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Minimizing the least eigenvalue of graphs with fixed order and size

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ABSTRACT

The problem of identifying those simple, undirected graphs with n vertices and k edges that have the smallest minimum eigenvalue of the adjacency matrix is considered. Several general properties of the minimizing graphs are described. These strongly suggest bipartition, to the extent possible for the number of edges. In the bipartite case, the precise structure of the minimizing graphs is given for a number of infinite classes.

1. Introduction

Among simple, undirected graphs on n vertices with k edges, we are interested in identifying the graph (or, in some cases, graphs) having the smallest minimum eigenvalue for the usual adjacency matrix. Our initial interest was due to a connection with invincibly positive semi-definite matrices, which are defined as positive semidefinite (PSD) matrices that remain PSD after changing any symmetrically placed pairs of off-diagonal entries to 0 [7]; such matrices arise in semi-definite programming. The problem studied is also interesting from an algebraic graph theory, taxonomy point of view.

The question of bounding $\lambda_{\min}(G)$ – the least eigenvalue of a graph G – has a long history, and here are some of its milestones:

$$\lambda_{\min}(G) \geq -\sqrt{MaxCut} \quad \text{(see [6])}$$

$$\lambda_{\min}(G) \geq -\sqrt{\left\lfloor \frac{n^2}{4} \right\rfloor} \quad \text{(see [4])}$$

$$\lambda_{\min}(G) \geq -\sqrt{k} \quad \text{(see [11])}$$

where *MaxCut* is the maximum size of a bipartite subgraph of G. Tight relations between the least eigenvalue of a graph and its *MaxCut* are described in [12]. The main result of [12] is a new approximation algorithm for *MaxCut*, which can be seen

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as an analogue of Cheeger's inequality [1] for the smallest eigenvalue of the adjacency matrix of a d-regular graph

$$\sqrt{2(1-|\lambda_{\min}(G)|)} \geq \beta(G) \geq \frac{1}{2}(1-|\lambda_{\min}(G)|),$$

where

$$\beta(G) = \min_{y \in \{-1,0,1\}^V} \frac{\sum_{i,j} |y_i + y_j|}{2d \sum_i |y_i|}.$$

First we focus on the structure of minimizing graphs in the general case. Let $A(G)$ (or simply A) be the adjacency matrix of a simple, undirected graph G. Thus, A is a symmetric (0, 1)-matrix with zero diagonal. Since our adjacency matrix A is a symmetric, real matrix, all its eigenvalues are real and can be ordered increasingly

$$\lambda_{\min}(A) = \lambda_n(A) \leq \lambda_{n-1}(A) \leq \cdots \leq \lambda_2(A) \leq \lambda_1(A) = \lambda_{\max}(A),$$

where $\lambda_{\max}$ is the largest eigenvalue (Perron root or index) of graph G. Since $\sum_{i=1}^n \lambda_i = 0$ (and we do not consider graphs without edges), some of these are negative, and some are positive. Thus, $\lambda_{\max}(A)$ is greater than 0, and $\lambda_{\min}(A)$, the eigenvalue of interest to us, is less than 0. (Since $\lambda_{\max}(A)$ is the spectral radius, it is greater than the most negative eigenvalue, therefore $\lambda_{\min}(A) + \lambda_{\max}(A) \geq 0$, with equality for bipartite graphs.)

Let $\mathcal{G}_{n,k}$ be the collection of simple undirected graphs with n vertices and k edges; let also $\mathcal{A}_{n,k} = \{A(G) : G \in \mathcal{G}_{n,k}\}$. We are interested in $\min_{A \in \mathcal{A}_{n,k}} \lambda_{\min}(A)$ and in the subset of $\mathcal{G}_{n,k}$ on which this minimum is attained.

Since in the next section we show that our problem leads to the question of the largest singular value among a class of (0, 1)-matrices, now we recall the singular value decomposition and a related definition.

Let $M_{m,n}$ ($M_{m,n}(\mathbb{R})$, resp). be the set of all m-by-n complex (real, resp.) matrices, let $M_{m,n}(\{0, 1\})$ be the set of all m-by-n (0, 1)-matrices, and abbreviate $M_{n,n}$ to M_n.

Theorem 1 (*[9], p. 144*)**.** *Let $A \in M_{m,n}$ be given, and let $q = \min\{m, n\}$. There are unitary matrices $V \in M_m$ and $W \in M_n$ such that $A = V\Sigma W^*$, where Σ is a matrix in $M_{m,n}$ whose entries $\sigma_{i,j}$ are 0 when $i \neq j$ and satisfy $\sigma_{1,1} \geq \cdots \geq \sigma_{q,q} \geq 0$. If $A \in M_{m,n}(\mathbb{R})$, then V and W may be taken to be real orthogonal matrices.*

Definition 1 (*[9], p. 146*)**.** Let $A \in M_{m,n}$, and let $q = \min\{m, n\}$. Given the singular value decomposition guaranteed by Theorem 1, let $\sigma_i(A)$ denote $\sigma_{i,i}$. The values $\sigma_1(A), \ldots, \sigma_q(A)$ are the *singular values* of A, and σ_1 is called the largest singular value of A. The number of positive singular values of A is equal to the rank of A. The columns of the unitary matrix W are the right singular vectors of A; the columns of V are the left singular vectors of A.

The largest singular value of A is the *spectral norm* of A.

In [2], graph modifications are used to show that a graph in $\mathcal{G}_{n,k}$ with smallest least eigenvalue (a *minimizing graph*) is either bipartite or a join of two threshold graphs. This can be seen as follows.

For $G \in \mathcal{G}_{n,k}$, $\lambda_{\min}(A(G))$ satisfies ([10], p. 176):

$$\min_{x \in \mathbb{R}^n : x^T x = 1} x^T A(G) x = \lambda_{\min}(A(G)).$$

Let y be a minimizing vector, i.e. $\lambda_{\min}(A(G)) = y^T A(G) y$. We henceforth suppose without loss of generality that the vertices of G are labeled so that

$$y_1 \leq \cdots \leq y_n.$$

Since $A(G) \geq 0$ (entry-wise) and $\lambda_{\min}(A(G)) < 0$, if all entries of y are nonnegative (or nonpositive), the equality $A(G)y = \lambda_{\min}(A(G))y$ could not hold. Hence y includes at least one negative entry and at least one positive entry, i.e. $y_1 < 0$ and $y_n > 0$. This means that there is a subscript p with $0 < p < n$ such that $y_i < 0$ for all $i \leq p$ and $y_i \geq 0$ for $i > p$. Thus, y may be partitioned as

$$y = \begin{pmatrix} y^{(1)} \\ y^{(2)} \end{pmatrix},$$

with $y^{(1)} < 0$, $y^{(2)} \geq 0$, and $y^{(1)} \in \mathbb{R}^p$, $y^{(2)} \in \mathbb{R}^{n-p}$. Similarly, partition A as

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{1,2}^T & A_{2,2} \end{pmatrix},$$

with $A_{1,1} \in M_p(\mathbb{R})$ and $A_{2,2} \in M_{n-p}(\mathbb{R})$. All matrix and vector inequalities are to be interpreted entry-wise.

