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Smoothness of generalized inverses☆

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Dedicated to the memory of Israel Gohberg, our teacher and mentor

Abstract

The paper is largely expository. It is shown that if $a(x)$ is a smooth unital Banach algebra valued function of a parameter x, and if $a(x)$ has a locally bounded generalized inverse in the algebra, then a generalized inverse of $a(x)$ exists which is as smooth as $a(x)$ is. Smoothness is understood in the sense of having a certain number of continuous derivatives, being real-analytic, or complex holomorphic. In the complex holomorphic case, the space of parameters is required to be a Stein manifold. Local formulas for the generalized inverses are given. In particular, the Moore–Penrose and the generalized Drazin inverses are studied in this context.

Keywords: Banach algebra; C^*-algebra; Generalized inverse; Drazin inverse; Moore–Penrose inverse; Smoothness; Analytic dependence

1. Introduction

Let $\mathfrak{B}$ be a complex unital Banach algebra. It is a well-known and often useful fact that the inverse a^{-1} of an invertible element $a \in \mathfrak{B}$ is a holomorphic function of a, i.e. a^{-1} admits

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a (noncommutative) power series expansion in a neighborhood of every invertible element; indeed,

$$a_0^{-1}\left(\sum_{j=0}^{\infty}((a_0-a)a_0^{-1})^j\right)=a^{-1}=\left(\sum_{j=0}^{\infty}(a_0^{-1}(a_0-a))^j\right)a_0^{-1} \tag{1.1}$$

for all $a \in \mathfrak{B}$ sufficiently close to an invertible element $a_0 \in \mathfrak{B}$. The same expansion holds for one-sided inverses; thus, the right, and left equality in (1.1) are valid for all $a \in \mathfrak{B}$ sufficiently close to a right-invertible, resp. left-invertible, element $a_0 \in \mathfrak{B}$, where now a^{-1} is understood as a right inverse, resp. left inverse, of a. Since one-sided inverses are generally speaking not unique, we may say that a one-sided inverse of a one-sided invertible element $a \in \mathfrak{B}$ locally can be *chosen* a holomorphic function of a.

This statement breaks down once we consider generalized inverses. An element $b \in \mathfrak{B}$ is said to be a *generalized inverse*, in short GI, of $a \in \mathfrak{B}$ if the equalities

$$aba = a \quad \text{and} \quad bab = b \tag{1.2}$$

hold. Indeed, for $\mathfrak{B} = \mathbb{C}$, the complex field, the (unique) GI of $z \in \mathbb{C}$ is z^{-1} if $z \neq 0$, and 0 if $z = 0$. Thus, the GI function is discontinuous at zero.

Therefore, additional hypotheses are needed to ensure smooth behavior of GIs, suitably chosen. In the literature, these additional hypotheses often take the form of assuming Fredholm type properties of operators and invariance of dimension of certain subspaces.

For example, let $\phi : \Omega \to L(E)$ be a holomorphic function on a domain $\Omega \subseteq \mathbb{C}$, all values of which are Fredholm operators. Recall that then, by Gohberg's theorem [12, Theorem 1], there is a subset Λ of Ω, which is discrete and relatively closed in Ω, such that $n_0 := \dim \operatorname{Ker} \phi(z)$ is constant for $z \in \Omega \setminus \Lambda$, whereas $\dim \operatorname{Ker} \phi(z) > n_0$ if $z \in \Lambda$. The following local fact is an immediate corollary of the local Gohberg–Sigal factorization theorem [17, Theorem 3.1]: *For each $z_0 \in \Omega$, there exists a neighborhood $U \subseteq \Omega$ of z_0 and a holomorphic function $\psi : U \setminus \{z_0\} \to L(E)$ such that $\psi(z)$ is a GI of $\phi(z)$ for all $z \in U \setminus \{z_0\}$, and, moreover, the associated projection functions $\phi\psi$ and $\psi\phi$ (see* Remark 4.4*) admit holomorphic extensions to z_0, whereas ψ itself admits a holomorphic extension to z_0 if and only if $z_0 \notin \Lambda$.* Bart [4, Theorem 5.2] and Shubin [47, Corollary 4 on p. 419] independently proved that *there exists a global holomorphic function $\psi : \Omega \setminus \Lambda \to L(E)$ such that $\psi(z)$ is a GI of $\phi(z)$ for all $z \in \Omega \setminus \Lambda$.* This result then was complemented by Bart et al. [5, Theorem 2.2] proving that *this function ψ can be chosen so that the associated projection functions $\phi\psi$ and $\psi\phi$ admit holomorphic extensions to Λ.*

Browder's theorem [6], in the context of $L(H)$, where H is a Hilbert space, also follows the approach of Fredholm type properties and invariance of dimension.

In this paper, we prove results in which the additional hypotheses assert local boundedness, as follows: *If $a = a(x) \in \mathfrak{B}$ is a smooth (in the sense of having a certain number of continuous derivatives, being real-analytic, or (complex-) holomorphic) function of a parameter x, and $a(x)$ has a GI for every x which can be chosen bounded (possibly not continuous) in a neighborhood of x, then there exists a GI of $a(x)$ as smooth as $a(x)$ is.* (In the matrix case, i.e. when $\mathfrak{B}$ is finite dimensional and therefore can be identified with an algebra of matrices, the local boundedness condition amounts to the rank of $a(x)$ being locally constant in x.) We make this statement precise in various contexts and for several classes of GIs in Sections 3–5, including the Moore–Penrose inverse in Section 3. In Section 7, these questions are studied for (generalized) Drazin inverses.

We intend the exposition to be reasonably self-contained and accessible for a wide audience of mathematicians, including non-experts. The present paper is largely expository, although we do include several seemingly new results. Thus, in Section 2 we proceed with preparatory material concerning continuous families of subspaces of a Hilbert space (which will be generalized to Banach spaces in Section 4).

To review the main results of Sections 3 and 4, let $X \subseteq \mathbb{R}^n$ be an open set. Let us say that a function $a : X \to \mathfrak{B}$ satisfies condition (C) if

- for each $x_0 \in X$, there exist a neighborhood U of x_0 and a continuous function $b : U \to \mathfrak{B}$ such that $aba = a$ on U,

and let us say that it satisfies condition (B) if

- for each $x_0 \in X$, there exist a neighborhood U of x_0 and a bounded (possibly not continuous) function $b : U \to \mathfrak{B}$ such that $aba = a$ on U.

(Note that functions satisfying condition (B) later will be called *locally boundedly generalized invertible*— Definition 5.1.)

If $\mathfrak{B} = L(E)$, where E is a Banach space, it is easy to see (cf. the proof of Proposition 4.7) that condition (B) implies a certain other condition called in the literature *uniform regularity*— Definition 4.5. By a result of Markus [43] it is known that, assuming continuity of a, uniform regularity is equivalent to condition (B). So, in the $L(E)$ case, assuming continuity of a, the apparently much weaker condition (B) is actually equivalent to (C). (In the matrix case, this is easy to see—for a continuous matrix function, each condition means that the matrix function has locally constant rank.) For convenience of the reader, in Section 4, we will prove this result of Markus— Proposition 4.7.

Then in Section 5, we obtain this equivalence also in the case of a general Banach algebra $\mathfrak{B}$— Corollary 5.6.

First, in Section 3, we consider the case when $\mathfrak{B}$ is a C^*-algebra. Then each generalized invertible element $a \in \mathfrak{B}$ has a canonical GI, the *Moore–Penrose inverse*, a^+, which is uniquely determined (in the set of all GIs of a) by the additional condition

$$(aa^+)^* = aa^+ \quad \text{and} \quad (a^+a)^* = a^+a.$$

The main result of Section 3 is Theorem 3.6, which says that if a function $a : X \to \mathfrak{B}$ is of class $\mathcal{C}^\alpha, 0 \leq \alpha \leq \infty$, (or real-analytic), and satisfies condition (C), then the Moore–Penrose inverse of a is also of class $\mathcal{C}^\alpha, 0 \leq \alpha \leq \infty$ (or real-analytic). (For the precise definition of $\mathcal{C}^\alpha$ see Section 2.) In the matrix case, Theorem 3.6 is well-known (see, for example, [10,48]). Together with the equivalence of (B) and (C), proved later in Section 5, Corollary 5.6, then we obtain that condition (B) can be replaced by (C)— Theorem 3.7. In particular: *If the Moore–Penrose inverse of a $\mathcal{C}^\alpha$ (or real-analytic) C^*-algebra valued function is locally bounded, then it actually is $\mathcal{C}^\alpha$ (or real-analytic).*

Note that, with a trivial exception, the Moore–Penrose inverse of a (complex-) holomorphic function is not holomorphic (see Remark 3.8). To get a holomorphic GI, we have to make another choice (explained in Sections 5 and 6).

In Section 5, we pass to the case of a general complex unital Banach algebra $\mathfrak{B}$. Here the main result is Theorem 5.2. Under the condition that a continuous function $a : X \to \mathfrak{B}$ satisfies condition (B), formula (5.2) – which we call the *Atkinson formula* – provides a “good local choice” of GIs for a. Immediate corollaries of this formula are:

(I) if $a : X \to \mathfrak{B}$ is of class $\mathcal{C}^\alpha, 0 \le \alpha \le \infty$, (or real-analytic) and satisfies condition (B), then, locally, a admits a GI, which is also $\mathcal{C}^\alpha$ (or real-analytic);

(II) If X is an open subset of $\mathbb{C}^n$, $a : X \to \mathfrak{B}$ is (complex-) holomorphic and satisfies condition (B), then, locally, a admits a GI, which is also holomorphic;

(III) setting $\alpha = 0$ in (I) it follows, assuming continuity of a, that conditions (B) and (C) are equivalent.

Section 6 is devoted to global GIs. Here the first result is Theorem 6.1: *For every* $\mathcal{C}^\alpha, 0 \le \alpha \le \infty$, (*or real-analytic*) *manifold with countable topology, and every* $\mathcal{C}^\alpha$ (*or real-analytic*) *function* $a : X \to \mathfrak{B}$ *satisfying condition* (B), *there exists a global GI for* a *on* X, *which is also* $\mathcal{C}^\alpha$ (*or real-analytic*). In the $\mathcal{C}^\alpha$ case, the proof is simple, because the local GIs which we have from Theorem 5.2 can be easily glued, using a $\mathcal{C}^\alpha$ partition of unity. In the real-analytic case, this simple proof does not work, because real-analytic partitions of unity do not exist.

Therefore we first prove the following Theorem 6.3: *If* X *is a Stein manifold, then each holomorphic function* $a : X \to \mathfrak{B}$ *satisfying condition* (B) *admits a global holomorphic GI on* X. From this theorem we then deduce the real-analytic case of Theorem 6.1, using Grauert's tube theorem [21]—a well-known method, employed in this context first by Gramsch [19, Section 2.3] (see Theorem 6.7).

In Section 7 we consider briefly generalized Drazin inverses (GDI). Here again, the smoothness of the GDI's of a smooth unital Banach algebra valued function is guaranteed provided they exist and are locally bounded (Theorem 7.1). We leave aside the theory of inverses, one-sided inverses and generalized inverses of meromorphic functions with values in a Banach algebra. Including some of this theory would take us too far afield, and we only mention here key references [17,5,19,20,18] (for one variable, see also the book [15]).

We conclude the introduction with two simple but useful remarks. For Banach spaces E, F, we denote by $L(E, F)$ the Banach space (algebra if $E = F$) of all bounded linear operators $E \to F$; $L(E, E)$ will be often abbreviated to $L(E)$.

Remark 1.1. A generalized inverse $b \in L(F, E)$ of an operator $a \in L(E, F)$ is defined by the same equalities (1.2). Let $a \in \mathfrak{B}$ or $a \in L(E, F)$. It is well known that a has a generalized inverse if and only if $aba = a$ holds for some $b \in \mathfrak{B}$ or $b \in L(F, E)$, as the case may be. Indeed, the "only if" part is trivial, and if $aba = a$ holds, then a straightforward computation shows that the element $b' := bab$ satisfies the two relations $ab'a = a$ and $b'ab' = b'$.

Remark 1.2. Let $a \in L(E, F)$, and consider

$$\widehat{a} = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \in L(E \oplus F, E \oplus F),$$

where the operator matrix is represented with respect to the direct sum decomposition $E \oplus F$. Then a has a GI if and only if $\widehat{a}$ does. Indeed, one easily verifies that if $\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ is a GI of $\widehat{a}$, then b_2 is a GI of a. It also follows that $\begin{bmatrix} 0 & b_2 \\ 0 & 0 \end{bmatrix}$ is a GI of $\widehat{a}$ as well.

2. Continuous and smooth families of closed subspaces of a Hilbert space

Here we collect some well-known facts on continuous families of subspaces of a Hilbert space.

Throughout this and the next section, H is a Hilbert space, $L(H)$ is the Banach algebra of bounded linear operators on H, endowed with the operator norm, and $X \subseteq \mathbb{R}^n$ is an open set. All projections (idempotents) are assumed to be linear and bounded. We denote by $\Pi_{H_0} \in L(H)$ the orthogonal projection on a closed subspace $H_0 \subseteq H$.

There are different equivalent definitions for the continuity of a family of subspaces of a Banach space. In the case of a Hilbert space, the following one is especially convenient.

A family $\{M(x)\}_{x\in X}$ of closed subspaces of a Hilbert space H is called *continuous* if the map which assigns $\Pi_{M(x)}$ to each $x \in X$ is continuous as an $L(H)$-valued map.

The following simple lemma provides the connection with non-orthogonal projections.

Lemma 2.1. *Let P be a projection, and let $Q = I - P$. Then:*

(i) *the restriction of PP^* to* $\operatorname{Im} P$ *is an isomorphism of* $\operatorname{Im} P$*;*
(ii) *the restriction of Q^*Q to* $(\operatorname{Im} P)^\perp$ *is an isomorphism of* $(\operatorname{Im} P)^\perp$*;*
(iii) *$PP^* + Q^*Q$ is a isomorphism of H;*
(iv) $\Pi_{\operatorname{Im} P} = (PP^* + Q^*Q)^{-1} PP^*$.

Proof. Since $\operatorname{Im} P \oplus \operatorname{Ker} P^* = H$ and $\operatorname{Im} P^* \oplus \operatorname{Ker} P = H$, we see that P^* maps $\operatorname{Im} P$ isomorphically onto $\operatorname{Im} P^*$, and P maps $\operatorname{Im} P^*$ isomorphically onto $\operatorname{Im} P$, which proves (i). Replacing P by Q^* in (i), we get (ii). As $H = \operatorname{Im} P \oplus (\operatorname{Im} P)^\perp$, (iii) follows from (i) and (ii). To prove (iv), note that $\Pi_{\operatorname{Im} P} P = P$ and therefore $P^* \Pi_{\operatorname{Im} P} = P^*$. Moreover $Q\Pi_{\operatorname{Im} P} = 0$, as $\operatorname{Ker} Q = \operatorname{Im} P = \operatorname{Im} \Pi_{\operatorname{Im} P}$. Hence

$$(PP^* + Q^*Q)\Pi_{\operatorname{Im} P} = PP^*,$$

which implies that $\Pi_{\operatorname{Im} P} = (PP^* + Q^*Q)^{-1} PP^*$. □

The following two propositions were obtained independently by different authors. To our knowledge, Proposition 2.2 and its generalization to Banach spaces (see Proposition 4.1) was observed for the first time by Gohberg and Markus [16], whereas Proposition 2.3 and its generalization to Banach spaces (the equivalence of conditions (i)–(iv) in Proposition 4.7) was observed for the first time by Markus [43]. For convenience of the reader, we supply proofs.

Proposition 2.2. *Let $\{M(x)\}_{x\in X}$ be a family of closed subspaces of H. Then the following are equivalent:*

(i) *the family $\{M(x)\}_{x\in X}$ is continuous;*
(ii) *for each $x_0 \in X$, there exist a neighborhood $U \subset X$ of x_0 and a continuous function $P : U \to L(H)$ all values of which are projections (not necessarily orthogonal) such that* $\operatorname{Im} P(x) = M(x)$ *for all $x \in U$;*
(iii) *for each $x_0 \in X$, there exist a neighborhood $U \subset X$ of x_0 and a continuous function $A : U \to L(H)$ all values of which are invertible such that $A(x_0) = I$ and $M(x) = A(x)M(x_0)$ for all $x \in U$.*
(iv) *for each $x_0 \in X$ and each complement N_0 of $M(x_0)$ in H, there exist a neighborhood $U \subset X$ of x_0 such that N_0 is a complement also for each $M(x)$ with $x \in U$, and, moreover, the projection $P(x)$ defined by*

$$\operatorname{Im} P(x) = M(x) \quad \text{and} \quad \operatorname{Ker} P(x) = N_0 \tag{2.1}$$

depends continuously on $x \in U$.

