Totally positive shapes and TPk-completable patterns

S. Nasserasr

D. Hoff

C. R. Johnson
William & Mary, crjohn@WM.EDU

Linear Algebra and its Applications 436 (2012) 4412–4422

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Totally positive shapes and TP_k-completable patterns ☆

D. Hoff [a], C.R. Johnson [b], S. Nasserasr [c,*]

[a] Department of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church St. SE, Minneapolis, MN 55455, USA
[b] Department of Mathematics, College of William and Mary, P.O. Box 8795, Williamsburg, VA 23187-8795, USA
[c] Department of Mathematics and Statistics, University of Regina, College West 307.14, Regina, Sask., Canada S4S 0A2

ARTICLE INFO

Article history:
Received 25 February 2011
Accepted 9 May 2011
Available online 31 August 2011

Submitted by S. Fallat

AMS classification:
15A83
15B48

Keywords:
Matrix completion problem
Totally positive matrix
TP_k matrix

ABSTRACT

The notions of total positivity and of TP_k are generalized to "shapes" (a generalization of matrices). In particular, the relationship between positivity of "contiguous" minors and all minors is characterized for general shapes and for certain special types of shapes. This and other ideas are used to address the TP_k-completion problem and TP_k-completable patterns. In case $k = 2$, a near characterization of TP_2-completable patterns is given.

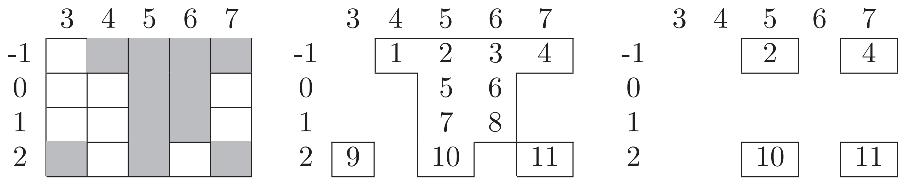
1. Introduction

An m-by-n matrix A is called TP_k (*totally positive k-by-k*) if every minor of size at most k is positive. If $k = \min\{m, n\}$, the matrix is simply called *TP* (*totally positive*). A *partial matrix* is one in which some of the entries are specified while the remaining, unspecified, entries are free to be chosen. A *completion* of a partial matrix is a choice of values for the unspecified entries resulting in a conventional matrix. A *TP_k-completion* of a partial matrix $\mathcal{P}$ is a completion of $\mathcal{P}$ such that the result is a TP_k matrix.

The TP_k-completion problem asks which partial matrices have a TP_k-completion. In this paper, the *pattern* of specified entries is considered, that is, the arrangement of the specified entries. An obvious necessary condition for TP_k-completability is that every ℓ-by-ℓ fully specified submatrix has a positive determinant, for $\ell = 1, \ldots, k$. Matrices satisfying this condition are called *partial TP_k*. A pattern is

☆ All authors were supported in part by NSF Grant #DMS-0751964.
* Corresponding author.
E-mail addresses: hoffx118@umn.edu (D. Hoff), crjohnso@math.wm.edu (C.R. Johnson), snaseras@uvic.ca (S. Nasserasr).

0024-3795/$ - see front matter
http://dx.doi.org/10.1016/j.laa.2011.05.027



(a) $\mathcal{S}$ (b) $A[\mathcal{S}]$ (c) $S[\{-1, 2\} \times \{5, 7\}]$

Fig. 1. (a) A shape $\mathcal{S}$ represented as the shaded subset of the cells of a matrix. (b) A shape $A[\mathcal{S}]$ formed by associating integers to each $(i, j) \in \mathcal{S}$. (c) A square subshape (submatrix) $A[\{-1, 2\} \times \{5, 7\}]$ of $A[\mathcal{S}]$. The determinant of this square subshape is a 2 *by*2 minor of $A[\mathcal{S}]$ with value -18.

called *TP$_k$-completable* if this condition is also sufficient for any matrix with that pattern. In other words, a pattern $\mathcal{P}$ is TP$_k$-completable if every partial TP$_k$ matrix with pattern $\mathcal{P}$ has a TP$_k$-completion. We are particularly interested in *TP$_2$*-completable patterns.

A series of intersecting horizontal and vertical axes in the plane forms a *grid* and its *cells*. By a *shape* we mean an arbitrary subset of the cells of a grid. Following the matrix notations, a horizontal set of cells in a shape $\mathcal{S}$ is called a *row* and a vertical set of cells in $\mathcal{S}$ is called a *column* of $\mathcal{S}$. A shape with m rows and n columns is said to be of dimension m-by-n. If the number of rows or columns of a shape $\mathcal{S}$ is not finitely many, then $\mathcal{S}$ is said to be infinite dimensional. We allow the rows and columns of a shape to be labeled from $\mathbb{Z}$. Thus a shape may be identified with an index set $\mathcal{S}$ of distinct ordered pairs $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. For example, in Fig. 1(a), the shaded cells in the 4-by-5 matrix constitute a shape with $\mathcal{S} = \{(-1, 4), (-1, 5), (-1, 6), (-1, 7), (0, 5), (0, 6), (1, 5), (1, 6), (2, 3), (2, 5), (2, 7)\}$.

Typically, we associate a shape $\mathcal{S}$ with a field $\mathbb{F}$, and each cell $(i, j) \in \mathcal{S}$ with a value $a_{ij} \in \mathbb{F}$ as shown in Fig. 1(b), writing $A[\mathcal{S}] = \{a_{ij}\}$. We will call the resulting object a shape as well. A shape is *rectangular* if its index set can be written as $\mathcal{S} = I \times J$ for some sets $I, J \subset \mathbb{Z}$. If, in addition, $|I| = |J|$, the shape is said to be *square*. A shape $\mathcal{S}'$ is a *subshape* of $\mathcal{S}$ if $\mathcal{S}' \subset \mathcal{S}$, and a shape $A[\mathcal{S}']$ is a *subshape* of $B[\mathcal{S}]$ if $\mathcal{S}' \subset \mathcal{S}$ and $a_{ij} = b_{ij}$ for all $(i, j) \in \mathcal{S}'$.

Note that a traditional matrix is a rectangular shape, and that the notion of a rectangular subshape corresponds exactly with that of a submatrix. As such, we may refer to a rectangular subshape as a submatrix. We have chosen notation with every effort to extend that already accepted for matrices. Given a matrix A, for instance, it is traditional to denote the submatrix indexed by rows α and columns β by $A[\alpha, \beta]$. This notation can only refer to a submatrix and not a more general shape, so we use the similar notation $A[\alpha \times \beta]$ to refer to that submatrix, allowing us to speak of more general subshapes as $A[\mathcal{S}]$ for a more general index set $\mathcal{S}$.

In this way, the notions of TP$_k$ may be extended to shapes by simply thinking of an ℓ-by-ℓ minor as the determinant of an ℓ-by-ℓ square submatrix such as that in Fig. 1(c). The notions of a *partial shape* and *pattern shape* extend from matrices in the natural way, and we may therefore consider TP$_k$-completability in shapes.