Since $A \geq 0$ and y achieves $\min_{x^T x=1} x^T A x$, if G were a minimizing graph in $\mathcal{G}_{n,k}$, then either $A_{1,1}$ and $A_{2,2}$ would be 0 matrices (i.e., G is bipartite) or all entries of $A_{1,2}$ would be 1, since

$$y^T A y = y^{(1)T} A_{1,1} y^{(1)} + y^{(2)T} A_{2,2} y^{(2)} + 2y^{(1)T} A_{1,2} y^{(2)} \tag{1}$$

and the first two terms consist entirely of nonnegative summands. Otherwise, symmetrically placed entries equal to 1 in $A_{1,1}$ or $A_{2,2}$ could be shifted to $A_{1,2}$ and $A_{1,2}^T$ to produce another matrix $A' \in \mathcal{A}_{n,k}$ whose quadratic form, evaluated at y, would give a smaller value than $\lambda_{\min}(A)$, contradicting the minimality of $\lambda_{\min}(A)$.

Here we consider just the bipartite graphs in $\mathcal{G}_{n,k}$; these have adjacency matrices of the form

$$\begin{pmatrix} 0 & A_{1,2} \\ A_{1,2}^T & 0 \end{pmatrix},$$

where $A_{1,2}$ is a p-by-q (0, 1)-matrix with $p + q = n$ and k entries equal to 1. We shall see that when the least eigenvalue is smallest (that is, when the index is maximal) $A_{1,2}$ may be taken to be a *normalized, left-justified* matrix; such a matrix is called a *stepwise* matrix in [5]. For connected bipartite graphs with maximal index, this property of $A_{1,2}$ was established in [2, Theorem 2.1]. We conjecture that if G is a bipartite minimizing graph, then $A_{1,2}$ has rank 1 or 2 (Conjecture 1). We prove several results (Theorems 8–10 and 13, 14, Corollary 15) to support this conjecture by considering the singular values of $A_{1,2}$. The results provide various conditions on $A_{1,2}$ ensuring that the index of G is not maximal; a special case was treated in [3].

2. Minimization of the smallest eigenvalue

The main goal of this section is to show that a minimizing graph G is either bipartite or has a spanning complete bipartite subgraph. In the non-bipartite case, we give information about the structure of the minimizing graphs.

Theorem 2. *A minimizing graph G is either bipartite or has a spanning complete bipartite subgraph. In the bipartite case only, some vertices could be isolated. In the non-bipartite case, any edges not in the spanning complete bipartite subgraph correspond to the smallest positive products of eigenvector components.*

Proof. The first claim was shown in the introduction. Moreover, if $A_{1,2}$ in (1) is all-1, then any 1 in $A_{1,1}$ (or $A_{2,2}$) must lie in such entry of $A_{1,1}$ (or $A_{2,2}$) that $y^{(1)T}A_{1,1}y^{(1)}$ (or $y^{(2)T}A_{2,2}y^{(2)}$) is minimal. If every entry of $A_{1,1}$ is 0 except for two entries $a_{i,j}$ and $a_{j,i}$ equal to 1, where $i, j \in 1, \dots, p$, then $y^{(1)T}A_{1,1}y^{(1)} = 2y_iy_j$. Therefore any 1 in $A_{1,1}$ must be placed in such position (i, j) of $A_{1,1}$ that $\min_{i,j} y_iy_j$ is attained. It is easy to see that if there are more 1's in $A_{1,1}$, then any 1 must lie in a place that gives the smallest sum of products of components from $y^{(1)}$. The case of $A_{2,2}$ is similar. □

Of course, when G is bipartite, it is a particular choice from among (possibly) many bipartite graphs in $\mathcal{G}_{n,k}$.

Let $\mathcal{B}_{n,k}$ be the subset of $\mathcal{G}_{n,k}$ consisting of bipartite graphs. The set $\mathcal{B}_{n,k}$ is nonempty if and only if $k \le \frac{n^2}{4}$. If $G \in \mathcal{B}_{n,k}$, then $A(G)$, with vertices labeled according to the bipartition, appears as

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{1,2}^T & A_{2,2} \end{pmatrix}, \tag{2}$$

with $A_{1,1}$ and $A_{2,2}$ being all-0. Suppose, without loss of generality, that $A_{1,2}$ is p-by-q, with $q \le p$ and $p + q = n$, and let

$$\sigma_1(A_{1,2}), \dots, \sigma_q(A_{1,2})$$

be the singular values of $A_{1,2}$. The (possibly) nonzero eigenvalues of A are then

$$-\sigma_1(A_{1,2}), -\sigma_2(A_{1,2}), \dots, -\sigma_q(A_{1,2}), \sigma_q(A_{1,2}), \dots, \sigma_2(A_{1,2}), \sigma_1(A_{1,2}),$$

in numerical order, so that $\lambda_{\min}(A(G)) = -\sigma_1(A_{1,2})$ and $|\lambda_{\min}(A(G))| = \lambda_{\max}(A(G))$. Moreover, an eigenvector y corresponding to $\lambda_{\min}(A)$ is composed of left and right singular vectors $y^{(1)}$ and $y^{(2)}$ of $A_{1,2}$, properly signed. Hence using the Perron–Frobenius Theorem ([8], p. 329) we can suppose that $y^{(1)} < 0$ and $y^{(2)} \ge 0$; by proper labeling of the vertices, our bipartition of A is consistent with the earlier $-/+$ partition of y.

We conclude that to minimize $\lambda_{\min}(A)$ within $\mathcal{B}_{n,k}$, it suffices to maximize the largest singular value of a p-by-q (with $p + q = n$) (0, 1)-matrix with k entries equal to 1. Our minimizing bipartite graph is, then, a graph whose adjacency matrix is in the form (2), where $A_{1,2}$ maximizes σ_1 among (0, 1)-matrices with k entries equal to 1 whose numbers of rows and columns sum to n. Such matrices we call, for short, *maximizing matrices*.

3. Singular value maximization in (0, 1)-matrices

At the beginning of this section we show that if there is a rank-1 matrix among the p-by-q (0, 1)-matrices with k entries equal to 1, then it has the greatest largest singular value. Next we introduce the definition of a normalized, left-justified matrix and show that if there is no rank-1 matrix among the p-by-q (0, 1)-matrices with k entries equal to 1, then the matrix maximizing the largest singular value is, up to permutation equivalence, a normalized, left-justified matrix.

Let $S_{p,q,k} = \{C \in M_{p,q}(\{0, 1\}) : e_p^T C e_q = k\}$, where e_p and e_q denote the all-1 vectors in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, and $\bigcup_{p+q=n} S_{p,q,k} = \mathcal{S}_{n,k}$. We will often think of a matrix in $S_{p,q,k}$ with a 0 row or column as having such lines deleted; the new matrix $C' \in M_{p',q'}(\{0, 1\})$ with $p' \le p$ and $q' \le q$ has the same singular values as C, and we may view it as an element of $\mathcal{S}_{n,k}$ without difficulty.

Our interest here is to describe the matrices in $S_{p,q,k}$ for which the largest singular value σ_1 is a maximum. Let

$$m_{p,q,k} = \max_{C \in S_{p,q,k}} \sigma_1(C) \quad \text{and} \quad m_{n,k} = \max_{C \in \mathcal{S}_{n,k}} \sigma_1(C).$$

Finally, let $M_{p,q,k}$ denote the subset of $S_{p,q,k}$ of matrices for which $m_{p,q,k}$ is attained and $\mathcal{M}_{n,k}$ denote the subset of $\mathcal{S}_{n,k}$ of matrices for which $m_{n,k}$ is attained.

Since the singular values are unitary equivalence invariant ([10], p. 296), we have the following lemma.

