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A diffusive Holling–Tanner predator–prey model with no-flux boundary condition is considered, and it is proved that the unique constant equilibrium is globally asymptotically stable under a new simpler parameter condition.

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1. Introduction

In this work, we revisit a reaction–diffusion Holling–Tanner predator–prey model in the form given in [1]:

\begin{equation}
\begin{aligned}
\frac{\partial u}{\partial t} &= d_1 \Delta u + au - u^2 - \frac{uv}{m + u}, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial v}{\partial t} &= d_2 \Delta v + bv - \frac{v^2}{\gamma u}, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial u(x, t)}{\partial \nu} &= \frac{\partial v(x, t)}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t > 0, \\
u(x, 0) &= u_0(x) > 0, \quad v(x, 0) = v_0(x) \geq (\neq) 0, \quad x \in \Omega.
\end{aligned}
\end{equation}

Here $u(x, t)$ and $v(x, t)$ represent the density of prey and predators; respectively, $x \in \Omega \subset \mathbb{R}^n, n \geq 1$, and $\Omega$ is a bounded domain with a smooth boundary $\partial \Omega$; $d_1, d_2$ are the diffusion coefficients of prey and predators respectively; and parameters $a, m, b$ and $\gamma$ are all positive constants; a no-flux boundary condition is imposed on $\partial \Omega$ so that the ecosystem is closed to the exterior environment.

The (non-spatial) kinetic equation of system (1.1) was first proposed by Tanner [2] and May [3], while Leslie [4] and Leslie and Gower [5] consider a similar equation with unbounded predation rate. In (1.1), the predator functional response is of Holling type II as in Holling [6]. The Holling–Tanner system is regarded as one of the prototypical predator–prey models in several classical mathematical biology books; see, for example, May [3, p. 84] and Murray [7, pp. 88–94].
Hsu and Huang [8] dealt with the question of global stability of the positive equilibrium in a class of predator–prey systems including the ODE version of system (1.1) with certain conditions on the parameters, and in [9], they proved the uniqueness of the limit cycle when the unique positive equilibrium is unstable. For diffusive system (1.1), Peng and Wang [10] studied the existence/nonexistence of positive steady state solutions, and they [1] also proved a result on the global stability of the positive constant steady state. Li et al. [11] considered the Turing and Hopf bifurcations in (1.1). Related work on a similar diffusive Leslie–Gower system can also be found in Du and Hsu [12], Chen et al. [13].

In this note, we prove a new global stability result for the constant positive equilibrium by using a comparison method, and our result significantly improves the earlier one given in [10] which was established with the Lyapunov method.

2. The main results

It is easy to verify that system (1.1) has a unique positive equilibrium \((u_*, v_*)\), where

\[
 u_* = \frac{1}{2}(a - m + b \gamma + \sqrt{(a - m - b \gamma)^2 + 4am}), \quad v = b \gamma u_*. 
\]

We recall the following known result from [1].

**Theorem 2.1.** Assume that the parameters \(m, a, b, \gamma, d_1, d_2\) are all positive. Then for system (1.1):

1. The positive equilibrium \((u_*, v_*)\) is locally asymptotically stable if

\[
 m^2 + 2(a + b \gamma)m + a^2 - 2ab \gamma \geq 0. \tag{2.1}
\]

2. The positive equilibrium \((u_*, v_*)\) is globally asymptotically stable if

\[
 m > b \gamma, \quad \text{and} \quad (m + K)[b \gamma + 2(m + u_* + K - a)] > (a + m)b \gamma, \tag{2.2}
\]

where

\[
 K = \frac{1}{2} \left( a - m + \sqrt{(a - m)^2 + 4a(m - b \gamma)} \right). 
\]

In [1], the local stability was established through a standard linearization procedure, and the global stability was proved by using a Lyapunov functional. In this note, we prove the global stability under only the condition \(m > b \gamma\) but without the second condition in (2.2); thus our result improves on the one in [1]. Our proof is based on the upper and lower solution method in [14,15]. Our main result is stated as:

**Theorem 2.2.** Assume that the parameters \(m, a, b, \gamma, d_1, d_2\) are all positive. Then for system (1.1), the positive equilibrium \((u_*, v_*)\) is globally asymptotically stable, that is, for any initial values \(u_0(x) > 0, v_0(x) \geq \epsilon \neq 0\),

\[
 \lim_{t \to \infty} u(t, x) = u_*, \quad \lim_{t \to \infty} v(t, x) = v_*, \quad \text{uniformly for } x \in \Omega. 
\]

if

\[
 m > b \gamma. \tag{2.3}
\]

**Proof.** It is well known that if \(c > 0\), and \(w(x, t) > 0\) satisfies the equation

\[
 \begin{align*}
 &\frac{\partial w}{\partial t} = D \Delta w + w(c - w), \quad x \in \Omega, \ t > 0, \\
 &\frac{\partial w(t, x)}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
 &w(x, 0) \geq \epsilon \neq 0, \quad x \in \Omega,
\end{align*}
\]

then \(w(x, t) \to c\) uniformly for \(x \in \overline{\Omega}\) as \(t \to \infty\). Since (2.3) holds, we can choose an \(\epsilon_0\) satisfying

\[
 0 < \epsilon_0 < \frac{b \gamma(m - b \gamma)a}{b \gamma(m - b \gamma)a + m}. \tag{2.4}
\]

