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Stability and Hopf bifurcation in a diffusive logistic population model with nonlocal delay effect ☆

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ABSTRACT

A reaction–diffusion model with logistic type growth, nonlocal delay effect and Dirichlet boundary condition is considered, and combined effect of the time delay and nonlocal spatial dispersal provides a more realistic way of modeling the complex spatiotemporal behavior. The stability of the positive spatially nonhomogeneous positive equilibrium and associated Hopf bifurcation are investigated for the case of near equilibrium bifurcation point and the case of spatially homogeneous dispersal kernel.

1. Introduction

Partial functional-differential equations have been proposed as mathematical models for various biological phenomena by many researchers in recent years. And the theory of partial functional-differential equations and related bifurcation theory have been developed to analyze various mathematical questions arisen from models of population biology, biochemical reactions, neural conduction and other applications [8,22,27,33,40].

For the models with a single population, the global stability and the Hopf bifurcation of the diffusive Nicholson's blowflies equation have been investigated by many researchers (see Refs. [34,35,37,42,44]). Another prototypical delayed reaction–diffusion equation is the diffusive Hutchinson equation (or diffusive logistic equation with delay effect) following the pioneering work of Hutchinson [25]. For the Neumann boundary value problem, the diffusive Hutchinson equation has been considered in [29,43], and they considered the stability and related Hopf bifurcation from the homogeneous

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equilibrium. Busenberg and Huang [4] studied the Hopf bifurcation of the diffusive logistic equation with delay effect and Dirichlet boundary condition proposed in Green and Stech [21]:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + \lambda u(x,t)\big(1 - u(x,t-\tau)\big), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.1}$$

For a one-dimensional spatial domain $\Omega = (0,\pi)$, they showed that when $\lambda > d$ but close to d, the unique spatially nonhomogeneous positive equilibrium loses the stability for a large delay τ and a Hopf bifurcation occurs so that the system exhibits oscillatory pattern. Su, Wei and Shi [36] studied the Hopf bifurcation of a delayed reaction–diffusion population model with more general growth rate per capita, which generalized the work of [4], and see also Yan and Li [41] for the higher-dimensional case.

It has been pointed out by several authors that, in a reaction–diffusion model with time-delay effect, the effects of diffusion and time delays are not independent of each other, and the individuals which were at location x at previous times may not be at the same point in space presently. Hence the localized density-dependent growth rate per capita $1 - u(x,t-\tau)$ in (1.1) is not realistic. Instead, following the approach in [3,18–20], it is more reasonable to consider the diffusive logistic population model with nonlocal delay effect as follows:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + \lambda u(x,t)\left(1 - \displaystyle\int_\Omega K(x,y)u(y,t-\tau)\,dy\right), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.2}$$

where $u(x,t)$ is the population density at time t and location x, $d > 0$ is the diffusion coefficient, $\tau > 0$ is the time delay representing the maturation time, and $\lambda > 0$ is a scaling constant; Ω is a connected bounded open domain in $\mathbb{R}^n$ $(n \geqslant 1)$, with a smooth boundary $\partial\Omega$, and Dirichlet boundary condition is imposed so the exterior environment is hostile; $K(x,y)$ is a kernel function which describes the dispersal behavior of the population. The nonlocal growth rate per capita in (1.2) incorporates the possible dispersal of the individuals during the maturation period, hence it is a more realistic model than (1.1).

We consider Eq. (1.2) with the following initial condition:

$$u(x,s) = \eta(x,s), \quad x \in \Omega,\ t \in [-\tau, 0], \tag{1.3}$$

where $\eta \in C := C([-\tau,0],Y)$ and $Y = L^2(\Omega)$. Then from [24,30], the operator $d\Delta$ generates an analytic strongly positive semigroup $T(t)$ on Y with the domain $\mathscr{D}(d\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. Throughout the paper, we impose the following assumption on the dispersal kernel function $K(x,y)$:

(**A**) The kernel function $K(x,y)$ is a continuous and nonnegative function on $\overline{\Omega} \times \overline{\Omega}$, and the linear Fredholm integral operator

$$L\big(\phi(x)\big) := \int_\Omega K(x,y)\phi(y)\,dy$$

is strictly positive on $C_+(\overline{\Omega})$, which is the space of positive continuous functions, in the sense that

$$L\big(C_+(\overline{\Omega}) \setminus \{0\}\big) \subset C_+(\overline{\Omega}) \setminus \{0\}.$$

Define $F : C \to Y$ by

$$F(\phi)(x) = \lambda\phi(0)\left(1 - \int_{\Omega} K(x, y)\phi(-\tau)(y)\,dy\right), \tag{1.4}$$

then F is locally Lipschitz continuous. Therefore, from [24,40], for each $\phi \in C$, there exists a maximum $t_\phi > 0$ such that

$$\begin{cases} u(t) = T(t)\phi(0) + \displaystyle\int_0^t T(t-s)F(u_s)\,ds, & t > 0, \\ u(0) = \phi, \end{cases} \tag{1.5}$$

has a unique solution $u^\phi(t)$ which exists on $[-\tau, t_\phi)$ and $u^\phi(t)$ is a classical solution of (1.2) for $t > \tau$. This shows the local existence of the solutions to (1.2).

Define by λ_* the principal eigenvalue of the following eigenvalue problem

$$\begin{cases} -d\Delta u(x) = \lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

and let ϕ be the corresponding eigenfunction of λ_* such that $\phi(x) > 0$. Our main results in this paper is in the spirit of [4,36] for the local growth rate case: for a general bounded domain Ω, there exists a λ^* satisfying $0 < \lambda^* - \lambda_* \ll 1$, such that for any $\lambda \in (\lambda_*, \lambda^*]$, Eq. (1.2) has a positive spatially nonhomogeneous equilibrium solution u_λ and there exists a $\tau_0(\lambda) > 0$ such that u_λ is locally asymptotically stable when $\tau \in [0, \tau_0(\lambda))$ and it is unstable when $\tau > \tau_0(\lambda)$. Moreover, there exists a sequence of values $\{\tau_n(\lambda)\}_{n=0}^{\infty}$, such that for Eq. (1.2), a Hopf bifurcation occurs at $\tau = \tau_n(\lambda)$ from the positive equilibrium u_λ. On the other hand, for a special case that the kernel function $K(x, y) \equiv 1$ and $\Omega = (0, \pi)$ (for convenience), we show that the above stability/instability result and Hopf bifurcation can be proved for *any* $\lambda > \lambda_*$, not just when $0 < \lambda - \lambda_* \ll 1$. For the original diffusive Hutchinson equation (1.1), it was conjectured and showed by numerical simulation that such stability/instability result and Hopf bifurcation indeed occur for all $\lambda > \lambda_*$. Our result here for the case $K(x, y) \equiv 1$ further verifies this conjecture. We conjecture that such results hold for all kernels satisfying the assumption (**A**) and general bounded domain Ω.

It is known that nonzero equilibrium and periodic solutions in a Dirichlet boundary value problem are all spatially nonhomogeneous [21,24], hence it usually poses more difficulties for the stability and bifurcation analysis. Nonlocal delay effect brings some more technical hurdles as the resulting linearized equation is not self-adjoint when the delay $\tau \neq 0$, and some *a priori* estimates for the nonlocal equations are also considerably harder. Here we further develop the methods in [4,36,41] to overcome these difficulties. On the other hand, for the special case $K(x, y) \equiv 1$, we find that although the equilibrium and periodic solutions are both spatially nonhomogeneous, but they can be *explicitly* solved, which makes it possible to consider the stability and bifurcation for all $\lambda > \lambda_*$. Indeed in this case, we find that the periodic solutions could have a fixed spatial profile with a temporal oscillation (see the remark at the end of Section 3).

The traveling wave solutions for an equation in form of Eq. (1.2) have been considered in many papers, for example, [1,2,9,17,39] and the references therein. On the other hand, spatiotemporal pattern formation for the nonlocal Fisher–KPP type equation (again without delay effect) has been studied in [15,16,26,28]. It is shown in [5] that a Hopf bifurcation can occur for a reaction–diffusion equation with a nonlinear and nonlocal boundary condition. But there are very few stability/bifurcation results for the emergence of spatially nonhomogeneous time-periodic patterns for reaction–diffusion model with combined nonlocal and delay effect.

The rest of the paper is organized as follows. In Section 2, we study the stability/instability of the positive spatially nonhomogeneous equilibrium solution and the associated Hopf bifurcation of

Eq. (1.2) when $\lambda > \lambda_*$ and is close to λ_*. In Section 3, we consider the case of $K(x, y) \equiv 1$ and spatial dimension $n = 1$ but for any $\lambda > \lambda_*$; and we consider the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits in Section 4. Some numerical simulations are given at the end.

Throughout the paper, we denote the spaces $X = H^2(\Omega) \cap H_0^1(\Omega)$, $Y = L^2(\Omega)$, $C = C([-\tau, 0], Y)$, and $\mathcal{C} = C([-1, 0], Y)$. For any subspace Z of X, Y, C or $\mathcal{C}$, we also define the complexification of Z to be $Z_{\mathbb{C}} := Z \oplus iZ = \{x_1 + ix_2 \mid x_1, x_2 \in Z\}$. For a linear operator $L : Z_1 \to Z_2$, we denote the domain of L by $\mathscr{D}(L)$ and the range of L by $\mathscr{R}(L)$. For the complex-valued Hilbert space $Y_{\mathbb{C}}$, we use the standard inner product $\langle u, v\rangle = \int_\Omega \bar{u}(x)v(x)\,dx$.

2. Nonlocal equation with general kernel function

2.1. Existence of positive equilibrium

In this subsection we consider the existence of the positive equilibrium solutions of Eq. (1.2), which satisfy the following equation:

$$\begin{cases} d\Delta u(x) + \lambda u(x)\left(1 - \int_\Omega K(x, y)u(y)\,dy\right) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

It is well known that we have the following decompositions:

$$X = \mathscr{N}(d\Delta + \lambda_*) \oplus X_1,$$
$$Y = \mathscr{N}(d\Delta + \lambda_*) \oplus Y_1,$$

where

$$\mathscr{N}(d\Delta + \lambda_*) = \operatorname{span}\{\phi\},$$
$$X_1 = \left\{y \in X\colon \int_\Omega \phi(x)y(x)\,dx = 0\right\},$$

and

$$Y_1 = \mathscr{R}(d\Delta + \lambda_*) = \left\{y \in Y\colon \int_\Omega \phi(x)y(x)\,dx = 0\right\}.$$

Then we have the following result on the existence of the positive equilibrium solutions of Eq. (1.2).

Theorem 2.1. *There exist $\lambda^* > \lambda_*$ and a continuously differential mapping $\lambda \mapsto (\xi_\lambda, \alpha_\lambda)$ from $[\lambda_*, \lambda^*]$ to $X_1 \times \mathbb{R}^+$ so Eq. (1.2) has an equilibrium solution*

$$u_\lambda = \alpha_\lambda(\lambda - \lambda_*)\left[\phi + (\lambda - \lambda_*)\xi_\lambda\right], \quad \lambda \in \left[\lambda_*, \lambda^*\right], \tag{2.2}$$

where

$$\alpha_{\lambda_*} = \frac{\int_\Omega \phi^2(x)\,dx}{\lambda_* \int_\Omega \int_\Omega K(x, y)\phi^2(x)\phi(y)\,dy\,dx}, \tag{2.3}$$

and $\xi_{\lambda_*} \in X_1$ is the unique solution of the equation

$$(d\Delta + \lambda_*)\xi + \phi\left(1 - \lambda_*\alpha_{\lambda_*}\int_\Omega K(\cdot, y)\phi(y)\,dy\right) = 0. \tag{2.4}$$

Proof. Since the kernel function $K(x, y)$ satisfies the assumption (**A**), then

$$\int_\Omega\int_\Omega K(x, y)\phi^2(x)\phi(y)\,dy\,dx > 0, \tag{2.5}$$

and α_{λ_*} is well defined. Because $d\Delta + \lambda_*$ is bijective from X_1 to $\mathscr{R}(d\Delta + \lambda_*)$, we also have ξ_{λ_*} is well defined. Define $m: X_1 \times \mathbb{R} \times \mathbb{R} \to Y$ by

$$m(\xi, \alpha, \lambda) = (d\Delta + \lambda_*)\xi + \phi + (\lambda - \lambda_*)\xi - \lambda\big[\phi + (\lambda - \lambda_*)\xi\big]m_1(\xi, \alpha, \lambda),$$

where

$$m_1(\xi, \alpha, \lambda) = \alpha\int_\Omega K(\cdot, y)\big[\phi(y) + (\lambda - \lambda_*)\xi(y)\big]\,dy. \tag{2.6}$$

From Eqs. (2.3) and (2.4), we see that $m(\xi_{\lambda_*}, \alpha_{\lambda_*}, \lambda_*) = 0$, and

$$D_{(\xi,\alpha)}m(\xi_{\lambda_*}, \alpha_{\lambda_*}, \lambda_*)[\eta, \epsilon] = (d\Delta + \lambda_*)\eta - \lambda_*\epsilon\phi\int_\Omega K(\cdot, y)\phi(y)\,dy.$$

Here $D_{(\xi,\alpha)}m(\xi_{\lambda_*}, \alpha_{\lambda_*}, \lambda_*)[\eta, \epsilon]$ is the Fréchet derivative of m with respect to (ξ, α). From Eq. (2.5) we have

$$\phi\int_\Omega K(\cdot, y)\phi(y)\,dy \notin \mathscr{R}(d\Delta + \lambda_*).$$

So $D_{(\xi,\alpha)}m(\xi_{\lambda_*}, \alpha_{\lambda_*}, \lambda_*)$ is bijective from $X_1 \times \mathbb{R}$ to Y. Then from the implicit function theorem, there exist a $\lambda^* > \lambda_*$ and a continuously differentiable mapping $\lambda \mapsto (\xi_\lambda, \alpha_\lambda) \in X_1 \times \mathbb{R}^+$ such that

$$m(\xi_\lambda, \alpha_\lambda, \lambda) = 0, \quad \lambda \in \big[\lambda_*, \lambda^*\big].$$

Hence $\alpha_\lambda(\lambda - \lambda_*)[\phi + (\lambda - \lambda_*)\xi_\lambda] \in X$ solves Eq. (2.1). □

Remark 2.2. The existence of positive equilibrium solutions near $(\lambda, u) = (\lambda_*, 0)$ can also be obtained by using the "bifurcation from simple eigenvalue theorem" of Crandall and Rabinowitz [6]. Moreover one can apply the global bifurcation theorem in Rabinowitz [31] to show that the curve $\{(\lambda, u_\lambda): \lambda \in (\lambda_*, \lambda^*]\}$ belongs to a global continuum which is unbounded in $\mathbb{R} \times X^+$, where X^+ is the positive cone in X.

2.2. Eigenvalue problems

Let $\lambda \in (\lambda_*, \lambda^*]$, and let u_λ be the positive equilibrium solution of Eq. (1.2) obtained in Theorem 2.1. In the following, we will always assume $\lambda \in (\lambda_*, \lambda^*]$ unless otherwise specified, and $0 < \lambda^* - \lambda_* \ll 1$. But the value of λ^* may be chosen smaller than the one in Theorem 2.1 when further perturbation arguments are used. Linearizing system (1.2) at u_λ, we have

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} = d\Delta v(x,t) + \lambda v(x,t)\left(1 - \displaystyle\int_\Omega K(x,y)u_\lambda(y)\,dy\right) & \\ \qquad - \lambda u_\lambda(x) \displaystyle\int_\Omega K(x,y)v(y,t-\tau)\,dy, & x \in \Omega,\ t > 0, \\ v(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.7}$$

Define a linear operator $A(\lambda) : \mathscr{D}(A(\lambda)) \to Y$ by

$$A(\lambda) = d\Delta + \lambda\left(1 - \int_\Omega K(\cdot,y)u_\lambda(y)\,dy\right),$$

with domain $\mathscr{D}(A(\lambda)) = X$, then $A(\lambda)$ is the infinitesimal generator of a compact C_0-semigroup [30]. From [40], the semigroup induced by the solutions of Eq. (2.7) has the infinitesimal generator $A_\tau(\lambda)$ given by

$$A_\tau(\lambda)\psi = \dot{\psi}, \tag{2.8}$$

where

$$\mathscr{D}\big(A_\tau(\lambda)\big) = \left\{\psi \in C_{\mathbb{C}} \cap C^1_{\mathbb{C}}:\ \psi(0) \in X_{\mathbb{C}},\ \dot{\psi}(0) = A(\lambda)\psi(0) - \lambda u_\lambda \int_\Omega K(\cdot,y)\psi(-\tau)(y)\,dy\right\},$$

and $C^1_{\mathbb{C}} = C^1([-\tau,0], Y_{\mathbb{C}})$. The spectral set of $A_\tau(\lambda)$ is

$$\sigma\big(A_\tau(\lambda)\big) = \big\{\mu \in \mathbb{C}:\ \Delta(\lambda,\mu,\tau)\psi = 0,\ \text{for some } \psi \in X_{\mathbb{C}} \setminus \{0\}\big\},$$

where

$$\Delta(\lambda,\mu,\tau)\psi := A(\lambda)\psi - \lambda u_\lambda \int_\Omega K(\cdot,y)\psi(y)\,dy\, e^{-\mu\tau} - \mu\psi. \tag{2.9}$$

Then $A_\tau(\lambda)$ has a purely imaginary eigenvalue $\mu = i\nu$ ($\nu \neq 0$) for some $\tau \geqslant 0$ if and only if

$$A(\lambda)\psi - \lambda u_\lambda \int_\Omega K(\cdot,y)\psi(y)\,dy\, e^{-i\theta} - i\nu\psi = 0, \quad \psi(\neq 0) \in X_{\mathbb{C}}, \tag{2.10}$$

is solvable for some value of $\nu > 0$ and $\theta \in [0, 2\pi)$. So if there exists a pair (ν,θ) such that Eq. (2.10) has a solution ψ, then

$$\Delta(\lambda, i\nu, \tau_n)\psi = 0, \qquad \tau_n = \frac{\theta + 2n\pi}{\nu}, \quad n = 0,1,2,\cdots.$$

Next we shall show that, for $\lambda \in (\lambda_*, \lambda^*]$, there exists a unique pair (ν, θ) which solves Eq. (2.10). First we give two lemmas which will be used later.

