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Every invertible matrix is diagonally equivalent to a matrix with distinct eigenvalues

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\textbf{ABSTRACT}

We show that for every invertible $n \times n$ complex matrix $A$ there is an $n \times n$ diagonal invertible $D$ such that $AD$ has distinct eigenvalues. Using this result, we affirm a conjecture of Feng, Li, and Huang that an $n \times n$ matrix is not diagonally equivalent to a matrix with distinct eigenvalues if and only if it is singular and all its principal minors of size $n-1$ are zero.

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1. Introduction

Denote by $M_n$ the set of $n \times n$ complex matrices. In [1], the authors pointed out that matrices with distinct eigenvalues have many nice properties. They then raised the question whether every invertible matrix in $M_n$ is diagonally equivalent to a matrix with distinct eigenvalues, and conjectured that a matrix in $M_n$ is not diagonally equivalent to a matrix with distinct eigenvalues if and only if it is singular and every principal minor of size $n-1$ is zero. They provided a proof for matrices in $M_n$ with...
Proof. Suppose $A \in M_n$ is invertible. There is an invertible diagonal $D \in M_n$ such that $AD$ has distinct eigenvalues.

Once this result is proved, we have the following corollary.

Corollary 1.2. Let $A \in M_n$. The following are equivalent.

(a) $A$ is not diagonally equivalent to a matrix with distinct eigenvalues.
(b) There is no diagonal matrix $D$ such that $AD$ has distinct eigenvalues.
(c) The matrix $A$ is singular and all principal minors of size $n−1$ are zero.

Proof. The implication (a) $\implies$ (b) is clear. Suppose condition (c) does not hold. Then either $A$ is invertible or $A$ has an invertible principal submatrix of size $n−1$. Assume the former case holds. There is an invertible diagonal matrix $D$ such that $AD$ has distinct eigenvalues by Theorem 1.1. If the latter case holds, we may assume without loss of generality that the leading principal submatrix $A_1 \in M_{n−1}$ is invertible. By Theorem 1.1, there is an invertible diagonal matrix $D_1 \in M_{n−1}$ such that $A_1D_1$ has distinct (nonzero) eigenvalues. Let $D = D_1 \oplus [0]$. Then $AD$ has distinct eigenvalues including zero as an eigenvalue. Thus, (b) cannot hold. So, we have proved (a) $\implies$ (b) $\implies$ (c).

Recall that the characteristic polynomial of a matrix $B \in M_n$ has the form $\det(xI_n - B) = x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$, where $(-1)^jb_{n−j}$ is the sum of $j \times j$ principal minors of $B$. Suppose condition (c) holds. Since the principal minors of $D_1AD_2$ are scalar multiples of the corresponding principal minors of $A$, then $D_1AD_2$ has characteristic polynomial of the form $\det(xI_n - D_1AD_2) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2$ so that 0 is a root with multiplicity at least two. Thus, $D_1AD_2$ cannot have $n$ distinct eigenvalues. So, the implication (c) $\implies$ (a) is proved.

Note that the set of diagonal matrices is an $n$-dimensional subspace in $M_n$. We can extend Theorem 1.1 to the following.

Corollary 1.3. Suppose $\mathcal{V}$ is a subspace of matrices in $M_n$.

(a) If there are invertible matrices $R$ and $S$ such that $RV = \{RX : X \in \mathcal{V}\}$ contains the subspace of diagonal matrices, then for any invertible $A \in M_n$ there is $X \in \mathcal{V}$ such that $AX$ has distinct eigenvalues.
(b) If there are invertible matrices $R$ and $S$ such that $RXS$ has zero first row and zero last column for every $X \in \mathcal{V}$, then $A = SR$ is invertible and $AX$ is similar to $RXS$ which cannot have distinct eigenvalues for any $X \in \mathcal{V}$.

Proof. (a) Suppose $A$ is invertible. Then there is a diagonal matrix $D$ such that $S^{-1}AR^{-1}D$ has distinct eigenvalues by Theorem 1.1. Set $X = R^{-1}DS^{-1} \in \mathcal{V}$. Notice that $AX$ has distinct eigenvalues as $S^{-1}(AX)S = S^{-1}(AR^{-1}DS^{-1})S = (S^{-1}AR^{-1})D$.

Assertion (b) can be verified readily.

2. Proof of Theorem 1.1

We will prove Theorem 1.1 by induction on $n$. The result is clear if $A \in M_1$. Assume that the result holds for all $k \times k$ invertible matrices with $1 \leq k < n$. Suppose $A \in M_n$ is invertible. We consider two cases.

