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Group inversion in certain finite-dimensional algebras generated by two idempotents

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In memory of Israel Gohberg, whose mathematics and personality have left an everlasting imprint on us

Abstract

Invertibility in Banach algebras generated by two idempotents can be checked with the help of a theorem by Roch, Silbermann, Gohberg, and Krupnik. This theorem cannot be used to study generalized invertibility. The present paper is devoted to group invertibility in two types of finite-dimensional algebras which are generated by two idempotents, algebras generated by two tightly coupled idempotents on the one hand and algebras of dimension at most four on the other. As a side product, the paper gives the classification of all at most four-dimensional algebras which are generated by two idempotents.

Keywords: Idempotent; Two projections; Finite-dimensional algebra; Group inversion

1. Introduction

One of the great achievements of Israel Gohberg, gained in collaboration with Nahum Krupnik, is the Fredholm theory for Banach algebras of singular integral operators with piecewise continuous coefficients [16]. Ronald Douglas [13] was the first to understand that this theory can very elegantly be based on the two projections theorem by Halmos. At that time one did not have two projections theorems for Banach algebras and hence Douglas had to restrict himself to the

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Hilbert space case. The idea to extend two projections theorems from C^*-algebras to Banach algebras in order to tackle singular integral operators in the fashion of Douglas was developed by Roch and Silbermann [21,22]. The theorem of [21] is as follows.

Let $\mathcal{A}$ be a unital Banach algebra and let P and Q be two idempotents in $\mathcal{A}$, that is, elements satisfying $P^2 = P$ and $Q^2 = Q$. Let $\mathcal{B} = \mathrm{alg}(I, P, Q)$ stand for the smallest closed subalgebra of $\mathcal{A}$ which contains P, Q, and the unit I. Given $A \in \mathcal{A}$, we denote by $\sigma(A)$ the spectrum of A in $\mathcal{A}$. Finally, put $T = (P - Q)^2$.

Theorem 1.1 (*Roch and Silbermann*)**.** *Suppose* 0 *and* 1 *are cluster points of* $\sigma(T)$*. Then for each point* $\lambda \in \sigma(T)$ *the map* $F_\lambda : \{I, P, Q\} \to \mathbb{C}^{2\times 2}$ *given by*

$$F_\lambda(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad F_\lambda(P) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad F_\lambda(Q) = \begin{pmatrix} 1-\lambda & w(\lambda) \\ w(\lambda) & \lambda \end{pmatrix}, \tag{1}$$

where $w(\lambda) := \sqrt{\lambda(1-\lambda)}$ *denotes any number the square of which equals* $\lambda(1-\lambda)$*, extends to a continuous homomorphism of* $\mathcal{B}$ *to* $\mathbb{C}^{2\times 2}$*, and an element* $A \in \mathcal{B}$ *is invertible in* $\mathcal{A}$ *if and only if* $\det F_\lambda(A) \neq 0$ *for all* $\lambda \in \sigma(T)$.

The assumption that 0 and 1 are cluster points of $\sigma(T)$ is fortunately always satisfied when applying the theorem to singular integral operators [2,3]. However, when dealing with finite idempotent matrices, this assumption cannot hold because in that case the spectra are finite sets. This defect was remedied by Gohberg and Krupnik [15,17], who realized that one has to add still four one-dimensional representations. The ultimate result is as follows.

Theorem 1.2 (*Roch, Silbermann, Gohberg, Krupnik*)**.** *For each point* $\lambda \in \sigma(T)$ *the map* $F_\lambda : \{I, P, Q\} \to \mathbb{C}^{2\times 2}$ *given by* (1) *extends to a continuous homomorphism of* $\mathcal{B}$ *to* $\mathbb{C}^{2\times 2}$*, and for each point* $\lambda \in \sigma(P + 2Q) \cap \{0, 1, 2, 3\}$ *the map* $G_\lambda : \{I, P, Q\} \to \mathbb{C}$ *given by* $G_\lambda(I) = 1$ *and*

$$\begin{aligned} &G_0(P) = 0, \qquad G_0(Q) = 0, \qquad G_1(P) = 1, \qquad G_1(Q) = 0, \\ &G_2(P) = 0, \qquad G_2(Q) = 1, \qquad G_3(P) = 1, \qquad G_3(Q) = 1 \end{aligned}$$

extends to a continuous homomorphism of $\mathcal{B}$ *to* $\mathbb{C}$*. An element* $A \in \mathcal{B}$ *is invertible in* $\mathcal{A}$ *if and only if* $\det F_\lambda(A) \neq 0$ *for all* λ *belonging to* $\sigma(T) \setminus \{0, 1\}$ *and* $G_\lambda(A) \neq 0$ *for all* λ *in* $\sigma(P + 2Q) \cap \{0, 1, 2, 3\}$.

Full proofs and further extensions of this theorem are also in [2,3,20]. Paper [5] contains detailed notes on the history of two projections theorems.

In recent times, several authors [8,10–12,19,23] turned attention to generalized invertibility of elements in $\mathcal{B} = \mathrm{alg}(I, P, Q)$. A major part of these results concerns either very special elements in $\mathcal{B}$, for example linear combinations $aP + bQ$, or is based on extra assumptions, such as $(PQ)^2 = (QP)^2$ or $(PQ)^2 = 0$. Sufficiently general results for selfadjoint idempotents are in [24] (Moore–Penrose inversion) and [4] (Drazin inversion). The study of generalized invertibility in $\mathcal{B}$ requires some understanding of the structure of the algebra $\mathcal{B}$.

We here consider the case where P and Q are finite idempotent matrices. Then $\mathcal{B}$ is a finite-dimensional algebra. While finite-dimensional linear spaces can completely be classified by their dimension, a complete classification of finite-dimensional associative algebras (= finite-dimensional Banach algebras) is much more involved and seems to be out of reach [9,14]. Dana-Picard and Schaps begin their paper [9] as follows. "The classical problem of classifying n-dimensional algebras suffers from being too easy. Once the ground rules are explained, a competent algebraist with time and patience can sit down and generate multiplication tables for associative algebras, but the activity becomes unilluminating around dimension six and has not been carried much further. Such calculations flourished for a while at the end of the last century [26] but more or less died out in the face of more general structure theorems, particularly the Wedderburn theorems".

One might hope to be able to classify at least the finite-dimensional associative algebras which are generated by two idempotents, but we realized that this eventually also becomes unpleasant. In [6,7], we classified the finite-dimensional associative algebras which are generated by two tightly coupled idempotents. Such pairs of idempotents are close to commuting idempotents; the precise definition will follow below. Here we give the classification for associative algebras which are generated by two idempotents and which are of dimension at most four.

Our insights into the structure of the two types of algebras mentioned in the preceding paragraph allow us to establish criteria for invertibility and group invertibility in these algebras. Recall that a complex matrix A is group invertible if and only if $\operatorname{rank} A = \operatorname{rank} A^2$, which is equivalent to the condition that the Jordan blocks for the eigenvalue zero are at most of dimension 1. This in its turn is equivalent to the existence of a matrix X such that $AX = XA$, $XAX = X$, and $AXA = A$; see, for example, [1]. If it exists, this matrix X is unique. It is called the group inverse of A and is denoted by A^g. Note that if $A = CJC^{-1}$ where J is the Jordan canonical form, then $A^g = CJ^gC^{-1}$ where J^g results from J by inverting the invertible Jordan blocks and keeping the non-invertible ones as they are. A moment's thought reveals that J^g is a polynomial in J whose constant term is zero.

Every matrix algebra $\mathcal{A}$, whether unital or not, is *group inverse closed*, that is, if $A \in \mathcal{A}$ is group invertible, then A^g automatically belongs to $\mathcal{A}$. To see this, it suffices to prove that A^g is a polynomial in A with a vanishing constant term. This follows directly from the formula $A^g = CJ^gC^{-1}$. Another argument is as follows. The group inverse X of $A \in \mathcal{A}$ is a polynomial of A and hence of the form $X = \mu I + B$ with $B \in \mathcal{A}$, where I is the identity matrix. But this implies that

$$X = XAX = (\mu I + B)A(\mu I + B) = \mu^2 A + \mu(AB + BA) + BAB \in \mathcal{A}.$$

Thus, when speaking of group invertibility of matrices $A \in \operatorname{alg}(P, Q)$, it does not matter whether we consider A as an element of $\operatorname{alg}(P, Q)$, of $\operatorname{alg}(I, P, Q)$, or any superalgebra thereof.

