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Existence of a positive solution to Kirchhoff type problems without compactness conditions ☆

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ABSTRACT

The existence of a positive solution to a Kirchhoff type problem on $\mathbb{R}^N$ is proved by using variational methods, and the new result does not require usual compactness conditions. A cut-off functional is utilized to obtain the bounded Palais–Smale sequences.

1. Introduction

In this paper, we consider the positive solutions to the following nonlinear Kirchhoff type problem

$$\left(a+\lambda\int_{\mathbb{R}^N}|\nabla u|^2+\lambda b\int_{\mathbb{R}^N}u^2\right)[-\Delta u+bu]=f(u),\quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N\geqslant 3$, and a, b are positive constants, $\lambda\geqslant 0$ is a parameter. Kirchhoff type problem on a bounded domain $\Omega\subset\mathbb{R}^N$

$$\begin{cases} -\left(a+b\displaystyle\int_{\Omega}|\nabla u|^2\right)\Delta u=f(u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

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has been studied by many authors, for example [2,4–6,8,12,13,19,20]. Many solvability conditions with f near zero and infinity for problem (1.2) have been considered, such as the superlinear case [12]; and asymptotical linear case [16]. In addition, the following growth condition on f is often assumed:

(f) $f(t)t \geqslant 4F(t)$ for $|t|$ large, where $F(t) = \int_0^t f(s)\,ds$,

which assures the boundedness of any (PS) or Cerami sequence. Indeed the condition (f) may appear in different forms as follows:

(f_0) there exists $\theta \geqslant 1$ such that $\theta G(t) \geqslant G(st)$ for all $t \in \mathbb{R}$ and $s \in [0,1]$, where $G(t) = f(t) - 4F(t)$ (see [16]);
(f_1) $\lim_{|t|\to\infty}[f(t)t - 4F(t)] = \infty$ (see [19]); or
(f_2) $\lim_{|t|\to\infty} G(t) = \infty$ and there exists $\sigma > \max\{1, N/2\}$ such that $|f(t)|^\sigma \leqslant CG(t)|t|^\sigma$ for $|t|$ large (see [12]).

In the above papers, each of the conditions (f_0)–(f_2) implies that condition (f) holds. On the other hand, the condition (f) is sufficient to show the boundedness of any (PS) or Cerami sequence, which has been proved in [18].

There are few papers considering Kirchhoff type problems on $\mathbb{R}^N$ except [18]. In [18], the author studied the problem

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\right)\Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^N.$$

The existence of nontrivial solutions was proved in [18] under the condition (f) and

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V(x) > 0$ and for each $M > 0$, meas $\{x \in \mathbb{R}^N\colon\ V(x) \leqslant M\} < \infty$;
(H_1) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $|f(t)| \leqslant C(|t| + |t|^{p-1})$ for all $t \in \mathbb{R}_+ = [0,\infty)$ and some $p \in (2, 2^*)$, where $2^* = 2N/(N-2)$ for $N \geqslant 3$;
(H_2) $\lim_{t\to 0} \frac{f(t)}{t} = 0$;
(f_3) $\lim_{t\to\infty} \frac{F(t)}{t^4} = \infty$.

In this paper, we prove the existence of positive solutions of (1.1) without the condition (f) (or (f_0)–(f_2)), and we use a cut-off functional to obtain bounded (PS) sequences. We assume the following weaker condition:

(H_3) $\lim_{t\to\infty} \frac{f(t)}{t} = \infty$.

Our main result is as follows:

Theorem 1.1. *Assume that $N \geqslant 3$, and a, b are positive constants, $\lambda \geqslant 0$ is a parameter. If the conditions* (H_1), (H_2) *and* (H_3) *hold, then there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0)$,* (1.1) *has at least one positive solution.*

Theorem 1.1 appears to be the first existence result for Eq. (1.1). We also remark that the condition (H_3) is weaker than the ones in the above mentioned papers, in which $\lim_{|t|\to\infty} f(t)/t^3 = \infty$ or a constant (which implies (H_3)) was assumed. Since the result in Theorem 1.1 holds for $\lambda = 0$, then we have the following corollary regarding the well-known semilinear equation.

Corollary 1.2. *Assume that $N \geqslant 3$, and b is a positive constant. If the conditions* (H_1), (H_2) *and* (H_3) *hold, then the problem*

$$-\Delta u + bu = f(u), \quad in\ \mathbb{R}^N \tag{1.3}$$

has at least one positive solution.