Proof. (i) ⇒ (ii) is trivial, and (ii) ⇒ (i) follows from Lemma 2.1. (iv) ⇒ (ii) is also trivial. It remains to prove that (ii) ⇒ (iii) ⇒ (iv).

(ii) ⇒ (iii): Let $x_0 \in X$ be given, and let U and P be as in (ii). Set $Q = I - P$ and $A(x) = P(x)P(x_0) + Q(x)Q(x_0)$ for $x \in U$. Then A is continuous and $A(x_0) = I$. Hence, after shrinking U (if necessary), we may assume that, for all $x \in U$, $A(x)$ is invertible and, hence, H is the direct sum of $A(x)\mathrm{Im}\, P(x_0)$ and $A(x)\mathrm{Im}\, Q(x_0)$. Since H is also the direct sum of $\mathrm{Im}\, P(x)$ and $\mathrm{Im}\, Q(x)$, and since $A(x)\mathrm{Im}\, P(x_0) \subseteq \mathrm{Im}\, P(x)$ and $A(x)\mathrm{Im}\, Q(x_0) \subseteq \mathrm{Im}\, Q(x)$, this implies that $A(x)\mathrm{Im}\, P(x_0) = \mathrm{Im}\, P(x) = M(x)$ for all $x \in U$.

(iii) ⇒ (iv): Let $x_0 \in X$ and a complement N_0 of $M(x_0)$ be given, and let P_0 be the projection defined by

$$\mathrm{Im}\, P_0 = M(x_0) \quad \text{and} \quad \mathrm{Ker}\, P_0 = N_0. \tag{2.2}$$

Further, let U and A be as in condition (iii). Then we define a continuous function $\widetilde{A} : U \to L(E)$, setting

$$\widetilde{A}(x) = A(x)P_0 + I - P_0, \quad x \in U.$$

Since $A(x_0) = I$ and therefore also $\widetilde{A}(x_0) = I$, after shrinking U if necessary, we may assume that the values of $\widetilde{A}$ are invertible. Moreover, since $\mathrm{Im}\, P_0 = M(x_0)$, $\mathrm{Ker}\, P_0 = N_0$, and $A(x)M(x_0) = M(x)$, we see that

$$\widetilde{A}(x)M(x_0) = M(x) \quad \text{and} \quad \widetilde{A}(x)N_0 = N_0, \quad x \in U. \tag{2.3}$$

Since the values of $\widetilde{A}$ are invertible and N_0 is a complement of $M(x_0)$, this in particular implies that N_0 is a complement of each $M(x)$, $x \in U$. Define a continuous function, setting

$$P(x) = \widetilde{A}(x)P_0\widetilde{A}(x)^{-1}, \quad x \in U.$$

Obviously, the values of this functions are projections. Therefore, now it is sufficient to show that P is the function defined by (2.1).

Let $x \in U$ be given. Then we see from (2.2) and (2.3) that

$$P(x)M(x) = \widetilde{A}(x)P_0M(x_0) = \widetilde{A}(x)M(x_0) = M(x)$$

and

$$P(x)N_0 = \widetilde{A}(x)P_0N_0 = \{0\},$$

i.e. $M(x) = \mathrm{Im}\, P(x)$ and $N_0 \subseteq \mathrm{Ker}\, P(x)$. Since $P(x)$ is a projection and we already know that N_0 is a complement of $M(x)$, this is possible only if we have equality also in the second relation, i.e. if we have (2.1). □

Proposition 2.3. *Let H, K be Hilbert spaces, and let $A : X \to L(H, K)$ be a continuous map such that, for all $x \in X$, $\mathrm{Im}\, A(x)$ is closed, i.e. $A(x)$ admits a generalized inverse. Then the following are equivalent:*

(i) *the family $\{\mathrm{Ker}\, A(x)\}_{x \in X}$ is continuous;*
(ii) *the family $\{\mathrm{Im}\, A(x)\}_{x \in X}$ is continuous;*
(iii) *for each $x_0 \in X$, there exists a neighborhood $U \subseteq X$ of x_0 and a continuous function $B : U \to L(K, H)$ such that $ABA = A$ and $BAB = B$ on U;*
(iv) *for each $x_0 \in X$, there exists a neighborhood $U \subseteq X$ of x_0 and a continuous function $B : U \to L(K, H)$ such that $ABA = A$ on U.*

Proof. By considering

$$\widehat{A} = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \in L(H \oplus K)$$

as in Remark 1.2, we easily reduce the proof to the case $K = H$. Thus, assume $K = H$.

(i) ⇒ (ii): Let $x_0 \in X$ be given. As $\{\operatorname{Ker} A(x)\}_{x \in X}$ is continuous, then we have (by criterion (iii) in Proposition 2.2) a neighborhood U of x_0 and a continuous map $S : U \to L(H)$ all values of which are invertible such that $S(x)\operatorname{Ker} A(x_0) = \operatorname{Ker} A(x)$ for all $x \in U$. Then, for all $x \in U$,

$$\operatorname{Ker} A(x)S(x) = \operatorname{Ker} A(x_0) \quad \text{and} \quad \operatorname{Im} A(x)S(x) = \operatorname{Im} A(x). \tag{2.4}$$

Let M be a complement of $\operatorname{Ker} A(x_0)$ (for example, $M = \operatorname{Ker} A(x_0)^{\perp}$), and define a continuous function $\widetilde{A} : U \to L\big(M, H\big)$, setting

$$\widetilde{A}(x) = A(x)S(x)|_M, \quad x \in U.$$

Then by (2.4)

$$\operatorname{Ker} \widetilde{A}(x) = \{0\} \quad \text{and} \quad \operatorname{Im} \widetilde{A}(x) = \operatorname{Im} A(x) \quad \text{for all } x \in U.$$

Since the spaces $\operatorname{Im} A(x)$ are closed and hence (H is a Hilbert space), complemented, this implies that the values of $\widetilde{A}$ are left-invertible. Therefore (cf. (1.1)—it applies also to situations when $a_0 \in L(M, H)$ is left-invertible, with the understanding that a_0^{-1}, a^{-1} stand for left inverses of a_0, a, respectively), after shrinking of U if necessary, we can find a continuous function $B : U \to L(H, M)$ such that $B(x)\widetilde{A}(x) = I_M$ for all $x \in U$, where I_M is the identity operator of M. Then $\widetilde{A}B$ is a continuous $L(H)$-valued function such that each $\widetilde{A}(x)B(x)$ is a projection onto $\operatorname{Im} \widetilde{A}(x) = \operatorname{Im} A(x)$. By criterion (ii) in Proposition 2.2 this proves the continuity of $\{\operatorname{Im} A(x)\}_{x \in X}$.

(ii) ⇒ (i): We proceed similarly as in the proof of (i) ⇒ (ii), with the difference that now we reduce the problem to the special case of *right*-invertible functions.

Here are the details. Let $x_0 \in X$ be given. As now $\{\operatorname{Im} A(x)\}_{x \in X}$ is continuous, then we can find a neighborhood U of x_0 and a continuous function $S : U \to L(H)$ all values of which are invertible such that $S(x_0) = I$ and $S(x)\operatorname{Im} A(x_0) = \operatorname{Im} A(x)$ for all $x \in U$ (Proposition 2.2(iii)). Then, setting

$$\widetilde{A}(x) = S(x)^{-1}A(x), \quad x \in U,$$

we define a continuous function $\widetilde{A} : U \to L\big(H, \operatorname{Im} A(x_0)\big)$ such that

$$\operatorname{Im} \widetilde{A}(x) = \operatorname{Im} A(x_0) \quad \text{and} \quad \operatorname{Ker} \widetilde{A}(x) = \operatorname{Ker} A(x) \quad \text{for all } x \in U.$$

Since the spaces $\operatorname{Ker} \widetilde{A}(x) = \operatorname{Ker} A(x)$ are closed and complemented, this implies that the values of $\widetilde{A}$ are right-invertible. Therefore (by (1.1)), after shrinking of U if necessary, we can find a continuous function $B : U \to L(\operatorname{Im} A(x_0), H)$ such that $\widetilde{A}(x)B(x) = I_{\operatorname{Im} A(x_0)}$ for all $x \in U$. Then $Q := B\widetilde{A}$ is a continuous $L(H)$-valued function such that each $Q(x)$ is a projection with $\operatorname{Ker} Q(x) = \operatorname{Ker} \widetilde{A}(x) = \operatorname{Ker} A(x)$. Therefore $P = I - Q$ is a continuous function whose values are projections with $\operatorname{Im} P(x) = \operatorname{Ker} A(x)$ for all $x \in U$. By criterion (i) Proposition 4.1 this proves the continuity of $\{\operatorname{Ker} A(x)\}_{x \in X}$.

(iii) ⇒ (i) and (ii): Let $x_0 \in X$ be given, and let U and B be as in condition (iii). Then $P_2 := AB$ and $P_1 := I - BA$ are continuous functions on U all values of which are projections, and such that $\operatorname{Im} P_2(x) = \operatorname{Im} A(x)$ and $\operatorname{Im} P_1(x) = \operatorname{Ker} A(x)$ for all $x \in U$. Hence, the families $\{\operatorname{Im} A(x)\}_{x \in U}$ and $\{\operatorname{Ker} A(x)\}_{x \in U}$ are continuous.

(i) and (ii) ⇒ (iii): Let $x_0 \in X$ be given. Choose a projection $P_2 \in L(H)$ onto $\operatorname{Im} A(x_0)$ and a projection $Q_1 \in L(H)$ onto $\operatorname{Ker} A(x_0)$. Set

$$P_1 = I - Q_1 \quad \text{and} \quad Q_2 = I - P_2.$$

As (i) and (ii) are satisfied, we have (by criterion (iii) in Proposition 2.2) a neighborhood U of x_0 and continuous functions $T_1, T_2 : U \to L(H)$ all values of which are invertible such that $T_1(x)\operatorname{Ker} A(x_0) = \operatorname{Ker} A(x)$ and $T_2(x)\operatorname{Im} A(x_0) = \operatorname{Im} A(x)$ for all $x \in U$. Set $\widehat{A} = T_2^{-1} A T_1$ on U. Then $\widehat{A}$ is a continuous $L(H)$-valued function on U which has the constant kernel $\operatorname{Ker} P_1 = \operatorname{Im} Q_1$ and the constant image $\operatorname{Im} P_2 = \operatorname{Ker} Q_2$, i.e.

$$P_2\widehat{A}(x) = \widehat{A}(x)P_1 = \widehat{A}(x), \quad x \in U, \tag{2.5}$$

and each $\widehat{A}(x)$ maps $\operatorname{Im} P_1$ isomorphically onto $\operatorname{Im} P_2$. Therefore, setting

$$\widetilde{A}(x) := \widehat{A}(x)|_{\operatorname{Im} P_1}, \quad x \in U,$$

we get a continuous $L(\operatorname{Im} P_1, \operatorname{Im} P_2)$-valued function all values of which are invertible and which satisfies

$$\widehat{A}(x) = P_2\widetilde{A}(x)P_1 = \widetilde{A}(x)P_1, \quad x \in U. \tag{2.6}$$

Let

$$\widehat{B}(x) := P_1\widetilde{A}(x)^{-1}P_2 = \widetilde{A}(x)^{-1}P_2, \quad x \in U. \tag{2.7}$$

Then $\widehat{B}$ is a continuous $L(H)$-valued function $\widehat{B}$ on U such that, by (2.6) and (2.7),

$$\widehat{B}\widehat{A}\widehat{B} = P_1\widetilde{A}^{-1}P_2\widetilde{A}P_1\widetilde{A}^{-1}P_2 = P_1\widetilde{A}^{-1}\widetilde{A}\widetilde{A}^{-1}P_2 = P_1\widetilde{A}^{-1}P_2 = \widehat{B} \tag{2.8}$$

and

$$\widehat{A}\widehat{B}\widehat{A} = P_2\widetilde{A}P_1\widetilde{A}^{-1}P_2\widetilde{A}P_1 = P_2\widetilde{A}\widetilde{A}^{-1}\widetilde{A}P_1 = P_2\widetilde{A}P_1 = \widehat{A}. \tag{2.9}$$

Now we set $B = T_1^{-1}\widehat{B}T_2$ on U. Since $A = T_2\widehat{A}T_1^{-1}$ (by definition of $\widehat{A}$), then we see from (2.8) and (2.9) that $ABA = A$ and $BAB = B$.

(iii) ⇒ (iv) ⇒ (v) is obvious, whereas (iv) ⇒ (iii) follows from Remark 1.1. □

Denote by $\mathcal{C}^{\aleph}$ one of the symbols $\mathcal{C}^{\omega}$ or $\mathcal{C}^{\alpha}$, $0 \le \alpha \le \infty$, where

- $\mathcal{C}^{\omega}$ means "real-analytic";
- if $\alpha = 0$, then $\mathcal{C}^{\alpha}$ means "continuous";
- if $0 < \alpha < 1$, then $\mathcal{C}^{\alpha}$ means "locally Hölder continuous with exponent α";
- if $\alpha \in \mathbb{N}^*$, $\mathbb{N}^* := \{1, 2, \ldots, \}$, then $\mathcal{C}^{\alpha}$ means "α times continuously differentiable";
- if $\alpha = k + \varepsilon$ with $k \in \mathbb{N}^*$ and $0 < \varepsilon < 1$, then $\mathcal{C}^{\alpha}$ means "$\mathcal{C}^k$ and the derivatives of order k are of class $\mathcal{C}^{\varepsilon}$".

Propositions 2.2 and 2.3 can be generalized to $\mathcal{C}^{\aleph}$ functions. Note that the proofs of these generalizations are repetitions of the proofs of Propositions 2.2 and 2.3, just replacing everywhere "continuous" with $\mathcal{C}^{\aleph}$. We therefore only state these generalizations, without proofs.

Proposition 2.4. *Let $\{M(x)\}_{x \in X}$ be a family of closed subspaces of H. Then the following are equivalent:*

(i) *The map which assigns $\Pi_{M(x)}$ to each $x \in X$ is $\mathcal{C}^{\aleph}$ as an $L(H)$-valued map.*

(ii) *For each* $x_0 \in X$, *there exist a neighborhood* $U \subset X$ *of* x_0 *and a* $\mathcal{C}^{\aleph}$ *function* $P : U \to L(H)$ *all values of which are projections (not necessarily orthogonal) such that* $\operatorname{Im} P(x) = M(x)$ *for all* $x \in U$.

(iii) *For each* $x_0 \in X$, *there exist a neighborhood* $U \subset X$ *of* x_0 *and a* $\mathcal{C}^{\aleph}$ *function* $A : U \to L(H)$ *all values of which are invertible such that* $A(x_0) = I$ *and* $M(x) = A(x)M(x_0)$ *for all* $x \in U$.

(iv) *for each* $x_0 \in X$ *and each complement* N_0 *of* $M(x_0)$ *in* H, *there exist a neighborhood* $U \subset X$ *of* x_0 *such that* N_0 *is a complement also for each* $M(x)$ *with* $x \in U$, *and, moreover, the projection* $P(x)$ *defined by*

$$\operatorname{Im} P(x) = M(x) \quad \text{and} \quad \operatorname{Ker} P(x) = N_0$$

is of class $\mathcal{C}^{\aleph}$ *on* U.

A family $\{M(x)\}_{x \in X}$ of closed subspaces of H will be called $\mathcal{C}^{\aleph}$ if the four equivalent conditions in Proposition 2.4 are satisfied.

Proposition 2.5. *Let* $A : X \to L(H, K)$ *be of class* $\mathcal{C}^{\aleph}$ *such that, for all* $x \in X$, $\operatorname{Im} A(x)$ *is closed, i.e.* $A(x)$ *admits a generalized inverse. Then the following are equivalent:*

(i) *the family* $\{\operatorname{Ker} A(x)\}_{x \in X}$ *is* $\mathcal{C}^{\aleph}$;

(ii) *the family* $\{\operatorname{Im} A(x)\}_{x \in X}$ *is* $\mathcal{C}^{\aleph}$;

(iii) *for each* $x_0 \in X$, *there exists a neighborhood* $U \subseteq X$ *of* x_0 *and a* $\mathcal{C}^{\aleph}$ *function* $B : U \to L(K, H)$ *such that* $ABA = A$ *and* $BAB = B$ *on* U.