By the spectrum $\sigma(A)$ of a matrix $A \in \operatorname{alg}(P, Q)$ we mean its spectrum as an element of $\operatorname{alg}(P, Q)$ if the algebra $\operatorname{alg}(P, Q)$ is unital and the spectrum in $\operatorname{alg}(I, P, Q) = \mathbb{C}I + \operatorname{alg}(P, Q)$ if $\operatorname{alg}(P, Q)$ is not unital. Note that if $\operatorname{alg}(P, Q)$ is unital but the unit is different from the identity matrix I, then the spectrum of a matrix $A \in \operatorname{alg}(P, Q)$ in $\operatorname{alg}(I, P, Q)$ is just the union of the spectrum in $\operatorname{alg}(P, Q)$ and $\{0\}$, that is, $\sigma(A) \cup \{0\}$ is equal to the set of the eigenvalues of A. Indeed, letting E be the unit of a matrix algebra $\mathcal{A} \subset \mathbb{C}^{N\times N}$ and I be the N-by-N identity matrix, it can be checked straightforwardly that if B is the inverse of $A - \lambda E$ in $\mathcal{A}$ and $\lambda \neq 0$, then $B - \lambda^{-1}(I - E)$ is the inverse of $A - \lambda I$ in $\mathbb{C}^{N\times N}$.

We will show that if P and Q are tightly coupled, then $\sigma(T) = \sigma(P - Q)^2$ is a subset of $\{0, 1\}$ (Corollary 2.5). Thus, if P, Q are tightly coupled, then in order to use Theorem 1.2 to decide whether A is invertible, we merely need to know $\sigma(P + 2Q)$. The theorem then contracts to the following: we have $\sigma(A) \subset \{G_0(A), G_1(A), G_2(A), G_3(A)\}$, and $G_\lambda(A) \in \sigma(A)$ if and only if $\lambda \in \sigma(P + 2Q)$. One task of this paper is to identify $\sigma(P + 2Q)$ in the case of tightly coupled idempotents. It turns out that our approach yields not only $\sigma(P + 2Q)$ but even $\sigma(A)$ for every $A \in \operatorname{alg}(P, Q)$, so that in the end we need not have any recourse to Theorem 1.2.

Since F_λ and G_λ are homomorphisms, they preserve group invertibility. Consequently, if A is group invertible, then so are also $F_\lambda(A)$ and $G_\lambda(A)$. However, the scalars $G_\lambda(A)$ are always group invertible. Moreover, if P, Q are tightly coupled, we have only the matrices $F_0(A)$ and $F_1(A)$ at our disposal, and these are diagonal matrices and hence always group invertible as well. Therefore Theorem 1.2 delivers neither necessary nor sufficient conditions for group invertibility. This motivates our study of group invertibility.

The paper is organized as follows. Theorem 2.1 presents the classification of the algebras generated by two tightly coupled idempotents which was obtained in [7]. On the basis of this theorem, taking advantage of the fact that the group invertibility is preserved under algebra isomorphisms, we then study group invertibility in these algebras. The big building blocks of the algebras are certain algebras Z_m. The main result of Section 2 is Theorem 2.2, which provides us with a criterion for group invertibility in Z_m. Two corollaries to this theorem then settle group invertibility in the remaining algebras of tightly coupled idempotents. Section 3 contains the classification of at most four-dimensional algebras generated by two idempotents (Theorem 3.4), a criterion for group invertibility (Theorem 3.5), and the values of the Drazin index (Theorem 3.6) in those algebras.

2. Tightly coupled idempotents

Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$ and let $P, Q \in \mathcal{A}$ be idempotents. We denote by $\mathrm{alg}(P, Q)$ the smallest subalgebra of $\mathcal{A}$ which contains P and Q, that is, the set of all finite linear combinations which can be formed with the elements of the array

$$\begin{array}{llllll} P & PQ & PQP & PQPQ & PQPQP & \cdots \\ Q & QP & QPQ & QPQP & QPQPQ & \cdots . \end{array} \tag{2}$$

The number of factors in a product in (2) will be called its *order*. Equivalently, a product has the order j if and only if it stands in the jth column of array (2). We say that P and Q are tightly coupled if array (2) contains two products whose orders differ by at most 1 and which take the same value. This notion was introduced in [7]. In other words, P and Q are tightly coupled if and only if the two products in one column of (2) or two products standing in neighboring columns of (2) assume the same value. In a sense, the property of being tightly coupled means that the idempotents are close to commuting ones.

It is easily seen that if P and Q are tightly coupled, then the number of different values taken by the products in (2) is finite, that is, $\mathrm{alg}(P, Q)$ is a finite-dimensional algebra. As every finite-dimensional algebra may be realized as an algebra of finite matrices, we may and will henceforth suppose that P and Q are finite idempotent matrices.

In [25] it was shown that if P and Q are Hermitian idempotents and two different products of the list

$$PQ,\ QP,\ PQP,\ QPQ,\ PQPQ,\ QPQP,\ PQPQP,\ QPQPQ, \ldots$$

take the same value, then all products of this list coincide, that is, it follows that $PQ = QP$. It is also known that this may no longer happen if P and Q are not Hermitian. This phenomenon is another source of motivation for the investigations in [7] and in the present paper.

The algebras generated by two tightly coupled idempotents were completely classified in [7]. We define concrete algebras U_1, U_2, D_2, D_2^* and Z_m $(m \ge 0)$ as follows. Let

$$U_1 = \mathrm{alg}((1), (1)), \qquad U_2 = \mathrm{alg}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

$$D_2 = \mathrm{alg}\left(\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\right), \qquad D_2^* = \mathrm{alg}\left(\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\right),$$

and put

$$Z_0 = \mathrm{alg}((0),(0)), \qquad Z_1 = \mathrm{alg}((0),(1)), \qquad Z_2 = \mathrm{alg}\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right).$$

For $m \geq 3$, we define

$$Z_m = \mathrm{alg}\left(\begin{pmatrix}I & C\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ B & I\end{pmatrix}\right)$$

where I is the identity matrix of appropriate order and C, B are square matrices chosen in dependence of m. We write $m = 4n-k$ with $n \geq 2$ and $k \in \{2,3,4,5\}$. If $k = 2$ (resp. $k = 4$), we take B and C as the Jordan block of order $2n-1$ (resp. $2n-2$) with zeros on the main diagonal,

$$C = B = \begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1\\ & & & 0\end{pmatrix}.$$

For $k = 3$, we choose $C, B \in \mathbb{R}^{n\times n}$ by

$$C = \begin{pmatrix} & & 1\\ & \iddots & \\ 1 & & \\ 0 & & \end{pmatrix}, \qquad B = \begin{pmatrix} & & 1 & 0\\ & \iddots & \iddots & \\ 1 & 0 & & \\ 0 & & & \end{pmatrix}.$$

If $k = 5$, we let $C, B \in \mathbb{R}^{(n-1)\times(n-1)}$ be the matrices

$$C = \begin{pmatrix} & & 1 & 0\\ & \iddots & \iddots & \\ 1 & 0 & & \\ 0 & & & \end{pmatrix}, \qquad B = \begin{pmatrix} & & 1\\ & \iddots & \\ & 1 & \\ 1 & & \end{pmatrix},$$

with $C = (0)$ and $B = (1)$ if $n = 2$. Given two algebras $\mathcal{B}_1$ and $\mathcal{B}_2$, we denote by $\mathcal{B}_1 \oplus \mathcal{B}_2$ the algebra of all matrices $\mathrm{diag}(A, B)$ with $A \in \mathcal{B}_1$ and $B \in \mathcal{B}_2$.

