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# Every invertible matrix is diagonally equivalent to a matrix with distinct eigenvalues

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ABSTRACT

We show that for every invertible $n \times n$ complex matrix $A$ there is an $n \times n$ diagonal invertible $D$ such that $AD$ has distinct eigenvalues. Using this result, we affirm a conjecture of Feng, Li, and Huang that an $n \times n$ matrix is not diagonally equivalent to a matrix with distinct eigenvalues if and only if it is singular and all its principal minors of size $n - 1$ are zero.



## 1. Introduction

Denote by $M_n$ the set of $n \times n$ complex matrices. In [1], the authors pointed out that matrices with distinct eigenvalues have many nice properties. They then raised the question whether every invertible matrix in $M_n$ is diagonally equivalent to a matrix with distinct eigenvalues, and conjectured that a matrix in $M_n$ is not diagonally equivalent to a matrix with distinct eigenvalues if and only if it is singular and every principal minor of size $n - 1$ is zero. They provided a proof for matrices in $M_n$ with

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$n \leq 3$, and demonstrated the complexity of the problem for matrices in $M_4$ using their approach. In this note, we affirm their conjecture by proving the following theorem.

**Theorem 1.1.** *Suppose $A \in M_n$ is invertible. There is an invertible diagonal $D \in M_n$ such that $AD$ has distinct eigenvalues.*

Once this result is proved, we have the following corollary.

**Corollary 1.2.** *Let $A \in M_n$. The following are equivalent.*

(a) *A is not diagonally equivalent to a matrix with distinct eigenvalues.*
(b) *There is no diagonal matrix D such that AD has distinct eigenvalues.*
(c) *The matrix A is singular and all principal minors of size $n - 1$ are zero.*

**Proof.** The implication (a) $\Rightarrow$ (b) is clear. Suppose condition (c) does not hold. Then either $A$ is invertible or $A$ has an invertible principal submatrix of size $n - 1$. Assume the former case holds. There is an invertible diagonal matrix $D$ such that $AD$ has distinct eigenvalues by Theorem 1.1. If the latter case holds, we may assume without loss of generality that the leading principal submatrix $A_1 \in M_{n-1}$ is invertible. By Theorem 1.1, there is an invertible diagonal matrix $D_1 \in M_{n-1}$ such that $A_1D_1$ has distinct (nonzero) eigenvalues. Let $D = D_1 \oplus [0]$. Then $AD$ has distinct eigenvalues including zero as an eigenvalue. Thus, (b) cannot hold. So, we have proved (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Recall that the characteristic polynomial of a matrix $B \in M_n$ has the form $\det(xI_n - B) = x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$, where $(-1)^j b_{n-j}$ is the sum of $j \times j$ principal minors of $B$. Suppose condition (c) holds. Since the principal minors of $D_1AD_2$ are scalar multiples of the corresponding principal minors of $A$, then $D_1AD_2$ has characteristic polynomial of the form $\det(xI_n - D_1AD_2) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2$ so that 0 is a root with multiplicity at least two. Thus, $D_1AD_2$ cannot have $n$ distinct eigenvalues. So, the implication (c) $\Rightarrow$ (a) is proved. $\square$

Note that the set of diagonal matrices is an $n$-dimensional subspace in $M_n$. We can extend Theorem 1.1 to the following.

**Corollary 1.3.** *Suppose $\mathcal{V}$ is a subspace of matrices in $M_n$.*

(a) *If there are invertible matrices R and S such that $R\mathcal{V}S = \{RXS : X \in \mathcal{V}\}$ contains the subspace of diagonal matrices, then for any invertible $A \in M_n$ there is $X \in \mathcal{V}$ such that AX has distinct eigenvalues.*
(b) *If there are invertible matrices R and S such that RXS has zero first row and zero last column for every $X \in \mathcal{V}$, then $A = SR$ is invertible and AX is similar to RXS which cannot have distinct eigenvalues for any $X \in \mathcal{V}$.*

**Proof.** (a) Suppose $A$ is invertible. Then there is a diagonal matrix $D$ such that $S^{-1}AR^{-1}D$ has distinct eigenvalues by Theorem 1.1. Set $X = R^{-1}DS^{-1} \in \mathcal{V}$. Notice that $AX$ has distinct eigenvalues as $S^{-1}(AX)S = S^{-1}(AR^{-1}DS^{-1})S = (S^{-1}AR^{-1})D$.

Assertion (b) can be verified readily. $\square$

## 2. Proof of Theorem 1.1

We will prove Theorem 1.1 by induction on $n$. The result is clear if $A \in M_1$. Assume that the result holds for all $k \times k$ invertible matrices with $1 \leq k < n$. Suppose $A \in M_n$ is invertible. We consider two cases.

**Case 1.** If all $k \times k$ principal minors of $A$ are singular for $k = 1, \ldots, n - 1$, then the characteristic polynomial of $A$ has the form $x^n - a_0$ and has $n$ distinct roots. So, the result holds with $D = I_n$.

