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# Covariant nucleon wave function with $S$ -, $D$ -, and $P$ -state components

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Expressions for the nucleon wave functions in the covariant spectator theory are derived. The nucleon is described as a system with a off-mass-shell constituent quark, free to interact with an external probe, and two spectator constituent quarks on their mass shell. Integrating over the internal momentum of the on-mass-shell quark pair allows us to derive an effective nucleon wave function that can be written only in terms of the quark and diquark (quark-pair) variables. The derived nucleon wave function includes contributions from  $S$ -,  $P$ - and  $D$ -waves.

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## I. INTRODUCTION

Early phenomenological studies of the structure of the nucleon include the Isgur-Karl model [1] and the MIT and cloudy bag models reviewed in Ref. [2]. These models include confinement and assume a spin-flavor symmetry broken by the color hyperfine interaction. The confined quarks in bag models are Dirac spinors with lower  $P$ -wave components which substantially alter predictions that would follow from the leading  $S$ -waves [2,3]. The Isgur-Karl model has been extended to include small radial and angular momentum excitations, including  $D$ -waves [4], and relativistic corrections [5]. Wave functions on the light front have been introduced [6]. Reviews of the literature on early quark models can be found in Refs. [7,8]. More recently, Juliá-Díaz, Riska and Coester have calculated the nucleon form factors using phenomenological wave functions in all three forms of Hamiltonian dynamics [9] and have extended these calculations to include  $D$ -wave admixtures in the wave functions [10,11]. Recent reviews of the nucleon form factor measurements and theoretical models can be found in Refs. [12,13].

In this work, we use the covariant spectator theory (CST) [14–18] to determine the structure of the valence quark contributions to the wave functions of the nucleon. CST is a manifestly covariant model, with the novel feature that all four components of the energy-momentum four-vector are always conserved. In the CST, baryon systems consist of an off-mass-shell constituent quark, free to interact with electromagnetic fields or other probes, and two noninteracting on-mass-shell constituent quarks that are spectators to the interaction. Since the interaction does not depend on the internal momentum of these on-shell spectators, we can integrate over their internal momentum and express the effective matrix element in terms of a nucleon composed of an off-shell quark and an on-shell quark pair (or diquark) with an average mass  $m_s$ , which becomes a parameter in the wave function.

Previously, we have assumed a pure  $S$ -wave structure for the wave functions; here, we add contributions associated with  $D$ - and  $P$ -wave components. We begin by discussing the general form of the CST matrix elements. Then, for each angular momentum component, nonrelativistic wave functions are constructed first and then generalized to relativistic form.

## II. CST MATRIX ELEMENTS

### A. Relativistic impulse approximation

In the relativistic impulse approximation (RIA), it is assumed that the interaction is well-described by a single quark operator  $O^\alpha$  and that interactions involving a pair of quarks (i.e. exchange or interaction currents) can be neglected. In this case, the baryonic matrix elements in the CST can be written (for a brief discussion of corrections to the RIA, see Ref. [19])

$$\mathcal{O}_{\lambda_+ \lambda_-}^\mu(P_+, P_-) = \sum_{i=1}^3 \sum_{\lambda_j \lambda_\ell} \int_{k_j k_\ell} \langle O^\mu \rangle_i, \quad (2.1)$$

where the first sum is over all three possible choices for the interacting quark, and the three-momentum integrations and helicity sums are over the momenta and helicities of the on-shell spectator quarks, with  $i, j, \ell$  in cyclic order, so that, for example, if the third quark is interacting, then the (12) pair are the spectators. The matrix element is

$$\langle O^\mu \rangle_i = \bar{\Psi}_{\lambda_j \lambda_\ell, \lambda_+}(P_+, k_j k_\ell) O^\mu \Psi_{\lambda_j \lambda_\ell, \lambda_-}(P_-, k_j k_\ell), \quad (2.2)$$

with  $P_-$  ( $P_+$ ) and  $\lambda_-$  ( $\lambda_+$ ) the four-momenta and helicity of the incoming (outgoing) baryon, and  $\Psi$  is the baryon wave function. We have suppressed the Dirac indices of the off-shell quark. The matrix element is illustrated in Fig. 1. In the CST, the spectator quarks are constrained to their positive-energy mass shell, so the covariant volume integral is (with  $j = 1$  and  $\ell = 2$ )

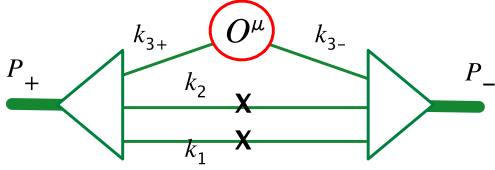


FIG. 1 (color online). Diagrammatic representation of the RIA approximation to the CST matrix element of the operator  $O^\mu$ . The quarks with four-momenta  $k_1$  and  $k_2$  are on-shell spectators (represented by the  $\times$ ).

$$\begin{aligned} \int_{k_1 k_2} &\equiv \int \frac{d^4 k_1 d^4 k_2}{(2\pi)^6} \delta_+(m_1^2 - k_1^2) \delta_+(m_2^2 - k_2^2) \\ &= \int \frac{d^3 k_1 d^3 k_2}{(2\pi)^6 4E_1 E_2}, \end{aligned} \quad (2.3)$$

where the four-momenta of the on-shell quarks are

$$k_1 = \{E_1, \mathbf{k}_1\} \quad k_2 = \{E_2, \mathbf{k}_2\}, \quad (2.4)$$

with  $E_i = \sqrt{m_i^2 + \mathbf{k}_i^2}$ .

In this paper, we will first assume the dressed masses of the three quarks are equal, so that  $m_1 = m_2 = m_3 = m$ . (Later, we will consider the case when  $m_u \neq m_d$ .) Since the color factor (suppressed) is fully antisymmetric, when the particles are identical, this means that the remaining matrix element must be symmetric under the interchange of the three quarks. In this case, the three terms for different  $i$  are identical, and we may write the full result as 3 times the result for the (arbitrary) choice of  $k_1$  and  $k_2$  as spectators transforming Eq. (2.1) to

$$O_{\lambda_+ \lambda_-}^\mu(P_+, P_-) = 3 \sum_{\lambda_1 \lambda_2} \int_{k_1 k_2} \langle O^\mu \rangle_3. \quad (2.5)$$

This means that it is only necessary for the wave function  $\Psi(P, k_1, k_2)$  to be symmetric under the interchange of quarks 1 and 2; symmetry under the interchanges of the other quarks,  $1 \leftrightarrow 3$  or  $2 \leftrightarrow 3$ , was used when the full result (2.1) is simplified to Eq. (2.5).

## B. Relativistic definition of the quark-diquark wave function

As shown in Fig. 1, the dependence of the matrix element on the relative momentum of the two on-shell quarks is determined by the wave function only, and not by the structure of the operator  $O^\mu$ , which depends only on the momenta  $k_{3\pm}$  of the initial and final off-shell quark

$$k_{3\pm} = P_\pm - (k_1 + k_2). \quad (2.6)$$

Since the wave function will be determined phenomenologically, very little is lost by averaging over the relative momenta of the two on-shell quarks and introducing a new wave function that depends only on the *total* four-momentum of the on-shell quark pair (which will be called a diquark). In order to do this, we introduce the diquark momentum variables,

$$k = (k_1 + k_2) \quad r = \frac{1}{2}(k_1 - k_2), \quad (2.7)$$

which, together with the fixed total momentum  $P = k_1 + k_2 + k_3$ , are a complete set. (If  $k \rightarrow k_1 + k_2 - \frac{2}{3}P$ , these would be the Jacobi coordinates.) The diquark four-momentum  $k = k_1 + k_2$  has a mass  $s = (k_1 + k_2)^2$ , which is initially unconstrained. In this notation, the integrals (2.3) can be reexpressed as integrals over  $\mathbf{k}$ ,  $s$ , and the direction of the internal relative three-momentum  $\mathbf{r} = \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2)$  of the diquark

$$\int_{sk} = \underbrace{\int \frac{d\Omega_{\mathbf{r}}}{4(2\pi)^3} \int_{4m_q^2}^\infty ds \sqrt{\frac{s - 4m_q^2}{s}}}_{\int_s} \underbrace{\int \frac{d^3 k}{(2\pi)^3 2E_s}}_{\int_k}, \quad (2.8)$$

where  $m_q$  is the dressed quark mass, and  $E_s = \sqrt{s + \mathbf{k}^2}$  is the energy of a diquark of mass  $\sqrt{s}$ , and the angular integrals  $d\Omega_{\mathbf{r}}$  are written in the *rest frame of the diquark*. The relation (2.8) is discussed in Appendix A.

Using the relation (2.8), the matrix element (2.5) can be written

$$O_{\lambda_+ \lambda_-}^\mu(P_+, P_-) = 3 \sum_{\Lambda} \int_k \bar{\Psi}_{\Lambda \lambda_+}(P_+, k) O^\mu \Psi_{\Lambda \lambda_-}(P_-, k), \quad (2.9)$$

where the sum over  $\Lambda$  includes all possible polarization pairs  $\lambda_1, \lambda_2$  of the diquark, and the new density in Eq. (2.9) is related to the density in Eq. (2.5) by

$$\begin{aligned} &\sum_{\lambda_1 \lambda_2} \int_s \bar{\Psi}_{\lambda_1 \lambda_2, \lambda_+}(P_+, k_1 k_2) \otimes \Psi_{\lambda_1 \lambda_2, \lambda_-}(P_-, k_1 k_2) \\ &\equiv \sum_{\Lambda} \bar{\Psi}_{\Lambda \lambda_+}(P_+, k) \otimes \Psi_{\Lambda \lambda_-}(P_-, k)|_{s=m_s^2}, \end{aligned} \quad (2.10)$$

where the operator  $\otimes$  shows where the Dirac operator  $O^\mu$  is to be inserted. This equation gives the precise relationship between the quark-diquark wave function, denoted by  $\Psi_{\Lambda, \lambda}(P, k)$ , and the full three-quark wave function,  $\Psi_{\lambda_1 \lambda_2, \lambda}(P, k_1 k_2)$ . (Note that these two wave functions are distinguished from each other only by their list of arguments.) Equation (2.10) shows that the quark-diquark wave function is obtained from the full three-quark wave function by averaging over the directions of the relative three-momentum  $\mathbf{r}$  (in the rest system) and replacing the integral over the continuous diquark mass,  $\sqrt{s}$ , by its mean value,  $m_s$ , which now becomes a parameter of the theory. All remaining factors from the integral  $\int_s$  are absorbed into the normalization of the wave function. After the average has been carried out, the diquark energy becomes  $E_s = \sqrt{m_s^2 + \mathbf{k}^2}$ .

