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Numerical ranges of Toeplitz operators with matrix symbols ☆

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ABSTRACT

For Toeplitz operators acting on the vector Hardy space H^2 with definite or indefinite metric, the closure of the respective numerical range is completely described. In the definite case, some observations regarding its boundary are also made.

1. Introduction

Let A be a bounded linear operator acting on a Hilbert space H endowed with the scalar product $\langle ., . \rangle$. The classical *numerical range* is the subset of the complex plane $\mathbb{C}$ defined by

$$W(A) = \{\langle Af, f\rangle / \langle f, f\rangle : f \in H,\ f \neq 0\}.$$

This concept is a useful tool in the study of matrices and operators and has been extensively investigated, see e.g. [1,2]. In particular, it is known that $W(A)$ is a convex set (the Toeplitz–Hausdorff theorem) whose closure contains the spectrum $\sigma(A)$ of A. So, in particular

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$$\operatorname{clos} W(A) \supset \operatorname{conv} \sigma(A),$$

where of course clos and conv stand for the operations of taking closure and convex hull, respectively.

There are several classes of operators A for which a complete description of $W(A)$ is known. We mention here the case of low (2- or 3-) dimensional underlying space H on the one side, and of normal and quadratic operators on infinite-dimensional H, on the other. However, in most of the cases there is not much information beyond the general properties mentioned above.

In this paper we are concerned with the numerical ranges of *Toeplitz operators*. Let us introduce the pertinent notation.

For any vector space X below, X_n ($X_{n\times n}$) will stand for the space of n-columns (respectively, $n \times n$ matrices) with entries in X. The role of X will be played, in particular, by the Lebesgue spaces L^2 and L^∞ on the unit circle $\mathbb{T}$ and their subspaces (the Hardy spaces) H^2 and H^∞ consisting of functions analytically extendable into the unit disk $\mathbb{D}$. In addition, L^2_n will be supplied with the standard Hilbert space structure, and P will stand for the *Riesz projection*, that is, the orthogonal projection of L^2_n onto H^2_n, acting entry-wise.

With any $a \in L^\infty_{n\times n}$ there is associated the *multiplication operator* M_a acting on L^2_n according to

$$(M_a f)(t) = a(t)f(t) \ a.e.$$

and the *Toeplitz operator* T_a defined on H^2_n by the formula

$$T_a = PM_aP.$$

The (matrix) function a is called the *symbol* of T_a.

For $n = 1$, the numerical range of T_a was characterized by Klein in [3]. Namely, $W(T_a)$ is the relative interior of $\operatorname{conv} \sigma(T_a)$. By Brown–Halmos theorem the latter set coincides with $\operatorname{conv} \mathcal{R}(a)$, where $\mathcal{R}(a)$ stands for the essential range of a. (Recall that the latter by definition consists of z such that the preimage of any neighborhood of z under a has positive measure.) In this form, Klein's result to some extent can be carried over to the case $n > 1$. This is discussed in Section 2. In Section 3 we obtain a parallel result for the indefinite numerical range.

2. The definite case

A moment's thought reveals that the numerical ranges of Toeplitz operators with $n > 1$ do not have to be (relatively) open. Consider for example a 2×2 diagonal matrix function a with the diagonal entries a_1, a_2 such that $\mathcal{R}(a_1)$ is a triple of non-collinear points z_1, z_2, z_3 while $\mathcal{R}(a_2) = \{z_1, z_2\}$. Since $T_a = T_{a_1} \oplus T_{a_2}$,

$$W(T_a) = \operatorname{conv}\{W(T_{a_1}), W(T_{a_2})\}$$

is the triangle with the vertices z_1, z_2, z_3 with the side (z_1, z_2) included and the other two excluded. Note that in this example $\mathcal{R}(a) = \mathcal{R}(a_1)$, so that $W(T_a)$ cannot be characterized completely only in terms of $\mathcal{R}(a)$. However, its closure still can.

Theorem 1. *The closures of the sets $W(T_a)$ and $W(M_a)$ are the same, and coincide with*

$$\operatorname{conv}\{W(A) \colon A \in \mathcal{R}(a)\}. \tag{1}$$

Proof. To show that

$$\operatorname{clos} W(T_a) = \operatorname{clos} W(M_a), \tag{2}$$

one might proceed as follows. Since the sets in question are convex, it suffices to show that they have the same supporting lines in every direction. Multiplying the symbol a by $e^{i\theta}$, we may without loss of generality consider vertical supporting lines only, lying to the right of the respective sets. Their location corresponds to the rightmost point of the spectrum of $\operatorname{Re} T_a = T_{\operatorname{Re} a}$ and $\operatorname{Re} M_a = M_{\operatorname{Re} a}$, respectively.

