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The Nevanlinna–Pick problem on the closed unit disk: Minimal norm rational solutions of low degree☆

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ABSTRACT

For the Nevanlinna–Pick interpolation problem with n interpolation conditions (interior and boundary), we construct a family of rational solutions of degree at most $n-1$. We also establish necessary and sufficient conditions for the existence and the uniqueness of a solution with the minimally possible H^∞-norm and construct a family of minimal-norm rational solutions of degree at most $n-1$ in the indeterminate case. Finally, we supplement a result of Ruscheweyh and Jones showing that in case the interpolation nodes and the target values are all unimodular, any rational solution of degree at most $n-1$ is necessarily a finite Blaschke product.

1. Introduction

Let $\mathcal{S}$ denote the Schur class of functions analytic and bounded by one in modulus on the open unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ (in other words, $\mathcal{S}$ is the closed unit ball of the Hardy space H^∞). It follows from the maximum modulus principle that every Schur-class functions is either analytic self-mapping of $\mathbb{D}$ or is a unimodular constant. We denote by $\mathcal{RS}$ the set of all *rational* Schur-class functions and more specifically we write $\mathcal{RS}_k$ and $\mathcal{RS}_{\le k}$ for the sets of $\mathcal{RS}$-functions of degree k and of degree at most k, respectively. The degree of a rational function $f=p/q$ is defined to be the maximum of the degrees of p and q, where p and q are polynomials in their lowest terms. The functions $f\in\mathcal{RS}_k$ which are unimodular on the unit circle $\mathbb{T}=\{z:|z|=1\}$ are of special interest; they are necessarily of the form

$$f(z)=c\prod_{i=1}^{k}\frac{z-a_i}{1-z\bar{a}_i},\quad \text{where } |c|=1,\qquad |a_i|<1\quad \text{for } i=1,\dots,k,$$

and are called *finite Blaschke products*. We will denote by $\mathcal{B}_k$ and $\mathcal{B}_{\le k}$ the set of all Blaschke product of degree k and of degree at most k, respectively.

In this paper we will discuss the following Nevanlinna–Pick type problem **NP**: *given n distinct points $z_1,\dots,z_n\in\overline{\mathbb{D}}$ together with n complex numbers $w_1,\dots,w_n$, find a Schur-class function f such that*

$$f(z_i)=w_i\quad \text{for } i=1,\dots,n. \tag{1.1}$$

We will call the problem *determinate* if it has a unique solution. If the problem has more than one solution, it has infinitely many of them by the evident convexity of the solution set; in this case, the problem will be termed *indeterminate*.

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If $|w_i| = 1$ for some $z_i \in \mathbb{D}$, then the problem may have only one solution $f \equiv w_i$ which is the case if and only if all target values w_j are equal. Excluding this trivial case we may say that there are only three types of conditions in (1.1) where (z_i, w_i) belongs to $\mathbb{D} \times \mathbb{D}$, to $\mathbb{T} \times \mathbb{D}$ or to $\mathbb{T} \times \mathbb{T}$. It seems convenient to rearrange interpolation conditions so that

$$\begin{aligned} \mathcal{I}_1 &= \{i : |z_i| < 1, |w_i| < 1\} = \{1, \ldots, n_1\}, \\ \mathcal{I}_2 &= \{i : |z_i| = 1, |w_i| < 1\} = \{n_1 + 1, \ldots, n_1 + n_2\}, \\ \mathcal{I}_3 &= \{i : |z_i| = 1, |w_i| = 1\} = \{n_1 + n_2 + 1, \ldots, n_1 + n_2 + n_3 = n\}. \end{aligned} \tag{1.2}$$

The standard questions appearing in any norm-constrained interpolation problem include the solvability and the determinacy criteria as well as the existence of certain good solutions. Here we will be particularly interested in rational (and more specifically, finite Blaschke products) solutions f of degree at most $n - 1$ (for some problems this complexity is the minimally possible) and/or with the minimally possible H^∞-norm $\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| \le 1$.

Since finite Blaschke products are unimodular on $\mathbb{T}$, the problem **NP** may have such solutions only if $\mathcal{I}_2 = \emptyset$. On the other hand, the question about minimal norm solutions is nontrivial only if $\mathcal{I}_3 = \emptyset$ (otherwise, every solution has the unit H^∞-norm). Assuming that $\mathcal{I}_2 = \mathcal{I}_3 = \emptyset$, we get the classical Nevanlinna–Pick problem which we denote by $\mathbf{NP}(\mathcal{I}_1)$. The results on this problem collected below are due to Nevanlinna [1] and Pick [2].

Theorem 1.1. *The problem* $\mathbf{NP}(\mathcal{I}_1)$ *has a solution if and only if the Pick matrix*

$$P_1 = \left[\frac{1 - w_i\overline{w}_j}{1 - z_i\overline{z}_j}\right]_{i,j=1}^{n_1} \tag{1.3}$$

is positive semidefinite. Furthermore:

1. *The problem is determinate if and only if* $P_1 \ge 0$ *is singular. The unique solution of a determinate problem is a Blaschke product of degree equal to the rank of* P_1.
2. *The indeterminate problem has infinitely many solutions in* $\mathcal{RS}_{\le n_1-1}$. *All finite Blaschke product solutions are of degree at least* n_1.
3. *For every solution* f *to the problem,* $\|f\|_\infty \ge \lambda_{\min}$, *where* $\lambda_{\min}$ *is the maximal solution to the equation*

$$\det P_1(\lambda) = 0, \quad \textit{where } P_1(\lambda) = \left[\frac{\lambda^2 - w_i\overline{w}_j}{1 - z_i\overline{z}_j}\right]_{i,j=1}^{n_1}. \tag{1.4}$$

4. *There exists a unique solution* $f_{\min}$ *to the problem with the minimally possible norm* $\lambda_{\min}$. *This function is of the form* $f_{\min}(z) = \lambda_{\min} \cdot b(z)$, *where* $b(z)$ *is a Blaschke product of degree equal to* rank $(P_1(\lambda_{\min})) \le n_1 - 1$.

A nice Nevanlinna's linear fractional parametrization of the solution set in the indeterminate case $P_1 > 0$ (recalled in Theorem 2.1 below) is easily adapted to describe all rational solutions and all solutions in $\mathcal{B}_k$ for every fixed $k \ge n_1$. The description of all rational solutions of degree at most $n_1 - 1$ was obtained more recently in [3–5]. The question of finding a solution of the minimally possible complexity (and even finding the value of this complexity) is still open. Two other particular cases of the problem (1.1) are the "boundary-to-interior" problem $\mathbf{NP}(\mathcal{I}_2)$ (where $\mathcal{I}_1 = \mathcal{I}_3 = \emptyset$) and the "boundary-to-boundary" problem $\mathbf{NP}(\mathcal{I}_3)$ (where $\mathcal{I}_1 = \mathcal{I}_2 = \emptyset$). Both of them are indeterminate. The next result should be well known although we did not find an appropriate reference for it. In any event, it is a particular case of Theorem 1.6.

Theorem 1.2. *The problem* $\mathbf{NP}(\mathcal{I}_2)$ *is indeterminate and for every solution* f *to the problem,* $\|f\|_\infty \ge \delta_{\min} := \max_{i\in\mathcal{I}_2} |w_i|$. *Furthermore, there are infinitely many rational solutions* $f \in \mathcal{RS}_{\le n_2-1}$ *with* $\|f\|_\infty = \delta_{\min}$.

In contrast to this case, the "boundary-to-boundary"problem $\mathbf{NP}(\mathcal{I}_3)$ can be solved by finite Blaschke products. The following theorem is due to Ruscheweyh and Jones [6].

Theorem 1.3. *The problem* $\mathbf{NP}(\mathcal{I}_3)$ *has infinitely many solutions in* $\mathcal{B}_{\le n_3-1}$.

The next supplement to the Ruscheweyh–Jones theorem shows that in the "boundary-to-boundary" case, the minimally possible complexity of rational solutions is attained just on finite Blaschke products.

Theorem 1.4. *Every solution* $f \in \mathcal{RS}$ *to the problem* $\mathbf{NP}(\mathcal{I}_3)$ *with* deg $f < n_3$ *is necessarily a finite Blaschke product. Every solution* $f \in \mathcal{RS}$ *to the general problem* (1.1) *which is not a finite Blaschke product is subject to* deg $f \ge n_3$.

The second statement in Theorem 1.4 refers to the problem (1.1) for which at least two of the three sets in (1.2) are not empty. Now we will discuss these "combined" problems in some more details.

The combined problem $\mathbf{NP}(\mathcal{I}_{13})$ (with $\mathcal{I}_2 = \emptyset$) was studied in [7] in a general meromorphic setting. It was shown that the problem is indeterminate if and only if the Pick matrix P_1 (1.3) is positive definite, in which case there are infinitely many solutions in $\mathcal{B}_{n_1+n_3-1}$. The uniqueness occurs if and only if $P_1 \ge 0$ is singular and the unique solution $\hat{f}$ to the subproblem $\mathbf{NP}(\mathcal{I}_1)$ also satisfies equalities $\hat{f}(z_i) = w_i$ for all $i \in \mathcal{I}_3$; due to the known explicit formula for $\hat{f}$ in terms of $\{z_j,\ w_j : j \in \mathcal{I}_1\}$, the latter equalities together with $P_1 \geq 0$ establish explicit criterion for the determinacy of the problem. Unlike to the problem $\mathbf{NP}(\mathcal{I}_3)$, the finite Blaschke product are not the lowest degree interpolants for the problem $\mathbf{NP}(\mathcal{I}_{13})$. For example, the problem with interpolation conditions

$$f(0) = 1/2, \qquad f(1/2) = 3/4, \qquad f(1) = 1$$

has a degree one rational solution $f(z) = (z+1)/2$ and infinitely many solutions in $\mathcal{B}_2$ as well as in $\mathcal{RS}_2 \setminus \mathcal{B}_2$. What we can guarantee in the present setting is that every solution f to the problem $\mathbf{NP}(\mathcal{I}_{13})$ which is not a finite Blaschke product, is subject to $\deg f \geq n_3$.

