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Steady states and dynamics of an autocatalytic chemical reaction model with decay ☆

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ABSTRACT

The dynamics and steady state solutions of an autocatalytic chemical reaction model with decay in the catalyst are considered. Nonexistence and existence of nontrivial steady state solutions are shown by using energy estimates, upper–lower solution method, and bifurcation theory. The effects of decay order, decay rate and diffusion rates to the dynamical behavior are discussed.

1. Introduction

Autocatalytic chemical reactions have been identified as one of main nonlinear mechanisms in biochemical procedures [8,22]. Several canonical forms of the reactions have been proposed as cornerstones of more complicated chemical reaction chains or networks. Well-known examples include Lotka–Volterra model [19,34], Brusselator model [24], and Gray–Scott model [9].

A simplest autocatalytic chemical reaction is of the following form:

$$A + pB \to (p+1)B, \tag{1.1}$$

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where A is the reactant and B is the autocatalyst, and the integer $p \geqslant 1$ is the order of the reaction (number of autocatalyst molecules involved in a reaction). While the most common case is $p = 1$, the higher order reactions with $p \geqslant 2$ have been considered in recent years [8,9].

For a general $p \geqslant 2$, Jakab et al. [12] proposed a reaction–diffusion model based on the single autocatalytic reaction (1.1):

$$\begin{cases} \dfrac{\partial a}{\partial t} = D_A \Delta a - k_1 a b^p, & \dfrac{\partial b}{\partial t} = D_B \Delta b + k_1 a b^p, \\ \lim\limits_{|x| \to \infty} a(x,t) = a_0, \quad \text{and} & \lim\limits_{|x| \to \infty} b(x,t) = 0, \end{cases} \tag{1.2}$$

where $a(x,t)$ and $b(x,t)$ are the concentrations of the reactant and the autocatalyst respectively, D_A and D_B are diffusion coefficients of A and B respectively, and k_1 is the reaction rate. The boundary condition means that there is plenty supply of the reactant, but only limited supply of the catalyst. It was discovered that a threshold phenomenon exists for $n \geqslant 3$ that only when the initial concentration is over some critical mass, the chemical reaction wave can propagate in a form of traveling wave. The threshold phenomenon was rigorously proved in Shi and Wang [32] with a careful study of steady state solutions and comparison methods. It was shown that both the spatial dimension n and reaction order p play important roles in the threshold dynamics. The threshold set was proved to be "thin" so it is called hair-triggered instability following Aronson and Weinberger [1]. The dynamics of (1.2) with zero boundary condition was considered in [2,17], in which the self-similarity of the profile of the solution (approaching to zero) was studied. The traveling wave solution of (1.2) in one-dimensional space was studied by Chen and Qi [4–6].

In (1.2), the spatial domain is assumed to be the whole $\mathbb{R}^n$. This is an idealized assumption when the chemical reactor is sufficiently large or the boundary effect can be ignored. On the other hand, when the boundary effect of the reactor is included, one can assume the reactor Ω to be a bounded region in $\mathbb{R}^n$. The chemicals A and B can diffuse from a reservoir of constant composition across the boundary $\partial\Omega$ into Ω, thus the boundary conditions of A and B are assumed to be

$$a(x,t) = a_0, \quad \text{and} \quad b(x,t) = 0, \quad x \in \partial\Omega. \tag{1.3}$$

The typical geometry of the reactor Ω is spherical ($n = 3$, and $\Omega = B^3$), or cylindrical ($n = 2$, and $\Omega = B^2$). Hence it is reasonable to consider $\Omega = B^n$, the unit ball in $\mathbb{R}^n$. The bifurcation of steady state solutions of the equations in (1.2) on a ball domain with boundary condition (1.3) have been considered in Ouyang and Shi [25,26], and also in Zhao et al. [36] for the case of $b(x,t) = b_0 > 0$ small (see also [15]).

The non-constant steady state solutions found in [32] are unstable. But if there is a decay of the autocatalyst B, then the steady states may be stabilized. Hence in Jakab et al. [13] (see also [10]), an additional decay step was proposed as

$$B \to C, \tag{1.4}$$

and the decay rate is $k_2 b^q$ for some $q \geqslant 1$ and $k_2 > 0$. Thus the equations governing the reaction and diffusion of the reactants A and B are

$$\frac{\partial a}{\partial t} = D_A \Delta a - k_1 a b^p, \qquad \frac{\partial b}{\partial t} = D_B \Delta b + k_1 a b^p - k_2 b^q, \tag{1.5}$$

subject to the boundary conditions (1.3). In this paper we consider the case of Ω being a bounded domain in $\mathbb{R}^n$. Systems with similar form as (1.5) have arisen from many other applications, such as interface growth and pattern formation in bacterial colonies [20,21].

With a rescaling

$$\bar{a} = \frac{a}{a_0}, \qquad \bar{b} = \frac{b}{a_0}, \quad \text{and} \quad \bar{t} = D_A t,$$

and dropping the bars of the new variables for convenience, we obtain the dimensionless equations with boundary and initial conditions:

$$\begin{cases} \dfrac{\partial a}{\partial t} = \Delta a - \lambda a b^p, & x \in \Omega,\ t > 0, \\ \dfrac{\partial b}{\partial t} = D \Delta b + \lambda a b^p - k b^q, & x \in \Omega,\ t > 0, \\ a(x,t) = 1, \quad b(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ a(x,0) = a^0(x) \geqslant 0, \quad b(x,0) = b^0(x) \geqslant 0, & x \in \Omega, \end{cases} \tag{1.6}$$

where

$$D = \frac{D_B}{D_A}, \qquad \lambda = \frac{k_1 a_0^p}{D_A}, \quad \text{and} \quad k = \frac{k_2 a_0^{q-1}}{D_A}. \tag{1.7}$$

The steady state solution of (1.6) satisfies

$$\begin{cases} \Delta a - \lambda a b^p = 0, & x \in \Omega, \\ D \Delta b + \lambda a b^p - k b^q = 0, & x \in \Omega, \\ a(x) = 1, \quad b(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.8}$$

Because of the nonhomogeneous boundary condition of $a(x,t)$, sometimes it is more convenient to consider the equivalent problems with homogeneous boundary condition: let $u(x,t) = 1 - a(x,t)$, $v(x,t) = b(x,t)$, then (u,v) satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \lambda (1-u) v^p, & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = D \Delta v + \lambda (1-u) v^p - k v^q, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, \quad v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u^0(x) \equiv 1 - a^0(x), \quad v(x,0) = v^0(x) \equiv b^0(x) \geqslant 0, & x \in \Omega, \end{cases} \tag{1.9}$$

and the corresponding steady state equation is

$$\begin{cases} \Delta u + \lambda (1-u) v^p = 0, & x \in \Omega, \\ D \Delta v + \lambda (1-u) v^p - k v^q = 0, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.10}$$

In this paper we consider the basic dynamics of (1.9) and the steady state problem (1.10) with the assumption $p, q > 1$. The behavior of the chemical reaction system can be classified by two cases:

$$q > p: \quad \text{strong decay case}; \quad \text{and} \quad q < p: \quad \text{weak decay case}.$$

In both cases we show that the positive solution of (1.9) exists globally, and all steady state solutions are bounded (see results in Section 2). In the strong decay case, for fixed reaction order p and decay order q, we prove that the asymptotical dynamical behavior of (1.9) depends on the quantity

$r = \lambda^{q-1}k^{1-p}D^{p-q}$. When r is small, then all solutions converge to the trivial steady state $(0,0)$, hence no pattern formation is possible, and the chemical conversion from A to B is not successful (see Theorem 2.4). This is different from the system (1.2) (or the case $k = 0$ or equivalently $r = \infty$). For (1.2), it has been shown that a bistable dynamics always exists, and the chemical reaction can successfully reach a positive steady state when the initial value of the catalyst B is large enough, see [14,32]. Here we show that a strong decay order q and a large decay rate k can inhibit the generation of chemical B, so that the autocatalytic chemical reaction cannot proceed. The same effect can be achieved if λ is small or D is large. From the change of variables (1.7), small λ can be realized with a small reaction rate k_1, and large D is possible if the catalyst B diffuses much faster than A.