Thus, it suffices to show that for a Hermitian symbol h the rightmost point of the spectrum of T_h and M_h is the same. Shifting h by a positive multiple of the identity, we may without loss of generality suppose that it is positive definite. But then the rightmost point of the spectrum is indeed the same and equals ess sup $\|h(t)\|$.

It remains to compare the closure of say $W(M_a)$ with the set (1). Any point of $W(M_a)$, by definition, is of the form

$$\int_{\mathbb{T}} x^*(t)a(t)x(t)\,dt, \tag{3}$$

where x is a unit vector in L_n^2. Approximating x and a by vector- (respectively, matrix-) functions with finitely many values and keeping the values A_j of the approximation of a in the essential range of a, we see that this approximation is a convex combination of expressions of the form $x_j^*A_jx_j$, with x_j being unit vectors in $\mathbb{C}_n$. Since

$$x_j^*A_jx_j \in W(A_j),$$

their convex combinations lie in (1). Considering that convex hulls of compact sets in $\mathbb{R}^n$ are compact, the integral (3) itself lies there. Thus,

$$\operatorname{clos} W(M(a)) \subseteq \operatorname{conv}\{W(A) : A \in \mathcal{R}(a)\}.$$

To prove the converse inclusion, we just need to show that for any $A \in \mathcal{R}(a)$ the set $W(A)$ lies in the closure of $W(M_a)$, since the latter is convex. To this end, take $z = x^*Ax$ (where x is an arbitrary constant unit vector in $\mathbb{C}_n$) and let

$$x_s(t) = \begin{cases} x & \text{if } \|A - a(t)\| < s, \\ 0 & \text{otherwise.} \end{cases}$$

Normalizing this vector-function in L_n^2 (which we can do, because it differs from zero on a set of positive measure for any $s > 0$, due to the definition of the essential range) and letting $s \to 0$ we see that the corresponding points in $W(M_a)$ converge to z. $\square$

Note that the first part of the proof of Theorem 1 among other things makes use of the fact that the norms of T_h and M_h are the same. Mimicking the proof of this fact, instead of simply using it, yields the following.

Alternative proof of (2). Since T_a is a compression of M_a, the inclusion $W(T_a) \subseteq W(M_a)$ holds. Of course, this implies one of the inclusions in (2).

To prove the reverse inclusion, consider $z \in W(M_a)$. Then $z = \langle af, f\rangle/\langle f, f\rangle$ for some $f \in L_n^2$. Approximating f by trigonometric polynomials g, we see that in any neighborhood of z there are points of the form $\langle ag, g\rangle/\langle g, g\rangle$. In its turn, $g(z) = z^{-k}h(z)$ for some $k \in \mathbb{N}$ and $h \in H_2^n$. Since multiplication by z is a unitary operator on L_n^2 commuting with M_a,

$$\frac{\langle ag, g\rangle}{\langle g, g\rangle} = \frac{\langle ah, h\rangle}{\langle h, h\rangle} = \frac{\langle aPh, Ph\rangle}{\langle h, h\rangle} = \frac{\langle PaPh, h\rangle}{\langle h, h\rangle} \in W(T_a).$$

Consequently, $W(M_a) \subseteq \operatorname{clos} W(T_a)$.

We chose to present the alternative proof here because it is more universal, and therefore useful in the indefinite setting of Section 3.

To illustrate Theorem 1, consider

$$a = \begin{bmatrix} 0 & 2\phi \\ 0 & 0 \end{bmatrix}, \tag{4}$$

where $\phi \in L^\infty$ is such that $\mathcal{R}(\phi) \subset \mathbb{T}$.

Theorem 2. *Let a be given by (4). Then $W(T_a)$ is either the closure of the unit disk $\mathbb{D}$ or $\mathbb{D}$ itself, depending on whether or not ϕ is the ratio of two inner functions.*

As it was observed in [4], due to the Bourgain's result [5] all unimodular functions on $\mathbb{T}$ are representable as such ratios, up to the so called "trivial" factors. Nevertheless, both possibilities occur.