Lemma 3. *If P is a p-by-p permutation matrix and Q is a q-by-q permutation matrix, then $C \in M_{p,q,k}$ if and only if $PCQ \in M_{p,q,k}$.*

Proof. Since $m_{p,q,k} = \max_{C \in S_{p,q,k}} \sigma_1(C)$, $\sigma_1^2(C) = \lambda_{\max}(CC^T)$, and $(PCQ)(PCQ)^T = P(CC^T)P^T$ we have $C \in M_{p,q,k}$ if and only if $PCQ \in M_{p,q,k}$. $\square$

Note that a (0, 1)-matrix C with k entries equal to 1 is a rank-1 matrix if and only if all its entries equal to 1 occur in a p'-by-q' submatrix for which $k = p'q'$. In this case, $\sigma_1(C)$ is the only nonzero singular value of C and, since the sum of squares of all singular values of C is k, i.e. $\sigma_1(C)^2 = k$, we obtain $\sigma_1(C) = \sqrt{k}$. No other matrix with k entries equal to 1 and of rank greater than 1 has greater σ_1. We conclude the following.

Lemma 4. *If there is a rank-1 matrix in $S_{p,q,k}$, then $m_{p,q,k} = \sqrt{k}$ and $M_{p,q,k}$ consists of all rank-1 matrices in $S_{p,q,k}$.*

The next lemma follows directly from the previous lemma.

Lemma 5. *If there is a rank-1 matrix in $\mathcal{S}_{n,k}$, then $m_{n,k} = \sqrt{k}$ and $\mathcal{M}_{n,k}$ consists of all rank-1 matrices in $\mathcal{S}_{n,k}$.*

Observe that there is a rank-1 matrix in $\mathcal{S}_{n,k}$ if and only if k factors into a product of integers whose sum is at most n. For example, this condition does not hold for $n = 5$ and $k = 5$. Thus, we have to consider all cases for which the construction of a rank-1 matrix is impossible.

Among the permutation equivalences of C there is (at least) one matrix in which the row or column sums are in nonincreasing order. We call such a representative of the permutation equivalence class of C *row-normalized* or *column-normalized*. Of course, both normalizations may occur simultaneously, in which case we call the matrix *normalized*. We call a (0, 1)-matrix *left-justified* if all 1's (if any) in each row are to the left of all 0's. Note that a left-justified, row-normalized matrix is normalized, but a normalized matrix may not be left-justified. Therefore, a normalized, left-justified (0, 1)-matrix is a matrix with all 1's in each row to the left of all 0's and with all 1's in each column above all 0's. Observe that a normalized, left-justified (0, 1)-matrix appears "upper-left triangular", and it may have some rows repeated.

We call the number of 1's in the i-th row of C the *content* of the i-th row. We may now prove the following key lemma.

Lemma 6. *For each p, q, k, with $k \le pq$, $M_{p,q,k}$ contains a normalized, left-justified matrix.*

Proof. Suppose that $C \in M_{p,q,k}$. Without loss of generality we may assume C is normalized. Suppose that $\hat{C} \in M_{p,q,k}$ is the left justification of C (move all 1's to the left in each row). The matrix $\hat{C}$ remains normalized. Note that $\hat{C}\hat{C}^T$ is an irreducible matrix and that by inspection of matrix multiplication, $\hat{C}\hat{C}^T \ge CC^T \ge 0$. By the Perron–Frobenius theory this implies $\sigma_1(\hat{C}) \ge \sigma_1(C)$. Among normalized matrices, equality occurs only for $C = \hat{C}$. $\square$

4. The role of a single additional row matrix

We call a matrix in $S_{p,q,k}$, whose first $p - 1$ rows have content q and whose last row has content strictly between 0 and q, a *single additional row matrix* or *SARM*. Such a matrix appears as

$$\begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ \vdots & & & & & \vdots \\ 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

It is easy to see that if $S_{p,q,k}$ for $p + q \le n$ contains no rank-1 matrix, it must contain a SARM. We have been led to the following conjecture.

Conjecture 1. *The value of $m_{p,q,k}$ ($m_{n,k}$) is always attained by either a rank-1 matrix, or by a SARM. The latter occurs only when there is no rank-1 matrix in $S_{p,q,k}$ ($S_{n,k}$).*

The intuition is strong that the maximum singular value should be attained by a matrix having low rank, thus that when a rank-1 matrix is not possible, a rank-2 matrix should attain the maximum. We have found, though, that this intuition is not easy to confirm, in general. We can determine the maximizer among SARM's and show that a SARM is maximal among rank-2 matrices, which we do next. Then, we give a number of situations in which our conjecture is correct.

First consider a normalized, left-justified (0, 1)-matrix of rank 2, i.e. of the following form:

$$C = \begin{pmatrix} J_{p_1q_1} & J_{p_1q_2} \\ J_{p_2q_1} & 0_{p_2q_2} \end{pmatrix} \tag{3}$$

with $p = p_1 + p_2$, $q = q_1 + q_2$, and $p_2, q_2 \geq 2$. Because C has $p_1p_2q_1q_2$ nonzero 2-by-2 minors, each with determinant 1 in absolute value, a simple calculation shows that the largest eigenvalue of CC^T

$$\sigma_1^2(C) = \lambda_{\max}(CC^T) = \frac{k + \sqrt{k^2 - 4p_1p_2q_1q_2}}{2}. \tag{4}$$

Let S be a matrix with the same number of columns (and, possibly, fewer rows) as in C, obtained from C by moving all 1's such that only one row has 1's and 0's. The matrix S is a SARM of the following form

$$S = \begin{pmatrix} J_{p'_1q'_1} & J_{p'_1q'_2} \\ e^T_{q'_1} & 0^T_{q'_2} \end{pmatrix} \tag{5}$$

with $q'_1 + q'_2 = q$ and $p'_1 + p'_2 \leq p$. According to (4) we have

$$\sigma_1^2(S) = \lambda_{\max}(SS^T) = \frac{k + \sqrt{k^2 - 4p'_1q'_1q'_2}}{2}. \tag{6}$$

In other words, when divided by the appropriate power of λ, the characteristic polynomial of SS^T has the form:

$$\lambda^2 - k\lambda + p'_1q'_1q'_2,$$

in which k is the number of 1's in S.

Recall that C is of the form (3). We define the *disparity* $D(C)$ of C to be the product of the number of 1's in its upper right block and the number of 1's in its lower left block.

Therefore, the largest singular value of S can be written in the form

$$\sigma_1^2(S) = \lambda_{\max}(SS^T) = \frac{k + \sqrt{k^2 - 4D(S)}}{2}. \tag{7}$$

Our goal is to show that there is always an S such that $\sigma_1(S)$ is greater than or equal to $\sigma_1(C)$, with equality if and only if $C = S$.

First, we show which SARM's, among all possible SARM's for the given n and k, have the greatest largest singular value.

Theorem 7. *Suppose that $\mathcal{S}_{n,k}$ has no rank-1 matrix. A SARM in $\mathcal{S}_{n,k}$ has the maximum largest singular value among all possible SARM's in $\mathcal{S}_{n,k}$ if and only if it has minimum disparity.*

Proof. From (7) we see that the smaller $D(S)$ is, the larger is $\sqrt{k^2 - 4D(S)}$ and the larger is $\sigma_1(S)$. Thus, among SARM's $\sigma_1(S)$ is decreasing in $D(S)$, and the maximum $\sigma_1(S)$ occurs for the smallest $D(S)$. □

Now we show that if C is rank-2 matrix and is not a SARM, then there is either a rank-1 matrix or a SARM in the same $S_{n,k}$ as C with greater largest singular value than $\sigma_1(C)$.