**Case 2.** Suppose $A$ has an invertible $k \times k$ principal minor. Without loss of generality, we may assume that $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ such that $A_{11} \in M_k$ is invertible for some $1 \le k < n$. Then the Schur complement of $A_{22}$ equals $B = A_{22} - A_{21}A_{11}^{-1}A_{12}$ which is invertible; see [2, pp. 21–22]. By induction assumption, there are diagonal invertible $D_1 \in M_k$ and $D_2 \in M_{n-k}$ such that each of $A_{11}D_1$ and $BD_2$ has distinct nonzero eigenvalues, say, $\lambda_1, \ldots, \lambda_k$ and $\lambda_{k+1}, \ldots, \lambda_n$, respectively. Thus, $A_{11}D_1$ and $BD_2$ are diagonalizable and there are invertible $S_1 \in M_k$ and $S_2 \in M_{n-k}$ such that $S_1A_{11}D_1S_1^{-1} = \Lambda_1 = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_k)$ and $S_2BD_2S_2^{-1} = \Lambda_2 = \mathrm{diag}\,(\lambda_{k+1}, \ldots, \lambda_n)$. Let $D_{r,s} = rD_1 \oplus sD_2$. The proof is complete if one can find some suitable $r$ and $s$ so that $AD_{r,s}$ has distinct eigenvalues. Notice that $AD_{r,s}$ has the same eigenvalues as

$$\begin{aligned}\tilde{A} &= \begin{bmatrix} S_1 & 0 \\ 0 & sS_2 \end{bmatrix} \begin{bmatrix} I_k & 0 \\ -A_{21}A_{11}^{-1} & I_{n-k} \end{bmatrix} AD_{r,s} \begin{bmatrix} I_k & 0 \\ A_{21}A_{11}^{-1} & I_{n-k} \end{bmatrix} \begin{bmatrix} S_1^{-1} & 0 \\ 0 & s^{-1}S_2^{-1} \end{bmatrix} \\ &= \begin{bmatrix} r\Lambda_1 + sS_1A_{12}D_2A_{21}A_{11}^{-1}S_1^{-1} & S_1A_{12}D_2S_2^{-1} \\ s^2S_2BD_2A_{21}A_{11}^{-1}S_1^{-1} & s\Lambda_2 \end{bmatrix}.\end{aligned}$$

Denote by $D(a, d)$ the closed disk in $\mathbb{C}$ centered at $a$ with radius $d \ge 0$. Suppose the $k \times k$ matrix $S_1A_{12}D_2A_{21}A_{11}^{-1}S_1^{-1}$ has diagonal entries $\mu_1, \ldots, \mu_k$ and let

$$d_1 = k\|S_1A_{12}D_2A_{21}A_{11}^{-1}S_1^{-1}\|, \quad d_2 = (n-k)\|S_1A_{12}D_2S_2^{-1}\|, \quad \text{and} \quad d_3 = k\|S_2BD_2A_{21}A_{11}^{-1}S_1^{-1}\|,$$

where $\|\cdot\|$ is the operator norm. By Geršgorin disk result (see [2, pp. 344–347]), the eigenvalues of $\tilde{A}$ must lie in the union of the $n$ Geršgorin disks, which is a subset of the union of $n$ disks

$$D(r\lambda_1 + s\mu_1, sd_1 + d_2), \ldots, D(r\lambda_k + s\mu_k, sd_1 + d_2),\ D(s\lambda_{k+1}, s^2d_3), \ldots, D(s\lambda_n, s^2d_3).$$

We can choose sufficiently large $r > 0$ and sufficiently small $s > 0$ so that these disks are disjoint, and hence $\tilde{A}$ has $n$ disjoint Geršgorin disks. Then $\tilde{A}$ has distinct eigenvalues. □

We thank Editor Zhan for sending us the two related Refs. [3,4]. In these papers, the author proved following. Suppose $A$ is an $n \times n$ matrix and $a_1, \ldots, a_n$ are complex numbers. Then there is a diagonal matrix $E$ such that $A + E$ has eigenvalues $a_1, \ldots, a_n$. Moreover, if all principal minors of $A$ are nonzero, then there is a diagonal matrix $D$ such that $AD$ has eigenvalues $a_1, \ldots, a_n$.

Note that the assumption on the principal minors of $A$ is important in the second assertion. Obviously, if $\det(A) = 0$, then one cannot find diagonal $D$ such that $AD$ has $n$ nonzero eigenvalues. Even if we remove this obvious obstacle and assume that $A$ is invertible, one may not be able to find diagonal $D$ so that $AD$ has prescribed eigenvalues. For example, if $\{E_{1,1}, E_{1,2}, \ldots, E_{n,n}\}$ is the standard basis for $M_n$ and $A = E_{1,2} + \cdots + E_{n-1,n}$, then the eigenvalues of $AD$ always have the form $z, zw, \ldots, zw^{n-1}$ for some $z \in \mathbb{C}$, where $w$ is the primitive $n$th root of unity.

It is interesting to determine the condition on $A$ so that for any complex numbers $a_1, \ldots, a_n$, one can find a diagonal $D$ such that $AD$ has $a_1, \ldots, a_n$ as eigenvalues.

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## References

[1] X.-L. Feng, Z. Li, T.-Z. Huang, Is every nonsingular matrix diagonally equivalent to a matrix with all distinct eigenvalues? Linear Algebra Appl., 436 (2012) 120–125.
[2] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[3] S. Friedland, On inverse multiplicative eigenvalue problems for matrices, Linear Algebra Appl. 12 (1975) 127–137.
[4] S. Friedland, Inverse eigenvalue problems, Linear Algebra Appl. 17 (1977) 15–51.