricing function starts to change, since the value of the modified model is generally smaller than the original model, such turning point may happen to be earlier than the strike price.

There are some limitations to our model. The major one is that the lower bound for the binary options are negatives values, which makes the range of the modified solution is not as effective as the case for the call and the put options. Furthermore, we did not fit the model with actual data. It is desirable to find some real-world datasets for each case in the future so that we can estimate the values of each parameter. Other future works may include the consideration of dividends, volatility with the transaction cost for more accurate studies.

Chapter 4

Acknowledgements

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