Case 1. If all $k \times k$ principal minors of $A$ are singular for $k = 1, \ldots, n−1$, then the characteristic polynomial of $A$ has the form $x^n - a_0$ and has $n$ distinct roots. So, the result holds with $D = I_n$. 
Case 2. Suppose $A$ has an invertible $k \times k$ principal minor. Without loss of generality, we may assume that $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ such that $A_{11} \in M_k$ is invertible for some $1 \leq k < n$. Then the Schur complement of $A_{22}$ equals $B = A_{22} - A_{21}A_{11}^{-1}A_{12}$ which is invertible; see [2, pp. 21–22]. By induction assumption, there are diagonal invertible $D_1 \in M_k$ and $D_2 \in M_{n-k}$ such that each of $A_{11}D_1$ and $BD_2$ has distinct nonzero eigenvalues, say, $\lambda_1, \ldots, \lambda_k$ and $\lambda_{k+1}, \ldots, \lambda_n$, respectively. Thus, $A_{11}D_1$ and $BD_2$ are diagonalizable and there are invertible $S_1 \in M_k$ and $S_2 \in M_{n-k}$ such that $S_1A_{11}D_1S_1^{-1} = \Lambda_1 = \text{diag} (\lambda_1, \ldots, \lambda_k)$ and $S_2BD_2S_2^{-1} = \Lambda_2 = \text{diag} (\lambda_{k+1}, \ldots, \lambda_n)$. Let $D_{r,s} = rD_1 \oplus sD_2$. The proof is complete if one can find some suitable $r$ and $s$ so that $AD_{r,s}$ has distinct eigenvalues. Notice that $AD_{r,s}$ has the same eigenvalues as

\[
\tilde{A} = \begin{bmatrix} S_1 & 0 \\ 0 & sS_2 \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \begin{bmatrix} I_k \\ A_{21}A_{11}^{-1}I_{n-k} \end{bmatrix} \begin{bmatrix} S_1^{-1} & 0 \\ 0 & s^{-1}S_2^{-1} \end{bmatrix} = \begin{bmatrix} r\Lambda_1 + sS_1A_{12}D_2A_{21}A_{11}^{-1}S_1^{-1}S_1A_{12}D_2S_2^{-1} \\ s^2S_2BD_2A_{21}A_{11}^{-1}S_1^{-1} \end{bmatrix}.
\]

Denote by $D(a, d)$ the closed disk in $\mathbb{C}$ centered at $a$ with radius $d \geq 0$. Suppose the $k \times k$ matrix $S_1A_{12}D_2A_{21}A_{11}^{-1}S_1^{-1}$ has diagonal entries $\mu_1, \ldots, \mu_k$ and let

\[
d_1 = k\|S_1A_{12}D_2A_{21}A_{11}^{-1}S_1^{-1}\|, \quad d_2 = (n-k)\|S_1A_{12}D_2S_2^{-1}\|, \quad \text{and} \quad d_3 = k\|S_2BD_2A_{21}A_{11}^{-1}S_1^{-1}\|,
\]

where $\| \cdot \|$ is the operator norm. By Geršgorin disk result (see [2, pp. 344–347]), the eigenvalues of $\tilde{A}$ must lie in the union of the $n$ Geršgorin disks, which is a subset of the union of $n$ disks

\[
D(r\lambda_1 + s\mu_1, s\mu_1 + d_1 + d_2), \ldots, D(r\lambda_k + s\mu_k, s\mu_k + d_1 + d_2), \quad D(s\lambda_{k+1}, s^2d_3), \ldots, D(s\lambda_n, s^2d_3).
\]

We can choose sufficiently large $r > 0$ and sufficiently small $s > 0$ so that these disks are disjoint, and hence $\tilde{A}$ has $n$ disjoint Geršgorin disks. Then $\tilde{A}$ has distinct eigenvalues. □

We thank Editor Zhan for sending us the two related Refs. [3,4]. In these papers, the author proved following. Suppose $A$ is an $n \times n$ matrix and $a_1, \ldots, a_n$ are complex numbers. Then there is a diagonal matrix $E$ such that $A + E$ has eigenvalues $a_1, \ldots, a_n$. Moreover, if all principal minors of $A$ are nonzero, then there is a diagonal matrix $D$ such that $AD$ has eigenvalues $a_1, \ldots, a_n$.

Note that the assumption on the principal minors of $A$ is important in the second assertion. Obviously, if $\det(A) = 0$, then one cannot find diagonal $D$ such that $AD$ has $n$ nonzero eigenvalues. Even if we remove this obvious obstacle and assume that $A$ is invertible, one may not be able to find diagonal $D$ so that $AD$ has prescribed eigenvalues. For example, if $[E_{1,1}, E_{1,2}, \ldots, E_{n,n}]$ is the standard basis for $M_n$ and $A = E_{1,2} + \cdots + E_{n-1,n}$, then the eigenvalues of $AD$ always have the form $z, zw, \ldots, zw^{n-1}$ for some $z \in \mathbb{C}$, where $w$ is the primitive $n$th root of unity.

It is interesting to determine the condition on $A$ so that for any complex numbers $a_1, \ldots, a_n$, one can find a diagonal $D$ such that $AD$ has $a_1, \ldots, a_n$ as eigenvalues.

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