We emphasize that, in the RIA, the replacement (2.10) is exact, as long as the effective diquark mass  $m_s$  is treated as a parameter. Later, we will constrain the values of this parameter, and in so doing, we make an approximation.

### C. Nonrelativistic definition of the quark-diquark wave function

The nonrelativistic limit of Eq. (2.8) is obtained by assuming the quark and diquark masses are very large and absorbing the mass factors into the wave function normalization. In this case,

$$s = \left( \sqrt{m_q^2 + k_1^2} + \sqrt{m_q^2 + k_2^2} \right)^2 - \mathbf{k}^2 \simeq 4m_q^2 + 4\mathbf{r}^2. \quad (2.11)$$

For very large masses ( $E_s \rightarrow m_s$ ;  $s \rightarrow 4m_q^2$ ), we can write

$$\begin{aligned} m_q m_s \int_{sk} &\rightarrow \frac{1}{16} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d\Omega_{\mathbf{r}}}{(2\pi)^3} \int_{4m_q^2}^{\infty} ds \sqrt{s - 4m_q^2} \\ &= \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 r}{(2\pi)^3}, \end{aligned} \quad (2.12)$$

where, in the last step, the integral over  $s$  has been replaced by an integral over the magnitude of the relative momentum,  $r = |\mathbf{r}|$ , using Eq. (2.11).

The result (2.12) also follows directly from the transformation from  $\mathbf{k}_1, \mathbf{k}_2$  to  $\mathbf{k}, \mathbf{r}$ , which gives

$$\int \frac{d^3 k_1 d^3 k_2}{(2\pi)^6} = \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 r}{(2\pi)^3} \equiv \int_{k,r}^{\text{NR}}. \quad (2.13)$$

Hence, the nonrelativistic definition of the diquark wave function is identical to Eq. (2.10), with

$$\frac{m_q}{2} \int_s \rightarrow \int \frac{d^3 r}{(2\pi)^3}. \quad (2.14)$$

Since the details of how the averaging over the internal diquark structure depends on the individual angular momentum components of the wave function, this discussion is postponed until Sec. IV.

### III. ELECTROMAGNETIC INTERACTION AND NORMALIZATION OF THE WAVE FUNCTION

The interaction of the photon with a constituent quark is decomposed into Dirac and Pauli components.

$$j^\mu(Q^2) = j_1 \left( \gamma^\mu - \frac{\not{q} q^\mu}{q^2} \right) + j_2 \frac{i \sigma^{\mu\nu} q_\nu}{2M}, \quad (3.1)$$

where  $M$  is the nucleon mass. Note that the constituent quarks can have anomalous magnetic moments and that we add the term  $-\not{q} q^\mu / q^2$  to insure that the current is always conserved [20]. In this paper, we will limit discussion to  $u$  ( $I_z \equiv t = +\frac{1}{2}$ ) and  $d$  ( $t = -\frac{1}{2}$ ) quarks only, in which case, the quark current  $j^\mu$  and the quark form factors  $j_i$  are operators in isospin space. The Dirac ( $j_1$ ) and Pauli ( $j_2$ ) form factors are therefore decomposed into two independent isospin structures,

$$j_i(Q^2) = \frac{1}{6} f_{i+}(Q^2) + \frac{1}{2} f_{i-}(Q^2) \tau_3, \quad (3.2)$$

where  $\tau_3$  is the  $z$ -projection of the isospin operator. This equation defines the isoscalar and isovector form factors

which can also be expressed as the  $u$ - and  $d$ - quark form factors. In the CST quark model, the functions  $f_{i\pm}(Q^2)$  are parametrized using a vector meson dominance representation [15].

Using Eq. (2.9) and the operator (3.1), the baryon form factors become

$$J_{\lambda_+ \lambda_-}^\mu(P_+, P_-) = 3 \sum_{\Lambda} \int_k \bar{\Psi}_{\Lambda \lambda_+}(P_+, k) j^\mu(Q^2) \Psi_{\Lambda \lambda_-}(P_-, k). \quad (3.3)$$

For the nucleon, at  $Q^2 = 0$ , when  $P_+ = P_- = (M, 0, 0, 0)$  and

$$j^0(0) = e_0 \left( \frac{1}{6} + \frac{1}{2} \tau_3 \right) \gamma^0 = j_1(0) \gamma^0 \quad (3.4)$$

(because of the presence of the factor  $e_0$ , this operator at  $Q^2 = 0$  gives the renormalized quark charge), we require that the wave function be diagonal in the nucleon polarization and normalized to the correct nucleon charge:

$$J_{\lambda_+ \lambda_-}^0(P, P) \equiv Q = \frac{1}{2} (1 + \tau_3) \delta_{\lambda_+, \lambda_-}. \quad (3.5)$$

It is always possible to normalize the wave functions so that the proton charge is unity, but to obtain zero for the neutron charge places constraints on the structure of the wave function.

## IV. WAVE FUNCTIONS OF THE NUCLEON

### A. Total relativistic wave function

The total relativistic quark-diquark wave function will be written as a sum of  $S$ -,  $P$ -, and  $D$ -wave components,

$$\Psi_{\Lambda \lambda}(P, k) = n_S \Psi_{\Lambda \lambda}^S(P, k) + n_P \Psi_{\Lambda \lambda}^P(P, k) + n_D \Psi_{\Lambda \lambda}^D(P, k), \quad (4.1)$$

where  $\Psi_{\Lambda \lambda}^L(P, k)$  is the  $L = S, P$ -, or  $D$ -state wave function to be defined in the next subsections.

We will require that *each* of these components be separately normalized to the nucleon charge, Eq. (3.5). This requirement will be decisive in fixing the structure of the wave functions. When each component has been separately normalized, the  $S$ -state normalization constant,  $n_S$ , will be determined by the  $P$ -state mixing parameter,  $n_P$ , and the  $D$ -state mixing parameter  $n_D$ ,

$$n_S = \sqrt{1 - n_P^2 - n_D^2}. \quad (4.2)$$

### B. $S$ -state component

#### 1. Spin-isospin part of the nonrelativistic wave function

The construction of the nonrelativistic  $S$ -state proton wave function was given previously [15], but here the construction will be done differently. We obtain the same result as we did before, but this way of doing things will be helpful when we extend the discussion to the  $D$  state.

The wave function is written as a product of a momentum space wave function symmetric in  $k_1$  and  $k_2$  (this is the only possibility for an  $S$  state) and a spin-flavor part also symmetric under the interchange of quarks 1 and 2. There are two possible structures which contribute to the symmetric spin-flavor part: the (0,0) component which is a direct product of an operator antisymmetric in the flavor (isospin 0) and antisymmetric in the spin (spin 0), and the (1,1) component symmetric in the flavor (isospin 1) and symmetric in the spin (spin 1). The isospin 0 and 1 components can be written in the following form:

$$\phi^0 \chi^t = \chi^t \quad \phi_\ell^1 \chi^t = -\frac{1}{\sqrt{3}}(\tau \cdot \xi_\ell^*) \chi^t, \quad (4.3)$$

with  $\chi^t$  the two-component isospinor of the nucleon

$$\chi^{+(1/2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi^{-(1/2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (4.4)$$

and  $\xi_\ell$  the isospin vector of the isospin-one diquark, defined in the usual way (in rectangular coordinates),

$$\xi_\pm = \mp \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}, \quad \xi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (4.5)$$

(This notation differs slightly from Ref. [15], but the results are the same.) The spin-1 vectors are normalized to

$$(\xi_{\ell'})^\dagger \cdot \xi_\ell = \delta_{\ell',\ell} \quad \sum_\ell (\xi_\ell)_i (\xi_\ell)_j^\dagger = \delta_{ij}, \quad (4.6)$$

where the  $i$ th component of the vector  $\xi_\ell$  is denoted  $(\xi_\ell)_i$ . In the nonrelativistic limit, the spin wave functions and operators have precisely the same form.

Using the operator notation (4.3), the total wave function of the nucleon can now be written in a general form which displays its dependence on all three quark momenta, but still describes the flavor and spin of the three quarks in terms of the quark-diquark description already developed. By separating explicitly the spin-0 isoscalar diquark contribution,  $\Psi_\lambda^{S,0}$ , from the spin 1 vector diquark contribution,  $\Psi_{\Lambda\lambda}^{S,1}$ , we have

$$\Psi_{\Lambda\lambda}^S(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2) = \cos\theta_S \phi^0 \Psi_\lambda^{S,0}(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2) + \sin\theta_S \phi^1 \Psi_{\Lambda\lambda}^{S,1}(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2), \quad (4.7)$$

where  $\lambda$  is the spin projection of the nucleon,  $\Lambda$  the spin projection of the spin-1 diquark, and  $\Psi_{\Lambda\lambda}^{S,s}(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2)$  is the  $S$ -state spin-space part of the wave functions (with  $s$  the spin of the diquark and no  $\Lambda$  dependence in the spin-0 component). Explicitly,

$$\begin{aligned} \Psi_\lambda^{S,0}(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2) &= \Phi_S(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2) \left| \frac{1}{2}, \lambda \right\rangle \\ \Psi_{\Lambda\lambda}^{S,1}(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2) &= -\frac{1}{\sqrt{3}} \Phi_S(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2) \sigma \cdot \varepsilon_\Lambda^* \left| \frac{1}{2}, \lambda \right\rangle, \end{aligned} \quad (4.8)$$

where we have introduced our notation for the diquark spin functions, modeled after the isospin operators (4.3), with  $\varepsilon_\Lambda$  as the spin vector for the spin-1 diquark [with its components in spin-space defined as in Eq. (4.5)], and  $|\frac{1}{2}, \lambda\rangle$  as the initial nucleon spin state with polarization  $\lambda$ . At this point, we assume that the same momentum space wave function,  $\Phi_S$ , accompanies both of the spin parts of the wave function; later, this assumption will be relaxed as the wave function is determined from phenomenological fits to the deep inelastic scattering (DIS) data. We emphasize that, at this stage, the two components of the wave function have spins equal to their isospins, but it is best to label these components by their spin because later (in Sec. V), we will break the isospin invariance.