The following familiar fact will be employed several times in the proof.

Fact 1. *If $0 < a' \leq a \leq b \leq b'$, then $a + b = a' + b'$ implies $a'b' \leq ab$.*

Theorem 8. *If $C \in S_{p,q,k}$ satisfies $\operatorname{rank}(C) = 2$ with $p + q = n$, and C is not a SARM, then there is an $S \in S_{p',q',k}$, with $p' + q' \leq n$ that is either a rank-1 matrix or a SARM, such that $\sigma_1(C) < \sigma_1(S)$.*

Proof. Suppose that $C \in S_{p,q,k}$ satisfies $\operatorname{rank}(C) = 2$ with $p + q = n$, and C is not a SARM. According to Lemma 4 if there exists a rank-1 matrix S in $\mathcal{S}_{n,k}$, then it has the greatest singular value among matrices in $\mathcal{S}_{n,k}$ and hence $\sigma_1(C) < \sigma_1(S)$.

Otherwise, there is in $\mathcal{S}_{n,k}$ a p'-by-q' matrix S, with $p' + q' \leq n$, that is a SARM. Without loss of generality, we may suppose that C is of the form

$$C = \begin{pmatrix} J_{p_1q_1} & J_{p_1q_2} \\ J_{p_2q_1} & 0_{p_2q_2} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & 0 \end{pmatrix}$$

with $p = p_1 + p_2$, $q = q_1 + q_2$, and $p_2, q_2 \geq 2$. Via transposition, without loss of generality we may assume that $p_2q_1 \leq p_1q_2$. We move 1's from the block C_{21} to the 0 block, one row at a time, starting at the bottom, until we achieve S of the form

$$S = \begin{pmatrix} J_{p'_1q'_1} & J_{p'_1q'_2} \\ e_{q'_1} & 0_{q'_2} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & 0 \end{pmatrix}$$

with $p'_2 = 1$. The matrix S consists of k 1's and $\sigma_1(C) < \sigma_1(S)$ if and only if $p_1p_2q_1q_2 > p'_1q'_1q'_2$, by formulas (4) and (6).

Consider five exhaustive possibilities for q_1, q_2, q_1', q_2'':

Case 1. $q_1' < q_1 \leq q_2$.
Case 2. $q_1 \leq q_2$ and $q_1 \leq q_1'$.
Case 3. $q_1' \leq q_2 < q_1$.
Case 4. $q_2 < q_1' < q_1$.
Case 5. $q_2 < q_1 \leq q_1'$.
Of course, $p_1' \geq p_1$.

In Case 2 and Case 5, the blocks S_{12} and S_{21} together contain no more 1's than did C_{12} and C_{21}, while S_{12} contains fewer 1's than did C_{12}. Application of Fact 1 shows that the desired inequality holds.

In Cases 1 and Case 3 we have $p_1' < p_1p_2$, because $p_1' < p_1' + p_2' \leq p_1 + p_2 \leq p_1p_2$. Since $q_1' < q_1 \leq q_2$ in Case 1 and $q_1' \leq q_2 < q_1$ in Case 3 and $q_1' + q_2' = q_1 + q_2$, Fact 1, again, implies that $q_1'q_2' \leq q_1q_2$, which gives the desired inequality in these cases as well.

In Case 4, if $p_1'q_1' \geq p_1q_1$, then the blocks S_{12} and S_{21} together contain no more 1's than have C_{12} and C_{21}. Since S_{21} has fewer 1's than C_{21} (which has no more than C_{12}), Fact 1 again gives the desired inequality. It is easy to see that if $p_1'q_1' < p_1q_1$, then $\frac{p_1q_1}{p_1'q_1'} > 1$. But, since the number of 0's in S (q_2') is not greater than in C (p_2q_2), we have $\frac{p_2q_2}{q_2'} \geq 1$. Together, these imply $\frac{p_1q_1p_2q_2}{p_1'q_1'q_2'} > 1$, which is the desired inequality completing the proof. $\square$

5. The bled matrix as a tool

To address our Conjecture 1 by ruling out (0, 1)-matrices of rank greater than 2, we use a "bled matrix" B and show that it gives us the lower bound for the largest singular of the given normalized, left-justified (0, 1)-matrix. Next we show that for any rank-2 (0, 1)-matrix that is not a SARM, there is either a rank-1 matrix or a SARM that has greater largest singular value than the largest singular value of a given (0, 1)-matrix.

First, we define a process of "bleeding" a matrix, and then we use singular value inequalities involving bled matrices to show that if A is a matrix of rank more than 2 of a certain form, then there is a SARM with a greater largest singular value.

Given $C = (c_{ij}) \in M_{p,q}$, let $B_i(C)$ be the matrix in $M_{p,q}$ that agrees with C in rows $1, 2, \ldots, i-1, i+1, \ldots, p$, while the i-th row sum of $B_i(C)$ is the same as that of C, and all entries in row i of $B_i(C)$ are equal. Thus every entry in the i-th row of C is replaced by the average entry in row i. This averaging process we call "bleeding" a matrix. If we bleed every row of C we obtain the (fully) *bled matrix*

$$B = B(C) = B_1(B_2(\ldots B_p(C))\ldots).$$

A bled matrix necessarily has rank 1, and its largest singular value is easy to calculate:

$$\sigma_1(B) = \left(\sum_{i=1}^{p} \left| \frac{1}{q} \sum_{j=1}^{q} c_{ij} \right|^2 \right)^{\frac{1}{2}}.$$

For our purposes, we need only consider fully bled matrices obtained from normalized, left-justified (0, 1)-matrices by bleeding each of its rows. Note that the permuting that produces row normalization commutes with bleeding and that left justification does not change the result of bleeding.

The following theorem gives a lower bound for the largest singular value of matrix C.

Theorem 9. *If $C \in S_{p,q,k}$ is a normalized, left-justified matrix that is not rank-1 and $B(C)$ is its bled matrix, then $\sigma_1(C) > \sigma_1(B(C))$.*

Proof. Let $C \in S_{p,q,k}$ be a normalized, left-justified matrix of the form

$$C = \begin{pmatrix} J_{p_1q_1} & J_{p_1q_2} & J_{p_1q_3} & \cdots & J_{p_1q_n} \\ J_{p_2q_1} & J_{p_2q_2} & \cdots & J_{p_2q_{n-1}} & O \\ J_{p_3q_1} & \cdots & J_{p_3q_{n-2}} & O & O \\ \vdots & ⋰ & O & O & O \\ J_{p_nq_1} & O & O & O & O \end{pmatrix}. \tag{8}$$

Let $B = B(C)$ be the bled matrix of C of the following form

$$B = \begin{pmatrix} J_{p_1q_1} & J_{p_1q_2} & J_{p_1q_3} & \cdots & J_{p_1q_n} \\ v_1J_{p_2q_1} & v_1J_{p_2q_2} & v_1J_{p_2q_3} & \cdots & v_1J_{p_2q_n} \\ v_2J_{p_3q_1} & v_2J_{p_3q_2} & v_2J_{p_3q_3} & \cdots & v_2J_{p_3q_n} \\ \vdots & & & & \vdots \\ v_{n-1}J_{p_nq_1} & v_{n-1}J_{p_nq_2} & v_{n-1}J_{p_nq_3} & \cdots & v_{n-1}J_{p_nq_n} \end{pmatrix},$$

in which $v_i = \frac{q_1+\cdots+q_{n-i}}{q}$.