Theorem 2.1 ([7]). *Every algebra generated by two tightly coupled idempotents is isomorphic to exactly one algebra of the array*

$$\begin{array}{cccccccc}
Z_0 & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & Z_7 \quad \cdots\\
 & & D_2 & Z_1\oplus D_2 & Z_2\oplus D_2 & Z_3\oplus D_2 & Z_4\oplus D_2 & Z_5\oplus D_2 \quad \cdots\\
 & & D_2^* & Z_1\oplus D_2^* & Z_2\oplus D_2^* & Z_3\oplus D_2^* & Z_4\oplus D_2^* & Z_5\oplus D_2^* \quad \cdots\\
 & & & Z_2\oplus U_1 & Z_3\oplus U_1 & Z_4\oplus U_1 & Z_5\oplus U_1 & Z_6\oplus U_1 \quad \cdots.
\end{array} \tag{3}$$

The dimension of each algebra in the mth column of this array ($m \geq 0$) is m.

It is well known and will be shown again below that the algebras U_1 and U_2 are isomorphic to Z_1 and Z_2, respectively, $U_1 \cong Z_1$ and $U_2 \cong Z_2$. However, sometimes it is more natural to work with U_1, U_2, while on other occasions Z_1, Z_2 is the better choice. This minor freedom enters atlas (3), where we preferred Z_1, Z_2 in the first row and U_1 in the last.

The algebras $U_1, U_2, D_2, D_2^*, Z_m$ are of the form alg(first matrix, second matrix), and we denote the first and the second matrix by $\mathbf{P}$ and $\mathbf{Q}$, respectively. The algebra $Z_0 = \{0\}$ is the zero algebra and will be excluded in what follows. Thus, let $m \geq 1$. In [7], we showed that if $\mathcal{B}_1 \oplus \mathcal{B}_2$ is from (3) and $\mathcal{B}_1 = \mathrm{alg}(\mathbf{P}_1, \mathbf{Q}_1)$, $\mathcal{B}_2 = \mathrm{alg}(\mathbf{P}_2, \mathbf{Q}_2)$, then

$$\mathcal{B}_1 \oplus \mathcal{B}_2 = \mathrm{alg}\left(\begin{pmatrix} \mathbf{P}_1 & 0 \\ 0 & \mathbf{P}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{Q}_1 & 0 \\ 0 & \mathbf{Q}_2 \end{pmatrix}\right).$$

It follows that each algebra in (3) is of the form $\mathrm{alg}(\mathbf{P}, \mathbf{Q})$ with well-defined concrete idempotent matrices $\mathbf{P}$ and $\mathbf{Q}$.

There is a subtlety we want to point out. Let P, Q be any tightly coupled idempotent matrices and let $m \geq 1$ be the dimension of $\mathrm{alg}(P, Q)$. If $m \geq 3$, then $\mathrm{alg}(P, Q)$ is isomorphic to exactly one of the four algebras in the mth column of (3), and this algebra is $\mathrm{alg}(\mathbf{P}, \mathbf{Q})$ with specific matrices $\mathbf{P}, \mathbf{Q}$. The result of [7] shows that there is an isomorphism $\Phi : \mathrm{alg}(P, Q) \to \mathrm{alg}(\mathbf{P}, \mathbf{Q})$ such that either $\Phi(P) = \mathbf{P}$, $\Phi(Q) = \mathbf{Q}$ or $\Phi(P) = \mathbf{Q}$, $\Phi(Q) = \mathbf{P}$. In other words, everything depends only on which of the two idempotents is denoted by P and which by Q. Things are a little more involved for $m = 1, 2$. Let first $m = 1$. Then there are essentially two possibilities for array (2):

$$\begin{matrix} P & P & P & \cdots \\ P & P & P & \cdots, \end{matrix} \qquad \begin{matrix} 0 & 0 & 0 & \cdots \\ Q & 0 & 0 & \cdots, \end{matrix}$$

which correspond to U_1 and Z_1, respectively. An isomorphism $\Phi : \mathrm{alg}(P, Q) \to Z_1$ is given by $P = Q$ and $\Phi(P) = (1)$ in the first case and by $P = 0$ and $\Phi(Q) = (1)$ in the second. In particular, $U_1 \cong Z_1$. In case $m = 2$, we have essentially the following four possibilities for array (2):

$$\begin{matrix} P & P & P & \cdots \\ Q & P & P & \cdots, \end{matrix} \qquad \begin{matrix} P & 0 & 0 & \cdots \\ Q & 0 & 0 & \cdots, \end{matrix} \qquad \begin{matrix} P & P & P & \cdots \\ Q & Q & Q & \cdots, \end{matrix} \qquad \begin{matrix} P & Q & P & \cdots \\ Q & P & Q & \cdots. \end{matrix}$$

In the last three cases, $\Phi(P) = \mathbf{P}$, $\Phi(Q) = \mathbf{Q}$ extends to an isomorphism of $\mathrm{alg}(P, Q)$ onto Z_2, D_2, D_2^*, respectively. In the first case, this is an isomorphism of $\mathrm{alg}(P, Q)$ onto U_2. An isomorphism $\Phi : \mathrm{alg}(P, Q) \to Z_2$ is given by $\Phi(\alpha P + \beta Q) = (\alpha + \beta)\mathbf{P} + \beta\mathbf{Q}$ in the first case. In this way we also see that $U_2 \cong Z_2$.

We also showed in [7] that the first m products of the list

$$\mathbf{Q}, \mathbf{P}, \mathbf{QP}, \mathbf{PQ}, \mathbf{QPQ}, \mathbf{PQP}, \ldots$$

form a basis in $\mathrm{alg}(\mathbf{P}, \mathbf{Q})$ for all algebras (3). For $A \in \mathrm{alg}(\mathbf{P}, \mathbf{Q})$, let

$$A = a_1\mathbf{P} + b_1\mathbf{Q} + a_2\mathbf{PQ} + b_2\mathbf{QP} + a_3\mathbf{PQP} + b_3\mathbf{QPQ} + \cdots \tag{4}$$

be the representation in this basis. We introduce polynomials φ_{ij} by

$$\begin{aligned}
\varphi_{00}(t) &= a_1 + (a_2 + a_3)t + (a_4 + a_5)t^2 + \cdots, \\
\varphi_{01}(t) &= (a_1 + a_2) + (a_3 + a_4)t + \cdots, \\
\varphi_{11}(t) &= b_1 + (b_2 + b_3)t + (b_4 + b_5)t^2 + \cdots, \\
\varphi_{10}(t) &= (b_1 + b_2) + (b_3 + b_4)t + \cdots
\end{aligned}$$

and we also put

$$\ell(m, a_1) = \begin{cases} \lceil m/4 \rceil - 1 & \text{if } m = 1 \mod 4 \text{ and } a_1 = 0, \\ \lceil m/4 \rceil & \text{otherwise,} \end{cases} \tag{5}$$

where $\lceil m/4 \rceil$ is the smallest integer greater than or equal to $m/4$. We remark that the concrete choice of the algebras U_2, D_2, D_2^*, Z_m $(m \geq 1)$ is not completely symmetric, but depends on which idempotent is $\mathbf{P}$ and which is $\mathbf{Q}$. For example, Z_1 could equally well be replaced by $Z_1 = \text{alg}((1), (0))$. Our concrete choice of the algebras implies that formula (5) involves a_1 and not b_1.

In the following proof, we make use of the Schur complement. Recall that if X, Y, Z, W are square matrices of the same size and W invertible, then

$$A := \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} I & YW^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} X - YW^{-1}Z & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} I & 0 \\ W^{-1}Z & I \end{pmatrix}. \tag{6}$$

The matrix $X - YW^{-1}Z$ is called the Schur complement of W in A. From (6) we see that if W is invertible, then the invertibility of A is equivalent to the invertibility of the Schur complement of W. Moreover, factorization (6) implies that if we are given two block matrices A_1 and A_2 of the same shape with invertible 2,2 blocks W_1 and W_2, then $\text{rank}\, A_1 = \text{rank}\, A_2$ if and only if the Schur complements of W_1 and W_2 have equal rank.

For $A \in Z_1$, representation (4) is $A = b_1\mathbf{Q}$ and a_1 does not occur. We make the convention to put $a_1 := b_1$ in this case. We call A properly group invertible if A is group invertible but not invertible. Recall that group invertibility does not depend on the surrounding algebra and that invertibility and thus the spectrum are well-defined and independent of the algebra if the identity matrix is contained in it.