On the other hand, when r is large, then some nontrivial steady state solutions of (1.9) exist. We prove the existence of positive steady state solutions with two different approaches: (i) assuming strong decay $q > p$, we prove the existence of a positive steady state solution when D is small by using upper–lower solution method for the quasimonotone system (see Theorem 3.4); (ii) for both strong and weak decay cases, but with domain being a large ball, we prove the existence of two positive steady state solutions for small k by using implicit function theorem, and we also show that these two solutions indeed belong to the same connected component of the solution set in the space of (k, u, v) by using a global bifurcation method (see Theorem 4.4).

Our results here are the first rigorous ones for the autocatalytic reaction system with decay. There are many open questions about (1.5) remain to be study in the future:

1. Existence and uniqueness of steady state solution for the whole space $\Omega = \mathbb{R}^n$;
2. Existence and multiplicity of positive steady state solution in the weak decay case;
3. Existence and wave speed of traveling wave solution.

Note that all these questions have been answered positively when there is no decay ($k = 0$). However the decay case appears to be much more difficult. Solving these questions will help us to have a better understanding of the pattern formation mechanism of (1.5) shown in [20,21].

Throughout the paper, for a bounded domain Ω, we use $\|u\|_p$ to denote the L^p norm of a function $u \in L^p(\Omega)$ where $1 \leqslant p \leqslant \infty$, and we use $|\Omega|$ to denote the Lebesgue measure of Ω; and we denote (λ_1, ϕ_1) to be the principal eigen-pair of

$$-\Delta\phi = \lambda\phi, \quad x \in \Omega, \qquad \phi(x) = 0, \quad x \in \partial\Omega, \tag{1.11}$$

such that $\phi_1(x) > 0$ in Ω and $\|\phi_1\|_\infty = 1$.

2. *A priori* estimates and nonexistence

In this section, we first establish some *a priori* estimates for the steady state solutions of (1.6), and we also prove some nonexistence results for certain parameter ranges.

Lemma 2.1. *Suppose the parameters $p, q, k, D > 0$. If $(a(x), b(x))$ is a positive solution of* (1.8)*, then $a(x) + Db(x) < 1$ for any $x \in \Omega$. In particular,*

$$0 < a(x) < 1, \qquad 0 < b(x) < \frac{1}{D}, \quad x \in \Omega. \tag{2.1}$$

If in addition $q > p$, then

$$0 < b(x) < \min\left\{\left(\frac{\lambda}{k}\right)^{\frac{1}{q-p}}, \frac{1}{D}\right\}, \quad x \in \Omega. \tag{2.2}$$

Proof. By adding the two equations in (1.8), we obtain

$$\begin{cases} \Delta\big(a(x)+Db(x)\big)=kb^q(x), & x\in\Omega, \\ a(x)+Db(x)=1, & x\in\partial\Omega. \end{cases} \tag{2.3}$$

From the strong maximum principle of elliptic equations, we have $a(x)+Db(x)<1$ and (2.1), for any $x\in\Omega$. On the other hand, from the equation of $b(x)$ in (1.8) and the fact $a(x)<1$, we have

$$0=D\Delta b+\lambda ab^p-kb^q\leqslant D\Delta b+\lambda b^p-kb^q. \tag{2.4}$$

If $b(x_0)=\max_{x\in\overline{\Omega}} b(x)$ for some $x_0\in\Omega$, then from (2.4) we get $\lambda b^p(x_0)-kb^q(x_0)>0$. Hence if $q>p$, then $b(x)<(\frac{\lambda}{k})^{1/(q-p)}$, $x\in\Omega$. From the strong maximum principle, we actually obtain that $b(x)<\min\{(\frac{\lambda}{k})^{1/(q-p)},\frac{1}{D}\}$, $x\in\Omega$. □

From Lemma 2.1, we also have the estimates for solution (u,v) of (1.10):

$$Dv(x)<u(x)<1,\quad 0<v(x)<D^{-1},\ x\in\Omega. \tag{2.5}$$

Now from (1.10) and (2.5), we can obtain some energy estimates for the positive solutions of (1.8).

Lemma 2.2. *For any positive solution* $(a(x),b(x))$ *of* (1.8), *there hold the estimates*:

(i) $\|\nabla a\|_2\leqslant\lambda D^{-p}\lambda_1^{-1/2}\|a\|_2\leqslant\lambda D^{-p}\lambda_1^{-1/2}|\Omega|^{1/2}$,
(ii) $0<\|\nabla a\|_2-D\|\nabla b\|_2\leqslant kD^{-q}\lambda_1^{-1/2}|\Omega|^{1/2}$,

where λ_1 *is defined as in* (1.11).

Proof. Multiplying the first equation in (1.10) with u and integrating over Ω, we get

$$\begin{aligned}\int_\Omega|\nabla u|^2=\int_\Omega\lambda(1-u)uv^p&\leqslant\lambda D^{-p}\int_\Omega(1-u)u\\ &\leqslant\lambda D^{-p}\bigg[\int_\Omega(1-u)^2\bigg]^{1/2}\cdot\bigg(\int_\Omega u^2\bigg)^{1/2}\\ &\leqslant\lambda D^{-p}\bigg[\int_\Omega(1-u)^2\bigg]^{1/2}\cdot\lambda_1^{-1/2}\bigg(\int_\Omega|\nabla u|^2\bigg)^{1/2},\end{aligned}$$

by the Cauchy–Schwartz inequality and the Poincaré inequality. That is to say

$$\bigg(\int_\Omega|\nabla u|^2\bigg)^{1/2}\leqslant\lambda D^{-p}\lambda_1^{-1/2}\cdot\bigg[\int_\Omega(1-u)^2\bigg]^{1/2}.$$

Noted that $u=1-a$, and $\nabla u=-\nabla a$, we also get

$$\bigg(\int_\Omega|\nabla a|^2\bigg)^{1/2}\leqslant\lambda D^{-p}\lambda_1^{-1/2}\cdot\bigg(\int_\Omega a^2\bigg)^{1/2},$$

which finishes the proof of (i), together with Lemma 2.1.