(iv) *for each* $x_0 \in X$, *there exists a neighborhood* $U \subseteq X$ *of* x_0 *and a* $\mathcal{C}^{\aleph}$ *function* $B : U \to L(K, H)$ *such that* $ABA = A$ *on* U.

3. The Moore–Penrose inverse

In this section, $\mathfrak{B}$ is a complex unital C^*-algebra. An element $b \in \mathfrak{B}$ is said to be a *Moore–Penrose inverse*, notation $b = a^+$, of $a \in \mathfrak{B}$ if the following conditions hold:

$$aba = a, \qquad bab = b, \qquad (ab)^* = ab, \qquad (ba)^* = ba. \tag{3.1}$$

Clearly, the Moore–Penrose inverse is a GI, but, in distinction to general GIs, it is uniquely determined (if it exists).

Indeed, assume $b, b' \in \mathfrak{B}$ are Moore–Penrose inverses of some element $a \in \mathfrak{B}$. By using the standard representation of $\mathfrak{B}$ as a norm closed *-subalgebra of a $L(H)$ for some Hilbert space H, we may assume that a, b, b' are bounded linear operators in a Hilbert space. Then, from $aba = a$ and $ab'a = a$ we see that ab and ab' are projections with

$$\operatorname{Im} ab = \operatorname{Im} a = \operatorname{Im} ab',$$

and from $bab = b$ and $b'ab' = b'$ it follows that and ba and $b'a$ are projections with

$$\operatorname{Ker} ba = \operatorname{Ker} a = \operatorname{Ker} b'a.$$

As $(ab)^* = ab$, $(ba)^* = ba$, $(ab')^* = ab'$, $(b'a)^* = b'a$, all these projections are orthogonal. Since orthogonal projections are uniquely determined by either their image or their kernel, it follows that $ab = ab'$ and $ba = b'a$. Hence, $b' = b'ab' = bab = b$.

If $a \in L(H, K)$, where H, K are Hilbert spaces, then a Moore–Penrose inverse $b \in L(K, H)$ of a is defined by the same equalities (3.1). It is unique (if exists); see the proof above.

The literature on Moore–Penrose inverses is extensive, especially in linear algebra where the Moore–Penrose inverses of matrices are used to compute least squares solutions of systems of linear equations (to mention just one application out of many). In the matrix case, it is well known that the Moore–Penrose inverse is generally not continuous, but it is continuous, even real-analytic, on the set of $m \times n$ matrices having fixed rank (see, for example, [10,48]). In the abstract framework of C^*-algebras, the Moore–Penrose inverses have been studied in [22,23,33, 34]. In particular, it is proved by Koliha [34] that a^+ is a differentiable function of a provided a^+ is continuous.

Existence criteria for the Moore–Penrose inverse are given in the following proposition.

Proposition 3.1. (a) *$a \in \mathfrak{B}$ has a Moore–Penrose inverse if and only if $aba = a$ for some $b \in \mathfrak{B}$;*

(b) *An operator $A \in L(H, K)$ has the Moore–Penrose inverse if and only if the image (range)* $\operatorname{Im} A$ *of A is a closed subspace.*

For the proof of (a) see [22, Theorem 6] or [33] (the "only if" part is trivial). Statement (b) is standard in operator theory.

It will be advantageous to consider first the case when $\mathfrak{B} = L(H)$, as it affords more informative statements of results (see Theorem 3.4).

We continue to use the notation introduced in Section 2.

Lemma 3.2. *Let $A : X \to L(H, K)$ be of class $\mathcal{C}^{\aleph}$ such that* $\operatorname{Im} A(x)$ *is closed for all $x \in X$ and the four equivalent conditions in Proposition* 2.3 *are satisfied. Then also the four stronger conditions in Proposition* 2.5 *are satisfied.*

Proof. By Proposition 2.5 we only have to prove one of the four conditions. We prove condition (ii) in Proposition 2.5. For this it is sufficient to prove that, for each $x_0 \in X$, there exists a neighborhood U of x_0 such that the family $\{\operatorname{Im} A(x)\}_{x\in U}$ is of class $\mathcal{C}^{\aleph}$. Let $x_0 \in X$ be given.

Let P_1 be a projection onto a direct complement of $\operatorname{Ker} A(x_0)$, and let P_2 be a projection onto $\operatorname{Im} A(x_0)$. Then the operator $A_0 \in L(\operatorname{Im} P_1, \operatorname{Im} P_2)$ defined by

$$A_0 = P_2 A(x_0)|_{\operatorname{Im} P_1}$$

is invertible. Now we define a $\mathcal{C}^{\aleph}$ function $T : U \to L(K)$, setting

$$T(x) = A(x) P_1 A_0^{-1} P_2 + I - P_2 \quad \text{for } x \in U.$$

Then $T(x_0) = I$. Therefore, we can find a neighborhood U of x_0 such that $T(x)$ is invertible for all $x \in U$. Then (by criterion (iii) in Proposition 2.4) the family of subspaces $\{T(x)\operatorname{Im} P_2\}_{x\in U}$ is of class $\mathcal{C}^{\aleph}$.

In particular, it is continuous. Moreover, from $T(x_0)\operatorname{Im} P_2 = \operatorname{Im} P_2 = \operatorname{Im} A(x_0)$ we see that $\operatorname{Ker} P_2$ is a complement of both $T(x_0)\operatorname{Im} P_2$ and $\operatorname{Im} A(x_0)$. Since, by hypothesis, also the family $\{\operatorname{Im} A(x)\}_{x\in X}$ is continuous, this implies by criterion (iv) in Proposition 2.2 that, after shrinking U if necessary, for each $x \in U$, $\operatorname{Ker} P_2$ is a complement of both $T(x)\operatorname{Im} P_2$ and $\operatorname{Im} A(x)$. Since, obviously,

$$T(x)\operatorname{Im} P_2 \subseteq \operatorname{Im} A(x) \quad \text{for all } x \in U,$$

this is possible only if

$$T(x)\operatorname{Im} P_2 = \operatorname{Im} A(x) \quad \text{for all } x \in U.$$

As $\{T(x)\operatorname{Im} P_2\}_{x\in U}$ is $\mathcal{C}^{\aleph}$, this completes the proof. □

In the proof of the following theorem, we will use also the following trivial fact.

Lemma 3.3. *Let $P, R \in L(H)$ be two projections such that* $\operatorname{Im} P = \operatorname{Im} R$. *Further assume that $R = PAP$ for a certain operator $A \in L(H)$. Then $R = P$.*

Proof. As we already have $\operatorname{Im} P = \operatorname{Im} R$, we must only prove that $\operatorname{Ker} P = \operatorname{Ker} R$. From $R = PAP$ it follows that $\operatorname{Ker} P \subseteq \operatorname{Ker} R$. Since the spaces $\operatorname{Ker} P$ and $\operatorname{Ker} R$ both are direct complements of the same subspace $\operatorname{Im} P = \operatorname{Im} R$, this is possible only for $\operatorname{Ker} P = \operatorname{Ker} R$. □

Theorem 3.4. *Let $A : X \to L(H, K)$ be of class $\mathcal{C}^{\aleph}$ such that* $\operatorname{Im} A(x)$ *is closed for all $x \in X$ and the four equivalent conditions in Proposition 2.3 are satisfied. Let $A^+ : X \to L(K, H)$ be the function which assigns to each $x \in X$ the Moore–Penrose inverse of $A(x)$. Then A^+ is of class $\mathcal{C}^{\aleph}$.*

Proof. Since $\mathcal{C}^{\aleph}$ is a local property, we only have to prove that each $x_0 \in X$ has a neighborhood such that A^+ is of class $\mathcal{C}^{\aleph}$ on U. Let $x_0 \in X$ be given.

By Lemma 3.2, then there exist a neighborhood U of x_0 and a $\mathcal{C}^{\aleph}$ function $B : U \to L(K, H)$ such that $ABA = A$ and $BAB = B$ on U. Moreover, again by Lemma 3.2, the functions $P_1 := \Pi_{(\operatorname{Ker} A(x))^{\perp}}$ and $P_2 := \Pi_{\operatorname{Im} A(x)}$ are of class $\mathcal{C}^{\aleph}$ on X.

Hence, the function $\widetilde{B} := P_1 B P_2$ is of class $\mathcal{C}^{\aleph}$ on U. Therefore, to complete the proof, it is sufficient to show that $\widetilde{B} = A^+$ on U. Since $P_2 A = A$ and $AP_1 = A$, we see that, on U,

$$A\widetilde{B}A = AP_1 B P_2 A = ABA = A$$

and

$$\widetilde{B}A\widetilde{B} = P_1 B P_2 A P_1 B P_2 = P_1 BAB P_2 = P_1 B P_2 = \widetilde{B}.$$

Therefore, for each $x \in U$, $A(x)\widetilde{B}(x)$ is a projection onto $\operatorname{Im} P_2(x)$, and $\widetilde{B}(x)A(x)$ is a projection onto $\operatorname{Im} P_1(x)$. Since

$$A(x)\widetilde{B}(x) = P_2(x)\Big(P_1(x)B(x)\Big)P_2(x),$$
$$\widetilde{B}(x)A(x) = P_1(x)\Big(B(x)P_2(x)A(x)\Big)P_1(x),$$

this implies $A\widetilde{B} = P_2$, $\widetilde{B}A = P_1$ on U, by Lemma 3.3. □

Corollary 3.5. *Let $A : X \to L(H, K)$ be of class $\mathcal{C}^{\aleph}$ such that at least one of the following conditions is fulfilled:*

(i) $\operatorname{Im} A(x)$ *is closed for all $x \in X$, and the dimension of* $\operatorname{Ker} A(x)$ *is finite for all $x \in X$ and is independent of $x \in X$,*
(ii) *the codimension of* $\operatorname{Im} A(x)$ *is finite*[1] *for all $x \in X$ and is independent of $x \in X$.*

Then A^+ is of class $\mathcal{C}^{\aleph}$.

Proof. It is well-known (see, e.g., Theorem 6.2.8 in [15]) that each of the conditions (i) and (ii) implies the four equivalent conditions of Proposition 2.3. Therefore, the corollary follows from Theorem 3.4. □

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[1] It is a simple consequence of the Banach open mapping theorem, that then $\operatorname{Im} A(x)$ is automatically closed, see, e.g., [13, Chapter XI, Corollary 2.3].

We now return to C^*-algebras.

Theorem 3.6. *Let $A : X \to \mathfrak{B}$ be of class $\mathcal{C}^{\aleph}$, and assume that for every $x_0 \in X$ there exist a neighborhood U and a continuous function $B : U \to \mathfrak{B}$ such that $A(x)B(x)A(x) = A(x)$ for every $x \in U$. Then $A(x)$ has a Moore–Penrose inverse (in $\mathfrak{B}$) for every $x \in X$, and the function $A^+ : X \to \mathfrak{B}$ that assigns to $x \in X$ the Moore–Penrose inverse $A(x)^+$ is of class $\mathcal{C}^{\aleph}$.*

Proof. The existence of $A(x)^+$ follows from Proposition 3.1(a).

We may assume that $\mathfrak{B}$ is a norm closed *-subalgebra of $L(H)$. By Theorem 3.4 the Moore–Penrose inverse of $A(x)$ (as an element of $L(H)$) is of class $\mathcal{C}^{\aleph}$. But by uniqueness of the Moore–Penrose inverse we actually have that the Moore–Penrose inverse of $A(x)$ belongs to $\mathfrak{B}$, and the proof is complete. □

In Section 5 (see Corollary 5.6) we in particular obtain that the condition (C) (stated in the introduction and assumed in Theorem 3.6) is equivalent to the apparently much weaker condition (B) (also stated in the introduction). So, Theorem 3.6 admits the following stronger formulation:

Theorem 3.7. *Let $A : X \to \mathfrak{B}$ be of class $\mathcal{C}^{\aleph}$, and assume that for every $x_0 \in X$ there exist a neighborhood U and a bounded (possibly not continuous) function $B : U \to \mathfrak{B}$ such that $A(x)B(x)A(x) = A(x)$ for every $x \in U$. Then $A(x)$ has a Moore–Penrose inverse (in $\mathfrak{B}$) for every $x \in X$, and the function $A^+ : X \to \mathfrak{B}$ that assigns to $x \in X$ the Moore–Penrose inverse $A(x)^+$ is of class $\mathcal{C}^{\aleph}$.*

We remark that the results of Theorems 3.4, 3.6 and 3.7 extend to more general classes of $\mathfrak{B}$-valued or $L(H, K)$-valued functions (with essentially the same proofs). We shall define these classes for $L(H, K)$; extension to C^*-algebras-valued functions is immediate upon representation of C^*-algebras as norm closed *-subalgebras of $L(H)$. In what follows, we denote by $H_1, H_2, \ldots$ Hilbert spaces. For every open subset $U \subseteq X$ let $\mathcal{C}(U, L(H_1, H_2))$ be a (complex) vector space of continuous functions $U \to L(H_1, H_2)$ subject to the following conditions (analogous to those specified in [6]):

(a) $\mathcal{C}(U, L(H_1, H_2))$ contains all constant functions;
(b) $\mathcal{C}(U, L(H_1, H_2))$ is closed under *-operation: $f \in \mathcal{C}(U, L(H_1, H_2))$ implies that the function $f^*(x) := (f(x))^*$, $x \in U$, is in $\mathcal{C}(U, L(H_2, H_1))$;
(c) $\mathcal{C}(U, L(H_1, H_2))$ is defined locally: If $V \subseteq X$ is open, and if $V = \cup_j U_j$ is an open cover of V, then $f \in \mathcal{C}(V, L(H_1, H_2))$ if and only if the restriction of f to U_j is in $\mathcal{C}(U_j, L(H_1, H_2))$, for every index j;
(d) if $f \in \mathcal{C}(U, L(H_1, H_2))$ takes invertible values, then the function $f^{-1}(x) = (f(x))^{-1}$, $x \in U$, is in $\mathcal{C}(U, L(H_2, H_1))$;
(e) if $f_1 \in \mathcal{C}(U, L(H_1, H_2))$ and $f_2 \in \mathcal{C}(U, L(H_2, H_3))$, then the composite function $f_2 \circ f_1(x) = f_2(f_1(x))$, $x \in U$, belongs to $\mathcal{C}(U, L(H_1, H_3))$.

Then Theorems 3.4, 3.6 and 3.7 are valid with $\mathcal{C}^{\aleph}$ replaced with $\mathcal{C}(X, L(H, K))$, resp. $\mathcal{C}(X, \mathfrak{B})$.

Furthermore, we observe that Theorems 3.4, 3.6 and 3.7 are valid also for operators between real Hilbert spaces and for real C^*-algebras, respectively, with essentially the same proofs. Recall that a real unital C^*-algebra $\mathfrak{B}$ is a real unital Banach algebra with an involution * such that $\|aa^*\| = \|a\|^2$ for all $a \in \mathfrak{B}$ and $1 + aa^*$ is invertible for every $a \in \mathfrak{B}$. Such algebras are (isometrically *-isomorphic to) norm closed subalgebras of linear operators on a real Hilbert space, see [45], for example.

We conclude this section with the following remark, which shows that, except for a trivial case, the Moore–Penrose inverse of a (complex-) holomorphic function is not holomorphic.

Remark 3.8. Assume that X is an open subset of $\mathbb{C}^n$, $n \geq 1$. Let H be a Hilbert space, and let $A : X \to L(H)$ be a (complex-) holomorphic function such that the values of A are generalized invertible. If the functions

$$X \ni z \mapsto \operatorname{Im} A(z) \quad \text{and} \quad X \ni z \mapsto \operatorname{Ker} A(z) \tag{3.2}$$

are locally constant, then it is trivial that also the Moore–Penrose inverse of A holomorphic. This is the only possibility.

Indeed, suppose the Moore–Penrose inverse A^+ is holomorphic. Then also the functions AA^+ and A^+A are holomorphic. This implies that the functions $(AA^+)^*$ and $(A^+A)^*$ are anti-holomorphic. In view of the relations

$$AA^+ = (AA^+)^* \quad \text{and} \quad A^+A = (A^+A)^*,$$

this further implies that the functions AA^+ and A^+A are also anti-holomorphic. Hence AA^+ and A^+A are locally constant. As $A(x)A(x)^+$ is the orthogonal projection onto $\operatorname{Im} A(x)$ and $A(x)^+A(x)$ is the orthogonal projection onto $\operatorname{Ker} A(x)^\perp$, this means that the functions (3.2) are locally constant.