Another combined problem $\mathbf{NP}(\mathcal{I}_{23})$ (with $\mathcal{I}_1 = \emptyset$) turns out to be always indeterminate with infinitely many solutions in $\mathcal{RS}_{\leq n_2+n_3-1}$ and with the estimate $\deg f \geq n_3$ for every solution f; see Theorem 1.4. This problem is perhaps the least interesting: it has no finite Blaschke product solutions and all solutions have the unit H^∞-norm.

Our main results below are concerned about the two remaining cases: the general problem $\mathbf{NP}(\mathcal{I}_{123})$ involving interpolation conditions (1.2) of all three types and its special case $\mathbf{NP}(\mathcal{I}_{12})$ containing no "boundary-to-boundary" condition.

Theorem 1.5. *The problem $\mathbf{NP}(\mathcal{I}_{123})$ (the problem $\mathbf{NP}(\mathcal{I}_{12})$) is solvable if and only if the Pick matrix P_1 (1.3) is positive definite, in which case the problem has infinitely many solutions in $\mathcal{RS}_{n_1+n_2+n_3-1}$ (respectively, in $\mathcal{RS}_{n_1+n_2-1}$).*

The necessity part is immediate: by Theorem 1.1, the condition $P_1 \geq 0$ is necessary for both problems to have a solution. If P_1 is singular, a unique solution of the "interior-to-interior" subproblem $\mathbf{NP}(\mathcal{I}_1)$ is a finite Blaschke product which cannot satisfy conditions (1.1) for every $i \in \mathcal{I}_2$. Thus, P_1 cannot be singular and the condition $P_1 > 0$ is in fact necessary for problems $\mathbf{NP}(\mathcal{I}_{12})$ and $\mathbf{NP}(\mathcal{I}_{123})$ to have a solution. The sufficiency part will be justified in Section 3 via explicit constructing a family of rational solutions to the problem $\mathbf{NP}(\mathcal{I}_{123})$.

The next theorem discusses the existence of solutions of the problem $\mathbf{NP}(\mathcal{I}_{12})$ with the minimally possible norm. The value of this minimal norm is suggested by Theorems 1.1 and 1.2.

Theorem 1.6. *Let us assume that the Pick matrix P_1 is positive definite (so that the problem $\mathbf{NP}(\mathcal{I}_{12})$ is indeterminate) and let $\mu := \inf \|f\|_\infty$ where the infimum is taken over all solutions f to the problem. Then*

$$\mu = \max\{\lambda_{\min}, \delta_{\min}\}, \quad \text{where } \delta_{\min} := \max_{i\in\mathcal{I}_2} |w_i| \tag{1.5}$$

and where $\lambda_{\min}$ is the maximal solution of the Eq. (1.4). Furthermore:

1. *If $\lambda_{\min} < \delta_{\min}$, then there are infinitely many solutions $f \in \mathcal{RS}_{\leq n_1+n_2-1}$ to the problem such that $\|f\|_\infty = \mu$.*
2. *If $\lambda_{\min} > \delta_{\min}$, then there are no minimal norm solutions.*
3. *If $\lambda_{\min} = \delta_{\min}$, then there are no minimal norm solutions, unless the boundary target values $\{w_i\}_{i\in\mathcal{I}_2}$ are very special (see Lemma 4.2) in which case the minimal norm solution is unique.*

Note that a result of this type (in a less explicit form and for the case $n_2 = 1$) has recently appeared in [8]. In case $n_2 > 1$, it was shown that $\inf \|f\|_\infty$ (that is, μ defined above) exists.

The outline of the paper is the following. Some needed background on Nevanlinna–Pick interpolation is presented in Section 2 as well as the proof of Theorem 1.4. In Section 3, we develop an idea from [9], introducing and studying perturbed Pick matrices and associated matrix pencils. In Section 4 we present several algorithms producing low-degree and/or minimal norm solutions for various Nevanlinna–Pick problems discussed in this introduction. The proofs of Theorems 1.5 and 1.6 are obtained then as byproducts of these algorithms. The paper is concluded by an illustrative example.

2. Preliminaries

In this section we prove Theorem 1.4 and present some auxiliary material needed for proving Theorems 1.5 and 1.6. The results presented in Sections 2.1 and 2.2 can be found in [9].

2.1. The interior problem $\mathbf{NP}(\mathcal{I}_1)$

Here we recall a linear fractional parametrization of the solution set for the indeterminate problem $\mathbf{NP}(\mathcal{I}_1)$. We thus assume that the Pick matrix P_1 is positive definite. Observe that P_1 satisfies the Stein identity

$$P_1 - T_1 P_1 T_1^* = E_1 E_1^* - M_1 M_1^* \tag{2.1}$$

where the matrix $T_1 \in \mathbb{C}^{n_1\times n_1}$ and the columns $M_1, E_1 \in C^{n_1\times 1}$ are give by

$$T_1 = \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_{n_1} \end{bmatrix}, \qquad M_1 = \begin{bmatrix} w_1 \\ \vdots \\ w_{n_1} \end{bmatrix}, \qquad E_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{2.2}$$

For the latter objects we will use a more compact notation

$$T_1 = \mathrm{diag}_{i\in\mathcal{I}_1}\{z_i\}, \qquad M_1 = \mathrm{Col}_{i\in\mathcal{I}_1}\{w_i\}, \qquad E_1 = \mathrm{Col}_{i\in\mathcal{I}_1}\{1\}. \tag{2.3}$$

We next introduce the 2×2 matrix-function

$$\Psi(z) = \begin{bmatrix} \psi_{11}(z) & \psi_{12}(z) \\ \psi_{21}(z) & \psi_{22}(z) \end{bmatrix} = I - (1 - z\overline{\mu}) \begin{bmatrix} E_1^* \\ M_1^* \end{bmatrix} (I - zT_1^*)^{-1} P_1^{-1} (I - \overline{\mu} T_1)^{-1} \begin{bmatrix} E_1 & -M_1 \end{bmatrix}, \tag{2.4}$$

where μ is an arbitrary point in $\mathbb{T}$ and where I denotes the identity matrix of an appropriate size. It is readily seen that Ψ is a rational function having simple poles at $1/\overline{z}_i$ ($i \in \mathcal{I}_1$). A straightforward calculation based solely on the equality (2.1) verifies the identity

$$J - \Psi(z) J \Psi(\zeta)^* = (1 - z\overline{\zeta}) \begin{bmatrix} E_1^* \\ M_1^* \end{bmatrix} (I - zT_1^*)^{-1} P_1^{-1} (I - \overline{\zeta} T_1)^{-1} \begin{bmatrix} E_1 & M_1 \end{bmatrix} \tag{2.5}$$

for every z, ζ where Ψ is analytic, where J is the signature matrix given by

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.6}$$

Identity (2.4) implies in particular, that $\Psi(z)$ is J-unitary on $\mathbb{T}$, i.e.,

$$\Psi(z) J \Psi(z)^* = J \quad \text{for all } z \in \mathbb{T}. \tag{2.7}$$

Theorem 2.1. *Let $P_1 > 0$ and let Ψ be defined as in (2.4). A function f belongs to $\mathcal{RS}$ and satisfies conditions (1.1) for all $i \in \mathcal{I}_1$ if and only if it is of the form*

$$f = \frac{\psi_{11}\mathcal{E} + \psi_{12}}{\psi_{21}\mathcal{E} + \psi_{22}} \quad \textit{for some } \mathcal{E} \in \mathcal{RS}. \tag{2.8}$$

Since $\Psi(\mu) = I$, it follows that $\det \Psi(z) \not\equiv 0$, so that formula (2.8) establishes a one-to-one correspondence between $\mathcal{RS}$ and the set of all rational solutions to the problem $\mathbf{NP}(\mathcal{I}_1)$.

Corollary 2.2. *A function f belongs to $\mathcal{RS}$ and satisfies conditions (1.1) if and only if it is of the form (2.8) for some $\mathcal{E} \in \mathcal{RS}$ subject to conditions*

$$\mathcal{E}(z_i) = v_i := \frac{\psi_{12}(z_i) - \psi_{22}(z_i) w_i}{\psi_{21}(z_i) w_i - \psi_{11}(z_i)} \quad \textit{for all } i \in \mathcal{I}_2 \cup \mathcal{I}_3. \tag{2.9}$$

Furthermore, $|v_i| = 1 \Longleftrightarrow |w_i| = 1$ and $|v_i| < 1 \Longleftrightarrow |w_i| < 1$.

Proof. The statement follows from (2.8) once we evaluate the latter one at z_i, replace $f(z_i)$ by the target value w_i and solve the obtained equality for $\mathcal{E}(z_i)$. We just need to be sure that the denominator in (2.9) is not equal to zero. To this end, let us recall that the adjoint of a J-unitary matrix is J-unitary so that the equality $\Psi(z)^* J \Psi(z) = J$ holds for all $z \in \mathbb{T}$ due to (2.7). Equating the upper-left entries in the latter equality gives $|\psi_{11}(z)|^2 - |\psi_{21}(z)|^2 = 1$ for all $z \in \mathbb{T}$. Therefore $|\psi_{11}(z)| > |\psi_{21}(z)|$ and also $\psi_{21}(z_i) w_i - \psi_{11}(z_i) \neq 0$ for every $z_i \in \mathbb{T}$ and $w_i \in \overline{\mathbb{D}}$. The last statement follows from (2.5), (2.7) and (2.9):

$$\begin{aligned} |\psi_{21}(z_i) w_i - \psi_{11}(z_i)|^2 \cdot (1 - |v_i|^2) &= \begin{bmatrix} 1 & -w_i \end{bmatrix} \Psi(z_i) J \Psi(z_i)^* \begin{bmatrix} 1 \\ -\overline{w}_i \end{bmatrix} \\ &= \begin{bmatrix} 1 & -w_i \end{bmatrix} J \begin{bmatrix} 1 \\ -\overline{w}_i \end{bmatrix} = 1 - |w_i|^2, \end{aligned}$$

which completes the proof. □

Conclusion. Corollary 2.2 shows how to reduce the general problem (1.1) to a problem containing no interior interpolation conditions. Due to the second statement in Corollary 2.2, the reduced problem has the same "boundary-to-interior" and "boundary-to-boundary" components (with recalculated target values of course) as the original problem.