To prove (ii) we define $w = u - Dv$, then from (1.10) we have

$$-\Delta w = -\Delta(u - Dv) = kv^q. \tag{2.6}$$

Multiplying (2.6) with w and integrating over Ω, we get

$$\int_\Omega |\nabla w|^2 = k \int_\Omega v^q w,$$

which yields

$$\int_\Omega |\nabla u|^2 - D^2 \int_\Omega |\nabla v|^2 - 2Dk \int_\Omega v^{q+1} = k \int_\Omega v^q u - Dk \int_\Omega v^{q+1},$$

i.e.

$$\int_\Omega |\nabla u|^2 - D^2 \int_\Omega |\nabla v|^2 = Dk \int_\Omega v^q v + k \int_\Omega v^q u. \tag{2.7}$$

By the Hölder inequality, we also get

$$\int_\Omega v^q v \leqslant \left[\int_\Omega v^{2q}\right]^{1/2} \cdot \left[\int_\Omega v^2\right]^{1/2} \leqslant D^{-q}|\Omega|^{1/2} \cdot \left(\frac{\int_\Omega |\nabla v|^2}{\lambda_1}\right)^{1/2},$$

$$\int_\Omega v^q u \leqslant \left[\int_\Omega v^{2q}\right]^{1/2} \cdot \left[\int_\Omega u^2\right]^{1/2} \leqslant D^{-q}|\Omega|^{1/2} \cdot \left(\frac{\int_\Omega |\nabla u|^2}{\lambda_1}\right)^{1/2}.$$

Hence (2.7) becomes

$$\|\nabla u\|_2^2 - D^2\|\nabla v\|_2^2 \leqslant kD^{-q}|\Omega|^{\frac{1}{2}}\lambda_1^{-\frac{1}{2}} \cdot \big(\|\nabla u\|_2 + D\|\nabla v\|_2\big),$$

i.e.

$$\|\nabla u\|_2 - D\|\nabla v\|_2 \leqslant kD^{-q}|\Omega|^{1/2}\lambda_1^{-1/2}. \tag{2.8}$$

Then (ii) follows from (2.7) and (2.8). □

From the estimates given in Lemma 2.2, we easily obtain the following energy bounds of all positive steady state solutions $(a(x), b(x))$:

Corollary 2.3. *Suppose the parameters $p, q > 1$ and $k, \lambda, D > 0$, and let λ_1 be the principal eigenvalue of $-\Delta$ defined in* (1.11)*. Then for any positive solution $(a(x), b(x))$ of* (1.8)*,*

$$\|a\|_{H^1(\Omega)} + D\|b\|_{H^1(\Omega)} \leqslant \big(\lambda D^{-p}\lambda_1^{-1/2} + 1\big)|\Omega|^{1/2}. \tag{2.9}$$

In particular, the bound is independent of q and k.

Next we consider the parabolic system (1.9), and we have the following *a priori* estimates.

Theorem 2.4. *Suppose the parameters* $p, q > 1$ *and* $k, \lambda, D > 0$. *Let* $(u(x,t), v(x,t))$ *be a positive solution of the system* (1.9).

1. *The solution* $(u(x,t), v(x,t))$ *exists for all* $t \geqslant 0$, *and the solution is bounded in* $L^\infty(\Omega)$. *Moreover if* $q > p$, *then the bound can be expressed explicitly*:

$$\limsup_{t\to\infty} u(x,t) \leqslant 1, \qquad \limsup_{t\to\infty} v(x,t) \leqslant \left(\frac{\lambda}{k}\right)^{1/(q-p)}. \tag{2.10}$$

2. *If* $q > p$, *and* p, q, λ, D, k *satisfy*

$$(q-p)\left(\frac{\lambda}{q-1}\right)^{\frac{q-1}{q-p}}\left(\frac{p-1}{k}\right)^{\frac{p-1}{q-p}} < D\lambda_1, \tag{2.11}$$

then $(0,0)$ *is globally asymptotically stable for* (1.9), i.e. *for any nonnegative initial condition* $(u^0(x), v^0(x)) \not\equiv (0,0)$, *we have*

$$\lim_{t\to\infty}\big(u(x,t), v(x,t)\big) = (0,0), \quad \textit{uniformly for } x \in \overline{\Omega}. \tag{2.12}$$

3. *There exists* $m = m(D\lambda, D) > 0$ *such that for initial value* (u^0, v^0) *satisfying* $\|v^0\|_\infty < m$, (2.12) *holds. Hence* $(0,0)$ *is locally asymptotically stable for any parameters* $p, q > 1$ *and* $k, \lambda, D > 0$.

Proof. 1. The global existence and boundedness follow from Hollis, Martin and Pierre [11, Theorems 1 and 2]. For the case of $q > p$, first we consider an auxiliary problem:

$$\begin{cases} \dfrac{\partial v}{\partial t} = D\Delta v + \lambda v^p - kv^q, & x \in \Omega, \\ v(x,t) = 0, & x \in \partial\Omega, \\ v(x,0) = v^0(x) \geqslant 0. \end{cases} \tag{2.13}$$

Define

$$M = \left(\frac{\lambda}{k}\right)^{1/(q-p)}.$$

Then it is well known that the solution $v_1(x,t)$ of (2.13) satisfies $\limsup_{t\to\infty} v_1(x,t) \leqslant M$ as the function $g(v) = \lambda v^p - kv^q$ is negative for $v > M$. From the comparison principle of parabolic equations, $v(x,t) \leqslant v_1(x,t)$ for all $t \geqslant 0$, hence the bound for v holds. Now $u(x,t)$ satisfies $u_t \leqslant \Delta u + \lambda(M+\epsilon)^p(1-u)$ for large enough t, then we also obtain the estimate for $u(x,t)$, since any solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \lambda(M+\epsilon)^p(1-u), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, \\ u(x,0) = u^0(x) \geqslant 0, \end{cases} \tag{2.14}$$

is bounded by 1 eventually.

2. Assume that $q > p$. Multiplying the second equation of (1.9) with v and integrating over Ω, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 &= -D\int_\Omega |\nabla v|^2 + \lambda\int_\Omega v^{p+1} - k\int_\Omega v^{q+1}\\ &\leqslant -D\lambda_1\int_\Omega v^2 + \lambda\int_\Omega v^{p+1} - k\int_\Omega v^{q+1}\\ &= -D\lambda_1\int_\Omega v^2 + \int_\Omega g(v)v, \end{aligned}\tag{2.15}$$

where λ_1 is the principal eigenvalue of $-\Delta$ defined in (1.11), and $g(v) = \lambda v^p - kv^q$. Then from a simple calculation, we see that

$$g(v) \leqslant g'(A)v, \quad v \in [0,\infty), \tag{2.16}$$

where A is the unique solution of $g(v) - g'(v)v = 0$. In fact, from

$$\left[\frac{g(v)}{v}\right]' = \frac{g(v) - g'(v)v}{v^2} = \lambda(p-1)v^{p-2} - k(q-1)v^{q-2},$$

hence

$$g(v) - g'(v)v = \lambda(p-1)v^p - k(q-1)v^q.$$

Since $q > p > 0$, we can compute that

$$\left[\frac{g(v)}{v}\right]' > 0 \quad \text{for } 0 < v < A, \qquad \left[\frac{g(v)}{v}\right]' < 0 \quad \text{for } v > A,$$

and $A = [\frac{\lambda(p-1)}{k(q-1)}]^{\frac{1}{q-p}}$. Hence (2.16) holds. One can easily compute that

$$g'(A) = (q-p)\left(\frac{\lambda}{q-1}\right)^{\frac{q-1}{q-p}}\left(\frac{p-1}{k}\right)^{\frac{p-1}{q-p}}. \tag{2.17}$$

Therefore, combining (2.16) and (2.17), the inequality (2.15) is reduced to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 \leqslant \left[(q-p)\left(\frac{\lambda}{q-1}\right)^{\frac{q-1}{q-p}}\left(\frac{p-1}{k}\right)^{\frac{p-1}{q-p}} - D\lambda_1\right]\int_\Omega v^2. \tag{2.18}$$

Hence if (2.11) holds, then

$$\lim_{t\to\infty}\int_\Omega v^2 = 0,$$

which implies $v(x,t) \to 0$ almost everywhere for $x \in \Omega$. Combining the equation of u in (1.9) and the convergence of v, we obtain the convergence of u as well.