4. Continuous families of complemented subspaces of a Banach space

Here we collect some well-known facts on continuous families of complemented subspaces of a Banach space.

A subspace E_0 of a Banach space E will be called *complemented* if it is closed and if there exists a second closed subspace E_1 of E, called a *complement* of E_0, such that E is the (algebraically) direct sum of E_0 and E_1. Recall that by Banach's open mapping theorem this is the case if and only if there exists a (bounded) projection P from E onto E_0.

To define the notion of continuity for families of complemented subspaces, we first recall the following proposition, obtained by Gohberg and Markus in [16].

Proposition 4.1. *Let E be a Banach space, let X be a topological space, and let $\{M(x)\}_{x\in X}$ be a family of complemented subspaces of a Banach space E. Then the following conditions are equivalent:*

(i) *for each $x_0 \in X$, there exist a neighborhood $U \subset X$ of x_0 and a continuous function $P : U \to L(E)$ all values of which are projections such that* $\operatorname{Im} P(x) = M(x)$ *for all $x \in U$;*

(ii) *for each $x_0 \in X$, there exist a neighborhood $U \subset X$ of x_0 and a continuous function $A : U \to L(E)$ all values of which are invertible such that $A(x_0) = I$ and $M(x) = A(x)M(x_0)$ for all $x \in U$.*

(iii) *for each $x_0 \in X$ and each complement N_0 of $M(x_0)$ in E, there exist a neighborhood $U \subset X$ of x_0 such that N_0 is a complement also for each $M(x)$ with $x \in U$, and, moreover, the projection $P(x)$ defined by*

$$\operatorname{Im} P(x) = M(x) \quad \textit{and} \quad \operatorname{Ker} P(x) = N_0 \tag{4.1}$$

depends continuously on $x \in U$.

Proof. We can use the same arguments as in the proof of the equivalence of conditions (ii), (iii), and (iv) in Proposition 2.2, since, in that proof, the *Hilbert* space setting is used only for the conclusion that the spaces $M(x)$ are *complemented* because they are closed by hypothesis. Here, in Proposition 4.1, the spaces $M(x)$ are complemented by hypothesis. □

Definition 4.2. A family $\{M(x)\}_{x\in X}$ of complemented subspaces of a Banach space E will be called *continuous* if conditions (i)–(iii) in Proposition 4.1 are satisfied.

We note in passing that the continuity of $\{M(x)\}_{x\in X}$ is equivalent to the continuity of the mapping $x \mapsto M(x)$ with respect to the so-called *gap metric* introduced and studied in [14,16,38] (see also [31, Chapter IV, Section 2] and [15, Sections 6.1, 6.2]). The gap metric is defined on the set of all closed subspaces of E (not only the complemented ones).

For later reference we observe the following simple uniqueness result.

Proposition 4.3. *Let X be a connected topological space, and let $\{M(x)\}_{x\in X}$ and $\{N(x)\}_{x\in X}$ be two continuous families of complemented subspaces of a Banach space E such that*

$$N(x) \subseteq M(x)$$

for all $x \in X$, where for at least one point we have equality. Then we have equality for all $x \in X$.

Proof. Let X' be the set of all $x \in X$ such that $N(x) = M(x)$. By hypothesis, $X' \neq \emptyset$. It remains to prove that both X' and $X \setminus X'$ are open.

Openness of X': Let $x_0 \in X'$ be given. Take a closed subspace K_0 of E which is a complement of $M(x_0) = N(x_0)$. Then, by criterion (iii) in Proposition 4.1, there exists a neighborhood U of x_0 such that, for each $x \in U$, K_0 is a complement of both $M(x)$ and $N(x)$. Since $N(x)$ is contained in $M(x)$, this is possible only if $N(x) = M(x)$.

Openness of $X \setminus X'$: Let $x_0 \in X \setminus X'$ be given. Take a closed subspace K_0 of E which is a complement of $M(x_0)$. Then, by criterion (iii) in Proposition 4.1, there exists a neighborhood U of x_0 such that, for each $x \in U$, K_0 is a complement of $M(x)$. Moreover, by criterion (i) in Proposition 4.1, after shrinking U if necessary, we can find continuous functions $P_M, P_N : U \to L(E)$ all values of which are projections such that $\operatorname{Im} P_M(x) = M(x)$ and $\operatorname{Im} P_N(x) = N(x)$ for all $x \in U$. Since $N(x_0) \subsetneqq M(x_0)$, we can choose a vector $v_0 \in M(x_0) \setminus N(x_0)$. Set $v(x) = P_M(x)v_0$ for $x \in U$. Then $v(x) \in M(x)$ for all $x \in U$, and, after shrinking U if necessary, $v(x) \neq 0$ for all $x \in U$. On the other hand,

$$P_N(x_0)v(x_0) = P_N(x_0)v_0 \neq v_0 = v(x_0)$$

and therefore, by continuity of P_N and v,

$$P_N(x)v(x) \neq v(x)$$

for all x in some neighborhood $V \subseteq U$ of x_0. Hence $v(x) \notin N(x)$ for all $x \in V$, and therefore $V \subseteq X \setminus X'$. □

Now we pass to families of complemented subspaces which appear as images or kernels of continuous operator functions. Let E and F be two (complex) Banach spaces, and recall that an operator $B \in L(F, E)$ is called a GI (generalized inverse) of $A \in L(E, F)$ if $ABA = A$ and $BAB = B$.

Remark 4.4. An operator $A \in L(E, F)$ admits a GI if and only if $\operatorname{Im} A$ and $\operatorname{Ker} A$ are complemented subspaces of F and E, respectively. Indeed, if we have projections $Q_1 \in L(E)$ and $P_2 \in L(F)$ with $\operatorname{Im} P_2 = \operatorname{Im} A$ and $\operatorname{Im} Q_1 = \operatorname{Ker} A$, then A defines an invertible operator $A_0 \in L(\operatorname{Ker} Q_1, \operatorname{Im} P_2)$, and if A_0^{-1} is the inverse of A_0, then $B := A_0^{-1}P_2$ is a GI of A. Conversely, if $A \in L(E, F)$ and $B \in L(F, E)$ are such that B is a GI of A (or, equivalently, A is a GI of B), then AB and BA are projections with

$$\operatorname{Im} AB = \operatorname{Im} A, \qquad \operatorname{Ker} AB = \operatorname{Ker} B, \qquad \operatorname{Im} BA = \operatorname{Im} B, \qquad \operatorname{Ker} BA = \operatorname{Ker} A.$$

This implies that a GI of an operator $A \in L(E, F)$ is uniquely determined by its image and kernel, i.e. if $B, B' \in L(F, E)$ are two generalized inverses of A with

$$\operatorname{Im} B = \operatorname{Im} B' \quad \text{and} \quad \operatorname{Ker} B = \operatorname{Ker} B', \tag{4.2}$$

then $B = B'$. Indeed, then BA and $B'A$ are projections with

$$\operatorname{Ker} BA = \operatorname{Ker} A = \operatorname{Ker} B'A \quad \text{and} \quad \operatorname{Im} BA = \operatorname{Im} B = \operatorname{Im} B' = \operatorname{Im} B'A,$$

and AB and AB' are projections with

$$\operatorname{Im} AB = \operatorname{Im} A = \operatorname{Im} AB' \quad \text{and} \quad \operatorname{Ker} AB = \operatorname{Ker} B = \operatorname{Ker} B' = \operatorname{Ker} AB'.$$

Hence $AB = AB'$ and $BA = B'A$, which implies that

$$B = BAB = BAB' = B'AB' = B'.$$

Let E and F be Banach spaces and $T \in L(E, F)$. Recall that the element $\gamma(T) \in [0, \infty]$ defined by

$$\gamma(T) = \inf_{v \in E, \operatorname{dist}(v, \operatorname{Ker} T) \geq 1} \|Tv\| \tag{4.3}$$

is called the *reduced minimum modulus* of T [31, Chapter IV, Section 5].

Note that $\gamma(T) > 0$ if and only if $\operatorname{Im} T$ is a closed subspace of F. For $T = 0$ this holds by definition (as $\inf \emptyset = \infty$), and for $T \neq 0$ this follows from Banach's open mapping theorem. In particular, if T admits a GI, then always $\gamma(T) > 0$, where $\gamma(T) = \infty$ is equivalent to $T = 0$.

Note also that

$$\gamma(T) \leq \|T\| \quad \text{if and only if} \quad T \neq 0. \tag{4.4}$$

Indeed, if $T = 0$, then $\gamma(T) = \infty > 0 = \|T\|$; if $T \neq 0$, then for each $\varepsilon > 0$ we can find a vector $v \in E$ with $\operatorname{dist}(v, \operatorname{Ker} T) \geq 1$ and $\|v\| \leq 1 + \varepsilon$, which implies that

$$\gamma(T) \leq \|Tv\| \leq \|T\| \, \|v\| \leq \|T\|(1 + \varepsilon).$$

We adapt the following definition introduced by Kaballo and Thijsse [29, Definition 1.1].

Definition 4.5. Let E, F be Banach spaces and X a topological space. An operator function $A : X \to L(E, F)$ is called *uniformly regular* if, for each $x_0 \in X$, there exists a neighborhood U of x_0 such that

$$\inf_{x \in U} \gamma\big(A(x)\big) > 0. \tag{4.5}$$

We do not assume in this definition that A is continuous, although the benefit of it arises only for continuous functions A (so far as we know). Note that uniform regularity is important in the study of the so-called lifting problem (see the comments following Theorem 6.10).

Remark 4.6. Let E, F be Banach spaces, X a topological space, and $A : X \to L(E, F)$ uniformly regular. If A is continuous and $A(x_0) = 0$ for some $x_0 \in X$, then $A(x) = 0$ for all x in some neighborhood of x_0. Indeed, otherwise we can find a sequence $x_n, n = 1, 2, \ldots,$ which converges to x_0 such that $A(x_n) \neq 0$ and hence, by (4.4), $\gamma\big(A(x_n)\big) \leq \|A(x_n)\|$ for all $n \geq 1$. By continuity of A, this implies that

$$\lim \gamma\big(A(x_n)\big) \leq \lim \|A(x_n)\| = \|A(x_0)\| = 0,$$

which is a contradiction to (4.5).

There are several conditions which are equivalent to uniform regularity. If the values of A are generalized invertible, one has the following proposition.

Proposition 4.7. *Let E, F be Banach spaces, X a topological space, and $A : X \to L(E, F)$ a continuous operator function such that, for all $x \in X$, $\operatorname{Im} A(x)$ and $\operatorname{Ker} A(x)$ are complemented subspaces, i.e. $A(x)$ admits a generalized inverse. Then the following are equivalent:*

(i) *A is uniformly regular;*
(ii) *the family $\{\operatorname{Ker} A(x)\}_{x\in X}$ is continuous;*
(iii) *the family $\{\operatorname{Im} A(x)\}_{x\in X}$ is continuous;*
(iv) *for each $x_0 \in X$, there exists a neighborhood U of x_0 and a continuous function $B : U \to L(F, E)$ such that $ABA = A$ and $BAB = B$ on U;*
(v) *for each $x_0 \in X$, there exists a neighborhood U of x_0 and a continuous function $B : U \to L(F, E)$ such that $ABA = A$ on U;*
(vi) *for each $x_0 \in X$, there exists a neighborhood U of x_0 and a closed subspace N_0 of E which is a complement of $\operatorname{Ker} A(x)$ for all $x \in U$;*
(vii) *for each $x_0 \in X$, there exists a neighborhood U of x_0 and a closed subspace M_0 of F which is a complement of $\operatorname{Im} A(x)$ for all $x \in U$;*
(viii) *for each $x_0 \in X$, there exists a neighborhood U of x_0 and a bounded (possibly not continuous) function $B : U \to L(F, E)$ such that $ABA = A$ on U.*

The equivalence of conditions (i)–(v) was established by Markus [43]. Shubin [47, pages 411–413, and the remark on p. 415], Gramsch [19, pages 135–137], and Thijsse [50, pages 12,13] proved (independently, so far as we know) that conditions (vi) and (vii) can be added. For condition (viii) we have no explicit reference, but (v) ⇒ (vi) is trivial, and (viii) ⇒ (i) is easy to show (see the proof below). For convenience of the reader we give a proof of the entire proposition.

Proof of Proposition 4.7. We first prove the equivalence of (ii)–(vii).

(ii) ⇔ (iii), (iv) ⇒ (ii) and (iii), (ii) and (iii) ⇒ (iv): This can be proved in the same way as the corresponding parts of the proof of Proposition 2.3, where the complementedness of $\operatorname{Im} A(x)$ and $\operatorname{Ker} A(x)$ is now assured by hypothesis.

(iv) ⇔ (v): This follows from Remark 1.1.

(ii) ⇔ (vi): Using criterion (iii) in Proposition 4.1, (ii) ⇒ (vi) is trivial. To prove (vi) ⇒ (ii), assume that (vi) is satisfied and a point $x_0 \in X$ is given. Then we have a neighborhood U of x_0 and a closed subspace N_0 of E which is a complement of $\operatorname{Ker} A(x)$ for all $x \in U$. Choose a projection $P_0 \in L(F)$ onto $\operatorname{Im} A(x_0)$, and consider the continuous operator function $\widetilde{A} : X \to L\big(E, \operatorname{Im} A(x_0)\big)$ defined by

$$\widetilde{A}(x) = P_0 A(x).$$

Then $\widetilde{A}(x_0)$ is right invertible, and it follows by continuity of $\widetilde{A}$ that, after shrinking U if necessary, $\widetilde{A}(x)$ is right invertible for all $x \in U$. In particular, the family $\big\{\operatorname{Im} \widetilde{A}(x) = \operatorname{Im} \widetilde{A}(x_0)\big\}_{x\in U}$ is constant and therefore continuous. Applying the already proved equivalence of conditions (ii) and (iii) to $\widetilde{A}$, this implies that the family $\big\{\operatorname{Ker} \widetilde{A}(x)\big\}_{x\in U}$ is continuous. To complete the proof of (ii), it is therefore sufficient to show that

$$\operatorname{Ker} A(x) = \operatorname{Ker} \widetilde{A}(x) \tag{4.6}$$

for all x in some neighborhood of x_0. Obviously,

$$\operatorname{Ker} A(x) \subseteq \operatorname{Ker} \widetilde{A}(x) \quad \text{for all } x \in X. \tag{4.7}$$

Since N_0 is a complement of $\operatorname{Ker} \widetilde{A}(x_0) = \operatorname{Ker} A(x_0)$ and the family $\{\operatorname{Ker} \widetilde{A}(x)\}_{x \in U}$ is continuous, by criterion (iii) in Proposition 4.1, we can find a neighborhood $V \subseteq U$ of x_0 such that N_0 is a complement of $\operatorname{Ker} \widetilde{A}(x)$ for all $x \in V$. So, for $x \in V$, N_0 is a complement for both $\operatorname{Ker} A(x)$ and $\operatorname{Ker} \widetilde{A}(x)$. Together with (4.7) this implies that (4.6) holds true for all $x \in V$.

(iii) $\Leftrightarrow$ (vii): Using criterion (iii) in Proposition 4.1, the direction (iii) $\Rightarrow$ (vii) is trivial. To prove the opposite direction, assume that (vii) is satisfied and a point $x_0 \in X$ is given. Then we have a neighborhood U of x_0 and a closed subspace M_0 of F which is a complement of $\operatorname{Im} A(x)$ for all $x \in U$. Choose a projection $P_0 \in L(E)$ onto a complement of $\operatorname{Ker} A(x_0)$, and consider the continuous operator function $\widetilde{A} : X \to L(\operatorname{Im} P_0, F)$ defined by

$$\widetilde{A}(x) = A(x)P_0.$$

Then $\widetilde{A}(x_0)$ is left invertible, and it follows by continuity of $\widetilde{A}$ that, after shrinking U if necessary, $\widetilde{A}(x)$ is left invertible for all $x \in U$. In particular, the family $\{\operatorname{Ker} \widetilde{A}(x) = \{0\}\}_{x \in U}$ is constant and therefore continuous. Applying (ii) $\Leftrightarrow$ (iii) to $\widetilde{A}$, we obtain that the family $\{\operatorname{Im} \widetilde{A}(x)\}_{x \in U}$ is continuous. It remains to show that

$$\operatorname{Im} A(x) = \operatorname{Im} \widetilde{A}(x) \tag{4.8}$$

for all x in some neighborhood of x_0. Obviously,

$$\operatorname{Im} A(x) \supseteq \operatorname{Im} \widetilde{A}(x) \quad \text{for all } x \in X. \tag{4.9}$$

Since M_0 is a complement of $\operatorname{Im} \widetilde{A}(x_0) = \operatorname{Im} A(x_0)$ and the family $\{\operatorname{Im} \widetilde{A}(x)\}_{x \in U}$ is continuous, by criterion (iii) in Proposition 4.1, we can find a neighborhood $V \subseteq U$ of x_0 such that M_0 is a complement of $\operatorname{Im} \widetilde{A}(x)$ for all $x \in V$. So, for $x \in V$, M_0 is a complement for both $\operatorname{Im} A(x)$ and $\operatorname{Im} \widetilde{A}(x)$. Together with (4.9) this implies that (4.8) holds true for all $x \in V$.