A *contiguous* submatrix M of a shape $\mathcal{S}$ is a submatrix whose rows and columns are indexed by consecutive integers. A shape $\mathcal{S}$ is called *TP$_k$-contiguous*, denoted by TP$_k$C, if its ℓ-by-ℓ contiguous submatrices have positive determinant for all $\ell = 1, \ldots, k$. By definition, every TP$_k$ shape is also TP$_k$C. The following lemma shows that the converse is true in traditional matrices; see [1].

Lemma 1.1. *An m-by-n matrix A is TP$_k$ if and only if it is TP$_k$-contiguous.*

For shapes, however, the analogous statement is not always true. In the next section we generalize Lemma 1.1 under a broader notion of contiguity and characterize the shapes for which traditional contiguity is sufficient for being TP$_k$. In Section 3, we introduce the notions of barriers and thickness, which allows us to apply these ideas to TP$_k$-completions and completable patterns in the final section.

2. Relating TP_k and TP_kC in shapes

In this section, we consider the set of conditions sufficient for a shape to be TP_k. As stated in Section 1, a matrix is TP_k if and only if it is TP_kC. We begin by proving an analogous result for shapes under a broadened notion of contiguity, and conclude the section by classifying the shapes for which classical contiguity is sufficient.

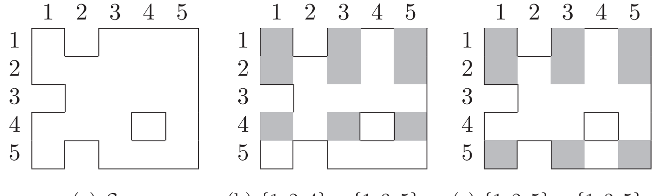
For a shape $\mathcal{S}$, consider a submatrix $M = I \times J \subset \mathcal{S}$. Then M is called *quasi-row-contiguous* (QRC) if $\min\{i \in I\} < i_0 < \max\{i \in I\}$ and $i_0 \times J \subset \mathcal{S}$ imply $i_0 \in I$. Similarly, M is called *quasi-column-contiguous* (QCC) if $\min\{j \in J\} < j_0 < \max\{j \in J\}$ and $I \times j_0 \subset \mathcal{S}$ imply $j_0 \in J$. A submatrix that is both QRC and QCC is simply called *quasi-contiguous* (QC). A minor is said to be quasi-contiguous if its corresponding submatrix is quasi-contiguous. Intuitively, quasi-contiguity is the property that the rows and columns of a submatrix, of a given size, cannot be any closer together without leaving the shape in which the submatrix resides.

Note that all contiguous submatrices are QC. In fact, given a matrix, the set of QC submatrices is exactly the set of contiguous submatrices. The same is not true for shapes in general. For instance, consider the submatrix in Fig. 2(b), which is quasi-contiguous as a submatrix of the shape $\mathcal{S}$ in 2(a), but not contiguous. It is important to note that the quasi-contiguity of a submatrix depends upon the shape from which it is taken. Indeed any submatrix is quasi-contiguous when considered as a subshape of itself. For reference, Fig. 2(c) shows an example of a submatrix of $\mathcal{S}$ which is not QC, because both its third row index and third column index could be decreased with a resulting submatrix still contained in the shape.

This broader notion of contiguity allows us to determine a set of sufficient conditions for a shape to be TP_k. The next theorem makes these conditions precise. Its proof, however, requires an additional definition. For a given m-by-n shape $\mathcal{S}$, there exists a minimal matrix $M \supset \mathcal{S}$ of dimension m'-by-n'. The *bulk* of $\mathcal{S}$ is $b = m' + n'$. It is important to note that the bulk of a submatrix does not always coincide with the sum of its dimensions. For instance, the shaded submatrix in Fig. 2(b) has bulk $b = 9$, whereas that shaded in 2(c) has bulk $b = 10$, even though these are both 3-by-3 submatrices.

Theorem 2.1. *For any $k \geq 1$, a shape $A[\mathcal{S}]$ is TP_k if and only if all of its ℓ-by-ℓ quasi-contiguous minors are positive for $\ell = 1, \ldots, k$.*

Proof. Necessity is immediate from the definition of TP_k. To prove sufficiency, we show that for a shape $A[\mathcal{S}]$ with positive QC minors, the determinant of any ℓ-by-ℓ submatrix is positive. The proof is by induction on b, the bulk of the square submatrices. All submatrices of bulk 2 have positive determinant, as they are contiguous, so the statement holds for $b = 2$. Now for $b \geq 2$, suppose all square submatrices of bulk b or less have positive determinant and consider an arbitrary submatrix $A[I \times J]$ with $|I| = |J| = \ell$ of bulk $b + 1$. If the submatrix $I \times J$ is QC, then $\det A[I \times J] > 0$ by assumption. Otherwise, it fails either row or column quasi-contiguity. We assume the former; the latter case follows from a symmetrical argument. Writing $I = \{i_1, i_2, ..., i_\ell\}$ in increasing order, this means that there exists $i_0 \notin I$ such that $i_0 \times J \subset \mathcal{S}$ and $i_h < i_0 < i_{h+1}$ for some $1 \leq h \leq \ell - 1$. Consider an $(\ell + 1)$-by-ℓ matrix B whose entries are those of $A[\{i_1, \ldots, i_h, i_0, i_{h+1}, \ldots, i_\ell\} \times J]$. Any contiguous square submatrix of B corresponds to a submatrix of $A[\mathcal{S}]$ with bulk b or less. These minors



(a) $\mathcal{S}$ (b) $\{1, 2, 4\} \times \{1, 3, 5\}$ (c) $\{1, 2, 5\} \times \{1, 3, 5\}$

Fig. 2. This figure depicts a shape $\mathcal{S}$ in (a). The shaded cells in (b) and (c) constitute 3-by-3 submatrices of $\mathcal{S}$. The submatrix in (b) is quasi-contiguous, while that in (c) is not as it fails both quasi-row-contiguity and quasi-column-contiguity. The bulk of $\mathcal{S}$ is $b = 10$.

are positive by the inductive hypothesis, and since TP_kC implies TP_k in matrices (Lemma 1.1), all minors of B are positive. In particular, $\det A[I \times J] = \det B[\{1, \ldots, h, h+2, \ldots, \ell+1\} \times \{1, \ldots, \ell\}] > 0$. □

Theorem 2.1 gives a set of minors sufficient for checking whether a shape is TP_k. The following theorem describes the shapes for which it is sufficient to check only contiguous minors.

Theorem 2.2. *For any $k \geq 1$, the set of shapes for which TP_kC implies TP_k is exactly the set of shapes whose ℓ-by-ℓ quasi-contiguous minors are contiguous, $\ell = 1, \ldots, k$.*

Proof. Suppose $\mathcal{S}$ is a shape whose quasi-contiguous minors are contiguous and suppose $\mathcal{S}$ is TP_kC. Then in fact all of its quasi-contiguous minors are positive (since they are contiguous), so using Theorem 2.1, $\mathcal{S}$ is TP_k.