Lemma 2.3. *If $z \in X_{\mathbb{C}}$ and $\langle \phi, z\rangle = 0$, then $|\langle (d\Delta + \lambda_*)z, z\rangle| \geqslant (\lambda_2 - \lambda_*)\|z\|^2_{Y_{\mathbb{C}}}$, where λ_2 is the second eigenvalue of (1.6).*

This is similar to Lemma 2.3 of [4] and we omit its proof here.

Lemma 2.4. *If (ν, θ, ψ) solves Eq. (2.10) with $\psi(\neq 0) \in X_{\mathbb{C}}$, then $\frac{\nu}{\lambda - \lambda_*}$ is bounded for $\lambda \in (\lambda_*, \lambda^*]$.*

Proof. We see that if (ν, θ, ψ) solves Eq. (2.10) with $\psi(\neq 0) \in X_{\mathbb{C}}$, then

$$\left\langle A(\lambda)\psi - \lambda u_\lambda \int_\Omega K(\cdot, y)\psi(y)\,dy\, e^{-i\theta} - i\nu\psi, \psi \right\rangle = 0, \tag{2.11}$$

and for some $\theta_1 \in [0, 2\pi)$,

$$\left\langle \lambda u_\lambda \int_\Omega K(\cdot, y)\psi(y)\,dy, \psi \right\rangle = \left|\left\langle \lambda u_\lambda \int_\Omega K(\cdot, y)\psi(y)\,dy, \psi \right\rangle\right| e^{i\theta_1}.$$

Since $A(\lambda)$ is self-adjoint, then separating the real and imaginary parts of Eq. (2.11), we have

$$\nu\langle \psi, \psi\rangle = -\lambda \sin(\theta + \theta_1)\left|\left\langle u_\lambda \int_\Omega K(\cdot, y)\psi(y)\,dy, \psi \right\rangle\right|.$$

Then

$$\begin{aligned} \frac{|\nu|}{\lambda - \lambda_*} &\leqslant \lambda\alpha_\lambda \left|\sin(\theta + \theta_1)\right| \left(\|\phi\|_\infty + (\lambda - \lambda_*)\|\xi_\lambda\|_\infty\right) \max_{\overline{\Omega}\times\overline{\Omega}} K(x, y)\|\psi\|^2_{L^1}/\|\psi\|^2_{Y_{\mathbb{C}}} \\ &\leqslant \lambda\alpha_\lambda \left(\|\phi\|_\infty + (\lambda - \lambda_*)\|\xi_\lambda\|_\infty\right) \max_{\overline{\Omega}\times\overline{\Omega}} K(x, y)|\Omega|. \end{aligned}$$

From the continuity of mapping $\lambda \mapsto (\|\xi_\lambda\|_\infty, \alpha_\lambda)$, we have the conclusion. $\square$

Now, for $\lambda \in (\lambda_*, \lambda^*]$, suppose that (ν, θ, ψ) is a solution of Eq. (2.10) with $\psi(\neq 0) \in X_{\mathbb{C}}$. Ignoring a scalar factor, we see that ψ can be represented as

$$\begin{aligned} &\psi = \beta\phi + (\lambda - \lambda_*)z, \quad \langle \phi, z\rangle = 0, \quad \beta \geqslant 0, \\ &\|\psi\|^2_{Y_{\mathbb{C}}} = \beta^2\|\phi\|^2_{Y_{\mathbb{C}}} + (\lambda - \lambda_*)^2\|z\|^2_{Y_{\mathbb{C}}} = \|\phi\|^2_{Y_{\mathbb{C}}}. \end{aligned} \tag{2.12}$$

Substituting (2.2), (2.12) and $\nu = (\lambda - \lambda_*)h$ into Eq. (2.10), we obtain the following system equivalent to Eq. (2.10):

$$\begin{aligned} g_1(z, \beta, h, \theta, \lambda) := {} & (d\Delta + \lambda_*)z + \left[\beta\phi + (\lambda - \lambda_*)z\right]\left(1 - \lambda m_1(\xi_\lambda, \alpha_\lambda, \lambda) - ih\right) \\ & - \lambda\alpha_\lambda\left[\phi + (\lambda - \lambda_*)\xi_\lambda\right] \int_\Omega K(\cdot, y)\left[\beta\phi(y) + (\lambda - \lambda_*)z(y)\right] dy\, e^{-i\theta}, \end{aligned}$$

$$g_2(z,\beta,\lambda) := \left(\beta^2 - 1\right)\|\phi\|_{Y_{\mathbb{C}}}^2 + (\lambda - \lambda_*)^2\|z\|_{Y_{\mathbb{C}}}^2, \tag{2.13}$$

where $m_1(\xi,\alpha,\lambda)$ is defined in (2.6). We define $G:(X_1)_{\mathbb{C}} \times \mathbb{R}^3 \times \mathbb{R} \to Y_{\mathbb{C}} \times \mathbb{R}$ by $G = (g_1, g_2)$ and note that

$$G(z_{\lambda_*}, \beta_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*}, \lambda_*) = 0,$$

where

$$z_{\lambda_*} = (1-i)\xi_{\lambda_*}, \qquad \beta_{\lambda_*} = 1, \qquad h_{\lambda_*} = 1, \qquad \theta_{\lambda_*} = \frac{\pi}{2}, \tag{2.14}$$

and ξ_{λ_*} is defined as in Theorem 2.1. Then we have the following result on the solvability of $G = 0$.

Theorem 2.5. *There exists a continuously differentiable mapping $\lambda \mapsto (z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda)$ from $[\lambda_*, \lambda^*]$ to $X_{\mathbb{C}} \times \mathbb{R}^3$ such that $G(z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda, \lambda) = 0$. Moreover, if $\lambda \in (\lambda_*, \lambda^*)$, and $(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda, \lambda)$ solves the equation $G = 0$ with $h^\lambda > 0$, and $\theta^\lambda \in [0, 2\pi)$, then $(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda) = (z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda)$.*

Proof. Let $T = (T_1, T_2) : (X_1)_{\mathbb{C}} \times \mathbb{R}^3 \mapsto Y_{\mathbb{C}} \times \mathbb{R}$ be defined by

$$T = D_{(z,\beta,h,\theta)} G(z_{\lambda_*}, \beta_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*}, \lambda_*).$$

Thus, we have

$$\begin{aligned} T_1(\chi,\kappa,\epsilon,\vartheta) &= (d\Delta + \lambda_*)\chi - i\epsilon\phi + \lambda_*\vartheta\alpha_{\lambda_*}\phi \int_\Omega K(\cdot,y)\phi(y)\,dy \\ &\quad + \kappa(1-i)\phi\left(1 - \lambda_*\alpha_{\lambda_*}\int_\Omega K(\cdot,y)\phi(y)\,dy\right), \\ T_2(\kappa) &= 2\kappa\|\phi\|_{Y_{\mathbb{C}}}^2. \end{aligned}$$

Since α_{λ_*} is defined as in Eq. (2.3), then T is bijective from $(X_1)_{\mathbb{C}} \times \mathbb{R}^3$ to $Y_{\mathbb{C}} \times \mathbb{R}$. Then from the implicit function theorem, we see that there exists a continuously differentiable mapping $\lambda \mapsto (z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda)$ from $[\lambda_*, \lambda^*]$ to $X_{\mathbb{C}} \times \mathbb{R}^3$ such that $G(z_\lambda, \beta_\lambda, h_\lambda, \theta_\lambda, \lambda) = 0$. Hence the existence is proved, and it remains to prove the uniqueness. From the implicit function theorem, we need to verify that if $G(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda, \lambda) = 0$, $h^\lambda > 0$ and $\theta^\lambda \in [0, 2\pi)$, then

$$\left(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda\right) \to (z_{\lambda_*}, \beta_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*}) = \left((1-i)\xi_{\lambda_*}, 1, 1, \frac{\pi}{2}\right)$$

as $\lambda \to \lambda_*$ in the norm of $X_{\mathbb{C}} \times \mathbb{R}^3$. From Lemma 2.4 and Eq. (2.13), we see that $\{h^\lambda\}, \{\beta^\lambda\}$ and $\{\theta^\lambda\}$ are bounded. From Lemma 2.3 and the first equation of Eq. (2.13), we have

$$\begin{aligned} \left\|z^\lambda\right\|_{Y_{\mathbb{C}}}^2 &\leqslant \frac{1}{\lambda_2 - \lambda_*}\left|\left\langle\left(1 - \lambda m_1(\alpha_\lambda, \xi_\lambda, \lambda) - ih^\lambda\right)\left[\beta^\lambda\phi + (\lambda - \lambda_*)z^\lambda\right], z^\lambda\right\rangle\right| \\ &\quad + \left|\left\langle\lambda\alpha_\lambda\left[\phi + (\lambda - \lambda_*)\xi_\lambda\right]\int_\Omega K(x,y)\left[\beta^\lambda\phi + (\lambda - \lambda_*)z^\lambda\right], z^\lambda\right\rangle\right|. \end{aligned}$$

The boundedness of $\{h^\lambda\}$, $\{\alpha_\lambda\}$ and $\{\xi_\lambda\}$ implies that there exists $M > 0$ such that $\|1-\lambda m_1(\alpha_\lambda,\xi_\lambda,\lambda)-ih^\lambda\|_\infty \leqslant M$, and

$$\big\|\lambda\alpha_\lambda\big[\phi+(\lambda-\lambda_*)\xi_\lambda\big]\big\|_\infty \max_{\overline{\Omega}\times\overline{\Omega}} K(x,y) \leqslant M$$

for $\lambda \in [\lambda_*, \lambda^*]$. Thus we have

$$\|z^\lambda\|_{Y_{\mathbb{C}}}^2 \leqslant \frac{M|\beta^\lambda|}{\lambda_2-\lambda_*}\big(1+|\Omega|\big)\|\phi\|_{Y_{\mathbb{C}}}\|z^\lambda\|_{Y_{\mathbb{C}}} + \frac{1+|\Omega|}{\lambda_2-\lambda_*}M(\lambda-\lambda_*)\|z^\lambda\|_{Y_{\mathbb{C}}}^2.$$

Hence for sufficiently small λ^*, $\{z^\lambda\}$ is bounded in $Y_{\mathbb{C}}$ when $\lambda \in [\lambda_*, \lambda^*]$. Since the operator $d\Delta+\lambda_*: (X_1)_{\mathbb{C}} \mapsto (Y_1)_{\mathbb{C}}$ has a bounded inverse, by applying $(d\Delta+\lambda_*)^{-1}$ on $g_1(z^\lambda,\beta^\lambda,h^\lambda,\theta^\lambda,\lambda)=0$, we find that $\{z^\lambda\}$ is also bounded in $X_{\mathbb{C}}$, and hence $\{(z^\lambda,\beta^\lambda,h^\lambda,\theta^\lambda)\colon \lambda\in(\lambda_*,\lambda^*]\}$ is precompact in $Y_{\mathbb{C}}\times\mathbb{R}^3$. Therefore, there is a subsequence $\{(z^{\lambda^n},\beta^{\lambda^n},h^{\lambda^n},\theta^{\lambda^n})\}$ such that

$$\big(z^{\lambda^n},\beta^{\lambda^n},h^{\lambda^n},\theta^{\lambda^n}\big) \to \big(z^{\lambda_*},\beta^{\lambda_*},h^{\lambda_*},\theta^{\lambda_*}\big), \quad \lambda^n \to \lambda_* \quad \text{as } n\to\infty.$$

By taking the limit of the equation $(d\Delta+\lambda_*)^{-1}G(z^{\lambda^n},\beta^{\lambda^n},h^{\lambda^n},\theta^{\lambda^n},\lambda^n)=0$ as $n\to\infty$, we have that $G(z^{\lambda_*},\beta^{\lambda_*},h^{\lambda_*},\theta^{\lambda_*},\lambda_*)=0$. Also, we can verify that

$$G(z,\beta,h,\theta,\lambda_*)=0$$

has a unique solution given by $(z,\beta,h,\theta)=(z_{\lambda_*},\beta_{\lambda_*},h_{\lambda_*},\theta_{\lambda_*})$ defined in (2.14), thus

$$\big(z^{\lambda_*},\beta^{\lambda_*},h^{\lambda_*},\theta^{\lambda_*}\big)=(z_{\lambda_*},\beta_{\lambda_*},h_{\lambda_*},\theta_{\lambda_*}).$$

Hence, $(z^\lambda,\beta^\lambda,h^\lambda,\theta^\lambda)\to(z_{\lambda_*},\beta_{\lambda_*},h_{\lambda_*},\theta_{\lambda_*})$ as $\lambda\to\lambda_*$ in the norm of $X_{\mathbb{C}}\times\mathbb{R}^3$. □

Hence we have the following conclusion about the eigenvalue problem:

Corollary 2.6. *For each $\lambda\in(\lambda_*,\lambda^*]$, the eigenvalue problem*

$$\Delta(\lambda,i\nu,\tau)\psi=0, \quad \nu\geqslant 0,\ \tau\geqslant 0,\ \psi(\neq 0)\in X_{\mathbb{C}},$$

has a solution, or equivalently, $i\nu\in\sigma(A_\tau(\lambda))$ if and only if

$$\nu=\nu_\lambda=(\lambda-\lambda_*)h_\lambda, \qquad \tau=\tau_n=\frac{\theta_\lambda+2n\pi}{\nu_\lambda}, \quad n=0,1,2,\cdots, \tag{2.15}$$

and

$$\psi=r\psi_\lambda, \quad \psi_\lambda=\beta_\lambda\phi+(\lambda-\lambda_*)z_\lambda,$$

where r is a nonzero constant, and $z_\lambda,\beta_\lambda,h_\lambda,\theta_\lambda$ are defined as in Theorem 2.5.