As the equivalence of (ii)–(vii) is established, and (v) $\Rightarrow$ (viii) is trivial, now the proof of the proposition can be completed by proving that (viii) $\Rightarrow$ (i) $\Rightarrow$ (ii).

(viii) $\Rightarrow$ (i): Assume (viii) is satisfied, and let $x_0 \in X$ be given. We have to find a neighborhood U of x_0 such that

$$\inf_{x \in U} \gamma\big(A(x)\big) > 0. \tag{4.10}$$

From (viii) we get a neighborhood U of x_0 and a bounded function $B : U \to L(F, E)$ such that $ABA = A$ on U. As B is bounded, for the proof of (4.10) it is sufficient that, for each $x \in U$,

$$\gamma\big(A(x)\big) \geq \frac{1}{\|B(x)\|}.$$

If $A(x) = 0$, this is trivial, since then $\gamma\big(A(x)\big) = \infty$. Let $x \in U$ such that $A(x) \neq 0$. Then for all $v \in E$

$$\|B(x)\| \, \|A(x)v\| \geq \|B(x)A(x)v\| = \left\| v - \big(v - B(x)A(x)v\big) \right\|.$$

Since $A = ABA$ and therefore $v - B(x)A(x)v \in \operatorname{Ker} A(x)$, this implies that

$$\|B(x)\| \, \|A(x)v\| \geq \operatorname{dist}\big(v, \operatorname{Ker} A(x)\big)$$

and hence

$$\|A(x)v\| \geq \frac{\operatorname{dist}\big(v, \operatorname{Ker} A(x)\big)}{\|B(x)\|} \tag{4.11}$$

for all $v \in E$. Since $A(x) \neq 0$ the set of all $v \in E$ with $\operatorname{dist}\big(v, \operatorname{Ker} A(x)\big) \geq 1$ is not empty. Therefore we see from (4.11) that

$$\gamma\big(A(x)\big) = \inf_{v\in E, \operatorname{dist}(v, \operatorname{Ker} A(x))\geq 1} \|A(x)v\| \geq \frac{1}{\|B(x)\|}.$$

(i) ⇒ (ii): Assume (i) is satisfied, and let $x_0 \in X$ be given. Choose projections $P_0 \in L(E)$ and $Q_0 \in L(F)$ with $\operatorname{Im} P_0 = \operatorname{Ker} A(x_0)$ and $\operatorname{Im} Q_0 = \operatorname{Im} A(x_0)$, and define a continuous operator function $\widetilde{A} : X \to L(E, \operatorname{Im} Q_0)$, setting

$$\widetilde{A}(x) = Q_0 A(x), \quad x \in X.$$

Then $\widetilde{A}(x_0)$ is right invertible. Since $\widetilde{A}$ is continuous, this implies that $\widetilde{A}(x)$ is also right invertible for all x in some neighborhood U of x_0. Arguing as in the proof of (vi) ⇒ (ii), it is sufficient to prove that

$$\operatorname{Ker} A(x) = \operatorname{Ker} \widetilde{A}(x)$$

for all x in some neighborhood of x_0.

Assume this is not the case. As, obviously, $\operatorname{Ker} A(x) \subseteq \operatorname{Ker} \widetilde{A}(x)$, then we can find a sequence $x_n \in U$, $n = 1, 2, \ldots$, which converges to x_0 such that

$$\operatorname{Ker} A(x_n) \subsetneq \operatorname{Ker} \widetilde{A}(x_n) \quad \text{for } n = 1, 2, \ldots.$$

Therefore, we can find vectors $v_n \in \operatorname{Ker} \widetilde{A}(x_n)$, $n = 1, 2, \ldots$, such that $\|v_n\| = 2$ and

$$\operatorname{dist}\big(v_n, \operatorname{Ker} A(x_n)\big) \geq 1 \quad \text{for } n = 1, 2, \ldots.$$

The last inequality implies that

$$\gamma\big(A(x_n)\big) \leq \|A(x_n)v_n\| \quad \text{for } n = 1, 2, \ldots. \tag{4.12}$$

Since $\operatorname{Ker} P_0$ is a complement of $\operatorname{Ker} \widetilde{A}(x_0)$ and the family $\big\{\operatorname{Ker} \widetilde{A}(x)\big\}_{x\in U}$ is continuous, it follows from criterion (iii) in Proposition 4.1 that there exist a neighborhood $V \subseteq U$ of x_0 such that $\operatorname{Ker} P_0$ is a complement also for all $\operatorname{Ker} \widetilde{A}(x)$ with $x \in V$, and, moreover, the projection $P(x)$ defined by

$$\operatorname{Im} P(x) = \operatorname{Ker} \widetilde{A}(x) \quad \text{and} \quad \operatorname{Ker} P(x) = \operatorname{Ker} P_0$$

depends continuously on $x \in V$. Note that $P(x_0) = P_0$ and therefore $A(x_0)P(x_0) = 0$. Take n_0 so large that $x_n \in V$ for $n \geq n_0$. Since $v_n \in \operatorname{Ker} \widetilde{A}(x_n) = \operatorname{Im} P(x_n)$ for all $n \geq n_0$ and $A(x_0)P(x_0) = 0$, we have

$$A(x_n)v_n = A(x_n)P(x_n)v_n = \Big(A(x_n)P(x_n) - A(x_0)P(x_0)\Big)v_n$$

for all $n \geq n_0$. As $\|v_n\| = 2$, this implies that

$$\|A(x_n)v_n\| \leq 2\|A(x_n)P(x_n) - A(x_0)P(x_0)\| \quad \text{for } n \geq n_0,$$

and further, by continuity of A and P, $\lim_{n\to\infty} A(x_n)v_n = 0$. Together with (4.12) this implies that, for each neighborhood W of x_0,

$$\inf_{x\in W} \gamma\big(A(x)\big) \leq \limsup_{n\to\infty} \gamma\big(A(x_n)\big) = 0,$$

which is a contradiction to the uniform regularity of A. □

Remark 4.8. If, in Proposition 4.7, the values of A are semi-Fredholm or finite dimensional operators and X is connected, then the equivalent conditions (i)–(viii) can be completed by especially convenient conditions. Namely, assume that under the hypotheses of Proposition 4.7 there exists $n_0 \in \{0, 1, 2, \ldots\}$ such at least one of the following holds:

(α) $\dim \operatorname{Ker} A(x) = n_0$ for all $x \in X$;
(β) $\dim F/\operatorname{Im} A(x) = n_0$ for all $x \in X$;
(γ) $\dim \operatorname{Im} A(x) = n_0$ (or, equivalently, $\dim F/\operatorname{Ker} A(x) = n_0$) for all $x \in X$.

Since, clearly, (α) $\Rightarrow$ (vi), (β) $\Rightarrow$ (vii), and (γ) $\Rightarrow$ (vi) and (vii), then A is uniformly regular. Conversely, if A is uniformly regular and therefore conditions (ii) and (iii) are satisfied, then we see from criterion (ii) in Proposition 4.1 that the functions

$$X \ni x \mapsto \dim \operatorname{Ker} A(x), \qquad X \ni x \mapsto \dim \operatorname{Im} A(x)$$
$$X \ni x \mapsto \dim F/\operatorname{Im} A(x), \qquad X \ni x \mapsto \dim E/\operatorname{Ker} A(x)$$

are constant.

5. The Atkinson formula

Everywhere in this section, $\mathfrak{B}$ stands for a complex Banach algebra with unity, denoted by 1, and X is a topological space.

Conditions (iv), (v), and (viii) in Proposition 4.7 can be formulated also for $\mathfrak{B}$-valued functions. But it is not immediately clear that these conditions stay equivalent in this general setting (actually they do— Corollary 5.6). We take (the apparently weakest) condition (viii) for the following definition.

Definition 5.1. A function $a : X \to \mathfrak{B}$ is called *locally boundedly generalized invertible* if, for each $x_0 \in X$, there exist a neighborhood U of x_0 and a *bounded* function $b : U \to \mathfrak{B}$ such that $aba = a$ on U.[2]

We indulge with slight imprecisions of language in Definition 5.1. Namely, $b(x), x \in X$, need not be a generalized inverse of $a(x)$ because the equality $b(x)a(x)b(x) = b(x)$ is not required. On the other hand, Remark 1.1 guarantees that $a(x)$ is generalized invertible for every $x \in X$. However, the generalized inverse of $a(x)$ given there, namely $b(x)a(x)b(x)$, need not be locally bounded as function of $x \in X$, unless a itself is locally bounded.

If $\mathfrak{B} = L(E)$, where E is a Banach space, then locally boundedly generalized invertibility implies uniform regularity (the continuity of A is not used in the proof of (viii) $\Rightarrow$ (i) in Proposition 4.7). Of course, the opposite is not true, for the values of uniformly regular functions need not be generalized invertible. But there exist also uniformly regular functions with generalized invertible values, which are not locally boundedly generalized invertible. By Proposition 4.7 such functions cannot be continuous.

A basic tool for a "good local choice" of GIs is the following theorem.

Theorem 5.2. *Let $a : X \to \mathfrak{B}$ be a continuous function which is locally boundedly generalized invertible. Then, for each point $x_0 \in X$ and each element $b_0 \in \mathfrak{B}$ which is a generalized*

[2] If $\mathfrak{B}$ is an algebra of matrices and if a is continuous, then this means that the values of a have constant rank on each connected component of X.

inverse of $a(x_0)$*, there exists a neighborhood* U *of* x_0 *such that, for all* $x \in U$*, the elements* $1 - \big(a(x_0) - a(x)\big)b_0$ *and* $1 - b_0\big(a(x_0) - a(x)\big)$ *are invertible, we have*

$$b_0\big[1 - \big(a(x_0) - a(x)\big)b_0\big]^{-1} = \big[1 - b_0\big(a(x_0) - a(x)\big)\big]^{-1}b_0, \tag{5.1}$$

and the element

$$b(x) := b_0\big[1 - \big(a(x_0) - a(x)\big)b_0\big]^{-1} = \big[1 - b_0\big(a(x_0) - a(x)\big)\big]^{-1}b_0 \tag{5.2}$$

is a generalized inverse of $a(x)$*.*

If $\mathfrak{B} = L(E)$, where E is a Banach space, and the function a satisfies at least one of the conditions (α) or (β) in Remark 4.8, then the claim of this theorem was proved by Atkinson [3]. Therefore we call (5.2) the *Atkinson formula*.

We begin the proof of Theorem 5.2 with the observation of Gramsch [20, p. 45, Lemma 4.1 and its proof] that a part of Theorem 5.2 is of purely algebraic nature, namely:

Lemma 5.3. *Let* R *be an arbitrary unital ring, let* $a_0, b_0, a \in R$ *be such that* $b_0a_0b_0 = b_0$ *and* $1 - (a_0 - a)b_0$ *is invertible. Then also* $1 - b_0(a_0 - a)$ *is invertible, we have*

$$b_0\big(1 - (a_0 - a)b_0\big)^{-1} = \big(1 - b_0(a_0 - a)\big)^{-1}b_0, \tag{5.3}$$

and if

$$b := b_0\big(1 - (a_0 - a)b_0\big)^{-1} = \big(1 - b_0(a_0 - a)\big)^{-1}b_0, \tag{5.4}$$

then $bab = b$*.*

Proving this lemma, Gramsch uses the following well-known algebraic lemma whose proof is standard.

Lemma 5.4. *Let* R *be a unital ring, and* $a, b \in R$*. Then* $1 + ab$ *is invertible if and only if* $1 + ba$ *is invertible. If this is the case, then*

$$a(1 + ba)^{-1} = (1 + ab)^{-1}a. \tag{5.5}$$

Indeed, if say $1+ab$ is invertible, then a straightforward verification shows $1-b(1+ab)^{-1}a = (1 + ba)^{-1}$, and (5.5) is obvious upon pre- and post-multiplying the equality $(1 + ab)a = a(1 + ba)$ by $(1 + ab)^{-1}$ and by $(1 + ba)^{-1}$, respectively.

Proof of Lemma 5.3 (*[20, p. 45]*)**.** From Lemma 5.4 we see that $1 - b_0(a_0 - a)$ is invertible and (5.3) holds true. Moreover, as $1 - b_0(a_0 - a)$ is invertible, we can write

$$\begin{aligned} 1 &= \big(1 - b_0(a_0 - a)\big)^{-1}\big(1 - b_0(a_0 - a)\big) = \big(1 - b_0(a_0 - a)\big)^{-1}\big((1 - b_0a_0) + b_0a\big) \\ &= \big(1 - b_0(a_0 - a)\big)^{-1}(1 - b_0a_0) + \big(1 - b_0(a_0 - a)\big)^{-1}b_0a, \end{aligned}$$

i.e.

$$\big(1 - b_0(a_0 - a)\big)^{-1}b_0a = 1 - \big(1 - b_0(a_0 - a)\big)^{-1}\big(1 - b_0a_0\big).$$

Since, by definition (5.4), we have both $\big(1 - b_0(a_0 - a)\big)^{-1}b_0 = b$ and $b_0\big(1 - (a_0 - a)b_0\big)^{-1} = b$, this implies that

$$ba = 1 - \big(1 - b_0(a_0 - a)\big)^{-1}\big(1 - b_0a_0\big),$$

and further

$$bab = b - \big(1 - b_0(a_0 - a)\big)^{-1}\big(1 - b_0 a_0\big) b_0 \big(1 - (a_0 - a) b_0\big)^{-1}$$
$$= b - \big(1 - b_0(a_0 - a)\big)^{-1}\big(b_0 - b_0 a_0 b_0\big)\big(1 - (a_0 - a) b_0\big)^{-1}.$$

As, by hypothesis, $b_0 - b_0 a_0 b_0 = 0$, this implies $bab = b$. □

Observe also the following supplement to Lemma 5.3.

Proposition 5.5. *Under the hypotheses and notation of Lemma 5.3, we have: $aba = a$ if and only if $aR = ab_0R$.*

Proof. First let $aba = a$. Since, by definition (5.4), b is of the form $b = b_0 g$, where g is invertible, then $aR = abaR \subseteq abR = ab_0gR = ab_0R$. The relation $aR \supseteq ab_0R$ is trivial.

Now let $aR = ab_0R$. Again using that b is of the form $b = b_0 g$, where g is invertible, then we obtain $aR = ab_0R = ab_0gR = abR$. In particular, $a \in abR$, i.e. a is of the form $a = abc$ for some $c \in R$. Since, by the claim of Lemma 5.3, $bab = b$, this implies $aba = ababc = abc = a$. □

Proof of Theorem 5.2. Let $x_0 \in X$ and $b_0 \in \mathfrak{B}$ be given such that b_0 is a GI of $a(x_0)$. Since a is continuous, we can find a neighborhood U of x_0 such that, for each $x \in U$, the elements $1 - \big(a(x_0) - a(x)\big)b_0$ and $1 - b_0\big(a(x_0) - a(x)\big)$ are invertible. Then it follows from Lemma 5.3 that, for all $x \in U$, we have (5.1) and the element $b(x)$ defined by (5.2) satisfies $b(x)a(x)b(x) = b(x)$. It remains to prove that, after shrinking U if necessary, also $a(x)b(x)a(x) = a(x)$ for all $x \in U$.

By Proposition 5.5, for this purpose it is sufficient to prove that

$$a(x)\mathfrak{B} = a(x)b_0\mathfrak{B} \tag{5.6}$$

for all x in some neighborhood of x_0.