2.2. Boundary rational interpolation with prescribed derivatives

Boundary interpolation by rational Schur-class functions with unimodular target values becomes much more transparent if, in addition to conditions $f(z_i) = w_i$, one prescribes the values of f' at each interpolation node z_i. To given $z_i, w_i \in \mathbb{T}$ ($i \in \mathcal{I}_3$)

we attach a tuple $\boldsymbol{\gamma} = \{\gamma_i : i \in \mathcal{I}_3\}$ and construct the Hermitian matrix

$$P_{3,\boldsymbol{\gamma}} = \left[p_{ij}\right]_{i,j\in\mathcal{I}_3}, \quad \text{where } p_{ij} = \begin{cases} \dfrac{1-w_i\overline{w}_j}{1-z_i\overline{z}_j} & \text{if } i \neq j, \\ \gamma_i & \text{if } i = j. \end{cases} \tag{2.10}$$

Definition 2.3. We will say that the tuple $\boldsymbol{\gamma}$ is *admissible* if the matrix $P_{3,\boldsymbol{\gamma}}$ is positive definite.

We next introduce the matrix $T_3 \in \mathbb{C}^{n_3\times n_3}$ and the columns $M_3, E_3 \in C^{n_1\times 1}$ by the formulas similar to those in (2.3):

$$T_3 = \text{diag}_{i\in\mathcal{I}_3}\{z_i\}, \qquad M_3 = \text{Col}_{i\in\mathcal{I}_3}\{w_i\}, \qquad E_3 = \text{Col}_{i\in\mathcal{I}_3}\{1\}, \tag{2.11}$$

and observe the identity

$$P_{3,\boldsymbol{\gamma}} - T_3P_{3,\boldsymbol{\gamma}}T_3^* = E_3E_3^* - M_3M_3^*, \tag{2.12}$$

similar to (2.1). The difference between the Stein equations (2.1) and (2.12) is that the first has a unique solution P_1 whereas the second has infinitely many solutions which may differ, however, only by their diagonal entries. In any event, for an admissible tuple $\boldsymbol{\gamma}$ we may introduce the rational matrix-function

$$\begin{aligned}\Theta^{\boldsymbol{\gamma}}(z) &= \begin{bmatrix}\theta_{11}^{\boldsymbol{\gamma}}(z) & \theta_{12}^{\boldsymbol{\gamma}}(z)\\ \theta_{21}^{\boldsymbol{\gamma}}(z) & \theta_{22}^{\boldsymbol{\gamma}}(z)\end{bmatrix}\\ &= I - (1-z\overline{\mu})\begin{bmatrix}E_3^*\\ M_3^*\end{bmatrix}(I-zT_3^*)^{-1}P_{3,\boldsymbol{\gamma}}^{-1}(I-\overline{\mu}T_3)^{-1}\begin{bmatrix}E_3 & -M_3\end{bmatrix},\end{aligned} \tag{2.13}$$

where μ is an arbitrary point in $\mathbb{T}\setminus\{z_i : i \in \mathcal{I}_3\}$. The function $\Theta^{\boldsymbol{\gamma}}$ has simple poles at z_i $(i \in \mathcal{I}_3)$ and it is readily seen that its scalar multiple

$$\widehat{\Theta}^{\boldsymbol{\gamma}}(z) = \begin{bmatrix}\widehat{\theta}_{11}^{\boldsymbol{\gamma}}(z) & \widehat{\theta}_{12}^{\boldsymbol{\gamma}}(z)\\ \widehat{\theta}_{21}^{\boldsymbol{\gamma}}(z) & \widehat{\theta}_{22}^{\boldsymbol{\gamma}}(z)\end{bmatrix} := \prod_{i\in\mathcal{I}_3}(1-z\overline{z}_i)\cdot\Theta^{\boldsymbol{\gamma}}(z) \tag{2.14}$$

is a matrix polynomial of degree equal $|\mathcal{I}_3| = n_3$. Due to identity (2.12), the function $\Theta^{\boldsymbol{\gamma}}(z)$ is J-unitary on $\mathbb{T}\setminus\{z_i : i \in \mathcal{I}_3\}$ and another consequence of (2.12) is that $\det\Theta^{\boldsymbol{\gamma}}(z) = 1$ for all $z \in \mathbb{T}\setminus\{z_i : i \in \mathcal{I}_3\}$. Indeed, using the determinantal equality $\det(I-AB) = \det(I-BA)$ we have from (2.12) and (2.13)

$$\begin{aligned}\det\Theta^{\boldsymbol{\gamma}}(z) &= \det\left(I-(1-z\overline{\mu})\begin{bmatrix}E_3^*\\ M_3^*\end{bmatrix}(I-zT_3^*)^{-1}P_{3,\boldsymbol{\gamma}}^{-1}(I-\overline{\mu}T_3)^{-1}\begin{bmatrix}E_3 & -M_3\end{bmatrix}\right)\\ &= \det\left(I-(1-z\overline{\mu})(I-zT_3^*)^{-1}P_{3,\boldsymbol{\gamma}}^{-1}(I-\overline{\mu}T_3)^{-1}\begin{bmatrix}E_3 & -M_3\end{bmatrix}\begin{bmatrix}E_3^*\\ M_3^*\end{bmatrix}\right)\\ &= \det\left(I-(1-z\overline{\mu})(I-zT_3^*)^{-1}P_{3,\boldsymbol{\gamma}}^{-1}(I-\overline{\mu}T_3)^{-1}\left(P_{3,\boldsymbol{\gamma}}-T_3P_{3,\boldsymbol{\gamma}}T_3^*\right)\right)\\ &= \det\left((I-zT_3^*)^{-1}P_{3,\boldsymbol{\gamma}}^{-1}(I-\overline{\mu}T_3)^{-1}\right)\times\det\left((I-\overline{\mu}T_3)P_{3,\boldsymbol{\gamma}}(I-zT_3^*)-(1-z\overline{\mu})\left(P_{3,\boldsymbol{\gamma}}-T_3P_{3,\boldsymbol{\gamma}}T_3^*\right)\right)\\ &= \det\left((I-zT_3^*)^{-1}P_{3,\boldsymbol{\gamma}}^{-1}(I-\overline{\mu}T_3)^{-1}\right)\cdot\det\left((zI-T)P_{3,\boldsymbol{\gamma}}(\overline{\mu}I-T^*)\right)\\ &= \prod_{i\in\mathcal{I}_3}\frac{(z-z_i)(\overline{\mu}-\overline{z}_i)}{(1-z\overline{z}_i)(1-\overline{\mu}z_i)} = 1,\end{aligned}$$

where the two last equalities hold due to the diagonal structure of T_3 and since $|z_i| = 1$. Now it follows from (2.14) that

$$\det\widehat{\Theta}^{\boldsymbol{\gamma}}(z) = \prod_{i\in\mathcal{I}_3}(1-z\overline{z}_i)^2 \quad \text{for all } z \in \mathbb{C}. \tag{2.15}$$

Theorem 2.4. *Let $\boldsymbol{\gamma}$ be an admissible tuple and let $\widehat{\Theta}^{\boldsymbol{\gamma}}$ be defined as in (2.14). A function f belongs to $\mathcal{RS}$ and satisfies conditions*

$$f(z_i) = w_i, \qquad |f'(z_i)| \le \gamma_i \quad \text{for } i \in \mathcal{I}_3 \tag{2.16}$$

if and only if it is of the form

$$f = \frac{\widehat{\theta}_{11}^{\boldsymbol{\gamma}}\mathcal{E}+\widehat{\theta}_{12}^{\boldsymbol{\gamma}}}{\widehat{\theta}_{21}^{\boldsymbol{\gamma}}\mathcal{E}+\widehat{\theta}_{22}^{\boldsymbol{\gamma}}} \tag{2.17}$$

for some $\mathcal{E} \in \mathcal{RS}$. Moreover, a function f of the form (2.8) meets the condition $|f'(z_i)| = \gamma_i$ if and only if

$$\widehat{\theta}_{21}^{\boldsymbol{\gamma}}(z_i)\mathcal{E}(z_i)+\widehat{\theta}_{22}^{\boldsymbol{\gamma}}(z_i) \neq 0. \tag{2.18}$$

Remark 2.5. By the converse to the Carathéodory–Julia theorem (see [10, Chapter 4] or [11, Chapter 6]), whenever a function $f \in \mathcal{RS}$ takes a unimodular value at a boundary point $t \in \mathbb{T}$, the following equalities hold

$$\lim_{z\to t} \frac{1-|f(z)|^2}{1-|z|^2} = tf'(t)\overline{f(t)} = |f'(t)|. \tag{2.19}$$

Therefore, conditions $|f'(z_i)| = \gamma_i$ and $|f'(z_i)| \le \gamma_i$ in Theorem 2.4 can be equivalently replaced by $f'(z_i) = \bar{z}_i w_i \gamma_i$ and $z_i\overline{w}_i f'(z_i) \le \gamma_i$ respectively. Thus, interpolation conditions involving $|f'|$ are in fact concerned about f' itself.

Remark 2.6. It is worth mentioning that the boundary interpolation problem (2.16) can be considered for all Schur-class functions (not only rational) in which case $f(z_i)$ and $f'(z_i)$ should be interpreted as the non-tangential boundary limits of $f(z)$ and $f'(z)$ as z tends to a boundary point z_i non-tangentially. It is quite remarkable, that the first statement in Theorem 2.4 is still true in this more general setting once we allow for the parameter $\mathcal{E}$ to run through the whole class $\mathcal{S}$ rather than through $\mathcal{RS}$. However, the characterization of the parameters $\mathcal{E}$ leading to the equality $|f'(z_i)| = \gamma_i$ rather than to the inequality in (2.16) is more tricky in this more general context; we refer to [12–14].