Using this wave function, the nonrelativistic matrix element of the charge operator, Eq. (3.5) becomes

$$Q = \frac{1}{2}(1 + \tau_3) \delta_{\lambda_+, \lambda_-} = \{\cos^2\theta_S j^A + \sin^2\theta_S j^S\} \delta_{\lambda_+, \lambda_-} e_0 \mathcal{N}_S, \quad (4.9)$$

where the isospin matrix elements are

$$\begin{aligned} j^A &\equiv 3(\phi^0)^\dagger j_1(0) \phi^0 = e_0 \left( \frac{1}{2} + \frac{3}{2} \tau_3 \right) \\ j^S &\equiv 3 \sum_\ell (\phi_\ell^1)^\dagger j_1(0) \phi_\ell^1 = \sum_\ell (\xi_\ell \cdot \tau) j_1(0) (\xi_\ell^* \cdot \tau) \\ &= \frac{1}{2} e_0 (1 - \tau_3), \end{aligned} \quad (4.10)$$

where  $j_1(0)$  is the quark charge operator (3.4), the normalization integral is

$$\mathcal{N}_S = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3r}{(2\pi)^3} \Phi_S^2(\mathbf{0}; \mathbf{k}_1, \mathbf{k}_2), \quad (4.11)$$

and we used the fact that the spin operators in Eq. (4.8) are renormalized. Choosing  $\mathcal{N}_S = 1/e_0$ , the charge operator (4.9) becomes

$$\begin{aligned} Q &= \{\cos^2\theta_S [\frac{1}{2} + \frac{3}{2} \tau_3] + \sin^2\theta_S [\frac{1}{2} - \frac{1}{2} \tau_3]\} \delta_{\lambda_+, \lambda_-} \\ &= \frac{1}{2} [1 + \tau_3 [3\cos^2\theta_S - \sin^2\theta_S]] \delta_{\lambda_+, \lambda_-}. \end{aligned} \quad (4.12)$$

Hence, an equal mixture of diquark spin-0 and spin-1 components ( $\theta_S = \pi/4$ ), which we used in our previous work, is required by the demand that the neutron charge be zero. We will find this requirement useful in the construction of the  $D$  state below.

The integral (4.11) provides an explicit example of how the quark-diquark wave function emerges from an average over the internal momenta of the diquark, as shown in general in Eq. (2.10). Since the momentum wave function must be symmetric in  $k_1$  and  $k_2$ , we may choose its argument to be



$$\begin{aligned}
\chi_{nr}(k, P, r) &= (\mathbf{k}_1 - \frac{1}{3}\mathbf{P})^2 + (\mathbf{k}_2 - \frac{1}{3}\mathbf{P})^2 \\
&= \frac{1}{2}\mathbf{k}^2 + 2\mathbf{r}^2 - \frac{2}{3}\mathbf{k} \cdot \mathbf{P} + \frac{2}{9}\mathbf{P}^2 \\
&= \frac{1}{2}(\mathbf{k} - \frac{2}{3}\mathbf{P})^2 + 2\mathbf{r}^2.
\end{aligned} \quad (4.13)$$

With this choice, the quark-diquark wave function in an arbitrary frame is defined by the relation

$$\begin{aligned}
&\int \frac{d^3k}{(2\pi)^3} \int \frac{d^3r}{(2\pi)^3} \Phi_S[\chi_{nr}(k, P_+, r)] \Phi_S[\chi_{nr}(k, P_-, r)] \\
&= \int \frac{d^3k}{(2\pi)^3} \Phi_S[\chi_{nr}(k, P_+, \bar{r})] \Phi_S[\chi_{nr}(k, P_-, \bar{r})] \\
&\equiv \int \frac{d^3k}{(2\pi)^3} \phi_S(\mathbf{P}_+, \mathbf{k}) \phi_S(\mathbf{P}_-, \mathbf{k}),
\end{aligned} \quad (4.14)$$

where, in the second step, we replace  $r$  by an average value  $\bar{r}$ , and the last step gives the precise relation between the quark-diquark wave function,  $\phi_S(\mathbf{P}, \mathbf{k})$ , and the three-quark wave function,  $\Phi_S(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2)$ , analogous to the definition given in Eq. (2.10). With this definition, the normalization of the quark-diquark wave function is

$$e_0 \mathcal{N}_S \equiv e_0 \int \frac{d^3k}{(2\pi)^3} \phi_S^2(\mathbf{0}, \mathbf{k}) = 1. \quad (4.15)$$

Note that the normalization of the wave function compensates for the renormalization of the quark charge, ensuring that the proton charge is correct.

## 2. Relativistic wave function

The relativistic generalization of the nonrelativistic wave function is straightforward and was discussed extensively in Ref. [15]. The wave function we use is

$$\Psi_{\Lambda\lambda}^S(P, k) = \frac{1}{\sqrt{2}} [\phi^0 u(P, \lambda) - \phi^1 (\varepsilon_\Lambda^\alpha)^* U_\alpha(P, \lambda)] \psi_S(P, k), \quad (4.16)$$

where the *relativistic quark-diquark* wave function,  $\Psi_{\Lambda\lambda}^S(P, k)$ , is distinguished from the *nonrelativistic, three-quark* wave function,  $\Psi_{\Lambda\lambda}^S(P; k, r)$  only by its arguments, and  $U_\alpha$  is

$$U_\alpha(P, \lambda) = \frac{1}{\sqrt{3}} \gamma_5 \left( \gamma_\alpha - \frac{P_\alpha}{M} \right) u(P, \lambda), \quad (4.17)$$

with  $u(P, \lambda)$  the free nucleon spinor with four-momentum  $P$  and spin projection  $\lambda$ ,  $\varepsilon_\Lambda$  is the polarization vector of the spin-1 diquark state with polarization  $\Lambda$  in the direction of  $\mathbf{P}$  (and subject to the constraint  $\varepsilon_\Lambda^\mu P_\mu = 0$ ), and the  $\phi^{0,1}$  are the flavor wave functions of the quarks in a (12)3 configuration with the (12) pair in an isospin zero or one state [see Eq. (4.3)]. The polarization vectors  $\varepsilon^\mu$  are discussed in detail in Refs. [15,21]. The states have been normalized so the superposition (4.16) corresponds to an equal mixture of (0,0) and (1,1) components, as discussed above.

The spatial part of the  $S$ -state wave function will be fixed by comparison with the DIS data as discussed in the accompanying paper [22]. The model used in earlier work had the form

$$\psi_S(P, k) = \frac{N_0}{m_s(\beta_1 + \chi)(\beta_2 + \chi)}, \quad (4.18)$$

where

$$\chi = \frac{(M - m_s)^2 - (P - k)^2}{M m_s} = \frac{2\mathbf{P} \cdot \mathbf{k}}{M m_s} - 2 \quad (4.19)$$

and  $\beta_2 > \beta_1 > 0$  are range parameters with the normalization constant  $N_0$  chosen so that

$$1 = e_0 \int_k |\psi_S(P, k)|^2, \quad (4.20)$$

with the covariant integration defined by Eq. (2.8) (with  $s = m_s^2$ ). Note that, in the nonrelativistic limit,  $\chi$  reduces to

$$\begin{aligned}
\chi &= 2\sqrt{1 + \frac{\mathbf{k}^2}{m_s^2}} \sqrt{1 + \frac{\mathbf{P}^2}{M^2}} - 2 \frac{\mathbf{k} \cdot \mathbf{P}}{M m_s} - 2 \\
&\rightarrow \left( \frac{\mathbf{k}}{m_s} - \frac{\mathbf{P}}{M} \right)^2 \rightarrow \frac{1}{4m_q^2} \left( \mathbf{k} - \frac{2}{3}\mathbf{P} \right)^2 \\
&= \frac{1}{2m_q^2} \chi_{nr}(k, P, 0),
\end{aligned} \quad (4.21)$$

where in the last step, to display the relation between  $\chi$  and  $\chi_{nr}$ , we made the approximation  $m_s \simeq 2m_q$  and  $M \simeq 3m_q$ , true when all momenta (and binding energies) are much smaller than  $m_q$ . For future reference, we note now that

$$\begin{aligned}
\tilde{k}^2 &= \left( k - \frac{(P \cdot k)P}{P^2} \right)^2 = m_s^2 - \frac{(P \cdot k)^2}{M^2} \\
&= -m_s^2 \left( \chi + \frac{1}{4}\chi^2 \right).
\end{aligned} \quad (4.22)$$

With the normalization (4.20), the nucleon charge is

$$Q = 3 \sum_\Lambda \int_k \bar{\Psi}_{\Lambda\lambda}^S(P, k) j_1(0) \gamma^0 \Psi_{\Lambda\lambda}^S(P, k), \quad (4.23)$$

where  $j_1(0)$  is the quark charge operator defined in Eq. (3.4). This agrees with Eq. (2.14) of Ref. [23] (with the sum over the diquark polarization  $\Lambda$  extended to include the spin-1 diquark with polarization  $\Lambda$  and the spin-0 diquark as a separate term).

We now discuss the construction of the  $D$ -state wave function.

## C. $D$ -state component

### 1. Three-quark nonrelativistic wave function

In order to produce a spin 1/2 state through the coupling of a  $D$ -wave operator with a state of total quark spin  $S$ , the quark spin  $S$  must be 3/2. This is a purely symmetric spin state. Hence, maintaining the requirement that the wave

function be symmetric under the interchange of quarks 1 and 2 (recall that the other symmetries are handled as discussed in Sec. II A), the flavor-space part of the wave function can be constructed from the sum of only two components. One will be the product of components antisymmetric under 12 interchange in both flavor and momentum space, and the other will be the product of symmetric components. In analogy with Eq. (4.7), the flavor-space wave function will then be a superposition

$$\Psi_{m_\ell}^D(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2) = \cos\theta_D \boldsymbol{\phi}^0 \Psi_{m_\ell}^{Da}(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2) + \sin\theta_D \boldsymbol{\phi}^1 \Psi_{m_\ell}^{Ds}(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2), \quad (4.24)$$

where  $m_\ell$  is the projection of the spin-2 spatial wave function in some (arbitrary) fixed direction, and the  $\boldsymbol{\phi}^{0,1}$  are the isospin operators (4.3). In parallel with Eq. (4.8), the spatial wave functions will be written

$$\begin{aligned} \Psi_{m_\ell}^{Da} &= \sqrt{\frac{4\pi}{2}} [\mathbf{k}_1^2 Y_{m_\ell}^2(\hat{\mathbf{k}}_1) - \mathbf{k}_2^2 Y_{m_\ell}^2(\hat{\mathbf{k}}_2)] \Phi_D \\ \Psi_{m_\ell}^{Ds} &= \sqrt{4\pi} [\cos\phi \mathbf{k}^2 Y_{m_\ell}^2(\hat{\mathbf{k}}) + \sin\phi \mathbf{r}^2 Y_{m_\ell}^2(\hat{\mathbf{r}})] \Phi_D, \end{aligned} \quad (4.25)$$

where  $\Psi^{Da}$  is the most general  $L = 2$  function depending on  $\mathbf{k}_1$  and  $\mathbf{k}_2$  which is antisymmetric under the interchange  $\mathbf{k}_1 \leftrightarrow \mathbf{k}_2$ , and  $\Psi^{Ds}$  is the most general symmetric function, and  $\Phi_D$  is a symmetric  $D$ -state counterpart to the  $S$ -state function  $\Phi_S$ . Note that the mixing between the two terms in  $\Psi^{Da}$  is fixed by the antisymmetry requirement but that the requirement of symmetry alone cannot fix the relative contributions of the two terms in  $\Psi^{Ds}$ . We will fix these below.