Conversely, suppose a shape $\mathcal{S}$ has the property that TP_kC implies TP_k. If $k = 1$ we are done, since all 1-by-1 minors are contiguous. For $k \geq 2$ we will show that all the quasi-contiguous minors of $\mathcal{S}$ up to size k-by-k are contiguous. Suppose there exists an ℓ-by-ℓ QC submatrix $I \times J \subset \mathcal{S}$, with $2 \leq \ell \leq k$, which is not contiguous. We will find a contradiction by constructing a shape $B[\mathcal{S}]$ that is TP_kC but not TP_k. We know that TP_kC matrices exist in any dimension, so considering $\mathcal{S}$ as a subshape of some TP_kC matrix, we know that there exists a set of values $A[\mathcal{S}] = \{a_{ij}\}$ such that $A[\mathcal{S}]$ is TP_kC.

There exists a quasi-contiguous 2-by-2 submatrix $\{i_1, i_2\} \times \{j_1, j_2\} \subset I \times J$ that is not contiguous, since otherwise, $I \times J$ is contiguous, which contradicts our assumption. Since $\{i_1, i_2\} \times \{j_1, j_2\}$ is not contiguous, there exists either i_0 with $i_1 < i_0 < i_2$ or j_0 with $j_1 < j_0 < j_2$. Assume the former; the latter case follows from a symmetrical argument. Since $\{i_1, i_2\} \times \{j_1, j_2\}$ is QC, we have either $(i_0, j_1) \notin \mathcal{S}$ or $(i_0, j_2) \notin \mathcal{S}$. Again without loss of generality assume the former. We now define the shape $B[\mathcal{S}] = \{b_{ij}\}$ as follows.

For any constants $c_1, c_2 > 0$ and $(i, j) \in \mathcal{S}$ we set

$$b_{ij} = \begin{cases} a_{ij}, & \text{if } j \neq j_1; \\ c_1 a_{ij}, & \text{if } j = j_1, i \leq i_1; \\ c_2 a_{ij}, & \text{if } j = j_1, i \geq i_2. \end{cases} \tag{1}$$

The contiguous minors are then affected in a predictable way. For $j \geq i$, we will use the notation $\{i : j\} = \{n \in \mathbb{Z} \mid i \leq n \leq j\}$. Then

$$\det B[\{i_3 : i_4\} \times \{j_3 : j_4\}] = \begin{cases} \det A[\{i_3 : i_4\} \times \{j_3 : j_4\}], & \text{if } j_1 \notin \{j_3 : j_4\}; \\ c_1 \det A[\{i_3 : i_4\} \times \{j_3 : j_4\}], & \text{if } j_1 \in \{j_3 : j_4\} \text{ and } i_4 \leq i_1; \\ c_2 \det A[\{i_3 : i_4\} \times \{j_3 : j_4\}], & \text{if } j_1 \in \{j_3 : j_4\} \text{ and } i_3 \geq i_2. \end{cases} \tag{2}$$

In any case, the sign of the contiguous minor is unchanged. Note that these three are the only cases because no contiguous minor can include elements $(i, j_1), (i', j_1)$ with $i \leq i_1, i' \geq i_2$ since it would then contain $(i_0, j_1) \notin \mathcal{S}$. Thus for any $c_1, c_2 > 0$, $B[\mathcal{S}]$ defined in this way is still in TP_kC. Notice, however, that $\det B[\{i_1, i_2\} \times \{j_1, j_2\}] = c_1 a_{i_1 j_1} a_{i_2 j_2} - c_2 a_{i_2 j_1} a_{i_1 j_2}$. Thus by choosing c_1 and c_2 properly, this minor can be made arbitrarily negative, making $B[\mathcal{S}]$ TP_kC but not TP_k. □

Theorem 2.2 completely classifies the shapes for which TP_kC implies TP_k. There exists another classification that is very useful. To illustrate it, we make a number of definitions.

A *path* of length η in a shape $\mathcal{S}$ is an ordered set of index pairs $\Gamma = \{(i_k, j_k)\}_{k=1}^{\eta} \subset \mathcal{S}$ such that $|i_{k+1} - i_k| + |j_{k+1} - j_k| = 1$ for all $k \in [1, \eta - 1]$. A path is called *closed* if $(i_1, j_1) = (i_\eta, j_\eta)$. A shape $\mathcal{S}$ is called *path connected* if for any $(i_1, j_1), (i_2, j_2) \in \mathcal{S}$ there exists a path $\Gamma \subset \mathcal{S}$ such that $(i_1, j_1), (i_2, j_2) \in \Gamma$. Finally, a shape $\mathcal{S}$ is *Manhattan convex* if it is path connected and if for any non-negative integers k_1, k_2, we have $(i, j) \in \mathcal{S}$ whenever $(i+k_1, j), (i-k_2, j) \in \mathcal{S}$ or $(i, j+k_1), (i, j-k_2) \in \mathcal{S}$. We consider the empty set to be Manhattan convex. Fig. 3(a) shows a Manhattan convex shape while 3(b) shows a shape that fails this condition.

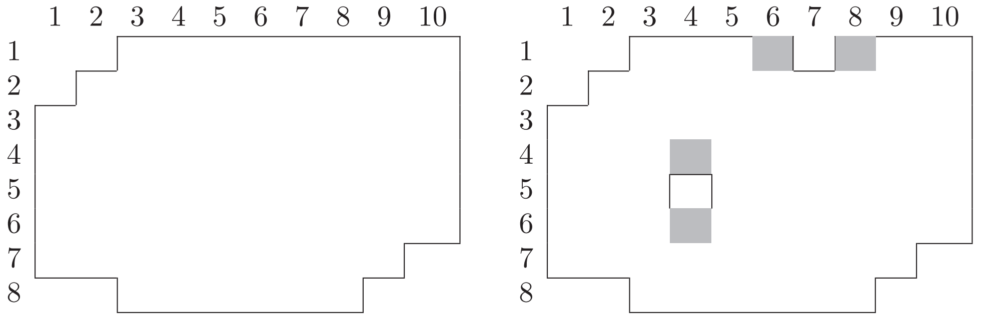


Fig. 3. This figure depicts (a) a Manhattan convex shape, and (b) a not Manhattan convex shape. The shape in (b) fails the convexity conditions at the two pairs of shaded cells.

Theorem 2.3. *For any $k \geq 2$, the set of path connected shapes whose ℓ-by-ℓ quasi-contiguous minors are contiguous, for $\ell = 1, \ldots, k$, is exactly the set of Manhattan convex shapes.*

Proof. We first show that in any Manhattan convex shape $\mathcal{S}$, all quasi-contiguous minors are contiguous. Consider an ℓ-by-ℓ quasi-contiguous submatrix of $\mathcal{S}$, say $I \times J$, with $I = \{i_1, \ldots, i_\ell\}, J = \{j_1, \ldots, j_\ell\}$. Since the array is Manhattan convex, $(i, j_1), (i, j_l) \in \mathcal{S}$ for any $i_1 \leq i \leq i_\ell$. Then, applying convexity to each of $i_\ell - i_1$ rows, we have $(i, j) \in \mathcal{S}$ for all $i_1 \leq i \leq i_\ell$ and $j_1 \leq j \leq j_\ell$. Therefore, for any i with $i_1 < i < i_\ell$ (or j with $j_1 < j < j_\ell$) we have $i \times J \subset S$ $(I \times j \subset \mathcal{S})$. Since $I \times J$ is QRC (QCC) this implies that $i \in I$ whenever $i_1 \leq i \leq i_\ell$ ($j \in J$ whenever $j_1 \leq j \leq j_\ell$). Thus, $I \times J$ is a contiguous submatrix.