2.3. Proof of Theorem 1.4

Let us assume that $f \in \mathcal{RS}$ is not a finite Blaschke product and satisfies conditions (1.1) for all $i \in \mathcal{I}_3$. Let us define the tuple $\boldsymbol{\gamma} = \{\gamma_i : i \in \mathcal{I}_3\}$ by letting $\gamma_i = |f'(z_i)|$. Then the matrix $P_{3,\boldsymbol{\gamma}}$ defined as in (2.10) is positive definite by [15, Lemma 2.1] ($P_{3,\boldsymbol{\gamma}}$ is positive semidefinite since f belongs to the Schur class, and it is singular only if f is a finite Blaschke product of degree less than n_3, the dimension of $P_{3,\boldsymbol{\gamma}}$). Thus, the tuple $\boldsymbol{\gamma}$ is admissible and it follows from Theorem 2.4 that f admits a representation (2.17) for some $\mathcal{E} \in \mathcal{RS}$ subject to constraint (2.18) for every $i \in \mathcal{I}_3$. Writing $\mathcal{E} = P/Q$ as a ratio of two polynomials in the lowest terms we get from (2.17) a representation of f as a ratio of two polynomials

$$f = \frac{\widetilde{\theta}^{\boldsymbol{\gamma}}_{11}P + \widetilde{\theta}^{\boldsymbol{\gamma}}_{12}Q}{\widetilde{\theta}^{\boldsymbol{\gamma}}_{21}P + \widetilde{\theta}^{\boldsymbol{\gamma}}_{22}Q} = \frac{N}{D}. \tag{2.20}$$

The numerator N and the denominator D in the latter representation may have common zeros only at the zeros of $\det \widetilde{\Theta}^{\boldsymbol{\gamma}}$, that is, at $\{z_i\}_{i\in\mathcal{I}_3}$. However, conditions (2.18) tell us that $D(z_i) \neq 0$ for all $i \in \mathcal{I}_3$. Thus the representation (2.20) is coprime and therefore, $\deg f = \max\{\deg N, \deg D\}$. On the other hand, it follows from (2.13), (2.14) that the leading coefficient A_{n_3} of the matricial polynomial

$$\widetilde{\Theta}^{\boldsymbol{\gamma}}(z) = A_{n_3}z^{n_3} + \cdots + A_1 z + A_0$$

is an invertible matrix. Indeed, by (2.13), (2.14),

$$(-1)^{n_3}\left(\prod_{i=1}^{n_3} z_i\right)\cdot A_{n_3} = I - \begin{bmatrix} E_3^* \\ M_3^* \end{bmatrix} T_3 P_{3,\boldsymbol{\gamma}}^{-1}(\mu I - T_3)^{-1}\begin{bmatrix} E_3 & -M_3 \end{bmatrix}.$$

On the other hand, a computation similar to the one used to get (2.15) shows that

$$\begin{aligned}
&\det\left(I - \begin{bmatrix} E_3^* \\ M_3^* \end{bmatrix} T_3 P_{3,\boldsymbol{\gamma}}^{-1}(\mu I - T_3)^{-1}\begin{bmatrix} E_3 & -M_3 \end{bmatrix}\right) \\
&\quad = \det\left(I - T_3 P_{3,\boldsymbol{\gamma}}^{-1}(\mu I - T_3)^{-1}\begin{bmatrix} E_3 & -M_3 \end{bmatrix}\begin{bmatrix} E_3^* \\ M_3^* \end{bmatrix}\right) \\
&\quad = \det\left(I - T_3 P_{3,\boldsymbol{\gamma}}^{-1}(\mu I - T_3)^{-1}\left(P_{3,\boldsymbol{\gamma}} - TP_{3,\boldsymbol{\gamma}}T^*\right)\right) \\
&\quad = \det\left(T_3 P_{3,\boldsymbol{\gamma}}^{-1}(\mu I - T_3)^{-1}\left[(\mu I - T_3)P_{3,\boldsymbol{\gamma}}T_3^* - P + TP_{3,\boldsymbol{\gamma}}T^*\right]\right) \\
&\quad = \det\left(T_3 P_{3,\boldsymbol{\gamma}}^{-1}(\mu I - T_3)^{-1}\right)\cdot \det\left(P_{3,\boldsymbol{\gamma}}(\mu T_3^* - I)\right) = \prod_{i=1}^{n_3}\frac{z_i(\mu\bar{z}_i - 1)}{\mu - z_i} = 1
\end{aligned}$$

and thus, A_{n_3} is invertible. Then one can conclude from (2.20) that

$$\deg f = \max\{\deg N, \deg D\} = n + \max\{\deg P, \deg Q\} = n_3 + \deg \mathcal{E} \ge n_3$$

which completes the proof of the second statement in Theorem 1.4. The first statement now follows immediately. □

3. Perturbed Pick matrices

A family of rational solutions to the problem $\mathbf{NP}(\mathcal{I}_2)$ can be constructed as follows: for a fixed $r \in [0, 1)$, find a function $g \in \mathcal{RS}$ such that $g(rz_i) = w_i$ for all $i \in \mathcal{I}_2$ (if r is close enough to one, then there are infinitely many such functions) and then let $f(z) = g(rz)$ to get a solution f to the problem $\mathbf{NP}(\mathcal{I}_2)$. This idea was outlined in [9] and will be applied here to the combined problem $\mathbf{NP}(\mathcal{I}_{12})$.

Assuming throughout this section that the necessary condition $P_1 > 0$ for the problem $\mathbf{NP}(\mathcal{I}_2)$ to have a solution is in force, we introduce the matrix

$$\mathbf{P}_r = \left[\frac{1 - w_i\overline{w}_j}{1 - r^2 z_i\overline{z}_j}\right]_{i,j=1}^{n_1+n_2} = \begin{bmatrix} P_{1,r} & P_{12,r} \\ P_{12,r}^* & P_{2,r}\end{bmatrix}, \quad r \in [0, 1), \tag{3.1}$$

where the block $P_{1,r} \in \mathbb{C}^{n_1\times n_1}$ corresponds to the index set $\mathcal{I}_1$ and the block $P_{2,r} \in \mathbb{C}^{n_2\times n_2}$ corresponds to $\mathcal{I}_2$. Define the number

$$q = \min_{i,j\in\mathcal{I}_1\cup\mathcal{I}_2}\{q_{ij} : i \neq j, \}, \quad \text{where } q_{ij} = \begin{cases} |\mathrm{Im}(z_i\overline{z}_j)| & \text{if } \mathrm{Re}(z_i\overline{z}_j) > 0, \\ 1 & \text{if } \mathrm{Re}(z_i\overline{z}_j) \leq 0,\end{cases}$$

and observe that for all $i \neq j$ ($i, j \in \mathcal{I}_1 \cup \mathcal{I}_2$) and all $r \in (0, 1)$,

$$0 < q \leq q_{ij} < |1 - r^2 z_i\overline{z}_j|.$$

Therefore, all non-diagonal entries in $\mathbf{P}_r$ are bounded by $2/q$ in modulus. On the other hand, all the diagonal entries in the block $P_{2,r}$ have the form

$$\frac{1 - |w_i|^2}{1 - r^2|z_i|^2} = \frac{1 - |w_i|^2}{1 - r^2} \quad (|z_i| = 1, |w_i| < 1),$$

and are as large as we wish if r is close enough to one. We also know that the entry-wise limit of $P_{1,r}$ as $r \to 1$ is equal to the matrix $P_1 > 0$ and thus $P_{1,r} > 0$ if r is close to one. Combining all the above information we conclude by the standard Schur complement argument that there exists $r_0 \in [0, 1)$ so that $\mathbf{P}_r > 0$ for every $r \in (r_0, 1)$. We next describe all r such that $\mathbf{P}_r$ is positive definite.

Proposition 3.1. *Let $r_0 = \max\{r \in [0, 1) : \det \mathbf{P}_r = 0\}$. Then*

$$\mathbf{P}_{r_0} \geq 0, \qquad \mathbf{P}_r \ngeq 0 \quad \text{for } r \in (0, r_0) \quad \text{and} \quad \mathbf{P}_r > 0 \quad \text{for } r \in (r_0, 1).$$

Proof. The function $d(r) = \det \mathbf{P}_r$ is rational, so it has finitely many zeros, and thus r_0 is well defined. Let us assume that $\mathbf{P}_{r_0}$ has a negative eigenvalue. Since for all r sufficiently close to one, all eigenvalues of $\mathbf{P}_r$ are positive, it follows by continuity of eigenvalues that for some $r' \in (r_0, 1)$, the matrix $\mathbf{P}_{r'}$ has zero eigenvalue so that $\det \mathbf{P}_{r'} = 0$ which contradicts the definition of r_0. Therefore, all eigenvalues of $\mathbf{P}_{r_0}$ are nonnegative so that $\mathbf{P}_{r_0} \geq 0$.

Let us fix $r_1, r_2 \in [r_0, 1)$ and assume that $r_2 > r_1$. We have

$$\mathbf{P}_{r_2} - \mathbf{P}_{r_1} = \left[\frac{1 - w_i\overline{w}_j}{1 - r_2^2 z_i\overline{z}_j} - \frac{1 - w_i\overline{w}_j}{1 - r_1^2 z_i\overline{z}_j}\right]_{i,j=1}^{n_1+n_2} = \left[\frac{1 - w_i\overline{w}_j}{1 - r_1^2 z_i\overline{z}_j} \cdot \frac{(r_2^2 - r_1^2)z_i\overline{z}_j}{1 - r_2^2 z_i\overline{z}_j}\right]_{i,j=1}^{n_1+n_2}$$

and thus, $\mathbf{P}_{r_2} - \mathbf{P}_{r_1}$ is equal to the Hadamard product

$$\mathbf{P}_{r_2} - \mathbf{P}_{r_1} = \mathbf{P}_{r_1} \circ ((r_2^2 - r_1^2)T\Gamma_{r_2}T^*), \quad \text{where } \Gamma_r = \left[\frac{1}{1 - r^2 z_i\overline{z}_j}\right]_{i,j=1}^{n_1+n_2} \tag{3.2}$$

and where $T = \mathrm{diag}_{i\in\mathcal{I}_1\cup\mathcal{I}_2}\{z_i\}$. The positivity of Γ_r for any $r \in (0, 1)$ is well-known. Therefore the second factor on the right hand side of (3.2) is positive semidefinite and we conclude by the Schur product theorem, that $\mathbf{P}_{r_2} \geq \mathbf{P}_{r_1}$ if $\mathbf{P}_{r_1} \geq 0$.