For later application, it is also useful to consider the adjoint operator of $A_\tau(\lambda)$. Since the domain of $\Delta(\lambda,i\nu,\tau)$ is $X_{\mathbb{C}}$, which is dense in $Y_{\mathbb{C}}$, and for $\psi,\tilde\psi\in X_{\mathbb{C}}$,

$$\big\langle\tilde\psi,\Delta(\lambda,i\nu,\tau)\psi\big\rangle=\big\langle\tilde\Delta(\lambda,i\nu,\tau)\tilde\psi,\psi\big\rangle, \tag{2.16}$$

where

$$\tilde{\Delta}(\lambda, i\nu, \tau)\tilde{\psi} = A(\lambda)\tilde{\psi} + i\nu\tilde{\psi} - \lambda \int_{\Omega} K(y,\cdot)u_\lambda(y)\tilde{\psi}(y)\,dy\,e^{i\nu\tau}.$$

Then from [30], we have that $\tilde{\Delta}(\lambda, i\nu, \tau)$ is the adjoint operator of $\Delta(\lambda, i\nu, \tau)$, and its point spectrum is the same as that of $\Delta(\lambda, i\nu, \tau)$:

$$\sigma_p\big(\Delta(\lambda, i\nu, \tau)\big) = \sigma_p\big(\tilde{\Delta}(\lambda, i\nu, \tau)\big).$$

Similar to the study of Eq. (2.10), we can conclude that if the corresponding adjoint equation

$$A(\lambda)\tilde{\psi} - \lambda \int_{\Omega} K(y,\cdot)u_\lambda(y)\tilde{\psi}(y)\,dy\,e^{i\tilde{\theta}} + i\tilde{\nu}\tilde{\psi} = 0, \quad \tilde{\psi}(\neq 0) \in X_{\mathbb{C}}, \tag{2.17}$$

is solvable for some value of $\tilde{\nu} > 0$, $\tilde{\theta} \in [0, 2\pi)$, then

$$\tilde{\Delta}(\lambda, i\tilde{\nu}, \tilde{\tau}_n)\tilde{\psi} = 0, \qquad \tilde{\tau}_n = \frac{\tilde{\theta} + 2n\pi}{\tilde{\nu}}, \quad n = 0, 1, 2, \cdots.$$

Similar to Theorem 2.5, we can show that, for $\lambda \in (\lambda_*, \lambda^*]$, there is a unique $(\tilde{\nu}, \tilde{\theta}, \tilde{\psi})$ which solves Eq. (2.17) with $\tilde{\psi}(\neq 0) \in X_{\mathbb{C}}$. Ignoring a scalar factor, we see that $\tilde{\psi}$ can be represented as

$$\begin{aligned}
&\tilde{\psi} = \tilde{\beta}\phi + (\lambda - \lambda_*)\tilde{z}, \quad \langle \tilde{\phi}, \tilde{z}\rangle = 0, \quad \tilde{\beta} \geqslant 0, \\
&\|\tilde{\psi}\|^2_{Y_{\mathbb{C}}} = \tilde{\beta}^2\|\phi\|^2_{Y_{\mathbb{C}}} + (\lambda - \lambda_*)^2\|\tilde{z}\|^2_{Y_{\mathbb{C}}} = \|\phi\|^2_{Y_{\mathbb{C}}}.
\end{aligned} \tag{2.18}$$

Substituting (2.18) and $\nu = (\lambda - \lambda_*)\tilde{h}$ into Eq. (2.17), we obtain the following system equivalent to Eq. (2.17):

$$\begin{aligned}
\tilde{g}_1(\tilde{z}, \tilde{\beta}, \tilde{h}, \tilde{\theta}, \lambda) := {} & (d\Delta + \lambda_*)\tilde{z} + \big[\tilde{\beta}\tilde{\phi} + (\lambda - \lambda_*)\tilde{z}\big]\big(1 - \lambda m_1(\xi_\lambda, \alpha_\lambda, \lambda) + i\tilde{h}\big) \\
& - \lambda\alpha_\lambda \int_{\Omega} K(y,\cdot)\big[\phi(y) + (\lambda - \lambda_*)\xi_\lambda(y)\big]\big[\tilde{\beta}\phi(y) + (\lambda - \lambda_*)\tilde{z}(y)\big]\,dy\,e^{i\tilde{\theta}},
\end{aligned}$$

$$\tilde{g}_2(\tilde{z}, \tilde{\beta}, \lambda) := \big(\tilde{\beta}^2 - 1\big)\|\phi\|^2_{Y_{\mathbb{C}}} + (\lambda - \lambda_*)^2\|\tilde{z}\|^2_{Y_{\mathbb{C}}},$$

where $m_1(\xi, \alpha, \lambda)$ is defined in Eq. (2.6). We define $\tilde{G} : (X_1)_{\mathbb{C}} \times \mathbb{R}^3 \times \mathbb{R} \to Y_{\mathbb{C}} \times \mathbb{R}$ by $\tilde{G} = (\tilde{g}_1, \tilde{g}_2)$ and note that

$$\tilde{G}(\tilde{z}_{\lambda_*}, \tilde{\beta}_{\lambda_*}, \tilde{h}_{\lambda_*}, \tilde{\theta}_{\lambda_*}, \lambda_*) = 0,$$

where

$$\tilde{\beta}_{\lambda_*} = 1, \qquad \tilde{h}_{\lambda_*} = 1, \qquad \tilde{\theta}_{\lambda_*} = \frac{\pi}{2}, \tag{2.19}$$

and $\tilde{z}_{\lambda_*} \in (X_1)_{\mathbb{C}}$ is the unique solution of equation

$$\begin{cases} (d\Delta + \lambda_*)z = -\phi\left[1 - \lambda_*\alpha_{\lambda_*}\displaystyle\int_\Omega K(\cdot, y)\phi(y)\,dy\right] - i\phi + i\lambda_*\alpha_{\lambda_*}\displaystyle\int_\Omega K(y, \cdot)\phi^2(y)\,dy, \\ \langle \phi, z\rangle = 0. \end{cases} \tag{2.20}$$

Since the right-hand side term of the first equation in Eq. (2.20) belongs to $(Y_1)_{\mathbb{C}}$, then the uniqueness of the solution of Eq. (2.20) can be easily obtained. Hence we have the following result similar to Theorem 2.5 and Corollary 2.6 (with a similar proof):

Theorem 2.7.

1. *There exists a continuously differentiable mapping $\lambda \mapsto (\tilde{z}_\lambda, \tilde{\beta}_\lambda, \tilde{h}_\lambda, \tilde{\theta}_\lambda)$ from $[\lambda_*, \lambda^*]$ to $X_{\mathbb{C}} \times \mathbb{R}^3$ such that $\tilde{G}(\tilde{z}_\lambda, \tilde{\beta}_\lambda, \tilde{h}_\lambda, \tilde{\theta}_\lambda, \lambda) = 0$. Moreover, if $\lambda \in (\lambda_*, \lambda^*)$, and $(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda, \lambda)$ solves the equation $\tilde{G} = 0$ with $h^\lambda > 0$, and $\theta^\lambda \in [0, 2\pi)$, then $(z^\lambda, \beta^\lambda, h^\lambda, \theta^\lambda) = (\tilde{z}_\lambda, \tilde{\beta}_\lambda, \tilde{h}_\lambda, \tilde{\theta}_\lambda)$.*
2. *For each $\lambda \in (\lambda_*, \lambda^*]$, the eigenvalue problem*

$$\tilde{\Delta}(\lambda, i\tilde{\nu}, \tilde{\tau})\tilde{\psi} = 0, \quad \tilde{\nu} \geqslant 0,\ \tilde{\tau} \geqslant 0,\ \tilde{\psi}(\neq 0) \in X_{\mathbb{C}},$$

has a solution, if and only if

$$\tilde{\nu} = \tilde{\nu}_\lambda = (\lambda - \lambda_*)\tilde{h}_\lambda, \qquad \tilde{\tau} = \tilde{\tau}_n = \frac{\tilde{\theta}_\lambda + 2n\pi}{\tilde{\nu}_\lambda}, \quad n = 0, 1, 2, \cdots, \tag{2.21}$$

and

$$\tilde{\psi} = r\tilde{\psi}_\lambda, \quad \tilde{\psi}_\lambda = \tilde{\beta}_\lambda\phi + (\lambda - \lambda_*)\tilde{z}_\lambda,$$

where r is a nonzero constant, and $\tilde{z}_\lambda, \tilde{\beta}_\lambda, \tilde{h}_\lambda, \tilde{\theta}_\lambda$ are defined as above.

Remark 2.8. For a fixed $\lambda \in (\lambda_*, \lambda^*)$, if $0 \in \sigma_p(\Delta(\lambda, i\nu_\lambda, \tau_n))$, then we have that $0 \in \sigma_p(\tilde{\Delta}(\lambda, i\nu_\lambda, \tau_n))$. From the uniqueness of $(h_\lambda, \theta_\lambda)$ and $(\tilde{h}_\lambda, \tilde{\theta}_\lambda)$ in Theorems 2.5 and 2.7, we must have that $h_\lambda = \tilde{h}_\lambda$ and $\theta_\lambda = \tilde{\theta}_\lambda$, and consequently $\nu_\lambda = \tilde{\nu}_\lambda$ and $\tilde{\tau}_n = \tau_n$. Therefore in the following we will use $(h_\lambda, \theta_\lambda, \nu_\lambda, \tau_n)$ only and not the ones with tilde. On the other hand, the corresponding eigenfunction and its components $(\beta_\lambda, z_\lambda, \psi_\lambda)$ of $\Delta(\lambda, i\nu_\lambda, \tau_n)$ are possibly different from the ones for the adjoint operator $\tilde{\Delta}(\lambda, i\nu_\lambda, \tau_n)$.

2.3. Stability and Hopf bifurcations

We first analyze the stability of the positive equilibrium u_λ of Eq. (1.2) when $\tau = 0$.

Proposition 2.9. *For each $\lambda \in (\lambda_*, \lambda^*]$, all the eigenvalues of $A_\tau(\lambda)$ have negative real parts when $\tau = 0$, and hence the positive equilibrium u_λ of Eq. (1.2) is locally asymptotically stable when $\tau = 0$.*

Proof. If the conclusion is not true, then there exists a sequence $\{\lambda^n\}_{n=1}^\infty$, such that $\lambda^n > \lambda_*$ for $n \geqslant 1$, $\lim_{n\to\infty} \lambda^n = \lambda_*$, and for $n \geqslant 1$, the corresponding eigenvalue problem

$$\begin{cases} A(\lambda^n)\psi - \lambda^n u_{\lambda^n}\displaystyle\int_\Omega K(x, y)\psi(y)\,dy = \mu\psi, & x \in \Omega, \\ \psi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.22}$$

has an eigenvalue μ_{λ^n} with nonnegative real part and the eigenfunction ψ_{λ^n} satisfying $\|\psi_{\lambda^n}\|_{Y_{\mathbb{C}}} = 1$. For each $n \geqslant 1$, we write ψ_{λ^n} as $\psi_{\lambda^n} = c_{\lambda^n} u_{\lambda^n} + \phi_{\lambda^n}$, where $c_{\lambda^n} \in \mathbb{C}$ and $c_{\lambda^n} = \langle u_{\lambda^n}, \psi_{\lambda^n}\rangle / \langle u_{\lambda^n}, u_{\lambda^n}\rangle$. Here u_{λ^n} is the positive solution of Eq. (1.2) when $\lambda = \lambda^n$ satisfying Eq. (2.2), and $\phi_{\lambda^n} \in X_{\mathbb{C}}$ satisfies $\langle \phi_{\lambda^n}, u_{\lambda^n}\rangle = 0$. If $\phi_{\lambda^n} \equiv 0$, then substituting $\psi_{\lambda^n} = c_{\lambda^n} u_{\lambda^n}$ and $\mu = \mu_{\lambda^n}$ into the first equation of Eq. (2.22), we have

$$-\mu_{\lambda^n} u_{\lambda^n} = \lambda^n u_{\lambda^n} \int_\Omega K(\cdot, y) u_{\lambda^n}(y)\,dy. \tag{2.23}$$

Since the kernel function K satisfies assumption (**A**), then we have a contradiction. Hence $\phi_{\lambda^n} \not\equiv 0$ for each $n \geqslant 1$. Since

$$\bigl\langle A\bigl(\lambda^n\bigr)\phi_{\lambda^n}, u_{\lambda^n}\bigr\rangle = \bigl\langle \phi_{\lambda^n}, A\bigl(\lambda^n\bigr)u_{\lambda^n}\bigr\rangle \quad \text{and} \quad A\bigl(\lambda^n\bigr)u_{\lambda^n} = 0,$$

multiplying by $\psi_{\lambda^n} = u_{\lambda^n} + \phi_{\lambda^n}$ the first equation of Eq. (2.22) when $\mu = \mu_{\lambda^n}$, we have that

$$\bigl\langle A\bigl(\lambda^n\bigr)\phi_{\lambda^n}, \phi_{\lambda^n}\bigr\rangle = \lambda^n \biggl\langle \psi_{\lambda^n}, u_{\lambda^n} \int_\Omega K(\cdot, y)\psi_{\lambda^n}(y)\,dy \biggr\rangle + \mu_{\lambda^n}. \tag{2.24}$$

Since u_{λ^n} is the principal eigenfunction of $A(\lambda^n)$ with principal eigenvalue 0, then $\langle A(\lambda^n)\phi_{\lambda^n}, \phi_{\lambda^n}\rangle < 0$, and consequently

$$0 \leqslant \mathcal{R}e(\mu_{\lambda^n}) \leqslant \mathcal{R}e\biggl[-\lambda^n \biggl\langle \psi_{\lambda^n}, u_{\lambda^n} \int_\Omega K(\cdot, y)\psi_{\lambda^n}(y)\,dy \biggr\rangle\biggr] \leqslant \sigma \|u_{\lambda^n}\|_\infty \max_{\overline{\Omega}\times\overline{\Omega}} K(x, y)|\Omega|,$$

where $\sigma = \max_n \lambda^n$. Hence $\lim_{n\to\infty} \mathcal{R}e(\mu_{\lambda^n}) = 0$. Similarly, we have that

$$\bigl|\mathcal{I}m(\mu_{\lambda^n})\bigr| = \biggl|\mathcal{I}m\biggl[-\lambda^n \biggl\langle \psi_{\lambda^n}, u_{\lambda^n} \int_\Omega K(\cdot, y)\psi_{\lambda^n}(y)\,dy \biggr\rangle\biggr]\biggr| \leqslant \sigma \|u_{\lambda^n}\|_\infty \max_{\overline{\Omega}\times\overline{\Omega}} K(x, y)|\Omega|,$$

thus $\lim_{n\to\infty} \mathcal{I}m(\mu_{\lambda^n}) = 0$. Similar to the proof of Lemma 2.3, we have that

$$\bigl|\bigl\langle A\bigl(\lambda^n\bigr)\phi_{\lambda^n}, \phi_{\lambda^n}\bigr\rangle\bigr| \geqslant \bigl|\lambda_2\bigl(\lambda^n\bigr)\bigr| \cdot \|\phi_{\lambda^n}\|^2_{Y_{\mathbb{C}}},$$

where $\lambda_2(\lambda^n)$ is the second eigenvalue of $A(\lambda^n)$. So

$$\bigl|\lambda_2\bigl(\lambda^n\bigr)\bigr| \cdot \|\phi_{\lambda^n}\|^2_{Y_{\mathbb{C}}} \leqslant \biggl|\lambda^n \biggl\langle \psi_{\lambda^n}, u_{\lambda^n} \int_\Omega K(\cdot, y)\psi_{\lambda^n}(y)\,dy \biggr\rangle\biggr| + |\mu_{\lambda^n}|. \tag{2.25}$$

Since all the eigenvalues of $A(\lambda)$ continuously depend on λ, then we have that

$$\lim_{n\to\infty} \lambda_2\bigl(\lambda^n\bigr) = \lambda_2 - \lambda_* > 0,$$

where λ_2 is the second eigenvalue of Eq. (1.6). Since

$$\biggl|\lambda^n \biggl\langle \psi_{\lambda^n}, u_{\lambda^n} \int_\Omega K(\cdot, y)\psi_{\lambda^n}(y)\,dy \biggr\rangle\biggr| \leqslant \sigma \|u_{\lambda^n}\|_\infty \max_{\overline{\Omega}\times\overline{\Omega}} K(x, y)|\Omega|,$$

then we have that $\lim_{n\to\infty}\|\phi_{\lambda^n}\|_{Y_{\mathbb{C}}}=0$. Since $\psi_{\lambda^n}=c_{\lambda^n}u_{\lambda^n}+\phi_{\lambda^n}$ and $\|\psi_{\lambda^n}\|_{L^2}=1$, then we have that

$$\lim_{n\to\infty}|c_{\lambda^n}|\left(\lambda^n-\lambda_*\right)\lim_{n\to\infty}\left\|\frac{u_{\lambda^n}}{\lambda^n-\lambda_*}\right\|_{Y_{\mathbb{C}}}=1,$$

and hence $\lim_{n\to\infty}|c_{\lambda^n}|(\lambda^n-\lambda_*)>0$ from Theorem 2.1. We denote

$$\lambda^n\left\langle\psi_{\lambda^n},u_{\lambda^n}\int_\Omega K(\cdot,y)\psi_{\lambda^n}(y)\,dy\right\rangle$$

in Eq. (2.25) by D_{λ^n}, then

$$\begin{aligned}\frac{D_{\lambda^n}}{\lambda^n-\lambda_*}&=\frac{1}{\lambda^n-\lambda_*}\lambda^n\left\langle c_{\lambda^n}u_{\lambda^n}+\phi_{\lambda^n},u_{\lambda^n}\int_\Omega K(\cdot,y)(c_{\lambda^n}u_{\lambda^n}+\phi_{\lambda^n})\,dy\right\rangle\\&=|c_{\lambda^n}|^2\left(\lambda^n-\lambda_*\right)^2\lambda^n\int_\Omega\int_\Omega K(x,y)\frac{u^2_{\lambda^n}(x)u_{\lambda^n}(y)}{(\lambda^n-\lambda_*)^3}\,dx\,dy\\&\quad+c_{\lambda^n}\left(\lambda^n-\lambda_*\right)\lambda^n\int_\Omega\int_\Omega K(x,y)\frac{\overline{\phi_{\lambda^n}}(x)u_{\lambda^n}(x)u_{\lambda^n}(y)}{(\lambda^n-\lambda_*)^2}\,dx\,dy\\&\quad+\overline{c_{\lambda^n}}\left(\lambda^n-\lambda_*\right)\lambda^n\int_\Omega\int_\Omega K(x,y)\frac{\phi_{\lambda^n}(y)u^2_{\lambda^n}(x)}{(\lambda^n-\lambda_*)^2}\,dx\,dy\\&\quad+\lambda^n\int_\Omega\int_\Omega K(x,y)\frac{\phi_{\lambda^n}(x)\overline{\phi_{\lambda^n}}(y)u_{\lambda^n}(x)}{\lambda^n-\lambda_*}\,dx\,dy.\end{aligned}\tag{2.26}$$