To complete the proof we show that any path-connected shape whose quasi-contiguous minors are all contiguous is a Manhattan convex. It is sufficient to show that any shape that is not Manhattan convex has a QC minor of size at most k that is not contiguous. Since we assume $k \geq 2$ we can cover all cases by showing the existence of a 2-by-2 minor of this type.

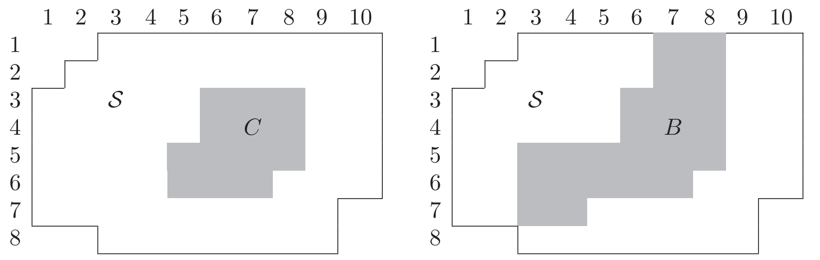
Consider therefore a path connected shape $\mathcal{S}$ that is not Manhattan convex. Then there exist either $(i_1, j_1), (i_2, j_1) \in \mathcal{S}$ with $(i, j_1) \notin \mathcal{S}$ for all $i_1 + 1 \leq i < i_2$, or $(i_1, j_1), (i_1, j_2) \in \mathcal{S}$ with $(i_1, j) \notin \mathcal{S}$ for all $j_1 + 1 \leq j < j_2$. Assume the former; the latter case follows by a similar argument.

By assumption, $\mathcal{S}$ is path connected. Let Γ be a path in $\mathcal{S}$ containing (i_1, j_1) and (i_2, j_1). The path will at some point pass through the $i_1 + 1$ row since $i_1 < i_1 + 1 < i_2$. Since $(i_1 + 1, j_1) \notin I$, Γ contains a point $(i_1 + 1, j_2)$ with $j_2 \neq j_1$. Therefore, starting at (i_1, j_1), at some point the path passes from that column towards the j_2 column, that is, there exist $(i_1 - \delta_1, j_1), (i_1 - \delta_1, j_1 \pm 1) \in \Gamma$, for some $\delta_1 \in \mathbb{Z}_{\geq 0}$ where the sign in $j_1 \pm 1$ is given by sign$(j_2 - j_1)$. The path will reenter the j_1 column for some $i \geq i_2$ as well, so there exist $(i_2 + \delta_2, j_1 \pm 1), (i_2 + \delta_2, j_1) \in \Gamma$ for some $\delta_2 \in \mathbb{Z}_{\geq 0}$. Thus, $\{i_1 - \delta_1, i_2 + \delta_2\} \times \{j_1, j_1 \pm 1\}$ is a 2-by-2 submatrix in $\Gamma \subset \mathcal{S}$. If this submatrix is not QRC, it can be reduced to QRC by decreasing δ_1 or δ_2. The submatrix will be QRC for some non-negative δ_1 and δ_2 because $(i, j_1) \notin \mathcal{S}$ whenever $i_1 < i < i_2$ and so no such i could have $i \times \{j_1, j_1 \pm 1\} \subset \mathcal{S}$. The resulting submatrix, say M, is QCC (and hence QC) because there cannot exist j_0 strictly between j_1 and $j_1 \pm 1$. However, since $i_2 - i_1 \geq 2$, M is not contiguous. $\square$

The following corollary follows from Theorems 2.2 and 2.3.

Corollary 2.4. *For any $k \geq 2$, the set of path connected shapes for which TP_kC implies TP_k is exactly the set of Manhattan convex shapes.*

This classification offers an alternative to that given by Theorem 2.2. It is valuable because with the additional assumption of path-connectivity (which is satisfied by most shapes we study), we can concern ourselves with the more intuitive notion of manhattan convexity instead of examining whether or not all QC minors are contiguous.



(a) A 3-thick chunk $C \subset \mathcal{S}$ (b) A 3-thick barrier $B \subset \mathcal{S}$

Fig. 4. A Manhattan convex shape $\mathcal{S}$. The shaded cells represent (a) a 3-thick chunk C, and (b) a 3-thick barrier B.

3. *k*-Thick chunks and barriers

This section introduces a way of splitting a shape into sup-shapes with a well defined intersection. The theorem that concludes this section illustrates how this splitting respects the notion of total positivity. The results in Section 4 also illustrate the value of this tool.

A *k-thick chunk* in a shape $\mathcal{S}$ is a subshape C of $\mathcal{S}$ such that for any contiguous ℓ-by-ℓ submatrix M of $\mathcal{S}$, $\ell = 1, \ldots, k$, the shape $M \setminus (M \cap C)$ is Manhattan convex. That is, the subshape of M obtained by deleting the cells that lie in both M and C is Manhattan convex. For the most part we are only concerned with a certain class of chunks that we call barriers. To define them, we consider a decomposition of non-path connected shapes.

Let a *component* of a shape $\mathcal{S}$ to be a path-connected subset $\mathcal{S}_1 \subset \mathcal{S}$ such that there exist no path $\Gamma \subset \mathcal{S}$ and cells $(i,j) \in \mathcal{S} \setminus \mathcal{S}_1$, $(i_1,j_1) \in \mathcal{S}_1$ with (i,j), $(i_1,j_1) \in \Gamma$. Thus a component is a maximal path-connected subset of a shape. A *k-thick barrier* in a shape $\mathcal{S}$ is a k-thick chunk B such that $\mathcal{S} \setminus B$ consists of components $\mathcal{S}_1, \ldots, \mathcal{S}_{n_B}$ with n_B strictly greater than the number of components of $\mathcal{S}$. Fig. 4(a) and 4(b) are examples of 3-thick chunk and 3-thick barrier, respectively.

Theorem 3.1. *Let $\mathcal{S}$ be a shape whose ℓ-by-ℓ quasi-contiguous minors are contiguous, for $\ell = 1, \ldots, k$ (in particular, $A[\mathcal{S}]$ may be any matrix or other Manhattan convex shape). Suppose that a k-thick barrier $B \subset \mathcal{S}$ exists. Then $A[\mathcal{S}]$ is TP_k if and only if the resulting components $\mathcal{S}_1, \ldots, \mathcal{S}_{n_B} \subset \mathcal{S} \setminus B$ are such that $A[\mathcal{S}_r \cup B] \in TP_k$ for $r = 1, \ldots, n_B$.*

Proof. We have necessity because every subshape of a TP_k shape is TP_k.