3. It is well known (see [23] or [28, Theorem 19.2]) that for the Fujita type equation

$$\begin{cases} \dfrac{\partial v}{\partial t} = D\Delta v + \lambda v^p, & x \in \Omega, \\ v(x,t) = 0, & x \in \partial\Omega, \\ v(x,0) = v^0(x) \geqslant 0, \end{cases} \tag{2.19}$$

the solution $v(x,t) \to 0$ uniformly for $x \in \overline{\Omega}$ if $\|v^0\|_\infty$ is small enough. Then the result for (1.9) follows easily from the comparison principle. □

When (2.11) is satisfied, (1.9) only has the trivial steady state, and no any spatial patterns can be generated in that case. One can easily see that for fixed $q > p > 0$, the condition (2.11) holds, if D is large, or λ is small, or k is large. For later application, we state the following nonexistence results for the steady state solutions of (1.6) or (1.9).

Corollary 2.5. *Suppose the parameters satisfy $p, q > 1$ and $k, \lambda, D > 0$, and let λ_1 be the principal eigenvalue of $-\Delta$ defined in* (1.11)*.*

1. *If $p \geqslant q$, then Eq.* (1.8) *or* (1.10) *has no positive solution for $k > \lambda D^{q-p}$.*
2. *If $p < q$, then* (1.8) *or* (1.10) *has no positive solution for $k > k_1$, where*

$$k_1 \equiv k_1(p,q,D,\Omega) = (p-1)\left(\frac{\lambda}{q-1}\right)^{(q-1)/(p-1)}\left(\frac{q-p}{D\lambda_1}\right)^{(q-p)/(p-1)}. \tag{2.20}$$

Proof. We only need to prove part 1 as part 2 has been proved in Theorem 2.4 part 2. Integrating the second equation of (1.10), we obtain

$$D\int_\Omega |\nabla v|^2 + k\int_\Omega v^{q+1} = \lambda\int_\Omega (1-u)v^{p+1}. \tag{2.21}$$

Hence from (2.21) and (2.5),

$$\begin{aligned} k\int_\Omega v^{q+1} &\leqslant \lambda\int_\Omega v^{p+1} = \lambda\int_\Omega v^{p-q}\cdot v^{q+1} \\ &\leqslant \lambda D^{q-p}\int_\Omega v^{q+1}, \end{aligned}$$

which implies that

$$k \leqslant \lambda D^{q-p}. \qquad \square$$

3. Existence by upper–lower solution methods

In this section we use comparison method to establish the existence of positive steady state solutions of (1.9). For that purpose, we recall the following setup (see [16,27]): consider a semilinear elliptic system

$$\begin{cases} d_1\Delta u + f(u,v) = 0, & x \in \Omega, \\ d_2\Delta v + g(u,v) = 0, & x \in \Omega, \\ u(x) = v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

where $f, g : O \to \mathbb{R}$ are continuously differentiable, and $O \subseteq \overline{\mathbb{R}^2_+}$ is an open subset. The vector field $(f, g) : O \to \mathbb{R}^2$ is called mixed quasimonotone in O if

$$\frac{\partial f}{\partial v}(u,v) > 0, \qquad \frac{\partial g}{\partial u}(u,v) < 0, \quad (u,v) \in O. \tag{3.2}$$

Suppose (f, g) is mixed quasimonotone in O, then the two pairs of functions $\overline{U}(x) = (\bar{u}(x), \bar{v}(x))$ and $\underline{U}(x) = (\underline{u}(x), \underline{v}(x))$ are called an ordered upper–lower solution pair of (3.1) if they satisfy

$$\begin{cases} d_1\Delta\bar{u} + f(\bar{u},\bar{v}) \leqslant 0, & x \in \Omega, \\ d_2\Delta\bar{v} + g(\underline{u},\bar{v}) \leqslant 0, & x \in \Omega, \\ d_1\Delta\underline{u} + f(\underline{u},\underline{v}) \geqslant 0, & x \in \Omega, \\ d_2\Delta\underline{v} + g(\bar{u},\underline{v}) \geqslant 0, & x \in \Omega, \\ \bar{u}(x) \geqslant \underline{u}(x), \quad \bar{v}(x) \geqslant \underline{v}(x), & x \in \Omega, \\ \bar{u}(x) \geqslant 0 = \underline{u}(x), \quad \bar{v}(x) \geqslant 0 = \underline{v}(x), & x \in \partial\Omega. \end{cases} \tag{3.3}$$

With the existence of an ordered upper–lower solution pair, the existence of a positive steady state of (3.1) is the consequence of an iteration process (see [16,27]):

Theorem 3.1. *Suppose that O is an open subset of $\overline{\mathbb{R}^2_+}$, (f, g) is mixed quasimonotone in O, and f, g are Lipschitz continuous in O. If there exist an ordered upper–lower solution pair $\overline{U}(x) = (\bar{u}(x), \bar{v}(x))$ and $\underline{U}(x) = (\underline{u}(x), \underline{v}(x))$, then there exists a positive solution $(u(x), v(x))$ of (3.1) satisfying*

$$\bar{u}(x) \geqslant u(x) \geqslant \underline{u}(x), \qquad \bar{v}(x) \geqslant v(x) \geqslant \underline{v}(x), \quad x \in \Omega. \tag{3.4}$$

It is easy to see that for $O = [0,1) \times [0,\infty) \subseteq \overline{\mathbb{R}^2_+}$, the vector field $(f(u,v), g(u,v)) = (\lambda(1-u)v^p, \lambda(1-u)v^p - kv^q)$ is mixed quasimonotone in O. Hence to apply the upper–lower solution method in Theorem 3.1, we next construct an ordered upper–lower solution pair for (1.10). To achieve this goal, we first consider some auxiliary equations, and we prove the following two lemmas:

Lemma 3.2. *Consider*

$$\begin{cases} \Delta u + \lambda\varphi^p(1-u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.5}$$

where $\varphi \in \mathcal{C}_1 = \{\omega \in C(\overline{\Omega}) \mid \omega(x) \geqslant 0,\ x \in \overline{\Omega}\}$. Then (3.5) has a unique positive solution u_φ for any $\varphi \in \mathcal{C}_1$ and $\varphi \not\equiv 0$. Moreover,

(i) *$0 < u_\varphi(x) < 1$, for $x \in \Omega$;*
(ii) *u_φ is increasing with respect to $\varphi \in \mathcal{C}_1$, in the sense that, if $\varphi_1, \varphi_2 \in \mathcal{C}_1$, and $\varphi_1(x) \geqslant \varphi_2(x)$ for $x \in \Omega$, then $u_{\varphi_1}(x) \geqslant u_{\varphi_2}(x)$.*

Proof. We prove the existence of positive solution to (3.5) by using the upper–lower solution method. It is easy to see that a constant $\underline{u}(x) \equiv 1$ is an upper solution, and $\underline{u}(x) \equiv 0$ is a lower solution for (3.5). Hence, (3.5) has a positive solution u_φ which satisfies $0 \leqslant u_\varphi(x) < 1$ for $x \in \Omega$ and any $\varphi \in \mathcal{C}_1$. If $\varphi \not\equiv 0$, then $0 < u_\varphi(x) < 1$, for $x \in \Omega$ from the strong maximum principle. The uniqueness of u_φ and the increasing property in (ii) follow from Theorem 2 in [31]. □

Lemma 3.3. *Consider*

$$\begin{cases} D\Delta v + \lambda(1-\psi)v^p - kv^q = 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases} \tag{3.6}$$

where $\psi \in \mathcal{C}_2 = \{\omega \in C(\overline{\Omega}) \mid 0 \leqslant \omega(x) \leqslant 1,\ x \in \overline{\Omega}\}$. *Suppose that* $1 < p < q$, *and* $\psi \in \mathcal{C}_2$, $\psi \not\equiv 1$. *Then for any* $\lambda > 0$, $k > 0$, *there exists some constant* $D_0 > 0$ *such that for* $0 < D < D_0$, *Eq.* (3.6) *has two positive solutions. Moreover,*

(i) *for* $0 < D < D_0$, (3.6) *has a maximal solution* $\widetilde{v}_{\psi,D}$, i.e. *for any positive solution* v_D *of* (3.6),

$$\widetilde{v}_{\psi,D}(x) \geqslant v_D(x), \quad x \in \Omega;$$

(ii) $\widetilde{v}_{\psi,D}$ *is strictly decreasing in* D;
(iii) *for* $x \in \Omega$, $\widetilde{v}_{\psi,D}(x) \leqslant (\frac{\lambda}{k})^{\frac{1}{q-p}}$;
(iv) $\widetilde{v}_{\psi,D}$ *is decreasing with respect to* $\psi \in \mathcal{C}_2$.

