2012

Global stability in a diffusive Holling-Tanner predator-prey model

Shanshan Chen
William & Mary

Junping Shi
William & Mary, jxshix@wm.edu

Shanshan Chen

Junping Shi

Recommended Citation

Chen, S., & Shi, J. (2012). Global stability in a diffusive Holling–Tanner predator–prey model. Applied Mathematics Letters, 25(3), 614-618.

Applied Mathematics Letters 25 (2012) 614–618

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Global stability in a diffusive Holling–Tanner predator–prey model[☆]

Shanshan Chen[a,b], Junping Shi[b,c,*]

[a] Department of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang, 150001, PR China
[b] Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA
[c] School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi, 030006, PR China

ARTICLE INFO

Article history:
Received 12 February 2011
Received in revised form 5 August 2011
Accepted 27 September 2011

Keywords:
Reaction–diffusion system
Predator–prey
Holling–Tanner
Global stability

ABSTRACT

A diffusive Holling–Tanner predator–prey model with no-flux boundary condition is considered, and it is proved that the unique constant equilibrium is globally asymptotically stable under a new simpler parameter condition.

1. Introduction

In this work, we revisit a reaction–diffusion Holling–Tanner predator–prey model in the form given in [1]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + au - u^2 - \dfrac{uv}{m+u}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + bv - \dfrac{v^2}{\gamma u}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) \geq (\not\equiv) 0, & x \in \Omega. \end{cases} \tag{1.1}$$

Here $u(x,t)$ and $v(x,t)$ represent the density of prey and predators; respectively, $x \in \Omega \subset \mathbf{R}^n$, $n \geq 1$, and Ω is a bounded domain with a smooth boundary $\partial\Omega$; d_1, d_2 are the diffusion coefficients of prey and predators respectively; and parameters a, m, b and γ are all positive constants; a no-flux boundary condition is imposed on $\partial\Omega$ so that the ecosystem is closed to the exterior environment.

The (non-spatial) kinetic equation of system (1.1) was first proposed by Tanner [2] and May [3], while Leslie [4] and Leslie and Gower [5] consider a similar equation with unbounded predation rate. In (1.1), the predator functional response is of Holling type II as in Holling [6]. The Holling–Tanner system is regarded as one of the prototypical predator–prey models in several classical mathematical biology books; see, for example, May [3, p. 84] and Murray [7, pp. 88–94].

☆ This work was partially supported by a grant from China Scholarship Council, NSF grant DMS-1022648.
* Corresponding author at: Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA.
E-mail address: shij@math.wm.edu (J. Shi).

0893-9659/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2011.09.070

Hsu and Huang [8] dealt with the question of global stability of the positive equilibrium in a class of predator–prey systems including the ODE version of system (1.1) with certain conditions on the parameters, and in [9], they proved the uniqueness of the limit cycle when the unique positive equilibrium is unstable. For diffusive system (1.1), Peng and Wang [10] studied the existence/nonexistence of positive steady state solutions, and they [1] also proved a result on the global stability of the positive constant steady state. Li et al. [11] considered the Turing and Hopf bifurcations in (1.1). Related work on a similar diffusive Leslie–Gower system can also be found in Du and Hsu [12], Chen et al. [13].

In this note, we prove a new global stability result for the constant positive equilibrium by using a comparison method, and our result significantly improves the earlier one given in [10] which was established with the Lyapunov method.

2. The main results

It is easy to verify that system (1.1) has a unique positive equilibrium (u_*, v_*), where

$$u_* = \frac{1}{2}\left(a - m + b\gamma + \sqrt{(a-m-b\gamma)^2 + 4am}\right), \quad v = b\gamma u_*.$$

We recall the following known result from [1].

Theorem 2.1. *Assume that the parameters $m, a, b, \gamma, d_1, d_2$ are all positive. Then for system (1.1):*

1. *The positive equilibrium (u_*, v_*) is locally asymptotically stable if*
$$m^2 + 2(a + b\gamma)m + a^2 - 2ab\gamma \geq 0. \tag{2.1}$$
2. *The positive equilibrium (u_*, v_*) is globally asymptotically stable if*
$$m > b\gamma, \quad \text{and} \quad (m+K)[b\gamma + 2(m + u_* + K - a)] > (a+m)b\gamma, \tag{2.2}$$
where
$$K = \frac{1}{2}\left(a - m + \sqrt{(a-m)^2 + 4a(m - b\gamma)}\right).$$

In [1], the local stability was established through a standard linearization procedure, and the global stability was proved by using a Lyapunov functional. In this note, we prove the global stability under only the condition $m > b\gamma$ but without the second condition in (2.2); thus our result improves on the one in [1]. Our proof is based on the upper and lower solution method in [14,15]. Our main result is stated as:

Theorem 2.2. *Assume that the parameters $m, a, b, \gamma, d_1, d_2$ are all positive. Then for system (1.1), the positive equilibrium (u_*, v_*) is globally asymptotically stable, that is, for any initial values $u_0(x) > 0$, $v_0(x) \geq (\not\equiv) 0$,*

$$\lim_{t\to\infty} u(t,x) = u_*, \qquad \lim_{t\to\infty} v(t,x) = v_*, \quad \text{uniformly for } x \in \overline{\Omega},$$

if

$$m > b\gamma. \tag{2.3}$$

Proof. It is well known that if $c > 0$, and $w(x,t) > 0$ satisfies the equation

$$\begin{cases} \dfrac{\partial w}{\partial t} = D\Delta w + w(c - w), & x \in \Omega,\ t > 0, \\ \dfrac{\partial w(t,x)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ w(x,0) \geq (\not\equiv) 0, & x \in \Omega, \end{cases}$$

then $w(x,t) \to c$ uniformly for $x \in \overline{\Omega}$ as $t \to \infty$.

Since (2.3) holds, we can choose an ϵ_0 satisfying

$$0 < \epsilon_0 < \frac{b\gamma(m - b\gamma)a}{b\gamma(b\gamma + 1) + mb\gamma + m}. \tag{2.4}$$

Because $u(x,t)$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1\Delta u + au - u^2 - \frac{uv}{m+u} \\ &\leq d_1\Delta u + au - u^2, \end{aligned}$$

and the Neumann boundary condition, then from comparison principle of parabolic equations, there exists t_1 such that for any $t > t_1$, $u(x,t) \le \overline{c}_1$, where $\overline{c}_1 = a + \epsilon_0$. This in turn implies

$$\begin{aligned}\frac{\partial v}{\partial t} &= d_2\Delta v + bv - \frac{v^2}{\gamma u}\\ &\le d_2\Delta v + v\left(b - \frac{v}{\gamma(a+\epsilon_0)}\right)\end{aligned}$$

for $t > t_1$. Hence there exists $t_2 > t_1$ such that for any $t > t_2$, $v(x,t) \le \overline{c}_2$, where $\overline{c}_2 = b\gamma(a+\epsilon_0) + \epsilon_0$. Again this implies

$$\begin{aligned}\frac{\partial u}{\partial t} &= d_1\Delta u + au - u^2 - \frac{uv}{m+u}\\ &\ge d_1\Delta u + au - u^2 - \frac{b\gamma(a+\epsilon_0)+\epsilon_0}{m}u,\end{aligned}$$

for $t > t_2$. Since $m > b\gamma$, then for ϵ_0 chosen as in (2.4),

$$a - \frac{b\gamma(a+\epsilon_0)+\epsilon_0}{m} > 0, \quad \text{and} \quad a - \frac{b\gamma(a+\epsilon_0)+\epsilon_0}{m} - \epsilon_0 > 0.$$

Hence there exists $t_3 > t_2$ such that for any $t > t_3$, $u(x,t) \ge \underline{c}_1 > 0$, where

$$\underline{c}_1 = a - \frac{b\gamma(a+\epsilon_0)+\epsilon_0}{m} - \epsilon_0.$$

Finally we apply the lower bound of u to the equation of v, and we have

$$\begin{aligned}\frac{\partial v}{\partial t} &= d_2\Delta v + bv - \frac{v^2}{\gamma u}\\ &\ge d_2\Delta v + v\left(b - \frac{v}{\gamma \underline{c}_1}\right)\end{aligned}$$

for $t > t_3$. Since for the ϵ_0 chosen above in (2.4),

$$b\gamma\left(a - \frac{b\gamma(a+\epsilon_0)+\epsilon_0}{m} - \epsilon_0\right) - \epsilon_0 > 0,$$

then there exists $t_4 > t_3$ such that for any $t > t_4$, $v(x,t) \ge \underline{c}_2 > 0$, where

$$\underline{c}_2 = b\gamma\left(a - \frac{b\gamma(a+\epsilon_0)+\epsilon_0}{m} - \epsilon_0\right) - \epsilon_0.$$