In particular, $\mathbf{P}_r \geq \mathbf{P}_{r_0} \geq 0$ for every $r \in (r_0, 1)$. Therefore, the function $d(r)$ is non-decreasing on $[r_0, 1)$. Since $d(r)$ is rational and since $d(r_0) = 0$, it follows that $d(r) > 0$ for every $r \in (r_0, 1)$. Therefore, we have in fact $\mathbf{P}_r > 0$ for all $r \in (r_0, 1)$. Finally if we had $\mathbf{P}_{r'} \geq 0$ for some $r' \in (0, r_0)$, the above arguments would show that $\mathbf{P}_{r_0}$ is positive definite which would contradict the choice of r_0. Thus, $\mathbf{P}_r \ngeq 0$ for $r \in (0, r_0)$, which completes the proof. Observe that $r_0 = 0$ if and only if $w_1 = \cdots = w_{n_1+n_2}$. □

The matrix $\mathbf{P}_r$ is the Pick matrix of the "interior" interpolation problem with interpolation conditions

$$g(rz_i) = w_i \quad \text{for } i \in \mathcal{I}_1 \cup \mathcal{I}_2. \tag{3.3}$$

Combining Proposition 3.1 and Theorem 1.1 we conclude that for every $r \in [r_0, 1)$, there exist infinitely many functions $g \in \mathcal{RS}$ satisfying conditions (3.3); for every such g, the function $f(z) = g(rz)$ solves the original problem $\mathbf{NP}(\mathcal{I}_{12})$. There are two reasons to consider a solution $g \in \mathcal{S}$ to the problem (3.3) with the minimally possible H^∞-norm: (1) this construction will be used to prove Theorem 1.6 and (2) this g is a rational function of degree at most $n_1 + n_2 - 1$. The construction is suggested by Theorem 1.1, part (4).

For every fixed $r \in (r_0, 1)$, let us introduce the pencil

$$\mathbf{P}_r(\lambda) = \left[\frac{\lambda^2 - w_i\overline{w}_j}{1 - r^2 z_i\overline{z}_j}\right]_{i,j=1}^{n_1+n_2} = \lambda^2\Gamma_r - W\Gamma_r W^*, \tag{3.4}$$

where $W = \mathrm{diag}_{i\in\mathcal{I}_1\cup\mathcal{I}_2}\{w_i\}$ and where Γ_r is defined in (3.2).

Proposition 3.2. *Let $r_0 = \max\{r \in [0, 1) : \det \mathbf{P}_r = 0\}$ and let λ_r denote the maximal solution of the equation*

$$\det \mathbf{P}_r(\lambda) = \det(\lambda^2 \Gamma_r - W\Gamma_r W^*) = 0. \tag{3.5}$$

Then the function $r \to \lambda_r$ decreases on $[r_0, 1)$.

Proof. Take r_1, r_2 so that $r_0 < r_1 < r_2 < 1$ and let us consider the following modification of $\mathbf{P}_r$:

$$\mathbf{P}'_r = \left[\frac{1 - w'_i\overline{w}'_j}{1 - r^2 z_i \bar{z}_j}\right]_{i,j=1}^{n_1+n_2}, \quad \text{where } w'_i = \frac{w_i}{\lambda_{r_1}}.$$

By the very definition of λ_{r_1}, the matrix $\mathbf{P}'^{r_1} \geq 0$ is singular. By (the proof of) Proposition 3.1, $\mathbf{P}'^{r} > 0$ for every $r \in (r_1, 1)$ and in particular, $\mathbf{P}'^{r_2} > 0$. Then the maximal solution λ' to the equation $\det(\mathbf{P}'^{r_2}(\lambda)) = 0$ (where $\mathbf{P}'^{r_2}(\lambda)$ is defined by formula (3.4) with w'_i instead of w_i) is less than one. By the definition of λ_{r_2}, it follows that $\lambda_{r_2} = \lambda' \cdot \lambda_{r_1}$ and thus, $\lambda_{r_2} < \lambda_{r_1}$. □

Since $\mathbf{P}_r(1) = \mathbf{P}_r$ (see formula (3.1)) is positive definite, it is clear that $\lambda_r < 1$ and that the matrix $\mathbf{P}_r(\lambda_r)$ is positive semidefinite (singular). Let us denote its rank by $\rho := \operatorname{rank}(\mathbf{P}_r(\lambda_r)) \leq n_1 + n_2 - 1$. By Theorem 1.1, there is a unique Schur-class function b_r such that

$$b_r(rz_i) = \frac{w_i}{\lambda_r} \quad \text{for } i = 1, \ldots, n_1 + n_2. \tag{3.6}$$

This function is a Blaschke product of degree ρ and it can be constructed from interpolation data as follows. It turns out that any $\rho \times \rho$ principal submatrix of $\mathbf{P}_r(\lambda_r)$ is positive definite. We fix one such submatrix by choosing the index set $\mathcal{I} \subset \mathcal{I}_1 \cup \mathcal{I}_2$ of cardinality $|\mathcal{I}| = \rho$ and let $E_\rho \in \mathbb{C}^{\rho\times 1}$ to be the column with all entries equal one, and let

$$P_\rho = \left[\frac{\lambda_r^2 - w_i\overline{w}_j}{1 - r^2 z_i \bar{z}_j}\right]_{i,j\in\mathcal{I}}, \qquad T_\rho = \operatorname{diag}_{i\in\mathcal{I}}\{z_i\}, \qquad M_\rho = \operatorname{Col}_{i\in\mathcal{I}}\{w_i\}.$$

Then the desired b_r is defined by the formula

$$b_r(z) = \lambda_r \cdot \frac{1 - (1 - zr\bar{z}_j)E_\rho^*(I - zrT_\rho^*)^{-1}G}{\overline{w}_j - (1 - zr\bar{z}_j)M_\rho^*(I - zrT_\rho^*)^{-1}G}, \tag{3.7}$$

where j is *any* index from $(\mathcal{I}_1 \cup \mathcal{I}_2) \setminus \mathcal{I}$ and where

$$G = P_\rho^{-1}(I - r^2\bar{z}_j T_\rho)^{-1}(\lambda_r^2 E_\rho - M_\rho \overline{w}_j).$$

We refer to [16] for details. The function $g(z) = \lambda_r b_r(z)$ is the minimal norm solution of the interpolation problem (3.3) whereas the function

$$f_r(z) = \lambda_r b_r(rz) \tag{3.8}$$

is a rational solution of the original problem $\mathbf{NP}(\mathcal{I}_{12})$ of degree $\rho \leq n_1 + n_2 - 1$. Combining (3.8) and (3.7) gives

$$f_r(z) = \lambda_r^2 \cdot \frac{1 - (1 - zr^2\bar{z}_j)E_\rho^*(I - zrT_\rho^*)^{-1}G}{\overline{w}_j - (1 - zr^2\bar{z}_j)M_\rho^*(I - zrT_\rho^*)^{-1}G}. \tag{3.9}$$

We summarize: for every $r \in [r_0, 1)$ we constructed a solution $f_r \in \mathcal{RS}_{\leq n_1+n_2-1}$ to the problem $\mathbf{NP}(\mathcal{I}_{12})$. To make sure that we got an infinite family of solutions we need the injectivity of the map $r \to f_r$.

Proposition 3.3. *The correspondence $r \to f_r$ established by formula (3.9) is either one-to-one or its range is a singleton.*

Proof. Let us assume that the numbers $r_1 < r_2$ lead via formula (3.9) (or (3.8)) to the same function $f_{r_1} = f_{r_2}$, so that

$$\lambda_1 b_1(r_1 z) = \lambda_2 b_2(r_2 z) \tag{3.10}$$

where $b_1 = b_{r_1}$ and $b_2 = b_{r_2}$ are Blaschke products of degree at most $n_1 + n_2 - 1$ and where we have set for short $\lambda_1 = \lambda_{r_1}$ and $\lambda_2 = \lambda_{r_2}$. We have from (3.10),

$$b_2(z) = \frac{\lambda_1}{\lambda_2} \cdot b_1\left(\frac{r_1 z}{r_2}\right). \tag{3.11}$$

Since b_1 and b_2 are unimodular on $\mathbb{T}$, we have by the symmetry principle and (3.11)

$$\frac{\lambda_1}{\lambda_2} \cdot b_1\left(\frac{r_1 z}{r_2}\right) = b_2(z) = \frac{1}{\overline{b_2(1/\bar{z})}} = \frac{\lambda_2}{\lambda_1} \cdot \frac{1}{\overline{b_1\left(\frac{r_1}{r_2\bar{z}}\right)}} = \frac{\lambda_2}{\lambda_1} \cdot b_1\left(\frac{r_2 z}{r_1}\right),$$

which eventually gives

$$b_1\left(\left(\frac{r_1}{r_2}\right)^2 z\right) = \left(\frac{\lambda_2}{\lambda_1}\right)^2 \cdot b_1(z)$$

for every $z \in \mathbb{C}$ at which b_1 is analytic. Iterating the latter identity leads us to

$$b_1\left(\left(\frac{r_1}{r_2}\right)^{2k} z\right) = \left(\frac{\lambda_2}{\lambda_1}\right)^{2k} \cdot b_1(z) \quad \text{for all } k = 1, 2, \ldots. \tag{3.12}$$

