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Linear colorings of subcubic graphs

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A R T I C L E   I N F O

A B S T R A C T

A linear coloring of a graph is a proper coloring of the vertices of the graph so that each pair of color classes induces a union of disjoint paths. In this paper, we prove that for every connected graph with maximum degree at most three and every assignment of lists of size four to the vertices of the graph, there exists a linear coloring such that the color of each vertex belongs to the list assigned to that vertex and the neighbors of every degree-two vertex receive different colors, unless the graph is C_5 or $K_{3,3}$. This confirms a conjecture raised by Esperet, Montassier and Raspaud [L. Esperet, M. Montassier, and A. Raspaud, Linear choosability of graphs, Discrete Math. 308 (2008) 3938–3950]. Our proof is constructive and yields a linear-time algorithm to find such a coloring.

1. Introduction

A *proper coloring* of a graph is an assignment of colors to the vertices of the graph so that adjacent vertices receive different colors. Graph coloring is an important topic in graph theory and has wide applications in scheduling and partitioning problems.

Yuster (1998, [20]) introduced the notion of *linear coloring*, which is a proper coloring such that each pair of color classes induces a linear forest, where a linear forest is a union of disjoint paths. This notion combines the well-studied *acyclic colorings* (which are proper colorings so that each pair of color classes induces a forest) introduced by Grünbaum (1973, [9]) and the frugal colorings (a proper coloring is *k-frugal* if the subgraph induced by each pair of color classes has maximum degree less than *k*) introduced by Hind, Molloy, and Reed (1997, [10]).

We write lc(*G*) to denote the *linear chromatic number* of *G*, which is the smallest integer *k* such that *G* has a linear coloring with *k* colors. Yuster [20] constructed an infinite family of graphs such

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that $lc(G) \geq C_1 \Delta(G)^{3/2}$, for some constant C_1. He also proved an upper bound of $lc(G) \leq C_2 \Delta(G)^{3/2}$, for some constant C_2 and for sufficiently large $\Delta(G)$.

As most coloring problems, it is hard in general to determine the linear chromatic number. For example, Esperet, Montassier, and Raspaud [8] proved that deciding whether a bipartite subcubic graph is linearly 3-colorable is an NP-complete problem. On the other hand, there are some easy upper and lower bounds on $lc(G)$ for every graph G. Let G be a graph with maximum degree $\Delta(G)$. Then $lc(G) \geq \lceil \Delta(G)/2 \rceil + 1$, since each color can appear on at most two neighbors of a vertex of maximum degree, and $lc(G) \leq \chi(G^2) \leq \Delta(G^2) + 1 \leq \Delta(G)^2 + 1$, where $\chi(G)$ denotes the chromatic number of G, and G^2 is the graph obtained from G by adding edges xy for each pair of vertices x, y with distance two. Li, Wang, and Raspaud [13] improved the easy upper bound to $lc(G) \leq (\Delta(G)^2 + \Delta(G))/2$.

For every family of lists $L = \{L(v) : v \in V(G)\}$ of size k, we say that a proper coloring f is a *proper L-coloring* of G if $f(v) \in L(v)$ for every vertex v of G. General list-coloring was first introduced by Erdös, Rubin, and Taylor [7] and independently by Vizing [18] in the 1970s, and it has been well-explored since then [11]. In particular, the following analog of Brooks' Theorem for list-coloring was proven [7,18], and Skulrattanakulchai [16] gave a linear-time algorithm to find a proper L-coloring when the family of lists $\{L(v) : v \in V(G)\}$ is given.

Theorem 1. *Every connected graph with maximum degree $\Delta(G)$ is $\Delta(G)$-choosable unless G is an odd cycle or a complete graph.*

The list-version of linear coloring was first studied by Esperet, Montassier, and Raspaud [8]. We say that a proper L-coloring of G is *linear* if the subgraph of G induced by each two color classes is a linear forest. A graph G is *linearly k-choosable* if for every family of lists $(L(v) : v \in V(G))$ of size k, the graph G has a linear L-coloring. When all the lists are the same, it is the same as linear k-coloring. We denote by $lc_l(G)$ the smallest k so that G is linearly k-choosable.