For the converse, we first show that under the stated conditions $A[\mathcal{S}]$ is TP_kC. To do so, consider any ℓ-by-ℓ contiguous square submatrix $M \subset \mathcal{S}$. Now, since $\mathcal{S} = (\bigcup_r \mathcal{S}_r) \cup B$, either $M \subset \mathcal{S}_r \cup B$ for some r or there exist $(i_1,j_1), (i_2,j_2) \in M$ with $(i_1,j_1) \in \mathcal{S}_{r_1}$, $(i_2,j_2) \in \mathcal{S}_{r_2}$ for some $r_1 \neq r_2$. Suppose the latter. Then, since B is k-thick, $M \setminus B$ is Manhattan convex. In particular, it is path connected, so there exists a path $\Gamma \subset M \setminus B$ such that $(i_1,j_1), (i_2,j_2) \in \Gamma$. This, however, is a contradiction, since $\mathcal{S}_{r_1}$ and $\mathcal{S}_{r_2}$ are different components of $\mathcal{S} \setminus B$ and thus no path in $\mathcal{S} \setminus B \supset M \setminus B$ can connect $(i_1,j_1) \in \mathcal{S}_{r_1}$ and $(i_2,j_2) \in \mathcal{S}_{r_2}$. Thus we must have $M \subset \mathcal{S}_r \cup B$ for some r. Further, $\det A[M] > 0$ since by assumption $\mathcal{S}_r \cup B \in TP_k$.

Since M was an arbitrary contiguous square submatrix of size at most k, we have shown that all contiguous minors of $A[\mathcal{S}]$ up to size k are positive, so $A[\mathcal{S}]$ is TP_kC. By assumption, all quasi-contiguous minors up to size k are contiguous, so using Theorem 2.2, $\mathcal{S}$ is TP_k. $\square$

4. The TP_2-completion problem for some patterns

In this section, TP_k-completion for matrices are considered. In the remainder of this paper, *pattern* is used for the arrangement of the specified entries of partial matrices, unless otherwise indicated. A case for which the TP_k-completability of some subpatterns implies the TP_k-completability of the original pattern is given. Moreover, an explicit combinatorial condition for a large class of patterns to be TP_2-completable is given.

Suppose the partial shape S has a k-thick barrier of specified entries B. For a component S_r of $S \setminus B$, we use the following notation

$$S'_r = S_r \cup \{a_{ij} \in B \text{ such that } \exists\, a_{pq} \in S_r \text{ with } |p-i| \leqslant k \text{ and } |q-j| \leqslant k\}.$$

Lemma 4.1. *Suppose the partial shape S has a k-thick barrier of specified entries B. Let $S_1, S_2, \ldots, S_r$ be the components of $S \setminus B$. Then S has a TP$_k$-completion if and only if each S'_i has a TP$_k$-completion, for $i = 1, 2, \ldots, r$.*

Proof. Suppose the partial shape S has a TP$_k$-completion, then every contiguous subshape, in particular S'_i, for $i = 1, 2, \ldots, r$, has a TP$_k$-completion. For the converse, suppose every S'_i, $i = 1, 2, \ldots, r$, has a TP$_k$-completion. Consider an arbitrary shape $A[S]$ and let $A[S'_i]$ be a TP$_k$-completion for S'_i. Since the components $A[S'_i]$ share the same specified entries of the k-thick barrier, using Theorem 3.1, they form a TP$_k$-completion for $A[S]$. Therefore, there is a TP$_k$-completion for every shape $A[S]$ which implies that S is TP$_k$-completable. □

An immediate consequence of Lemma 4.1 is for patterns. That is, if $\mathcal{P}$ is an m-by-n pattern with a k-thick barrier of specified entries B and if $S_1, S_2, \ldots, S_r$ are the components of $S \setminus B$, then $\mathcal{P}$ is TP$_k$-completable if and only if S'_i is TP$_k$-completable, for each $i = 1, 2, \ldots, r$.

Lemma 4.2. *For $n \geqslant 1$, every pattern of size 2-by-n, is TP$_2$-completable.*

Proof. Let $\mathcal{P}$ be a pattern of size $2 \times n$ and consider a partial TP$_2$ matrix $\mathcal{T}$ with pattern $\mathcal{P}$ as follows

$$\mathcal{T} = \begin{pmatrix} t_{11} & t_{12} & \ldots & t_{1k} \\ t_{21} & t_{22} & \ldots & t_{2k} \end{pmatrix},$$

where the entries are either specified or unspecified. Since $\mathcal{T}$ is partial TP$_2$, the specified entries are positive. Note that a minor lying in columns j_1, j_2 is positive if and only if $\frac{t_{1j_1}}{t_{2j_1}} > \frac{t_{1j_2}}{t_{2j_2}}$. This together with Lemma 1.1 implies that $\mathcal{T}$ is TP$_2$-completable if and only if there exist values for the unspecified entries such that

$$\frac{t_{11}}{t_{21}} > \frac{t_{12}}{t_{22}} > \ldots > \frac{t_{1k}}{t_{2k}}. \tag{3}$$

Since every entry appears only once in the sequence of inequalities in (3), there exist values for the unspecified entries so that the inequalities in (3) hold. This implies that, there is a TP$_2$-completion for $\mathcal{T}$. Since $\mathcal{T}$ is arbitrary, the pattern $\mathcal{P}$ is TP$_2$-completable. □

By a similar proof to the proof of Lemma 4.2, we can show that every pattern of size n-by-2, for $n \geqslant 1$, is also TP$_2$-completable. Note that, since the largest minor in a 2-by-n matrix is 2-by-2, the above statement is automatically implies that every 2-by-n, is TP$_k$-completable, for $k \geqslant 1$.

Lemma 4.3. *Every pattern $\mathcal{P}$ of size m-by-n with only one unspecified entry is TP$_2$-completable.*

Proof. First consider an m-by-n pattern $\mathcal{P}_1$ in which the only unspecified entry lies in the (k, ℓ) position with at least one of k or ℓ in the set $\{1, m, n\}$. Using Lemmas 1.1 and 4.2, the pattern $\mathcal{P}_1$ is TP$_2$-completable. Now consider an m-by-n pattern $\mathcal{P}$ in which the only unspecified entry is in the (k, ℓ) position with neither k nor ℓ in the set $\{1, m, n\}$. Therefore, there is a 3-by-3 contiguous subpattern of $\mathcal{P}$, say $\mathcal{P}_2$, of the following form. By repeatedly using Lemma 4.1, $\mathcal{P}$ is TP$_2$-completable if and only if $\mathcal{P}_2$ is TP$_2$-completable. Let $\mathcal{T}_2$ be an arbitrary partial TP$_2$ matrix with pattern $\mathcal{P}_2$, with specified entries t_{ij} and unspecified entries x_{ij}.

$$\mathcal{P}_2 = \begin{pmatrix} \times & \times & \times \\ \times & ? & \times \\ \times & \times & \times \end{pmatrix}, \quad \mathcal{T}_2 = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & x_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}.$$

Using Lemma 1.1, $\mathcal{T}_2$ is TP$_2$-completable if and only if

$$\frac{t_{12}t_{21}}{t_{11}} < x_{22} < \frac{t_{12}t_{23}}{t_{13}}, \quad \text{and} \quad \frac{t_{23}t_{32}}{t_{33}} < x_{22} < \frac{t_{21}t_{32}}{t_{31}}.$$