Note that the existence result like the one in Corollary 1.2 has been obtained by many authors, for example, [1,3,10,11,14]. Hence our result in Theorem 1.1 can be regarded as an extension of the classical result for the semilinear equation (1.3) to the case of the nonlinear Kirchhoff type problem (1.1). On the other hand, it is not clear whether the result in Theorem 1.1 still holds for large $\lambda > 0$. In our result, the choice of λ_0 depends on the nonlinearity f, constants N, a and b, Sobolev embedding constant, several test functions and constants used in the proof.

We recall some preliminaries and prove some lemmas in Section 2, and we give the proof of Theorem 1.1 in Section 3.

2. Preliminaries

Let $H^1(\mathbb{R}^N)$ be the usual Sobolev space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^N} [\nabla u \cdot \nabla v + buv], \qquad \|u\| = (u, u)^{1/2}.$$

We denote by $|\cdot|_q$ the usual $L^q(\mathbb{R}^N)$ norm. Then we have that $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ continuously for $q \in [2, 2^*]$. Let $H = H^1_r(\mathbb{R}^N)$ be the subspace of $H^1(\mathbb{R}^N)$ containing only the radial functions. Then $H \hookrightarrow L^q(\mathbb{R}^N)$ compactly for $q \in (2, 2^*)$ [17, Corollary 1.26, p. 18]. In this paper, we consider positive solutions to (1.1), then we assume that $f(t) = 0$ for $t < 0$.

Define a functional J_λ on the space H by

$$J_\lambda(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda\|u\|^4 - \int_{\mathbb{R}^N} F(u), \quad u \in H.$$

Then we have from (H_1) that J_λ is well defined on H and is of C^1 for all $\lambda \geqslant 0$, and

$$\big(J'_\lambda(u), v\big) = a(u, v) + \lambda\|u\|^2(u, v) - \int_{\mathbb{R}^N} f(u)v, \quad u, v \in H.$$

It is standard to verify that the weak solutions of (1.1) correspond to the critical points of the functional J_λ.

To overcome the difficulty of finding bounded Palais–Smale sequences for the associated functional J_λ, following [7,9], we use a cut-off function $\psi \in C^\infty(\mathbb{R}_+, \mathbb{R})$ satisfying

$$\begin{cases} \psi(t) = 1, & t \in [0, 1], \\ 0 \leqslant \psi(t) \leqslant 1, & t \in (1, 2), \\ \psi(t) = 0, & t \in [2, \infty), \\ \|\psi'\|_\infty \leqslant 2, \end{cases}$$

and study the following modified functional $J^T_\lambda : H \to \mathbb{R}$ defined by

$$J^T_\lambda(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda h_T(u)\|u\|^4 - \int_{\mathbb{R}^N} F(u), \quad u \in H,$$

where for every $T > 0$,

$$h_T(u) = \psi\left(\frac{\|u\|^2}{T^2}\right).$$

With this penalization, for $T > 0$ sufficiently large and for λ sufficiently small, we are able to find a critical point u of J_λ^T such that $\|u\| \leqslant T$ and so u is also a critical point of J_λ. We recall the following result. The "monotonicity trick" at the core of this theorem was invented by Struwe (see [15]).

Theorem 2.1. *(See [6].) Let $(X, \|\cdot\|)$ be a Banach space and $I \subset \mathbb{R}_+$ an interval. Consider the family of C^1 functionals on X*

$$J_\mu(u) = A(u) - \mu B(u), \quad \mu \in I,$$

with B nonnegative and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$ and such that $J_\mu(0) = 0$.

For any $\mu \in I$ we set

$$\Gamma_\mu = \{\gamma \in C([0,1], X)\colon\ \gamma(0) = 0,\ J_\mu(\gamma(1)) < 0\}.$$

If for every $\mu \in I$ the set Γ_μ is nonempty and

$$c_\mu = \inf_{\gamma \in \Gamma_\mu} \max_{t \in [0,1]} J_\mu(\gamma(t)) > 0,$$

then for almost every $\mu \in I$ there is a sequence $\{u_n\} \subset X$ such that

(i) *$\{u_n\}$ is bounded;*
(ii) *$J_\mu(u_n) \to c_\mu$;*
(iii) *$J'_\mu(u_n) \to 0$ in the dual X^{-1} of X.*

In our case, $X = H$,

$$A(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda h_T(u)\|u\|^4, \qquad B(u) = \int_{\mathbb{R}^N} F(u).$$

