2012

Submatrix monotonicity of the Perron root

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It is known that increasing an entry of a nonnegative matrix non-decreases (and generally increases) its Perron root. Motivated by a question raised by José Dias da Silva, we study the partial order on $k$-by-$k$ nonnegative matrices in which $A \preceq_{DS} B$ if whenever $A$ and $B$ occur as submatrices in the same position in otherwise equal non-negative matrices $F$ and $G$, $\rho(F) \leq \rho(G)$. We find that this partial order is equivalent to the entry-wise partial order. This is proven with some asymptotic results about the Perron root that may be of independent interest.

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a larger Perron root. Otherwise, it may or may not. Examples are easily produced in which a larger
Perron root for the submatrix results in a smaller Perron root for the full matrix (as was also noted in
[3]) in all cases except when the submatrix is 1-by-1 or n-by-n.

Here, our purpose is to give an answer to Dias da Silva’s question by studying the following preorder
on k-by-k nonnegative matrices. We say that \( A \preceq_{DS} B \) ("A is Dias da Silva less than or equal to B") if,
when two n-by-n nonnegative matrices \( F \) and \( G \) are equal, except in a k-by-k principal submatrix, with
\( 0 < k < n \), in which \( F \) is A and \( G \) is B, we necessarily have \( \rho(F) \leq \rho(G) \). Here, as usual, \( \rho \) denotes
the spectral radius or Perron root. We denote the entry-wise (weak) partial order on nonnegative matrices
(domination) by \( A \preceq B \) if \( a_{ij} \leq b_{ij} \) for all \( i, j \) in \( A = (a_{ij}) \) and \( B = (b_{ij}) \). Our main result, surprising to
both Dias da Silva and us, is that the Dias da Silva partial order and the domination partial order are
equivalent.

**Theorem 1.** If \( A \) and \( B \) are k-by-k nonnegative matrices, then \( A \preceq_{DS} B \) if and only if \( A \preceq B \).

Of course, if \( A \preceq B \), then \( A \preceq_{DS} B \) by the facts mentioned earlier (and continuity of the Perron root).
In the remainder of this work, we present the necessary analytical results about asymptotic behaviour
of the Perron root (which may be of independent interest) that are necessary to prove the reverse
implication.

### 2. Domination of diagonal entries

Throughout this section, let

\[
F(x, y) = \begin{pmatrix} A_{11} & a_{12} & 0 \\ a_{21}^T & a_{22} & x \\ 0 & y & 0 \end{pmatrix}, \quad G(x, y) = \begin{pmatrix} B_{11} & b_{12} & 0 \\ b_{21}^T & b_{22} & x \\ 0 & y & 0 \end{pmatrix},
\]

with \( A_{11} \) and \( B_{11} \) \((n - 2)\)-by-\((n - 2)\), \( a_{12}, a_{21}, b_{12} \) and \( b_{21} \) \((n - 2)\)-by-1, and \( a_{22}, x \) and \( y \) scalars. Let

\[
A = \begin{pmatrix} A_{11} & a_{12} \\ a_{21}^T & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & b_{12} \\ b_{21}^T & b_{22} \end{pmatrix}.
\]

Our goal is to first show that for a nonnegative matrix of the form \( F(x, y) \), the Perron root is arbitrarily
approximated by that of

\[
\begin{pmatrix} a_{22} & x \\ y & 0 \end{pmatrix}
\]

for sufficiently large \( x \) and \( y \). Given the explicit value of the latter Perron root, this will mean that
\( \rho(F(x, y)) < \rho(G(x, y)) \) whenever \( a_{22} < b_{22} \) and \( x \) and \( y \) are sufficiently large.

**Lemma 2.** For \( a_{22}, x, y \geq 0 \)

\[
\rho \left( \begin{pmatrix} a_{22} & x \\ y & 0 \end{pmatrix} \right) = \frac{a_{22}}{2} + \sqrt{\left( \frac{a_{22}}{2} \right)^2 + xy}.
\]

If \( y = x + a_{22} \), then

\[
\rho \left( \begin{pmatrix} a_{22} & x \\ y & 0 \end{pmatrix} \right) = a_{22} + x.
\]
**Proof.** The proof is a standard calculation. □

**Lemma 3.** For any \( \epsilon > 0 \), there are numbers \( X, Y > 0 \) such that for all \( x > X \) and \( y > Y \)

\[
\rho(F(x, y)) \leq \epsilon + \left| \frac{a_{22}}{2} \right| + \sqrt{\left( \frac{a_{22}}{2} \right)^2 + xy}.
\]