Since $\lim_{n\to\infty}\|\phi_{\lambda^n}\|_{Y_{\mathbb{C}}}=0$, then $\lim_{n\to\infty}\|\phi_{\lambda^n}\|_{L^1}=0$, and hence each of the last three terms of Eq. (2.26) goes to zero as $n\to\infty$. Since $K(x,y)$ satisfies assumption (**A**) and u_{λ^n} satisfies Eq. (2.2), then

$$\lim_{n\to\infty}\int_\Omega\int_\Omega K(x,y)\frac{u^2_{\lambda^n}(x)u_{\lambda^n}(y)}{(\lambda^n-\lambda_*)^3}\,dx\,dy=\alpha^3_{\lambda_*}\int_\Omega\int_\Omega K(x,y)\phi^2(x)\phi(y)\,dx\,dy>0,$$

and hence the first term of Eq. (2.26) tends to a positive constant as $n\to\infty$. So there exists $N_*\in\mathbb{N}$ such that for each $n\geqslant N_*$, $\mathcal{R}e(D_{\lambda^n})>0$, which implies that

$$\mathcal{R}e(\mu_{\lambda^n})=\left\langle A\left(\lambda^n\right)\phi_{\lambda^n},\phi_{\lambda^n}\right\rangle-\mathcal{R}e(D_{\lambda^n})<0.\tag{2.27}$$

This is a contradiction with $\mathcal{R}e(\mu_{\lambda^n})\geqslant 0$ for $n\geqslant 1$. Therefore all the eigenvalues of $A_\tau(\lambda)$ have negative real parts when $\tau=0$. $\square$

Next we prove some estimates needed for stability and bifurcation results.

Lemma 2.10. *Assume that $\lambda\in(\lambda_*,\lambda^*]$, and let ψ_λ and $\tilde{\psi}_\lambda$ be the eigenfunctions defined as in Corollary 2.6 and Theorem 2.7 respectively, then for $n=0,1,2,\cdots$,*

$$S_n(\lambda):=\int_\Omega\overline{\tilde{\psi}_\lambda}(y)\psi_\lambda(y)\,dy-\lambda\tau_n\int_\Omega\int_\Omega u_\lambda(x)K(x,y)e^{-i\theta_\lambda}\overline{\tilde{\psi}_\lambda}(x)\psi_\lambda(y)\,dx\,dy\neq 0.\tag{2.28}$$

Proof. From the expressions of u_λ, ψ_λ, $\tilde{\psi}_\lambda$, τ_n, and θ_λ, $\tilde{\theta}_\lambda \to \pi/2$ as $\lambda \to \lambda_*$, we obtain from the Dominated Convergence Theorem that

$$\lim_{\lambda\to\lambda_*}\int_\Omega \overline{\tilde{\psi}_\lambda}(y)\psi_\lambda(y)\,dy = \int_\Omega \phi^2(y)\,dy$$

and

$$\begin{aligned}&\lim_{\lambda\to\lambda_*}\lambda\tau_n\int_\Omega\int_\Omega u_\lambda(x)K(x,y)e^{-i\theta_\lambda}\overline{\tilde{\psi}_\lambda}(x)\psi_\lambda(y)\,dx\,dy\\ &\quad= -i\alpha_{\lambda_*}\lambda_*\left(\frac{\pi}{2}+2n\pi\right)\int_\Omega\int_\Omega K(x,y)\phi^2(x)\phi(y)\,dy\,dx.\end{aligned} \tag{2.29}$$

Then from (2.3),

$$S_n(\lambda) \to \left[1+i\left(\frac{\pi}{2}+2n\pi\right)\right]\int_\Omega \phi^2(x)\,dx, \quad \text{as } \lambda\to\lambda_*. \tag{2.30}$$

So we have $S_n(\lambda) \neq 0$ for $\lambda \in (\lambda_*, \lambda^*]$ and $n = 0, 1, 2, \cdots$. □

Theorem 2.11. *Assume that $\lambda \in (\lambda_*, \lambda^*]$, then $\mu = i\nu_\lambda$ is a simple eigenvalue of A_{τ_n} for $n = 0, 1, 2, \cdots$.*

Proof. From Corollary 2.6 we have $\mathscr{N}[A_{\tau_n}(\lambda) - i\nu_\lambda] = \mathrm{Span}[e^{i\nu_\lambda\cdot}\psi_\lambda]$. Suppose that for some $\phi_1 \in \mathscr{D}(A_{\tau_n}(\lambda)) \cap \mathscr{D}([A_{\tau_n}(\lambda)]^2)$, we have

$$\left[A_{\tau_n}(\lambda) - i\nu_\lambda\right]^2\phi_1 = 0.$$

Then

$$\left[A_{\tau_n}(\lambda) - i\nu_\lambda\right]\phi_1 \in \mathscr{N}\left[A_{\tau_n}(\lambda) - i\nu_\lambda\right] = \mathrm{Span}\left[e^{i\nu_\lambda\cdot}\psi_\lambda\right].$$

So there is a constant a such that

$$\left[A_{\tau_n}(\lambda) - i\nu_\lambda\right]\phi_1 = ae^{i\nu_\lambda\cdot}\psi_\lambda.$$

Hence

$$\begin{aligned}&\dot{\phi}_1(\theta) = i\nu_\lambda\phi_1(\theta) + ae^{i\nu_\lambda\theta}\psi_\lambda, \quad \theta\in[-\tau_n,0],\\ &\dot{\phi}_1(0) = A(\lambda)\phi_1(0) - \lambda u_\lambda\int_\Omega K(\cdot,y)\big(\phi_1(-\tau_n)\big)(y)\,dy.\end{aligned} \tag{2.31}$$

The first equation of Eq. (2.31) yields

$$\begin{aligned}\phi_1(\theta) &= \phi_1(0)e^{i\nu_\lambda\theta} + a\theta e^{i\nu_\lambda\theta}\psi_\lambda,\\ \dot{\phi}_1(0) &= i\nu_\lambda\phi_1(0) + a\psi_\lambda.\end{aligned} \tag{2.32}$$

From Eqs. (2.31) and (2.32) we have

$$\Delta(\lambda, i\nu, \tau_n)\phi_1(0) = \left[A(\lambda) - i\nu_\lambda\right]\phi_1(0) - \lambda u_\lambda \int_\Omega K(\cdot, y)\phi_1(0)(y)\,dy\, e^{-i\theta_\lambda}$$
$$= a\left(\psi_\lambda - \lambda\tau_n u_\lambda \int_\Omega K(\cdot, y)\psi_\lambda(y)\,dy\, e^{-i\theta_\lambda}\right). \tag{2.33}$$

From Eq. (2.16) and Remark 2.8, we have

$$\begin{aligned} 0 &= \left\langle \tilde{\Delta}(\lambda, i\tilde{\nu}, \tilde{\tau}_n)\tilde{\psi}_\lambda, \phi_1(0)\right\rangle = \left\langle \tilde{\Delta}(\lambda, i\nu, \tau_n)\tilde{\psi}_\lambda, \phi_1(0)\right\rangle \\ &= \left\langle \tilde{\psi}_\lambda, \Delta(\lambda, i\nu, \tau_n)\phi_1(0)\right\rangle \\ &= a\left(\int_\Omega \overline{\tilde{\psi}_\lambda}(y)\psi_\lambda(y)\,dy - \lambda\tau_n \int_\Omega\int_\Omega u_\lambda(x)K(x, y)e^{-i\theta_\lambda}\,\overline{\tilde{\psi}_\lambda}(x)\psi_\lambda(y)\,dx\,dy\right). \end{aligned}$$

As a consequence of Lemma 2.10 we have $a = 0$, which leads to $\phi_1 \in \mathcal{N}[A_{\tau_n}(\lambda) - i\nu_\lambda]$. By induction we obtain

$$\mathcal{N}\left[A_{\tau_n}(\lambda) - i\nu_\lambda\right]^j = \mathcal{N}\left[A_{\tau_n}(\lambda) - i\nu_\lambda\right], \quad j = 1, 2, 3, \cdots, \ n = 0, 1, 2, \cdots.$$

Therefore, $\lambda = i\nu_\lambda$ is a simple eigenvalue of A_{τ_n} for $n = 0, 1, 2, \cdots$. □

Since $\mu = i\nu_\lambda$ is a simple eigenvalue of A_{τ_n}, from the implicit function theorem, there are a neighborhood $O_n \times D_n \times H_n \subset \mathbb{R} \times \mathbb{C} \times X_{\mathbb{C}}$ of $(\tau_n, i\nu_\lambda, \psi_\lambda)$ and a continuously differential function $(\mu, \psi) : O_n \to D_n \times H_n$ such that for each $\tau \in O_n$, the only eigenvalue of $A_\tau(\lambda)$ in D_n is $\mu(\tau)$, and

$$\mu(\tau_n) = i\nu_\lambda, \qquad \psi(\tau_n) = \psi_\lambda,$$
$$\Delta\big(\lambda, \mu(\tau), \tau\big) = \left[A(\lambda) - \mu(\tau)\right]\psi(\tau) - \lambda u_\lambda \int_\Omega K(x, y)\big(\psi(\tau)\big)(y)\,dy\, e^{-\mu(\tau)\tau} = 0, \quad \tau \in O_n. \tag{2.34}$$

Then we have the following transversality condition.

Theorem 2.12. *Assume that $\lambda \in (\lambda_*, \lambda^*]$, and $\mu(\tau)$ is the eigenvalue of A_τ defined as above, then*

$$\frac{d\mathcal{R}e(\mu(\tau_n))}{d\tau} > 0, \quad n = 0, 1, 2, \cdots.$$

Proof. Differentiating Eq. (2.34) with respect to τ at $\tau = \tau_n$, we have

$$\frac{d\mu(\tau_n)}{d\tau}\left[-\psi_\lambda + \lambda\tau_n u_\lambda e^{-i\theta_\lambda} \int_\Omega K(\cdot, y)\psi_\lambda(y)\,dy\right]$$
$$+ \Delta(\lambda, i\nu_\lambda, \tau_n)\frac{d\psi(\tau_n)}{d\tau} + i\nu_\lambda\lambda u_\lambda \int_\Omega K(\cdot, y)\psi_\lambda(y)\,dy\, e^{-i\theta_\lambda} = 0. \tag{2.35}$$

Multiplying the equation by $\overline{\tilde{\psi}_\lambda}(x)$ and integrating on Ω, we see that

$$\frac{d\mu(\tau_n)}{d\tau} = \frac{i\nu_\lambda \lambda e^{-i\theta_\lambda} \int_\Omega \int_\Omega u_\lambda(x) K(x, y)\psi_\lambda(y)\overline{\tilde{\psi}_\lambda}(x)\,dx\,dy}{\int_\Omega \overline{\tilde{\psi}_\lambda}(x)\psi_\lambda(x)\,dx - \lambda\tau_n e^{-i\theta_\lambda} \int_\Omega \int_\Omega u_\lambda(x) K(x, y)\psi_\lambda(y)\overline{\tilde{\psi}_\lambda}(x)\,dx\,dy}$$
$$= \frac{1}{|S_n|^2}\Biggl(i\nu_\lambda \lambda e^{-i\theta_\lambda} \int_\Omega \tilde{\psi}_\lambda(x)\overline{\psi_\lambda}(x)\,dx \int_\Omega\int_\Omega u_\lambda(x) K(x, y)\psi_\lambda(y)\overline{\tilde{\psi}_\lambda}(x)\,dx\,dy$$
$$- i\nu_\lambda \lambda^2 \tau_n \left| \int_\Omega\int_\Omega u_\lambda(x) K(x, y)\psi_\lambda(y)\overline{\tilde{\psi}_\lambda}(x)\,dx\,dy \right|^2 \Biggr). \tag{2.36}$$

From the expressions of u_λ, τ_n, the fact that

$$\theta_\lambda, \tilde{\theta}_\lambda \to \frac{\pi}{2}, \qquad \psi_\lambda \to \phi, \qquad \tilde{\psi}_\lambda \to \phi, \qquad (\lambda - \lambda_*)\nu_\lambda = \theta_\lambda \to \frac{\pi}{2} \quad \text{as } \lambda \to \lambda_*,$$

and the Dominated Convergence Theorem, we have that

$$\lim_{\lambda\to\lambda_*} \frac{d\mathcal{R}e(\mu(\tau_n))}{d\tau} = \frac{\alpha_{\lambda_*}\pi\lambda_*}{2\lim_{\lambda\to\lambda_*}|S_n(\lambda)|^2} \int_\Omega \phi^2(x)\,dx \int_\Omega\int_\Omega K(x, y)\phi^2(x)\phi(y)\,dx\,dy > 0,$$

where $\alpha_{\lambda_*} > 0$ is defined in Eq. (2.3). □

From Corollary 2.6, Proposition 2.9, and Theorem 2.12 we see that:

Theorem 2.13. *For $\lambda \in (\lambda_*, \lambda^*]$, the infinitesimal generator $A_\tau(\lambda)$ has exactly $2(n+1)$ eigenvalues with positive real parts when $\tau \in (\tau_n, \tau_{n+1}]$, $n = 0, 1, 2, \cdots$.*

Then we have the following results on the stability and the associated Hopf bifurcations of the positive steady state solution u_λ.

Theorem 2.14. *For $\lambda \in (\lambda_*, \lambda^*]$, the positive equilibrium solution u_λ of Eq. (1.2) is locally asymptotically stable when $\tau \in [0, \tau_0)$ and is unstable when $\tau \in (\tau_0, \infty)$. Moreover at $\tau = \tau_n$ $(n = 0, 1, 2, \cdots)$, a Hopf bifurcation occurs so that a branch of spatially nonhomogeneous periodic orbits of Eq. (1.2) emerges from (τ_n, u_λ).*

More precisely, there exist $\varepsilon_0 > 0$ and continuously differentiable function $[-\varepsilon_0, \varepsilon_0] \mapsto (\tau_n(\varepsilon), T_n(\varepsilon), u_n(\varepsilon, x, t)) \in \mathbb{R} \times \mathbb{R} \times X$ satisfying $\tau_n(0) = \tau_n$, $T_n(0) = 2\pi/\nu_\lambda$, and $u_n(\varepsilon, x, t)$ is a $T_n(\varepsilon)$-periodic solution of Eq. (1.2) such that $u_n = u_\lambda + \varepsilon v_n(\varepsilon, x, t)$ where v_n satisfies $v_n(0, x, t)$ is a $2\pi/\nu_\lambda$-periodic solution of (2.7). Moreover there exists $\delta > 0$ such that if Eq. (1.2) has a nonconstant periodic solution $u(x, t)$ of period T for some $\tau > 0$ with

$$|\tau - \tau_n| < \delta, \qquad \left| T - \frac{2\pi}{\nu_\lambda} \right| < \delta, \qquad \max_{t\in\mathbb{R},\, x\in\overline{\Omega}} \bigl|u(x, t) - u_\lambda(x)\bigr| < \delta,$$

then $\tau = \tau_n(\varepsilon)$ and $u(x, t) = u_n(\varepsilon, x, t + \theta)$ for some $|\varepsilon| < \varepsilon_0$ and some $\theta \in \mathbb{R}$.

We comment that local Hopf bifurcation theorems for evolution equation in a Banach space with delays have been proved in [40] (see Theorem 4.6 on page 211). With Corollary 2.6, Proposition 2.9, and Theorem 2.12, all conditions in the result of [40] are verified, hence the conclusions in Theorem 2.14 hold. Note that the direction of the Hopf bifurcation curve $\tau_n(\varepsilon)$ can be calculated from the first Lyapunov coefficient μ_2, which will be done in Section 4. The nonlinear terms in the equation play an important role for the direction of Hopf bifurcation. If the first Lyapunov coefficient $\mu_2 \neq 0$, then a family of periodic orbits exists for a left-hand side or right-hand side neighborhood of $\tau = \tau_n$.