Since $r_1 < r_2$, we have $\lambda_1 > \lambda_2$, by Proposition 3.2. Letting $k \to \infty$ in (3.12) we conclude that $b_1(0) = 0$. Therefore b_1 is of the form $b_1(z) = zb_1^{(1)}(z)$ and substituting this product into (3.12) one gets a similar identity for $b_1^{(1)}$:

$$b_1^{(1)}\left(\left(\frac{r_1}{r_2}\right)^{2k} z\right) = \left(\frac{r_2\lambda_2}{r_1\lambda_1}\right)^{2k} \cdot b_1^{(1)}(z). \tag{3.13}$$

The ratio $\frac{r_2\lambda_2}{r_1\lambda_1}$ cannot exceed one, since in this case we would have concluded from (3.13) that $b_1^{(1)}(z)$ tends to infinity as $z \to 0$ which is impossible as $b_1^{(1)}$ is a finite Blaschke product. If $\frac{r_2\lambda_2}{r_1\lambda_1} = 1$, we conclude from (3.13) that $b_1^{(1)}$ takes the same value on an infinite sequence of points converging to the origin and therefore, $b_1^{(1)}$ is a unimodular constant by the uniqueness theorem. If $\frac{r_2\lambda_2}{r_1\lambda_1} < 1$, then we conclude as before, that $b_1^{(1)} = 0$ and therefore, $b_1^{(1)}(z) = zb_1^{(2)}(z)$ for a finite Blaschke product $b_1^{(2)}$ subject to identity

$$b_1^{(2)}\left(\left(\frac{r_1}{r_2}\right)^{2k} z\right) = \left(\frac{r_2^2\lambda_2}{r_1^2\lambda_1}\right)^{2k} \cdot b_1^{(2)}(z).$$

We continue this procedure which will stop after $m \le n_1 + n_2 - 1$ steps with a unimodular constant $b_1^{(m)}$ showing therefore, that (3.11) is possible only if $b_1(z) = b_2(z) = z^m$. In this case, the target values w_i are very special

$$w_i = \delta z_i^m \quad \text{for some } m \le n_1 + n_2 - 1 \text{ and } \delta \in \mathbb{D}, \tag{3.14}$$

and it is clear that in this case, $\lambda_r = \frac{|\delta|}{r^m}$ and $f_r(z) = \delta z^m$ for every $r \in (r_0, 1)$. □

Corollary 3.4. *The correspondence $r \to f_r$ is one-to-one in the following two cases:*

1. $|w_i| \ne |w_j|$ *for some* $i, j \in \mathcal{I}_2$;
2. $w_i = 0 \ne z_i$ *for some* $i \in \mathcal{I}_{12}$.

For the proof, it suffices to observe that both assumptions exclude (3.14).

4. Construction of low-degree solutions

In this section we develop several algorithms producing low degree solutions to the problem (1.1) as well as to several particular cases of this problem. Corollary 3.4 suggests the following procedure to get an infinite family of low-degree solutions to the problem $\mathbf{NP}(\mathcal{I}_{12})$ regardless the target values w_i are special as in (3.14) or not.

Algorithm 1. $\mathcal{RS}_{\le n_1+n_2-1}$*-solutions to the problem* $\mathbf{NP}(\mathcal{I}_{12})$.

Case 1: *$z_i \ne 0$ and $w_i = 0$ for some $i \in \mathcal{I}_{12}$.*

Step 1: Construct $\mathbf{P}_r$ as in (3.1) and find $r_0 = \max\{r \in [0, 1) : \det \mathbf{P}_r = 0\}$.

Step 2: For every $r \in (r_0, 1)$, find λ_r, the maximal solution of Eq. (3.5).

Step 3: Construct the function f_r as in (3.9).

For every $r \in (r_0, 1)$, the function f_r belongs to $\mathcal{RS}_{\le n_1+n_2-1}$ and solves the problem $\mathbf{NP}(\mathcal{I}_{12})$. Different parameters r lead to different functions f_r by Corollary 3.4.

Case 2: *The target values w_i are all non-zero.*

Step 1: Modify the target values letting

$$\widetilde{w}_i = \frac{w_i - w_1}{1 - w_i\overline{w}_1} \quad \text{for } i \in \mathcal{I}_1 \cup \mathcal{I}_2.$$

Then $|\widetilde{w}_i| < 1$ for all $i \in \mathcal{I}_1 \cup \mathcal{I}_2$ and in addition, $\widetilde{w}_1 = 0$.

Step 2: Apply Case 1 to the modified problem with interpolation conditions

$$\widetilde{f}(z_i) = \widetilde{w}_i \quad \text{for } i \in \mathcal{I}_1 \cup \mathcal{I}_2. \tag{4.1}$$

For every solution $\widetilde{f}_r$ of this problem obtained by Step 1, the function

$$f_r(z) = \frac{\widetilde{f}_r(z) + w_1}{1 + \widetilde{f}_r(z)\overline{w}_1} \tag{4.2}$$

belongs to $\mathcal{RS}_{\le n_1+n_2-1}$ and solves the original problem $\mathbf{NP}(\mathcal{I}_{12})$. The transformation $\widetilde{f}_r \to f_r$ established by formula (4.2) is one-to-one which together with Corollary 3.4 implies that $f_{r_1} \not\equiv f_{r_2}$ whenever $r_1 \neq r_2$.

Observe that the latter algorithm applies to problems $\mathbf{NP}(\mathcal{I}_1)$ and $\mathbf{NP}(\mathcal{I}_2)$. In the first case we get explicit formulas for a family of $\mathcal{RS}_{\le n_1-1}$-solutions to the classical Nevanlinna–Pick problem; recall that the complete characterization of all such solutions as solutions of certain extremal problem has been obtained in [3–5]. In the second case we get a family of $\mathcal{RS}_{\le n_1-1}$-solutions to the "boundary-to-interior" problem $\mathbf{NP}(\mathcal{I}_2)$. Although this algorithm is a literal repetition of Algorithm 1, we display it here for the convenience of future references.

Algorithm 2. *$\mathcal{RS}_{\le n_2-1}$-solutions to the problem* $\mathbf{NP}(\mathcal{I}_2)$.

Case 1: *The target values w_i are not all of the same modulus.*

Step 1: Construct $P_{2,r}$ as in (3.1) and find $r_0 = \max\{r \in [0, 1) : \det P_{2,r} = 0\}$.

Step 2: For every $r \in (r_0, 1)$, find λ_r, the maximal solution of the equation

$$\det P_{2,r}(\lambda) = \det\left(\left[\frac{\lambda^2 - w_i\overline{w}_j}{1 - r^2 z_i\bar{z}_j}\right]_{i,j\in\mathcal{I}_2}\right) = 0,$$

and construct the function f_r as in (3.9). This function belongs to $\mathcal{RS}_{\le n_2-1}$ and solves the problem $\mathbf{NP}(\mathcal{I}_2)$.

Case 2: $|w_i| = |w_j|$ *for all* $i, j \in \mathcal{I}_2$. Since we excluded the case where all target values are the same, we have $w_i \neq 0$ for all $i \in \mathcal{I}_2$.

Step 1: Modify the target values w_i to $\widetilde{w}_i$ as in (4.1).

Step 2: Apply Case 1 to the modified problem with interpolation conditions

$$\widetilde{f}(z_i) = \widetilde{w}_i \quad \text{for } i \in \mathcal{I}_2.$$

For every solution $\widetilde{f}_r$ of this problem obtained by Step 1, the function f_r defined as in (4.2) belongs to $\mathcal{RS}_{\le n_2-1}$ and solves the original problem $\mathbf{NP}(\mathcal{I}_2)$.

It is readily seen from formula (3.8) that all solutions f_r obtained by Algorithms 1 and 2 have the H^∞-norm strictly less than one. We now present an alternative algorithm for the problem $\mathbf{NP}(\mathcal{I}_{12})$ based on Algorithm 2 and Corollary 2.2.

Algorithm 3. *$\mathcal{RS}_{\le n_1+n_2-1}$-solutions to the problem* $\mathbf{NP}(\mathcal{I}_{12})$.

Step 1: Construct the function Ψ as in (2.4) and modify the target values w_i $(i \in \mathcal{I}_2)$ as in (2.9).

Step 2: Apply Algorithm 2 to the modified "boundary-to-interior" problem with interpolation conditions

$$\mathcal{E}(z_i) = v_i \quad \text{for } i \in \mathcal{I}_2.$$

This modified problem is indeed "boundary-to-interior", due to Corollary 2.2.

Step 3: For every solution $\mathcal{E}_r$ to the modified problem, the function

$$f_r = \frac{\psi_{11}\mathcal{E}_r + \psi_{12}}{\psi_{21}\mathcal{E}_r + \psi_{22}} \tag{4.3}$$

solves the problem $\mathbf{NP}(\mathcal{I}_{12})$ (by Corollary 2.2) and belongs to $\mathcal{RS}_{\le n_1+n_2-1}$, since $\deg \mathcal{E}_r \le n_2 - 1$ and since the MacMillan degree of Ψ equals n_1. The advantage of this algorithm is entirely computational: once the interior conditions are eliminated, we deal with $n_2 \times n_2$ matrices $P_{2,r}$ and $P_{2,r}(\lambda)$ to produce different solutions rather than $(n_1 + n_2) \times (n_1 + n_2)$ matrices $\mathbf{P}_r$ and $\mathbf{P}_r(\lambda)$.

Algorithm 4. *$\mathcal{RS}_{\le n_2+n_3-1}$-solutions to the problem* $\mathbf{NP}(\mathcal{I}_{23})$.