Theorem 2.2. *Let $m \geq 1$. The algebra Z_m contains the identity matrix I. If $A \in Z_m$ is of the form* (4)*, then $\sigma(A) = \{a_1, b_1\}$, and A is properly group invertible if and only if* (i) $A = 0$ *or* (ii) *exactly one of the coefficients a_1, b_1 is nonzero and the multiplicity of zero as a root of the polynomial*

$$\psi(t) = \varphi_{00}(t)\varphi_{11}(t) - t\varphi_{01}(t)\varphi_{10}(t) \tag{7}$$

is at least $\ell(m, a_1)$.

Proof. This is trivial for $m = 1, 2$. So let $m \geq 3$. By construction, Z_m is a subalgebra of $\mathbb{C}^{N\times N}$ for some N. It can be checked straightforwardly that the matrix CB is nilpotent of degree $\lceil m/4 \rceil - 1$ if $m = 1 \mod 4$ and that the matrices CB and BC are both nilpotent of the degree $\lceil m/4 \rceil$ in all other cases. A direct computation reveals that

$$A = \begin{pmatrix} \varphi_{00}(CB) & \varphi_{01}(CB)C \\ B\varphi_{10}(CB) & \varphi_{11}(BC) \end{pmatrix}.$$

If $b_1 = 0$, then the matrices $\text{diag}(I, B)$ and $\text{diag}(I, C)$ can be factored out of A to the left and to the right, respectively. Since at least one of the matrices B, C is singular, so also is A.

Consequently, condition $b_1 \neq 0$ is necessary for A to be invertible. Suppose it holds. Then the lower right block $\varphi_{11}(BC)$ of A is invertible, so that A is invertible only simultaneously with the Schur complement S of this block. The latter equals

$$\begin{aligned} S &= \varphi_{00}(CB) - \varphi_{01}(CB)C[\varphi_{11}(BC)]^{-1}B\varphi_{10}(CB) \\ &= \varphi_{00}(CB) - \varphi_{01}(CB)CB[\varphi_{11}(CB)]^{-1}\varphi_{10}(CB) =: \xi(CB), \end{aligned}$$

where the function $\xi = \psi/\varphi_{11}$ assumes the value a_1 at zero. Thus, S is invertible if and only if $a_1 \neq 0$ (in addition to already imposed condition $b_1 \neq 0$). We so arrive at the conclusion that $\sigma(A) = \{a_1, b_1\}$ in the algebra $\mathbb{C}^{N\times N}$.

In particular, the spectrum of $E := \mathbf{P} + \mathbf{Q}$ is $\sigma(E) = \{1\}$. Therefore E is invertible, that is, there is an $X \in \mathbb{C}^{N\times N}$ such that $EX = I$. As X is a polynomial of E, we may write $X = \mu I + B$ with $\mu \in \mathbb{C}$ and $B \in Z_m$, and this implies that $I = EX = \mu E + EB$ belongs to Z_m.

Suppose now that $b_1 \neq 0$ while $a_1 = 0$. We have

$$A^2 = \begin{pmatrix} (\varphi_{00}^2 + t\varphi_{01}\varphi_{10})(CB) & (\varphi_{00}\varphi_{01} + \varphi_{11}\varphi_{01})(CB)C \\ B(\varphi_{00}\varphi_{10} + \varphi_{11}\varphi_{10})(CB) & (\varphi_{11}^2 + t\varphi_{01}\varphi_{10})(BC) \end{pmatrix},$$

and since $\varphi_{11}^2(t) + t\varphi_{01}(t)\varphi_{10}(t)$ takes the value $b_1^2 \neq 0$ at $t = 0$, the lower right block is invertible. Its Schur complement is $T = \eta(CB)$, where $\eta = \psi^2/(\varphi_{11}^2 + t\varphi_{01}\varphi_{10})$.

For A to be group invertible, it is necessary and sufficient that the ranks of A and A^2 coincide, which in turn happens if and only if the Schur complements S and T have the same rank. Since $\operatorname{rank} S = \operatorname{rank} \xi(CB) = \operatorname{rank} \psi(CB)$ and $\operatorname{rank} T = \operatorname{rank} \eta(CB) = \operatorname{rank} \psi(CB)^2$, the matrix A is group invertible if and only if $\operatorname{rank} \psi(CB) = \operatorname{rank} \psi(CB)^2$, that is, if and only if $\psi(CB)$ is group invertible. As $\psi(0) = a_1 b_1 = 0$, the matrix $\psi(CB)$ is nilpotent. Therefore it is group invertible if and only if it is the zero matrix. But this is exactly the root multiplicity condition of the theorem.

The case $a_1 \neq 0, b_1 = 0$ can be treated along the same lines. The only difference is that this time we have to work with BC instead of CB, and since BC is nilpotent of degree $\lceil m/4 \rceil$ independently of whether $m = 1 \mod 4$ or not, we now do not encounter the defect -1 in (5).

We are eventually left with the situation $a_1 = b_1 = 0$. For $\lambda \neq 0$, both diagonal blocks of the matrix

$$A - \lambda I = \begin{pmatrix} (\varphi_{00} - \lambda)(CB) & \varphi_{01}(CB)C \\ B\varphi_{10}(CB) & (\varphi_{11} - \lambda)(BC) \end{pmatrix}$$

are invertible. Moreover, the Schur complement of say the lower right one,

$$\left(\varphi_{00} - \lambda - \frac{t\varphi_{01}\varphi_{10}}{\varphi_{11} - \lambda}\right)(CB),$$

is also invertible. Thus, the only eigenvalue of A is zero, and A is therefore nilpotent. Since the only group invertible nilpotent matrix is the zero matrix, this completes the proof. □

In [7, Lemma 4.2], we showed that

$$\mathbf{Q} + \mathbf{P} - \mathbf{QP} - \mathbf{PQ} + \mathbf{QPQ} + \mathbf{PQP} - \cdots \quad (m \text{ terms}) \tag{8}$$

is the unit in Z_m. Theorem 2.2 implies that (8) is the identity matrix in $\mathbb{C}^{N\times N}$.

We now turn to the remaining algebras in list (3). For A of the form (4), we put

$$a := \sum a_i \quad (= \varphi_{00}(1) = \varphi_{01}(1)), \qquad b := \sum b_i \quad (= \varphi_{11}(1) = \varphi_{10}(1)).$$

Corollary 2.3. *Let $m \geq 3$. The algebra $Z_{m-1} \oplus U_1$ contains the identity matrix I. If A in $Z_{m-1} \oplus U_1$ is given by (4), then $\sigma(A) = \{a_1, b_1, a+b\}$, and A is properly group invertible if and only if* (i) *all coefficients in (4) except for possibly the last one (that is, $b_{(m+1)/2}$ if m is odd and $a_{m/2}$ if m is even) are zero or* (ii) *$a_1 b_1 \neq 0$ and $a + b = 0$ or* (iii) *exactly one of the coefficients a_1, b_1 is nonzero and the root multiplicity condition of Theorem 2.2 is satisfied with $\ell(m-1, a_1)$.*

Proof. We have $A = \operatorname{diag}(A_0, a+b)$ with

$$A_0 = a_1\mathbf{P} + b_1\mathbf{Q} + a_2\mathbf{PQ} + b_2\mathbf{QP} + \cdots \tag{9}$$

and $\operatorname{alg}(\mathbf{P},\mathbf{Q}) = Z_{m-1}$, so that the right hand side of (9) contains $m-1$ terms. Since $\sigma(A_0) = \{a_1, b_1\}$ due to Theorem 2.2, it follows that $\sigma(A) = \sigma(A_0) \cup \{a+b\} = \{a_1, b_1, a+b\}$. The matrix A is properly group invertible if and only if (a) A_0 is properly group invertible or (b) A_0 is invertible and $a+b=0$. Condition (b) is equivalent to (ii). From Theorem 2.2 we infer that (a) happens if and only if (a1) $A_0 = 0$ or (a2) exactly one of the coefficients a_1, b_1 is nonzero and the root multiplicity condition is satisfied with $\ell(m-1, a_1)$. Clearly, (a2) is the same as (iii). On the other hand, all the products appearing in (9) are linearly independent, and thus (a1) is equivalent to condition (i) of the corollary. □

The algebra D_2 and hence also the algebras $Z_{m-2} \oplus D_2$ and $Z_{m-2} \oplus D_2^*$ are not unital. Hence, when speaking about the spectrum of matrices in $Z_{m-2} \oplus D_2$ and $Z_{m-2} \oplus D_2^*$, we mean their spectrum in $\operatorname{alg}(I, \mathbf{P}, \mathbf{Q}) = \mathbb{C}I + \operatorname{alg}(\mathbf{P},\mathbf{Q})$. Such matrices are in particular never invertible, which implies that they are properly group invertible if and only if they are group invertible. We use the abbreviation

$$M(\alpha,\beta) = \begin{pmatrix} \alpha & \alpha \\ \beta & \beta \end{pmatrix}.$$

The eigenvalues of $M(\alpha,\beta)$ are 0 and $\alpha+\beta$. This matrix is therefore never invertible. It is easily seen that this matrix is group invertible if and only if either $\alpha=\beta=0$ or $\alpha+\beta \neq 0$. Note also that a matrix A is in D_2^* if and only if its transpose $A^\top$ belongs to D_2. Furthermore, A and $A^\top$ are group invertible only simultaneously.