Proof. We prove the existence of positive solutions by using variational methods. Setting

$$X = H_0^1(\Omega), \qquad \|v\| = \left(\int_\Omega |\nabla v|^2\,dx\right)^{1/2},$$

and the energy functional $I : X \times \mathcal{C}_2 \to \mathbb{R}$ is defined by

$$I(v,\psi) = \frac{D}{2}\int_\Omega |\nabla v|^2 + \int_\Omega h(v,\psi),$$

where

$$h(v,\psi) = \begin{cases} \frac{k}{q+1}v^{q+1} - \frac{\lambda}{p+1}(1-\psi)v^{p+1}, & v \geqslant 0, \\ 0, & v < 0. \end{cases}$$

For a fixed $v \in X$ and $\psi \in \mathcal{C}_2$, define

$$\Omega_+ = \bigl\{x \in \Omega\colon h\bigl(v(x),\psi(x)\bigr) \geqslant 0\bigr\}, \qquad \Omega_- = \bigl\{x \in \Omega\colon h\bigl(v(x),\psi(x)\bigr) < 0\bigr\}.$$

Then

$$\begin{aligned} I(v,\psi) &= \frac{D}{2}\|v\|^2 + \int_{\Omega_-} h(v,\psi) + \int_{\Omega_+} h(v,\psi) \\ &\geqslant \frac{D}{2}\|v\|^2 + \int_{\Omega_-} h(v,\psi). \end{aligned} \tag{3.7}$$

For any $x \in \Omega_-$, we observe that

$$h(v,0) = \frac{k}{q+1}v^{q+1} - \frac{\lambda}{p+1}v^{p+1} \leqslant \frac{k}{q+1}v^{q+1} - \frac{\lambda}{p+1}(1-\psi)v^{p+1} < 0.$$

Hence for any $x \in \Omega_-$,

$$0 \leqslant v(x) < \left[\frac{\lambda(q+1)}{k(p+1)}\right]^{\frac{1}{q-p}} \equiv v_*,$$

where v_* is unique positive zero of $h(v, 0)$. We can choose constants $M_1 > 0$ and $1 < s < \min\{p, \frac{n+2}{n-2}\}$ so that for all $v \in [0, v_*]$,

$$\left|\frac{k}{q+1} v^{q+1} - \frac{\lambda}{p+1}(1-\psi)v^{p+1}\right| \leqslant M_1 v^{s+1}.$$

Therefore, (3.7) and the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{s+1}(\Omega)$ imply that

$$\begin{aligned} I(v, \psi) &\geqslant \frac{D}{2}\|v\|^2 - M_1 \int_\Omega v^{s+1} \\ &\geqslant \frac{D}{2}\|v\|^2 - M_1 C_1 \|v\|^{s+1}, \end{aligned}$$

where $C_1 > 0$ is the embedding constant. Since $s > 1$, we can choose constants $r > 0$, $\rho > 0$ such that

$$\begin{aligned} &I(v, \psi) > 0 \quad \text{for any } v \in B_r(0)\backslash\{0\}, \quad \text{and} \\ &I(v, \psi) \geqslant \rho \quad \text{for any } v \in \partial B_r(0), \end{aligned} \tag{3.8}$$

where $B_r(0) \equiv \{u \in X\colon\ \|u\| < r\}$.

Next we fix an $e \in X$ with $e(x) > 0$ for $x \in \Omega$, then for $t \in \mathbb{R}^+$,

$$\begin{aligned} I(te, \psi) &= \frac{D}{2} t^2 \|e\|^2 + \frac{k}{q+1} t^{q+1} \int_\Omega e^{q+1} - \frac{\lambda}{p+1} t^{p+1} \int_\Omega (1-\psi) e^{p+1} \\ &= t^2 H(t), \end{aligned}$$

where

$$H(t) = \frac{D}{2}\|e\|^2 + \frac{k}{q+1} t^{q-1} \int_\Omega e^{q+1} - \frac{\lambda}{p+1} t^{p-1} \int_\Omega (1-\psi) e^{p+1}.$$

We can calculate that

$$\begin{aligned} H(0) &= \frac{D}{2}\|e\|^2 > 0, \\ H'(t) &= \frac{k(q-1)}{q+1} t^{q-2} \int_\Omega e^{q+1} - \frac{\lambda(p-1)}{p+1} t^{p-2} \int_\Omega (1-\psi) e^{p+1}. \end{aligned}$$

Define $t_* > 0$ such that $H'(t_*) = 0$, then

$$t_* = \left[\frac{\lambda(p-1)(q+1)}{k(p+1)(q-1)} \frac{\int_\Omega (1-\psi) e^{p+1}}{\int_\Omega e^{q+1}}\right]^{\frac{1}{q-p}},$$

and we have $H'(t) < 0$ for $0 < t < t_*$ and $H'(t) > 0$ for $t > t_*$.

From computation, the inequality $H(t_*) < 0$ is equivalent to

$$D < \frac{2(q-p)k}{(p-1)(q+1)} t_*^{q-1} \int\limits_{\Omega} e^{q+1}$$
$$= 2(q-p)\left[\frac{\lambda \int_{\Omega}(1-\psi)e^{p+1}}{(p+1)(q-1)}\right]^{\frac{q-1}{q-p}} \left[\frac{(p-1)(q+1)}{k\int_{\Omega} e^{q+1}}\right]^{\frac{p-1}{q-p}}$$
$$\equiv D_1.$$

This means that there exists some constant $D_1 > 0$ such that $H(t_*) < 0$ holds only for $0 < D < D_1$. On the other hand, $H(t_*) < 0$ also means that there exists an element $t_* e \in X$ such that

$$I(t_* e, \psi) < 0. \tag{3.9}$$

So when $0 < D < D_1$, the functional I satisfies the Mountain Pass geometry.

2. Now we prove any (PS) sequence is bounded. Take a sequence $\{v_n\}$ satisfying

$$I(v_n, \psi) \to 0, \qquad I_v(v_n, \psi) \to 0, \quad \text{as } n \to \infty,$$

where I_v is the Fréchet derivative of I with respect to v. Then as $n \to \infty$,

$$0 \leftarrow I(v_n, \psi) \geqslant \frac{D}{2}\|v_n\|^2 - \frac{\lambda}{p+1}\|v_n\|_{p+1}^{p+1} + \frac{k}{q+1}\|v_n\|_{q+1}^{q+1} = \frac{D}{2}\|v_n\|^2 + K(v_n),$$

and here for any $v \in X$, the function K is defined as

$$K(v) = \frac{k}{q+1}\|v\|_{q+1}^{q+1} - \frac{\lambda}{p+1}\|v\|_{p+1}^{p+1}.$$

If $\{v_n\}$ is unbounded, then $K(v_n) \to -\infty$. But the assumption of $q > p$ leads to a contradiction. So, $\{v_n\}$ must be bounded. Since X is reflexive, from a standard proof we can get that $\{v_n\}$ has a convergent subsequence.