3. Eigenvalue problem with homogeneous kernel

In this section we analyze Eq. (1.2) when $K(x,y) \equiv 1$, $n=1$ and $\Omega = (0, L)$ where $L > 0$. Following the method of [4, Section 5], we obtain the following dimensionless form:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} + \lambda u(x,t)\left(1 - \displaystyle\int_0^\pi u(y, t-\tau)\,dy\right), & x \in (0,\pi),\ t > 0, \\ u(x,t) = 0, & x = 0,\ \pi, t > 0. \end{cases} \tag{3.1}$$

We can easily verify that Eq. (3.1) has a unique positive equilibrium solution $u_\lambda(x) = \frac{\lambda-1}{2\lambda}\sin x$ for any $\lambda > 1$ (here $\lambda_* = 1$). Linearizing Eq. (3.1) at u_λ, we have that:

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} = \dfrac{\partial^2 v(x,t)}{\partial x^2} + v - \dfrac{\lambda-1}{2}\sin x \displaystyle\int_0^\pi v(y, t-\tau)\,dy, & x \in (0,\pi),\ t > 0, \\ v(x,t) = 0, & x = 0,\ \pi, t > 0. \end{cases} \tag{3.2}$$

Following the approach in Section 2, we still denote the infinitesimal generator of Eq. (3.2) by $A_\tau(\lambda)$. Then μ is an eigenvalue of $A_\tau(\lambda)$ if and only if μ is an eigenvalue of the following nonlocal elliptic eigenvalue problem:

$$\begin{cases} \Delta(\lambda,\mu,\tau)\psi := \psi'' + \psi - \dfrac{\lambda-1}{2}e^{-\mu\tau}\sin x \displaystyle\int_0^\pi \psi(y)\,dy - \mu\psi = 0, \quad x \in (0,\pi), \\ \psi(0) = \psi(\pi) = 0. \end{cases} \tag{3.3}$$

Lemma 3.1. *Suppose that $\lambda > 1$ and $\tau \geqslant 0$. Then $\mu \in \mathbb{C}$ is an eigenvalue of the problem (3.3) if and only if one of the following is satisfied*:

1. $\mu = -n^2 + 1$ *for* $n = 2, 3, 4, \cdots$; *or*
2. μ *satisfies*

$$(\lambda - 1)e^{-\mu\tau} + \mu = 0. \tag{3.4}$$

Proof. Substituting the Fourier series $\psi = \sum_{n=1}^\infty c_n \sin nx$ into Eq. (3.3), we have:

$$\sum_{n=2}^\infty c_n\left(-n^2 + 1 - \mu\right)\sin nx - \left[(\lambda-1)\sum_{n=0}^\infty \frac{c_{2n+1}}{2n+1}e^{-\mu\tau} + \mu c_1\right]\sin x = 0. \tag{3.5}$$

Suppose that $\mu \in \mathbb{C}$ is an eigenvalue of (3.3), and $\mu \neq -n^2 + 1$ for each of $n = 2, 3, 4, \cdots$, then (3.5) implies each $c_n = 0$ for $n \geqslant 2$, and if $c_1 \neq 0$, then (3.4) is satisfied.

On the other hand, if (3.4) is not satisfied and for some $m = 2, 3, 4, \cdots$, $\mu = -m^2 + 1$, then $c_n = 0$ for $n \geqslant 2$ and $n \neq m$. If m is even, then $c_1 = 0$ as well, hence $\mu = -m^2 + 1$ is an eigenvalue with an eigenfunction $\phi_m(x) = \sin mx$; if m is odd, then $\mu = -m^2 + 1$ is an eigenvalue with an eigenfunction in form $\phi_m(x) = \sin x + c_m \sin mx$, where c_m satisfies

$$(\lambda - 1)\left(1 + \frac{c_m}{m}\right)e^{(-m^2+1)\tau} - m^2 + 1 = 0.$$

If μ satisfies (3.4), then μ is an eigenvalue with an eigenfunction $\phi_1(x) = \sin x$. $\square$

It is clear that $\mu = -n^2 + 1$, $n = 2, 3, \cdots$, are the fixed eigenvalues for all $\tau \geqslant 0$. For eigenvalues satisfying (3.4), we have the following further result:

Lemma 3.2. *Suppose that $\lambda > 1$ and $\tau \geqslant 0$. Then $\mu \in \mathbb{C}$ is an eigenvalue of (3.3) satisfying (3.4). Then either*

1. *$\mu \in \mathbb{R}$, and for each $\tau \in [0, \tau_*)$, there are exactly two such real-valued eigenvalues $\mu_1^{\pm}(\tau)$ satisfying $1 - \lambda \geqslant \mu_1^{+}(\tau) > \mu_1^{-}(\tau)$, where $\tau_* = \frac{1}{e(\lambda-1)}$. Moreover*

$$\lim_{\tau \to 0^+} \mu_1^{+}(\tau) = 1 - \lambda, \qquad \lim_{\tau \to 0^+} \mu_1^{-}(\tau) = -\infty, \quad \textit{and} \quad \lim_{\tau \to \tau_*^-} \mu_1^{\pm}(\tau) = -e(\lambda - 1),$$

which is the unique real-valued eigenvalue for $\tau = \tau_*$; or

2. *$\mu = \alpha \pm i\beta \in \mathbb{C}$ with $\beta > 0$, where α and β satisfy*

$$(\lambda - 1)e^{-\alpha\tau} \cos \beta\tau = -\alpha, \qquad (\lambda - 1)e^{-\alpha\tau} \sin \beta\tau = \beta. \tag{3.6}$$

Moreover for each $\tau > \tau_*$, there are infinitely many such complex-valued eigenvalues $\alpha_n \pm i\beta_n$ ($\beta_n > 0$), for $n \in \mathbb{N} \cup \{0\}$, where α_n satisfies

$$\tau(\lambda - 1)e^{-\alpha_n\tau}\left(1 - \frac{\alpha_n^2 e^{2\alpha_n\tau}}{(\lambda - 1)^2}\right)^{1/2} = \arccos \frac{-\alpha_n e^{\alpha_n\tau}}{\lambda - 1} + 2n\pi \tag{3.7}$$

and β_n satisfies

$$\beta_n = (\lambda - 1)e^{-\alpha_n\tau}\left(1 - \frac{\alpha_n^2 e^{2\alpha_n\tau}}{(\lambda - 1)^2}\right)^{1/2}. \tag{3.8}$$

Proof. In the case of $\mu \in \mathbb{R}$, since $\lambda > 1$, then $\mu = -(\lambda - 1)e^{-\mu\tau} < 0$. So from Eq. (3.4), we have that

$$\tau = \frac{\ln(-\mu) - \ln(\lambda - 1)}{-\mu}. \tag{3.9}$$

Since $\tau \geqslant 0$, then the domain of μ is $(-\infty, -(\lambda - 1)]$. Differentiating Eq. (3.9) with respect to μ, we have that

$$\tau'(\mu) = \frac{\ln(-\mu) - 1 - \ln(\lambda - 1)}{\mu^2}. \tag{3.10}$$

From Eq. (3.10), we have that there exists $\mu_* = -e(\lambda - 1)$ such that $\tau'(\mu_*) = 0$, $\tau'(\mu) > 0$ when $\tau \in (-\infty, \mu_*)$, and $\tau'(\mu) < 0$ when $\tau \in (\mu_*, -(\lambda - 1))$. Hence when $\tau \in (-\infty, \mu_*)$, $\tau(\mu)$ is strictly increasing, when $\tau \in (\mu_*, -(\lambda - 1))$, $\tau(\mu)$ is strictly decreasing, and when $\mu = \mu_*$, $\tau(\mu)$ obtains its maximum value. Setting $\tau_* = \tau(\mu_*) = \frac{1}{e(\lambda-1)}$, we obtain the first result.

If $\mu = \alpha \pm i\beta \in \mathbb{C}$ with $\beta > 0$, substituting $\mu = \alpha + i\beta$ into Eq. (3.4), we obtain that α and β satisfy Eq. (3.6). Since $\beta > 0$, then from the second equation of Eq. (3.6), we have that $\beta\tau \in (2n\pi, (2n + 1)\pi)$ for $n \in \mathbb{N} \cup \{0\}$. Hence solving the first equation of Eq. (3.6), we have that

$$\beta = \frac{1}{\tau}\left(\arccos \frac{-\alpha e^{\alpha\tau}}{\lambda - 1} + 2n\pi\right), \tag{3.11}$$

for some $n \in \mathbb{N}$, and

$$\sin \beta\tau = \left(1 - \cos^2 \beta\tau\right)^{1/2} = \left(1 - \frac{\alpha^2 e^{2\alpha\tau}}{(\lambda - 1)^2}\right)^{1/2}. \tag{3.12}$$

Substituting Eqs. (3.11) and (3.12) into the second equation of Eq. (3.6), we have Eq. (3.7). $\square$

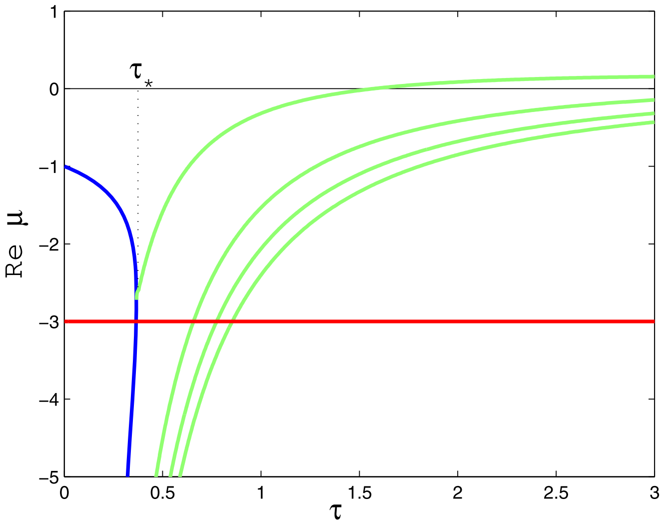

Fig. 1. Relation between $\mathcal{R}e(\mu)$ and τ for Eq. (3.4). Here $\lambda = 2$. $\mu = -3$ is a fixed real-valued eigenvalue; on the left side of $\tau = \tau_*$ is the curve of real-valued eigenvalues μ satisfying $(\lambda - 1)e^{-\mu\tau} + \mu = 0$; and on the right side of $\tau = \tau_*$ are the curves of real part α_n of complex-valued eigenvalues $\alpha_n \pm i\beta_n$. The curve $\alpha_0(\tau)$ connects with the curve of real eigenvalues at $\tau = \tau_*$, and at $\tau = \pi/2$, $\alpha_0(\tau) = 0$ which gives rise of the first Hopf bifurcation point.

Lemmas 3.1 and 3.2 completely classify the eigenvalues of the nonlocal eigenvalue problem (3.3), and the variation of the eigenvalues with respect to the delay τ is shown in Fig. 1. It can be shown that $\alpha_n(\tau)$ is strictly increasing in τ, and

$$\lim_{\tau \to \tau_*^+} \alpha_0(\tau) = -e(\lambda - 1) \quad \text{and} \quad \lim_{\tau \to \tau_*^+} \alpha_n(\tau) = -\infty, \quad \text{for } n \in \mathbb{N}.$$

The spectral properties of nonlocal linear elliptic eigenvalue problem (without delay effect) have been studied in [7,13,14]. It is known that such problem may have different spectral properties compared to the linear elliptic eigenvalue problem without integral nonlocal terms. The following can be noticed for the nonlocal eigenvalue problem (3.3) even with $\tau = 0$:

1. The eigenspace of (3.3) may not be one-dimensional. When $\mu = -n^2 + 1$ is also a root of (3.4), the eigenspace is two-dimensional. However as shown in [7], usually the eigenspace of such nonlocal problem is at most two-dimensional.
2. The eigenvalue problem (3.3) with $\tau = 0$ always has a principal eigenvalue μ_0 satisfying (3.4) with a positive eigenfunction $\sin x$. But μ_0 may not be the largest eigenvalue of (3.3). For example when $\tau = 0$ and $\lambda < 4$, the maximum eigenvalue of (3.3) is $1 - \lambda$ which is also the principal eigenvalue; but when $\tau = 0$ and $\lambda \geqslant 4$, then the maximum eigenvalue is -3 with the corresponding eigenfunction $\sin 2x$, and hence the maximum eigenvalue is not the principal eigenvalue.

We can now state our main result for the Hopf bifurcations along the unique positive equilibrium $u_\lambda(x) = \frac{\lambda-1}{2\lambda}\sin x$ for *any* $\lambda > 1$:

Theorem 3.3. *For each $\lambda > 1$, there exist*

$$\tau_n(\lambda) = \frac{(4n+1)\pi}{2(\lambda - 1)}, \quad n = 0, 1, 2, \cdots, \tag{3.13}$$

such that when $\tau = \tau_n(\lambda)$, $n = 0, 1, 2, \cdots$, $A_\tau(\lambda)$ has a pair of simple purely imaginary roots $\pm i\nu_\lambda = \pm i(\lambda - 1)$. Moreover when $\tau < \tau_0$, all the eigenvalues of (3.3) have negative real parts, and when $\tau \in (\tau_n, \tau_{n+1}]$ $(n = 0, 1, 2, \cdots)$, the eigenvalue problem (3.3) has exactly $2n + 2$ eigenvalues with positive real parts.

Proof. When $\tau = 0$, from Lemma 3.1 we obtain that all the eigenvalues of characteristic equation (3.3) have negative real parts. For any $\tau \geqslant 0$, from Lemmas 3.1 and 3.2, we also have that 0 is not an eigenvalue of (3.3). If $\mu = \pm i\beta$ $(\beta > 0)$ is a pair of purely imaginary eigenvalue, then

$$(\lambda - 1)\cos\beta\tau = 0, \qquad (\lambda - 1)\sin\beta\tau = \beta.$$

Hence only when $\tau = \tau_n(\lambda)$ defined as in (3.13), the characteristic equation (3.3) has a pair of purely imaginary root $\pm i\nu_\lambda = \pm i(\lambda - 1)$, and $\Delta(\lambda, i\nu_\lambda, \tau_n(\lambda))\sin x = 0$. Then in this case the adjoint equation of $\Delta(\lambda, \mu, \tau)$ becomes:

$$\begin{cases} \tilde{\Delta}(\lambda, \mu, \tau)\tilde{\psi} := \tilde{\psi}'' + \tilde{\psi} - \dfrac{\lambda - 1}{2}e^{\mu\tau}\displaystyle\int_0^\pi \sin y\tilde{\psi}(y)\,dy + \mu\tilde{\psi} = 0, \quad x \in (0, \pi), \\ \tilde{\psi}(0) = \tilde{\psi}(\pi) = 0. \end{cases} \tag{3.14}$$

Substituting $\mu = i\nu_\lambda = i(\lambda - 1)$, $\tau = \tau_n(\lambda)$, and $\tilde{\psi} = \sum_{n=1}^\infty \tilde{c}_n \sin nx$ into Eq. (3.14), we have that

$$\begin{aligned} &\tilde{c}_1 \sin x\big[-(\lambda - 1)e^{i\nu_\lambda\tau_n(\lambda)} + i\nu_\lambda\big] + \sum_{n=1}^\infty \tilde{c}_{2n}\sin 2nx\big[1 - (2n)^2 + i\nu_\lambda\big] \\ &\quad = -\sum_{n=1}^\infty \left[\big(1 - (2n+1)^2 + i\nu_\lambda\big)\tilde{c}_{2n+1} - \frac{\lambda - 1}{2n+1}\tilde{c}_1 e^{i\nu_\lambda\tau_n(\lambda)}\right]\sin(2n+1)x. \end{aligned}$$

Hence in this case we can solve that

$$\tilde{\psi}_\lambda(x) = \sin x + \sum_{n=1}^\infty \frac{i\nu_\lambda}{(2n+1)(1 + i\nu_\lambda - (2n+1)^2)}\sin(2n+1)x, \tag{3.15}$$

and $\tilde{\Delta}(\lambda, i\nu_\lambda, \tau_n(\lambda))\tilde{\psi} = 0$. Substituting $\psi_\lambda = \sin x$, $\tilde{\psi}_\lambda$ into Eq. (2.28), we have

$$S_n(\lambda) = \frac{\pi}{2} + \left(\frac{\pi^2}{4} + n\pi^2\right)i \neq 0.$$

Using the same method in Theorem 2.11, we can prove that $\pm i\nu_\lambda$ is a pair of simple purely imaginary roots of $A_{\tau_n}(\lambda)$. By using the implicit function theorem, then there is a continuously differential function $(\mu(\tau), \psi(\tau))$, which is defined in a neighborhood of τ_n, such that

$$\mu(\tau_n) = i\nu_\lambda, \qquad \psi(\tau_n) = \psi_\lambda, \qquad \Delta\big(\lambda, \mu(\tau), \tau\big)\psi(\tau) = 0.$$

Then using the same method as in Theorem 2.12, we have $\frac{d\mathcal{R}e(\mu(\tau_n))}{d\tau} > 0$. Then the conclusions in the theorem follow. $\square$

We can now state the result on the stability of positive equilibrium and the associated Hopf bifurcation for Eq. (3.1) with any $\lambda > 1$.