Therefore for $t > t_4$ we obtain that

$$\underline{c}_1 \le u(x,t) \le \overline{c}_1, \qquad \underline{c}_2 \le v(x,t) \le \overline{c}_2,$$

and $\underline{c}_1$, $\underline{c}_2$, $\overline{c}_1$, $\overline{c}_2$ satisfy

$$\begin{aligned} &0 \ge a - \overline{c}_1 - \frac{\underline{c}_2}{m+\overline{c}_1}, \qquad 0 \ge b - \frac{\overline{c}_2}{\gamma\overline{c}_1},\\ &0 \le a - \underline{c}_1 - \frac{\overline{c}_2}{m+\underline{c}_1}, \qquad 0 \le b - \frac{\underline{c}_2}{\gamma\underline{c}_1}.\end{aligned} \tag{2.5}$$

The inequalities (2.5) show that $(\overline{c}_1, \overline{c}_2)$ and $(\underline{c}_1, \underline{c}_2)$ are a pair of coupled upper and lower solutions of system (1.1) as in the definition in [14,15] (see also [16]), as the nonlinearities in (1.1) are mixed quasimonotone. It is clear that there exists $K > 0$ such that for any $(\underline{c}_1, \underline{c}_2) \le (u_1, v_1), (u_2, v_2) \le (\overline{c}_1, \overline{c}_2)$,

$$\begin{aligned}&\left|au_1 - u_1^2 - \frac{u_1v_1}{m+u_1} - au_2 + u_2^2 + \frac{u_2v_2}{m+u_2}\right| \le K(|u_1-u_2| + |v_1-v_2|),\\ &\left|bv_1 - \frac{v_1^2}{\gamma u_1} - bv_2 + \frac{v_2^2}{\gamma u_2}\right| \le K(|u_1-u_2| + |v_1-v_2|).\end{aligned}$$

We define two iteration sequences $(\overline{c}_1^{(m)}, \overline{c}_2^{(m)})$ and $(\underline{c}_1^{(m)}, \underline{c}_2^{(m)})$ as follows: for $m \ge 1$,

$$\overline{c}_1^{(m)} = \overline{c}_1^{(m-1)} + \frac{1}{K}\left(a\overline{c}_1^{(m-1)} - (\overline{c}_1^{(m-1)})^2 - \frac{\overline{c}_1^{(m-1)}\underline{c}_2^{(m-1)}}{m+\overline{c}_1^{(m-1)}}\right),$$

$$\overline{c}_2^{(m)} = \overline{c}_2^{(m-1)} + \frac{1}{K}\left(b\overline{c}_2^{(m-1)} - \frac{(\overline{c}_2^{(m-1)})^2}{\gamma \overline{c}_1^{(m-1)}}\right),$$
$$\underline{c}_1^{(m)} = \underline{c}_1^{(m-1)} + \frac{1}{K}\left(a\underline{c}_1^{(m-1)} - (\underline{c}_1^{(m-1)})^2 - \frac{\underline{c}_1^{(m-1)}\overline{c}_2^{(m-1)}}{m+\underline{c}_1^{(m-1)}}\right),$$
$$\underline{c}_2^{(m)} = \underline{c}_2^{(m-1)} + \frac{1}{K}\left(b\underline{c}_2^{(m-1)} - \frac{(\underline{c}_2^{(m-1)})^2}{\gamma \underline{c}_1^{(m-1)}}\right),$$

where $(\overline{c}_1^0, \overline{c}_2^0) = (\overline{c}_1, \overline{c}_2)$ and $(\underline{c}_1^0, \underline{c}_2^0) = (\underline{c}_1, \underline{c}_2)$. Then for $m \geq 1$, $(\underline{c}_1, \underline{c}_2) \leq (\underline{c}_1^{(m)}, \underline{c}_2^{(m)}) \leq (\underline{c}_1^{(m+1)}, \underline{c}_2^{(m+1)}) \leq (\overline{c}_1^{(m+1)}, \overline{c}_2^{(m+1)}) \leq (\overline{c}_1^{(m)}, \overline{c}_2^{(m)}) \leq (\overline{c}_1, \overline{c}_2)$, and there exists $(\tilde{c}_1, \tilde{c}_2)$ and $(\check{c}_1, \check{c}_2)$ such that $(\underline{c}_1, \underline{c}_2) \leq (\check{c}_1, \check{c}_2) \leq (\tilde{c}_1, \tilde{c}_2) \leq (\overline{c}_1, \overline{c}_2)$, so $\lim_{m\to\infty} \overline{c}_1^m = \tilde{c}_1$, $\lim_{m\to\infty} \overline{c}_2^{(m)} = \tilde{c}_2$, $\lim_{m\to\infty} \underline{c}_1^{(m)} = \check{c}_1$, $\lim_{m\to\infty} \underline{c}_2^{(m)} = \check{c}_2$ and

$$\begin{aligned} 0 &= a - \tilde{c}_1 - \frac{\check{c}_2}{m+\tilde{c}_1}, & 0 &= b - \frac{\tilde{c}_2}{\gamma \tilde{c}_1}, \\ 0 &= a - \check{c}_1 - \frac{\tilde{c}_2}{m+\check{c}_1}, & 0 &= b - \frac{\check{c}_2}{\gamma \check{c}_1}. \end{aligned} \tag{2.6}$$