Next we prove I is bounded from bellow and $\inf_{v \in X} I(v, \psi) < 0$. Noticed that there holds the inequality (3.9) in step 1, we next only focus on the boundedness from bellow. In fact, by using (3.7) and $0 \leqslant v(x) \leqslant v_*$, $x \in \Omega_-$, we obtain that for any $v \in X$,

$$I(v, \psi) \geqslant \int\limits_{\Omega_-} h(v, 0)$$
$$\geqslant \int\limits_{\Omega} -\frac{\lambda}{p+1} v_*^{p+1}$$
$$= -\left(\frac{\lambda}{p+1}\right)^{\frac{q+1}{q-p}} \left(\frac{q+1}{k}\right)^{\frac{p+1}{q-p}} |\Omega|$$
$$\equiv -M_2|\Omega|.$$

So, I is bounded from bellow, and $\inf I(v, \psi)$ is a critical value. Combining parts 1 and 2, and by using the well-known minimization theorem and the Mountain Pass Lemma [30], (3.6) has two distinct positive solutions.

3. At last, we turn to the proof of the existence of the maximal solutions. Let $\mu = D^{-1}$ and rewrite (3.6) to be

$$\begin{cases} \Delta v + \mu v f(v) = 0, & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases} \tag{3.10}$$

where $f(v) = \lambda(1-\psi)v^{p-1} - kv^{q-1}$. Then we can compute that

$$f'(v) = \lambda(p-1)(1-\psi)v^{p-2} - k(q-1)v^{q-2}.$$

Setting $v_1 = \sqrt[q-p]{\frac{\lambda(p-1)}{k(q-1)}(1-\psi)}$, then $f(v)$ is increasing on $[0, v_1]$, and decreasing on $[v_1, \infty)$, *i.e.* f behaves as a function of weak Allee effect type (see [31]). Then by using the same proof as in Theorem 3 in [31], there exists $\mu_* > 0$ such that (3.6) has a maximal solution $\widetilde{v}_{\psi,D}$ when $\mu \geqslant \mu_*$, and $\widetilde{v}_{\psi,D}$ is increasing in μ. Since $\mu = D^{-1}$, then parts (i) and (ii) follow by letting

$$D_0 \equiv \min\{D_1, \mu_*^{-1}\}.$$

The proof of part (iii) is from the maximum principle. Suppose that (iii) is false, and let $x_0 \in \Omega$ be the maximum point of v_D satisfying

$$v_D(x_0) > \left(\frac{\lambda}{k}\right)^{\frac{1}{q-p}}.$$

Noted that v_D acts as a solution, so we get the following

$$\begin{aligned} 0 &= D\Delta v_D(x_0) + \lambda\big(1-\psi(x_0)\big)v_D^p(x_0) - kv_D^q(x_0) \\ &\leqslant \lambda v_D^p(x_0) - kv_D^q(x_0) \\ &< 0, \end{aligned}$$

which is a contradiction.

For the proof of (iv) we employ the arguments of upper–lower solutions. Let $\psi_1, \psi_2 \in \mathcal{C}_2$, $\psi_1 \geqslant \psi_2$, then

$$\begin{aligned} &D\Delta\widetilde{v}_{\psi_1,D} + \lambda(1-\psi_2)\widetilde{v}_{\psi_1,D}^p - k\widetilde{v}_{\psi_1,D}^q \\ &\quad\geqslant D\Delta\widetilde{v}_{\psi_1,D} + \lambda(1-\psi_1)\widetilde{v}_{\psi_1,D}^p - k\widetilde{v}_{\psi_1,D}^q \\ &\quad= 0. \end{aligned}$$

This means that $\widetilde{v}_{\psi_1,D}$ is a lower solution of (3.6) when $\psi = \psi_2$. It is easy to see that $(\frac{\lambda}{k})^{\frac{1}{q-p}}$ is an upper solution, and (iii) shows that $\widetilde{v}_{\psi_1,D} \leqslant (\frac{\lambda}{k})^{\frac{1}{q-p}}$. So from the well-known existence of solution via upper–lower solution method, there exists a solution v_D when $\psi = \psi_2$ satisfying

$$\widetilde{v}_{\psi_1,D}(x) \leqslant v_D(x) \leqslant \left(\frac{\lambda}{k}\right)^{\frac{1}{q-p}}, \quad x \in \Omega,$$

which implies that

$$\widetilde{v}_{\psi_1,D} \leqslant \widetilde{v}_{\psi_2,D}. \qquad \square$$

Now we are ready to state and prove the main existence result in this section:

Theorem 3.4. *Suppose the parameters $1 < p < q$ and $k, \lambda > 0$. Then there exists $D_* > 0$ such that for $0 < D < D_*$, (1.10) possesses at least one positive solution.*

Proof. We define an ordered upper–lower solution pair as follows. From Lemma 3.2, we define $\bar{u}$ to be the unique positive solution of

$$\begin{cases} \Delta u + \lambda D^{-p}(1-u) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.11}$$

Also from Lemma 3.2, $0 < \bar{u}(x) < 1$ for $x \in \Omega$ then $\bar{u} \in \mathcal{C}_2$. Hence for $0 < D < D_0(\bar{u})$, the equation

$$\begin{cases} D\Delta v + \lambda(1-\bar{u})v^p - kv^q = 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.12}$$

has a maximal solution $\widetilde{v}_{\bar{u},D}$ from Lemma 3.3. We denote $\underline{v} \equiv \widetilde{v}_{\bar{u},D}$.

Next from Lemma 3.2, we define $\underline{u}$ to be unique positive solution of

$$\begin{cases} \Delta u + \lambda \underline{v}^p(1-u) = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.13}$$

Then similarly from Lemma 3.2, $0 < \underline{u}(x) < 1$ for $x \in \Omega$ then $\underline{u} \in \mathcal{C}_2$. Hence for $0 < D < D_0(\underline{u})$, the equation

$$\begin{cases} D\Delta v + \lambda(1-\underline{u})v^p - kv^q = 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.14}$$

has a maximal solution $\widetilde{v}_{\underline{u},D}$ from Lemma 3.3. We denote $\bar{v} \equiv \widetilde{v}_{\underline{u},D}$. Now we define $D_* = \min\{D_0(\underline{u}), D_0(\bar{u})\}$, then the pair $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ are well defined for $0 < D < D_*$.

To show that $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ defined above is an ordered upper–lower solution pair, we prove that $\bar{u}(x) \geqslant \underline{u}(x)$ and $\bar{v}(x) \geqslant \underline{v}(x)$ for $x \in \Omega$. First similar to the proof of Lemma 2.1, since $\bar{a} = 1 - \bar{u} \geqslant 0$ in Ω, then $\bar{a} + D\underline{v}$ satisfies

$$\begin{cases} \Delta(\bar{a} + D\underline{v}) = k\underline{v}^q(x), & x \in \Omega, \\ \bar{a}(x) + D\underline{v}(x) = 1, & x \in \partial\Omega. \end{cases} \tag{3.15}$$

Hence $D\underline{v}(x) < \bar{a}(x) + D\underline{v}(x) < 1$ for $x \in \Omega$. Now since $\underline{v} < D^{-1}$, then $\bar{u}(x) \geqslant \underline{u}(x)$ from Lemma 3.2(ii). This in turn also implies that $\bar{v}(x) \geqslant \underline{v}(x)$ from Lemma 3.3(iii). Therefore from Theorem 3.1, (1.10) has a positive solution (u, v) satisfying (3.4). $\square$

4. A global bifurcation theorem

In this section, we consider the existence of positive solutions to (1.8) or (1.10) via the bifurcation or continuation methods. We use the decay coefficient k as bifurcation parameter. When $k = 0$ (there is no decay for the catalyst), the system (1.8) has been considered in [32,14], and the structure of the steady state solutions is known in that case. Here we use the information known for $k = 0$ to consider the case of $k > 0$.