Clearly, $lc_l(G) \geq lc(G)$. Substantial work has been done on the study of graphs whose linear (list) chromatic number is close to the easy lower bound $\lceil \Delta/2 \rceil + 1$, see [2,4,5,8,13,15,19]. On the other hand, a little more is known when a graph has small maximum degree. Li, Wang, and Raspaud [13] showed that $lc(G) \leq 8$ if $\Delta(G) \leq 4$ and $lc(G) \leq 14$ if $\Delta(G) \leq 5$. Esperet, Montassier, and Raspaud [8] proved that $lc_l(G) \leq 9$ if $\Delta(G) \leq 4$ and $lc_l(G) \leq 5$ if G is subcubic (i.e., $\Delta(G) \leq 3$). Note that $K_{3,3}$ is not linearly 4-colorable, so their upper bound on subcubic graph is tight, but they conjectured $K_{3,3}$ is the only subcubic graph which is not linearly 4-choosable. In this paper, we confirm this conjecture. As a matter of fact, we prove the following slightly stronger result.

We say that a linear L-coloring is *superlinear* if the neighbors of every vertex of degree two receive different colors. We say that a graph G is *superlinearly k-choosable* if it is linearly k-choosable in such a way that the corresponding linear coloring can be chosen to be superlinear.

Theorem 2. *Let G be a subcubic graph which has no component isomorphic to $K_{3,3}$ or C_5. Then G is superlinearly 4-choosable.*

Note that C_5 is linearly 4-choosable but not superlinearly 4-choosable. However, there is a superlinear L-coloring of C_5 when some vertices u and v have the different lists of colors $L(u)$ and $L(v)$. In addition, our proof of Theorem 2 is constructive and yields a linear-time algorithm to find a superlinear L-coloring when the family of lists $\{L(v) : v \in V(G)\}$ is given. This generalizes an algorithm of Skulrattanakulchai [17] to acyclically color subcubic graphs with four colors.

As an additional remark, coloring of subcubic graphs has been an interesting research subject, see for example [1,3,6,12,14,17].

2. Proof of Theorem 2

For every vertex v, denote the degree of v by $\deg(v)$, and denote the set of neighbors of v by $N(v)$, and write $N(v) \cup \{v\}$ as $N[v]$. The *order* of a graph is the number of its vertices.

Theorem 2 holds for graphs of order at most 4 since we can color each vertex by a different color. We say that G is a *minimum counterexample* if it is a subcubic graph without $K_{3,3}$ or C_5 as components

and there is a family L of lists of size 4 such that G has no superlinear L-coloring, but every subcubic graph with fewer vertices than G and with no component isomorphic to $K_{3,3}$ or C_5 is superlinearly 4-choosable. Thus every minimum counterexample has at least five vertices.

In the rest of this section, we assume that G is a minimum counterexample and L is a family of lists of size four such that G has no superlinear L-coloring.

Lemma 3. *G is a connected graph of minimum degree at least two, and G is not a cycle.*

Proof. If G is not connected, then G has a component that is not superlinearly 4-choosable, contradicting the assumption that G is a minimum counterexample. Suppose that G contains a vertex v of degree one, and let u be the neighbor of v in G. Then $G - v$ is either C_5 or a graph that has a superlinear L-coloring f. For the latter, we can extend f to G by coloring v with a color different from $f(u)$ and the colors on the neighbors of u. For the former and the case that the C_5 does not have a superlinear L-coloring, we can first define f on the C_5 such that the two neighbors of u receive the same color, but all other vertices in C_5 receive the different colors, and then define $f(v)$ to be the color that is different from $f(u)$ and the colors of neighbors of u. So G has minimum degree at least two.