Now we pass to the Banach algebra $L(\mathfrak{B})$ of bounded linear operators in $\mathfrak{B}$. Let $B_0 \in L(\mathfrak{B})$ be the operator defined by multiplication by b_0 from the left, $B_0 v := b_0 v$, $v \in \mathfrak{B}$, and let $A : X \to L(\mathfrak{B})$ be the operator function defined by $A(x)v = a(x)v$, $x \in X$, $v \in \mathfrak{B}$. Then A is continuous, B_0 is a GI of $A(x_0)$, and (5.6) means that

$$\operatorname{Im} A(x) = \operatorname{Im} A(x)B_0. \tag{5.7}$$

Thus, it is sufficient to prove that (5.7) holds true for all x in some neighborhood of x_0.

Since $A(x_0)B(x_0)$ is a projection onto $\operatorname{Im} A(x_0)$, this is case for $x = x_0$. Moreover, it is trivial that $\operatorname{Im} A(x) \supseteq \operatorname{Im} A(x)B_0$ for all $x \in X$. Therefore, by the uniqueness criterion given by Proposition 4.3, it is sufficient to prove that the families $\{\operatorname{Im} A(x)\}$ and $\{\operatorname{Im} A(x)B_0\}$ are continuous in some neighborhood of x_0.

The continuity of $\{\operatorname{Im} A(x)B_0\}$ at x_0: Consider the continuous operator function $\widetilde{A} : X \to L\big(\operatorname{Im} A(x_0), \mathfrak{B}\big)$ defined by

$$\widetilde{A}(x) = A(x)B_0|_{\operatorname{Im} A(x_0)}.$$

Since B_0 is a GI of $A(x_0)$, $\operatorname{Ker} B_0$ is a complement of $\operatorname{Im} A(x_0)$ in $L(\mathfrak{B})$. Therefore $B_0 \operatorname{Im} A(x_0) = B_0\mathfrak{B}$ and

$$\operatorname{Im} \widetilde{A}(x) = A(x)B_0 \operatorname{Im} A(x_0) = A(x)B_0\mathfrak{B} = \operatorname{Im} A(x)B_0 \quad \text{for all } x \in X. \tag{5.8}$$

Moreover, as $A(x_0)B_0$ is a projection onto $\operatorname{Im} A(x_0)$, the operator $\widetilde{A}(x_0)$ is left invertible. By continuity of $\widetilde{A}$, it follows that $\widetilde{A}(x)$ is left invertible for all x in some neighborhood of x_0, which yields, by Proposition 4.7, that the family $\{\operatorname{Im} \widetilde{A}(x)\}$ is continuous in a neighborhood of x_0. By (5.8) this means that $\{\operatorname{Im} A(x)B_0\}$ is continuous in this neighborhood.

The continuity of $\{\operatorname{Im} A(x)\}$ *at* x_0: Here we use (for the first time in this proof) the hypothesis that a is locally boundedly generalized invertible. By this hypothesis we can find a neighborhood $V \subseteq U$ of x_0 and a bounded function $b : V \to \mathfrak{B}$ such that $aba = a$ on V. Let $B : V \to L(\mathfrak{B})$ be the operator function defined by $B(x)v = b(x)v$, $v \in \mathfrak{B}$, $x \in V$. Then also B is bounded and $ABA = A$ on V. Hence, A is locally boundedly generalized invertible on V, i.e. by Proposition 4.7, the family $\{\operatorname{Im} A(x)\}_{x\in V}$ is continuous. □

Since the function b defined by the Atkinson formula (5.2) is continuous (as a is continuous), Theorem 5.2 immediately implies the following corollary.

Corollary 5.6. *Let $a : X \to \mathfrak{B}$ be a continuous function. Then the following are equivalent:*

(i) *the function a is locally boundedly generalized invertible;*
(ii) *for each $x_0 \in X$, there exist a neighborhood U of x_0 and a continuous function $b : U \to \mathfrak{B}$ such that $aba = a$ on U;*
(iii) *for each $x_0 \in X$, there exist a neighborhood U of x_0 and a continuous function $b : U \to \mathfrak{B}$ such that $aba = a$ and $bab = b$ on U.*

Moreover, the map $g \to g^{-1}$ defined on the group of invertible elements of $\mathfrak{B}$ is holomorphic. Therefore, if $R = \mathfrak{B}$ in Lemma 5.3, then the Atkinson formula (5.4) defines a holomorphic map on the open set of all $b \in \mathfrak{B}$ such that $1 - b_0(a_0 - b)$ is invertible. This implies that Theorem 5.2 also has the following two corollaries.

Corollary 5.7. *Let X be an open subset of $\mathbb{C}^n$, and let $a : X \to \mathfrak{B}$ be a holomorphic function which is locally boundedly generalized invertible.*

Then, for each point $x_0 \in X$, there exists a neighborhood $U \subseteq X$ of x_0 and a holomorphic function $h : U \to \mathfrak{B}$ such that $h(x)$ is a GI of $a(x)$ for all $x \in U$.

Corollary 5.8. *Let X be an open subset of $\mathbb{R}^n$, and let $a : X \to \mathfrak{B}$ be a function of class $\mathcal{C}^\aleph$ (where $\mathcal{C}^\aleph$ has the same meaning as in Section 2) which is locally boundedly generalized invertible.*

Then, for each point $x_0 \in X$, there exists a neighborhood $U \subseteq X$ of x_0 and a function $h : U \to \mathfrak{B}$ of class $\mathcal{C}^\aleph$ such that $h(x)$ is a GI of $a(x)$ for all $x \in U$.

Consider the case $\mathfrak{B} = L(E)$, where E is a Banach space. Then, provided one is aware of the equivalence of conditions (i)–(viii) in Proposition 4.7, Corollaries 5.7 and 5.8 are well known. The claim of Corollary 5.7 was first proved by Atkinson [3], using his formula, in the case when at least one of the conditions (α) and (β) in Remark 4.8 are satisfied. In the general case, by a different method not using the Atkinson formula, Shubin [47, Proposition 4] proved that the claim of Corollary 5.7 is equivalent to each of the conditions (vi) and (vii) in Proposition 4.7. This proof works also in the situation of Corollary 5.8.

Remark 5.9. Assume that $\mathfrak{B} = L(E)$ in Theorem 5.2. Then the image and the kernel of the generalized inverse $b(x)$ defined by the Atkinson formula (5.2) do not depend on x, namely

$$\operatorname{Im} b(x) = \operatorname{Im} b_0 \quad \text{and} \quad \operatorname{Ker} b(x) = \operatorname{Ker} b_0 \quad \text{for all } x \in U. \tag{5.9}$$

This follows from the fact that, by (5.2), $b(x)$ is both of the form $b(x) = b_0 f(x)$ and of the form $b(x) = g(x)b_0$, where $f(x)$ and $g(x)$ are invertible operators.

If $E = H$ is a Hilbert space, this shows that, in general, the GI defined by the Atkinson formula (5.2) is not the Moore–Penrose inverse $a(x)^+$ of $a(x)$, since the latter is the GI of $a(x)$ which is uniquely determined by

$$\operatorname{Im} a(x)^+ = \operatorname{Ker} a(x)^\perp \quad \text{and} \quad \operatorname{Ker} a(x)^+ = \operatorname{Im} a(x)^\perp. \tag{5.10}$$

Nevertheless, using Lemma 2.1, we can derive from the Atkinson formula (5.2) also a useful formula for the Moore–Penrose inverse. Namely, let b be defined by (5.2), and let $\Pi_{\operatorname{Im} a}$ and $\Pi_{\operatorname{Ker} a^\perp}$ be the functions which assign to each $x \in U$ the orthogonal projections onto $\operatorname{Im} a(x)$ and $\operatorname{Ker} a(x)^\perp$, respectively. Since $a(x)b(x)$ is a projection onto $\operatorname{Im} a(x)$ and $I - b(x)a(x)$ is a projection onto $\operatorname{Ker} a(x)$, then it follows from Lemma 2.1 that

$$\Pi_{\operatorname{Im} a} = \Big(abb^*a^* + (I - b^*a^*)(I - ab)\Big)^{-1} abb^*a^* \tag{5.11}$$

and

$$\Pi_{\operatorname{Ker} a^\perp} = I - \Big((I - ba)(I - a^*b^*) + a^*b^*ba\Big)^{-1} (I - ba)(I - a^*b^*). \tag{5.12}$$

Since (cf. the arguments given in the proof of Theorem 3.4)

$$a^+ = \Pi_{\operatorname{Ker} a^\perp} b \Pi_{\operatorname{Im} a} \tag{5.13}$$

this gives a formula for the Moore–Penrose inverse. As this formula is a composition of holomorphic maps (the algebraic operations in $L(H)$ and the map $A \mapsto A^{-1}$ defined on the group of invertible elements of $L(H)$) and the anti-holomorphic map $L(H) \ni A \mapsto A^*$, this proves again Theorem 3.7.

For $u \in \mathfrak{B}$, let $M_u \in L(\mathfrak{B})$ be the operator defined by $M_u v := uv$, $v \in \mathfrak{B}$. Then it is easy to see that the map $\mathfrak{B} \ni u \mapsto M_u$ is an isometric isomorphism from $\mathfrak{B}$ onto a closed subalgebra of $L(\mathfrak{B})$, and that an element $u \in \mathfrak{B}$ is invertible if and only if the operator M_u is invertible. Note also the following consequence of Theorem 5.2.

Proposition 5.10. *For each continuous $a : X \to \mathfrak{B}$, the following are equivalent:*

(i) *the function a is locally boundedly generalized invertible;*
(ii) *the function M_a is locally boundedly generalized invertible.*

Proof. (i) $\Rightarrow$ (ii) is trivial. Assume (ii), and let $x_0 \in X$ be given. It is sufficient to find a neighborhood U of x_0 and a continuous function $b : U \to \mathfrak{B}$ such that, for all $x \in U$,

$$a(x)b(x)a(x) = a(x). \tag{5.14}$$

Choose a GI, b_0, of $a(x_0)$. Then M_{b_0} is a GI of $M_{a(x_0)}$, and from Theorem 5.2 applied to the algebra $L(\mathfrak{B})$ we get a neighborhood U of x_0 such that, for all $x \in U$, the operator $I_{L(\mathfrak{B})} - \big(M_{a(x_0)} - M_{a(x)}\big)M_{b_0}$ is invertible, and the operator

$$B(x) := M_{b_0}\big[I_{L(\mathfrak{B})} - \big(M_{a(x_0)} - M_{a(x)}\big)M_{b_0}\big]^{-1}$$

is a generalized inverse of $M_{a(x)}$. Clearly, shrinking U if necessary, for all $x \in U$ we have that for all $x \in U$, $1 - \big(a(x_0) - a(x)\big)b_0$ is an invertible element of $\mathfrak{B}$. Therefore we can define a continuous function $b : U \to \mathfrak{B}$, setting

$$b(x) = b_0\big[1 - \big(a(x_0) - a(x)\big)b_0\big]^{-1}, \quad x \in U.$$

Then, for all $x \in U$, $M_{b(x)} = B(x)$. As $B(x)$ is a GI of $M_{a(x)}$, it follows that, for all $x \in U$,

$$M_{a(x)b(x)a(x)} = M_{a(x)}M_{b(x)}M_{a(x)} = M_{a(x)}B(x)M_{a(x)} = M_{a(x)},$$

i.e. we have (5.14). □

If E and F are (possibly different) Banach spaces, then we say that a function $a : X \to L(E, F)$ is *locally boundedly generalized invertible* if, for each $x_0 \in X$, there exist a neighborhood U of x_0 and a bounded function $b : U \to L(F, E)$ such that $aba = a$ on U.

The $L(E, F)$ version of Theorem 5.2 and Proposition 5.10 runs as follows.

Theorem 5.11. *Let E, F be Banach spaces, and let $a : X \to L(E, F)$ be a continuous function which satisfies either of the following two equivalent conditions:*

(a) *the function a is locally boundedly generalized invertible;*
(b) *the function*

$$M_a : X \to L\Big(L(E \oplus F, E), L(E \oplus F, F)\Big)$$

defined by $M_a(x)v = a(x)v$, $v \in L(E \oplus F, E)$, $x \in X$, is locally boundedly generalized invertible.

Then, for each point $x_0 \in X$ and each operator $b_0 \in L(F, E)$ which is a generalized inverse of $a(x_0)$, there exists a neighborhood U of x_0 such that, for all $x \in U$, the operators $1 - \big(a(x_0) - a(x)\big)b_0 \in L(F)$ and $1 - b_0\big(a(x_0) - a(x)\big) \in L(E)$ are invertible, we have

$$b_0\big[1 - \big(a(x_0) - a(x)\big)b_0\big]^{-1} = \big[1 - b_0\big(a(x_0) - a(x)\big)\big]^{-1}b_0, \tag{5.15}$$

and the operator

$$b(x) := b_0\big[1 - \big(a(x_0) - a(x)\big)b_0\big]^{-1} = \big[1 - b_0\big(a(x_0) - a(x)\big)\big]^{-1}b_0 \tag{5.16}$$

is a generalized inverse of $a(x)$.

Proof. Assume (a) holds. Let $x_0 \in X$ and a generalized inverse $b_0 \in L(F, E)$ of $a(x_0)$ be given. Let

$$\widehat{a}(x) := \begin{bmatrix} 0 & 0 \\ a(x) & 0 \end{bmatrix}, \qquad \widehat{b_0} := \begin{bmatrix} 0 & b_0 \\ 0 & 0 \end{bmatrix} \in L(E \oplus F).$$

Then (see Remark 1.2) the hypotheses of Theorem 5.2 are satisfied for $\widehat{a}(x)$ and $\widehat{b_0}$, therefore by Theorem 5.2, there exists a neighborhood U of x_0 such that, for all $x \in U$, the operators

$$\widehat{g}(x) := I_{L(E\oplus F)} - \big[\widehat{a}(x_0) - \widehat{a}(x)\big]\widehat{b_0} \quad \text{and} \quad \widehat{f}(x) := I_{L(E\oplus F)} - \widehat{b_0}\big[\widehat{a}(x_0) - \widehat{a}(x)\big]$$

are invertible, we have $\widehat{b_0}\widehat{g}(x)^{-1} = \widehat{f}(x)^{-1}\widehat{b_0}$, and

$$\widehat{b}(x) := \widehat{b_0}\widehat{g}(x)^{-1} = \widehat{f}(x)^{-1}\widehat{b_0}$$

is a generalized inverse of $\widehat{a}(x)$. A straightforward computation shows that

$$\widehat{g}(x) = \begin{bmatrix} I_E & 0 \\ 0 & I_F - \big[a(x_0) - a(x)\big]b_0 \end{bmatrix}, \qquad \widehat{f}(x) = \begin{bmatrix} I_E - b_0\big[a(x_0) - a(x)\big] & 0 \\ 0 & I_F \end{bmatrix},$$

hence invertibility of $\widehat{g}(x)$ and $\widehat{f}(x)$ is equivalent to that of

$$g(x) := I_F - \big[a(x_0) - a(x)\big]b_0 \quad \text{and} \quad f(x) := I_E - b_0\big[a(x_0) - a(x)\big],$$

respectively, and moreover,

$$\widehat{b}(x) = \begin{bmatrix} 0 & b_0 g(x)^{-1} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & f(x)^{-1} b_0 \\ 0 & 0 \end{bmatrix}.$$

Thus (again by Remark 1.2), $b(x) := b_0 g(x)^{-1} = f(x)^{-1} b_0$ is a generalized inverse of $a(x)$, as claimed. □

Furthermore, by Remark 1.2, condition (a) is easily seen to be equivalent to the locally bounded generalized invertibility of $\widehat{a}$. Proposition 5.10 applied to the function $\widehat{a}$ now yields that (a) is equivalent to

$\widehat{(\mathrm{b})}$ the function

$$M_{\widehat{a}} : X \to L(E \oplus F)$$

defined by $M_{\widehat{a}}(x)\widehat{v} = \widehat{a}(x)\widehat{v}, \widehat{v} \in L(E \oplus F), x \in X$, is locally boundedly generalized invertible.

Since

$$\widehat{a}L(E \oplus F) = \begin{bmatrix} 0 & 0 \\ aL(E \oplus F, E) \end{bmatrix},$$

clearly $\widehat{(\mathrm{b})}$ is equivalent to (b), and (a) $\Leftrightarrow$ (b) follows.

6. Global generalized inverses

We continue to assume in this section that $\mathfrak{B}$ be a unital Banach algebra, and assume in addition that X is a manifold (to be further specified). Let $a : X \to \mathfrak{B}$ belongs to a certain class of functions defined on X. We develop here results concerning existence of a generalized inverse of a in the same class of $\mathfrak{B}$-valued functions.