Since $\mathcal{T}_2$ is partial TP$_2$, $\frac{t_{21}}{t_{11}} < \frac{t_{23}}{t_{13}}$, and $\frac{t_{23}}{t_{33}} < \frac{t_{21}}{t_{31}}$, and $\frac{t_{12}}{t_{11}} < \frac{t_{32}}{t_{31}}$, and $\frac{t_{32}}{t_{33}} < \frac{t_{12}}{t_{13}}$. Therefore, every lower bound for x_{22} is less than every upper bound for it. Thus, there is a value for x_{22}, say t_{22}, that replacing t_{22} with x_{22} in $\mathcal{T}_2$ results a TP$_2$ matrix, which means there is a TP$_2$-completion for $\mathcal{T}$. Since $\mathcal{T}_2$ was arbitrary, it implies that the pattern $\mathcal{P}_2$ and therefore, $\mathcal{P}$ is TP$_2$-completable. □

An unspecified entry x_{ij} in a pattern $\mathcal{P}$ is said to be *surrounded* by specified entries, if every entry $p_{k\ell}$ of $\mathcal{P}$, with $\max\{|i-k|, |j-\ell|\} = 1$, is specified.

Corollary 4.4. *If every unspecified entry in an m-by-n pattern $\mathcal{P}$ is surrounded by specified entries, then $\mathcal{P}$ is TP$_2$-completable.*

Proof. If $\min\{m, n\} \leqslant 2$, or $m = n = 3$, the result is implied by Lemmas 4.2 and 4.3. Otherwise, eight specified entries that surround an unspecified entry form a 2-thick barrier. Using Lemma 4.1 and induction on the number of unspecified entries, the pattern $\mathcal{P}$ is TP$_2$ completable. □

Lemma 4.5. *For a* 3*-by-n pattern $\mathcal{P}$, if the set of unspecified entries lie only in the middle row, then the pattern $\mathcal{P}$ is TP$_2$-completable.*

Proof. First, suppose that every entry in the middle row of $\mathcal{P}$ is unspecified. Consider a partial TP$_2$ matrix $\mathcal{T}$ with pattern $\mathcal{P}$ and let $\mathcal{T}_1$ be obtained from $\mathcal{T}$ by replacing the unspecified entry in the position (2, 1) with an arbitrary positive number. Then $\mathcal{T}_1$ is also a partial TP$_2$ matrix. Using Lemma 4.3, the unspecified entry in the position $(2, n)$ of $\mathcal{T}_1$ can be specified such that the 3-by-2 submatrix of $\mathcal{T}_1$ lying in columns 1 and n is TP$_2$. Suppose the resulting partial matrix is called $\mathcal{T}_2$, since the unspecified entries (2, 1) and $(2, n)$ do not complete any other 2-by-2 submatrix with three specified entries, $\mathcal{T}_2$ is still partial TP$_2$. Let the pattern of $\mathcal{T}_2$ be $\mathcal{P}_2$. Thus, the pattern $\mathcal{P}$ is TP$_2$-completable if and only if $\mathcal{P}_2$ is TP$_2$-completable. Next, suppose that in pattern $\mathcal{P}$ there is a fully specified column j with $j \neq 1, n$, then using Lemma 4.1, the pattern $\mathcal{P}$ is TP$_2$-completable if and only if both of the subpatterns lying in columns $1, 2, \ldots, j-1, j$ and $j, j+1, \ldots, n$ are TP$_2$-completable. Repeating this allows us to only consider the pattern $\mathcal{P}$ that the unspecified entries are in the positions $(2, 2), (2, 3), \ldots, (2, n-1)$ and every other entry is specified. By Lemma 4.3, the 3-by-3 subpattern $\mathcal{P}_1 = \mathcal{P}(\{1, 2, 3\}, \{1, 2, n\})$ is TP$_2$-completable. Consider a partial TP$_2$ matrix $\mathcal{T}'$ with pattern $\mathcal{P}$ and suppose t_{22} is a value for the (2, 2) entry that completes the 3-by-3 submatrix $\mathcal{T}_1' = \mathcal{T}'(\{1, 2, 3\}, \{1, 2, n\})$. Since t_{22} does not complete any other 2-by-2 submatrix of $\mathcal{T}'$ that involves other unspecified entries, the resulting 3-by-n matrix, $\mathcal{T}_1'$ is a partial TP$_2$ matrix. Moreover, $\mathcal{T}_1'$ has exactly the same structure as $\mathcal{T}'$ with one fewer columns with an unspecified entry than $\mathcal{T}'$. Therefore, by reduction, $\mathcal{T}'$ has a TP$_2$-completion. By reduction and since $\mathcal{T}'$ was arbitrary, the pattern $\mathcal{P}$ is TP$_2$-completable. □

Throughout, a row or a column of a matrix (or pattern) is referred to as a *line* of that matrix (or pattern). Using Lemma 4.1, if the unspecified entries of an m-by-n pattern $\mathcal{P}$ lie in one line, then $\mathcal{P}$ is TP$_2$-completable. This also implies that, a line can be inserted into a TP$_2$ matrix to form a TP$_2$ matrix of larger size.

Lemma 4.6. *Suppose pattern $\mathcal{P}$ contains a line ℓ with all unspecified entries and let $\mathcal{P}'$ be obtained from $\mathcal{P}$ by deleting the line ℓ. Then $\mathcal{P}$ is TP$_2$-completable if and only if $\mathcal{P}'$ is TP$_2$-completable.*

Proof. Let $\mathcal{P}$ be TP$_2$-completable, and suppose $\mathcal{T}'$ is a partial TP$_2$ matrix with pattern $\mathcal{P}'$. Let $\mathcal{T}$ be a partial TP$_2$ matrix obtained from $\mathcal{T}'$ by inserting a line of fully unspecified entries in the position ℓ. Thus, $\mathcal{T}$ is a partial TP$_2$ matrix with pattern $\mathcal{P}$ and therefore, has a TP$_2$-completion, say $\mathcal{T}_c$. The corresponding submatirx of $\mathcal{T}_c$ is a TP$_2$-completion for T'. Since $\mathcal{T}'$ was arbitrary, it implies that $\mathcal{P}'$ is TP$_2$-completable. For the converse, let $\mathcal{P}'$ be TP$_2$-completable, and suppose $\mathcal{T}$ is a partial TP$_2$ matrix with pattern $\mathcal{P}$. Suppose $\mathcal{T}_c'$ is a TP$_2$-completion of the corresponding submatrix of $\mathcal{T}$. Using

Lemma 4.5 and the note after Lemma 1.1, a line can be inserted to the TP_2 matrix $\mathcal{T}_c'$ in the position ℓ so that the result is TP_2. This is a TP_2-completion for $\mathcal{T}$. Since $\mathcal{T}$ was arbitrary, it implies that $\mathcal{P}$ is TP_2-completable. □

Next we show some results when the unspecified entries do not just lie in one line.