If G is a cycle, then G has at least six vertices, and G^2 has maximum degree four but is not K_5, where G^2 is the graph that is obtained from G by adding edges uv for every pair of vertices u and v with distance two. So, G^2 is 4-choosable by Theorem 1. Since every proper L-coloring of G^2 is a superlinear L-coloring of G, we have that G is superlinearly 4-choosable. $\square$

Lemma 4. *G contains no adjacent vertices of degree two.*

Proof. Suppose that G contains adjacent vertices of degree two. Let $P = v_1 v_2 \cdots v_k$ be a maximal induced path of order at least two in G such that $\deg(v_i) = 2$ for $1 \leq i \leq k$. Let u_1 and u_k be the neighbor of v_1 and v_k other than v_2 and v_{k-1}, respectively. As P is maximal, $\deg(u_1) = \deg(u_k) = 3$. We claim that $G - P$ is the disjoint union of two 5-cycles.

Note that $G - P$ has no component isomorphic to $K_{3,3}$ as G does not. Suppose that $G - P$ contains at most one component isomorphic to C_5. Furthermore, we may assume that if $G - P$ has such a component, then it contains u_1. By induction, there is an L-coloring f defined on $G - P$ that is superlinear, except that the two neighbors of u_1 may receive the same color. We define v_0 to be u_1. Define $f(v_1)$ to be a color in $L(v_1)$ but different from $f(u_1)$ and the f-values of the other neighbors of u_1, and then define $f(v_i)$ to be a color in $L(v_i) - \{f(v_{i-2}), f(v_{i-1}), f(u_k)\}$ for $2 \leq i \leq k$. It is clear that the neighbors of every vertex of degree two have different colors, so no 2-colored cycle passes vertices in P, and hence there exists no 2-colored cycle. In addition, note that u_k has degree three, and u_k is adjacent to two vertices unless $u_k = u_1$, so it is clear that no vertex is adjacent to three vertices that have the same color. That is, f is a superlinear L-coloring of G, contradicting the assumption that G is a counterexample. Therefore, $G - P$ contains two components, and each of them is a 5-cycle.

As a result, G can be obtained from two disjoint 5-cycles by adding a path connecting them. However, there exists a maximal path P' of order four in a 5-cycle such that every vertex of P' is of degree two, but $G - P'$ does not contain two components, contradicting the claim. This proves that G contains no adjacent vertices of degree two. $\square$

We say that an induced path $P = v_1 v_2 \cdots v_k$ in G is *special* if $k \geq 3$, and $\deg(v_1) = \deg(v_k) = 2$ and $\deg(v_i) = 3$ for $2 \leq i \leq k - 1$, and the neighbor of v_1 (and v_k, respectively) outside P, denoted by u_1 (and u_k, respectively) has degree three. For $2 \leq i \leq k - 1$, consider the neighbor of v_i other than v_{i-1} and v_{i+1}. We denote the said neighbor by u_i if it has degree three; otherwise, we denote it by x_i, and we denote the neighbor of x_i other than v_i by u_i. For the former case, x_i is undefined. Note that when $x_i = x_j$ for some $2 \leq i < j \leq k - 1$, then $u_i = v_j$ and $u_j = v_i$.

Lemma 5. *Let $P = v_1 v_2 \cdots v_k$ be a special path in G, and define x_i and u_i as in the last paragraph. Let Q be the subgraph of G induced by $V(P) \cup \{x_i : 2 \leq i \leq k-1 \text{ and } x_i \text{ is defined}\}$. If $x_i \neq x_j$ for $2 \leq i < j \leq k-1$, and u_1 is not adjacent to three pairwise nonadjacent vertices of Q, and no u_i is adjacent to two nonadjacent vertices of Q unless $u_i = u_1$, then there is a component of $G - Q$ isomorphic to C_5 containing u_j for some $2 \leq j \leq k$ but not containing u_1.*