Proof. Observe first of all that $W(a(t)) = \operatorname{clos}\mathbb{D}$ a.e. on $\mathbb{T}$, so that the set (1) is the closed unit disk. By Theorem 1 then

$$\operatorname{clos} W(T_a) = \operatorname{clos}\mathbb{D}.$$

Being convex, the set $W(T_a)$ must therefore contain $\mathbb{D}$. Moreover, this set is rotationally invariant, since for any $\omega \in \mathbb{T}$

$$T_{\omega a} = \omega T_a = U^* T_a U,$$

where U is the unitary operator of multiplication by a constant matrix

$$\begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix}.$$

Consequently, either $W(T_a) = \mathbb{D}$ or $W(T_a) = \operatorname{clos}\mathbb{D}$, depending on whether or not $1 \in W(T_a)$. But the latter inclusion holds if and only if

$$\int_{\mathbb{T}} \phi(t)\overline{\xi_1(t)}\xi_2(t)\,dt = 1/2$$

for some $\xi_1, \xi_2 \in H^2$ such that

$$\int_{\mathbb{T}} \left(|\xi_1(t)|^2 + |\xi_2(t)|^2\right)\,dt = 1.$$

This is only possible if $|\xi_1| = |\xi_2|$ are constant a.e. on $\mathbb{T}$ (that is, ξ_1 and ξ_2 are inner functions, up to constant multiples), while ϕ therefore is the ratio of these inner functions. □

3. The indefinite case

Recall that a self-adjoint involution $J\colon H \to H$ generates an indefinite inner product on H according to the rule $[f,g] := \langle Jf, g\rangle$. A vector $f \in H$ is called *positive (negative, neutral)* if $[f,f] > 0$ (respectively, < 0, $= 0$).

The *indefinite numerical range* of an operator $A\colon H \to H$ is then defined as

$$W^J(A) = \{[Af,f]/[f,f] : f \in H, [f,f] \neq 0\},$$

see e.g. [6], where this concept was first introduced, and [7]. For convenience we also consider

$$W^J_+(A) = \{[Af,f]\,/\,[f,f] : f \text{ is positive}\},$$

$$W^J_-(A) = \{[Af,f]\,/\,[f,f] : f \text{ is negative}\}.$$

Clearly,

$$W^J(A) = W^J_+(A) \cup W^J_-(A). \tag{5}$$

Recall that a set $X \subseteq \mathbb{C}$ is *pseudo-convex* if for any pair of distinct points $x, y \in X$ either the line segment $[x,y]$ or the union of the rays $\{tx + (1-t)y\colon t \leqslant 0 \text{ or } t \geqslant 1\}$ is contained in X.

Theorem 3 [6]. *The sets $W^J_\pm(T)$ are convex while $W^J(T)$ is pseudo-convex.*

As in Section 2, we consider $H = L^2_n$. We also restrict ourselves to the case $[f,g] := \langle Jf, g\rangle = \int_{\mathbb{T}} g^*(t)Jf(t)\,dt$, where J is a self adjoint involution on $\mathbb{C}_n$.

In the notation above, we do not distinguish between J as an element of $\mathbb{C}_{n\times n}$ and the multiplication operator M_J.

Theorem 4. *The closures of the sets $W^J_+(T_a)$ and $W^J_+(M_a)$ are the same, and coincide with the closure of $Z_+ + K$. Here Z_+ is the union of all the rays of the lines passing through $z_\pm$ with $z_\pm \in \operatorname{conv}\{W^J_\pm(A)\colon A \in \mathcal{R}(a)\}$ having the endpoints z_+ and not containing the respective z_-, while K is the cone generated by $\{[Ax,x] : A \in \mathcal{R}(a),\ [x,x]=0\}$.*

Proof. The first part of the statement can be justified along the same lines as the alternative proof of (2) described in Section 2.