**Proof.** First, via appropriate positive diagonal similarity of \( A \), we may suppose without loss of generality that the sum of the absolute entries in \( a_{21}^T \) is no more than \( \epsilon \). By another diagonal similarity on \( F(x, y) \) via

\[
\begin{pmatrix}
I & 0 \\
0 & d
\end{pmatrix}
\]

with \( d > 0 \) scalar, we may suppose that both absolute row sums of

\[
\begin{pmatrix}
a_{22} & x \\
y & 0
\end{pmatrix}
\]

are

\[
\left| \frac{a_{22}}{2} \right| + \sqrt{\left( \frac{a_{22}}{2} \right)^2 + |xy|}
\]

(\text{use the Perron vector of} \( \begin{pmatrix} |a_{22}| & |x| \\ |y| & 0 \end{pmatrix} \)). The spectral radius of \( F(x, y) \) has not been changed. Now, for sufficiently large \( x, y \), all absolute row sums of \( F(x, y) \) are no more than

\[
\epsilon + \left| \frac{a_{22}}{2} \right| + \sqrt{\left( \frac{a_{22}}{2} \right)^2 + |xy|},
\]

and the claim follows from the fact that the spectral radius is no more than the maximum 1-norm of the rows [1]. □

We may now prove the main result of this section, that the Perron root of a nonnegative matrix \( F(x, y) \) is eventually (as \( x \) and \( y \) increase) approximated by that of its lower right 2-by-2 submatrix. Of course, \( \rho(F(x, y)) \) must always be at least the latter Perron root.

**Theorem 4.** Suppose that \( F(x, y) \geq 0 \). For any \( \epsilon > 0 \), there are numbers \( X, Y > 0 \) such that for all \( x \geq X \) and \( y \geq Y \),

\[
0 \leq \rho(F(x, y)) - \rho\left( \begin{pmatrix} a_{22} & x \\ y & 0 \end{pmatrix} \right) \leq \epsilon.
\]

**Proof.** This follows directly from Lemmas 2 and 3. □

Now, if we consider nonnegative matrices \( F(x, y) \) and \( G(x, y) \), of the same form, in terms of the relationship between \( a_{22} \) and \( b_{22} \), importantly the spectral radii will eventually (as \( x \) and \( y \) grow) follow the relationship between \( a_{22} \) and \( b_{22} \) irrespective of the relative sizes of the remaining entries \( a_{ij}, b_{ij} \).

**Corollary 5.** Suppose that \( F(x, y), G(x, y) \geq 0 \). If \( a_{22} < b_{22} \), then there are numbers \( X, Y > 0 \) such that for all \( x \geq X \) and \( y \geq Y \), we have

\[
\rho(F(x, y)) < \rho(G(x, y)).
\]
Proof. In view of Lemma 2,
\[ \rho \left( \begin{bmatrix} a_{22} & x \\ y & 0 \end{bmatrix} \right) < \rho \left( \begin{bmatrix} b_{22} & x \\ y & 0 \end{bmatrix} \right). \]
Pick
\[ 0 < \epsilon < \frac{1}{2} \left[ \rho \left( \begin{bmatrix} b_{22} & x \\ y & 0 \end{bmatrix} \right) - \rho \left( \begin{bmatrix} a_{22} & x \\ y & 0 \end{bmatrix} \right) \right] \]
and apply Theorem 4 to verify the claim. \[ \square \]

Applying Corollary 5 to our partial order, we may conclude:

Corollary 6. Let \( A = (a_{ij}) \geq 0 \) and \( B = (b_{ij}) \geq 0 \) be \( k \)-by-\( k \). If \( A \preceq_{DS} B \), then
\[ a_{ii} \leq b_{ii}, \quad i = 1, \ldots, n. \]

Proof. Suppose that \( a_{ii} > b_{ii} \) for some \( i \). By permutation similarity we may suppose without loss of generality that this entry is in the lower right corner. Embedding \( A \) in \( F(x, y) \) and \( B \) in \( G(x, y) \) and choosing \( x \) and \( y \) sufficiently large that Corollary 5 applies now contradicts the hypothesis that \( A \preceq_{DS} B \) and verifies Corollary 6. \[ \square \]

We close this section by noting that a related statement may be proven by similar means. If
\[ H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \geq 0 \]
and is \( n \)-by-\( n \), with \( H_{22} \) square and having positive Perron root, then for any \( \epsilon > 0 \) there is a number \( T > 0 \) such that for all \( t > T \)
\[ 0 \leq \rho \left( \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & t H_{22} \end{bmatrix} \right) - \rho (t H_{22}) \leq \epsilon. \]
This statement, while nicely general, is not sufficiently precise for our needs here.