Proof. Note that $G - Q$ has no component isomorphic to $K_{3,3}$ as G does not. We may assume that $G - Q$ has at most one component isomorphic to C_5, for otherwise the lemma holds. Furthermore, we may assume that if $G - Q$ has such a component, then it contains u_1. By induction, there is an L-coloring f defined on $G - Q$ that is superlinear, except that the two neighbors of u_1 may receive the same color. Note that every two vertices in $Q - P$ have degree two, so they are not adjacent, and $\deg(u_i) = 3$ for $1 \le i \le k$, by Lemma 4.

Define $f(v_1)$ to be a color in $L(v_1)$ but different from $f(u_1)$ and the f-values of the other neighbors of u_1, and then define $f(v_i)$ to be a color in $L(v_i) - \{f(v_{i-1}), f(u_{i-1}), f(u_i)\}$ for $2 \le i \le k - 2$, where $f(v_{k-2})$ is chosen such that $L(v_{k-1}) - \{f(v_{k-2}), f(u_{k-2}), f(u_{k-1}), f(u_k)\}$ is as large as possible. If $f(v_{k-2}), f(u_{k-1}), f(u_k)$ are pairwise distinct, then we define $f(v_{k-1})$ to be a color in $L(v_{k-1}) - \{f(v_{k-2}), f(u_{k-1}), f(u_k)\}$; otherwise, we define $f(v_{k-1})$ to be $L(v_{k-1}) - \{f(v_{k-2}), f(u_{k-2}), f(u_{k-1}), f(u_k)\}$. And finally define $f(v_k)$ to be a color in $L(v_k) - \{f(v_{k-2}), f(v_{k-1}), f(u_k)\}$, and define $f(x_i)$ to be a color in $L(x_i) - \{f(v_{i-1}), f(v_i), f(u_i)\}$ for those i such that x_i exists.

It is clear that the neighbors of every vertex of degree two have different colors, so no two-colored cycle passes vertex of degree two. Suppose there is a two-colored cycle, then it must pass some vertex v_i for $2 \le i \le k - 1$. Let t be the minimum number such that v_t is in a 2-colored cycle, so u_t is adjacent to v_t, and u_t, v_{t+1} are also in the 2-colored cycle, and $t \le k - 2$. However, it is impossible since $f(v_{t+1}) \ne f(u_t)$ when $t + 1 \le k - 2$. Also, if $t = k - 2$, then u_{k-1} is adjacent to v_{k-1}, and u_{k-1} is also in the 2-colored cycle. However, it is still impossible since either $f(v_{k-1}) \ne f(u_{k-2})$ or $f(v_{k-2}) \ne f(u_{k-1})$. Hence there are no 2-colored cycles.

In addition, it is clear that no vertex other than v_{k-2} is adjacent to three vertices that have the same color. Suppose that v_{k-2} is adjacent to three vertices that have the same color, then $f(v_{k-3}) = f(u_{k-2})$ and the three colors $f(v_{k-2}), f(u_{k-1}), f(u_k)$ are pairwise distinct. This implies that $L(v_{k-2}) - \{f(v_{k-3}), f(u_{k-3}), f(u_{k-2})\}$ has size two, and $L(v_{k-1}) = \{f(u_{k-2}), f(v_{k-2}), f(u_{k-1}), f(u_k)\}$. Hence, if we choose the other color in $L(v_i) - \{f(v_{i-1}), f(u_{i-1}), f(u_i)\}$ when we color v_{k-2}, $L(v_{k-1}) - \{f(v_{k-2}), f(u_{k-2}), f(u_{k-1}), f(u_k)\}$ is larger, a contradiction. Consequently, f is a superlinear L-coloring of G. □

Lemma 6. Let $Q = v_1 v_2 v_3 v_4$ be an induced path in G with $\deg(v_1) = \deg(v_4) = 2$, and let u_1 and u_4 be the neighbor of v_1 and v_4 outside Q, respectively. If u_1 is adjacent to v_3, then u_4 is not adjacent to v_2.