Theorem 3.4. *Consider the nonlocal problem (3.1). For each $\lambda > 1$ and $n \in \mathbb{N} \cup \{0\}$, there exists a $\tau_n(\lambda)$ defined as in (3.13) such that a Hopf bifurcation occurs for Eq. (3.1) at the unique positive equilibrium solution $u_\lambda = \frac{\lambda-1}{2\lambda}\sin x$ when $\tau = \tau_n(\lambda)$. Moreover, u_λ is locally asymptotically stable when $0 \leqslant \tau < \tau_0(\lambda)$, and it is unstable when $\tau > \tau_0(\lambda)$.*

In Theorem 3.4, the meaning of occurrence of a Hopf bifurcation is the same as that in Theorem 2.14, which is not repeated here. We remark that the results in Theorems 3.3 and 3.4 are proved for any $\lambda > 1$ because the equilibrium solution and associated eigenvalues are explicitly expressed, which is impossible for general kernel functions and general domains in higher dimension. It is also interesting to compare Eq. (3.1) and the classical Fisher–KPP equation with delay:

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} = \dfrac{\partial^2 v(x,t)}{\partial x^2} + \lambda v(x,t)\big(1 - v(x,t-\tau)\big), & x \in (0,\pi),\ t > 0, \\ v(x,t) = 0, & x = 0,\ \pi, t > 0. \end{cases} \tag{3.16}$$

It is well known that Eq. (3.16) has a unique positive equilibrium solution v_λ for $\lambda > 1$, and as $\lambda \to \infty$, $v_\lambda(x) \to 1$ uniformly on any compact subset of $(0,\pi)$. Hence the profiles of equilibrium solutions for Eqs. (3.1) and (3.16) are different. For Eq. (3.16), the transition to oscillatory pattern for large delay τ is only known for λ near $\lambda_* = 1$, and here we showed that such transition always occurs for the nonlocal equation (3.1).

Finally we make the following observation: suppose that a solution $u(x,t)$ of Eq. (3.1) is in a separable form

$$u(x,t) = \frac{\lambda - 1}{2\lambda}\sin x \cdot w(t). \tag{3.17}$$

Here we recall that $u_\lambda(x) = \frac{\lambda-1}{2\lambda}\sin x$ is the unique positive equilibrium of Eq. (3.1) for $\lambda > 1$. Then it is easy to verify that $w(t)$ satisfies the well-known (non-spatial) Hutchinson equation

$$\frac{dw}{dt} = (\lambda - 1)w(t)\big(1 - w(t-\tau)\big). \tag{3.18}$$

It is also well known that the Hopf bifurcation points of Eq. (3.18) are also given by (3.13) [27,32,33], hence all the bifurcating periodic orbits obtained in Theorem 3.4 are indeed in separable form (3.17). This shows that the dynamics of Eq. (3.18) is embedded in the dynamics of Eq. (3.1) if the initial value is also in separable form (3.17). This is interesting for a Dirichlet boundary value problem, while it is common for Neumann (no-flux) boundary value problem. It would be interesting to know the stability of periodic solution with such separable form for all $\lambda > 1$, and whether a symmetry-breaking bifurcation can occur so that non-separable periodic orbits can arise.

4. The direction of the Hopf bifurcation

In this section, we analyze the direction of the Hopf bifurcation of Eq. (1.2) obtained in Theorem 2.14 using τ as bifurcation parameter. Here we combine the methods in Faria [10–12] and Hassard et al. [23]. Similar approach has also been used in [38,41].

We first transform the equilibrium to the origin via the translations $U(t) = u(\cdot,t) - u_\lambda$ and $\tau = \tau_n + \gamma$, then $\gamma = 0$ is the Hopf bifurcation value of system (1.2). Re-scaling the time by $t \to \frac{t}{\tau}$ to normalize the delay, system (1.2) can be written in the following form

$$\frac{dU(t)}{dt} = \tau_n d\Delta U(t) + \tau_n L_0(U_t) + J(U_t,\gamma), \tag{4.1}$$

where

$$U_t \in \mathcal{C}, \qquad L_0(\psi) = \lambda\left(1 - \int_\Omega K(\cdot, y) u_\lambda(y)\,dy\right)\psi(0) - \lambda u_\lambda \int_\Omega K(\cdot, y)\psi(-1)(y)\,dy,$$

$$J(\psi, \gamma) = \gamma \Delta\psi(0) + \gamma L_0(\psi) - (\gamma + \tau_n)\lambda\psi(0)\int_\Omega K(\cdot, y)\psi(-1)(y)\,dy,$$

for $\psi \in \mathcal{C}$, and $\mathcal{C} = C([-1, 0], Y)$. Denote $\mathcal{A}_{\tau_n}$ to be the infinitesimal generator of the linearized equation

$$\frac{dU(t)}{dt} = \tau_n d\Delta U(t) + \tau_n L_0(U_t). \tag{4.2}$$

Then

$$\mathcal{A}_{\tau_n}\psi = \dot{\psi},$$

$$\mathscr{D}(\mathcal{A}_{\tau_n}) = \left\{\psi \in \mathcal{C}_\mathbb{C} \cap \mathcal{C}^1_\mathbb{C}\colon\ \psi(0) \in X_\mathbb{C},\ \dot{\psi}(0) = \tau_n A(\lambda)\psi(0) - \lambda\tau_n u_\lambda \int_\Omega K(\cdot, y)\psi(-1)(y)\,dy\right\},$$

where $\mathcal{C}^1_\mathbb{C} = C^1([-1, 0], Y_\mathbb{C})$. So Eq. (4.1) can be written in the following abstract form

$$\frac{dU_t}{dt} = \mathcal{A}_{\tau_n} U_t + X_0 J(U_t, \gamma), \tag{4.3}$$

where

$$X_0(\theta) = \begin{cases} 0, & \theta \in [-1, 0), \\ I, & \theta = 0. \end{cases}$$

From Theorem 2.14, we know that $\mathcal{A}_{\tau_n}$ has only one pair of purely imaginary eigenvalues $\pm i\nu_\lambda\tau_n$ which are simple. The corresponding eigenfunction with respect to $i\nu_\lambda\tau_n$ (or $-i\nu_\lambda\tau_n$) is $\psi_\lambda(x)e^{i\nu_\lambda\tau_n\theta}$ (or $\overline{\psi_\lambda}(x)e^{-i\nu_\lambda\tau_n\theta}$) for $\theta \in [-1, 0]$, where $\psi_\lambda(x)$ is defined in Corollary 2.6.

Following [11], we introduce the formal duality $\langle\langle\cdot,\cdot\rangle\rangle$ in $\mathcal{C}$ by

$$\langle\langle\tilde{\psi}, \psi\rangle\rangle = \langle\tilde{\psi}(0), \psi(0)\rangle - \lambda\tau_n \int_{-1}^{0}\left\langle\tilde{\psi}(s+1), u_\lambda \int_\Omega K(\cdot, y)\psi(s)(y)\,dy\right\rangle ds, \tag{4.4}$$

for $\psi \in \mathcal{C}_\mathbb{C}$ and $\tilde{\psi} \in \mathcal{C}^*_\mathbb{C} := C([0, 1], Y_\mathbb{C})$. Using similar consideration in [22], we give two lemmas about the formal adjoint operator of $\mathcal{A}_{\tau_n}$.

Lemma 4.1. *Define an operator $\mathcal{A}^*_{\tau_n} : \mathscr{D}(\mathcal{A}^*_{\tau_n}) \to \mathcal{C}^*$ by $\mathcal{A}^*_{\tau_n}\tilde{\psi}(s) = -\dot{\tilde{\psi}}(s)$ with*

$$\mathscr{D}(\mathcal{A}^*_{\tau_n}) = \left\{\tilde{\psi} \in \mathcal{C}^*_\mathbb{C} \cap (\mathcal{C}^*_\mathbb{C})^1\colon\ \tilde{\psi}(0) \in X_\mathbb{C},\ \dot{\tilde{\psi}}(0) = \tau_n A(\lambda)\tilde{\psi}(0) - \lambda\tau_n \int_\Omega K(y, \cdot)u_\lambda(y)\tilde{\psi}(1)(y)\,dy\right\},$$

where $(\mathcal{C}^*_\mathbb{C})^1 = C^1([0, 1], Y_\mathbb{C})$. Then $\mathcal{A}^*_{\tau_n}$ and $\mathcal{A}_{\tau_n}$ satisfy

$$\langle\langle\mathcal{A}^*_{\tau_n}\tilde{\psi}, \psi\rangle\rangle = \langle\langle\tilde{\psi}, \mathcal{A}_{\tau_n}\psi\rangle\rangle, \quad \textit{for } \psi \in \mathscr{D}(\mathcal{A}_{\tau_n}) \textit{ and } \tilde{\psi} \in \mathscr{D}(\mathcal{A}^*_{\tau_n}). \tag{4.5}$$

Proof. For $\psi \in \mathscr{D}(\mathcal{A}_{\tau_n})$ and $\tilde{\psi} \in \mathscr{D}(\mathcal{A}^*_{\tau_n})$,

$$\begin{aligned}
\langle\!\langle \tilde{\psi}, \mathcal{A}_{\tau_n}\psi \rangle\!\rangle = {} & \bigl\langle \tilde{\psi}(0), (\mathcal{A}_{\tau_n}\psi)(0)\bigr\rangle - \lambda\tau_n \int_{-1}^{0} \Bigl\langle \tilde{\psi}(s+1), u_\lambda \int_\Omega K(\cdot,y)\dot{\psi}(s)(y)\,dy \Bigr\rangle ds \\
= {} & \Bigl\langle \tilde{\psi}(0), \tau_n A(\lambda)\psi(0) - \lambda\tau_n u_\lambda \int_\Omega K(\cdot,y)\psi(-1)(y)\,dy \Bigr\rangle \\
& - \lambda\tau_n \Bigl[\Bigl\langle \tilde{\psi}(s+1), u_\lambda \int_\Omega K(\cdot,y)\psi(s)(y)\,dy \Bigr\rangle \Bigr]_{-1}^{0} \\
& + \lambda\tau_n \int_{-1}^{0} \Bigl\langle \dot{\tilde{\psi}}(s+1), u_\lambda \int_\Omega K(\cdot,y)\psi(s)(y)\,dy \Bigr\rangle ds \\
= {} & \bigl\langle \tau_n A(\lambda)\tilde{\psi}(0), \psi(0)\bigr\rangle - \lambda\tau_n \Bigl\langle \tilde{\psi}(1), u_\lambda \int_\Omega K(\cdot,y)\psi(0)(y)\,dy \Bigr\rangle \\
& - \lambda\tau_n \int_{-1}^{0} \Bigl\langle -\dot{\tilde{\psi}}(s+1), u_\lambda \int_\Omega K(\cdot,y)\psi(s)(y)\,dy \Bigr\rangle ds \\
= {} & \Bigl\langle \tau_n A(\lambda)\tilde{\psi}(0) - \lambda\tau_n \int_\Omega K(y,\cdot)u_\lambda(y)\tilde{\psi}(1)(y)\,dy, \psi(0) \Bigr\rangle \\
& - \lambda\tau_n \int_{-1}^{0} \Bigl\langle -\dot{\tilde{\psi}}(s+1), u_\lambda \int_\Omega K(\cdot,y)\psi(s)(y)\,dy \Bigr\rangle ds \\
= {} & \langle\!\langle \mathcal{A}^*_{\tau_n}\tilde{\psi}, \psi \rangle\!\rangle. \qquad \square
\end{aligned}$$

Lemma 4.2. *The operator $\mathcal{A}^*_{\tau_n}$ has only one pair of purely imaginary eigenvalues $\pm i\nu_\lambda\tau_n$ which are simple, and the corresponding eigenfunction with respect to $-i\nu_\lambda\tau_n$ (or $i\nu_\lambda\tau_n$) is $\tilde{\psi}_\lambda(x)e^{i\nu_\lambda\tau_n s}$ (or $\overline{\tilde{\psi}_\lambda}(x)e^{i\nu_\lambda\tau_n s}$) for $s \in [0,1]$, where $\tilde{\psi}_\lambda$ is defined in Theorem 2.7.*

Proof. If μ is an eigenvalue of $\mathcal{A}^*_{\tau_n}$, then there exists $\tilde{\psi} \in \mathscr{D}(\mathcal{A}^*_{\tau_n})$ such that $\mathcal{A}^*_{\tau_n}\tilde{\psi} = \mu\tilde{\psi}$. From the definition of $\mathcal{A}^*_{\tau_n}$, we have that $-\dot{\tilde{\psi}} = \mu\tilde{\psi}$, and hence $\tilde{\psi}(s) = \tilde{\psi}(0)e^{-\mu s}$, where $\tilde{\psi}(0) \in X_{\mathbb{C}}$ satisfies

$$\tau_n A(\lambda)\tilde{\psi}(0) - \lambda\tau_n \int_\Omega K(y,\cdot)u_\lambda(y)\tilde{\psi}(0)(y)\,dx\,e^{-\mu} = \mu\tilde{\psi}(0).$$

Hence from Theorem 2.7 and Remark 2.8, we have that $\mathcal{A}^*_{\tau_n}$ has only one pair of purely imaginary eigenvalues $\pm i\nu_\lambda\tau_n$. The simplicity can be proved as in Theorem 2.11. $\square$

Lemma 4.2 implies that $\mathcal{A}_{\tau_n}$ and $\mathcal{A}^*_{\tau_n}$ are adjoint operators under the bilinear form (4.4). The center subspace of Eq. (4.1) is $P = \operatorname{span}\{p(\theta), \overline{p}(\theta)\}$, where $p(\theta) = \psi_\lambda e^{i\nu_\lambda\tau_n\theta}$ is the eigenfunction of $\mathcal{A}_{\tau_n}$ with respect to $i\nu_\lambda\tau_n$. Similarly the formal adjoint subspace of P with respect to the bilinear form (4.4) is $P^* = \operatorname{span}\{q(s), \bar{q}(s)\}$, where $q(s) = \tilde{\psi}_\lambda e^{i\nu_\lambda \tau_n s}$ is the eigenfunction of $\mathcal{A}^*_{\tau_n}$ with respect to $-i\nu_\lambda \tau_n$. Then $\mathcal{C}_{\mathbb{C}}$ can be decomposed as $\mathcal{C}_{\mathbb{C}} = P \oplus Q$, where

$$Q = \left\{\psi \in \mathcal{C}_{\mathbb{C}}\colon \langle\!\langle \tilde{\psi}, \psi \rangle\!\rangle = 0 \text{ for all } \tilde{\psi} \in P^*\right\}.$$

Let $\Phi = (p(\theta), \overline{p}(\theta))$, $\Psi = \frac{1}{\overline{S_n(\lambda)}}(q(s), \bar{q}(s))^T$, where $S_n(\lambda)$ is defined in Lemma 2.10, then $\langle\!\langle \Psi, \Phi \rangle\!\rangle = I$, where I is the identity matrix in $\mathbb{R}^{2\times 2}$.