As for the second part, consider any $z \in W^J_+(M_a)$. By definition, $z = [M_a f, f]\,/\,[f,f]$ for some positive f. Denote by $\mathbb{T}_\pm$ and $\mathbb{T}_0$ the subsets of $\mathbb{T}$ on which the vector $f(t)$ is positive/negative or neutral, respectively (naturally, these subsets are defined modulo measure zero), and let

$$s_\pm = \pm\int_{\mathbb{T}_\pm} [f(t), f(t)]\ dt.$$

Then $s_+ > s_- \geqslant 0$, and

$$z = \left(\frac{s_+}{s_+ - s_-} z_+ - \frac{s_-}{s_+ - s_-} z_-\right) + \frac{1}{s_+ - s_-} z_0, \tag{6}$$

where

$$z_+ = \frac{1}{s_+}\int_{\mathbb{T}_+} [a(t)f(t), f(t)]\ dt, \tag{7}$$

$$z_- = -\frac{1}{s_-}\int_{\mathbb{T}_-} [a(t)f(t), f(t)]\ dt \tag{8}$$

if $s_- > 0$ and $z_- = 0$ otherwise, while

$$z_0 = \int_{\mathbb{T}_0} [a(t)f(t), f(t)]\ dt.$$

Approximating a and f as in the respective part of the proof of Theorem 1, we conclude that $z_\pm$ given by (7), (8) belong to the closures of $\operatorname{conv}\{W^J_\pm(A)\colon A \in \mathcal{R}(a)\}$. Consequently, the expression in the parentheses in (6) is in the closure of Z_+, while the last summand in (6) obviously lies in the closure of K. This proves that $W^J_+(M_a) \subseteq \operatorname{clos}(Z_+ + K)$.

To proceed in reverse, note that points in $Z_+ + K$ have the form (6), in which $s_+ > s_- \geqslant 0$, $z_0 \in K$ and $z_\pm \in \operatorname{conv}\{W^J_\pm(A)\colon A \in \mathcal{R}(a)\}$. Let us approximate z_+ by a finite convex combination of $[A_jf_j, f_j]\,/\,[f_j, f_j]$ with some positive vectors f_j and $A_j \in \mathcal{R}(a)$, and let f take the value f_j on sufficiently small (and therefore non-overlapping) subsets $U_j \subset \mathbb{T}$ with positive measure such that $a|_{U_j}$ are close to A_j. Denote the union of these U_j by $\mathbb{T}_+$. Then

$$\int_{\mathbb{T}_+} [a(t)f(t), f(t)]\ dt \Big/ \int_{\mathbb{T}_+} [f(t), f(t)]\ dt \tag{9}$$

can be made arbitrarily close to z_+. Scaling f_j if needed, we may arrange for

$$\int_{\mathbb{T}_+} [f(t), f(t)]\ dt = s_+$$

to also hold, while not changing the value of (9). Treating z_- in a similar way, we construct a disjoint with $\mathbb{T}_+$ subset $\mathbb{T}_- \subset \mathbb{T}$ and extend f onto $\mathbb{T}_-$ in such a way that $\int_{\mathbb{T}_-} [a(t)f(t), f(t)]\ dt$ is arbitrarily close to $-s_-z_-$ (of course, for $s_- = 0$ it suffices to take $\mathbb{T}_-$ of measure zero). Finally, on $\mathbb{T}_0 = \mathbb{T} \setminus (\mathbb{T}_+ \cup \mathbb{T}_-)$ let f assume finitely many neutral values in order to approximate the last summand in (6). Then

$$\int_{\mathbb{T}} [a(t)f(t), f(t)]\ dt \Big/ \int_{\mathbb{T}} [f(t), f(t)]\ dt$$

is an element of $W^J_+(M_a)$ which can be made arbitrarily close to z of the form (6). Consequently, $Z_+ + K \subset \operatorname{clos} W^J_+(M_a)$. $\square$

The respective result for "-" sets can be proved similarly, or by switching from J to $-J$. Namely, taking into consideration that $W^J_\pm(T) = W^{-J}_\mp(T)$ for any operator T while K changes to $-K$, we obtain the following.

Theorem 5. *The closures of the sets $W^J_-(T_a)$ and $W^J_-(M_a)$ are the same, and coincide with the closure of $Z_- - K$. Here Z_+ is the union of all the rays of the lines passing through $z_\pm$ with $z_\pm \in \operatorname{conv}\{W^J_\pm(A) : A \in \mathcal{R}(a)\}$ having the endpoints z_- and not containing the respective z_+, while K is the same as in Theorem 4.*

The final result is obtained by combining Theorems 4, 5 and using (5).

Theorem 6. *Let $Z\pm$ and K be as defined in Theorems 4, 5. The closures of the sets $W^J(T_a)$ and $W^J(M_a)$ are the same, and coincide with the closure of $(Z_+ + K) \cup (Z_- - K)$.*

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