Lemma 4.7. *An m-by-n pattern $\mathcal{P}$ with exactly two unspecified entries is TP_2-completable if and only if they do not lie in the positions $(i,j), (i+1,j+1)$ with $(i,j) \neq (1,1), (m-1,n-1)$ or in positions $(i,j), (i-1,j+1)$ with $(i,j) \neq (m,1), (2,n-1)$.*

Proof. If the unspecified entries lie in contiguous rows and contiguous columns and not in the positions $(i,j), (i+1,j+1)$ with $(i,j) \neq (1,1), (m-1,n-1)$ or in positions $(i,j), (i-1,j+1)$ with $(i,j) \neq (m,1), (2,n-1)$, then the pattern contains a contiguous 3-by-3 subpattern that is not TP_2-completable, which implies that the pattern $\mathcal{P}$ is not TP_2-completable; see [2]. Otherwise, we consider two cases: (1) if the unspecified entries lie in contiguous rows and contiguous columns, and they lie in one of the positions in the statement, one can show that the 3-by-3 contiguous subpattern containing the unspecified entries is TP_2-completable; see [2] for details. Therefore, using Lemma 1.1, the pattern $\mathcal{P}$ is TP_2-completable; (2) if at least one of the rows or columns of the unspecified entries is not contiguous, then there are two cases: (i) they lie in the same line, in which case, using Lemmas 4.5 and 1.1, the pattern is TP_2-completable; (ii) they do not lie in the same line, in which case the pattern is TP_2-completable by Corollary 4.4. □

A shape pattern is said to have the *corner closure* (CC) property if whenever the entries (i_1,j_1) and (i_2,j_2), with $i_1 \neq i_2, j_1 \neq j_2$, are unspecified, then at least one of the entries (i_1,j_2) or (i_2,j_1) is also unspecified. Let p_{ij} denote the (i,j) entry of the pattern $\mathcal{P}$. For a given row i of the pattern $\mathcal{P}$, we denote the column index set of the unspecified entries lying in row i by J_i. That is,

$$J_i = \{j \in \mathbb{Z} \mid p_{ij} \text{ is unspecified}\}.$$

Lemma 4.8. *A pattern $\mathcal{P}$ satisfies the CC property if and only if for every pair of rows r_1, r_2 with $J_{r_1}, J_{r_2} \neq \emptyset$, either $J_{r_1} \subseteq J_{r_2}$ or $J_{r_2} \subseteq J_{r_1}$.*

Proof. Let pattern $\mathcal{P}$ satisfy the CC property and suppose there are rows i_1, i_2 such that $J_{r_1}, J_{r_2} \neq \emptyset$, $J_{i_1} \nsubseteq J_{i_2}$ and $J_{i_2} \nsubseteq J_{i_1}$. Therefore, there exist columns j_1 and j_2 such that the entries in the positions (i_1,j_1) and (i_2,j_2) are unspecified but the ones in the positions (i_1,j_2) and (i_2,j_1) are specified. This contradicts the assumption of satisfying the CC property. For the converse, suppose $\mathcal{P}$ does not satisfy the CC property. That is, there exist unspecified entries in the positions (i_1,j_1) and (i_2,j_2) such that the entries in the positions (i_1,j_2) and (i_2,j_1) are specified. Thus, we have rows i_1, i_2, with $J_{i_1}, J_{i_2} \neq \emptyset$, $J_{i_1} \nsubseteq J_{i_2}$ and $J_{i_2} \nsubseteq J_{i_1}$. □

Lemma 4.9. *Suppose $P[\mathcal{S}]$ is a pattern shape that has the CC property and contains at least two unspecified entries. Then there exists an unspecified entry, p_{i_0,j_0}, in $P[\mathcal{S}]$ such that the pattern shape $P'[\mathcal{S}]$ obtained by specifying p_{i_0,j_0} also satisfies the CC property.*

Proof. It is enough to show the existence of an unspecified entry p_{i_0,j_0} such that whenever $p_{i_0,j}$ and p_{i,j_0} are unspecified, so is $p_{i,j}$. Using Lemma 4.8, there exists an index i_0 such that J_{i_0} is nonempty and $J_{i_0} \subset J_i$ for any $i \in \mathbb{Z}$ with J_i nonempty. Consider (i_0,j_0) for a $j_0 \in J_{i_0}$. If $p_{i_0,j}$ is unspecified, then $j \in J_{i_0} \subset J_i$ for $J_i \neq \emptyset$. We are done because $j \in J_i$ implies $p_{i,j}$ unspecified. □

Theorem 4.10. *If a pattern $\mathcal{P}$ satisfies the CC property, then it is TP_2-completable.*

Proof. Suppose the pattern $\mathcal{P}$ satisfies the CC property, and let r_ℓ be a row with at least one unspecified entry that is minimal with respect to the inclusion in the CC property. That is, $J_{r_\ell} \neq \emptyset$ and $J_{r_\ell} \subseteq J_{r_i}$, for every row r_i with $J_{r_i} \neq \emptyset$. If row r_ℓ does not have any specified entry, then the pattern consists of some fully unspecified rows and possibly some fully specified rows. Such pattern is TP_2-completable by the note after Lemma 4.5. So suppose that row r_ℓ has at least one specified entry. Note that, for a column j_0, if the entry in the position (r_ℓ, j_0) of $\mathcal{P}$ is an unspecified entry, then every row r_i with $J_{r_i} \neq \emptyset$, has an unspecified entry in the position (r_i, j_0). We consider three possibilities; (1) there are at least two fully specified rows r_p, r_q with $r_p < r_\ell < r_q$; (2) the pattern has at least one fully specified row, say r_p, and they all satisfy $r_p < r_\ell$ (or they all satisfy $r_\ell < r_p$), (3) there is no fully specified row. Consider a partial TP_2 matrix $\mathcal{T}$ with pattern $\mathcal{P}$. The claim is that in each case the unspecified entries of the row r_ℓ of $\mathcal{T}$ can be specified so that the resulting matrix is partial TP_2 and satisfies the CC property. In case (1), suppose rows r_p, r_q are fully specified with $r_p < r_\ell < r_q$ such that for any other fully specified row r_s, either $r_s < r_p$ or $r_q < r_s$. Thus, in the contiguous submatrix of $\mathcal{T}$ lying in rows $r_p, r_{p+1}, r_{p+2}, \ldots, r_\ell, \ldots, r_{q-1}, r_q$, say $\mathcal{T}_1$, every column j_0 with unspecified entry in the position (r_ℓ, j_0) has unspecified entries in locations $(r_{p+1}, j_0), (r_{p+2}, j_0), \ldots, (r_\ell, j_0), \ldots, (r_{q-1}, j_0)$. Therefore, in $\mathcal{T}_1$, specifying (r_ℓ, j_0) does not complete any 2-by-2 submatrix that involves other unspecified entries in $\mathcal{T}_1$. Therefore, row r_ℓ can be specified such that the resulting matrix $\mathcal{T}'$ is partial TP_2 and satisfies the CC property. In case (2), suppose row r_p is the fully specified row with $r_p < r_\ell$ (resp. $r_\ell < r_p$) such that for any other fully specified row r_s, we have $r_s < r_p$ (resp. $r_p < r_s$). By a similar method to the part (1) and Lemma 4.2, we can show that the pattern $\mathcal{P}$ is TP_2-completable. In case (3), there is no restriction on the unspecified entries of row r_ℓ. Therefore, they can be specified so that the resulting matrix is still partial TP_2 and satisfies the CC property. Thus, in each of the above cases, any partial TP_2 matrix with pattern $\mathcal{P}$ has a TP_2-completion, which implies the pattern is TP_2-completable. □

A pattern that has specified and unspecified entries in each line alternatively, is called a *checkerboard pattern*. If the entry (1, 1) in a checkerboard pattern is specified, the pattern is called *odd checkerboard pattern*, otherwise it is called *even checkerboard pattern*. Note that, the converse of Theorem 4.10 is not true in general. By the following examples, both 4-by-4 checkerboard patterns are TP_2-completable, however, they do not satisfy the CC property.