To achieve this goal, we first prove a global bifurcation theorem which is of independent interest. Let E be a real-valued Banach space, and let O be an open subset of $\mathbb{R} \times E$. Suppose that $T : O \to E$ is a compact map, and consider the abstract equation

$$\phi(\lambda, u) \equiv u - T(\lambda, u) = 0. \tag{4.1}$$

Define the set of solutions to (4.1) to be

$$S = \{(\lambda, u) \in O\colon u = T(\lambda, u)\}.$$

We have the following abstract result:

Theorem 4.1. *Suppose that $(\lambda_0, u_0) \in S \cap O$, and $I - D_u T(\lambda_0, u_0)$ is invertible. Let Σ be the connected component of S containing (λ_0, u_0), and let $\Sigma_\pm = \Sigma \cap (\mathbb{R}^0_\pm \times E)$, where $\mathbb{R}^0_+ = \{\lambda > \lambda_0\}$ and $\mathbb{R}^0_- = \{\lambda < \lambda_0\}$. Then the following alternatives hold for each of Σ_+ and Σ_-: for $* = +, -$, either*

1. *Σ_* is unbounded; or*
2. *$\overline{\Sigma_*}$ contains another $(\lambda_0, u_*) \in S \cap O$ with $u_* \neq u_0$; or*
3. *$\overline{\Sigma_*} \cap \partial O \neq \emptyset$, where ∂O is the boundary of O.*

Before the proof of Theorem 4.1, we cite a useful topological lemma:

Lemma 4.2. *(See [35].) Let K be a compact metric space, and let A and B be disjoint closed subsets of K. Then either there exists a sub-continuum of K meeting both A and B or $K = K_A \cup K_B$, where K_A, K_B are disjoint compact subsets of K containing A and B, respectively.*

Proof of Theorem 4.1. We only need to consider Σ_+, and the proof for Σ_- is similar. In the proof we will apply various well-known properties of Leray–Schauder degree, which can be found in standard references of nonlinear analysis, see for example [3,7].

We prove by contradiction. Suppose that in the contrary,

$$\begin{aligned} &\Sigma_+ \quad \text{is bounded,} \;\; \overline{\Sigma_+} \cap \{(\lambda_0, u) \in O\} = \{(\lambda_0, u_0)\} \quad \text{and} \\ &\overline{\Sigma_+} \subset \mathring{O} \quad (\text{the interior of } O). \end{aligned}$$

Since Σ_+ is bounded, and T is compact, then $\overline{\Sigma_+}$ is also compact. Also, $\overline{\Sigma_+}$ is closed and $\overline{\Sigma_+} \cap \partial O = \emptyset$, then $\operatorname{dist}(\overline{\Sigma_+}, \partial O) > 0$. Define $S_+ = S \cap (\mathbb{R}^0_+ \times E)$. Let $0 < \varepsilon < \frac{1}{2}\operatorname{dist}(\overline{\Sigma_+}, \partial O)$, we define

$$\Sigma^\varepsilon_+ = \{(\lambda, u) \in O\colon \operatorname{dist}((\lambda, u), \overline{\Sigma_+}) < \varepsilon\},$$

and let $K = \overline{\Sigma^\varepsilon_+} \cap \overline{S_+}$, then K is a compact metric space. Setting

$$K_1 = \overline{\Sigma_+}, \quad \text{and} \quad K_2 = \partial \Sigma^\varepsilon_+ \cap \overline{S_+}.$$

Then K_1 and K_2 are disjoint closed subsets of K. From the definition of Σ_+, there is no sub-continuum of K meeting both K_1 and K_2. Hence according to Lemma 4.2 there exist disjoint compact subsets $\widehat{K_1}$ and $\widehat{K_2}$ of K satisfying

$$K_1 \subseteq \widehat{K_1}, \qquad K_2 \subseteq \widehat{K_2}, \quad \text{and} \quad K = \widehat{K_1} \cup \widehat{K_2}.$$

Choosing δ satisfying

$$0<\delta<\min\left\{\operatorname{dist}(\widehat{K_1},\widehat{K_2}),\operatorname{dist}\left(\widehat{K_1},\partial\Sigma_+^{\varepsilon}\right)\right\},$$

and letting $\mathcal{N}$ be a $\frac{\delta}{2}$-neighborhood of $\widehat{K_1}$, we have

$$\Sigma_+\subset\mathcal{N},\quad\text{and}\quad\partial\mathcal{N}\cap S_+=\emptyset.$$

We define a mapping $\Phi:\mathbb{R}\times E\times[0,1]\to\mathbb{R}\times E$ by

$$\Phi(\lambda,u,t)=\left(\lambda-\lambda_0-t(\lambda_*-\lambda_0),\phi(\lambda,u)\right),$$

where λ_* is an upper bound of the projection of $\mathcal{N}$ onto $\mathbb{R}_+^0$. Then, $\Phi(\lambda,u,t)=(0,0)$ if and only if $\lambda=\lambda_0+t(\lambda_*-\lambda_0)$ and $\phi(\lambda,u)=0$. Hence $(0,0)\notin\Phi(\partial\mathcal{N}\times[0,1])$.

By using the homotopic invariance of Leray–Schauder degree, we have

$$\deg\left(\Phi(\cdot,\cdot,0),\mathcal{N},(0,0)\right)=\deg\left(\Phi(\cdot,\cdot,1),\mathcal{N},(0,0)\right).\tag{4.2}$$

Since $\lambda\neq\lambda_*$ on $\mathcal{N}$, by the Kronecker existence theorem of Leray–Schauder degree we further have

$$\deg\left(\Phi(\cdot,\cdot,1),\mathcal{N},(0,0)\right)=0.\tag{4.3}$$

But the left-hand side of (4.2) does not equal zero. In fact, if $\lambda=\lambda_0$, then for all $t\in[0,1]$, $u\in\mathcal{N}$, $u-T(t\lambda+(1-t)\lambda_0,u)=0$ only if $u=u_0$. Since $I-D_uT(\lambda_0,u_0)$ is invertible, then from the excision property, and the Leray–Schauder index formula, we obtain that

$$\deg\left(\Phi(\cdot,\cdot,0),\mathcal{N},(0,0)\right)=\deg\left(\phi(\lambda_0,\cdot),B_{\frac{\delta}{2}}(u_0),0\right)=(-1)^{\beta},\tag{4.4}$$

where β is the sum of all $m(\mu)$, such that $m(\mu)$ is the algebraic multiplicity of $\mu\in\sigma(D_uT(\lambda_0,u_0))$ so that $\mu>1$. Thus a contradiction is reached from (4.2), (4.3) and (4.4). This contradiction implies that one of alternatives in the theorem holds. □

The result in Theorem 4.1 appears to be new, but it is in the same spirit of global bifurcation theorems proved by Rabinowitz [29]. We also point out that, since $I-D_uT(\lambda_0,u_0)$ is invertible, then the implicit function theorem can be applied to conclude that Σ is locally a curve near (λ_0,u_0) transversal to $\lambda=\lambda_0$, and Theorem 4.1 gives the information of the global nature of Σ. Indeed the result here gives more specific information on Σ_+ or Σ_-, hence it has the unilateral nature as in [29, Theorem 1.27] and related work in [18,33]. But here the "onesideness" of the global branch is for the parameter λ instead of variable u as in [18,29,33]. Compared with the unilateral global bifurcation theorems in [18,33], the result here excludes the possibility of Σ_+ meeting Σ_- as this case is included in the second alternatives in Theorem 4.1.

