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Submatrix monotonicity of the Perron root ☆

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ABSTRACT

It is known that increasing an entry of a nonnegative matrix nondecreases (and generally increases) its Perron root. Motivated by a question raised by José Dias da Silva, we study the partial order on k-by-k nonnegative matrices in which $A \precsim_{DS} B$ if whenever A and B occur as submatrices in the same position in otherwise equal nonnegative matrices F and G, $\rho(F) \leqslant \rho(G)$. We find that this partial order is equivalent to the entry-wise partial order. This is proven with some asymptotic results about the Perron root that may be of independent interest.

1. Introduction

It is well known [1] that if an entry of an n-by-n positive matrix is increased, the Perron root increases. This may be improved by replacing "positive" with irreducible after the increase, and, in general, for nonnegative matrices if an entry is non-decreased, the Perron root does not decrease. Suppose now that instead of replacing an entry by a larger one, we consider replacing a proper square submatrix by another one with larger Perron root. Recently [2], Dias da Silva raised the question of the circumstances under which such a replacement of a principal submatrix would increase the Perron root of the full matrix. Of course, if the matrix is positive and the replacing submatrix "dominates" the replaced one, in the sense that some entries are larger and none smaller, then the full matrix will have

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a larger Perron root. Otherwise, it may or may not. Examples are easily produced in which a larger Perron root for the submatrix results in a smaller Perron root for the full matrix (as was also noted in [3]) in all cases except when the submatrix is 1-by-1 or n-by-n.

Here, our purpose is to give an answer to Dias da Silva's question by studying the following preorder on k-by-k nonnegative matrices. We say that $A \precsim_{DS} B$ ("A is Dias da Silva less than or equal to B") if, when two n-by-n nonnegative matrices F and G are equal, except in a k-by-k principal submatrix, with $0 < k < n$, in which F is A and G is B, we necessarily have $\rho(F) \leqslant \rho(G)$. Here, as usual, ρ denotes the spectral radius or Perron root. We denote the entry-wise (weak) partial order on nonnegative matrices (domination) by $A \leqslant B$ if $a_{ij} \leqslant b_{ij}$ for all i, j in $A = (a_{ij})$ and $B = (b_{ij})$. Our main result, surprising to both Dias da Silva and us, is that the Dias da Silva partial order and the domination partial order are equivalent.

Theorem 1. *If A and B are k-by-k nonnegative matrices, then $A \precsim_{DS} B$ if and only if $A \leqslant B$.*

Of course, if $A \leqslant B$, then $A \precsim_{DS} B$ by the facts mentioned earlier (and continuity of the Perron root). In the remainder of this work, we present the necessary analytical results about asymptotic behaviour of the Perron root (which may be of independent interest) that are necessary to prove the reverse implication.

2. Domination of diagonal entries

Throughout this section, let

$$F(x,y) = \begin{pmatrix} A_{11} & a_{12} & 0 \\ a_{21}^T & a_{22} & x \\ 0 & y & 0 \end{pmatrix}, \quad G(x,y) = \begin{pmatrix} B_{11} & b_{12} & 0 \\ b_{21}^T & b_{22} & x \\ 0 & y & 0 \end{pmatrix},$$

with A_{11} and B_{11} $(n-2)$-by-$(n-2)$, a_{12}, a_{21}, b_{12} and b_{21} $(n-2)$-by-1, and a_{22}, x and y scalars. Let

$$A = \begin{pmatrix} A_{11} & a_{12} \\ a_{21}^T & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & b_{12} \\ b_{21}^T & b_{22} \end{pmatrix}.$$

Our goal is to first show that for a nonnegative matrix of the form $F(x,y)$, the Perron root is arbitrarily approximated by that of

$$\begin{pmatrix} a_{22} & x \\ y & 0 \end{pmatrix}$$

for sufficiently large x and y. Given the explicit value of the latter Perron root, this will mean that $\rho(F(x,y)) < \rho(G(x,y))$ whenever $a_{22} < b_{22}$ and x and y are sufficiently large.