Corollary 2.4. *Let $A \in Z_{m-2} \oplus D_2$ or $A \in Z_{m-2} \oplus D_2^*$ $(m \geq 2)$ be given by* (4).

(a) *If $m=2$, then $\sigma(A) = \{0, a+b\}$, and A is properly group invertible if and only if $M(a,b)$ is group invertible.*
(b) *If $m=3$, then $\sigma(A) = \{0, b_1, a+b\}$, and A is properly group invertible if and only if $M(a,b)$ is group invertible.*
(c) *If $m \geq 4$, then $\sigma(A) = \{0, a_1, b_1, a+b\}$, and A is properly group invertible if and only if $M(a,b)$ is group invertible and either* (i) *all coefficients in* (4) *except for possibly the last two (that is, $a_{(m-1)/2}$ and $b_{(m+1)/2}$ if m is odd and $a_{m/2}$ and $b_{m/2}$ if m is even) are zero or* (ii) *$a_1b_1 \neq 0$ or* (iii) *exactly one of the coefficients a_1, b_1 is nonzero and the root multiplicity condition of Theorem 2.2 is satisfied with $\ell(m-2, a_1)$.*

Proof. It suffices to consider the cases where $A \in Z_{m-2} \oplus D_m$, because the results for $Z_{m-2} \oplus D_2^*$ follow from passage to transposed matrices.

If $A \in D_2$, then $A = M(a_1, b_1) = M(a,b)$, which gives (a). For $A \in Z_1 \oplus D_2$ we have $A = \operatorname{diag}(b_1, M(a,b))$, which implies (b). So let $m \geq 4$. Then $A = \operatorname{diag}(A_0, M(a,b))$ where A_0 is given by (9), but this time with $\operatorname{alg}(\mathbf{P},\mathbf{Q}) = Z_{m-2}$ and thus only $m-2$ summands on the right hand side. Theorem 2.2 therefore shows that $\sigma(A) = \{0, a_1, b_1, a+b\}$. The matrix A is properly group invertible if and only if (A) A_0 is invertible and $M(a,b)$ is group invertible or (B) A_0 is properly group invertible and $M(a,b)$ is group invertible. By Theorem 2.2, condition (A) is equivalent to (ii). Also due to Theorem 2.2, condition (B) holds if and only if $M(a,b)$ is group invertible and (B1) $A_0 = 0$ or (B2) exactly one of the coefficients a_1 and b_1 is nonzero and the root multiplicity condition is satisfied with $\ell(m-2, a_1)$. Condition (B2) is the same as (iii), and it is readily seen that (B1) is equivalent to condition (i). □

In Theorem 1.2 we encounter the spectra of $T = (\mathbf{P} - \mathbf{Q})^2$ and $\mathbf{P} + 2\mathbf{Q}$. The following result identifies these spectra in the case where $\mathbf{P}$ and $\mathbf{Q}$ are tightly coupled.

Corollary 2.5. *We have*

$$\begin{aligned}
&\sigma(T) = \{1\}, && \sigma(\mathbf{P}+2\mathbf{Q}) = \{2\} \quad \text{in } Z_1,\\
&\sigma(T) = \{1\}, && \sigma(\mathbf{P}+2\mathbf{Q}) = \{1,2\} \quad \text{in } Z_m \ (m \ge 2),\\
&\sigma(T) = \{0,1\}, && \sigma(\mathbf{P}+2\mathbf{Q}) = \{1,2,3\} \quad \text{in } Z_{m-1} \oplus U_1 \ (m \ge 3),\\
&\sigma(T) = \{0\}, && \sigma(\mathbf{P}+2\mathbf{Q}) = \{0,3\} \quad \text{in } D_2,\\
&\sigma(T) = \{0,1\}, && \sigma(\mathbf{P}+2\mathbf{Q}) = \{0,2,3\} \quad \text{in } Z_1 \oplus D_2,\\
&\sigma(T) = \{0,1\}, && \sigma(\mathbf{P}+2\mathbf{Q}) = \{0,1,2,3\} \quad \text{in } Z_{m-2} \oplus D_2 \ (m \ge 4).
\end{aligned}$$

Proof. This can be derived from Theorem 2.2 and Corollaries 2.3 and 2.4 by straightforward inspection. □

Immediately from the construction of the algebras we also see that

$$\begin{aligned}
&\sigma(T) = \{0\}, && \sigma(\mathbf{P}+2\mathbf{Q}) = \{3\} \quad \text{in } U_1,\\
&\sigma(T) = \{0,1\}, && \sigma(\mathbf{P}+2\mathbf{Q}) = \{2,3\} \quad \text{in } U_2.
\end{aligned}$$

3. Algebras of dimension at most four

In this section, we classify all at most 4-dimensional algebras which are generated by two idempotents and establish criteria for invertibility and group invertibility in these algebras.

Theorem 2.1 provides us with all algebras of dimension at most 4 which are generated by two tightly coupled idempotents. Returning to the notation of [7], we now write

$$D_m = Z_{m-1} \oplus D_2, \qquad D_m^* = Z_{m-1} \oplus D_2^*, \qquad U_m = Z_{m-2} \oplus U_1. \tag{10}$$

With this notation, the algebras of the dimensions 1, 2, 3, 4 in the list (3) are

$$\begin{array}{cccc}
Z_1 & Z_2 & Z_3 & Z_4\\
 & D_2 & D_3 & D_4\\
 & D_2^* & D_3^* & D_4^*\\
 & & U_3 & U_4.
\end{array} \tag{11}$$

Two more algebras W_3 and W_4 were constructed in [7]. These are

$$W_3 = \operatorname{alg}(\mathbf{P}, \mathbf{Q}) = \operatorname{alg}\left(\left(\begin{array}{cc|cc} 1 & 0 & & \\ 1 & 0 & & \\ \hline & & 1 & 1 \\ & & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 1 & & \\ 0 & 1 & & \\ \hline & & 0 & 0 \\ & & 1 & 1 \end{array}\right)\right) \tag{12}$$

and

$$W_4 = \operatorname{alg}(\mathbf{P}, \mathbf{Q}) = \operatorname{alg}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right). \tag{13}$$

Note that W_3 and W_4 are the algebras of all matrices of the form

$$\left(\begin{array}{cc|cc} x & y & & \\ x & y & & \\ \hline & & z & z \\ & & x+y-z & x+y-z \end{array}\right) \quad \text{and} \quad \begin{pmatrix} 0 & x & y \\ 0 & z & w \\ 0 & 0 & 0 \end{pmatrix}, \tag{14}$$

respectively. Given an algebra $\mathcal{A}$, we denote by $\mathcal{N}(\mathcal{A})$ the set of nilpotent elements in $\mathcal{A}$.