So the perturbed functional which we study is

$$J_{\lambda,\mu}^T(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda h_T(u)\|u\|^4 - \mu \int_{\mathbb{R}^N} F(u),$$

and

$$\left(\left(J_{\lambda,\mu}^T\right)'(u), v\right) = a(u,v) + \lambda h_T(u)\|u\|^2(u,v) + \frac{\lambda}{2T^2}\psi'\left(\frac{\|u\|^2}{T^2}\right)\|u\|^4(u,v) - \mu \int_{\mathbb{R}^N} f(u)v. \tag{2.1}$$

The following Lemmas 2.2–2.4 imply that $J_{\lambda,\mu}^T$ satisfies the conditions of Theorem 2.1.

Lemma 2.2. *$\Gamma_\mu \neq \emptyset$ for all $\mu \in I = [\delta, 1]$, where $\delta \in (0,1)$ is a positive constant.*

Proof. We choose $\phi \in C_0^\infty(\mathbb{R}^N)$ with $\phi \geqslant 0$, $\|\phi\| = 1$ and $\operatorname{supp}(\phi) \subset B(0, R)$ for some $R > 0$. By (H_3), we have that for any $C_1 > 0$ with $C_1 \delta \int_{B(0,R)} \phi^2 > a/2$, there exists $C_2 > 0$ such that

$$F(t) \geqslant C_1 |t|^2 - C_2, \quad t \in \mathbb{R}_+. \tag{2.2}$$

Then for $t^2 > 2T^2$,

$$\begin{aligned} J^T_{\lambda,\mu}(t\phi) &= \frac{1}{2} a t^2 \|\phi\|^2 + \frac{1}{4} \lambda \psi\left(\frac{t^2 \|\phi\|^2}{T^2}\right) t^4 \|\phi\|^4 - \mu \int_{\mathbb{R}^N} F(t\phi) \\ &= \frac{1}{2} a t^2 - \mu \int_{\mathbb{R}^N} F(t\phi) \\ &\leqslant \frac{1}{2} a t^2 - \delta C_1 t^2 \int_{B(0,R)} \phi^2 + C_3. \end{aligned}$$

Then we can choose $t > 0$ large such that $J^T_{\lambda,\mu}(t\phi) < 0$. The proof is completed. □

Lemma 2.3. *There exists a constant $c > 0$ such that $c_\mu \geqslant c > 0$ for all $\mu \in I$.*

Proof. For any $u \in H$ and $\mu \in I$, using (H_1) and (H_2), for $\varepsilon \in (0, a/2)$, we have

$$\begin{aligned} J^T_{\lambda,\mu}(u) &\geqslant \frac{1}{2} a \|u\|^2 + \frac{1}{4} \lambda h_T(u) \|u\|^4 - \int_{\mathbb{R}^N} \left(\frac{1}{2} \varepsilon b u^2 + C_\varepsilon |u|^p\right) \\ &\geqslant \frac{1}{4} a \|u\|^2 - C_\varepsilon \int_{\mathbb{R}^N} |u|^p. \end{aligned}$$

By Sobolev's embedding theorem, we conclude that there exists $\rho > 0$ such that $J^T_{\lambda,\mu}(u) > 0$ for any $\mu \in I$ and $u \in H$ with $0 < \|u\| \leqslant \rho$. In particular, for $\|u\| = \rho$, we have $J^T_{\lambda,\mu}(u) \geqslant c > 0$. Fix $\mu \in I$ and $\gamma \in \Gamma_\mu$. By the definition of Γ_μ, $\|\gamma(1)\| > \rho$. By continuity, we deduce that there exists $t_\gamma \in (0, 1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore, for any $\mu \in I$,

$$c_\mu \geqslant \inf_{\gamma \in \Gamma_\mu} J^T_{\lambda,\mu}\big(\gamma(t_\gamma)\big) \geqslant c > 0.$$

The proof is completed. □

Lemma 2.4. *For any $\mu \in I$ and $8\lambda T^2 < a$, each bounded Palais–Smale sequence of the functional $J^T_{\lambda,\mu}$ admits a convergent subsequence.*