Lemma 2. *For $a_{22}, x, y \geqslant 0$*

$$\rho\left(\begin{bmatrix} a_{22} & x \\ y & 0 \end{bmatrix}\right) = \frac{a_{22}}{2} + \sqrt{\left(\frac{a_{22}}{2}\right)^2 + xy}.$$

If $y = x + a_{22}$, then

$$\rho\left(\begin{bmatrix} a_{22} & x \\ y & 0 \end{bmatrix}\right) = a_{22} + x.$$

Proof. The proof is a standard calculation. $\square$

Lemma 3. *For any $\epsilon > 0$, there are numbers $X, Y > 0$ such that for all $x > X$ and $y > Y$*

$$\rho(F(x,y)) \leqslant \epsilon + \left|\frac{a_{22}}{2}\right| + \sqrt{\left(\frac{a_{22}}{2}\right)^2 + xy}.$$

Proof. First, via appropriate positive diagonal similarity of A, we may suppose without loss of generality that the sum of the absolute entries in a_{21}^T is no more than ϵ. By another diagonal similarity on $F(x,y)$ via $\begin{pmatrix} I & 0 \\ 0 & d \end{pmatrix}$, with $d > 0$ scalar, we may suppose that both absolute row sums of

$$\begin{pmatrix} a_{22} & x \\ y & 0 \end{pmatrix}$$

are

$$\left|\frac{a_{22}}{2}\right| + \sqrt{\left(\frac{a_{22}}{2}\right)^2 + |xy|}$$

(use the Perron vector of $\begin{pmatrix} |a_{22}| & |x| \\ |y| & 0 \end{pmatrix}$). The spectral radius of $F(x,y)$ has not been changed. Now, for sufficiently large x, y, all absolute row sums of $F(x,y)$ are no more than

$$\epsilon + \left|\frac{a_{22}}{2}\right| + \sqrt{\left(\frac{a_{22}}{2}\right)^2 + |xy|},$$

and the claim follows from the fact that the spectral radius is no more than the maximum 1-norm of the rows [1]. $\square$

We may now prove the main result of this section, that the Perron root of a nonnegative matrix $F(x,y)$ is eventually (as x and y increase) approximated by that of its lower right 2-by-2 submatrix. Of course, $\rho(F(x,y))$ must always be at least the latter Perron root.

Theorem 4. *Suppose that $F(x,y) \geqslant 0$. For any $\epsilon > 0$, there are numbers $X, Y > 0$ such that for all $x \geqslant X$ and $y \geqslant Y$,*

$$0 \leqslant \rho(F(x,y)) - \rho\left(\begin{bmatrix} a_{22} & x \\ y & 0 \end{bmatrix}\right) \leqslant \epsilon.$$

Proof. This follows directly from Lemmas 2 and 3. $\square$

Now, if we consider nonnegative matrices $F(x,y)$ and $G(x,y)$, of the same form, in terms of the relationship between a_{22} and b_{22}, importantly the spectral radii will eventually (as x and y grow) follow the relatonship between a_{22} and b_{22} irrespective of the relative sizes of the remaining entries a_{ij}, b_{ij}.

Corollary 5. *Suppose that $F(x,y), G(x,y) \geqslant 0$. If $a_{22} < b_{22}$, then there are numbers $X, Y > 0$ such that for all $x \geqslant X$ and $y \geqslant Y$, we have*

$$\rho(F(x,y)) < \rho(G(x,y)).$$

Proof. In view of Lemma 2,

$$\rho\left(\begin{bmatrix} a_{22} & x \\ y & 0 \end{bmatrix}\right) < \rho\left(\begin{bmatrix} b_{22} & x \\ y & 0 \end{bmatrix}\right).$$

Pick

$$0 < \epsilon < \frac{1}{2}\left[\rho\left(\begin{bmatrix} b_{22} & x \\ y & 0 \end{bmatrix}\right) - \rho\left(\begin{bmatrix} a_{22} & x \\ y & 0 \end{bmatrix}\right)\right]$$

and apply Theorem 4 to verify the claim. □

Applying Corollary 5 to our partial order, we may conclude:

Corollary 6. *Let $A = (a_{ij}) \geqslant 0$ and $B = (b_{ij}) \geqslant 0$ be k-by-k. If $A \precsim_{DS} B$, then*

$$a_{ii} \leqslant b_{ii}, \quad i = 1, \ldots, n.$$

Proof. Suppose that $a_{ii} > b_{ii}$ for some i. By permutation similarity we may suppose without loss of generality that this entry is in the lower right corner. Embedding A in $F(x, y)$ and B in $G(x, y)$ and choosing x and y sufficiently large that Corollary 5 applies now contradicts the hypothesis that $A \precsim_{DS} B$ and verifies Corollary 6. □

We close this section by noting that a related statement may be proven by similar means. If

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \geqslant 0$$

and is n-by-n, with H_{22} square and having positive Perron root, then for any $\epsilon > 0$ there is a number $T > 0$ such that for all $t > T$

$$0 \leqslant \rho\left(\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & tH_{22} \end{bmatrix}\right) - \rho(tH_{22}) \leqslant \epsilon.$$

This statement, while nicely general, is not sufficiently precise for our needs here.