Step 1: Choose an admissible tuple $\boldsymbol{\gamma} = \{\gamma_i\}_{i\in\mathcal{I}_2}$ in the sense of Definition 2.3 and construct the matrix function $\widetilde{\Theta}^{\gamma}$ as in (2.14)

Step 2: Apply Algorithm 2 to get $\mathcal{RS}_{\le n_2-1}$-solutions to the modified "boundary-to-interior" problem with interpolation conditions

$$\mathcal{E}(z_i) = v_i := \frac{\widetilde{\theta}^{\gamma}_{12}(z_i) - \widetilde{\theta}^{\gamma}_{22}(z_i)w_i}{\widetilde{\theta}^{\gamma}_{21}(z_i)w_i - \widetilde{\theta}^{\gamma}_{11}(z_i)} \quad i \in \mathcal{I}_2. \tag{4.4}$$

Step 3: For every solution $\mathcal{E}_r$ to the modified problem (obtained by applying Algorithm 2), the function

$$f=\frac{\widetilde{\theta}_{11}^{\gamma}\mathcal{E}_r+\widetilde{\theta}_{12}^{\gamma}}{\widetilde{\theta}_{21}^{\gamma}\mathcal{E}_r+\widetilde{\theta}_{22}^{\gamma}} \tag{4.5}$$

solves the problem $\mathbf{NP}(\mathcal{I}_{23})$.

Justification: Due to the Stein identity (2.12), the function Θ^{γ} satisfies

$$J-\Theta^{\gamma}(z)J\Theta^{\gamma}(\zeta)^*=(1-z\bar{\zeta})\begin{bmatrix}E_3^*\\ M_3^*\end{bmatrix}(I-zT_3^*)^{-1}P_{3,\gamma}^{-1}(I-\bar{\zeta}T_3)^{-1}\begin{bmatrix}E_3 & M_3\end{bmatrix}$$

and therefore, Θ^{γ} is J-unitary on $\mathbb{T}\setminus\{z_i : i\in\mathcal{I}_3\}$. Then the arguments from the proof of Corollary 2.2 show that (1) the denominator in (4.4) does not vanish for every $w_i\in\mathbb{D}$ and (2) that $|v_i|<1$ for all $i\in\mathcal{I}_2$. Therefore, the modified problem (4.4) is indeed of the "boundary-to-interior" type and we may apply Algorithm 2 to get a family of solutions $\mathcal{E}_r\in\mathcal{RS}_{\le n_2-1}$ to this problem.

It is readily verified that equalities (4.4) are equivalent to f of the form (4.5) to satisfy conditions $f(z_i)=w_i$ for $i\in\mathcal{I}_2$. On the other hand, since $|\mathcal{E}_r(z_i)|\le\|\mathcal{E}_r\|_\infty<1$ and since $|\widetilde{\theta}_{21}^{\gamma}(z_i)|=|\widetilde{\theta}_{22}^{\gamma}(z_i)|\neq 0$ for every $i\in\mathcal{I}_2$ (the proof can be found in [13]; see Lemma 3.1 and Remark 3.2 there), it follows that conditions (2.18) are met and therefore, by Theorem 2.4, the function f of the form (4.5) satisfies conditions $f(z_i)=w_i$ for all $i\in\mathcal{I}_2\cup\mathcal{I}_3$ and $|f'(z_i)|=\gamma_i$ for all $i\in\mathcal{I}_3$. We also conclude from (4.5) that

$$\deg f\le\deg\Theta^{\gamma}+\deg\mathcal{E}_r\le n_3+n_2-1.$$

Since formula (4.5) fixes $|f'(z_i)|$, it follows that different choices of $\boldsymbol{\gamma}$ and r lead via (4.5) to different solutions to the problem $\mathbf{NP}(\mathcal{I}_{23})$.

Algorithm 5. *$\mathcal{RS}_{\le n_2-1}$-solutions to the problem* $\mathbf{NP}(\mathcal{I}_2)$ *with the minimally possible H^∞-norm.*

Step 1: Let $\delta_{\min}:=\max_{i\in\mathcal{I}_2}|w_i|$ and apply Algorithm 4 to the rescaled problem with interpolation conditions $g(z_i)=w_i':=\frac{w_i}{\delta_{\min}}$ for $i\in\mathcal{I}_2$. For every solution g to the rescaled problem, the function $f(z)=\delta_{\min}\cdot g(z)$ is a minimal norm solution to the problem $\mathbf{NP}(\mathcal{I}_2)$. It is obvious that $\|f\|_\infty\ge\delta_{\min}$ for every solution f to the problem $\mathbf{NP}(\mathcal{I}_2)$ so that $\delta_{\min}$ is indeed the minimally possible value of the norm of a solution.

Remark 4.1. Let us assume for the sake of definiteness that $|w_i|=\delta_{\min}$ for $i\in\mathcal{I}_2'$ and $|w_i|<\delta_{\min}$ for $i\in\mathcal{I}_2''=\mathcal{I}_2\backslash\mathcal{I}_2'$. Strictly speaking, Algorithm 4 applies to the rescaled problem only in the "generic" case where $\mathcal{I}_2''\neq\emptyset$. In this case, Theorem 1.4 tells us that every rational minimal-norm solution f to the problem $\mathbf{NP}(\mathcal{I}_2)$ is subject to $\deg f\ge|\mathcal{I}_2'|$. In case $\mathcal{I}_2''=\emptyset$ (that is, all the target values w_i are of the same modulus), the rescaled interpolation problem is of the "boundary-to-boundary" type and the existence of infinitely many solutions of degree at most n_2-1 follows from Ruscheweyh–Jones Theorem 1.3. We refer to [17] for the explicit construction. Observe that every such solution is a scaled finite Blaschke product (by Theorem 1.4).

Algorithm 6. *$\mathcal{RS}_{\le n_1+n_2+n_3-1}$-solutions to the problem* $\mathbf{NP}(\mathcal{I}_{123})$.

Step 1: Construct the function Ψ as in (2.4) and modify the target values w_i ($i\in\mathcal{I}_2\cup\mathcal{I}_3$) as in (2.9).

Step 2: Apply Algorithm 4 to the modified "$\mathbb{T}\to\overline{\mathbb{D}}$" problem with interpolation conditions $\mathcal{E}(z_i)=v_i$ for $i\in\mathcal{I}_2\cup\mathcal{I}_3$. For every solution $\mathcal{E}_r$ to the modified problem, the function f defined as in (4.3) solves the problem $\mathbf{NP}(\mathcal{I}_{123})$ (by Corollary 2.2) and belongs to $\mathcal{RS}_{\le n_1+n_2+n_3-1}$.

To conclude this section we present the proofs of Theorems 1.5 and 1.6.

Proof of Theorem 1.5. The necessity part was presented just below the formulation. The sufficiency part is justified by Algorithms 1 and 6. □

Proof of Theorem 1.6. Let us recall the notation $\mu:=\inf\|f\|_\infty$ where infimum is taken over all solutions to the problem $\mathbf{NP}(\mathcal{I}_{12})$. Since every solution f to this problem solves the subproblems $\mathbf{NP}(\mathcal{I}_1)$ and $\mathbf{NP}(\mathcal{I}_2)$, it follows from Theorems 1.1 and 1.2 that

$$\|f\|_\infty\ge\mu:=\max\{\lambda_{\min},\delta_{\min}\}, \tag{4.6}$$

where $\delta_{\min}:=\max\{|w_i| : i\in\mathcal{I}_2\}$ and $\lambda_{\min}$ is the maximal solution of the Eq. (1.4). Let us pick any $\mu'>\mu$ and let us consider the rescaled interpolation problem with interpolation conditions

$$g(z_i)=w_i':=\frac{w_i}{\mu'}\quad\text{for } i\in\mathcal{I}_1\cup\mathcal{I}_2. \tag{4.7}$$

The Pick matrix $P_1'=\left[\frac{1-w_i'\overline{w}_j'}{1-z_i\bar{z}_j}\right]_{i,j=1}^{n_1}$ is positive definite, since $\mu'>\lambda_{\min}$ and on the other hand, $|w_i'|<1$ for all $i\in\mathcal{I}_2$, since $\mu'>\delta_{\min}$. By Theorem 1.5, the problem (4.7) has infinitely many solutions $g\in\mathcal{RS}_{\le n_1+n_2-1}$ and Algorithm 1 produces an infinite family of such solutions. For every $g \in \mathcal{RS}_{\leq n_1+n_2-1}$ satisfying (4.7), the function $f(z) = \mu' \cdot g(z)$ belongs to $\mathcal{RS}_{\leq n_1+n_2-1}$, solves the original problem $\mathbf{NP}(\mathcal{I}_2)$ and satisfies $\|f\|_\infty \leq \mu'$. Since every solution f to the problem $\mathbf{NP}(\mathcal{I}_2)$ satisfies inequality (4.6) and since for every $\mu' > \mu$, there exists a solution f such that $\|f\|_\infty \leq \mu'$, the first statement in Theorem 1.6 (equality (1.5)) follows. □

By the previous arguments, every minimal-norm solution f to the problem $\mathbf{NP}(\mathcal{I}_2)$ is necessarily of the form $f(z) = \mu \cdot g(z)$ where μ is defined as in (4.6) and where g is a Schur-class function solving the rescaled problem

$$g(z_i) = w_i' := \frac{w_i}{\mu} \quad \text{for } i \in \mathcal{I}_1 \cup \mathcal{I}_2. \tag{4.8}$$

Thus, the minimal-norm solution to the problem $\mathbf{NP}(\mathcal{I}_{12})$ exists if and only if there exists a Schur-class solution g to the problem (4.8).

Case 1: Let us assume that $\lambda_{\min} < \delta_{\min}$ so that $\mu = \delta_{\min}$ and the Pick matrix P_1' corresponding to the "interior" conditions in (4.8) is positive definite. We assume (as in Remark 4.1) that

$$|w_i| = \delta_{\min} \quad \text{for } i \in \mathcal{I}_2' \quad \text{and} \quad |w_i| < \delta_{\min} \quad \text{for } i \in \mathcal{I}_2'' = \mathcal{I}_2 \setminus \mathcal{I}_2'.$$

If $\mathcal{I}_2'' \neq \emptyset$, then the rescaled problem (4.8) is of the same type as $\mathbf{NP}(\mathcal{I}_{123})$ (with $\mathcal{I}_2$ and $\mathcal{I}_3$ replaced by $\mathcal{I}_2''$ and $\mathcal{I}_2'$, respectively) and it is indeterminate (by Theorem 1.5) since $P_1' > 0$. We may apply Algorithm 6 to construct a family of $\mathcal{S}_{\leq n_1+n_2-1}$-solutions to this problem which in turn, will produce a family of low-degree minimal-norm solutions to the original problem $\mathbf{NP}(\mathcal{I}_{12})$. If $\mathcal{I}_2'' = \emptyset$, then the problem (4.8) is of the same type as $\mathbf{NP}(\mathcal{I}_{13})$. In this case, the explicit construction of an infinite family of solutions can be found in [7].