Theorem 3.1 ([7]). *Up to isomorphism, there exist exactly four algebras W_3, W_4, $W_3 \oplus Z_1$, $W_4 \oplus Z_1$ which are generated by two idempotents P, Q such that P, Q, PQ, QP are pairwise different and $PQP = P$. The sets of nilpotents in these algebras are linear subspaces. None of these algebras is unital and none of these algebras occurs in the list (3).*

(a) *If $QPQ = Q$ and $P + Q = PQ + QP$, then* $\text{alg}(P, Q) \cong W_3$, $\dim W_3 = 3$, *and we have* $\dim \mathcal{N}(W_3) = 2$.
(b) *In case $QPQ = Q$ and $P + Q \neq PQ + QP$, we have* $\text{alg}(P, Q) \cong W_4$, $\dim W_4 = 4$, *and* $\dim \mathcal{N}(W_4) = 3$.
(c) *If $QPQ \neq Q$ and $P + QPQ = PQ + QP$, then* $\text{alg}(P, Q) \cong W_3 \oplus Z_1$ *and we have* $\dim(W_3 \oplus Z_1) = 4$, *and* $\dim \mathcal{N}(W_3 \oplus Z_1) = 2$.
(d) *In the case where $QPQ \neq Q$ and $P + QPQ \neq PQ + QP$, we have* $\text{alg}(P, Q) \cong W_4 \oplus Z_1$ *with* $\dim(W_4 \oplus Z_1) = 5$, *and* $\dim \mathcal{N}(W_4 \oplus Z_1) = 3$.

The isomorphism may be chosen so that, after labeling P and Q appropriately, $\Phi(P) = \mathbf{P}$ and $\Phi(Q) = \mathbf{Q}$, where $\mathbf{P}, \mathbf{Q}$ are the concrete matrices in (12), (13), and in Z_1.

Here is another 4-dimensional algebra.

Proposition 3.2. *Up to isomorphism, there exists exactly one 4-dimensional algebra V_4 which is generated by two idempotents P, Q for which P, Q, PQ, QP are linearly independent and which satisfy*

$$PQP = PQ + QP - Q \quad \textit{and} \quad QPQ = PQ + QP - P. \tag{15}$$

The algebra is not unital, its set of nilpotent elements is a linear subspace of dimension 2, and V_4 is not isomorphic to Z_4, D_4, D_4^*, U_4, $W_3 \oplus Z_1$, W_4.

Proof. The linear independence assumption and the two defining relations (15) determine the multiplication table completely. Letting $S = PQ + QP$, it is

	P	Q	PQ	QP
P	P	PQ	PQ	$S - Q$
Q	QP	Q	$S - P$	QP
PQ	$S - Q$	PQ	$S + PQ - P - Q$	$S - Q$
QP	QP	$S - P$	$S - P$	$S + QP - P - Q.$

Thus, any two such algebras are isomorphic. To show that such an algebra exists, put

$$V_4 = \text{alg}(\mathbf{P}, \mathbf{Q}) = \text{alg}\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right). \tag{16}$$

It can be verified straightforwardly that $\mathbf{P}$ and $\mathbf{Q}$ are idempotents satisfying (15) and that

$$\alpha\mathbf{P}+\beta\mathbf{Q}+\gamma\mathbf{PQ}+\delta\mathbf{QP}=\begin{pmatrix}\alpha+\beta+\gamma+\delta & \gamma & \alpha+\delta\\ 0 & \alpha+\beta+\gamma+\delta & 0\\ 0 & \beta+\delta & 0\end{pmatrix}.$$

This reveals that $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{PQ}$, $\mathbf{QP}$ are linearly independent and that V_4 is the algebra of all matrices of the form

$$\begin{pmatrix}x & z & w\\ 0 & x & 0\\ 0 & y & 0\end{pmatrix}. \tag{17}$$

It is easily seen that this algebra is not unital and that a matrix of this form is nilpotent if and only if $x=0$. Thus, the nilpotents form a linear subspace of dimension 3. Since V_4 is not unital, it is not isomorphic to Z_4 or U_4. As the nilpotents in $D_4 = D_2 \oplus Z_2$ and $D_4^* = D_2^* \oplus Z_2$ have the dimension 1, we conclude that V_4 is not isomorphic to D_4 or D_4^*. Finally, all nilpotents A in W_4 or $W_3 \oplus Z_1$ satisfy $A^2=0$ while

$$A=\begin{pmatrix}0 & 1 & 1\\ 0 & 0 & 0\\ 0 & -1 & 0\end{pmatrix},$$

is a matrix in V_4 such that $A^3=0$ but $A^2\neq 0$. This proves that V_4 is not isomorphic to W_4 or $W_3 \oplus Z_1$. $\square$

The previous proof shows that if $\mathrm{alg}(P,Q)$ is isomorphic to V_4, then there is an isomorphism Φ such that $\Phi(P)=\mathbf{P}$ and $\Phi(Q)=\mathbf{Q}$ with $\mathbf{P}$, $\mathbf{Q}$ as in (16).

Lemma 3.3. *Let P and Q be two idempotents.*

(a) *If P, Q, PQ are linearly dependent, then P and Q are tightly coupled.*
(b) *If P, Q, PQ, QP are linearly dependent then P and Q are either tightly coupled or there is a scalar $\mu\neq 0$ such that $PQP=\mu P$ and $QPQ=\mu Q$.*
(c) *If P, Q, PQ, QP, PQP are linearly dependent then P and Q are either tightly coupled or $PQP=PQ+QP-Q$ or there is a scalar $\mu\neq 0$ such that $PQP=\mu P$.*

Proof. (a) Assume P and Q are linearly dependent, say $Q=\alpha P$. Then $\alpha^2=\alpha$ and hence $Q=0$ or $Q=P$, which implies that P and Q are tightly coupled. So suppose P and Q are linearly independent and $PQ=\alpha P+\beta Q$. Multiplying this equality from the right by Q we get $PQ=\alpha PQ+\beta Q$, and subtracting the two equalities for PQ we obtain $0=\alpha(P-PQ)$. If $P=PQ$, then the idempotents are tightly coupled. In case $P\neq PQ$, it follows that $\alpha=0$ and hence $PQ=\beta Q$. Multiplication by Q from the right gives $PQ=\beta PQ$. Consequently, $PQ=0$ or $\beta=1$. In either case P and Q are tightly coupled.

(b) By virtue of (a) we may assume that P, Q, PQ are linearly independent and $QP=\alpha P+\beta Q+\gamma PQ$. Multiplication by P from the left and the right yields $PQP=\alpha P+(\beta+\gamma)PQP$. If $\beta+\gamma\neq 1$, this gives $PQP=\mu P$ with $\mu=\alpha/(1-\beta-\gamma)$. (For $\mu=0$, this means that P and Q are tightly coupled.) Thus, let $\beta+\gamma=1$. Then $0=\alpha P$ and hence $\alpha=0$, which shows that $QP=\beta Q+\gamma PQ$. Multiplying this by P from the left we get $PQP=(\beta+\gamma)PQ=PQ$, which tells us that P and Q are tightly coupled. In summary, either $PQP=\mu P$ with $\mu\neq 0$ or P, Q are tightly coupled.

By symmetry it follows that either $QPQ = \lambda Q$ with $\lambda \neq 0$ or that P, Q are tightly coupled. But if $PQP = \mu P$ and $QPQ = \lambda Q$, then $PQPQ = \mu PQ$ and $PQPQ = \lambda PQ$, whence $(\mu - \lambda)PQ = 0$. If $PQ = 0$, then P and Q are tightly coupled, and otherwise we see that $\mu = \lambda$.

(c) Due to (b) it suffices to consider the case where P, Q, PQ, QP are linearly independent. We then have $PQP = \alpha P + \beta Q + \gamma PQ + \delta QP$. Multiplying this from the left and right by P we get $(1 - \beta - \gamma - \delta)PQP = \alpha P$.

Let first $\beta + \gamma + \delta = 1$. Then $\alpha = 0$ and hence $PQP = \beta Q + \gamma PQ + \delta QP$. Multiplying this from the left by P we obtain

$$PQP = \beta PQ + \gamma PQ + \delta PQP = \beta PQ + \gamma PQ + \delta(\beta Q + \gamma PQ + \delta QP).$$

Subtraction of the two equalities for PQP gives

$$0 = \beta(\delta - 1)Q + (\beta + \gamma\delta)PQ + \delta(\delta - 1)QP.$$

If $\delta \neq 1$, it follows that $\beta = \delta = 0$ and hence $\gamma = 1$ and $PQP = PQ$. This says that P and Q are tightly coupled. So let $\delta = 1$ and $PQP = \beta Q + \gamma PQ + QP$ with $\beta + \gamma = 0$. We multiply the equality for PQP from the right by P to get

$$PQP = \beta QP + \gamma PQP + QP = \beta QP + \gamma(\beta Q + \gamma PQ + QP) + QP.$$

Subtracting the two inequalities we arrive at

$$0 = \beta(\gamma - 1)Q + \gamma(\gamma - 1)PQ + (\beta + \gamma)QP.$$

If $\gamma \neq 1$, then $\beta = \gamma = 0$ and hence $PQP = QP$, that is, P and Q are tightly coupled. In the case where $\gamma = 1$, we get $\beta = -1$ and thus $PQP = PQ + QP - Q$.