Since $\sigma_1(C)^2 = \lambda_{\max}(CC^T)$ and $\sigma_1(B)^2 = \lambda_{\max}(BB^T)$, we focus on the structure of CC^T and BB^T. Observe that

$$CC^T = \begin{pmatrix} qJ_{p_1p_1} & \sum_{l=1}^{n-1} q_lJ_{p_1p_2} & \sum_{l=1}^{n-2} q_lJ_{p_1p_3} & \cdots & q_1J_{p_1p_n} \\ \sum_{l=1}^{n-1} q_lJ_{p_2p_1} & \sum_{l=1}^{n-1} q_lJ_{p_2p_2} & \sum_{l=1}^{n-2} q_lJ_{p_2p_3} & \cdots & q_1J_{p_2p_n} \\ \sum_{l=1}^{n-2} q_lJ_{p_3p_1} & \sum_{l=1}^{n-2} q_lJ_{p_3p_2} & \sum_{l=1}^{n-2} q_lJ_{p_3p_3} & \cdots & q_1J_{p_3p_n} \\ \vdots & & & & \vdots \\ q_1J_{p_np_1} & q_1J_{p_np_2} & q_1J_{p_np_3} & \cdots & q_1J_{p_np_n} \end{pmatrix} \tag{9}$$

and

$$BB^T = \begin{pmatrix} qJ_{p_1p_1} & \sum_{l=1}^{n-1} q_lJ_{p_1p_2} & \sum_{l=1}^{n-2} q_lJ_{p_1p_3} & \cdots & q_1J_{p_1p_n} \\ \sum_{l=1}^{n-1} q_lJ_{p_2p_1} & qv_1^2J_{p_2p_2} & qv_1v_2J_{p_2p_3} & \cdots & qv_1v_{n-1}J_{p_2p_n} \\ \sum_{l=1}^{n-2} q_lJ_{p_3p_1} & qv_1v_2J_{p_3p_2} & qv_2^2J_{p_3p_3} & \cdots & qv_2v_{n-1}J_{p_3p_n} \\ \vdots & & & & \vdots \\ q_1J_{p_np_1} & qv_1v_{n-1}J_{p_np_2} & qv_1v_{n-2}q_1J_{p_np_3} & \cdots & qv_n^2J_{p_np_n} \end{pmatrix}.$$

Let CC^T and BB^T be matrices with p_i-by-p_j blocks C_{ij} and B_{ij}, respectively, where $i, j \in 1, \ldots, n$. Since the first blocks in both matrices are equal, i.e. $C_{11} = B_{11}$, we focus on the other blocks of those matrices. Observe that all entries in block C_{ij} are the same and all entries in block B_{ij} are the same. Let c_{ij} be one of the entries in block C_{ij} in CC^T and b_{ij} be one of the entries in block B_{ij} in BB^T. We have

$$c_{ij} = \sum_{l=1}^{n+1-\min\{i,j\}} q_l \quad \text{and} \quad b_{ij} = qv_iv_j = \frac{1}{q}\sum_{l=1}^{n+1-i} q_l \sum_{l=1}^{n+1-j} q_l.$$

Since $q = \sum_{l=1}^{n} q_l$, we have

$$b_{ij} \le \frac{\sum_{l=1}^{n+1-i} q_l \sum_{l=1}^{n+1-j} q_l}{\sum_{l=1}^{n+1-\max\{i,j\}} q_l} = c_{ij}.$$

Therefore we have $BB^T \ge CC^T \ge 0$. Of course all entries in BB^T are not the same as in CC^T, and hence $BB^T \ne CC^T$. It is easy to see that BB^T is an irreducible matrix. Therefore $\rho(BB^T) > \rho(CC^T)$, which implies $\sigma_1(B) > \sigma_1(C)$. □

The biggest advantage of a bled matrix is that it is rank-1 matrix and is simpler to analyze than the original matrix.

6. (0, 1)-matrices of rank greater than 2

In the next theorem we consider (0, 1)-matrices of rank greater than 2. We show that depending on the rank and the number of 1's in the last row of the SARM considered, the SARM has a greater largest singular value than the given (0, 1)-matrix has.

Theorem 10. *Let T be an r-by-r normalized, left-justified matrix with $(r - i + 1)$ 1's in its i-th row. Suppose that $C \in S_{p,r,k}$ satisfies $p + r = n$, and that C includes all rows from T, together with repetitions. Suppose that an $S \in S_{p',q,k}$ is a SARM, with $p' + r \le n$ and with t 1's in the last row. If*

$$\sigma_2^2(T) + \cdots + \sigma_r^2(T) > t\left(1 - \frac{2t}{n}\right), \tag{10}$$

then $\sigma_1(C) < \sigma_1(S)$.

Proof. Suppose that C, S, and T are matrices satisfying the hypothesis of our theorem and (10) holds. For a p-by-q matrix C, without loss of generality, let $p \geq q$, so $q \leq \frac{n}{2}$. Let $B = B(S)$ be a bled matrix of S of the form

$$B = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \\ \frac{t}{q} & \cdots & \frac{t}{q} \end{pmatrix}.$$

By Horn and Johnson [10], for any matrix $C \in M_{p,q}$

$$\mathrm{tr}(CC^T) = \sigma_1^2(C) + \cdots + \sigma_{\min\{p,q\}}^2(C). \tag{11}$$

Note that since

$$SS^T = \begin{pmatrix} q & \cdots & q & t \\ \vdots & & \vdots & \\ q & \cdots & q & t \\ t & \cdots & t & t \end{pmatrix} \quad \text{and} \quad BB^T = \begin{pmatrix} q & \cdots & q & t \\ \vdots & & \vdots & \\ q & \cdots & q & t \\ t & \cdots & t & \frac{t^2}{q} \end{pmatrix},$$

the difference between the sum of all squared singular values of S and squared the largest singular of B is

$$\sum_{i=1}^{p} \sigma_i^2(S) - \sigma_1^2(B) = \mathrm{tr}(SS^T) - \mathrm{tr}(BB^T) = t\left(1 - \frac{t}{q}\right). \tag{12}$$

Since S is rank-2 matrix, by (11) we have

$$\sigma_1^2(S) + \sigma_2^2(S) = k. \tag{13}$$

Since B is rank-1 matrix, by (11) and (12) we have

$$\sigma_1^2(B) = k - t\left(1 - \frac{t}{q}\right). \tag{14}$$

From Theorem 9 we conclude that

$$\sigma_1^2(B) < \sigma_1^2(S).$$

Using this inequality and (14), we have

$$\sigma_1^2(S) > k - t\left(1 - \frac{t}{q}\right).$$

According to this inequality and (13) we obtain

$$\sigma_2^2(S) < t\left(1 - \frac{t}{q}\right),$$

and since $q \leq \frac{n}{2}$ we get

$$\sigma_2^2(S) < t\left(1 - \frac{2t}{n}\right).$$

Since $\mathrm{tr}(CC^T) = \mathrm{tr}(SS^T) = k$, (11) implies we have

$$\sigma_1^2(C) + \sigma_2^2(C) + \cdots + \sigma_q^2(C) = \sigma_1^2(S) + \sigma_2^2(S).$$

Therefore, to show that $\sigma_1^2(C) < \sigma_1^2(S)$ it suffices to show that

$$\sigma_2^2(S) < \sigma_2^2(C) + \cdots + \sigma_q^2(C).$$

Since T is a submatrix of C obtained by deleting $(p - r)$ rows, we have from Interlacing Theorem ([10], p. 419), that

$$\sigma_2^2(C) + \cdots + \sigma_q^2(C) \geq \sigma_2^2(T) + \cdots + \sigma_r^2(T),$$

and it is enough to show that

$$\sigma_2^2(T) + \cdots + \sigma_r^2(T) > t\left(1 - \frac{2t}{n}\right),$$

which is implied by the assumption of our theorem. $\square$

7. The main results

In this section we show that the SARM has a greater largest singular value than a certain (0, 1)-matrix of rank greater than 2 and with all rows different, and a (0, 1)-matrix with repeated rows, where the number of repeated rows depends on rank. We obtain similar results for matrices under some conditions depending on rank and the number of 1's in a SARM, that we want to compare to our (0, 1)-matrix.