To complete the construction of the  $D$ -state wave function, we combine the  $L = 2$  orbital part with the total quark spin 3/2 wave functions. For a spin-1/2 state with projection  $\lambda$ , this combination is

$$\Psi_\lambda^{D\eta} = \sum_{m_\ell=-1}^2 \left\langle 2m_\ell \frac{3}{2} \mu \left| \frac{1}{2} \lambda \right\rangle \Psi_{m_\ell}^{D\eta} \left| \frac{3}{2} \mu \right\rangle, \quad (4.26)$$

where  $\eta = \{a, s\}$ , and the value of  $\mu$  is fixed by the vector coupling coefficient, and the spin 3/2 state can be written as a direct product of a vector and a spin-1/2 spinor:

$$\left| \frac{3}{2} \mu \right\rangle = \sum_\nu \left\langle 1\Lambda \frac{1}{2} \nu \left| \frac{3}{2} \mu \right\rangle |1\Lambda\rangle \otimes \left| \frac{1}{2} \nu \right\rangle, \quad (4.27)$$

where  $|1\Lambda\rangle = \varepsilon_\Lambda^i$  is the spin function of the (12) pair (spin 1 and polarization  $\Lambda$ ), and  $|\frac{1}{2} \nu\rangle$  is the spin of the spectator quark 3. It is convenient to project out the polarization state of the diquark, and the wave function we will use in applications will be constructed from the functions

$$\begin{aligned} \Theta_{\Lambda\lambda}^D(\mathbf{k}_i) &= \sqrt{4\pi} \sum_{m_\ell=-1}^2 \left\langle 2m_\ell \frac{3}{2} \mu \left| \frac{1}{2} \lambda \right\rangle \mathbf{k}_i^2 Y_{2m_\ell}(\hat{\mathbf{k}}_i) \left\langle 1\Lambda \left| \frac{3}{2} \mu \right\rangle \right\rangle \\ &= \sqrt{4\pi} \sum_{m_\ell=-1}^2 \left\langle 2m_\ell \frac{3}{2} \mu \left| \frac{1}{2} \lambda \right\rangle \mathbf{k}_i^2 Y_{2m_\ell}(\hat{\mathbf{k}}_i) \right. \\ &\quad \times \left. \left\langle 1\Lambda \frac{1}{2} \nu \left| \frac{3}{2} \mu \right\rangle \left| \frac{1}{2} \nu \right\rangle \right\rangle. \end{aligned} \quad (4.28)$$

The complete three-quark nonrelativistic symmetric  $D$  state is then

$$\begin{aligned} \Psi_{\Lambda\lambda}^{Da}(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2) &= \frac{1}{\sqrt{2}} (\Theta_{\Lambda\lambda}^D(\mathbf{k}_1) - \Theta_{\Lambda\lambda}^D(\mathbf{k}_2)) \Phi_D \\ \Psi_{\Lambda\lambda}^{Ds}(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2) &= (\cos\phi \Theta_{\Lambda\lambda}^D(\mathbf{k}) + \sin\phi \Theta_{\Lambda\lambda}^D(\mathbf{r})) \Phi_D. \end{aligned} \quad (4.29)$$

Note that both of these components depend on the diquark (spin-1) polarization  $\Lambda$ .

Before moving on with our construction, we call attention to the work of Diaz and Riska [10], who present a construction of a  $D$ -state wave function along lines similar to ours. However, in place of our spatially *antisymmetric* function introduced in the first of Eqs. (4.25), they use the spatially *symmetric* function  $q^2 Y_{m_\ell}^2(\hat{\mathbf{q}})$  [where  $\mathbf{q} = \frac{1}{\sqrt{2}}(\mathbf{k}_1 - \mathbf{k}_2)$  is one of the Jacobi momenta of the three-quark system—see their Eq. (15)], giving results that differ from ours.

## 2. Alternative nonrelativistic form

To prepare for what follows, it is useful to rewrite the  $\Theta$  functions in Eq. (4.29) in an alternative form. In Appendix B, we show that

$$\Theta_{\Lambda\lambda}^D(\mathbf{k}_i) = \sqrt{\frac{3}{2}} (\varepsilon_\Lambda^*)_\ell D^{\ell\ell'}(\mathbf{k}_i) \sigma_{\ell'} \left| \frac{1}{2} \lambda \right\rangle, \quad (4.30)$$

where  $\varepsilon_\Lambda$  is a spin-1 polarization vector of the (12) pair with spin projection  $\Lambda$  (already introduced above), and  $|\frac{1}{2} \lambda\rangle$  is the two-component spinor with spin projection  $\lambda$ , both along the  $\hat{z}$  direction, and the angular momentum two  $D$  matrix is

$$D^{\ell m}(\mathbf{k}_i) = \mathbf{k}_i^\ell \mathbf{k}_i^m - \frac{1}{3} \mathbf{k}_i^2 \delta_{\ell m}. \quad (4.31)$$

## 3. Normalization

The wave function (4.29) depends on the momenta of all three of the quarks. Extracting the diquark content requires that we average over the internal momenta of the 12 pair (the diquark), just as we did in Eq. (4.14).

Using the new representations for the  $\Theta$ 's, it is now possible to separate the internal diquark variable ( $r$ ) from the quark variable ( $k$ ). Denoting  $\mathbf{k}_+ \equiv \mathbf{k}_+ = \frac{1}{2} \mathbf{k} + \mathbf{r}$  and  $\mathbf{k}_- \equiv \mathbf{k}_- = \frac{1}{2} \mathbf{k} - \mathbf{r}$ , the  $D$  of Eq. (4.31) separates

$$D^{\ell m}(\mathbf{k}_{\pm}) = \frac{1}{4}D^{\ell m}(\mathbf{k}) + D^{\ell m}(\mathbf{r}) \pm \frac{1}{2}G^{\ell m}(\mathbf{k}, \mathbf{r}), \quad (4.32)$$

with

$$G^{\ell m}(\mathbf{k}, \mathbf{r}) \equiv \mathbf{k}^{\ell} \mathbf{r}^m + \mathbf{r}^{\ell} \mathbf{k}^m - \frac{2}{3} \delta_{\ell m} \mathbf{k} \cdot \mathbf{r}. \quad (4.33)$$

With this substitution, the antisymmetric component becomes

$$\Psi_{\Lambda\lambda}^{Da}(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2) = \sqrt{\frac{3}{4}}(\boldsymbol{\epsilon}_{\Lambda}^*)_{\ell} G^{\ell m}(\mathbf{k}, \mathbf{r}) \sigma_m |\frac{1}{2}\lambda\rangle \Phi_D(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2). \quad (4.34)$$

Note that this component is linear in both  $\mathbf{r}$  and  $\mathbf{k}$ , describing the coupling of a diquark with an *internal*  $P$ -wave, or *internal angular momentum*-1 structure to the third quark in a relative  $P$ -wave, while the isovector component (4.29) is a sum of two  $D$ -wave components, one in the total diquark momentum  $\mathbf{k}$  and the other in the *internal* diquark momentum  $\mathbf{r}$ .

In matrix elements, when we average over the direction of the relative momentum,  $\mathbf{r}$ , these components will give a nonzero result only when multiplied by another component of the same type. Using the results from Appendix C, and the isospin matrix elements (4.10), the normalization integral becomes

$$\begin{aligned} Q &= e_0 \left[ \frac{1}{2} + \frac{3}{2}\tau_3 \right] \cos^2 \theta_D \langle |\Psi^{Da}|^2 \rangle_{\lambda'\lambda} \\ &\quad + e_0 \left[ \frac{1}{2} - \frac{1}{2}\tau_3 \right] \sin^2 \theta_D \langle |\Psi^{Ds}|^2 \rangle_{\lambda'\lambda} \\ &= \frac{1}{2} e_0 [1 + \tau_3 (3 \cos^2 \theta_D - \sin^2 \theta_D)] \delta_{\lambda'\lambda} \mathcal{N}_D, \end{aligned} \quad (4.35)$$

with the normalization constant  $\mathcal{N}_D$  defined in Eqs. (C2) and (C7). Once again, we observe that the correct neutron charge cannot be obtained unless  $\theta_D = \pi/4$ , implying an equal mixture of diquark symmetric and antisymmetric components, just as for the  $S$  state. The  $D$ -state wave function is then normalized to  $\mathcal{N}_D = 1/e_0$ .

#### 4. Defining the nonrelativistic quark-diquark wave function

The  $D$  state presents us with some new issues in extracting a quark-diquark wave function from the full three-quark wave function. Examination of Eqs. (4.29) and (4.34) shows that there are three orthogonal structures, each with a different kind of average over the diquark internal momentum variable  $\mathbf{r}$ . The first of these accompanies the square of the  $\Theta_{\Lambda\lambda}^D(\mathbf{k})$  term in the isovector term. Including an extra factor of  $1/4$  (to simplify the final normalization) gives a definition

$$\frac{1}{4} \int_{k,r}^{\text{NR}} \mathbf{k}^4 \Phi_D^2 = \int \frac{d^3 k}{(2\pi)^3} \mathbf{k}^4 \phi_D^2, \quad (4.36)$$

where the integral was defined in Eq. (2.13), and to simplify the notation, we let  $\Phi_D(\mathbf{P}; \mathbf{k}_1, \mathbf{k}_2) \rightarrow \Phi_D$ , and the new wave function is  $\phi_D = \phi_D(\mathbf{P}, \mathbf{k})$ . Except for the factor of  $1/4$ , this is precisely the same prescription used

for the  $S$ -state wave function [see Eq. (4.14) for the arguments and other details] and leads to the replacement

$$\frac{1}{2} \Phi_D \rightarrow \phi_D. \quad (4.37)$$

This function is normalized to

$$\int \frac{d^3 k}{(2\pi)^3} \mathbf{k}^4 \phi_D^2 = \mathcal{N}_D = \frac{1}{e_0}, \quad (4.38)$$

which, because of the definition (4.36), is equivalent to the normalization defined in Eqs. (C2) and (C4).