We are left with the case where $\beta + \gamma + \delta \neq 1$ and thus $PQP = \mu P$ with μ given by $\mu = \alpha/(1 - \beta - \gamma - \delta)$. If $\mu = 0$, then P, Q are tightly coupled. Otherwise $\mu \neq 0$. $\square$

Herewith the classification of at most 4-dimensional algebras which are generated by two idempotents.

Theorem 3.4. *Let P and Q be two idempotents and suppose* $\dim \operatorname{alg}(P, Q) = m \leq 4$. *Then* $\operatorname{alg}(P, Q)$ *is isomorphic to exactly one of the algebras*

$$\begin{aligned}
&Z_1 \quad \textit{for } m = 1,\\
&Z_2,\ D_2,\ D_2^* \quad \textit{for } m = 2,\\
&Z_3,\ D_3,\ D_3^*,\ U_3,\ W_3 \quad \textit{for } m = 3,\\
&Z_4,\ D_4,\ D_4^*,\ U_4,\ W_4,\ W_3 \oplus Z_1,\ V_4,\ \mathbb{C}^{2\times 2} \quad \textit{for } m = 4.
\end{aligned}$$

Proof. This is clear for $m = 1$. If $m = 2$, then P, Q, PQ are linearly dependent and Lemma 3.3(a) implies that P and Q are tightly coupled. From Theorem 2.1 we therefore deduce that $\operatorname{alg}(P, Q)$ is isomorphic to exactly one of the algebras Z_2, D_2, D_2^*. So let $m = 3$ or $m = 4$. If P, Q are tightly coupled, Theorem 2.1 shows that the algebra is isomorphic to exactly one of the algebras $Z_3, D_3, D_3^*, Z_4, D_4, D_4^*, U_4$. Thus, let us assume that P and Q are not tightly coupled.

Assume first that P, Q, PQ, QP are linearly dependent. Then, by Lemma 3.3(b), $PQP = \mu P$ and $QPQ = \mu Q$ with $\mu \neq 0$. If even P, Q, PQ or P, Q, QP are linearly dependent, we infer from Lemma 3.3(a) that P, Q are tightly coupled; this case was excluded. We may therefore suppose that both P, Q, PQ and P, Q, QP are linearly independent. We then have $QP = \alpha P + \beta Q + \gamma PQ$. Multiplying this from the left by P we get $PQP = \alpha P + (\beta + \gamma) PQ$ and thus $(\alpha - \mu)P + (\beta + \gamma)PQ = 0$. This implies that $\alpha = \mu$ and $\beta + \gamma = 0$. Multiplying the equality $QP = \alpha P + \beta Q + \gamma PQ$ from the right by Q, we obtain analogously $(\beta - \mu)Q + (\alpha + \gamma)PQ = 0$, which gives $\beta = \mu$ and $\alpha + \gamma = 0$. Consequently, $QP = \mu(P + Q - PQ)$. Multiplication of this equality from the left by Q yields $QP = \mu(QP + Q - \mu Q)$, that is, $\mu(1-\mu)Q + (\mu - 1)QP = 0$. As P, Q, QP are linearly independent, we conclude that $\mu = 1$. Hence $PQP = P$ and $QPQ = Q$, and Theorem 3.1 implies that $\mathrm{alg}(P, Q)$ is isomorphic to W_3 for $m = 3$ and to W_4 for $m = 4$.

It remains to study the case where $m = 4$ and P, Q, PQ, QP are linearly independent. Since we suppose that P, Q are not tightly coupled, we deduce from Lemma 3.3(c) that $PQP = PQ + QP - Q$ or $PQP = \mu P$ for some $\mu \neq 0$. The same lemma with P, Q replaced by Q, P shows that $QPQ = PQ + QP - P$ or $QPQ = \lambda Q$ with some $\lambda \neq 0$.

Suppose first that $PQP = \mu P$ and $QPQ = \lambda Q$. As then $PQPQ = \mu PQ$ and $PQPQ = \lambda PQ$, we see that actually $\mu = \lambda$. If $\mu = 1$, then Theorem 3.1 shows that the algebra is isomorphic to W_4. Thus, let $\mu \notin \{0, 1\}$. We may take $\{P, Q, PQ, QP\}$ as a basis of the algebra. The multiplication table is

$$\begin{array}{c|cccc} & P & Q & PQ & QP \\ \hline P & P & PQ & PQ & \mu P \\ Q & QP & Q & \mu Q & QP \\ PQ & \mu P & PQ & \mu PQ & \mu P \\ QP & QP & \mu Q & \mu Q & \mu QP. \end{array} \tag{18}$$

If P, Q are the idempotent matrices

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} \mu & \sqrt{\mu(1-\mu)} \\ \sqrt{\mu(1-\mu)} & 1-\mu \end{pmatrix}, \tag{19}$$

with an arbitrary choice of the square root, then the multiplication table is the same. Thus, our algebra is isomorphic to the algebra generated by the two matrices (19). However, it is easily seen that if P, Q are given by (19) with $\mu \notin \{0, 1\}$, then the linear hull of $\{P, Q, PQ, QP\}$ is all of $\mathbb{C}^{2\times 2}$.

Assume next that $PQP = \mu P$ and $QPQ = PQ + QP - P$. Multiplying the last equality by P from the left we get $PQPQ = PQ + PQP - P$ and hence $\mu PQ = PQ + \mu P - P$, that is, $(\mu - 1)(PQ - P) = 0$. This implies that $\mu = 1$, and from Theorem 3.1 it follows that the algebra is isomorphic to $W_3 \oplus Z_1$. The result is the same if $QPQ = \lambda Q$ and $PQP = PQ + QP - Q$.

We are left with the case where $PQP = PQ + QP - Q$ and $QPQ = PQ + QP - P$. Proposition 3.2 tells us that then $\mathrm{alg}(P, Q) \cong V_4$. It remains to note that V_4 is not isomorphic to $\mathbb{C}^{2\times 2}$ since V_4 is not unital. $\square$

In connection with the appearance of $\mathbb{C}^{2\times 2}$ in Theorem 3.4 we want to mention the following. Recall that an associative algebra $\mathcal{A}$ is said to be an F_k-algebra if

$$\sum_{\sigma \in S_k} (\operatorname{sign} \sigma) a_{\sigma(1)} \cdots a_{\sigma(k)} = 0$$

for all $a_1, \ldots, a_k \in \mathcal{A}$. Roch and Silbermann [21] proved that an algebra generated by two idempotents is always an F_4-algebra. The famous Amitsur–Levitski theorem says that $\mathbb{C}^{n\times n}$ is an F_{2n}-algebra but not an F_{2n-2}-algebra. It follows that if $n \geq 3$, then $\mathbb{C}^{n\times n}$ is never generated by two idempotents. See also [18,27].

Theorem 3.5. *Suppose* $\mathrm{alg}(P, Q)$ *has dimension* 3 *or* 4 *and let* $A \in \mathrm{alg}(P, Q)$ *be given by*

$$A = a_1\mathbf{P} + b_1\mathbf{Q} + a_2\mathbf{PQ} + b_2\mathbf{QP}, \tag{20}$$

with $a_2 = 0$ *in case* $\dim \mathrm{alg}(P, Q) = 3$. *Put* $a = a_1 + a_2$ *and* $b = b_1 + b_2$.