To show our main results about $\sigma_1(C)$ for the remaining matrices C in $M_{p,q}(\{0, 1\})$, we need to focus on the elementary symmetric functions of CC^T (see [10], p. 41).

Lemma 11. *If $C \in S_{p,q,k}$ is normalized and left-justified matrix with exactly r different rows of contents $c_1, \ldots, c_r$ indexed in decreasing order with p_i the number of repetitions of rows of content c_i, $i = 1, \ldots, r$, then the characteristic polynomial of CC^T has the form:*

$$\lambda^{p-r}(\lambda^r - k\lambda^{r-1} + E_2\lambda^{r-2} - E_3\lambda^{r-3} + \cdots \pm E_r),$$

in which the E_t are the t-th elementary symmetric functions of CC^T, which are of the form

$$E_t = \sum_{1\le i_1<\cdots<i_t\le r} c_{i_t}p_{i_t}\prod_{j=1}^{t-1}(c_{i_j} - c_{i_{j+1}})p_{i_j}, \quad 2 \le t \le r.$$

Proof. If $C \in S_{p,q,k}$ is normalized and left-justified matrix with exactly r different rows of contents $c_1, \ldots, c_r$ indexed in decreasing order with p_i the number of repetitions of rows of content c_i, $i = 1, \ldots, r$, then

$$CC^T = \begin{pmatrix} c_1J_{p_1p_1} & c_2J_{p_1p_2} & \cdots & c_rJ_{p_1p_n} \\ c_2J_{p_2p_1} & c_2J_{p_2p_2} & \cdots & c_rJ_{p_2p_n} \\ \vdots & & & \vdots \\ c_rJ_{p_np_1} & c_rJ_{p_np_2} & \cdots & c_rJ_{p_np_n} \end{pmatrix} \tag{15}$$

and the characteristic polynomial of CC^T is of the form:

$$\lambda^{p-r}(\lambda^r - k\lambda^{r-1} + E_2\lambda^{r-2} - E_3\lambda^{r-3} + \cdots \pm E_r),$$

in which the E_t are the t-th elementary symmetric functions of CC^T.

Recall (see [10], p. 42) that E_t is the sum of all t-by-t principal minors of CC^T. A principal minor consisting of t rows of t different content types $i_1, \ldots, i_t$ with $1 \le i_1 < \cdots < i_t \le r$ is equal to $c_{i_t}\prod_{j=1}^{t-1}(c_{i_j} - c_{i_{j+1}})$, and every such minor repeats $p_{i_j} \cdot p_{i_{j+1}} \cdot \ldots \cdot p_{i_t}$ times, and minors involving repeated content types are 0. Therefore we have

$$E_t = \sum_{1\le i_1<\cdots<i_t\le r} c_{i_t}p_{i_t}\prod_{j=1}^{t-1}(c_{i_j} - c_{i_{j+1}})p_{i_j}$$

for $2 \le t \le r$, as asserted. $\square$

In case $t = r$, the nonzero r-by-r principal minors of CC^T must contain one row of each content type, and we have the following special case of Lemma 11.

Corollary 12. *Under the hypothesis of the preceding lemma,*

$$E_r = c_r\prod_{i=1}^{r-1}(c_i - c_{i+1}) \cdot \prod_{j=1}^{r} p_j.$$

Observe that the disparity of a rank-2 matrix C is E_2. Now, we consider a matrix C of rank greater than 2 such that its row contents increase by 1 with each successive row.

Theorem 13. *Suppose that $C \in S_{r,q,k}$ satisfies $\operatorname{rank}(C) = r \ge 3$ and $r + q = n$, with i-th row of content $a - i$, where $a \in \mathbb{N}$ and $a \ge r + 1$. There exists $S \in S_{p',q',k}$, with $p' + q' \le n$, that is either a rank-1 matrix or a SARM, such that $\sigma_1(C) < \sigma_1(S)$.*

Proof. Suppose that $C \in S_{r,q,k}$ satisfies $\operatorname{rank}(C) = r \geq 3$ and $r + q = n$, with i-th row of content $a - i$, $a \in \mathbb{N}$ and $a \geq r + 1$. Let $b = a - r - 1$. Then,

$$k = (b+1) + (b+2) + \cdots + (b+r) = \frac{(2b+1+r)r}{2}.$$

If r is odd, then

$$k = \frac{(2b+1+r)}{2} \cdot r = k_1 \cdot k_2, \qquad k_1 + k_2 = \frac{(2b+1+r)}{2} + r = b + \frac{3}{2}r + \frac{1}{2} < 2r + b \leq n$$

and the k_1-by-k_2 rank-1 matrix can be constructed; thus there exists S such that $\sigma_1(C) < \sigma_1(S)$.

If r is even and $b \leq \frac{r}{2} - 1$, then $k = \frac{r}{2} \cdot (2b + 1 + r)$ and

$$k_1 + k_2 = \frac{r}{2} + (2b+1+r) = \frac{3}{2}r + 2b + 1 \leq 2r + b \leq n.$$

It follows that the k_1-by-k_2 rank-1 matrix can be constructed, and thus there exists S such that $\sigma_1(C) < \sigma_1(S)$.

Observe that for even r and $b > \frac{r}{2} - 1$ the rank-1 matrix may not exist.

Suppose that $k = rd + t$, with $t > 0$. We show that there exists a $(d+1)$-by-r SARM $S \in S_{p',q',k}$ with t 1's in the last row $(t < r)$, such that $\sigma_1(S) > \sigma_1(C)$. Since $d + 1 \leq p$ it is easy to see that $(d+1) + r \leq n$.