The second term to be defined depends linearly on  $\mathbf{r}$  and appears only in the isoscalar term (4.34). As previously mentioned, this linear  $\mathbf{r}$  dependence describes a diquark with an internal *angular momentum* dependent  $P$ -wave, which can be represented by a diquark with a polarization vector  $\boldsymbol{\zeta}_{\nu}$  (with  $\nu = \{\pm, 0\}$  the three independent polarization states). But this isoscalar diquark is described by *two* vectors: one due to its internal momentum,  $\boldsymbol{\zeta}_{\nu}$ , and the other due to its *spin*,  $\boldsymbol{\epsilon}_{\Lambda}$ . This term is orthogonal to all other terms, and the diquark content of this term can be extracted by introducing the following correspondence

$$\begin{aligned} \int_{k,r}^{\text{NR}} \mathbf{k}^2 \mathbf{r}^{\ell} \mathbf{r}^m \Phi_D^2 &= \delta_{\ell m} \frac{1}{3} \int_{k,r}^{\text{NR}} \mathbf{k}^2 \mathbf{r}^2 \Phi_D^2 \\ &= \delta_{\ell m} \frac{c_P^2}{3} \int_{k,r}^{\text{NR}} \mathbf{k}^4 \Phi_D^2 \\ &\rightarrow \frac{4c_P^2}{3} \sum_{\nu} \boldsymbol{\zeta}_{\nu}^{\ell} \boldsymbol{\zeta}_{\nu}^{m*} \int \frac{d^3 k}{(2\pi)^3} \mathbf{k}^4 \phi_D^2, \end{aligned} \quad (4.39)$$

where the factor  $c_P^2 = 1/A = 3/20$  was computed in Appendix C [see Eq. (C7)], and the extra factor of 4 from the correspondence (4.36) has been included. Note that the integral over the continuous variable  $\mathbf{r}$  is replaced by the sum of the complete set of polarization states  $\boldsymbol{\zeta}_{\nu}$ , and the average over  $\mathbf{r}^2$  is replaced by  $c_P^2 \mathbf{k}^2$ . With this choice, we obtain the correspondence

$$\frac{1}{2} \mathbf{r}^{\ell} \Phi_D \rightarrow \frac{c_P}{\sqrt{3}} |\mathbf{k}| \boldsymbol{\zeta}_{\nu}^{\ell} \phi_D, \quad (4.40)$$

with the sum over polarization states,  $\nu$ , to be carried out in the calculation of any matrix element, using the completeness relation

$$\sum_{\nu} \boldsymbol{\zeta}_{\nu}^{\ell} \boldsymbol{\zeta}_{\nu}^{m*} = \delta_{\ell m}, \quad (4.41)$$

and with the understanding that the terms linear in the total diquark momentum,  $\mathbf{k}$ , will be left unchanged. The reason for using the momentum  $\mathbf{k}$  (instead of  $\mathbf{r}$ ) on the right-hand side of Eq. (4.40) is to eliminate all dependence on  $\mathbf{r}$ , leaving only one function  $\phi_D$  with the same normalization condition (4.38).

Finally, the last term to be defined is the term depending on  $\Theta_{\Lambda\lambda}^D(\mathbf{r})$ . This term is a contraction of the spin-1 vector,  $\boldsymbol{\epsilon}_{\Lambda}$ , of the diquark with the  $D$ -wave internal *momentum* structure of the diquark. Its average can be represented by a



new effective diquark polarization vector,  $\epsilon_{D\Lambda}$ , with the familiar property (4.41). The average we need is

$$\begin{aligned} & \sum_{\Lambda} \int_{k,r}^{\text{NR}} (\epsilon_{\Lambda})_{\ell'} D^{\ell' m'}(\mathbf{r}) (\epsilon_{\Lambda}^*)_{\ell} D^{\ell m}(\mathbf{r}) \Phi_D^2 \\ &= \delta_{m'm} \frac{2}{9} \int_{k,r}^{\text{NR}} \mathbf{r}^4 \Phi_D^2 = \delta_{m'm} \frac{2c_D^2}{9} \int_{k,r}^{\text{NR}} \mathbf{k}^4 \Phi_D^2 \\ &\rightarrow \frac{8c_D^2}{9} \sum_{\Lambda} \epsilon_{D\Lambda}^{m'} \epsilon_{D\Lambda}^{m*} \int \frac{d^3 k}{(2\pi)^3} \mathbf{k}^4 \phi_D^2, \end{aligned} \quad (4.42)$$

where  $c_D^2 = B/A = 1/16$  was computed in Appendix C, and the factor of 4 from Eq. (4.36) has again been included. With this choice, we obtain the correspondence

$$\frac{1}{2} (\epsilon_{\Lambda}^*)_{\ell} D^{\ell m}(\mathbf{r}) \Phi_D \rightarrow \frac{\sqrt{2}c_D}{3} \epsilon_{D\Lambda}^{m*} \mathbf{k}^2 \phi_D. \quad (4.43)$$

With this notation, the quark-diquark wave functions corresponding to the three-quark wave functions given in Eqs. (4.29) and (4.34) become

$$\begin{aligned} \psi_{\Lambda\lambda}^{Da}(P, k) &= c_P (\epsilon_{\Lambda}^*)_{\ell} G^{\ell m}(\mathbf{k}, \zeta_{\nu}) \sigma_m \left| \frac{1}{2} \lambda \right\rangle |\mathbf{k}| \phi_D(P, k) \\ \psi_{\Lambda\lambda}^{Ds}(P, k) &= \frac{2}{\sqrt{5}} \Theta_{\Lambda\lambda}^D(\mathbf{k}) \phi_D(P, k) \\ &\quad + \frac{4c_D}{\sqrt{15}} \epsilon_{D\Lambda}^{m*} \sigma_m \left| \frac{1}{2} \lambda \right\rangle \mathbf{k}^2 \phi_D(P, k), \end{aligned} \quad (4.44)$$

where the value of the mixing angle  $\phi$  determined in Appendix C has been used.

It is straightforward to confirm that our substitutions preserve the normalizations given in Appendix C. We now turn to the relativistic generalization of Eq. (4.44).

### 5. Relativistic quark-diquark $D$ -state wave function

Using the ideas developed in Ref. [23,24], the relativistic analogue of the wave functions from the previous section are constructed from the four-vector

$$\tilde{k} = k - \frac{(k \cdot P)}{M^2} P, \quad (4.45)$$

where  $P$  is the total four-momentum of the nucleon. This insures that  $\tilde{k}$  will reduce to  $\mathbf{k}$  in the rest system. Similarly, all of the polarization four-vectors are chosen to be orthogonal to  $P$ , insuring that, in the nucleon rest frame, their time components are zero, and their spatial components are identical to their corresponding nonrelativistic three-vectors. In addition, we replace

$$\delta_{\ell\ell'} \rightarrow -\tilde{g}_{\alpha\beta} = -\left(g_{\alpha\beta} - \frac{P_{\alpha} P_{\beta}}{M^2}\right)$$

$$(\epsilon_{\Lambda}^*)_{\ell} \rightarrow (\epsilon_{\Lambda}^*)^{\alpha}$$

$$\sigma_{\ell'} \left| \frac{1}{2} \lambda \right\rangle \rightarrow \sqrt{3} U^{\beta}(P, \lambda)$$

$$G^{\ell\ell'}(\mathbf{k}, \zeta_{\nu}) \rightarrow G^{\alpha\beta}(k, \zeta_{\nu}) \equiv \tilde{k}^{\alpha} \zeta_{\nu}^{\beta} + \zeta_{\nu}^{\alpha} \tilde{k}^{\beta} - \frac{2}{3} \tilde{g}^{\alpha\beta} (\tilde{k} \cdot \zeta_{\nu})$$

$$D^{\ell\ell'}(\mathbf{k}) \rightarrow D^{\alpha\beta}(P, k) \equiv \tilde{k}^{\alpha} \tilde{k}^{\beta} - \frac{1}{3} \tilde{g}^{\alpha\beta} \tilde{k}^2$$

$$\phi_D(P, k) \rightarrow \psi_D(P, k), \quad (4.46)$$

where  $\ell \rightarrow \alpha$ ,  $\ell' \rightarrow \beta$  and  $U^{\beta}(P, k)$  were defined in Eq. (4.17). With these correspondences, the total  $D$ -state wave function of the nucleon for an on-shell diquark (composed of quarks 1 and 2) and an off-shell quark (3) with momentum  $k_3 = P - k$  is,

$$\Psi_{\Lambda\lambda}^D(P, k) = \frac{1}{\sqrt{2}} \{ \Phi^0 \psi_{\Lambda\lambda}^{Da}(P, k) + \Phi^1 \psi_{\Lambda\lambda}^{Ds}(P, k) \}, \quad (4.47)$$

where

$$\begin{aligned} \psi_{\Lambda\lambda}^{Da}(P, k) &= \sqrt{3} c_P (\epsilon_{\Lambda}^*)_{\alpha} G^{\alpha\beta}(\tilde{k}, \zeta_{\nu}) U_{\beta}(P, \lambda) |\tilde{k}| \psi_D(P, k) \\ \psi_{\Lambda\lambda}^{Ds}(P, k) &= \frac{2}{\sqrt{5}} \Theta_{\Lambda\lambda}^D(P, k) \psi_D(P, k) \\ &\quad - \frac{4c_D}{\sqrt{5}} \epsilon_{D\Lambda}^{\beta*} U_{\beta}(P, \lambda) \tilde{k}^2 \psi_D(P, k), \end{aligned} \quad (4.48)$$

where  $G^{\alpha\beta}(\tilde{k}, \zeta_{\nu})$  is the straightforward generalization of its nonrelativistic counterpart,  $|\tilde{k}| \equiv \sqrt{-\tilde{k}^2}$ ,

$$\Theta_{\Lambda\lambda}^D(P, k) = \frac{3}{\sqrt{2}} (\epsilon_{\Lambda}^*)_{\alpha} D^{\alpha\beta}(P, k) U_{\beta}(P, \lambda), \quad (4.49)$$

and  $\phi_D(P, k)$  is a spherically symmetric scalar function of the off-shell quark momenta  $k_3 = P - k$ .

### 6. Normalization of the relativistic wave function

The normalization of the relativistic  $D$ -state wave function, like the  $S$  state, is fixed by the charge (3.5):

$$Q = 3 \sum_{\Lambda} \int_k \bar{\Psi}_{\Lambda\lambda'}^D(P, k) j_q(0) \gamma^0 \Psi_{\Lambda\lambda}^D(P, k). \quad (4.50)$$

This condition is satisfied by the wave function (4.47), with components (4.48), provided

$$1 = e_o \int \frac{d^3 k}{(2\pi)^3 2E_k} \tilde{k}^4 \psi_D^2(P, k), \quad (4.51)$$

which is the relativistic analogue of the nonrelativistic norm (4.38).

From now on, we will use  $E_k = \sqrt{m_s^2 + \mathbf{k}^2}$  instead of  $E_s$ .

### D. $P$ -state component

The origin of the  $P$ -state component can be traced back to the general CST relation between the relativistic vertex function,  $\Gamma$ , and the relativistic wave function

$$\Psi(P, k) = \frac{1}{m_q - \not{k}_3} \Gamma(P, k) = \frac{m_q + \not{k}_3 \not{k}_3}{m_q^2 - k_3^2} \Gamma(P, k), \quad (4.52)$$

where  $(m_q - \not{k}_3)^{-1}$  is the propagator of the off-shell quark with four-momentum  $k_3 = P - k$  (when quarks 1 and 2 are the on-shell spectators). In the presence of confinement, the pole at  $m_q^2 = k_3^2$  is cancelled by a zero  $\Gamma$ , so the wave function can be modeled without regard to this singularity. For simplicity, in our previous work [15], we also absorbed the projection operator  $m_q + \not{k}_3$  into our model of  $\Psi$ , but the assumption that the resulting  $\Psi$  is a *pure*  $S$ -state ignores some additional structure which the projection operator could provide.