Proof. Let $\mu \in I$ and $\{u_n\}$ be a bounded (PS) sequence of $J^T_{\lambda,\mu}$, namely

$$\{u_n\} \text{ and } \big\{J^T_{\lambda,\mu}(u_n)\big\} \text{ are bounded,}$$

$$\big(J^T_{\lambda,\mu}\big)'(u_n) \to 0 \quad \text{in } H',$$

where H' is the dual space of H. Subject to a subsequence, we can assume that there exists $u \in H$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H, \\ u_n &\to u \quad \text{in } L^p(\mathbb{R}^N), \\ u_n &\to u \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

By (H_1) and (H_2), for any $\varepsilon \in (0, a/2)$, there exists $C_\varepsilon > 0$ such that

$$\left|f(t)\right| \leqslant b\varepsilon|t| + C_\varepsilon|t|^{p-1}, \quad t \in \mathbb{R}, \tag{2.3}$$

hence,

$$\begin{aligned} \left|\int_{\mathbb{R}^N} f(u_n)(u_n - u)\right| &\leqslant \int_{\mathbb{R}^N} \left|f(u_n)\right||u_n - u| \\ &\leqslant b\varepsilon|u_n|_2|u_n - u|_2 + C_\varepsilon \int_{\mathbb{R}^N} |u_n|^{p-1}|u_n - u| \\ &\leqslant \varepsilon C\|u_n\|\|u_n - u\| + C_\varepsilon|u_n|_p^{p-1}|u_n - u|_p. \end{aligned}$$

It follows that

$$\int_{\mathbb{R}^N} f(u_n)(u_n - u) \to 0.$$

Thus,

$$\begin{aligned} 0 \leftarrow \left(\left(J_{\lambda,\mu}^T\right)'(u_n), u_n - u\right) &= a(u_n, u_n - u) + \lambda h_T(u_n)\|u_n\|^2(u_n, u_n - u) \\ &\quad + \frac{\lambda}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4(u_n, u_n - u) - \mu\int_{\mathbb{R}^N} f(u_n)(u_n - u) \\ &= \left(a + \lambda h_T(u_n)\|u_n\|^2 + \frac{\lambda}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4\right)(u_n, u_n - u) + o(1), \end{aligned}$$

and then

$$\left(a + \lambda h_T(u_n)\|u_n\|^2 + \frac{\lambda}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4\right)(u_n, u_n - u) \to 0.$$

Since $|\psi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^4| \leqslant 8T^4$ and $8\lambda T^2 < a$, $\|u_n\| \to \|u\|$. This together with $u_n \rightharpoonup u$ shows that $u_n \to u$ in H. The proof is completed. □

Lemma 2.5. *Let $8\lambda T^2 < a$. For almost every $\mu \in I$, there exists $u^\mu \in H \setminus \{0\}$ such that $(J_{\lambda,\mu}^T)'(u^\mu) = 0$ and $J_{\lambda,\mu}^T(u^\mu) = c_\mu$.*

Proof. By Theorem 2.1, for almost every $\mu \in I$, there exists a bounded sequence $\{u_n^\mu\} \subset H$ such that

$$J^T_{\lambda,\mu}\big(u_n^\mu\big) \to c_\mu,$$
$$\big(J^T_{\lambda,\mu}\big)'\big(u_n^\mu\big) \to 0.$$

By Lemma 2.4, we can suppose that there exists $u^\mu \in H$ such that $u_n^\mu \to u^\mu$ in H, then the assertion follows from Lemma 2.3. □

According to Lemma 2.5, there exist sequences $\{\mu_n\} \subset I$ with $\mu_n \to 1^-$ and $\{u_n\} \subset H$ as $n \to \infty$ such that

$$J^T_{\lambda,\mu_n}(u_n) = c_{\mu_n}, \qquad \big(J^T_{\lambda,\mu_n}\big)'(u_n) = 0.$$

The Pohozaev identity is important for many problems. In this paper, we also use this identity to obtain $\|u_n\| \leqslant T$. In fact we have the next lemma.

Lemma 2.6. *Let $8\lambda T^2 < a$ and $N \geqslant 3$. If $u \in H$ is a weak solution of*

$$\left(a + \lambda h_T(u)\|u\|^2 + \frac{\lambda}{2T^2}\psi'\left(\frac{\|u\|^2}{T^2}\right)\|u\|^4\right)(-\Delta u + bu) = \mu f(u), \quad \textit{in } x \in \mathbb{R}^N, \tag{2.4}$$

then the following Pohozaev type identity holds

$$\left(\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{Nb}{2}\int_{\mathbb{R}^N} u^2\right)\left(a + \lambda h_T(u)\|u\|^2 + \frac{\lambda}{2T^2}\psi'\left(\frac{\|u\|^2}{T^2}\right)\|u\|^4\right) = \mu N \int_{\mathbb{R}^N} F(u). \tag{2.5}$$