Proof. Suppose that u_1 is adjacent to v_3, and u_2 is adjacent to v_4. Since v_1 and v_4 have degree two, u_1 and u_4 are of degree three by Lemma 4, so $G - Q$ does not contain a component isomorphic to C_5 or $K_{3,3}$. Therefore, there exists a superlinear L-coloring of $G - Q$. For $i = 2, 3$, define $f(v_i)$ to be a color in $L(v_i) - \{f(u_1), f(u_4)\}$ such that $f(v_2) \ne f(v_3)$. And then for $i = 1, 4$, define $f(v_i)$ to be a color in $L(v_i) - \{f(u_i), f(v_2), f(v_3)\}$. It is clear that f is a superlinear L-coloring of G, a contradiction. □

Lemma 7. Let $P = v_1 v_2 v_3 v_4 v_5$ be an induced path in G with $\deg(v_1) = \deg(v_5) = 2$, and let u_i be the neighbor of v_i outside P for $i = 1, 3, 5$. Let w_3 be a common neighbor of v_2 and v_4 other than v_3, and let x be the neighbor of w_3 other than v_2 and v_4. Let Q be the subgraph of G induced by $V(P) \cup \{w_3\}$. If $\deg(x) = \deg(u_3) = 3$, and $w_3 \notin \{u_1, u_5\}$, then there exists a component of $G - Q$ isomorphic to C_5 not containing u_1 or u_5.

Proof. Suppose that either $G - Q$ contains no component isomorphic to C_5, or each component of $G - Q$ isomorphic to C_5 contains u_1 or u_5. By induction, there is an L-coloring f defined on $G - Q$ that is superlinear, except that the colored neighbors of u_1 may receive the same color, and the colored neighbors of u_5 may receive the same color.

We shall consider two cases. The first case is that $(L(v_2) - \{f(u_1), f(u_3)\}) \cap (L(v_4) - \{f(u_5), f(x)\}) \ne \emptyset$. In this case, define $f(v_2) = f(v_4)$ to be a color in $(L(v_2) - \{f(u_1), f(u_3)\}) \cap (L(v_4) - \{f(u_5), f(x)\})$. And then define $f(v_1)$ to be a color in $L(v_1) - \{f(u_1), f(v_2)\}$ but different from the f-value of some neighbor of u_1, and define $f(v_5)$ to be a color in $L(v_5) - \{f(v_4), f(u_5)\}$ but different from the f-value of some neighbor of u_5. Finally, define $f(v_3)$ to be a color in $L(v_3) - \{f(v_2), f(v_3)\}$ but different from the f-value of some neighbor of u_3, and define $f(w_3)$ to be a color $L(w_3) - \{f(v_2), f(v_3), f(x)\}$.

The second case is that $(L(v_2) - \{f(u_1), f(u_3)\}) \cap (L(v_4) - \{f(u_5), f(x)\}) = \emptyset$. Define $f(v_3)$ to be a color in $L(v_3) - \{f(u_3)\}$ but different from the f-value of some neighbor of u_3. Since $(L(v_2) -$

$\{f(u_1), f(u_3)\}) \cap (L(v_4) - \{f(u_5), f(x)\}) = \emptyset$, without loss of generality, we may assume that $f(v_3)$ is not in $f(v_4) - \{f(u_5), f(x)\}$. Then, define $f(v_2)$ to be a color in $L(v_2) - \{f(u_1), f(u_3), f(v_3)\}$, and define $f(v_1)$ to be a color in $L(v_1) - \{f(u_1), f(v_2)\}$ but different from an f-value of some neighbor of u_1, and define $f(w_3)$ to be a color in $L(w_3) - \{f(v_2), f(v_3), f(x)\}$. Finally, define $f(v_4)$ to be a color in $L(v_4) - \{f(w_3), f(u_5), f(x)\}$, and define $f(v_5)$ to be a color in $L(v_5) - \{f(v_4), f(u_5)\}$ but different from an f-value of some neighbor of u_5. It is clear that f is a superlinear L-coloring in the both cases, a contradiction, so $G - Q$ contains a component isomorphic to C_5 not containing u_1 or u_5. $\square$