Let $B = B(S)$ be the bled matrix of S, i.e.

$$B = \begin{pmatrix} J_{dr} \\ \frac{t}{r} e_r^T \end{pmatrix}.$$

By Theorem 9, we have $\sigma_1(S) > \sigma_1(B)$. Moreover, since B is a rank-1 matrix, $\sigma_1(B) = \operatorname{tr}(BB^T)$ and

$$BB^T = \begin{pmatrix} rJ_{dd} & te_d \\ te_d^T & \frac{t^2}{r} \end{pmatrix}.$$

Since $k = rd + t$, we have $\operatorname{tr}(BB^T) = rd + \frac{t^2}{r} = k - \frac{t(r-t)}{r}$. Suppose, for purpose of contradiction, that

$$\lambda_{\max}(CC^T) \geq k - \frac{t(r-t)}{r} = \lambda_{\max}(BB^T). \tag{16}$$

Without loss of generality let C be normalized and left-justified (see Lemma 6). Since C is of rank r, $E_r = \lambda_1(CC^T) \cdot \ldots \cdot \lambda_r(CC^T)$ and $\operatorname{tr}(CC^T) = k$, and under the assumption (16) an upper bound for E_r is obtained when all the eigenvalues (except the largest one) are $\frac{t(r-t)}{r(r-1)}$, we have

$$E_r \leq \left(k - \frac{t(r-t)}{r}\right)\left(\frac{t(r-t)}{r(r-1)}\right)^{r-1}. \tag{17}$$

Consider the function $f(t) = \frac{t(r-t)}{r(r-1)}$, which appears in (17). It is easy to see that the function $f(t)$ achieves a maximum value $f_{\max}(\frac{r}{2}) = \frac{r}{4(r-1)}$ for $t = \frac{r}{2}$. Therefore, from inequality (17) it follows that

$$E_r \leq k\left(\frac{r}{4(r-1)}\right)^{r-1}. \tag{18}$$

Let $c_1, \ldots, c_r$ indexed in decreasing order be the contents of rows of C. By Corollary 12, since the product $\prod_{i=1}^{r-1}(c_i - c_{i+1})$ is an integer, we have

$$E_r = c_r \prod_{i=1}^{r-1}(c_i - c_{i+1}) \geq c_r = b + 1.$$

Now we prove that

$$b + 1 > \left(\frac{r}{4(r-1)}\right)^{r-1} k, \quad r \geq 3. \tag{19}$$

Since $k = \frac{(2b+1+r)r}{2}$, the above inequality is equivalent to

$$b\left(1 - r\left(\frac{r}{4(r-1)}\right)^{r-1}\right) + 1 > \frac{r(r+1)}{2}\left(\frac{r}{4(r-1)}\right)^{r-1}.$$

Under the assumption $b \geq \frac{r}{2} - 1$, it suffices to show that

$$\left(\frac{r}{4(r-1)}\right)^{r-1}(2r-1) < 1.$$

It is easy to see that

$$\left(\frac{r}{4(r-1)}\right)^{r-1}(2r-1) = \frac{2r-1}{4^{r-1}} \cdot \frac{1-\frac{1}{r}}{\left(1-\frac{1}{r}\right)^{r}}. \tag{20}$$

Let e be the base of the natural logarithm. Since $r \geq 3$, the right side of equality (20) is smaller than

$$\frac{(2r-1)}{e4^{r-1}}.$$

It is obvious that

$$\frac{(2r-1)}{e4^{r-1}} < 1$$

for $r \geq 3$. Hence,

$$E_r > \left(\frac{r}{4(r-1)}\right)^{r-1} k,$$

which contradicts (18), and we obtain the desired result. □

The next theorem applies to matrices $C \in S_{p,q,k}$ with "large" numbers of columns, i.e. at least $(2r-1)$ columns, when $r = \text{rank}(C)$. The matrices considered may have repeated row contents, but the contents must increase by more than 1 with each successive row. The number of rows or columns in the constructed SARM still depends on the rank of C.

Theorem 14. *Suppose that $C \in S_{p,q,k}$ satisfies $\text{rank}(C) = r \geq 3$ and $p + q = n$, with rows of contents $c_1 > \cdots > c_r$, such that $c_i - c_{i+1} > 1$ for $i = 1, \ldots, r-1$. Let p_i be the number of repetitions of rows of content c_i, $\sum_{i=1}^{r} p_i = p$. If there exists an index i such that $p_i \neq 1$, then suppose that m satisfies $c_m p_m = \max_i c_i p_i$ and that not all of: $m = 1$, $p_i = 1$ for $i > 1$, and $c_r = 1$ occur. Then there exists:*

(1) *a rank-1 matrix $S \in S_{p',q',k}$, with $p' + q' \leq n$, such that $\sigma_1(C) < \sigma_1(S)$*
or
(2) *if there exists s, such that $s + \lceil \frac{k}{s} \rceil \leq n$ and $s < \frac{4(r-1)}{\sqrt[r-1]{2r-1}}$, then there exists a SARM $S \in S_{b,s,k}$, with $b + s \leq n$, such that $\sigma_1(C) < \sigma_1(S)$.*

Proof. Suppose that $C \in S_{p,q,k}$ satisfies $\text{rank}(C) = r \geq 3$ and $p + q = n$, with rows of contents $c_1 > \cdots > c_r$, such that $c_i - c_{i+1} > 1$ for $i = 1, \ldots, r-1$. Let p_i be the number of repetitions of rows of content c_i, $\sum_{i=1}^{r} p_i = p$.

If the existence of a rank-1 matrix S is possible, than Lemma 4 implies $\sigma_1(C) < \sigma_1(S)$.

First observe that if $s = r$, then $s < \frac{4(r-1)}{\sqrt[r-1]{2r-1}}$, since it is equivalent to

$$\left(\frac{r}{4(r-1)}\right)^{r-1}(2r-1) < 1.$$

By the proof of preceding theorem,

$$\frac{(2r-1)}{e4^{r-1}} < 1$$

for $r \geq 3$, where e is the base of the natural logarithm.

If there exists an index i such that $p_i \neq 1$, then suppose there exist distinct s and r such that $s + \lceil \frac{k}{s} \rceil \leq n$ and $s < \frac{4(r-1)}{\sqrt[r-1]{2r-1}}$.

Suppose that $k = sd + t$ with $t > 0$. We show that there exists a $(d+1)$-by-s SARM $S \in S_{b,s,k}$ with t 1's in the last row ($t < s$), such that $\sigma_1(S) > \sigma_1(C)$. Observe that $s + (d+1) \leq n$, because $d + 1 = \lceil \frac{k}{s} \rceil$.

Since we may assume $s = r$, if $p_i = 1$ for $1 \leq i \leq r-1$, then the conditions for s in our theorem hold for any matrix and $(d+1) + r \leq n$ (where $d + 1 \leq p$), and $s < \frac{4(r-1)}{\sqrt[r-1]{2r-1}}$.

Let $B = B(S)$ be the bled matrix of S, i.e.

$$B = \begin{pmatrix} J_{ds} \\ \frac{t}{s} e_s^T \end{pmatrix}.$$

By Theorem 9, $\sigma_1(S) > \sigma_1(B)$. Moreover, since B has rank 1, $\sigma_1(B) = \operatorname{tr}(BB^T)$ and

$$BB^T = \begin{pmatrix} sJ_{dd} & te_d \\ te_d^T & \frac{t^2}{s} \end{pmatrix}.$$

Since $k = sd + t$, we have $\operatorname{tr}(BB^T) = sd + \frac{t^2}{s} = k - \frac{t(s-t)}{s}$. Suppose, for purpose of contradiction, that

$$\lambda_{\max}(CC^T) \geq k - \frac{t(s-t)}{s} = \lambda_{\max}(BB^T). \tag{21}$$

Without loss of generality let C be normalized and left-justified (see Lemma 6). Note that since C is of rank r, i.e. it has r nonzero eigenvalues, $E_r = \lambda_1(CC^T) \cdot \ldots \cdot \lambda_r(CC^T)$ and $\operatorname{tr}(CC^T) = k$. Under the assumption (21) an upper bound for E_r is obtained when all the eigenvalues (except the largest one) are $\frac{t(s-t)}{s(r-1)}$. Therefore

$$E_r \leq \left(k - \frac{t(s-t)}{s}\right)\left(\frac{t(s-t)}{s(r-1)}\right)^{r-1}. \tag{22}$$

Consider the function $f(t) = \frac{t(s-t)}{s(r-1)}$, which appears in (22). It is easy to see that $f(t)$ achieves a maximum value $f_{\max}(\frac{s}{2}) = \frac{s}{4(r-1)}$ for $t = \frac{s}{2}$. Therefore, from inequality (22) it follows that

$$E_r \leq k\left(\frac{s}{4(r-1)}\right)^{r-1}. \tag{23}$$

Let $c_1, \ldots, c_r$ indexed in decreasing order be contents of rows of C, p_i be the number of repetitions of rows of content c_i. Let $h = \frac{c_r p_r}{k}$, note that $0 < h < 1$.