Because \(u(x, t)\) satisfies

\[
 \frac{\partial u}{\partial t} = d_1 \Delta u + au - u^2 - \frac{uv}{m + u} \\
 \leq d_1 \Delta u + au - u^2,
\]
and the Neumann boundary condition, then from comparison principle of parabolic equations, there exists $t_1$ such that for any $t > t_1$, $u(x, t) \leq \bar{c}_1$, where $\bar{c}_1 = a + \epsilon_0$. This in turn implies
\[
\frac{\partial v}{\partial t} = d_2 \Delta v + bv - \frac{v^2}{\gamma u} \\
\leq d_2 \Delta v + v \left( b - \frac{v}{\gamma (a + \epsilon_0)} \right)
\]
for $t > t_1$. Hence there exists $t_2 > t_1$ such that for any $t > t_2$, $v(x, t) \leq \bar{c}_2$, where $\bar{c}_2 = b\gamma (a + \epsilon_0) + \epsilon_0$. Again this implies
\[
\frac{\partial u}{\partial t} = d_1 \Delta u + au - u^2 - \frac{uv}{m + u} \\
\geq d_1 \Delta u + au - u^2 - \frac{b\gamma (a + \epsilon_0) + \epsilon_0}{m} u,
\]
for $t > t_2$. Since $m > b\gamma$, then for $\epsilon_0$ chosen as in (2.4),
\[
a - \frac{b\gamma (a + \epsilon_0) + \epsilon_0}{m} > 0, \quad \text{and} \quad a - \frac{b\gamma (a + \epsilon_0) + \epsilon_0}{m} - \epsilon_0 > 0.
\]
Hence there exists $t_3 > t_2$ such that for any $t > t_3$, $u(x, t) \geq \underline{c}_1 > 0$, where
\[
\underline{c}_1 = a - \frac{b\gamma (a + \epsilon_0) + \epsilon_0}{m} - \epsilon_0.
\]
Finally we apply the lower bound of $u$ to the equation of $v$, and we have
\[
\frac{\partial v}{\partial t} = d_2 \Delta v + bv - \frac{v^2}{\gamma u} \\
\geq d_2 \Delta v + v \left( b - \frac{v}{\gamma \underline{c}_1} \right)
\]
for $t > t_3$. Since for the $\epsilon_0$ chosen above in (2.4),
\[
b\gamma \left( a - \frac{b\gamma (a + \epsilon_0) + \epsilon_0}{m} - \epsilon_0 \right) - \epsilon_0 > 0,
\]
then there exists $t_4 > t_3$ such that for any $t > t_4$, $v(x, t) \geq \underline{c}_2 > 0$, where
\[
\underline{c}_2 = b\gamma \left( a - \frac{b\gamma (a + \epsilon_0) + \epsilon_0}{m} - \epsilon_0 \right) - \epsilon_0.
\]
Therefore for $t > t_4$ we obtain that
\[
\underline{c}_1 \leq u(x, t) \leq \bar{c}_1, \quad \underline{c}_2 \leq v(x, t) \leq \bar{c}_2,
\]
and $\underline{c}_1$, $\underline{c}_2$, $\bar{c}_1$, $\bar{c}_2$ satisfy
\[
0 \geq a - \bar{c}_1 - \frac{\underline{c}_2}{m + \bar{c}_1}, \quad 0 \geq b - \frac{\bar{c}_2}{\gamma \bar{c}_1}, \\
0 \leq a - \underline{c}_1 - \frac{\bar{c}_2}{m + \underline{c}_1}, \quad 0 \leq b - \frac{\underline{c}_2}{\gamma \underline{c}_1}.
\]
The inequalities (2.5) show that $(\bar{c}_1, \bar{c}_2)$ and $(\underline{c}_1, \underline{c}_2)$ are a pair of coupled upper and lower solutions of system (1.1) as in the definition in [14,15] (see also [16]), as the nonlinearities in (1.1) are mixed quasimonotone. It is clear that there exists $K > 0$ such that for any $(\underline{c}_1, \underline{c}_2) \leq (u_1, v_1), (u_2, v_2) \leq (\bar{c}_1, \bar{c}_2),$
\[
\begin{align*}
|au_1 - u_1^2 - \frac{u_1v_1}{m + u_1} - au_2 + u_2^2 + \frac{u_2v_2}{m + u_2}| &\leq K(|u_1 - u_2| + |v_1 - v_2|), \\
|bv_1 - \frac{v_1^2}{\gamma u_1} - bv_2 + \frac{v_2^2}{\gamma u_2}| &\leq K(|u_1 - u_2| + |v_1 - v_2|).
\end{align*}
\]
We define two iteration sequences $(\bar{c}_1^{(m)}, \bar{c}_2^{(m)})$ and $(\underline{c}_1^{(m)}, \underline{c}_2^{(m)})$ as follows: for $m \geq 1,$
\[
\bar{c}_1^{(m)} = \bar{c}_1^{(m-1)} + \frac{1}{K} \left( \alpha \bar{c}_1^{(m-1)} - (\bar{c}_1^{(m-1)})^2 - \frac{\bar{c}_2^{(m-1)} - \underline{c}_1^{(m-1)}}{m + \bar{c}_1^{(m-1)}} \right),
\]
\[
\underline{c}_1^{(m)} = \underline{c}_1^{(m-1)} + \frac{1}{K} \left( \alpha \underline{c}_1^{(m-1)} - (\underline{c}_1^{(m-1)})^2 - \frac{\bar{c}_2^{(m-1)} - \underline{c}_1^{(m-1)}}{m + \underline{c}_1^{(m-1)}} \right).
\]
\[ \xi_2^{(m)} = \xi_2^{(m-1)} + \frac{1}{K} \left( b\xi_2^{(m-1)} - \frac{(\xi_2^{(m-1)})^2}{\gamma} - \xi_2^{(m-1)} \right), \]
\[ \xi_1^{(m)} = \xi_1^{(m-1)} + \frac{1}{K} \left( \gamma \xi_1^{(m-1)} - \xi_1^{(m-1)} \right) - \frac{\xi_1^{(m-1)}\xi_2^{(m-1)}}{\gamma} + \frac{b}{m}\xi_2^{(m-1)}, \]
\[ \xi_2^{(m)} = \xi_2^{(m-1)} + \frac{1}{K} \left( b\xi_2^{(m-1)} - \gamma \xi_2^{(m-1)} \right), \]