Now, suppose we assume that the projection operator is *not* absorbed into the definition of the wave function. Then, the wave function would be

$$\Psi_0^S(P, k) \simeq \frac{\Gamma(P, k)}{m_q^2 - k_3^2}, \quad (4.53)$$

and the action of the quark projection operator on this wave function would give

$$\begin{aligned} (m_q + \not{k}_3) \Psi_0^S(P, k) &= (m_q + M - \not{k}) \Psi_0^S(P, k) \\ &= \left[ m_q + M - \frac{P \cdot k}{M} - \tilde{k} \right] \Psi_0^S(P, k) \\ &\simeq \Psi^S(P, k) + n_P \tilde{k} \Psi^P(P, k), \end{aligned} \quad (4.54)$$

where  $\tilde{k}$  was defined by Eq. (4.45), and the expressions were reduced using the Dirac equation,  $\not{P}\Psi = M\Psi$ , satisfied by the wave function. This clearly shows how the Dirac projection operator leads both to a redefinition of the  $S$  state and to a new  $P$ -state component. By allowing the mixing parameter,  $n_P$ , to be determined by the data, we free ourselves from any biases about the possible size of this effect.

A  $P$ -state component could have been constructed by starting from either the  $S$ - or  $D$ -state contributions, but in this paper, we limit the discussion to  $P$  states constructed from the dominant  $S$ -state component, as suggested by Eq. (4.54). This gives the following ansatz for the  $P$  state:

$$\Psi_{\Lambda\Lambda}^P(P, k) = \frac{1}{\sqrt{2}} \tilde{k} [\phi^0 u(P, \lambda) - \phi^1 \varepsilon_{\Lambda P}^{\alpha*} U_\alpha(P, \lambda)] \psi_P(P, k). \quad (4.55)$$

The normalization of the  $P$  state is

$$e_0 \int \frac{d^3 k}{(2\pi)^3 2E_k} (-\tilde{k}^2) \psi_P^2(P, k) = 1, \quad (4.56)$$

where the minus sign arises because  $\tilde{k} \gamma^0 \tilde{k} = -\tilde{k}^2 > 0$ .

Our  $S$ -state wave function is a Dirac spinor with the *lower* two components exactly zero in the nucleon rest frame. Conversely, the wave function (4.55) is a Dirac spinor with the *upper* two components exactly zero in the nucleon rest frame. For this reason, it has positive parity (the negative parity of a  $P$ -wave being cancelled by the negative sign from the Dirac parity operator  $\gamma^0$ ).

This  $P$ -wave component is of purely relativistic origin because it will only make nonzero contributions to matrix elements which have lower, relativistic components in Dirac space.

### V. BREAKING ISOSPIN SYMMETRY

In fitting the DIS cross sections, it will be necessary to include the possibility that the  $u$  and the  $d$  distributions are not identical. This assumption violates isospin invariance.

In the following discussion, the “ $u$  distribution” refers to the  $u$  distribution in protons and the  $d$  distribution in neutrons (we retain charge symmetry). Similarly, the “ $d$  distribution” refers to the  $d$  distribution in protons and the  $u$  distribution in neutrons.

To generalize the formalism to allow for this difference, look at the flavor wave functions (4.3) in more detail. Examination of their structure shows that some of the flavor functions describe  $u$  distributions and others  $d$  distributions. In particular, the isoscalar and the  $\ell = 0$  isovector components describe  $u$ -quark distributions, since the interacting quark is always a  $u$  quark in the proton or a  $d$  quark in the neutron. Returning to the full three-quark notation of Ref. [15], where the flavor wave functions were written as a direct product in the order 1,2,3, so that  $a_1 b_2 c_3 \rightarrow abc$ , and identifying the (12) pair with the diquark, so that the photon interacts with the third quark, it is easy to extract the separate  $u$  and  $d$  distributions,

$$\begin{aligned} \sqrt{\frac{1}{2}} (ud - du) \begin{Bmatrix} u & \text{proton} \\ d & \text{neutron} \end{Bmatrix} &\rightarrow \phi^0 \chi' \\ -\sqrt{\frac{1}{6}} (ud + du) \begin{Bmatrix} u & \text{proton} \\ d & \text{neutron} \end{Bmatrix} &\rightarrow \phi_{\ell=0}^1 \chi', \end{aligned} \quad (5.1)$$

while the isovector  $\ell \neq 0$  components describe the  $d$ -quark distribution in the proton and the  $u$ -quark distribution in the neutron:

$$\sqrt{\frac{2}{3}} \begin{Bmatrix} (uu)d & \text{proton} \\ -(dd)u & \text{neutron} \end{Bmatrix} \rightarrow \phi_{|\ell|=1}^1 \chi'. \quad (5.2)$$

Next, we introduce different wave functions for the  $u$  and  $d$  quarks for each angular momentum component,

$$\psi_L(P, k) \rightarrow \begin{cases} \psi_u^L(P, k) & u \text{ quark} \\ \psi_d^L(P, k) & d \text{ quark} \end{cases}, \quad (5.3)$$

where  $L = S, P, D$ . We assume that (for example) the  $u$ -quark distribution corresponding to the spin-0 and spin-1 diquarks are identical, but this could be relaxed

later, if necessary. Then, the matrix element of the charge accompanying the isoscalar diquark is unchanged except for replacing  $\psi_L^2(P, k)$  by  $[\psi_u^L(P, k)]^2$ . However, the isovector matrix element,  $j^S \psi_L^2$  [recall Eq. (4.10)], now separates into two independent contributions. Allowing for the fact that the renormalization of the  $u$ -quark charge ( $e_0 \rightarrow e_u^0$ ) and  $d$ -quark charge ( $e_0 \rightarrow e_d^0$ ) might be unequal, and dropping the index  $L$ , gives

$$\begin{aligned} j_0^S \psi_u^2(P, k) &= (\tau \cdot \xi_0) \left( \frac{1}{6} + \frac{1}{2} \tau_3 \right) (\tau \cdot \xi_0^*) e_u^0 \psi_u^2(P, k) \\ &= \left( \frac{1}{6} + \frac{1}{2} \tau_3 \right) e_u^0 \psi_u^2(P, k) \\ j_1^S \psi_d^2(P, k) &= (\tau \cdot \xi_+) \left( \frac{1}{6} + \frac{1}{2} \tau_3 \right) (\tau \cdot \xi_+^*) e_d^0 \psi_d^2(P, k) \\ &\quad + (\tau \cdot \xi_-) \left( \frac{1}{6} + \frac{1}{2} \tau_3 \right) (\tau \cdot \xi_-^*) e_d^0 \psi_d^2(P, k) \\ &= \left( \frac{1}{3} - \tau_3 \right) e_d^0 \psi_d^2(P, k). \end{aligned} \quad (5.4)$$

Hence, the generalized isovector charge operator is transformed (for any  $L$ ) to

$$\begin{aligned} j^S \psi^2(P, k) &\rightarrow j_0^S \psi_u^2(P, k) + j_1^S \psi_d^2(P, k) \\ &= \frac{1}{6} (1 - \tau_3) [e_u^0 \psi_u^2(P, k) + 2e_d^0 \psi_d^2(P, k)] \\ &\quad + \frac{2}{3} \tau_3 [e_u^0 \psi_u^2(P, k) - e_d^0 \psi_d^2(P, k)]. \end{aligned} \quad (5.5)$$

If  $\psi_d = \psi_u$  (and  $e_u^0 = e_d^0$ ), then the sum of these reduces to the previous result (4.10), but  $\psi_d \neq \psi_u$ , the last term, clearly gives a different result. When combined with the isoscalar contribution, the total is (choosing the mixing angle  $\theta_L = \pi/4$  as before)

$$\begin{aligned} &\frac{1}{2} j^A \psi_u^2(P, k) + \frac{1}{2} [j_0^S \psi_u^2(P, k) + j_1^S \psi_d^2(P, k)] \\ &= \frac{1}{6} (2e_u^0 \psi_u^2 + e_d^0 \psi_d^2) + \frac{1}{2} (2e_u^0 \psi_u^2 - e_d^0 \psi_d^2) \tau_3 \\ &= \left\{ \begin{array}{l} \frac{1}{3} (4e_u^0 \psi_u^2 - e_d^0 \psi_d^2) \\ -\frac{2}{3} (e_u^0 \psi_u^2 - e_d^0 \psi_d^2) \end{array} \right\} \begin{array}{l} P \\ n. \end{array} \end{aligned} \quad (5.6)$$

This gives the requirement that both  $u$  and  $d$  distributions be normalized to  $1/e_q^0$ . For arbitrary  $L$ , this gives

$$\begin{aligned} 1 &= e_u^0 \int \frac{d^3 k}{(2\pi)^3 2E_k} (-\tilde{k}^2)^L [\psi_u^L(P, k)]^2 \\ &= e_d^0 \int \frac{d^3 k}{(2\pi)^3 2E_k} (-\tilde{k}^2)^L [\psi_d^L(P, k)]^2. \end{aligned} \quad (5.7)$$

This result for the charge operator may be generalized. In particular, if  $O$  is any operator in isospin space, and the isospin-one matrix elements of  $O$  are defined by

$$3\phi_0^1 O \phi_0^1 \equiv O_0^1 \quad 3\phi_1^1 O \phi_1^1 + 3\phi_{-1}^1 O \phi_{-1}^1 \equiv O_1^1, \quad (5.8)$$

then for arbitrary angular momentum components  $L$  and  $L'$ , we may make the replacement

$$\begin{aligned} \langle O \rangle_{L', L} &\rightarrow (O + O_0^1) \psi_u^{L'}(P, k) \psi_u^L(P, k) \\ &\quad + O_1^1 \psi_d^{L'}(P, k) \psi_d^L(P, k). \end{aligned} \quad (5.9)$$

## VI. SUMMARY AND OVERVIEW

In this paper, we present relativistic CST wave function for the nucleon with  $S$ -,  $P$ -, and  $D$ -state components. These wave functions are designed to be used in calculations where two of the quarks are noninteracting on-shell spectators, with the third off-shell quark interacting with an external probe. In such a situation, the full dependence of the matrix element on the relative momentum,  $r$ , of the two on-shell spectators is contained in the three-quark wave function. Integrating over  $r$  will then lead to a new effective wave function with the two noninteracting quarks replaced by a quark pair (an effective diquark) with a mass,  $m_s$ , which may be treated as a parameter. The resulting quark-diquark wave function contains all of the information originally included in the three-quark wave function and may be used in a variety of calculations.