We start with the $\mathcal{C}^{\aleph}$ classes as defined in Section 2.

Theorem 6.1. *Let X be a $\mathcal{C}^{\aleph}$-manifold with countable topology, and let $a : X \to \mathfrak{B}$ be a $\mathcal{C}^{\aleph}$ function which is locally boundedly generalized invertible (Definition 5.1). Then there exists a $\mathcal{C}^{\aleph}$ function $b : X \to \mathfrak{B}$ such that $aba = a$ and $bab = b$ on X.*

Proof (*For the Case* $\aleph = \alpha$ *with* $0 \le \alpha \le \infty$)**.** From Corollary 5.8 we get an open covering $\{U_i\}_{i \in I}$ of X and a family $\{b_i\}_{i \in I}$ of $\mathcal{C}^{\alpha}$ functions $b_i : U_i \to \mathfrak{B}$ such that $ab_i a = a$ on U_j. Then we take a $\mathcal{C}^{\alpha}$ partition of unity, $\{U_i\}_{i \in I}$, subordinated to $\{U_i\}_{i \in I}$, and define a global $\mathcal{C}^{\alpha}$ function $u : X \to \mathfrak{B}$, setting

$$u = \sum_{i \in I} \chi_i b_i.$$

Then

$$aua = a\left(\sum_{i \in I} \chi_i b_i\right)a = \sum_{i \in I} \chi_i a b_i a = a \sum_{i \in I} \chi_i = a.$$

It remains to set $b = uau$ (cf. Remark 1.1). □

Possibly for the first time, this simple proof was noticed by Shubin [47, p. 415]. In the real-analytic case ($\aleph = \omega$), this does not work, because real-analytic partitions of unity do not exist. Therefore, we now first consider the holomorphic case. (The real-analytic case of Theorem 6.1 will be proved later.)

Global holomorphic generalized inverses do not always exist. Here is a counterexample, given (in a somewhat different context) in [19, p. 121].

Counterexample 6.2. Consider the holomorphic matrix function A defined on $\mathbb{C}^2$ by

$$A(z) = \begin{bmatrix} z_1 & 0 \\ z_2 & 0 \end{bmatrix}, \quad z = (z_1, z_2) \in \mathbb{C}^2.$$

Then A has the constant rank 1 on $\mathbb{C}^2 \setminus \{0\}$, but $A(0) = 0$. Hence, as a function on $\mathbb{C}^2$, A is not locally boundedly generalized invertible, but as a function defined only on $\mathbb{C}^2 \setminus \{0\}$ it is. Nevertheless, there does not exist a holomorphic matrix function

$$B(z) = \begin{bmatrix} b_1(z) & b_2(z) \\ b_3(z) & b_4(z) \end{bmatrix}, \quad z \in \mathbb{C}^2 \setminus \{0\}$$

such that $B(z)$ is a GI of $A(z)$ for all $z \in \mathbb{C}^2 \setminus \{0\}$. Indeed, assume such a function exists. Then

$$\begin{bmatrix} z_1 & 0 \\ z_2 & 0 \end{bmatrix} = \begin{bmatrix} z_1 & 0 \\ z_2 & 0 \end{bmatrix} \begin{bmatrix} b_1(z) & b_2(z) \\ b_3(z) & b_4(z) \end{bmatrix} \begin{bmatrix} z_1 & 0 \\ z_2 & 0 \end{bmatrix} = \begin{bmatrix} z_1^2 b_1(z) + z_1 b_2(z) z_2 & 0 \\ * & 0 \end{bmatrix}$$

for all $z \in \mathbb{C}^2$ with $z \neq 0$. In particular,

$$z_1 = z_1^2 b_1(z) + z_1 b_2 z_2 \quad \text{for } z \neq 0,$$

and, assuming $z_1 \neq 0$,

$$1 = z_1 b_1(z) + z_2 b_2(z). \tag{6.1}$$

By continuity, (6.1) holds for every $z \in \mathbb{C}^2 \setminus \{0\}$. However, by Hartogs' extension theorem, $b_1(z)$ and $b_2(z)$ admit holomorphic continuations to zero, and letting $z = 0$ in (6.1), a contradiction is obtained.

The punctured space $X = \mathbb{C}^2 \setminus \{0\}$ in this counterexample is not Stein. If X is a Stein manifold, then each holomorphic function on X, which is locally boundedly generalized invertible, admits a global holomorphic generalized inverse:

Theorem 6.3. *Let X be a Stein manifold, and $\mathfrak{B}$ a unital Banach algebra. Then, for each holomorphic function $a : X \to \mathfrak{B}$, which is locally boundedly generalized invertible, there exists a holomorphic function $b : X \to \mathfrak{B}$ such that $aba = a$ and $bab = b$ on X.*

If $\mathfrak{B} = L(E)$, where E is a Banach space, and if at least one of the conditions (vi) and (vii) in Proposition 4.7 is satisfied, the claim of this theorem (in a somewhat different formulation) was proved by Shubin [47, Corollary 1 on p. 418]. In the case when X is a domain in the complex plane and the values of a are Fredholm operators with constant kernel dimension, the claim of Theorem 6.3 was independently obtained also by Bart [4, Theorem 2.2]. Shubin proves that, in view of the local solvability of the problem (Corollary 5.7), the global solvability is equivalent to a certain Cousin problem, which can be solved on Stein manifolds by a result of Bungart [7]. In the case of one-sided invertible functions (and arbitrary $\mathfrak{B}$), Theorem 6.3 was already proved by

Allan [2], using a completely different method. Note that Allan has proved even the following more general result.[3]

Theorem 6.4. *Let X be a Stein manifold, and let $a_1, \dots, a_k : X \to \mathfrak{B}$ be holomorphic functions such that, for each $z \in X$, there exists a solution $u_1(z), \dots, u_k(z) \in \mathfrak{B}$ of the equation*

$$a_1(z)u_1(z) + \dots + a_k(z)u_k(z) = 1. \tag{6.2}$$

Then such a solution can be chosen holomorphically in $z \in X$.

Note also the recent work of Dineen and Venkova [11]. They consider the case $\mathfrak{B} = L(E)$ where E is a Banach space, and (as, by Propositions 4.1 and 4.7, locally boundedly generalized invertibility is equivalent to condition (4) from Theorem 2 in [11]) they prove the claim of Theorem 6.3 in the case when X is a pseudoconvex domain in an arbitrary Banach space with an unconditional basis.

The general case of Theorem 6.3 can be proved modifying Shubin's arguments and using the same result of Bungart, what we now explain. First recall that a special case of Bungart's result [7, 4.4 Remarks], which is sufficient for our purpose, can be stated as follows.

Theorem 6.5. *Let X be a Stein manifold and E a Banach space. Furthermore, let $\{M(z)\}_{z\in X}$ be a family of closed subspaces[4] of E satisfying the following condition.*

($*$) *For each $z_0 \in X$, there exist a neighborhood U of z_0 and a holomorphic operator function $T : U \to L(E)$ all values of which are invertible such that $T(z)M(z_0) = M(z)$ for all $z \in U$.*

Then, for each open covering $\{U_j\}_{j\in I}$ of X and each family $\{g_{ij}\}_{i,j\in I}$ of holomorphic vector functions $g_{ij} : U_i \cap U_j \to E$ such that

$$\begin{aligned} &g_{ij}(z) \in M(z) \quad \text{for all } z \in U_i \cap U_j, \quad \text{and} \\ &g_{ij} + g_{jk} = g_{ik} \quad \text{on } U_i \cap U_j \cap U_k, \end{aligned} \tag{6.3}$$

there exists a family $\{f_i\}_{i\in I}$ of holomorphic functions $f_i : U_i \to E$ such that

$$\begin{aligned} &f_i(z) \in M(z) \quad \text{for all } z \in U_i, \quad \text{and} \\ &g_{ij} = f_i - f_j \quad \text{on } U_i \cap U_j. \end{aligned} \tag{6.4}$$

Proof of Theorem 6.3. Consider the family $\{M(z)\}_{z\in X}$ of closed subspaces of $\mathfrak{B}$ defined by

$$M(z) = \Big\{ v \in \mathfrak{B} \mid a(z)\, v\, a(z) = 0 \Big\}.$$

We first prove that this family satisfies condition ($*$) in Bungart's Theorem 6.5. Let $z_0 \in X$ be given. We have to find a neighborhood U of z_0 and a holomorphic operator function $T : U \to L(\mathfrak{B})$ all values of which are invertible such that

$$T(z)M(z_0) = M(z) \quad \text{for all } z \in U. \tag{6.5}$$

[3] Actually, this more general result can be deduced also directly from its special case when $k = 1$ and, moreover, $\mathfrak{B} = L(E)$ for some Banach space E—see Remark 6.11.

[4] In our application these spaces will be complemented, but for the result of Bungart this hypothesis is not necessary.

By Corollary 5.7, we can find a neighborhood U of z_0 and a holomorphic function $h : U \to \mathfrak{B}$ such that

$$aha = a \quad \text{on } U. \tag{6.6}$$

Define a holomorphic operator function $P : U \to L(\mathfrak{B})$ by

$$P(z)v = h(z)a(z)\, v\, a(z)h(z), \qquad v \in \mathfrak{B},\ z \in U.$$

From (6.6) we see that $P(z)$ is a projection. Obviously, $M(z) \subseteq \operatorname{Ker} P(z)$. Conversely, if $v \in \operatorname{Ker} P(z)$, then, again by (6.6),

$$a(z)\, v\, a(z) = a(z)h(z)a(z)\, v\, a(z)h(z)a(z) = a(z)\big(P(z)v\big)a(z) = 0.$$

Hence

$$M(z) = \operatorname{Ker} P(z) = \operatorname{Im}\big(I - P(z)\big) \quad \text{for all } z \in U. \tag{6.7}$$

In particular, the family $\{M(z)\}_{z\in U}$ is continuous in the sense of Definition 4.2. Finally, define a holomorphic operator function $T : U \to L(\mathfrak{B})$ by

$$T(z) = P(z)P(z_0) + \big(I - P(z)\big)\big(I - P(z_0)\big), \qquad z \in U.$$

Since $T(z_0) = I$, after shrinking U if necessary, all values of T are invertible. From (6.7) we see that

$$T(z)M(z_0) = \big(I - P(z)\big)\big(I - P(z_0)\big)M(z_0) \subseteq \operatorname{Im}\big(I - P(z)\big) = M(z)$$

for all $z \in U$, and

$$T(z_0)M(z_0) = M(z_0).$$

Since also the family $\{T(z)M(z_0)\}_{z\in U}$ is continuous, after shrinking U if it is not connected, this yields (6.5) by Proposition 4.3.

To construct now the required function b, we observe that, again by Corollary 5.7, we can find an open covering $\{U_i\}_{i\in I}$ of X and a family $\{b_i\}_{i\in I}$ of holomorphic functions $b_j : U_j \to \mathfrak{B}$ such that $ab_ja = a$ on U_j. Then $a(b_i - b_j)a = a - a = 0$ on $U_i \cap U_j$, i.e. the family of functions $g_{ij} := b_i - b_j$ satisfies the first condition in (6.3). Moreover, $g_{ij} + g_{jk} = b_i - b_j + b_j - b_k = b_i - b_k = g_{ik}$ on $U_i \cap U_j$, i.e. also the second condition in (6.3) is satisfied. Therefore, by Bungart's Theorem 6.5, we can find a family $\{f_i\}_{i\in I}$ of holomorphic functions $f_i : U_i \to \mathfrak{B}$ satisfying (6.4). Therefore, we can define a global holomorphic function $h : X \to \mathfrak{B}$ by setting $h := b_i - f_i$ on U_i. Then the computation $aha = a(b_i - f_i)a = ab_ia = a$, carried out on each U_i, shows that $aha = a$ on X, and (cf. Remark 1.1) $b := hah$ satisfies $aba = a$ and $bab = b$ on X. □

The Steinness of X is not generally necessary for the claim of Theorem 6.3 (take for example a compact manifold). For domains in $\mathbb{C}^n$ however this is the case:

Theorem 6.6. *Let $X \subseteq \mathbb{C}^n$ be an open set. Then the following are equivalent:*

(i) *X is a domain of holomorphy.*

(ii) *For each point $a = (a_1, \ldots, a_n) \in \partial X$, for the $n \times n$ matrix function A defined on X by*

$$A(z) = \begin{bmatrix} z_1 - a_1 & 0 & \cdots & 0 \\ z_2 - a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ z_n - a_n & 0 & \cdots & 0 \end{bmatrix}, \tag{6.8}$$

there exists a holomorphic $n \times n$ matrix B on X such that $ABA = A$ and $BAB = B$ on X.

Proof. Since A has constant rank on X, (i) $\Rightarrow$ (ii) holds by Theorem 6.3.

To prove that (ii) $\Rightarrow$ (i), we assume that X is not a domain of holomorphy, i.e. (see, for example, Theorem 2.5.5 in [24]), there exists a compact subset K of X such that the hull

$$K_X := \left\{ z \in X \mid |f(z)| \le \max_{\zeta \in K} |f(\zeta)| \text{ for each holomorphic } f : X \to \mathbb{C} \right\}$$

is not compact. Since, on the other hand, K_X is bounded and relatively closed in X (see the text following Definition 2.5.2 in [24]), this implies that $\overline{K}_X \cap \partial X \neq \emptyset$, where $\overline{K}_X$ is the closure of K_X in $\mathbb{C}^n$. Choose a point $a = (a_1, \dots, a_n) \in \overline{K}_X \cap \partial X$ and a sequence

$$a^{(j)} = (a_1^{(j)}, \dots, a_n^{(j)}) \in K_X, \quad j = 1, 2, \dots,$$

which converges to a. Then

$$\sup_j \left| f\left(a^{(j)}\right) \right| \le \max_{\zeta \in K} |f(\zeta)| < \infty \quad \text{for each holomorphic } f : X \to \mathbb{C}. \tag{6.9}$$

We claim that, with this choice of the point a, the matrix A defined by (6.8) does not admit a global holomorphic generalized inverse B on X.

Indeed, assume the contrary: such a matrix function $B(z) = \left(b_{ij}(z)\right)$ exists. Then the relation $ABA = A$ in particular implies that

$$\begin{aligned} z_1 - a_1 = {} & \left(z_1 - a_1\right)^2 b_{11}(z) + \left(z_1 - a_1\right)\left(z_2 - a_2\right) b_{12}(z) \\ & + \cdots + \left(z_1 - a_1\right)\left(z_n - a_n\right) b_{1n}(z) \end{aligned}$$

for all $z \in X$, and, assuming $z_1 \neq a_1$,

$$1 = \left(z_1 - a_1\right) b_{11}(z) + \left(z_2 - a_2\right) b_{12}(z) + \cdots + \left(z_n - a_n\right) b_{1n}(z). \tag{6.10}$$

By continuity, (6.10) holds for every $z \in X$. (X is open, and therefore the set of points $z \in X$ with $z_1 \neq a_1$ is dense in X.) In particular,

$$1 = \left(a_1^{(j)} - a_1\right) b_{11}\left(a^{(j)}\right) + \left(a_2^{(j)} - a_2\right) b_{12}\left(a^{(j)}\right) + \cdots + \left(a_n^{(j)} - a_n\right) b_{1n}\left(a^{(j)}\right)$$

for all j. This is impossible, because the right hand side of this equality converges to zero, for $j \to \infty$ (which follows from (6.9) and $\lim_{j \to \infty} a^{(j)} = a$). $\square$

Finally, we prove Theorem 6.1 in the real-analytic case. Grauert [21] discovered a powerful tool to prove results for real-analytic functions on real-analytic manifolds which are already known for holomorphic functions on Stein manifolds, nowadays called the *Grauert tube theorem*. Grauert himself, in the same paper [21], deduced the fact that each (connected) real-analytic manifold with countable topology is a real-analytic submanifold of a certain $\mathbb{R}^N$ from the fact that each (connected) Stein manifold is a submanifold of a certain $\mathbb{C}^N$.

In our context, this method was first employed by Gramsch [19, Section 2.3], who proved the following.

Theorem 6.7. *Let X be an open subset of $\mathbb{R}^n$, and let $a_1, \dots, a_k : X \to \mathfrak{B}$ be real-analytic functions such that, for each $x \in X$, there exists a solution $u_1(x), \dots, u_k(x) \in \mathfrak{B}$ of the equation*

$$a_1(x) u_1(x) + \cdots + a_k(x) u_k(x) = 1. \tag{6.11}$$

Then such a solution can be chosen real-analytic in $x \in X$.