3. Domination of off-diagonal entries

Throughout this section let
\[ F(x, y) = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & a_{22} & a_{23} & 0 \\ & a_{32} & a_{33} & x \\ & & 0 & y & 0 \end{pmatrix}, \quad G(x, y) = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{21} & b_{22} & b_{23} & 0 \\ & b_{32} & b_{33} & x \\ & & 0 & y & 0 \end{pmatrix}, \]
with \( A_{11}, B_{11} \) \( (n - 3) \)-by-\( (n - 3) \), \( A_{12}, A_{21}, B_{12} \) and \( B_{21} \) \( (n - 3) \)-by-\( 2 \) and \( a_{ij}, b_{ij} \) with \( i, j \in \{2, 3\} \) and \( x, y \) scalars. Let
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & a_{22} & a_{23} \\ A_{32} & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & b_{22} & b_{23} \\ B_{32} & b_{32} & b_{33} \end{pmatrix}.

Our primary goal is to show that \( a_{23} < b_{23} \) if and only if \( \rho(F(x, y)) < \rho(G(x, y)) \) for sufficiently large \( x \) and \( y \), irrespective of the values of other entries besides \( x \) and \( y \). This is difficult to do as explicitly as in the case of diagonal entries, but complements the case of diagonal entries for the purpose of proving our main result.

**Lemma 7.** Assume that \( a_{23} \neq 0 \). For any given \( \delta > 0 \), there is a constant \( C \) and numbers \( X, Y > 0 \) such that

\[
\rho(F(x, y)) \leq C + \left( |a_{23}|^{1/3} + \delta \right) |xy|^{1/3}
\]

for all \( |x| > X \) and \( |y| > Y \).

**Proof.** Via positive diagonal similarity, we may suppose without loss of generality that each of the two absolute row sums of \( A_{21}^T \) is no more than any given \( \delta > 0 \). Now, let

\[
D_A = \begin{pmatrix} |x|^{1/3} |y|^{1/3} |a_{23}|^{-2/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |x|^{2/3} y^{-1/3} a_{23}^{-1/3} \end{pmatrix}
\]

Then diagonal similarity on \( F(x, y) \) via

\[
\begin{pmatrix} I & 0 \\ 0 & D_A \end{pmatrix}
\]

results in a matrix whose last three absolute row sums are no more than

\[
C + \left( |a_{23}|^{1/3} + \delta \right) |xy|^{1/3}
\]

for \( x \) and \( y \) sufficiently large and \( C \) a positive constant. For \( x \) and \( y \) sufficiently large, the remaining absolute row sums will be smaller. Since the maximum absolute row sum is an upper bound for the spectral radius [1], the claim follows. \( \square \)

**Lemma 8.** Let \( G(x, y) \geq 0 \) and suppose that \( b_{23} > 0 \). There are numbers \( X, Y > 0 \) such that

\[
\rho(G(x, y)) \geq b_{23}^{1/3} (xy)^{1/3}
\]

for all \( x > X \) and \( y > Y \).

**Proof.** Following the notation of the prior proof, the nonnegative matrix

\[
D_B = \begin{pmatrix} b_{22} & b_{23} & 0 \\ b_{32} & b_{33} & x \\ y & 0 & 0 \end{pmatrix} D_B^{-1}
\]

results in a matrix whose last three absolute row sums are no more than

\[
C + \left( |b_{23}|^{1/3} + \delta \right) |xy|^{1/3}
\]

for \( x \) and \( y \) sufficiently large and \( C \) a positive constant. For \( x \) and \( y \) sufficiently large, the remaining absolute row sums will be smaller. Since the maximum absolute row sum is an upper bound for the spectral radius [1], the claim follows. \( \square \)
has row sums at least $b_{23}^{1/3}(xy)^{1/3}$, as that value lies in each row as an entry. It follows that
\[
\rho\left(\begin{bmatrix} b_{22} & b_{23} & 0 \\ b_{32} & b_{33} & x \\ y & 0 & 0 \end{bmatrix}\right) \geq b_{23}^{1/3}(xy)^{1/3}
\]
and, as this matrix is a principal submatrix of $G(x, y)$, we have $\rho(G(x, y)) \geq b_{23}^{1/3}(xy)^{1/3}$ as well. □

**Theorem 9.** Suppose that $F(x, y) \geq 0$ and $G(x, y) \geq 0$. If $0 < a_{23} < b_{23}$, then there exist numbers $X, Y > 0$ such that
\[
\rho(F(x, y)) < \rho(G(x, y))
\]
for all $x > X$ and $y > Y$.