Now we are ready to consider the global bifurcation of positive solutions to system (1.8) or (1.10) with bifurcation parameter k. Let $E=C^{\alpha}(\overline{\Omega})\times C^{\alpha}(\overline{\Omega})$. We rewrite the system (1.10) into the form

$$\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}\lambda(-\Delta)^{-1}[(1-u)v^p]\\\frac{\lambda}{D}(-\Delta)^{-1}[(1-u)v^p]-\frac{k}{D}(-\Delta)^{-1}[v^q]\end{pmatrix},\tag{4.5}$$

where $(-\Delta)^{-1}:C^{\alpha}(\overline{\Omega})\to C^{\alpha}(\overline{\Omega})$ is defined by $(-\Delta)^{-1}(u)=v$, if v is the unique solution of

$$-\Delta v=u,\quad x\in\Omega,\qquad v(x)=0,\quad x\in\partial\Omega.$$

Define a mapping $T : \mathbb{R} \times E \to E$ by

$$T\big(k, (u, v)\big) = \left(\lambda(-\Delta)^{-1}\big[(1-u)v^p\big], \frac{\lambda}{D}(-\Delta)^{-1}\big[(1-u)v^p\big] - \frac{k}{D}(-\Delta)^{-1}\big[v^q\big]\right),$$

then T is compact, and the system (1.10) is equivalent to the abstract equation

$$\phi(k, u, v) \equiv (u, v) - T(k, u, v) = 0.$$

When $k = 0$, according to Theorem 2.1 in [14], we have the following existence result:

Theorem 4.3. *Suppose that $k = 0$ and $\Omega = B_R(0)$ is a ball with the radius $R > 0$. If $p > 1$, then there exists an $R_0 > 0$ such that the system (1.10) has the only nonnegative solution $(0, 0)$ when $R < R_0$, and has exactly two positive solutions when $R > R_0$. Moreover, when $R > R_0$, the two positive solutions are both radially symmetric, strictly decreasing along the radial direction, and non-degenerate; one of positive solution is locally asymptotically stable, and the other one is unstable with Morse index 1.*

Now we state the global bifurcation and multiplicity result for the positive solutions of system (1.8) or (1.10) when the domain Ω is a large ball.

Theorem 4.4. *Suppose the parameters $p > 1$, $q > 1$ and $\lambda, D > 0$, and assume that $\Omega = B_R(0)$ is a ball with the radius $R > 0$. Then there exists an $R_0 > 0$, such that for $R > R_0$,*

1. *There exists $0 < k_* < k^*$ such that (1.10) has no positive solution when $k > k^*$, has at least one positive solution when $k_* < k \leqslant k^*$, and has exactly two positive solutions when $0 \leqslant k \leqslant k_*$.*
2. *Let $S = \{(k, u, v)\colon k \geqslant 0,\ (u, v) \in E$ is a positive solution of (1.10)$\}$, and let Σ be the connected component of S containing $(0, u_0^1, v_0^1)$, where (u_0^1, v_0^1) is one of positive solution of (1.10) when $k = 0$. Then Σ also contains $(0, u_0^2, v_0^2)$, where (u_0^2, v_0^2) is the other positive solution of (1.10) when $k = 0$. That is, the solution branch of (1.10) emanating from $(0, u_0^1, v_0^1)$ connects to the one emanating from $(0, u_0^2, v_0^2)$.*

Proof. From Corollary 2.5, for both the cases of $p \geqslant q$ and $p < q$, there exists $k^* > 0$ such that (1.10) has no positive solution when $k > k^*$. On the other hand, recall $R_0 > 0$ from Theorem 4.3, then (1.10) has exactly two positive solutions (u_0^1, v_0^1) and (u_0^2, v_0^2) when $R > R_0$. Since both (u_0^1, v_0^1) and (u_0^2, v_0^2) are non-degenerate, then the implicit function theorem can be applied at $(k, u, v) = (0, u_0^i, v_0^i)$ for $i = 1, 2$, and there exists a $k_* > 0$ such that for $k \in [0, k_*]$, (1.10) has exactly two positive solutions (u_k^1, v_k^1) and (u_k^2, v_k^2) which are near (u_0^1, v_0^1) and (u_0^2, v_0^2) respectively.

Now let Σ be the connected component of S containing $\{(0, u_k^1, v_k^1)\colon\ k \in [0, k_*]\}$. Then $\Sigma \subseteq [0, \infty) \times E_+$, where $E_+ = \{(u, v) \in E\colon\ u \geqslant 0, v \geqslant 0\}$. We apply Theorem 4.1 with $\lambda = k$ and $O = [-\epsilon, \infty) \times E_+$ for some $\epsilon > 0$. Since (1.10) has exactly two positive solutions (u_0^1, v_0^1) and (u_0^2, v_0^2) when $k = 0$, and both of them are non-degenerate, then the conditions of Theorem 4.1 are satisfied at $(k, u, v) = (0, u_0^1, v_0^1)$, and one of the three alternatives must hold.

Since Lemma 2.1 guarantees the boundedness of all positive solutions (u, v) with a bound uniform for $k \geqslant 0$, and (1.10) has no positive solution when $k > k^*$, then Σ must be bounded in $[0, \infty) \times E_+$, which implies the alternative 1 cannot happen. Suppose that alternative 3 occurs, then there is $(\bar{k}, \bar{u}, \bar{v}) \in \Sigma \cap ([0, k^*) \times \partial E_+)$. From the maximum principle, we must have $\bar{u}(x) \equiv 0$ and $\bar{v}(x) \equiv 0$ for all $x \in \Omega$. But from Theorem 2.4, the constant steady state $(0, 0)$ is locally asymptotically stable, hence (1.9) does not have positive steady state solution with small amplitude. So Σ cannot connect to $(\bar{k}, 0, 0)$. This excludes the possibility of alternative 3. Therefore only alternative 2 in Theorem 4.1 is possible, *i.e.* there is another element $(0, u_*, v_*) \in \Sigma$. Since (1.10) has exactly two positive solutions, then $(0, u_*, v_*) = (0, u_0^2, v_0^2)$. Finally we can redefine k^* to be the upper bound of the projection of Σ onto k-axis. This completes the proof. □

The result in Theorem 4.4 holds for a bounded domain Ω other than a ball as long as the information of the solution set of $k = 0$ is known as in Theorem 4.3. That is, when $k = 0$, there are exactly two positive solutions of (1.10), and at least one of them is non-degenerate. For a general bounded smooth domain Ω, it is known that there is maximal solution (u_k^1, v_k^1) which is stable if the domain is "large". This can be achieved for an arbitrary domain Ω, with a rescaling $\Omega_R = \{x_0 + t(x - x_0)\colon\ 0 \leqslant t \leqslant R,\ x \in \Omega\}$ with a large $R > 0$. The existence of at least one unstable solution can also be proved by using the Mountain Pass Lemma. But in general (1.10) may have more than one unstable solutions. In this case, we can still conclude that each branch emanating from a solution with $k = 0$ must be bounded, and it connects to another solution with $k = 0$.

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