3. Domination of off-diagonal entries

Throughout this section let

$$F(x,y) = \left(\begin{array}{c|cc|c} A_{11} & \multicolumn{2}{c|}{A_{12}} & 0 \\ \hline A_{21}^T & a_{22} & a_{23} & 0 \\ & a_{32} & a_{33} & x \\ \hline 0 & y & 0 & 0 \end{array}\right), \quad G(x,y) = \left(\begin{array}{c|cc|c} B_{11} & \multicolumn{2}{c|}{B_{12}} & 0 \\ \hline B_{21}^T & b_{22} & b_{23} & 0 \\ & b_{32} & b_{33} & x \\ \hline 0 & y & 0 & 0 \end{array}\right),$$

with A_{11}, B_{11} $(n-3)$-by-$(n-3)$, A_{12}, A_{21}, B_{12} and B_{21} $(n-3)$-by-2 and a_{ij}, b_{ij} with $i, j \in \{2, 3\}$ and x, y scalars. Let

$$A = \left(\begin{array}{c|cc} A_{11} & \multicolumn{2}{c}{A_{12}} \\ \hline A_{21}^T & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{array}\right) \quad \text{and} \quad B = \left(\begin{array}{c|cc} B_{11} & \multicolumn{2}{c}{B_{12}} \\ \hline B_{21}^T & b_{22} & b_{23} \\ & b_{32} & b_{33} \end{array}\right).$$

Our primary goal is to show that $a_{23} < b_{23}$ if and only if $\rho(F(x, y)) < \rho(G(x, y))$ for sufficiently large x and y, irrespective of the values of other entries besides x and y. This is difficult to do as explicitly as in the case of diagonal entries, but complements the case of diagonal entries for the purpose of proving our main result.

Lemma 7. *Assume that $a_{23} \neq 0$. For any given $\delta > 0$, there is a constant C and numbers $X, Y > 0$ such that*

$$\rho(F(x, y)) \leqslant C + \left(|a_{23}|^{1/3} + \delta\right) |xy|^{1/3}$$

for all $|x| > X$ and $|y| > Y$.

Proof. Via positive diagonal similarity, we may suppose without loss of generality that each of the two absolute row sums of A_{21}^T is no more than any given $\delta > 0$. Now, let

$$D_A = \begin{pmatrix} |x|^{1/3}|y|^{1/3}|a_{23}|^{-2/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |x|^{2/3}y^{-1/3}a_{23}^{-1/3} \end{pmatrix}$$

Then diagonal similarity on $F(x, y)$ via

$$\begin{pmatrix} I & 0 \\ 0 & D_A \end{pmatrix}$$

results in a matrix whose last three absolute row sums are no more than

$$C + \left(|a_{23}|^{1/3} + \delta\right) |xy|^{1/3}$$

for x and y sufficiently large and C a positive constant. For x and y sufficiently large, the remaining absolute row sums will be smaller. Since the maximum absolute row sum is an upper bound for the spectral radius [1], the claim follows. $\square$

Lemma 8. *Let $G(x, y) \geqslant 0$ and suppose that $b_{23} > 0$. There are numbers $X, Y > 0$ such that*

$$\rho(G(x, y)) \geqslant b_{23}^{1/3}(xy)^{1/3}$$

for all $x > X$ and $y > Y$.

Proof. Following the notation of the prior proof, the nonnegative matrix

$$D_B \begin{pmatrix} b_{22} & b_{23} & 0 \\ b_{32} & b_{33} & x \\ y & 0 & 0 \end{pmatrix} D_B^{-1}$$

has row sums at least $b_{23}^{1/3}(xy)^{1/3}$, as that value lies in each row as an entry. It follows that

$$\rho\left(\begin{bmatrix} b_{22} & b_{23} & 0 \\ b_{32} & b_{33} & x \\ y & 0 & 0 \end{bmatrix}\right) \geqslant b_{23}^{1/3}(xy)^{1/3}$$

and, as this matrix is a principal submatrix of $G(x, y)$, we have $\rho(G(x, y)) \geqslant b_{23}^{1/3}(xy)^{1/3}$ as well. $\square$

Theorem 9. *Suppose that $F(x, y) \geqslant 0$ and $G(x, y) \geqslant 0$. If $0 < a_{23} < b_{23}$, then there exist numbers $X, Y > 0$ such that*

$$\rho(F(x, y)) < \rho(G(x, y))$$

for all $x > X$ and $y > Y$.