Proof. Since $u \in H$ is a weak solution of (2.4), by standard regularity results, $u \in H^2_{\mathrm{loc}}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. Let

$$g(u) = \frac{\mu f(u)}{a + \lambda h_T(u)\|u\|^2 + \frac{\lambda}{2T^2}\psi'(\frac{\|u\|^2}{T^2})\|u\|^4} - bu.$$

Then $u \in H$ is also a solution of

$$-\Delta u = g(u).$$

By [17, Corollary B.4, p. 138],

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 = N\int_{\mathbb{R}^N} G(u),$$

where $G(t) = \int_0^t g(s)\,ds$. Then the conclusion holds. □

The following lemma shows that $\|u_n\| \leqslant T$ which is the key for this paper.

Lemma 2.7. *Let u_n be a critical point of J^T_{λ,μ_n} at level c_{μ_n}. Then for $T > 0$ sufficiently large, there exists $\lambda_0 = \lambda_0(T)$ with $8\lambda_0 T^2 < a$ such that for any $\lambda \in [0, \lambda_0)$, subject to a subsequence, $\|u_n\| \leqslant T$ for all $n \in \mathbb{N}$.*

Proof. We argue by contradiction. Firstly, since $(J^T_{\lambda,\mu_n})'(u_n) = 0$, by (2.5), u_n satisfies the following Pohozaev identity

$$\left(\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2 + \frac{Nb}{2}\int_{\mathbb{R}^N}u_n^2\right)\left(a + \lambda h_T(u_n)\|u_n\|^2 + \frac{\lambda}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4\right)$$
$$= \mu_n N\int_{\mathbb{R}^N}F(u_n). \tag{2.6}$$

By using $J^T_{\lambda,\mu_n}(u_n) = c_{\mu_n}$, we have that

$$\frac{1}{2}aN\|u_n\|^2 + \frac{1}{4}\lambda N h_T(u_n)\|u_n\|^2 - \mu_n N\int_{\mathbb{R}^N}F(u_n) = c_{\mu_n}N. \tag{2.7}$$

Therefore, by (2.6) and (2.7), we can obtain that

$$\frac{1}{2}a\int_{\mathbb{R}^N}|\nabla u_n|^2 \leqslant \left(a + \lambda h_T(u_n)\|u_n\|^2 + \frac{\lambda}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^4\right)\int_{\mathbb{R}^N}|\nabla u_n|^2$$
$$= c_{\mu_n}N + \frac{1}{4}\lambda N h_T(u_n)\|u_n\|^4 + \frac{\lambda N}{4T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^6. \tag{2.8}$$

We estimate the right hand side of (2.8). By the min–max definition of the mountain pass level, Lemma 2.2 and (2.2), we have

$$\begin{aligned}
c_{\mu_n} &\leqslant \max_t J^T_{\lambda,\mu_n}(t\phi) \\
&\leqslant \max_t\left\{\frac{1}{2}at^2 - \mu_n\int_{\mathbb{R}^N}F(t\phi)\right\} + \max_t \frac{1}{4}\lambda\psi\left(\frac{t^2}{T^2}\right)t^4 \\
&\leqslant \max_t\left\{\frac{1}{2}at^2 - \delta C_1 t^2\int_{B(0,R)}\phi^2 + C_3\right\} + \max_t \frac{1}{4}\lambda\psi\left(\frac{t^2}{T^2}\right)t^4 \\
&= C_3 + A_1(T).
\end{aligned}$$

If $t^2 \geqslant 2T^2$, then $\psi(\frac{t^2}{T^2}) = 0$. Thus, we have that

$$A_1(T) \leqslant \lambda T^4.$$

We have also that

$$\frac{1}{4}\lambda N h_T(u_n)\|u_n\|^4 \leqslant \lambda N T^4,$$
$$\frac{\lambda N}{4T^2}\left|\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\right|\|u_n\|^6 \leqslant 4\lambda N T^4.$$

Then we have

$$\frac{1}{2}a\int_{\mathbb{R}^N}|\nabla u_n|^2 \leqslant NC_3 + 6\lambda NT^4.$$