where \((\xi_1^{(0)}, \xi_2^{(0)}) = (\xi_1, \xi_2)\) and \((\xi_1^{(0)}, \xi_2^{(0)}) = (\xi_1, \xi_2)\). Then for \(m \geq 1\), 
\[ (\xi_1^{(m)}, \xi_2^{(m)}) \leq (\xi_1^{(m)}, \xi_2^{(m)}) \leq (\xi_1^{(m+1)}, \xi_2^{(m+1)}) \leq (\xi_1^{(m+1)}, \xi_2^{(m+1)}) \leq (\xi_1^{(m+1)}, \xi_2^{(m+1)}) \leq \cdots \leq (\xi_1^{(1)}, \xi_2^{(1)}) \leq (\xi_1, \xi_2), \]
and there exists \((\xi_1^*, \xi_2^*)\) such that \(\lim_{m \to \infty} \xi_1^{(m)} = \xi_1^*, \lim_{m \to \infty} \xi_2^{(m)} = \xi_2^*, \lim_{m \to \infty} \xi_1^{(m)} = \xi_1^*, \lim_{m \to \infty} \xi_2^{(m)} = \xi_2^*\) and

\[ 0 = a - \tilde{c}_1 - \frac{\tilde{c}_2}{m + \tilde{c}_1}, \quad 0 = b - \frac{\tilde{c}_2}{\gamma \tilde{c}_1}, \quad \text{(2.6)} \]

Simplifying (2.6) we obtain

\[ (a - \tilde{c}_1)(m + \tilde{c}_1) = b\gamma \tilde{c}_1, \]
\[ (a - \tilde{c}_1)(m + \tilde{c}_1) = b\gamma \tilde{c}_1. \quad \text{(2.7)} \]

Subtracting the first equation of (2.7) from the second equation, we have

\[ (\tilde{c}_1 - \tilde{c}_1)(a - m + b\gamma - \tilde{c}_1 - \tilde{c}_1) = 0. \quad \text{(2.8)} \]

If we assume that \(\tilde{c}_1 \neq \tilde{c}_1\), then

\[ a - m + b\gamma = \tilde{c}_1 + \tilde{c}_1. \quad \text{(2.9)} \]

Substituting equation (2.9) into (2.7), we have

\[ (a - \tilde{c}_1)(m + \tilde{c}_1) = b\gamma (a - m + b\gamma - \tilde{c}_1), \]
\[ (a - \tilde{c}_1)(m + \tilde{c}_1) = b\gamma (a - m + b\gamma - \tilde{c}_1). \quad \text{(2.10)} \]

Hence the following equation:

\[ (a - x)(m + x) = b\gamma (a - m + b\gamma - x) \quad \text{(2.11)} \]

has two positive roots \(\tilde{c}_1\) and \(\tilde{c}_1\). Eq. (2.11) can be written as follows:

\[ x^2 + (m - a - b\gamma)x + (b\gamma + a)(b\gamma - m) = 0. \]

Since \(m > b\gamma\), Eq. (2.11) cannot have two positive roots. Hence \(\tilde{c}_1 = \tilde{c}_1\), and consequently, \(\tilde{c}_2 = \tilde{c}_2\). Then from the results in [14,15], the solution \((u(x, t), v(x, t))\) of system (1.1) satisfies

\[ \lim_{t \to \infty} u(t, x) = u_*, \quad \lim_{t \to \infty} v(t, x) = v_*, \quad \text{uniformly for } x \in \Omega. \]

The condition \(m > b\gamma\) implies that \(m^2 + 2(a + b\gamma)m + a^2 - 2ab\gamma \geq 0\). Hence from Theorem 2.1 and the above analysis, we can obtain that the constant equilibrium \((u_*, v_*)\) is globally asymptotically stable for system (1.1) if (2.3) holds. □

For the diffusive Holling–Tanner system with same kinetic equations, there are two other versions of nondimensionalized equations in [8,11]. Our result Theorem 2.2 can be applied to both equations with a conversion of the parameters. In [8] only a system of ordinary differential equations was considered, but adding diffusion will cast the system in [8] into the form