We next observe that if $\delta_{\min} \leq \lambda_{\min} = \mu$, then the matrix $P_1(\mu) = \left[\frac{\mu^2 - w_i\overline{w}_j}{1-z_i\bar{z}_j}\right]_{i,j\in\mathcal{I}_1}$ is positive semidefinite and singular. By Theorem 1.1 there exists a unique function $\hat{f} \in \mathcal{S}$ with $\|\hat{f}\|_\infty = \mu$ and satisfying conditions (1.1) for all $i \in \mathcal{I}_1$. This function is necessarily of the form $\hat{f}(z) = \mu \cdot b(z)$ where b is a Blaschke product of degree $\deg b = \rho := \operatorname{rank} P_1(\mu)$. Therefore $|\hat{f}(z)| = \mu$ for every $z \in \mathbb{T}$. The explicit formula for $\hat{f}$ is similar (3.9): we pick any subset $\mathcal{I} \subset \mathcal{I}_1$ with $|\mathcal{I}| = \rho$ and let

$$P_\rho = \left[\frac{\mu^2 - w_i\overline{w}_j}{1 - z_i\bar{z}_j}\right]_{i,j\in\mathcal{I}}, \qquad T_\rho = \operatorname{diag}_{i\in\mathcal{I}}\{z_i\}, \qquad M_\rho = \operatorname{Col}_{i\in\mathcal{I}}\{w_i\}.$$

We also let $E_\rho \in \mathbb{C}^{\rho\times 1}$ to be the column with all entries equal one. Then P_ρ is positive definite and $\hat{f}$ can be written as

$$\hat{f}(z) = \mu^2 \cdot \frac{1 - (1 - z\bar{z}_j)E_\rho^*(I - zT_\rho^*)^{-1}P_\rho^{-1}(I - \bar{z}_jT_\rho)^{-1}(\mu^2E_\rho - M_\rho\overline{w}_j)}{\overline{w}_j - (1 - z\bar{z}_j)M_\rho^*(I - zT_\rho^*)^{-1}P_\rho^{-1}(I - \bar{z}_jT_\rho)^{-1}(\mu^2E_\rho - M_\rho\overline{w}_j)}, \tag{4.9}$$

where j is *any* index from $\mathcal{I}_1 \setminus \mathcal{I}$. We now consider the two remaining cases in Theorem 1.6.

Case 2: If $\lambda_{\min} > \delta_{\min} = \max\{|w_i| : i \in \mathcal{I}_2\}$, then for every $i \in \mathcal{I}_2$, we have

$$|w_i| \leq \max\{|w_i| : i \in \mathcal{I}_2\} = \delta_{\min} < \lambda_{\min} = \mu$$

and since $|\hat{f}(z)| = \mu$ for every $z \in \mathbb{T}$, we conclude that $\hat{f}$ cannot satisfy condition (1.1) for every $i \in \mathcal{I}_2$. Therefore the problem $\mathbf{NP}(\mathcal{I}_{12})$ has no minimal-norm solutions.

Case 3: If $\mu = \lambda_{\min} = \delta_{\min}$, the unique candidate $\hat{f}$ might solve the problem $\mathbf{NP}(\mathcal{I}_{12})$. A necessary (but still not sufficient) condition for this to happen is that $|w_i| = \mu$ for all $i \in \mathcal{I}_2$. However, taking the advantage of the explicit formula (4.9) we can verify equalities $\hat{f}(z_i) = w_i$ for every $i \in \mathcal{I}_2$. These equalities provide the uniqueness criterion (in terms of interpolation data) for the minimal-norm solution to the problem $\mathbf{NP}(\mathcal{I}_{12})$. □

From the computational point of view, the uniqueness criterion presented in the proof of Case 3 above makes perfect sense. We conclude this section with its equivalent reformulation which is more consistent with the tradition of the norm-constraint interpolation theory to give necessary and sufficient conditions in terms of Pick matrices.

Lemma 4.2. *Let $\mu < 1$ be the maximal solution of the Eq.* (1.4) *and let $|w_i| = \mu$ for every $i \in \mathcal{I}_2$. Let $\mathbb{P} = [p_{ij}]_{i,\, j\in\mathcal{I}_1\cup\mathcal{I}_2}$ be the partially defined matrix whose entries are specified by the formula*

$$p_{ij} = \frac{\mu^2 - w_i\overline{w}_j}{1 - z_i\bar{z}_j} \quad \text{for all } i, j \in \mathcal{I}_1 \cup \mathcal{I}_2,$$

except for the diagonal entries p_{ii} ($i \in \mathcal{I}_2$) which are not specified. In particular, the matrix $P_1(\mu) = \left[\frac{\mu^2-w_i\overline{w}_j}{1-z_i\bar{z}_j}\right]_{i,j\in\mathcal{I}_1}$ is a completely specified principal submatrix of $\mathbb{P}$. The problem $\mathbf{NP}(\mathcal{I}_{12})$ has a unique minimal-norm solution if and only if the matrix $\mathbb{P}$ can be completed by an appropriate choice of p_{ii} ($i \in \mathcal{I}_2$) to a positive semidefinite matrix so that $\operatorname{rank}\mathbb{P} = \operatorname{rank} P_1(\mu)$.

Proof. The problem $\mathbf{NP}(\mathcal{I}_{12})$ has at most one minimal-norm solution since the Pick matrix $P_1(\mu)$ is singular. The only candidate $\hat{f}$ indeed satisfies conditions (1.1) for all $i \in \mathcal{I}_2$, we let $p_{ii} = |\hat{f}'(z_i)|$ for every $i \in \mathcal{I}_2$. The completed matrix $\mathbb{P}$ is positive semidefinite and its rank is equal to the degree of $\hat{f}$ (and therefore to $\operatorname{rank} P_1(\mu)$ by Lemma 2.1 in [15]). □

5. An example

In this concluding section, we illustrate some of the previous results by a simple example. Let us consider the two-point boundary Nevanlinna–Pick interpolation problem of the type $\mathbf{NP}(\mathcal{I}_2)$ with interpolation conditions

$$f(1) = \frac{1}{2} \quad \text{and} \quad f(-1) = \frac{1}{3}. \tag{5.1}$$

We now apply Algorithm 2 to get all $\mathcal{RS}_1$ solutions to this problem. We start with the perturbed Pick matrix

$$P_{2,r} = \begin{bmatrix} \dfrac{3/4}{1-r^2} & \dfrac{5/6}{1+r^2} \\ \dfrac{5/6}{1+r^2} & \dfrac{8/9}{1-r^2} \end{bmatrix}$$

and compute r_0, the maximal solution of the equation $\det P_{2,r} = 0$ on the interval $[0, 1)$. It turns out that $\boxed{r_0 = 5 - 2\sqrt{6}}$. We next write

$$P_{2,r}(\lambda) = \begin{bmatrix} \dfrac{\lambda^2 - 1/4}{1-r^2} & \dfrac{\lambda^2 - 1/6}{1+r^2} \\ \dfrac{\lambda^2 - 1/6}{1+r^2} & \dfrac{\lambda^2 - 1/9}{1-r^2} \end{bmatrix}$$

and for every $r \in (r_0, 1)$ we compute λ_r, the maximal solution of the equation $\det P_{2,r}(\lambda) = 0$. Some routine elementary algebra gives

$$\lambda_r = \frac{r^2 + 1 + \sqrt{r^4 + 98r^2 + 1}}{24r}. \tag{5.2}$$

We then use formula (3.9) and some elementary algebra to get a family of linear fractional solutions to the problem (5.1):

$$f_r(z) = \lambda_r^2 \cdot \frac{1 - G - zr(1+G)}{1/3 - G/2 - zr(1/3 + G/2)}, \quad \text{where } G = \frac{1-r^2}{1+r^2} \cdot \frac{\lambda_r^2 - 1/6}{\lambda_r^2 - 1/4}. \tag{5.3}$$

For any solution f of this problem, we have $\|f\|_\infty \geq \max\{1/2, 1/3\} = 1/2$. On the other hand, for f_r of the form (5.3), $\|f_r\|_\infty = \lambda_r$. By (5.2), λ_r increases to $1/2$ as $r \to 1$ and thus, for every r sufficiently close to one, the norm of $\|f\|_\infty$ will be close to the minimally possible value $1/2$.

Now we will apply Algorithm 4 to construct a family of low-degree minimal-norm solutions for the same problem (5.1). All such solutions are of the form $f(z) = g(z)/2$ where g is a $\mathcal{RS}_1$ functions such that

$$g(1) = 1 \quad \text{and} \quad g(-1) = \frac{2}{3}. \tag{5.4}$$

The latter problem is of the type $\mathbf{NP}(\mathcal{I}_{23})$. We fix a positive number γ, let $\mu = -1$ and use the formula (2.13) to compute

$$\Theta^\gamma(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1+z}{2\gamma(1-z)} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Then the function

$$g(z) = \frac{\frac{2}{3}\left(1 - \frac{1+z}{2\gamma(1-z)}\right) + \frac{1+z}{2\gamma(1-z)}}{-\frac{2}{3} \cdot \frac{1+z}{2\gamma(1-z)} + 1 + \frac{1+z}{2\gamma(1-z)}} = \frac{(1-4\gamma)z + 1 + 4\gamma}{(1-6\gamma)z + 1 + 6\gamma}$$

belongs to $\mathcal{RS}_1$ for every $\gamma > 0$ and satisfies conditions (5.4). Thus, the formula

$$f_\gamma(z) = \frac{1}{2} \cdot \frac{(1-4\gamma)z + 1 + 4\gamma}{(1-6\gamma)z + 1 + 6\gamma}$$

gives a family of minimal-norm low-degree solutions for the problem (5.1).

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