First, suppose that

$$h > \left(\frac{s}{4(r-1)}\right)^{r-1}.$$

By Corollary 12, since the product $\prod_{i=1}^{r-1}(c_i - c_{i+1}) \cdot \prod_{j=1}^{r-1} p_j$ is an integer, we have

$$E_r = c_r \prod_{i=1}^{r-1}(c_i - c_{i+1}) \cdot \prod_{j=1}^{r} p_j \geq c_r p_r = hk > k\left(\frac{s}{4(r-1)}\right)^{r-1}.$$

Contradiction with (23) implies the claimed result.

Now, suppose that

$$h \leq \left(\frac{s}{4(r-1)}\right)^{r-1}$$

and $c_i - c_{i+1} > 1$ for $i = 1, \ldots, r-1$.

If there exists an index i such that $p_i \neq 1$, then suppose m is such that $c_m p_m = \max_i c_i p_i$ and not all of: $m = 1$, $p_i = 1$ for $i > 1$, and $c_r = 1$ occur. Observe that

$$\prod_{i=1}^{r-1}(c_i - c_{i+1}) \geq \sum_{i=1}^{r-1}(c_i - c_{i+1}) = c_1 - c_r$$

and hence

$$E_r = c_r \prod_{i=1}^{r-1}(c_i - c_{i+1}) \cdot \prod_{j=1}^{r} p_j \geq (c_1 - c_r)c_r \cdot \prod_{j=1}^{r} p_j \geq c_1 p_m \geq c_m p_m > c_m p_m - c_r p_r.$$

It is easy to see that $\sum_{i=1}^{r} c_i p_i = k$, which implies $\sum_{i=1}^{r-1} c_i p_i = (1-h)k$. Now, one of the possible choices of $c_i p_i$ is $c_i p_i = \frac{1-h}{r-1}k$ for $1 \leq i \leq r-1$. However, $c_m p_m$ must be greater than the remaining $c_i p_i$, i.e. $c_m p_m > \frac{1-h}{r-1}k$. Thus

$$c_m p_m - c_r p_r > \frac{1-h}{r-1}k - hk = \frac{1-hr}{r-1}k$$

and

$$c_m p_m - c_r p_r > \frac{1 - r\left(\frac{s}{4(r-1)}\right)^{r-1}}{r-1}k.$$

Now, we prove that

$$\frac{1 - r\left(\frac{s}{4(r-1)}\right)^{r-1}}{r-1} k > \left(\frac{s}{4(r-1)}\right)^{r-1} k, \quad r \geq 3.$$

Observe that the above inequality is equivalent to

$$\left(\frac{s}{4(r-1)}\right)^{r-1} (2r-1) < 1,$$

which holds for $r \geq 3$ by the assumption $s < \frac{4(r-1)}{\sqrt[r-1]{2r-1}}$ (or if $s = r$).

Hence,

$$E_r > \left(\frac{s}{4(r-1)}\right)^{r-1} k$$

which contradicts (23), and we conclude that $\sigma_1(C) < \sigma_1(S)$. □

Corollary 15. *Suppose that $C \in S_{r,q,k}$ satisfies $\operatorname{rank}(C) = r \geq 3$ and $p + q = n$, with rows of contents $c_1 > \cdots > c_r$ such that $c_i - c_{i+1} > 1$ for $i = 1, \ldots, r-1$. Then, there exists a rank-1 matrix $S \in S_{p',q',k}$ with $p' + q' \leq n$ or a SARM $S \in S_{b,r,k}$ with $b + r \leq n$, such that $\sigma_1(C) < \sigma_1(S)$.*

The above corollary applies to (0, 1)-matrices such that for any two rows, the row contents differ by at least 2. In Theorem 14 are also allowed matrices with a "small" number of repeated rows (or columns).

Let C be a matrix satisfying the hypothesis of Theorem 14. Let s be such that $s \neq r$, $s + \lceil \frac{k}{s} \rceil \leq n$, and $s < \frac{4(r-1)}{\sqrt[r-1]{2r-1}}$.

For $r = 4$ we can construct a SARM with $s \leq 6$ rows,
for $r = 5$ we can construct a SARM with $s \leq 9$ rows,
for $r = 100$ we can construct a SARM with $s \leq 375$ rows.

Thus, if C is a matrix of rank 100 and there exists s with $s \leq 375$ such that $s + \lceil \frac{k}{s} \rceil \leq n$, then there is a rank-1 matrix or a SARM whose largest singular value beats that of C. It implies that if C has no more than 375 rows, then there is a rank-1 matrix or a SARM with the largest singular value greater then the largest singular value of C.

We have generated many examples that show, for example, limitations of our methods. However, we are always able to produce a winning SARM, when there is no rank-1 matrix, consistent with our conjecture.

Let $n = 16$, $k = 49$, and

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since C is a matrix of rank 4, we first construct a SARM S with four rows or four column, having 49 nonzero positions, such that the sum of its numbers of rows and columns is smaller than or equal to 16. Observe that this is impossible, because to obtain 49 nonzero positions, the matrix S must be a 4-by-13 matrix, so that the sum of its rows and columns exceeds 16. Using Theorem 14, we construct a SARM S with five rows having 49 nonzero positions, and it can be of the following form

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Here the sum of the numbers of rows and columns is 15, which is smaller than n. A calculations yields:

$$6.5944 = \sigma_1(C) < \sigma_1(S) = 6.9465.$$

8. Graph-theoretic interpretation of the results

Here we describe the structure of the minimizing graphs for some particular bipartite graphs. We introduce the notion of a nearly complete bipartite graph and indicate when the graph that minimizes the smallest eigenvalue of its adjacency matrix is a complete bipartite graph or a nearly complete bipartite graph. To ensure the existence of a bipartite graph, we assume $k \leq \frac{n^2}{4}$.

Definition 2. A graph G is a *nearly complete bipartite graph* if it is obtained from a complete bipartite graph by deleting some edges incident to a single vertex.

Note that the degree of the special vertex is the number of 1's in the last row of a SARM $S \in S_{p,q,k}$; hence it is $k - q(p-1)$.

Note that all theorems from Sections 3–7 can be interpreted graph-theoretically in the following way. If $G \in \mathcal{B}_{n,k}$ and its adjacency matrix satisfies the conditions of one of the theorems from Sections 3–7, then there exists a complete bipartite graph or a nearly complete bipartite graph with the least eigenvalue of its adjacency matrix smaller than the least eigenvalue of the adjacency matrix of G. For example, our last theorem is immediate consequence of Corollary 15.

Theorem 16. *If G is an n-vertex bipartite graph with k edges, such that the degrees of the vertices in one partite set differ by more than 1, then there exists a complete bipartite graph or a nearly complete bipartite graph with the least eigenvalue of its adjacency matrix smaller than the least eigenvalue of the adjacency matrix of G.*

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Further reading

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