**Proof.** By Lemmas 7 and 8, for any $\delta > 0$ and sufficiently large $x$ and $y$ with $x, y > 0$, we have both
\[
\rho(F(x, y)) \leq C + \left(a_{23}^{1/3} + \delta\right)(xy)^{1/3}
\]
and
\[
\rho(G(x, y)) \geq b_{23}^{1/3}(xy)^{1/3}.
\]
Since $a_{23}^{1/3} < b_{23}^{1/3}$, for $\delta < b_{23}^{1/3} - a_{23}^{1/3}$, we have that the upper bound for $\rho(F(x, y))$ is less than the lower bound for $G(x, y)$ for sufficiently large $x$ and $y$. The claim of the theorem then follows. □

We then have

**Corollary 10.** Let $A = (a_{ij}) \geq 0$ and $B = (b_{ij}) \geq 0$ be $k$-by-$k$. If $A \prec_{DS} B$, then
\[
a_{ij} \leq b_{ij}, \quad \text{for } i \neq j.
\]

**Proof.** Suppose that $a_{ij} > b_{ij}$ for some $i \neq j$. Via permutation similarity, we may suppose that $i = k - 1$ and $j = k$. Embedding $A$ in $F(x, y)$ and $B$ in $G(x, y)$ and choosing $x$ and $y$ sufficiently large that Theorem 9 applies now contradicts the hypothesis that $A \prec_{DS} B$ and verifies the claim of the corollary. □

4. **Proof of Theorem 1 and non-principal submatrices**

Theorem 1 now follows from Corollaries 6 (covering diagonal entries) and 10 (covering off-diagonal entries).

We close by making several observations.

First, our proof shows that, in the principal case, when $A \not\preceq_{DS} B$ an embedding in a matrix only one larger is necessary to show that $A \not\prec_{DS} B$.

Second, we may consider extending the partial order $\prec_{DS}$ to the non-principal case. If there is no limitation on the size of the matrix in which $A$ and $B$ (now not necessarily square) are embedded in a (fixed) non-principal position, then Theorem 1 is easily extended to the non-principal case. First, complete $A$ and $B$ to (minimal) square matrices $A'$ and $B'$, respectively, and then apply Theorem 1 to $A'$ and $B'$. Unless entry-wise (weak) domination is present, $A'$ and $B'$ can be further embedded to contradict $A' \not\prec_{DS} B'$, which means that $A \not\prec_{DS} B$ has been contradicted, unless $A \preceq B$. 
Third, if embedding is limited in size in the non-principal case, the story can be different. Consider

\[
A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 10 & 0 \\ 10 & 1 & 0 \end{pmatrix}
\]

to be embedded in rows 1, 2 and 3 and columns 2, 3 and 4 of a 4-by-4 nonnegative matrix. Because

\[
\rho \left( \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \right) = 4 \quad \text{and} \quad \rho \left( \begin{bmatrix} 1 & 10 \\ 10 & 1 \end{bmatrix} \right) = 11,
\]

\[\rho(F) \leq \rho(G),\]

whenever the embedding of \(A\) produces \(F\) and that of \(B\) produces \(G\), and often \(\rho(F) < \rho(G)\). This is because \(F\) and \(G\) are necessarily block-triangular with \(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\) being a block of \(F\) and \(\begin{bmatrix} 1 & 10 \\ 10 & 1 \end{bmatrix}\) being a block of \(G\), while the other two 1-by-1 blocks are the same in the two matrices. Of course

\[
\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \not\leq \begin{bmatrix} 1 & 10 \\ 10 & 1 \end{bmatrix}.
\]

Finally, if the two blocks \(A\) and \(B\) are to be embedded fully non-principally, only a minimum number of rows and columns (to result in a square matrix, possibly 1) is necessary to reach a contradiction \(A \not\leq_{DB} B\) when \(A \not\leq B\). Use all 0’s except for one sufficiently large entry in the symmetrically placed position to an improperly aligned entry. Similarly, for entries not lying in the principal part of non-principal submatrices to be embedded.

**Acknowledgement**

We wish to thank the referee for pointing out [3], which we did not previously know of, to us.

**References**