The extraction of a quark-diquark wave function is discussed in general terms in Sec. II B with the most general result for the extraction of the wave function presented in Eq. (2.9). In Sec. IV, the various component wave functions are constructed and presented. The relativistic  $S$ -state quark-diquark wave function is given in Eq. (4.16). This is identical to the  $S$ -state wave function used previously [15, 23], but the extraction of this wave function from the full three-quark wave function, culminating in the nonrelativistic correspondence (4.14), has never been discussed before. The relativistic  $P$ -state quark-diquark wave function, also new, is given in Eq. (4.55). This component vanishes in the nonrelativistic limit.

The bulk of this paper is devoted to obtaining the relativistic  $D$ -state quark-diquark wave function given in Eqs. (4.47) and (4.48). This component is a superposition of terms with an isoscalar  $P$ -wave diquark (represented by the polarization vector  $\zeta_\nu$ ) and two terms with an isovector diquark, one with an internal  $S$ -wave structure, and one with an internal  $D$ -wave structure. Although each of these diquarks has a different angular momentum structure, they are all derived from contributions in which the quarks are in a relative  $L = 2$  angular momentum state. Furthermore, each of these diquarks has spin 1, as required by the coupling of overall spin  $3/2$  to  $L = 2$  to produce a nucleon with spin- $1/2$ . These three components are orthogonal, and the *only* component which can interfere with the dominant  $S$ -state component is the latter, which is summarized by the contribution

$$\begin{aligned} \Psi_{\Lambda\lambda}^D(P, k) &\rightarrow \frac{3}{\sqrt{5}} \phi^1(\varepsilon_\Lambda^*)_\alpha D^{\alpha\beta}(P, k) U_\beta(P, \lambda) \psi_D(P, k) \\ &\equiv \Psi_{\Lambda\lambda}^{D,2}(P, k). \end{aligned} \quad (6.1)$$

In applications when we need terms linear in the  $D$ -state, this is the only term that need be considered, and it can contribute only to matrix elements which do not allow for a complete integration over all directions of  $\mathbf{k}$  (as in the important case of DIS).

The normalization condition for the component (6.1) is

$$3 \sum_{\Lambda} \int_k \bar{\Psi}_{\Lambda\lambda}^{D,2}(P, k) j_q(0) \gamma^0 \Psi_{\Lambda\lambda}^{D,2}(P, k) = j^S \delta_{\Lambda\lambda} \frac{2}{5}, \quad (6.2)$$

where  $j^S$  was defined in Eq. (4.10), and the  $D$ -state normalization condition (4.51) has been retained. Recalling that the contribution from both the isovector  $D$ -state components is  $1/2$ , we recover the previous result, also contained in Eq. (4.48), that the component (6.1) accounts for  $4/5$  of the total symmetric contribution.

To allow for the possibility that the  $u$  and  $d$  quarks have a different angular momentum distribution, we may break isospin symmetry as discussed in Sec. V. The symmetry can be broken and the shapes of the  $u$  and  $d$  quark distributions individually adjusted as long as their normalization remains fixed.

The wave functions derived in this paper are used in our discussion of DIS in the companion paper [22], which provides numerical calculations indicating that the  $L \neq 0$  wave function components described here make important contributions to the proton spin.

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## APPENDIX A: RELATIVISTIC DIQUARK AVERAGES

In this appendix, we discuss the reduction of the integral (2.3) to the form (2.8). Begin by writing the integral in a manifestly covariant form and replacing the integrals over  $k_1$  and  $k_2$  by integrals over the sums and differences,  $k$  and  $r$ , of these momenta:

$$\begin{aligned} (2\pi)^6 \int_{k_1 k_2} &= \int d^4 k_1 d^4 k_2 \delta_+(m_q^2 - k_1^2) \delta_+(m_q^2 - k_2^2) \\ &= \int d^4 k \int d^4 r \delta_+ \left( E_1^2 - \left( \frac{1}{2} k_0 + r_0 \right)^2 \right) \\ &\quad \times \delta_+ \left( E_2^2 - \left( \frac{1}{2} k_0 - r_0 \right)^2 \right). \end{aligned} \quad (A1)$$

Both integrals are covariant, so we will evaluate the integral over  $r$  in the *two-body rest system* and use the two  $\delta$  functions to fix  $r_0$  and  $|\mathbf{r}|$ . This will leave the integral over  $k$  completely unconstrained, except we *define*  $k^2 = s$ . Therefore, in the two-body rest system,  $k_0 = \sqrt{s}$  and the integral is

$$(2\pi)^6 \int_{k_1 k_2} = \frac{1}{4} \int d^4 k \int d\Omega_{\hat{\mathbf{r}}} \sqrt{\frac{s - 4m_q^2}{4s}}. \quad (A2)$$

Finally, we express the  $k_0$  integration in terms of  $s$  using  $k_0^2 = s + |\mathbf{k}|^2 \equiv E_s^2$ , giving the relation (2.8)

$$\int_{k_1 k_2} = \int_{sk}. \quad (A3)$$

In using this formula, we must be careful to evaluate angular integrals involving  $\mathbf{r}$  in the rest system of the two-body pair.

## APPENDIX B: ALTERNATIVE FORM FOR THE NON-RELATIVISTIC $D$ -WAVE

To show the equivalence of Eqs. (4.28) and (4.30), first introduce the spin-3/2 projection operator [obtained by using Eq. (4.27) to write each spin-3/2 state in terms of a direct product of spin-1 and spin-1/2 states],

$$\begin{aligned} \mathcal{P}_{3/2} &= \sum_{\mu} \left| \frac{3}{2} \mu \right\rangle \left\langle \frac{3}{2} \mu \right| \\ &= \sum_{\mu \nu \nu'} \left\langle 1 \Lambda_2 \frac{1}{2} \nu \left| \frac{3}{2} \mu \right\rangle \left\langle 1 \Lambda_1 \frac{1}{2} \nu' \left| \frac{3}{2} \mu \right\rangle \right| 1 \Lambda_2 \right\rangle \\ &\quad \times \left\langle 1 \Lambda_1 \left| \otimes \left| \frac{1}{2} \nu \right\rangle \left\langle \frac{1}{2} \nu' \right| \right|. \end{aligned} \quad (B1)$$

Now, starting with Eq. (4.30), insert this operator between the  $\varepsilon^*$  and  $D$  (we will show later that the contribution from the spin-1/2 projection operator is zero, and hence inserting the spin-3/2 projection operator is equivalent to inserting unity), use the orthogonality of the spin-1 polarization vectors ( $|1\Lambda\rangle = \varepsilon_{\Lambda}$ ), and insert a complete set of polarization vectors between the  $D$  and  $\sigma$ . This gives

$$\begin{aligned} \Theta_D^{\Lambda\lambda}(\mathbf{k}_i) &= \sqrt{\frac{3}{2}} \sum_{\mu \nu \nu'} \left\langle 1 \Lambda \frac{1}{2} \nu \left| \frac{3}{2} \mu \right\rangle \left\langle 1 \Lambda_1 \frac{1}{2} \nu' \left| \frac{3}{2} \mu \right\rangle \right| \frac{1}{2} \nu \right\rangle \\ &\quad \times \{ (\varepsilon_{\Lambda_1}^*)_{\ell} D^{\ell\ell'}(\mathbf{k}_i) (\varepsilon_{\Lambda'})_{\ell'} \} \left\langle \frac{1}{2} \nu' \left| \varepsilon_{\Lambda'}^* \cdot \sigma \right| \frac{1}{2} \lambda \right\rangle. \end{aligned} \quad (B2)$$

Use Eq. (2.20) from Ref. [23] (derived relativistically, but also true nonrelativistically, and with a sign correction) to evaluate the term in curly brackets,

$$\begin{aligned} D_{\Lambda_1, \Lambda'} &\equiv (\varepsilon_{\Lambda_1}^*)_{\ell} D^{\ell\ell'}(\mathbf{k}_i) (\varepsilon_{\Lambda'})_{\ell'} \\ &= -\frac{\sqrt{8\pi}}{3} \mathbf{k}_i^2 Y_{2m_{\ell}}(\hat{\mathbf{k}}_i) \langle 2m_{\ell} 1 \Lambda_1 | 1 \Lambda' \rangle, \end{aligned} \quad (B3)$$

(which fixes  $\Lambda' = m_{\ell} + \Lambda_1 = m_{\ell} + \mu - \nu'$ , allowing the replacement of the sum over  $\Lambda'$  by a sum over  $m_{\ell}$ ), and use a new identity,

$$\left\langle \frac{1}{2} \nu' \left| \varepsilon_{\Lambda'}^* \cdot \sigma \right| \frac{1}{2} \lambda \right\rangle = -\sqrt{3} \langle 1 \Lambda \frac{1}{2} \nu' | \frac{1}{2} \lambda \rangle, \quad (B4)$$



to fix  $\nu' = \lambda - \Lambda' = \lambda - m_\ell - \mu + \nu'$ , which requires that  $\mu = \lambda - m_\ell$  and removes the sum over  $\mu$ . With these substitutions,

$$\Theta_D^{\Lambda\lambda}(\mathbf{k}_i) = \sqrt{4\pi} \sum_{m_\ell} S_{m_\ell\lambda} \mathbf{k}_i^2 Y_{2m_\ell}(\hat{\mathbf{k}}_i) \left\langle 1\Lambda \frac{1}{2}\nu \left| \frac{3}{2}\mu \right\rangle \left| \frac{1}{2}\nu \right\rangle, \quad (\text{B5})$$

where

$$S_{m_\ell\lambda} = \sum_{\nu'} \langle 2m_\ell 1\Lambda_1 | 1\Lambda' \rangle \left\langle 1\Lambda' \frac{1}{2}\nu' \left| \frac{1}{2}\lambda \right\rangle \right. \\ \times \left. \left\langle 1\Lambda_1 \frac{1}{2}\nu' \left| \frac{3}{2}\mu \right\rangle \right\rangle. \quad (\text{B6})$$

This sum can be done using Racah coefficients<sup>1</sup> (from Rose [25]):

$$\sum_{\nu'} \langle 2m_\ell 1\Lambda_1 | 1\Lambda' \rangle \left\langle 1\Lambda' \frac{1}{2}\nu' \left| \frac{1}{2}\lambda \right\rangle \right\rangle \left\langle 1\Lambda_1 \frac{1}{2}\nu' \left| \frac{3}{2}\mu \right\rangle \right\rangle \\ = R_{3/21} \left\langle 2m_\ell \frac{3}{2}\mu \left| \frac{1}{2}\lambda \right\rangle \right\rangle. \quad (\text{B7})$$

The Racah coefficient, from Table 1.4 on page 227, is

$$R_{(3/2)1} = \sqrt{12} W\left(2, 1, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}\right) = \sqrt{12} \sqrt{\frac{5311}{4325 \frac{1}{2} \frac{3}{2}}} = 1. \quad (\text{B8})$$

Hence, the final answer for  $\Theta$  is

$$\Theta_D^{\Lambda\lambda}(\hat{\mathbf{k}}_i) = \sqrt{4\pi} \sum_{m_\ell=-1}^2 \left\langle 2m_\ell \frac{3}{2}\mu \left| \frac{1}{2}\lambda \right\rangle \right\rangle \mathbf{k}_i^2 Y_{2m_\ell}(\hat{\mathbf{k}}_i) \\ \times \left\langle 1\Lambda \frac{1}{2}\nu \left| \frac{3}{2}\mu \right\rangle \right\rangle \left| \frac{1}{2}\nu \right\rangle, \quad (\text{B9})$$

which is precisely Eq. (4.28).