Simplifying (2.6) we obtain

$$\begin{aligned} (a-\tilde{c}_1)(m+\tilde{c}_1) &= b\gamma\check{c}_1, \\ (a-\check{c}_1)(m+\check{c}_1) &= b\gamma\tilde{c}_1. \end{aligned} \tag{2.7}$$

Subtracting the first equation of (2.7) from the second equation, we have

$$(\tilde{c}_1 - \check{c}_1)(a - m + b\gamma - \tilde{c}_1 - \check{c}_1) = 0. \tag{2.8}$$

If we assume that $\tilde{c}_1 \neq \check{c}_1$, then

$$a - m + b\gamma = \tilde{c}_1 + \check{c}_1. \tag{2.9}$$

Substituting equation (2.9) into (2.7), we have

$$\begin{aligned} (a-\tilde{c}_1)(m+\tilde{c}_1) &= b\gamma(a-m+b\gamma-\tilde{c}_1), \\ (a-\check{c}_1)(m+\check{c}_1) &= b\gamma(a-m+b\gamma-\check{c}_1). \end{aligned} \tag{2.10}$$

Hence the following equation:

$$(a-x)(m+x) = b\gamma(a-m+b\gamma-x) \tag{2.11}$$

has two positive roots $\tilde{c}_1$ and $\check{c}_1$. Eq. (2.11) can be written as follows:

$$x^2 + (m-a-b\gamma)x + (b\gamma+a)(b\gamma-m) = 0.$$

Since $m > b\gamma$, Eq. (2.11) cannot have two positive roots. Hence $\tilde{c}_1 = \check{c}_1$, and consequently, $\tilde{c}_2 = \check{c}_2$. Then from the results in [14,15], the solution $(u(x,t), v(x,t))$ of system (1.1) satisfies

$$\lim_{t\to\infty} u(t,x) = u_*, \qquad \lim_{t\to\infty} v(t,x) = v_*, \quad \text{uniformly for } x \in \overline{\Omega}.$$

The condition $m > b\gamma$ implies that $m^2 + 2(a+b\gamma)m + a^2 - 2ab\gamma \geq 0$. Hence from Theorem 2.1 and the above analysis, we can obtain that the constant equilibrium (u_*, v_*) is globally asymptotically stable for system (1.1) if (2.3) holds. □

For the diffusive Holling–Tanner system with same kinetic equations, there are two other versions of nondimensionalized equations in [8,11]. Our result Theorem 2.2 can be applied to both equations with a conversion of the parameters. In [8] only a system of ordinary differential equations was considered, but adding diffusion will cast the system in [8] into the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + u(1-u) - \dfrac{uv}{a+u}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = d_2\Delta v + v\left(\delta - \beta\dfrac{v}{u}\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u(t,x)}{\partial \nu} = \dfrac{\partial v(t,x)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) \geq (\not\equiv)0, & x \in \Omega. \end{cases} \tag{2.12}$$

In [8], for the corresponding kinetic system, it was proved that the positive equilibrium (u_*, v_*) is globally asymptotically stable if one of the following assumptions is satisfied:

(C1) $a + \delta \geq 1$;
(C2) $a + \delta < 1$, $(1 - a - \delta)^2 - 8\delta \leq 0$;
(C3) $a + \delta < 1$, $(1 - a - \delta)^2 - 8\delta > 0$, $\beta > \beta_2$, where

$$\beta_2 = \frac{\delta a_2}{(1 - a_2)(a + a_2)}, \qquad a_2 = \frac{1}{4}(1 - a - \delta + \sqrt{(1 - a - \delta)^2 - 8a\delta}).$$

Theorem 2.2 implies that if $\beta > \frac{\delta}{a}$, then (u_*, v_*) is globally asymptotically stable for the diffusive Holling–Tanner system (2.12). One can show that the parameter region given by $\beta > \frac{\delta}{a}$ is contained in the set given by (C1)–(C3). If a and δ satisfy (C1) or (C2), then it is clear that $\beta > \frac{\delta}{a}$ is satisfied. If a and δ do not satisfy (C1) or (C2), then $0 < a + \delta < 1$, and $(1 - a - \delta)^2 - 8\delta > 0$. Hence

$$\begin{aligned} a_2 &= \frac{1}{4}(1 - a - \delta + \sqrt{(1 - a - \delta)^2 - 8a\delta}) \\ &\leq \frac{1}{2}(1 - a - \delta) \leq \frac{1}{2}. \end{aligned}$$

Consequently,

$$\beta_2 = \frac{\delta a_2}{(1 - a_2)(a + a_2)} < \frac{\delta}{a + a_2} < \frac{\delta}{a}.$$

Hence in this case, $\beta > \frac{\delta}{a}$ implies (C3).