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + u(1 - u) - \frac{uv}{a + u}, \quad x \in \Omega, \ t > 0, \\
\frac{\partial v}{\partial t} &= d_2 \Delta v + v \left( \delta - \frac{v}{u} \right), \quad x \in \Omega, \ t > 0, \\
\frac{\partial u(t, x)}{\partial t} &= \frac{\partial v(t, x)}{\partial t} = 0, \quad x \in \partial \Omega, \ t > 0, \\
u(x, 0) &= u_0(x) > 0, \ v(x, 0) = v_0(x) \geq (\neq)0, \quad x \in \Omega.
\end{align*}
\]
In [8], for the corresponding kinetic system, it was proved that the positive equilibrium \((u_*, v_*)\) is globally asymptotically stable if one of the following assumptions is satisfied:

\[(C1) \quad a + \delta \geq 1; \]
\[(C2) \quad a + \delta < 1, \quad (1 - a - \delta)^2 - 8\delta \leq 0; \]
\[(C3) \quad a + \delta < 1, \quad (1 - a - \delta)^2 - 8\delta > 0, \quad \beta > \beta_2, \]

where

\[\beta_2 = \frac{\delta a_2}{(1 - a_2)(a + a_2)}, \quad a_2 = \frac{1}{4}(1 - a - \delta + \sqrt{(1 - a - \delta)^2 - 8a\delta}).\]

Theorem 2.2 implies that if \(\beta > \frac{\delta}{a}\), then \((u_*, v_*)\) is globally asymptotically stable for the diffusive Holling–Tanner system (2.12). One can show that the parameter region given by \(\beta > \frac{\delta}{a}\) is contained in the set given by (C1)–(C3). If \(a\) and \(\delta\) satisfy (C1) or (C2), then it is clear that \(\beta > \frac{\delta}{a}\) is satisfied. If \(a\) and \(\delta\) do not satisfy (C1) or (C2), then \(0 < a + \delta < 1\), and \((1 - a - \delta)^2 - 8\delta > 0\). Hence

\[a_2 = \frac{1}{4}(1 - a - \delta + \sqrt{(1 - a - \delta)^2 - 8a\delta}) \leq \frac{1}{2} \leq \frac{1}{a}.\]

Consequently,

\[\beta_2 = \frac{\delta a_2}{(1 - a_2)(a + a_2)} < \frac{\delta}{a + a_2} < \frac{\delta}{a}.\]

Hence in this case, \(\beta > \frac{\delta}{a}\) implies (C3).

On the other hand, the diffusive Holling–Tanner system in [11] is in the form of

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + u(1 - \beta u) - \frac{mu}{1 + u}, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial v}{\partial t} &= d_2 \Delta v + s v \left(1 - \frac{v}{u}\right), \quad x \in \Omega, \quad t > 0, \\
\frac{\partial u(t, x)}{\partial v} &= 0, \quad x \in \partial \Omega, \quad t > 0, \\
\frac{\partial v(t, x)}{\partial v} &= 0, \quad x \in \partial \Omega, \quad t > 0, \\
u(x, 0) &= u_0(x) > 0, \quad v(x, 0) = v_0(x) \geq 0, \quad \forall x.
\end{align*}
\]

For the kinetics system corresponding to (2.13), it was shown in [11] (by using the result of [8]) that the positive equilibrium \((u_*, v_*)\) is globally asymptotically stable if

\[\beta \geq 1, \quad \text{or} \quad \beta < 1, \quad \text{and} \quad m \leq \frac{(1 + \beta)^2}{2(1 - \beta)^2}. \tag{2.14}\]

Now our Theorem 2.2 can be applied to (2.13), and we have proved that if \(\beta > m\), then \((u_*, v_*)\) is globally asymptotically stable for (2.13). The parameter region of global stability for the ODE in [8,11] is larger than the one proved in Theorem 2.2 for the PDE case (the diffusion coefficients \(d_1, d_2\) are arbitrary), but this is not unexpected as the global stability for an infinite dimensional dynamical system is much more complex, as demonstrated in [17]. The parameterization of the system in [11] is easier to show for the parameter regions of global stability in Theorem 2.2 and [11].

References