Repeating the derivation using a spin-1/2 projection operator in place of a spin-3/2 operator amounts to replacing the state  $|\frac{3}{2}\mu\rangle$  by  $|\frac{1}{2}\mu\rangle$  in the sum (B7), and in this case, the Racah coefficient [proportional to  $W(2, 1, \frac{1}{2}, \frac{1}{2}; 1, \frac{1}{2})$ ] is zero, proving, as asserted above, that the spin-1/2 contribution is zero.

<sup>1</sup>Rose, Eq. (6.5a), [25] gives

$$\sum_{m_2 \text{ or } m_3} \langle j_1 m_1 j_2 m_2 | j' \rangle \langle j' m_1 + m_2 j_3 m_3 | j \rangle \langle j_2 m_2 j_3 m_3 | j'' \rangle \\ = R_{j'j''} \langle j_1 m_1 j'' m_2 + m_3 | j \rangle.$$

Here, we identify  $j_1 = 2, m_1 = m_\ell, j_2 = 1, m_2 = \Lambda_1, j' = 1, m_1 + m_2 = \Lambda', j_3 = \frac{1}{2}, m_3 = \nu', j = \frac{1}{2}, m_1 + m_2 + m_3 = \lambda, j'' = \frac{3}{2}, m_2 + m_3 = \mu$ .

## APPENDIX C: NON-RELATIVISTIC AVERAGES OVER THE *D*-STATE MATRIX ELEMENTS

Summing over the diquark polarization and averaging over the direction of the relative momentum of the diquark,  $r$ , using the identities (where  $f$  is an arbitrary function)

$$\int d^3 r \mathbf{r}_i \mathbf{r}_j f(r^2) = \delta_{ij} \frac{1}{3} \int d^3 r \mathbf{r}^2 f(r^2) \sum_{\Lambda} (\boldsymbol{\varepsilon}_{\Lambda})_{\ell'} (\boldsymbol{\varepsilon}_{\Lambda}^*)_{\ell} \delta_{\ell'\ell} \quad (\text{C1})$$

gives the average of the isospin-0 component of the wave function (4.34)

$$\langle |\Psi^{Da}|^2 \rangle_{\lambda',\lambda} = \sum_{\Lambda} \int_{k,r}^{\text{NR}} \bar{\Psi}_{\Lambda\lambda'}^{Da}(\mathbf{0}; \mathbf{k}_1, \mathbf{k}_2) \Psi_{\Lambda\lambda}^{Da}(\mathbf{0}; \mathbf{k}_1, \mathbf{k}_2) \\ = \frac{3}{4} \int_{k,r}^{\text{NR}} \left\langle \frac{1}{2}\lambda' \left| \sigma_{\ell} \sum_{m'} G^{\ell m'}(\mathbf{k}, \mathbf{r}) G^{m' m}(\mathbf{k}, \mathbf{r}) \right. \right. \\ \times \left. \left. \sigma_m \left| \frac{1}{2}\lambda \right\rangle \Phi_D^2 \right\rangle \\ = \frac{5}{3} \delta_{\lambda'\lambda} \int_{k,r}^{\text{NR}} \mathbf{k}^2 \mathbf{r}^2 \Phi_D^2(\chi) \equiv \delta_{\lambda'\lambda} \mathcal{N}_D, \quad (\text{C2})$$

where the nonrelativistic volume integrals were defined in Eq. (2.13), and we assumed that  $\Phi_D$  has the same argument  $\chi = \frac{1}{2}\mathbf{k}^2 + 2\mathbf{r}^2$  [see Eq. (4.13)] as  $\Phi_S$ . The last line defines the *D*-state normalization constant.

To evaluate the isospin-1 average, first use the identity

$$\sum_{\ell} \int d^3 k_i D^{\ell m'}(\mathbf{k}_i) D^{\ell m}(\mathbf{k}_i) f(\mathbf{k}_i^2) = \delta_{m'm} \frac{2}{9} \int d^3 k_i \mathbf{k}_i^4 f(\mathbf{k}_i^2), \quad (\text{C3})$$

to derive the following convenient result for the normalization of the  $\Theta$ 's:

$$\sum_{\Lambda} \int d^3 k_i \Theta_{\Lambda\lambda'}^D(\mathbf{k}_i) \Theta_{\Lambda\lambda}^D(\mathbf{k}_i) f(\mathbf{k}_i^2) \\ = \frac{3}{2} \int d^3 k_i \sum_{\ell} \left\langle \frac{1}{2}\lambda' \left| \sigma_{m'} D^{\ell m'}(\mathbf{k}_i) D^{\ell m}(\mathbf{k}_i) \sigma_m \right| \frac{1}{2}\lambda \right\rangle f(\mathbf{k}_i^2) \\ = \delta_{\lambda'\lambda} \int d^3 k_i \mathbf{k}_i^4 f(\mathbf{k}_i^2). \quad (\text{C4})$$

Then, summing over the spin-1 diquark polarization vector and using the orthogonality and normalization properties of the  $\Theta$ 's gives

$$\langle |\Psi^{Ds}|^2 \rangle_{\lambda',\lambda} = \sum_{\Lambda} \int_{k,r}^{\text{NR}} \bar{\Psi}_{\Lambda\lambda'}^{Ds}(\mathbf{0}; \mathbf{k}_1, \mathbf{k}_2) \Psi_{\Lambda\lambda}^{Ds}(\mathbf{0}; \mathbf{k}_1, \mathbf{k}_2) \\ = \delta_{\lambda'\lambda} \int_{k,r}^{\text{NR}} (\cos^2 \phi \mathbf{k}^4 + \sin^2 \phi \mathbf{r}^4) \Phi_D^2. \quad (\text{C5})$$

Now, using the ansatz (4.13) (with  $\chi = \frac{1}{2}\mathbf{k}^2 + 2\mathbf{r}^2$  for  $\mathbf{P} = 0$ ), we can find a relation between the integrals of  $\mathbf{k}^4$ ,  $\mathbf{r}^4$ , and  $\mathbf{k}^2 \mathbf{r}^2$  over  $\Phi_D$ . Integrating over  $\mathbf{r}$  by parts first and then over  $\mathbf{k}$  gives



$$\begin{aligned}
\frac{3}{5} \mathcal{N}_D &= \langle \mathbf{k}^2 \mathbf{r}^2 \rangle \equiv \int_0^\infty \mathbf{k}^2 dk \int_0^\infty \mathbf{r}^2 dr (\mathbf{k}^2 \mathbf{r}^2) \Phi_D^2(\chi) \\
&= -\frac{4}{5} \int_0^\infty \mathbf{k}^4 dk \int_0^\infty dr r^6 \frac{d}{d\chi} \Phi_D^2(\chi) \\
&= \frac{12}{5} \int_0^\infty \mathbf{k}^2 dk \int_0^\infty \mathbf{r}^2 dr (\mathbf{r}^4) \Phi_D^2(\chi) = \frac{12}{5} \langle \mathbf{r}^4 \rangle.
\end{aligned} \tag{C6}$$

Doing the same steps, but in reverse order, gives a relation between  $\langle \mathbf{k}^4 \rangle$  and  $\langle \mathbf{k}^2 \mathbf{r}^2 \rangle$ . These results can be summarized

$$\begin{aligned}
\langle \mathbf{k}^4 \rangle &= A \langle \mathbf{k}^2 \mathbf{r}^2 \rangle = \frac{20}{3} \langle \mathbf{k}^2 \mathbf{r}^2 \rangle = 4 \mathcal{N}_D \\
\langle \mathbf{r}^4 \rangle &= B \langle \mathbf{k}^2 \mathbf{r}^2 \rangle = \frac{5}{12} \langle \mathbf{k}^2 \mathbf{r}^2 \rangle = \frac{1}{4} \mathcal{N}_D.
\end{aligned} \tag{C7}$$

Hence, both isospin averages can be written in terms of the single normalization constant,  $\mathcal{N}_D$ ,

$$\begin{aligned}
\langle |\Psi^{Da}|^2 \rangle_{\lambda', \lambda} &= \delta_{\lambda' \lambda} \mathcal{N}_D \\
\langle |\Psi^{Ds}|^2 \rangle_{\lambda', \lambda} &= \delta_{\lambda' \lambda} \mathcal{N}_D \left( 4 \cos^2 \phi + \frac{1}{4} \sin^2 \phi \right).
\end{aligned} \tag{C8}$$

We choose  $\phi$  so that both normalizations are equal, which gives

$$\cos \phi = \sqrt{\frac{1}{5}}, \quad \sin \phi = \sqrt{\frac{4}{5}}. \tag{C9}$$

Before leaving this section, we point out that the relations (C7) are essential for consistency. Using the original definition (4.29) for  $\Psi^{Da}$ , an alternative calculation of the norm using the orthogonality relations (C4) gives

$$\begin{aligned}
\langle |\Psi^{Da}|^2 \rangle_{\lambda', \lambda} &= \delta_{\lambda' \lambda} \frac{1}{2} [\langle \mathbf{k}_1^4 \rangle + \langle \mathbf{k}_2^4 \rangle] \\
&= \delta_{\lambda' \lambda} \frac{1}{2} [\langle (\frac{1}{2} \mathbf{k} + \mathbf{r})^4 \rangle + \langle (\frac{1}{2} \mathbf{k} - \mathbf{r})^4 \rangle] \\
&= \delta_{\lambda' \lambda} \langle \frac{1}{16} \mathbf{k}^4 + \mathbf{r}^4 + \frac{5}{6} \mathbf{k}^2 \mathbf{r}^2 \rangle \\
&= \delta_{\lambda' \lambda} \left( \frac{1}{4} + \frac{1}{4} + \frac{1}{2} \right) \mathcal{N}_D
\end{aligned} \tag{C10}$$

in agreement with (C2).

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