On the other hand, the diffusive Holling–Tanner system in [11] is in the form of

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u(1 - \beta u) - \dfrac{muv}{1 + u}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + sv\left(1 - \dfrac{v}{u}\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u(t,x)}{\partial \nu} = \dfrac{\partial v(t,x)}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) \geq 0, & \not\equiv 0. \end{cases} \tag{2.13}$$

For the kinetics system corresponding to (2.13), it was shown in [11] (by using the result of [8]) that the positive equilibrium (u_*, v_*) is globally asymptotically stable if

$$\beta \geq 1, \quad \text{or} \quad \beta < 1, \quad \text{and} \quad m \leq \frac{(1 + \beta)^2}{2(1 - \beta)^2}. \tag{2.14}$$

Now our Theorem 2.2 can be applied to (2.13), and we have proved that if $\beta > m$, then (u_*, v_*) is globally asymptotically stable for (2.13). The parameter region of global stability for the ODE in [8,11] is larger than the one proved in Theorem 2.2 for the PDE case (the diffusion coefficients d_1, d_2 are arbitrary), but this is not unexpected as the global stability for an infinite dimensional dynamical system is much more complex, as demonstrated in [17]. The parameterization of the system in [11] is easier to show for the parameter regions of global stability in Theorem 2.2 and [11].

References

[1] R. Peng, M.X. Wang, Global stability of the equilibrium of a diffusive Holling–Tanner prey–predator model, Appl. Math. Lett. 20 (6) (2007) 664–670.
[2] J.T. Tanner, The stability and the intrinsic growth rates of prey and predator populations, Ecology 56 (1975) 855–867.
[3] R.M. May, Stability and Complexity in Model Ecosystems, second ed., Princeton Univ. Press, 1974.
[4] P.H. Leslie, Some further notes on the use of matrices in population mathematics, Biometrika 35 (3–4) (1948) 213.
[5] P.H. Leslie, J.C. Gower, The properties of a stochastic model for the predator–prey type of interaction between two species, Biometrika 47 (3–4) (1960) 219.
[6] C.S. Holling, The components of predation as revealed by a study of small mammal predation of the European pine sawfly, Can. Entomol. 91 (5) (1959) 293–320.
[7] J.D. Murray, Mathematical Biology. I: An Introduction, third ed., in: Interdisciplinary Applied Mathematics, vol. 17, Springer-Verlag, New York, 2002.
[8] S.B. Hsu, T.W. Huang, Global stability for a class of predator–prey system, SIAM J. Appl. Math. 55 (3) (1995) 763–783.
[9] S.B. Hsu, T.W. Hwang, Uniqueness of limit cycles for a predator–prey system of Holling and Leslie type, Can. Appl. Math. Q. 6 (2) (1998) 91–117.
[10] R. Peng, M.X. Wang, Positive steady states of the Holling–Tanner prey–predator model with diffusion, Proc. Roy. Soc. Edinburgh Sect. A 135 (01) (2005) 149–164.
[11] X. Li, W.H. Jiang, J.P. Shi, Hopf bifurcation and Turing instability in the reaction–diffusion Holling–Tanner predator–prey model (2010) (submitted for publication).
[12] Y. Du, S.B. Hsu, A diffusive predator–prey model in heterogeneous environment, J. Differential Equations 203 (2) (2004) 331–364.
[13] S.S. Chen, J.P. Shi, J.J. Wei, Global stability and Hopf bifurcation in a delayed diffusive Leslie–Gower predator–prey system, Internat. J. Bifur. Chaos (2011) (in press).
[14] C.V. Pao, On nonlinear reaction–diffusion systems, J. Math. Anal. Appl. 87 (1) (1982) 165–198.
[15] C.V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Publishing Corporation, 1992.
[16] S.S. Chen, J.P. Shi, Global attractivity of equilibrium in Gierer–Meinhardt system with saturation and gene expression time delays (2011) (submitted for publication).
[17] S.B. Hsu, A survey of constructing Lyapunov functions for mathematical models in population biology, Taiwanese J. Math. 9 (2) (2005) 151–173.