Proof (*See [19, Section 2.3]*)**.** Consider $\mathbb{R}^n$ as

$$\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \operatorname{Im} z_1 = \dots = \operatorname{Im} z_n = 0\},$$

and let U be a neighborhood of X in $\mathbb{C}^n$ such that the functions $a_1, \dots, a_k$ admit holomorphic extensions, $\widetilde{a}_1, \dots, \widetilde{a}_k$, to U.

Now, if x is some point in X, and $u_1(x), \dots, u_k(x) \in \mathfrak{B}$ is a solution of (6.11), then we can find a neighborhood $V_x \subseteq U$ of x in $\mathbb{C}^n$ such that, for each $z \in V_x$, the element

$$b(z) := a_1(z)u_1(x) + \dots + a_k(z)u_k(x)$$

is still invertible, and, setting $u_i(z) = u_i(x)b(z)^{-1}$, we get a solution of

$$a_1(z)u_1(z) + \dots + a_k(z)u_k(z) = 1. \tag{6.12}$$

Then $V := \bigcup_{x \in X} V_x$ is a neighborhood of X in $\mathbb{C}^n$ such that (6.12) has a solution for all $z \in V$.

Furthermore, by Grauert's tube theorem [21, Section 3], X has a basis of Stein neighborhoods in $\mathbb{C}^n$. Hence, we can find a neighborhood $W \subseteq V$ of X in $\mathbb{C}^n$ which is Stein. By Allan's Theorem 6.4, then there exist holomorphic functions $h_1, \dots, h_k : W \to \mathfrak{B}$ such that $a_1h_1 + \dots + a_kh_k = 1$ on W, and the functions $b_1 := h_1|_X, \dots, b_k := h_k|_X$ have the required properties. □

We now use the same idea to prove the real-analytic case of Theorem 6.1.

Proof of Theorem 6.1 (*In the Real-Analytic Case*)**.** Let n be the real dimension of X. In [21, Section 3], Grauert first observes that, by a result of Whitney and Bruhat [51], there exists an n-dimensional complex manifold $\widetilde{X}$ with countable topology such that X is a real-analytic closed submanifold of $\widetilde{X}$ with the following property:

(∗) For each point $\zeta_0 \in X$, there exists a neighborhood U in $\widetilde{X}$ of ζ_0 and a system $z = (z_1, \dots, z_n)$ of holomorphic coordinates on U such that, if x_j and y_j are the underlying real coordinates with $z_j = x_j + iy_j$, then

$$U \cap X = \{\zeta \in U \mid y_1(\zeta) = \dots = y_n(\zeta) = 0\};$$

and then he proves that X admits a basis of Stein neighborhoods in $\widetilde{X}$, i.e.

(∗∗) for each open subset $\widetilde{W}$ of $\widetilde{X}$ with $X \subseteq \widetilde{W}$, there exists an open subset $\widetilde{V}$ of $\widetilde{X}$ such that $\widetilde{V}$ is Stein and $X \subseteq \widetilde{V} \subseteq \widetilde{W}$.

Using the fact that each real-analytic function f defined on some open subset U of $\mathbb{R}^n \subseteq \mathbb{C}^n$ admits a uniquely determined (complex-) holomorphic extension to some $\mathbb{C}^n$-neighborhood of U (depending on f), from (∗) we further obtain the following statement:

(∗ ∗ ∗) Let U be an open subset of X, and let $f : U \to \mathfrak{B}$ be real-analytic. Then there exist an open subset $\widetilde{U}$ of $\widetilde{X}$ and a holomorphic function $\widetilde{f} : \widetilde{U} \to \mathfrak{B}$ such that $\widetilde{U} \cap X = U$ and $\widetilde{f}|_U = f$.

To prove the claim of the theorem, now let a real-analytic and locally boundedly generalized invertible function $a : X \to \mathfrak{B}$ be given. Then, from statement (∗ ∗ ∗) (with $U = X$ and $f = a$) we get an open subset $\widetilde{U}$ of $\widetilde{X}$ and a holomorphic function $\widetilde{a} : \widetilde{U} \to \mathfrak{B}$ such that $X \subseteq \widetilde{U}$ and $\widetilde{a}|_X = a$.

Next we prove the following statement.

$(***\!*)$ There exists an open subset $\widetilde{V}$ of $\widetilde{X}$ such that $X \subseteq \widetilde{V} \subseteq \widetilde{U}$ and $\widetilde{a}|_{\widetilde{V}}$ is locally boundedly generalized invertible on $\widetilde{V}$.

To prove $(***\!*)$, it is sufficient to show that, for each $x_0 \in X$, there exists an open subset $\widetilde{V}_0$ of $\widetilde{X}$ such that $x_0 \in \widetilde{V}_0$, $\widetilde{V}_0 \subseteq \widetilde{U}$, and $\widetilde{a}|_{\widetilde{V}_0}$ is locally boundedly generalized invertible on $\widetilde{V}_0$. Let $x_0 \in X$ be given. Since a is locally boundedly generalized invertible, then we get from Corollary 5.8 an open subset V_0 of X and a real-analytic function $b_0 : V_0 \to \mathfrak{B}$ such that $x_0 \in V_0$ and

$$ab_0a = a \quad \text{on } V_0. \tag{6.13}$$

Moreover, by statement $(***)$ (with $U = V_0$ and $f = b_0$), then there exists an open subset $\widetilde{V}_0$ of $\widetilde{X}$ and a holomorphic function $\widetilde{b}_0 : \widetilde{V}_0 \to \mathfrak{B}$ such that $X \cap \widetilde{V}_0 = V_0$ and $\widetilde{b}_0|_{V_0} = b_0$. From (6.13) we see that, on $X \cap \widetilde{V}_0 = V_0$,

$$\widetilde{a}\widetilde{b}_0\widetilde{a} = \widetilde{a}. \tag{6.14}$$

Since holomorphic functions defined on a connected open subset W of $\mathbb{C}^n$ with $W \cap \mathbb{R}^n \neq \emptyset$ are uniquely determined by its values on $W \cap \mathbb{R}^n$, this further implies that, after shrinking $\widetilde{V}_0$ if necessary, (6.14) holds true everywhere on $\widetilde{V}$. In particular, this implies that $\widetilde{a}$ is locally boundedly generalized invertible on $\widetilde{V}_0$, and statement $(***\!*)$ is proved.

From statement $(**)$ it follows that the $\widetilde{V}$ in statement $(***\!*)$ can be chosen to be a Stein manifold. So finally we obtained

- a Stein open subset $\widetilde{V}$ of $\widetilde{X}$ such that $X \subseteq \widetilde{V}$, and
- a function $\widetilde{a} : \widetilde{V} \to \mathfrak{B}$ which is holomorphic and locally boundedly generalized invertible on $\widetilde{V}$ such that $\widetilde{a}|_X = a$.

Now, by Theorem 6.3 (with $X = \widetilde{V}$ and $a = \widetilde{a}$), we can find a holomorphic function $\widetilde{b} : \widetilde{V} \to \mathfrak{B}$ such that $\widetilde{a}\widetilde{b}\widetilde{a} = \widetilde{a}$ and $\widetilde{b}\widetilde{a}\widetilde{b} = \widetilde{b}$ on $\widetilde{V}$. It remains to set $b = \widetilde{b}|_X$. □

By Remark 1.2, Theorems 6.1 and 6.3 admit the following $L(E, F)$-versions.

Theorem 6.8. *Let E, F be Banach spaces, let X be a $\mathcal{C}^{\aleph}$-manifold with countable topology, and let $A : X \to L(E, F)$ be a $\mathcal{C}^{\aleph}$ operator function which is locally boundedly generalized invertible (defined as in the holomorphic case before Theorem 5.11). Then there exists a $\mathcal{C}^{\aleph}$ operator function $B : X \to L(F, E)$ such that $ABA = A$ and $BAB = B$ on X.*

Theorem 6.9. *Let E, F be Banach spaces, let X be a Stein manifold, and let $A : X \to L(E, F)$ be a holomorphic operator function which is locally boundedly generalized invertible (see the definition before Theorem 5.11). Then there exists a holomorphic operator function $B : X \to L(F, E)$ such that $ABA = A$ and $BAB = B$ on X.*

An application of Theorems 6.8 and 6.9 is that they yield lifting results. For example, from Theorem 6.9 it follows:

Theorem 6.10. *Let E, F be Banach spaces, let X be a Stein manifold, and let $A : X \to L(E, F)$ be a holomorphic operator function which is locally boundedly generalized invertible. Then, for each holomorphic vector function $f : X \to F$ such that $f(z) \in \operatorname{Im} A(z)$ for all $z \in X$, there exists a holomorphic vector function $u : X \to E$ such that $Au = f$ on X.*

Proof. By hypothesis there is a (possibly even not continuous) map $\gamma : X \to F$ with $f = A\gamma$, and, by Theorem 6.9, we can find a holomorphic operator function $B : X \to L(F, E)$ with $ABA = A$. Then $u := Bf$ has the required property: $Au = ABf = ABA\gamma = A\gamma = f$. □

However, the claim of Theorem 6.10 holds under much weaker hypotheses. For example, the condition on the complementedness of Ker $A(z)$ and Im $A(z)$ can be dropped. First steps in this direction were done in [39,29]. The strongest result then was obtained by Janz [25,26]. He only requires that A is uniformly regular, which is much weaker than locally boundedly generalized invertibility.

Remark 6.11. Note that Allan's Theorem 6.4 can be viewed also as a corollary of the lifting Theorem 6.10. Indeed, let, with the notations and hypotheses of Theorem 6.4,

$$A : X \to L(\mathfrak{B}^k, \mathfrak{B}) \quad \mathfrak{B}^k := \underbrace{\mathfrak{B} \oplus \cdots \oplus \mathfrak{B}}_{k \text{ times}}$$

be the holomorphic operator function defined by

$$A(z)(v_1, \ldots, v_n) = a_1(z)v_1 + \cdots + a_k(z)v_k.$$

Then, for each $z \in X$, the operator $A(z)$ is right invertible. Indeed, by hypothesis, we have elements $u_1(z), \ldots, u_k(z) \in \mathfrak{B}$ such that (6.2) is satisfied, which yields that the operator $A(z)^{(-1)} \in L(\mathfrak{B}, \mathfrak{B}^k)$ defined by

$$A(z)^{(-1)}v = \big(u_1(z)v, \ldots, u_k(z)v\big)$$

is a right inverse of $A(z)$. It remains to apply Theorem 6.10 to $E = \mathfrak{B}^k$, $F = \mathfrak{B}$, and $f \equiv 1$.

We summarize: By Remark 1.2, the special case $k = 1$ and $\mathfrak{B} = L(E)$ in Allan's Theorem 6.4 implies Theorem 6.9, which further implies the lifting Theorem 6.10, which in turn implies the general case of Allan's theorem.

Finally note that, by Theorem 6.8, there is also a $\mathcal{C}^{\aleph}$ version of the lifting Theorem 6.10. But also then, the claim of the theorem is known under much weaker hypotheses [44,49,27,1,28,30, 40–42].

Of course, also Remark 6.11 has a real-analytic counterpart, i.e. Gramsch's Theorem 6.7 holds also with an arbitrary real-analytic manifold countable at infinity in place of X.

7. Generalized Drazin inverses

Let $\mathfrak{B}$ be a unital Banach algebra A *generalized Drazin inverse* a^D of $a \in \mathfrak{B}$ is defined as an element $b \in \mathfrak{B}$ with the properties that

$$ab = ba, \qquad b = bab, \quad \text{and} \quad a - aba \text{ is quasinilpotent, i.e. } \sigma(a - aba) = \{0\}.$$

This concept (in the context of Banach algebras) was studied in [32], where it is termed Drazin inverse. By [32, Theorem 4.1], a has a generalized Drazin inverse if and only if either a is invertible or zero is an isolated point of the spectrum of a, and in this case the Drazin inverse is unique. Roughly speaking, a^D is zero on the part of a that corresponds to the spectrum at zero, and a^D is the inverse of a on the part of a where a is invertible. We say that $a \in \mathfrak{B}$ is *generalized Drazin invertible*, in short GDI, if a has a generalized Drazin inverse. If $ab = ba$, $b = bab$, and $a - aba$ is nilpotent (i.e. $(a - aba)^k = 0$ for some positive integer k), then b is called the *Drazin inverse* of a. The *index* of the generalized Drazin inverse b is defined as zero if a is invertible, and otherwise as the minimal positive integer k such that $(a-aba)^k = 0$ (thus, the index is equal to infinity if $a - aba$ is quasinilpotent but not nilpotent).

Note that a^D is not, generally speaking, a generalized inverse of a, because $a = aba$ need not hold; moreover, a GDI a need not be generalized invertible. For example, a compact quasinilpotent operator a on an infinite dimensional Hilbert space is GDI with $a^D = 0$; however, a is generalized invertible if and only of a is of finite rank. The literature on Drazin and generalized Drazin inverses is extensive; for basic theory and applications (in the context of matrices) see [9], and for results on perturbations, continuity and differentiability properties of (generalized) Drazin inverses see [46,33,8,36,37].

In the following, we focus on generalized Drazin inverses. The result of Theorem 7.1 for the Drazin inverses is valid as well, and can be obtained as a particular case of Theorem 7.1.

If $a \in \mathfrak{B}$ is GDI, we let

$$P_0(a) := \frac{1}{2\pi i} \int_{\{|z|=\epsilon\}} (z-a)^{-1} dz, \quad \epsilon > 0 \text{ is sufficiently small,}$$

be the spectral projection corresponding to the zero part of $\sigma(a)$; $P_0(a) = 0$ if a is invertible.

Theorem 7.1. *Let X be a $\mathcal{C}^{\aleph}$-manifold (or complex manifold), and let $a : X \to \mathfrak{B}$ be a $\mathcal{C}^{\aleph}$-function (or holomorphic function) such that $a(x)$ is GDI for every $x \in X$. Then the following statements are equivalent:*

(a) *The generalized Drazin inverse $a^D(x)$ of $a(x)$ is a $\mathcal{C}^{\aleph}$-function (or holomorphic function) of $x \in X$;*
(b) *$a(x)^D$ is a locally bounded function of $x \in X$;*
(c) *$a(x)^D$ is a continuous function of $x \in X$;*
(d) *$P_0(a(x))$ is a continuous function of $x \in X$.*

Proof. Let $\chi : \mathfrak{B} \to L(\mathfrak{B})$ be the standard representation of $\mathfrak{B}$. Then clearly $\chi(a(x))$ is GDI, and each of the properties (b), (c), (d) is equivalent to the corresponding property of $\chi(a(x)^D)$. However, the equivalence of (b), (c), and (d) for $\chi(a(x)^D)$ follows from [35, Theorem 4.1] (see also [37]). Moreover, as proved in [36], in this case $\chi(a(x))^D$ admits the following integral expression for every $x \in X$ sufficiently close to a fixed $x_0 \in X$:

$$\chi(a(x))^D = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-1} (\lambda I - \chi(a(x)))^{-1} d\lambda, \tag{7.1}$$

where Γ is a suitable contour such that the nonzero part of the spectra of all $\chi(a(x)), x$ sufficiently close to x_0, is inside Γ, and zero is outside Γ (the existence of such Γ follows from continuity (or local boundedness) of $\chi(a(x))^D$ at x_0; see [36]). Now the implication (c) $\Longrightarrow$ (a) can be derived easily from (7.1). □

The paper [35] contains many other statements equivalent to continuity of the Drazin inverse. We mention here only one such statement. The *core* c of an GDI element $a \in \mathfrak{B}$ is defined as the unique GDI element such that $a = c + q$, where q is quasinilpotent, $cq = qc = 0$, and the index of the generalized Drazin inverse of c is either 0 or 1. Assuming $\mathfrak{B} = L(F)$ for some Banach space F, it is easy to see that the range and the kernel of the core c are closed complemented subspaces in F. Moreover, under the hypotheses of Theorem 7.1, $a(x)^D$ is continuous on X if and only if the range and the kernel of $c(x)$ are continuous on X families of subspaces of F. Thus:

Corollary 7.2. *Assume $\mathfrak{B} = L(F)$, and assume the hypotheses of Theorem 7.1. Then $a^D(x)$ of $a(x)$ is a $\mathcal{C}^{\aleph}$-function (or holomorphic function) of $x \in X$ if and only if the range and the kernel of $c(x)$ are continuous on X.*

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