As the formulas to be developed for the bifurcation direction and stability are all relative to $\gamma = 0$ only, we set $\gamma = 0$ in Eq. (4.1) and obtain a center manifold

$$w(z, \bar{z}) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta) z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \tag{4.6}$$

with the range in Q. The flow of Eq. (4.1) on the center manifold can be written as:

$$U_t = \Phi \cdot \big(z(t), \bar{z}(t)\big)^T + w\big(z(t), \bar{z}(t)\big),$$

where

$$\begin{aligned}
\dot{z}(t) &= \frac{d}{dt}\langle\!\langle q(s), U_t \rangle\!\rangle \\
&= \langle\!\langle q(s), \mathcal{A}_{\tau_n} U_t \rangle\!\rangle + \frac{1}{S_n(\lambda)}\langle\!\langle q(s), X_0 J(U_t, 0) \rangle\!\rangle \\
&= \langle\!\langle \mathcal{A}^*_{\tau_n} q(s), U_t \rangle\!\rangle + \frac{1}{S_n(\lambda)}\langle q(0), J(U_t, 0) \rangle \\
&= i\nu_\lambda \tau_n z(t) + \frac{1}{S_n(\lambda)}\Big\langle q(0), J\Big(\Phi\big(z(t), \bar{z}(t)\big)^T + w\big(z(t), \bar{z}(t)\big), 0\Big)\Big\rangle.
\end{aligned} \tag{4.7}$$

We rewrite (4.7) as

$$\dot{z}(t) = i\nu_\lambda \tau_n z(t) + g(z, \bar{z}) \tag{4.8}$$

with

$$\begin{aligned}
g(z, \bar{z}) &= \frac{1}{S_n(\lambda)}\Big\langle q(0), J\Big(\Phi\big(z(t), \bar{z}(t)\big)^T + w\big(z(t), \bar{z}(t)\big), 0\Big)\Big\rangle \\
&= g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots.
\end{aligned} \tag{4.9}$$

Hence we have that

$$\begin{aligned}
g_{20} &= -\frac{2\lambda\tau_n}{S_n(\lambda)} e^{-i\nu_\lambda \tau_n} \int_\Omega \int_\Omega \overline{\tilde{\psi}_\lambda}(x)\psi_\lambda(x) K(x, y)\psi_\lambda(y)\, dx\, dy, \\
g_{11} &= -\frac{\lambda\tau_n}{S_n(\lambda)} e^{i\nu_\lambda \tau_n} \int_\Omega \int_\Omega \overline{\tilde{\psi}_\lambda}(x)\psi_\lambda(x) K(x, y)\overline{\psi_\lambda}(y)\, dx\, dy \\
&\quad - \frac{\lambda\tau_n}{S_n(\lambda)} e^{-i\nu_\lambda \tau_n} \int_\Omega \int_\Omega \overline{\tilde{\psi}_\lambda}(x)\overline{\psi_\lambda}(x) K(x, y)\psi_\lambda(y)\, dx\, dy,
\end{aligned}$$

$$g_{02} = -\frac{2\lambda\tau_n}{S_n(\lambda)} e^{i\nu_\lambda\tau_n} \int_\Omega\int_\Omega \overline{\tilde{\psi}_\lambda}(x)\overline{\psi_\lambda}(x)K(x, y)\overline{\psi_\lambda}(y)\,dx\,dy,$$

$$\begin{aligned}
g_{21} = &-\frac{2\lambda\tau_n}{S_n(\lambda)} \int_\Omega\int_\Omega \overline{\tilde{\psi}_\lambda}(x)\psi_\lambda(x)K(x, y)w_{11}(-1)(y)\,dx\,dy \\
&- \frac{\lambda\tau_n}{S_n(\lambda)} \int_\Omega\int_\Omega \overline{\tilde{\psi}_\lambda}(x)\overline{\psi_\lambda}(x)K(x, y)w_{20}(-1)(y)\,dx\,dy \\
&- \frac{\lambda\tau_n}{S_n(\lambda)} e^{i\nu_\lambda\tau_n} \int_\Omega\int_\Omega \overline{\tilde{\psi}_\lambda}(x)w_{20}(0)(x)K(x, y)\overline{\psi_\lambda}(y)\,dx\,dy \\
&- \frac{2\lambda\tau_n}{S_n(\lambda)} e^{-i\nu_\lambda\tau_n} \int_\Omega\int_\Omega \overline{\tilde{\psi}_\lambda}(x)w_{11}(0)(x)K(x, y)\psi_\lambda(y)\,dx\,dy.
\end{aligned} \tag{4.10}$$

So in order to compute g_{21}, we need to compute $w_{20}(\theta)$ and $w_{11}(\theta)$.

Since $w(z(t), \bar{z}(t))$ satisfies

$$\begin{aligned}
\dot{w} &= \mathcal{A}_{\tau_n} w + X_0 J\big(\Phi(z, \bar{z})^T + w(z, \bar{z}), 0\big) - \Phi\big\langle\!\big\langle \Psi, X_0 J\big(\Phi(z, \bar{z})^T + w(z, \bar{z}), 0\big)\big\rangle\!\big\rangle \\
&= \mathcal{A}_{\tau_n} w + H_{20}\frac{z^2}{2} + H_{11} z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots,
\end{aligned} \tag{4.11}$$

then by using the chain rule

$$\dot{w} = \frac{\partial w(z, \bar{z})}{\partial z}\dot{z} + \frac{\partial w(z, \bar{z})}{\partial \bar{z}}\dot{\bar{z}},$$

we have that

$$\begin{cases}
(2i\nu_\lambda\tau_n - \mathcal{A}_{\tau_n})w_{20} = H_{20}, \\
-\mathcal{A}_{\tau_n} w_{11} = H_{11}, \\
(-2i\nu_\lambda\tau_n - \mathcal{A}_{\tau_n})w_{02} = H_{02}.
\end{cases} \tag{4.12}$$

Note that for $-1 \leqslant \theta < 0$,

$$-\Phi\big\langle\!\big\langle \Psi, X_0 J\big(\Phi(z, \bar{z})^T + w(z, \bar{z}), 0\big)\big\rangle\!\big\rangle = H_{20}\frac{z^2}{2} + H_{11} z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots,$$

and then we see that for $-1 \leqslant \theta < 0$,

$$H_{20}(\theta) = -\big(g_{20}p(\theta) + \bar{g}_{02}\bar{p}(\theta)\big), \tag{4.13}$$

$$H_{11}(\theta) = -\big(g_{11}p(\theta) + \bar{g}_{11}\bar{p}(\theta)\big). \tag{4.14}$$

Therefore from (4.12), w_{20} and w_{11} can be expressed as

$$w_{20}(\theta) = \frac{ig_{20}}{\nu_\lambda\tau_n}p(\theta) + \frac{i\bar{g}_{02}}{3\nu_\lambda\tau_n}\bar{p}(\theta) + Ee^{2i\nu_\lambda\tau_n\theta} \tag{4.15}$$

and

$$w_{11}(\theta) = -\frac{i g_{11}}{\nu_\lambda \tau_n} p(\theta) + \frac{i \bar{g}_{11}}{\nu_\lambda \tau_n} \bar{p}(\theta) + F. \tag{4.16}$$

From Eqs. (4.11) and (4.12) with $\theta = 0$, the definition of $\mathcal{A}_{\tau_n}$ and

$$H_{20}(0) = -\big(g_{20} p(0) + \bar{g}_{02} \bar{p}(0)\big) - 2\lambda \tau_n e^{-i\nu_\lambda \tau_n} \psi_\lambda \int\limits_\Omega K(\cdot, y) \psi_\lambda(y)\, dy,$$

we find that E satisfies

$$(2i\nu_\lambda \tau_n - \mathcal{A}_{\tau_n}) E e^{2i\nu_\lambda \tau_n \theta}\big|_{\theta=0} = -2\lambda \tau_n e^{-i\nu_\lambda \tau_n} \psi_\lambda \int\limits_\Omega K(\cdot, y) \psi_\lambda(y)\, dy,$$

that is,

$$\Delta(\lambda, 2i\nu_\lambda, \tau_n) E = 2\lambda e^{-i\nu_\lambda \tau_n} \psi_\lambda \int\limits_\Omega K(\cdot, y) \psi_\lambda(y)\, dy. \tag{4.17}$$

From Corollary 2.6, we have $2i\nu_\lambda$ is not the eigenvalue of $A_{\tau_n}(\lambda)$, and hence

$$E = 2\lambda e^{-i\nu_\lambda \tau_n} \Delta(\lambda, 2i\nu_\lambda, \tau_n)^{-1} \left(\psi_\lambda \int\limits_\Omega K(\cdot, y) \psi_\lambda(y)\, dy \right),$$

where $\Delta(\lambda, \mu, \tau)$ is defined in Eq. (2.9). Similarly,

$$F = \lambda \Delta(\lambda, 0, \tau_n)^{-1} \left(e^{i\nu_\lambda \tau_n} \psi_\lambda \int\limits_\Omega K(\cdot, y) \overline{\psi_\lambda}(y)\, dy + e^{-i\nu_\lambda \tau_n} \overline{\psi_\lambda} \int\limits_\Omega K(\cdot, y) \psi_\lambda(y)\, dy \right). \tag{4.18}$$

Now we compute the functions E and F in the following lemma.

Lemma 4.3. *For $\lambda \in (\lambda_*, \lambda^*]$, let E and F be defined as in (4.15) and (4.16). Then*

$$E = \frac{1}{\lambda - \lambda_*} (c_\lambda u_\lambda + \varphi_\lambda), \qquad F = \frac{\tilde{\varphi}_\lambda}{\lambda - \lambda_*}, \tag{4.19}$$

where u_λ is the positive solution of Eq. (1.2) satisfying (2.2), φ_λ, $\tilde{\varphi}_\lambda$ satisfy

$$\langle u_\lambda, \varphi_\lambda \rangle = 0, \qquad \lim_{\lambda \to \lambda_*} \|\varphi_\lambda\|_{Y_\mathbb{C}} = 0, \qquad \lim_{\lambda \to \lambda_*} \|\tilde{\varphi}_\lambda\|_{Y_\mathbb{C}} = 0,$$

and the constant c_λ satisfies $\lim_{\lambda \to \lambda_*} (\lambda - \lambda_*) c_\lambda = \frac{2i}{\alpha_{\lambda_*}^2 (2i-1)}$.

Proof. We only prove the estimate for E, and the one for F is similar. Substituting Eq. (4.19) into Eq. (4.17), we have that

$$A(\lambda)\varphi_\lambda - \lambda u_\lambda \int_\Omega K(\cdot, y)(c_\lambda u_\lambda + \varphi_\lambda)\,dy\, e^{-2i\nu_\lambda\tau_n} - 2i\nu_\lambda(c_\lambda u_\lambda + \varphi_\lambda)$$
$$= 2\lambda e^{-i\nu_\lambda\tau_n}(\lambda - \lambda_*)\psi_\lambda \int_\Omega K(\cdot, y)\psi_\lambda(y)\,dy. \tag{4.20}$$

Multiplying Eq. (4.20) by u_λ, we have that

$$c_\lambda\left(\lambda \int_\Omega\int_\Omega K(x, y)u_\lambda^2(x)u_\lambda(y)\,dx\,dy\, e^{-2i\nu_\lambda\tau_n} + 2i\nu_\lambda\|u_\lambda\|_{Y_\mathbb{C}}^2\right)$$
$$= -\lambda \int_\Omega\int_\Omega K(x, y)u_\lambda^2(x)\varphi_\lambda(y)\,dx\,dy\, e^{-2i\nu_\lambda\tau_n}$$
$$- 2\lambda e^{-i\nu_\lambda\tau_n}(\lambda - \lambda_*)\int_\Omega\int_\Omega K(x, y)u_\lambda(x)\psi_\lambda(x)\psi_\lambda(y)\,dx\,dy. \tag{4.21}$$

Multiplying Eq. (4.20) by φ_λ, we have that

$$\langle A(\lambda)\varphi_\lambda, \varphi_\lambda\rangle - \lambda c_\lambda \int_\Omega\int_\Omega K(x, y)\varphi_\lambda(x)u_\lambda(x)u_\lambda(y)\,dx\,dy\, e^{-2i\nu_\lambda\tau_n}$$
$$= \lambda \int_\Omega\int_\Omega K(x, y)u_\lambda(x)\varphi_\lambda(x)\varphi_\lambda(y)\,dx\,dy\, e^{-2i\nu_\lambda\tau_n} + 2i\nu_\lambda\|\varphi_\lambda\|_{Y_\mathbb{C}}^2$$
$$+ 2\lambda e^{-i\nu_\lambda\tau_n}(\lambda - \lambda_*)\int_\Omega\int_\Omega K(x, y)\varphi_\lambda(x)\psi_\lambda(x)\psi_\lambda(y)\,dx\,dy. \tag{4.22}$$

From the expression of ν_λ, u_λ, ψ_λ and τ_n, we have that

$$\psi_\lambda \to \phi, \qquad u_\lambda/(\lambda - \lambda_*) \to \alpha_{\lambda_*}\phi, \qquad \nu_\lambda/(\lambda - \lambda_*) \to 1, \quad \text{and} \quad \nu_\lambda\tau_n \to \frac{\pi}{2} + 2n\pi.$$

So from Eq. (4.21), we have that

$$(\lambda - \lambda_*)c_\lambda = -\frac{(\lambda - \lambda_*)\lambda\int_\Omega\int_\Omega K(x, y)u_\lambda^2(x)\varphi_\lambda(y)\,dx\,dy\, e^{-2i\nu_\lambda\tau_n}}{(\lambda\int_\Omega\int_\Omega K(x, y)u_\lambda^2(x)u_\lambda(y)\,dx\,dy + 2i\nu_\lambda\|u_\lambda\|_{Y_\mathbb{C}})}$$
$$- \frac{2\lambda e^{-i\nu_\lambda\tau_n}(\lambda - \lambda_*)^2\int_\Omega\int_\Omega K(x, y)u_\lambda(x)\psi_\lambda(x)\psi_\lambda(y)\,dx\,dy}{(\lambda\int_\Omega\int_\Omega K(x, y)u_\lambda^2(x)u_\lambda(y)\,dx\,dy + 2i\nu_\lambda\|u_\lambda\|_{Y_\mathbb{C}})},$$

and hence there exist constants $\tilde{\lambda} > \lambda_*$, $M_0, M_1 > 0$ such that for any $\lambda \in (\lambda_*, \tilde{\lambda})$, $|(\lambda - \lambda_*)c_\lambda| \leqslant M_0\|\varphi_\lambda\|_{Y_\mathbb{C}} + M_1$. From Eq. (4.22), and the expression of ν_λ, u_λ, ψ_λ and τ_n, we have that there exist constants $\check{\lambda} > \lambda_*$, $M_3, M_4, M_5 > 0$ such that for any $\lambda \in (\lambda_*, \check{\lambda})$,

$$|\lambda_2(\lambda)| \cdot \|\phi_\lambda\|_{Y_\mathbb{C}}^2 \leqslant M_3(\lambda - \lambda_*)\|\varphi_\lambda\|_{Y_\mathbb{C}}(M_0\|\varphi_\lambda\|_{Y_\mathbb{C}} + M_1)$$
$$+ (\lambda - \lambda_*)M_4\|\varphi_\lambda\|_{Y_\mathbb{C}}^2 + M_5(\lambda - \lambda_*)\|\varphi_\lambda\|_{Y_\mathbb{C}},$$

where $\lambda_2(\lambda)$ is the second eigenvalue of $A(\lambda)$. Since $\lim_{\lambda\to\lambda_*}\lambda_2(\lambda)=\lambda_2-\lambda_*>0$, where λ_2 is the second eigenvalue of Eq. (1.6), then we have that $\lim_{\lambda\to\lambda_*}\|\varphi_\lambda\|_{Y_{\mathbb{C}}}=0$. Together with (4.21), we have that

$$\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)c_\lambda=\frac{2i}{\alpha_{\lambda_*}^2(2i-1)}. \qquad \square$$

It is well known that the following quantities determine the direction and stability of bifurcating periodic orbits (see [23,40]):

$$C_1(0)=\frac{i}{2\nu_\lambda\tau_n}\left(g_{11}g_{20}-2|g_{11}|^2-\frac{|g_{02}|^2}{3}\right)+\frac{g_{21}}{2}, \qquad \mu_2=-\frac{\mathcal{R}e(C_1(0))}{\mathcal{R}e(\mu'(\tau_n))},$$
$$\beta_2=2\mathcal{R}e\big(C_1(0)\big), \qquad T_2=-\frac{\mathcal{I}m(C_1(0))+\mu_2\mathcal{I}m(\mu'(\tau_n))}{\tau_n}.$$

Here

1. μ_2 determines the direction of the Hopf bifurcation: if $\mu_2>0$ ($\mu_2<0$), then the bifurcating periodic solutions exist for $\tau>\tau_n$ ($\tau<\tau_n$), and the bifurcation is called forward (backward);
2. β_2 determines the stability of bifurcating periodic solutions: the bifurcating periodic solutions are orbitally asymptotically stable (unstable) if $\beta_2<0$ ($\beta_2>0$);
3. T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2>0$ ($T_2<0$).