Proof. By Lemmas 7 and 8, for any $\delta > 0$ and sufficiently large x and y with $x, y > 0$, we have both

$$\rho(F(x, y)) \leqslant C + \left(a_{23}^{1/3} + \delta\right)(xy)^{1/3}$$

and

$$\rho(G(x, y)) \geqslant b_{23}^{1/3}(xy)^{1/3}.$$

Since $a_{23}^{1/3} < b_{23}^{1/3}$, for $\delta < b_{23}^{1/3} - a_{23}^{1/3}$, we have that the upper bound for $\rho(F(x, y))$ is less than the lower bound for $G(x, y)$ for sufficiently large x and y. The claim of the theorem then follows. $\square$

We then have

Corollary 10. *Let $A = (a_{ij}) \geqslant 0$ and $B = (b_{ij}) \geqslant 0$ be k-by-k. If $A \precsim_{DS} B$, then*

$$a_{ij} \leqslant b_{ij}, \quad \text{for } i \neq j.$$

Proof. Suppose that $a_{ij} > b_{ij}$ for some $i \neq j$. Via permutation similarity, we may suppose that $i = k - 1$ and $j = k$. Embedding A in $F(x, y)$ and B in $G(x, y)$ and choosing x and y sufficiently large that Theorem 9 applies now contradicts the hypothesis that $A \precsim_{DS} B$ and verifies the claim of the corollary. $\square$

4. Proof of Theorem 1 and non-principal submatrices

Theorem 1 now follows from Corollaries 6 (covering diagonal entries) and 10 (covering off-diagonal entries).

We close by making several observations.

First, our proof shows that, in the principal case, when $A \nleq B$ an embedding in a matrix only one larger is necessary to show that $A \not\precsim_{DS} B$.

Second, we may consider extending the partial order $\precsim_{DS}$ to the non-principal case. If there is no limitation on the size of the matrix in which A and B (now not necessarily square) are embedded in a (fixed) non-principal position, then Theorem 1 is easily extended to the non-principal case. First, complete A and B to (minimal) square matrices A' and B', respectively, and then apply Theorem 1 to A' and B'. Unless entry-wise (weak) domination is present, A' and B' can be further embedded to contradict $A' \precsim_{DS} B'$, which means that $A \precsim_{DS} B$ has been contradicted, unless $A \leqslant B$.

Third, if embedding is limited in size in the non-principal case, the story can be different. Consider

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 10 & 0 \\ 10 & 1 & 0 \end{pmatrix}$$

to be embedded in rows 1, 2 and 3 and columns 2, 3 and 4 of a 4-by-4 nonnegative matrix. Because

$$\rho\left(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right) = 4 \quad \text{and} \quad \rho\left(\begin{bmatrix} 1 & 10 \\ 10 & 1 \end{bmatrix}\right) = 11,$$

$\rho(F) \leqslant \rho(G)$, whenever the embedding of A produces F and that of B produces G, and often $\rho(F) < \rho(G)$. This is because F and G are necessarily block-triangular with $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ being a block of F and $\begin{pmatrix} 1 & 10 \\ 10 & 1 \end{pmatrix}$ being a block of G, while the other two 1-by-1 blocks are the same in the two matrices. Of course

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \not\leq \begin{pmatrix} 1 & 10 \\ 10 & 1 \end{pmatrix}.$$

Finally, if the two blocks A and B are to be embedded fully non-principally, only a minimum number of rows and columns (to result in a square matrix, possibly 1) is necessary to reach a contradiction $A \precsim_{DS} B$ when $A \not\leq B$. Use all 0's except for one sufficiently large entry in the symmetrically placed position to an improperly aligned entry. Similarly, for entries not lying in the principal part of non-principal submatrices to be embedded.

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References

[1] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, 1985.
[2] Private communication of author Johnson with José Dias da Silva, March 27, 2011.
[3] J.A. Dias da Silva, P.J. Freitas, Counting spectral radii of matrices with positive entries, April 2012, preprint.