Example 4.11. Consider the odd checkerboard pattern of size 4-by-4, $\mathcal{P}$ and the partial TP_2 matrix $\mathcal{T}$ with pattern $\mathcal{P}$.

$$\mathcal{P} = \begin{pmatrix} \times & ? & \times & ? \\ ? & \times & ? & \times \\ \times & ? & \times & ? \\ ? & \times & ? & \times \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} t_{11} & x_{12} & t_{13} & x_{14} \\ x_{21} & t_{22} & x_{23} & t_{24} \\ t_{31} & x_{32} & t_{33} & x_{34} \\ x_{41} & t_{42} & x_{43} & t_{44} \end{pmatrix}.$$

It can be checked that partial positive matrix $\mathcal{T}$ has a TP_2-completion if and only if

$$t_{22}t_{44} > t_{24}t_{42}, \quad t_{11}t_{33} > t_{13}t_{31},$$

and note that partial TP_2 suffices for a TP_2-completion. This implies that the pattern $\mathcal{P}$ is TP_2-completable.

Example 4.12. Let $\mathcal{P}$ be the even checkerboard pattern of size 4-by-4.

$$\mathcal{P} = \begin{pmatrix} ? & \times & ? & \times \\ \times & ? & \times & ? \\ ? & \times & ? & \times \\ \times & ? & \times & ? \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} x_{11} & t_{12} & x_{13} & t_{14} \\ t_{21} & x_{22} & t_{23} & x_{24} \\ x_{31} & t_{32} & x_{33} & t_{34} \\ t_{41} & x_{42} & t_{43} & x_{44} \end{pmatrix}.$$

It can be checked that this pattern has a TP_2-completion if and only if

$$t_{21}t_{43} > t_{23}t_{41}, \quad t_{12}t_{34} > t_{14}t_{32},$$

and again partial TP_2 suffices for a TP_2-completion. Therefore, the even checkerboard pattern of size 4-by-4 is also TP_2-completable.

Since neither of the even nor the odd 4-by-4 checkerboard pattern satisfy the CC property, both examples show that the converse of Theorem 4.10 is not valid. However, Theorem 4.13 shows that this is true for a vast class of patterns.

We say an m-by-n pattern $\mathcal{P}$ is *bordered* from left, right, above, or below, if the first column, last column, first row, or last row is fully specified, respectively. If the pattern is bordered from each side, then it is simply called a *bordered pattern*. By a *line* of a pattern, we mean a row or column of the pattern.

Recall that, using Corollary 4.4, if there is a barrier of specified entries in a pattern $\mathcal{P}$, then in order to check for TP_2-completability, $\mathcal{P}$ can be divided into patterns of smaller size, and it is enough to check the TP_2-completability of the smaller patterns. Therefore, we consider patterns with no barrier of specified entries. For a bordered pattern $\mathcal{P}$, if no subpattern contains an interior k-thick barrier of specified entries, then $\mathcal{P}$ is said to have *k-connected unspecified entries*. We simply call such patterns *k-connected*.

Theorem 4.13. *A m-by-n bordered 1-connected pattern $\mathcal{P}$ is TP_2-completable if and only if it has the CC property.*

Proof. Using Lemma 4.6, it is enough to show the statement for the patterns with no fully unspecified line. If pattern $\mathcal{P}$ has the CC property, using Theorem 4.10 it is TP_2-completable. For the converse, suppose an m-by-n bordered 1-connected pattern $\mathcal{P}$ does not have the CC property. Then, there exist unspecified entries (i_1, j_1) and (i_2, j_2) such that the entries (i_1, j_2) and (i_2, j_1) are specified. Without loss of generality, suppose $i_1 < i_2$ and $j_1 < j_2$. If there is a column j_0 with $j_1 < j_0 < j_2$ such that the entries (i_1, j_0) and (i_2, j_0) are both specified, then the 3-by-3 subpattern $\mathcal{P}(\{i_1, i_2, m\}, \{j_1, j_0, j_2\})$ has an interior 1-thick barrier of specified entries, which is a contradiction. Similarly, if there is a row i_0 with $i_1 < i_0 < i_2$ such that the entries (i_0, j_1) and (i_0, j_2) are both specified, then the 3-by-3 subpattern $\mathcal{P}(\{i_1, i_0, i_2\}, \{1, j_1, j_2\})$ has an interior 1-thick barrier of specified entries, which is a contradiction. If none of these cases hold, then we show that there is a partial TP_2 matrix with pattern $\mathcal{P}$ and with no TP_2-completion. Suppose j_1 is the first column index that has an unspecified entry in row i_1, and let j_2 be the first column index with $j_2 > j_1$ that has an unspecified entry in row i_2. The 3-by-3 subpattern $\mathcal{P}_1 = \mathcal{P}(\{i_1, i_2, m\}, \{1, j_1, j_2\})$ is not TP_2-completable. Consider a partial TP_2 matrix $\mathcal{T}_1$ with pattern $\mathcal{P}_1$ with no TP_2-completion. We show that $\mathcal{T}_1$ can be extended to a partial TP_2 matrix $\mathcal{T}$ with pattern $\mathcal{P}$. Since there is no TP_2-completion for $\mathcal{T}_1$, there is no TP_2-completion for $\mathcal{T}$, and therefore, the pattern $\mathcal{P}$ is not TP_2-completable. For this, it is enough to show that the interior data lines of the corresponding pattern of the submatrix of $\mathcal{T}$ can be inserted to $\mathcal{T}_1$ to form a partial TP_2 matrix. This can be done by a applying a proof similar to the proof of Lemma 4.5. □

Remark 4.14. Note that the "1-connected" hypothesis in Theorem 4.13 is necessary (but inconsequential) as a pattern with two unspecified entries that are surrounded is TP_2-completable, but may not enjoy the CC property.

Remark 4.15. The condition of being "bordered" in Theorem 4.13 can be reduced. Depending on the position of the unspecified entries, bordered on one side serves the same purpose. The proof of Theorem 4.13 shows this, as the desired 3-by-3 subpattern may be found under these lesser assumptions.

Acknowledgment

The last-named author would like to thank Dr. Gary MacGillivray for his support during some of the research for this paper.

References

[1] S. Fallat, C.R. Johnson, Totally Nonnegative Matrices, Princeton University Press, New Jersey, 2011.
[2] S. Nasserasr, The Logarithmic Method and the Solution to the TP_2-Completion Problem, Ph.D. Dissertation, Department of Applied Science/Mathematics, The College of William and Mary, 2010.