From Eqs. (4.10), (4.15), (4.16) and (4.19), we can compute g_{20}, g_{11}, g_{02} and g_{21} for the periodic orbits emerging from the Hopf bifurcation of Eq. (1.2) obtained in Theorem 2.14. Since $\lim_{\lambda\to\lambda_*}\psi_\lambda(x)=\lim_{\lambda\to\lambda_*}\tilde{\psi}_\lambda(x)=\phi(x)$, then

$$\lim_{\lambda\to\lambda_*}S_n(\lambda)=\frac{1}{2}\big(2+i(\pi+4n\pi)\big)\int_\Omega\phi^2(x)\,dx, \qquad \lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)F=0,$$
$$\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)\tau_n=\lim_{\lambda\to\lambda_*}\nu_\lambda\tau_n=\frac{\pi}{2}+2n\pi, \qquad \lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)E=\frac{2i}{\alpha_{\lambda_*}(2i-1)}\phi.$$

So we compute that

$$\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)g_{20}=\frac{2i(\pi+4n\pi)}{\alpha_{\lambda_*}(2+i(\pi+4n\pi))}, \qquad \lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)g_{11}=0,$$
$$\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)g_{02}=\frac{-2i(\pi+4n\pi)}{\alpha_{\lambda_*}(2+i(\pi+4n\pi))},$$
$$\lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)^2g_{21}=\frac{2(\pi+4n\pi)(1-3i)}{\alpha_{\lambda_*}^2(10+5i(\pi+4n\pi))}+\frac{8i\pi(1+4n)}{3\alpha_{\lambda_*}^2|2\pi+i(\pi+4n\pi)|^2}.$$

Then we can compute that $\lim_{\lambda\to\lambda_*}\mathcal{R}e((\lambda-\lambda_*)^2g_{21})<0$ and $\lim_{\lambda\to\lambda_*}\mathcal{R}e((\lambda-\lambda_*)^2C_1(0))<0$. Hence we have the following results:

Theorem 4.4. *For $\lambda\in(\lambda_*,\lambda^*]$, let $\tau_n(\lambda)$ given as in (2.15) be the Hopf bifurcation points for Eq. (1.2) where spatially nonhomogeneous periodic orbits of Eq. (1.2) emerge from (τ_n,u_λ). Then for each $n\in\mathbb{N}\cup\{0\}$, the direction of the Hopf bifurcation at $\tau=\tau_n$ is forward and the bifurcating periodic solution from $\tau=\tau_0$ is locally asymptotically stable.*

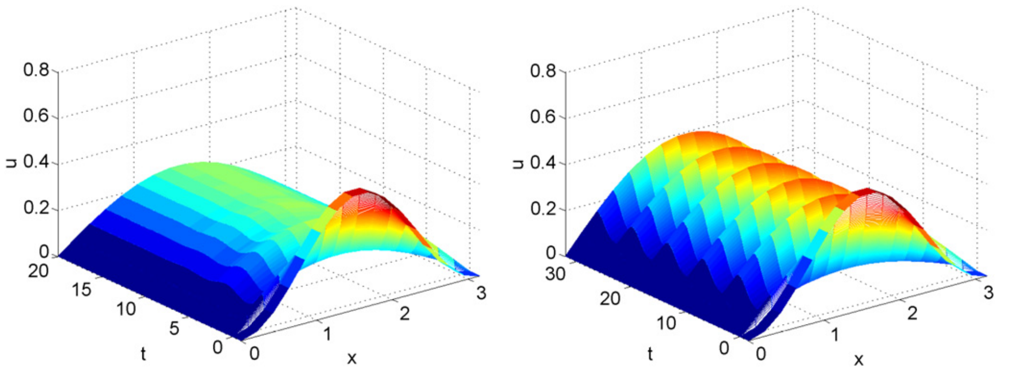

Fig. 2. Spatially homogeneous kernel $K(x, y) = 1$. Left: $\tau = 1$; right: $\tau = 1.6$.

Similarly we can consider the direction of Hopf bifurcations and stability of bifurcating periodic orbits of Eq. (3.1) for all $\lambda > 1$. In this case, from the proof of Theorem 3.3, ψ_λ, $\tilde{\psi}_\lambda$ and $S_n(\lambda)$ can all be explicitly calculated for all $\lambda > 1$, and the normal form calculation earlier also holds here. From Eqs. (4.17) and (4.18), we can also solve that

$$E = \frac{4i\lambda}{(2i-1)(\lambda-1)} \sin x, \qquad F = 0,$$

and by substituting the explicit form of ψ_λ, $\tilde{\psi}_\lambda$, and $S_n(\lambda)$ into Eq. (4.10), we have that

$$g_{20} = \frac{4i(\pi + 4n\pi)\lambda}{(\lambda-1)(2+i(\pi+4n\pi))}, \qquad g_{11} = 0,$$
$$g_{02} = \frac{-4i(\pi + 4n\pi)\lambda}{(\lambda-1)(2+i(\pi+4n\pi))},$$
$$g_{21} = \frac{8(\pi+4n\pi)(1-3i)\lambda^2}{(\lambda-1)^2(10+5i(\pi+4n\pi))} + \frac{32i\pi(1+4n)\lambda^2}{3(\lambda-1)^2|2+i(\pi+4n\pi)|^2}.$$

Then again we obtain that $\mathcal{R}e(g_{21}) < 0$ and $\mathcal{R}e(C_1(0)) < 0$. Hence we have the following results:

Theorem 4.5. *For each $\lambda > 1$, let $\tau_n(\lambda)$ given as in (3.13) be the Hopf bifurcation points for Eq. (3.1) where spatially nonhomogeneous periodic orbits of Eq. (3.1) emerge from (τ_n, u_λ). Then for each $n \in \mathbb{N} \cup \{0\}$, the direction of the Hopf bifurcation at $\tau = \tau_n$ is forward and the bifurcating periodic solution from $\tau = \tau_0$ is locally asymptotically stable.*

The results in Theorems 4.4 and 4.5 show that the Hopf bifurcation at τ_n is forward, hence for some $\varepsilon_n > 0$, there exists a spatially inhomogeneous periodic orbit for (1.2) (or (3.1) respectively) when $\tau \in (\tau_n, \tau_n + \varepsilon_n)$. This in a sense shows that a true Hopf bifurcation occurs at $\tau = \tau_n$. Finally we show two numerical simulations of Eq. (1.2) to demonstrate our results. In Fig. 2, the numerical simulations with a homogeneous kernel $K(x, y) \equiv 1$ are shown, and in Fig. 3, the ones with a nonhomogeneous kernel $K(x, y) = \frac{|x-y|}{\pi}$ are shown respectively. In each figure, $\lambda = 2$, $\Omega = (0, \pi)$, $d = 1$, and the initial value is $u(x, t) = 0.5 \sin^2 x$. In each case, the convergence to the spatially nonhomogeneous equilibrium u_λ occurs when τ is less than the first Hopf bifurcation point τ_0, and an oscillatory pattern emerges for $\tau > \tau_0$. While each simulation verifies the occurrence of spatially nonhomogeneous temporal oscillation, one can notice that the spatial profiles of the periodic solutions are different due to the different dispersal kernel. In particular, the spatial profile in Fig. 2 is concave (indeed it is $\sin x$ in this case), and the one in Fig. 3 is not.

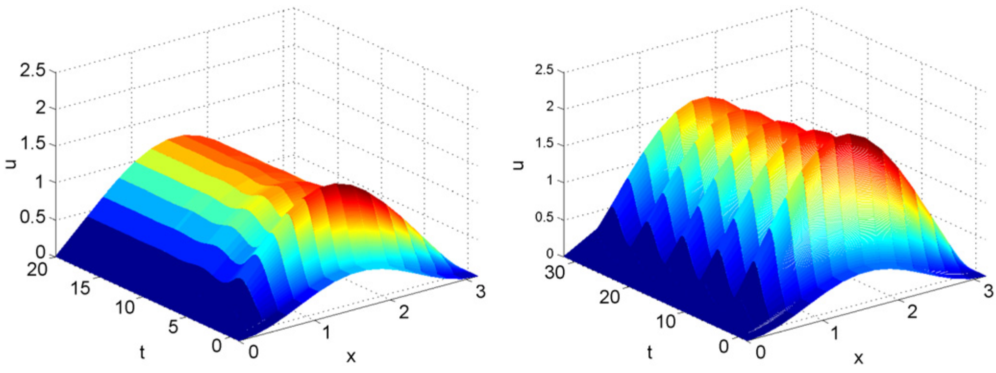

Fig. 3. Spatially nonhomogeneous kernel $K(x, y) = \frac{|x-y|}{\pi}$. Left: $\tau = 1$; right: $\tau = 1.6$.

References

[1] Shangbing Ai, Traveling wave fronts for generalized Fisher equations with spatio-temporal delays, J. Differential Equations 232 (1) (2007) 104–133.
[2] Henri Berestycki, Grégoire Nadin, Benoit Perthame, Lenya Ryzhik, The non-local Fisher–KPP equation: Travelling waves and steady states, Nonlinearity 22 (12) (2009) 2813–2844.
[3] N.F. Britton, Spatial structures and periodic travelling waves in an integro-differential reaction–diffusion population model, SIAM J. Appl. Math. 50 (6) (1990) 1663–1688.
[4] Stavros Busenberg, Wenzhang Huang, Stability and Hopf bifurcation for a population delay model with diffusion effects, J. Differential Equations 124 (1) (1996) 80–107.
[5] Jixun Chu, Arnaud Ducrot, Pierre Magal, Shigui Ruan, Hopf bifurcation in a size-structured population dynamic model with random growth, J. Differential Equations 247 (3) (2009) 956–1000.
[6] Michael G. Crandall, Paul H. Rabinowitz, Bifurcation from simple eigenvalues, J. Funct. Anal. 8 (1971) 321–340.
[7] Fordyce A. Davidson, Niall Dodds, Spectral properties of non-local differential operators, Appl. Anal. 85 (6–7) (2006) 717–734.
[8] Thomas Erneux, Applied Delay Differential Equations, Surv. Tutor. Appl. Math. Sci., vol. 3, Springer-Verlag, New York, 2009.
[9] Jian Fang, Xiaoqiang Zhao, Monotone wavefronts of the nonlocal Fisher–KPP equation, Nonlinearity 24 (2011) 3043–3054.
[10] Teresa Faria, Normal forms for semilinear functional differential equations in Banach spaces and applications. II, Discrete Contin. Dyn. Syst. 7 (1) (2001) 155–176.
[11] Teresa Faria, Wenzhang Huang, Stability of periodic solutions arising from Hopf bifurcation for a reaction–diffusion equation with time delay, in: Differential Equations and Dynamical Systems, Lisbon, 2000, in: Fields Inst. Commun., vol. 31, Amer. Math. Soc., Providence, RI, 2002, pp. 125–141.
[12] Teresa Faria, Wenzhang Huang, Jianhong Wu, Smoothness of center manifolds for maps and formal adjoints for semilinear FDEs in general Banach spaces, SIAM J. Math. Anal. 34 (1) (2002) 173–203.
[13] Pedro Freitas, A nonlocal Sturm–Liouville eigenvalue problem, Proc. Roy. Soc. Edinburgh Sect. A 124 (1) (1994) 169–188.
[14] Pedro Freitas, Guido Sweers, Positivity results for a nonlocal elliptic equation, Proc. Roy. Soc. Edinburgh Sect. A 128 (4) (1998) 697–715.
[15] M.A. Fuentes, M.N. Kuperman, V.M. Kenkre, Nonlocal interaction effects on pattern formation in population dynamics, Phys. Rev. Lett. 91 (2003) 158104.
[16] S. Genieys, V. Volpert, P. Auger, Pattern and waves for a model in population dynamics with nonlocal consumption of resources, Math. Model. Nat. Phenom. 1 (1) (2006) 65–82 (electronic).
[17] S.A. Gourley, Travelling front solutions of a nonlocal Fisher equation, J. Math. Biol. 41 (3) (2000) 272–284.
[18] S.A. Gourley, N.F. Britton, A predator–prey reaction–diffusion system with nonlocal effects, J. Math. Biol. 34 (3) (1996) 297–333.
[19] S.A. Gourley, J.W.-H. So, Dynamics of a food-limited population model incorporating nonlocal delays on a finite domain, J. Math. Biol. 44 (1) (2002) 49–78.
[20] S.A. Gourley, J.W.H. So, J.H. Wu, Nonlocality of reaction–diffusion equations induced by delay: Biological modeling and nonlinear dynamics, J. Math. Sci. 124 (4) (2004) 5119–5153.
[21] David Green Jr., Harlan W. Stech, Diffusion and hereditary effects in a class of population models, in: Differential Equations and Applications in Ecology, Epidemics, and Population Problems, Claremont, CA, 1981, Academic Press, New York, 1981, pp. 19–28.
[22] Jack Hale, Theory of Functional Differential Equations, second ed., Appl. Math. Sci., vol. 3, Springer-Verlag, New York, 1977.
[23] Brian D. Hassard, Nicholas D. Kazarinoff, Yieh Hei Wan, Theory and Applications of Hopf Bifurcation, London Math. Soc. Lecture Note Ser., vol. 41, Cambridge University Press, Cambridge, 1981.
[24] Wenzhang Huang, Global dynamics for a reaction–diffusion equation with time delay, J. Differential Equations 143 (2) (1998) 293–326.

[25] G.E. Hutchinson, Circular causal systems in ecology, in: Ann. New York Acad. Sci., vol. 50(4), 1948, pp. 221–246.
[26] V.M. Kenkre, N. Kumar, Nonlinearity in bacterial population dynamics: Proposal for experiments for the observation of abrupt transitions in patches, Proc. Natl. Acad. Sci. USA 105 (48) (2008) 18752.
[27] Yang Kuang, Delay Differential Equations with Applications in Population Dynamics, Math. Sci. Eng., vol. 191, Academic Press, Boston, MA, 1993.
[28] A.L. Lin, B.A. Mann, G. Torres-Oviedo, B. Lincoln, J. Käs, H.L. Swinney, Localization and extinction of bacterial populations under inhomogeneous growth conditions, Biophys. J. 87 (1) (2004) 75–80.
[29] M.C. Memory, Bifurcation and asymptotic behavior of solutions of a delay-differential equation with diffusion, SIAM J. Math. Anal. 20 (3) (1989) 533–546.
[30] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Appl. Math. Sci., vol. 44, Springer-Verlag, New York, 1983.
[31] Paul H. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Anal. 7 (1971) 487–513.
[32] S. Ruan, Delay differential equations in single species dynamics, in: Delay Differential Equations and Applications, in: NATO Sci. Ser. II Math. Phys. Chem., vol. 205, Springer-Verlag, Dordrecht, 2006, pp. 477–517.
[33] Hal Smith, An Introduction to Delay Differential Equations with Applications to the Life Sciences, Texts Appl. Math., vol. 57, Springer-Verlag, New York, 2011.
[34] Joseph W.-H. So, Jianhong Wu, Yuanjie Yang, Numerical steady state and Hopf bifurcation analysis on the diffusive Nicholson's blowflies equation, Appl. Math. Comput. 111 (1) (2000) 33–51.
[35] Joseph W.-H. So, Yuanjie Yang, Dirichlet problem for the diffusive Nicholson's blowflies equation, J. Differential Equations 150 (2) (1998) 317–348.
[36] Ying Su, Junjie Wei, Junping Shi, Hopf bifurcations in a reaction–diffusion population model with delay effect, J. Differential Equations 247 (4) (2009) 1156–1184.
[37] Ying Su, Junjie Wei, Junping Shi, Bifurcation analysis in a delayed diffusive Nicholson's blowflies equation, Nonlinear Anal. Real World Appl. 11 (3) (2010) 1692–1703.
[38] Ying Su, Junjie Wei, Junping Shi, Hopf bifurcation in a diffusive logistic equation with mixed delayed and instantaneous density dependence, J. Dynam. Differential Equations (2012), http://dx.doi.org/10.1007/s10884-012-9268-z, in press.
[39] Zhi-Cheng Wang, Wan-Tong Li, Shigui Ruan, Travelling wave fronts in reaction–diffusion systems with spatio-temporal delays, J. Differential Equations 222 (1) (2006) 185–232.
[40] Jianhong Wu, Theory and Applications of Partial Functional-Differential Equations, Appl. Math. Sci., vol. 119, Springer-Verlag, New York, 1996.
[41] Xiang-Ping Yan, Wan-Tong Li, Stability of bifurcating periodic solutions in a delayed reaction–diffusion population model, Nonlinearity 23 (6) (2010) 1413–1431.
[42] Taishan Yi, Xingfu Zou, Global attractivity of the diffusive Nicholson blowflies equation with Neumann boundary condition: A non-monotone case, J. Differential Equations 245 (11) (2008) 3376–3388.
[43] K. Yoshida, The Hopf bifurcation and its stability for semilinear diffusion equations with time delay arising in ecology, Hiroshima Math. J. 12 (2) (1982) 321–348.
[44] Xiao-Qiang Zhao, Global attractivity in a class of nonmonotone reaction–diffusion equations with time delay, Can. Appl. Math. Q. 17 (1) (2009) 271–281.