(a) *If* $\mathrm{alg}(P, Q)$ *is isomorphic to* W_3, *then* $\sigma(A) = \{0, a + b\}$ *and* A *is group invertible if and only if* (i) $A = 0$ *or* (ii) $a + b \neq 0$.
(b) *If* $\mathrm{alg}(P, Q)$ *is isomorphic to* W_4, *then* $\sigma(A) = \{0, a + b\}$ *and* A *is group invertible if and only if* (i) $A = 0$ *or* (ii) $a + b \neq 0$ *and* $ab_1 = a_2 b$ *(equivalently,* $a_1 b_1 = a_2 b_2$*).*
(c) *If* $\mathrm{alg}(P, Q)$ *is isomorphic to* V_4, *then* $\sigma(A) = \{0, a + b\}$ *and* A *is group invertible if and only if* (i) $A = 0$ *or* (ii) $a + b \neq 0$.
(d) *If* $\mathrm{alg}(P, Q)$ *is isomorphic to* $\mathbb{C}^{2\times 2}$, *then the multiplication table is of the form* (18) *with* $\mu \notin \{0, 1\}$ *and* A *is group invertible if and only if* (i) $A = 0$ *or* (ii) $a_1 + b_1 + (a_2 + b_2)\mu \neq 0$ *or* (iii) $a_1 b_1 - \mu a_2 b_2 \neq 0$.

Proof. (a) The matrix A is the first matrix in (14) with $x = a_1 + b_1, y = a_2 + b_2, z = a, x + y - z = b$. The spectrum of this matrix is $\{0, x + y\} = \{0, a + b\}$. A direct computation shows that $A^2 = (a + b)A$, which implies that $\operatorname{rank} A = \operatorname{rank} A^2$ if and only if (i) $A = 0$ or (ii) $a + b \neq 0$.

(b) This time A is the second matrix in (14) with $x = b, y = b_1, z = a + b, w = a_2 + b_1$. Thus,

$$A = \begin{pmatrix} 0 & x & y \\ 0 & z & w \\ 0 & 0 & 0 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 0 & xz & xw \\ 0 & z^2 & zw \\ 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

The spectrum of A is $\{0, z\} = \{0, a + b\}$, and it is easily seen that A has the same rank as A^2 if and only if (i) $A = 0$ or (ii) $z \neq 0$ and $xw - yz = 0$.

(c) Now A is the matrix (17) with $x = a + b, y = b, z = b_2, w = a_1 + b_2$, so that

$$A = \begin{pmatrix} x & z & w \\ 0 & x & 0 \\ 0 & y & 0 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} x^2 & 2xz + yw & xw \\ 0 & x^2 & 0 \\ 0 & xy & 0 \end{pmatrix}. \tag{22}$$

The spectrum of A equals $\{0, x\} = \{0, a + b\}$, and it can again readily be verified that the rank of A coincides with the rank of A^2 if and only (i) $A = 0$ or (ii) $x \neq 0$.

(d) The proof of Theorem 3.4 reveals that $\mathrm{alg}(P, Q)$ is isomorphic to $\mathbb{C}^{2\times 2}$ if and only if the multiplication table is (18) with some $\mu \notin \{0, 1\}$ and that therefore $\mathbf{P}$ and $\mathbf{Q}$ may be assumed to be given by (19). Then the matrix A becomes

$$A = \begin{pmatrix} a_1 + (b_1 + a_2 + b_2)\mu & (b_1 + a_2)\sqrt{\mu(1-\mu)} \\ (b_1 + b_2)\sqrt{\mu(1-\mu)} & b_1(1-\mu) \end{pmatrix}.$$

A nonzero 2×2 matrix is *not* group invertible if and only if both its eigenvalues are zero, that is, if and only if the trace and the determinant are zero. Consequently, our matrix A is group invertible if and only if (i) $A = 0$ or (ii) $\operatorname{trace} A = a_1 + b_1 + (a_2 + b_2)\mu \neq 0$ or (iii) $\det A = (1 - \mu)(a_1 b_1 - \mu a_2 b_2) \neq 0$. $\square$

We remark that invertibility and group invertibility in $W_3 \oplus Z_1$ may be checked by using part (a) of the previous theorem. Furthermore, with the help of part (b) one can also treat the questions for the 5-dimensional algebra $W_4 \oplus Z_1$ occurring in Theorem 3.1(d).

Finally, group invertibility is a special case of Drazin invertibility. Recall that an element A of an associative algebra $\mathcal{A}$ is said to be Drazin invertible if there exists an $X \in \mathcal{A}$ satisfying

$$AX = XA, \qquad XAX = X, \quad \text{and} \quad A^k XA = A^k \quad \text{for some natural } k. \tag{23}$$

Such X, when it exists, is defined uniquely and is called the Drazin inverse A^D of A. In its turn, the smallest value of k for which (23) holds is the Drazin index of A. Thus, group invertibility is just Drazin invertibility with Drazin index 1.

A finite matrix A is always Drazin invertible. Given the Jordan canonical form $A = CJC^{-1}$, we have $X = A^D = CJ^DC^{-1}$ where J^D is obtained from J by replacing the invertible Jordan blocks with their inverses and the singular Jordan blocks with the zero blocks of the same size. Consequently, the Drazin index k of a matrix A coincides with the size of its biggest Jordan block corresponding to the eigenvalue zero. Equivalently, k is the smallest non-negative integer for which $\operatorname{rank} A^k = \operatorname{rank} A^{k+1}$. As was the case for the group inverse, and for the same reasons, every matrix algebra is *Drazin inverse closed*, that is, contains Drazin inverses of all its elements. See, for example, [1].

It follows from the preceding paragraph that elements of finite-dimensional associative algebras $\mathcal{A}$ are always Drazin invertible. Moreover, as soon as a matrix representation of $\mathcal{A}$ is available, computation of the Drazin index and explicit formulas for the Drazin inverse become purely technical problems. In particular, the following statement holds, complementing Theorem 3.5.

Theorem 3.6. *Suppose* $\operatorname{alg}(P, Q)$ *has dimension not exceeding* 4 *and let A be an element of* $\operatorname{alg}(P, Q)$ *with no group inverse. Then the Drazin index of A equals* 2 *unless, in the notation* (20), *the algebra* $\operatorname{alg}(P, Q)$ *is isomorphic to W_4 or V_4, $a = -b \neq 0$ and $a_2 + b_1 (= a_1 + b_2) \neq 0$. In the latter case, the Drazin index of A equals* 3.

Proof. All the algebras under consideration are listed in Theorem 3.4. Of those, the 1-dimensional Z_1 consists of group invertible elements only. The 2-dimensional Z_2, D_2, D_2^* as well as the 3-dimensional Z_3 can be thought of as subalgebras of $\mathbb{C}^{2\times 2}$, and thus their elements have Drazin indices not exceeding 2. The same applies, of course, to the algebra $\mathbb{C}^{2\times 2}$ itself, as well as the algebras isomorphic to direct sums of the above mentioned algebras. This covers D_j, D_j^*, U_j $(j = 3, 4)$ according to (10), W_3 according to (12), and therefore also $W_3 \oplus Z_1$. The algebra Z_4, being generated by

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

falls into the same category. Indeed, under the permutation (4, 1, 2, 3) of their rows and columns, the latter pair of matrices becomes

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

It remains to consider W_4 and V_4.

If $A \in W_4$ is not group invertible then, by Theorem 3.5(b), we may suppose that A and A^2 are as in (21) with $z = 0$ or $xw \neq yz$. Suppose first that $z \neq 0$. Then A^2 has rank one and, since A is not nilpotent (z being its eigenvalue), all consequent powers of A are also of rank one. For $z = 0$, on the other hand, (21) yields $A^3 = 0$, while

$$\operatorname{rank} A^2 = \begin{cases} 1 & \text{if } xw \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, the Drazin index of A in this setting is 3 if $z = 0$, $xw \neq 0$ and 2 in all other cases. Since $z = a + b$, $x = b$ and $w = a_2 + b_1$, this agrees with the statement of the theorem.

Finally, let $A \in V_4$, that is (without loss of generality) be a matrix as in (22). By Theorem 3.5(c), the matrix A is not group invertible if and only if $x = 0$ but at least one of the entries y, z, w differs from zero. According to (22), the matrix A^2 then takes the form

$$A^2 = \begin{pmatrix} 0 & yw & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that $A^3 = 0$ and

$$\operatorname{rank} A^2 = \begin{cases} 1 & \text{if } yw \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the Drazin index of A is 3 if $x = 0$ and $yw \neq 0$. Since $w(=a_1 + b_2) = a_2 + b_1$ under the condition $x(=a + b) = 0$, this again agrees with the assertion. $\square$

Theorem 3.6 covers in particular the formulas for the Drazin index of linear combinations $a_1 P + b_1 Q$ obtained in [23].

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