On the other hand, by (2.1) and (2.3), we have that

$$a\|u_n\|^2 + \lambda h_T(u_n)\|u_n\|^4 + \frac{\lambda}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^6 = \mu_n\int_{\mathbb{R}^N} f(u_n)u_n \leqslant b\varepsilon|u_n|_2^2 + C_\varepsilon|u_n|_{2^*}^{2^*}.$$

So

$$\begin{aligned}(a-\varepsilon)\|u_n\|^2 &\leqslant C_\varepsilon|u_n|_{2^*}^{2^*} - \frac{\lambda}{2T^2}\psi'\left(\frac{\|u_n\|^2}{T^2}\right)\|u_n\|^6\\ &\leqslant C_4|\nabla u_n|_2^{2^*} + 8\lambda T^4\\ &\leqslant C_5\left(NC_3 + 6\lambda NT^4\right)^{2^*/2} + 8\lambda T^4.\end{aligned}$$

We suppose by contradiction that there exists no subsequence of $\{u_n\}$ which is uniformly bounded by T. Then we can assume that $\|u_n\| > T$, $n \in \mathbb{N}$. Then

$$T^2 < \|u_n\|^2 \leqslant C_6\left(NC_3 + 6\lambda NT^4\right)^{2^*/2} + C_7\lambda T^4,$$

which is not true for T large and $8\lambda T^4 < a$. So by setting $\lambda_0 < a/(8T^4)$, we obtain the conclusion. □

Remark 2.8. In Lemma 2.7, the choice of λ_0 depends on the nonlinearity f, constants N, a and b, Sobolev embedding constant γ_{2^*}, several test functions and constants used in the proof. So it is difficult to give explicitly the value of λ_0. However, for the special case $f(t) = at^2$, we can choose ϕ and δ in Lemma 2.2 to satisfy $\delta\int_{B(0,5)}\phi^2 > 1/4$. Similarly we can choose $C_1 = 2a$, $C_2 = 32a/3$, $C_3 = 32a/3|B(0,5)|$ in Lemma 2.2, where $|B(0,5)|$ is the volume of $B(0,5)$ in $\mathbb{R}^N$. Moreover, we choose $\varepsilon = a/2$ in Lemma 2.7, then $C_\varepsilon = a(2/b)^{1/(2^*-3)}$. Hence, we can compute a lower bound of λ_0 to be

$$\lambda_0 = \frac{a}{32[(2/b)^{1/(2^*-3)}\gamma_{2^*}(2/a)^{2^*/2}(32aN/3|B(0,5)| + a)^{2^*/2} + 1]^2},$$

where γ_{2^*} is the embedding constant in the embedding inequality $(\int_{\mathbb{R}^N}|\nabla u|^2)^{2^*/2} \leqslant \gamma_{2^*}\int_{\mathbb{R}^N}|u|^{2^*}$ for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. So the existence result in Theorem 1.1 holds for any $\lambda \in [0, \lambda_0)$ for this special case.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Let T, λ_0 be defined as in Lemma 2.7, and let u_n be a critical point for J_{λ,μ_n}^T at level c_{μ_n}. Then from Lemma 2.7 we may assume that

$$\|u_n\| \leqslant T.$$

So

$$J^T_{\lambda,\mu_n}(u_n) = \frac{1}{2}a\|u_n\|^2 + \frac{1}{4}\lambda\|u_n\|^4 - \mu_n \int\limits_{\mathbb{R}^N} F(u_n).$$

Since $\mu_n \to 1$, we can show that $\{u_n\}$ is a (PS) sequence of J_λ. Indeed, the boundedness of $\{u_n\}$ implies that $\{J_\lambda(u_n)\}$ is bounded. Also

$$\left(J'_\lambda(u_n), v\right) = \left(\left(J^T_{\lambda,\mu_n}\right)'(u_n), v\right) + (\mu_n - 1)\int\limits_{\mathbb{R}^N} f(u_n)v, \quad v \in H.$$

Thus $J'_\lambda(u_n) \to 0$, and then $\{u_n\}$ is a bounded (PS) sequence of J_λ. By Lemma 2.4, $\{u_n\}$ has a convergent subsequence. We may assume that $u_n \to u$. Consequently, $J'_\lambda(u) = 0$. According to Lemma 2.3, we have that $J_\lambda(u) = \lim_{n\to\infty} J_\lambda(u_n) = \lim_{n\to\infty} J^T_{\lambda,\mu_n}(u_n) \geqslant c > 0$ and u is a positive solution by the condition (H_1). The proof is completed. $\square$

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