Lemma 8. *The distance between any two vertices of G of degree two is at least five.*

Proof. Let a and b be two different vertices of degree two such that the distance between a and b is as small as possible. Let $P = v_1 v_2 \cdots v_{t+1}$ be a shortest path from a to b, where $a = v_1$, $b = v_{t+1}$ and t is the length of P. Since there are no adjacent vertices of degree two by Lemma 4, P is a special path, and we define vertices u_i, x_i and graph Q as in Lemma 5.

Suppose that $t = 2$. Since G contains at least five vertices, $G = K_{2,3}$ if u_1 or u_k is adjacent to three vertices of Q. However, it is impossible since $K_{2,3}$ is superlinearly 4-choosable: assigning two different colors c_1, c_2 to the two vertices of degree three in $K_{2,3}$, and assigning colors different from c_1, c_2 to the three vertices of degree two such that these three vertices do not receive the same color. On the other hand, if u_i is adjacent to two nonadjacent vertices of Q for some $i \geq 2$, then $u_1 \neq u_2 = u_3$, so we may assume that u_i is not adjacent to two nonadjacent vertices of Q unless $u_i = u_1$, by swapping v_1 and v_3. By Lemma 5, $G - Q$ contains a 5-cycle as a component which does not contain u_1, so there are at most two edges with one end in Q and one end in the component. This implies that two vertices of degree two in the 5-cycle are adjacent, a contradiction.

Hence, $t \geq 3$, and x_i does not exist for $2 \leq i \leq t$ as v_1, v_{t+1} is the closest pair of vertices of degree two. Similarly, $u_i = u_j$ implies that $|j - i| \leq 2$, so no u_i is adjacent to three pairwise nonadjacent vertices of Q.

If $t = 3$, then by Lemma 6 and by symmetry, we may assume that u_i is adjacent to two nonadjacent vertices of Q only when $u_i = u_1$. Therefore, by Lemma 5, there exists a component of $G - Q$ isomorphic to C_5 but not containing u_1. So there are at most three edges with one end in Q and one end in the 5-cycle, and hence there exist two vertices of degree two with distance 2, a contradiction. Similarly, when $t = 4$, $u_2 \neq u_4$ or by Lemma 7, $G - Q$ contains a 5-cycle C as a component which does not contain u_1, so C contains a vertex v of degree two in G. If v_5 is adjacent to some vertex of C, then the distance between v and v_5 is smaller than t, a contradiction. If v_5 is not adjacent to any vertex of C, then there are only three edges with one end in Q and one end in C, so C contains two vertices of degree two between distance less than t, a contradiction. This proves that $t \geq 5$. $\square$

Lemma 9. *G contains no triangle as a subgraph.*

Proof. Let $C = v_1 v_2 v_3 v_1$ be a triangle in G. By Lemma 8, there is at most one v_i of degree two, so we may assume that $3 = \deg(v_1) = \deg(v_2) \geq \deg(v_3)$. For $i = 1, 2, 3$, let x_i be the neighbor of v_i other than vertices in C such that $\deg(x_i) = 2$ if such vertex exists. Let $Q = \{v_i, x_i : 1 \leq i \leq 3\}$ (we only consider those x_i which are defined). Let u_i be the neighbor of x_i other than vertices in Q if x_i is defined, and let u_i be the neighbor of v_i other than vertices in Q if x_i is not defined and $\deg(v_i) = 3$. Note that $G - Q$ contains no $K_{3,3}$ as a component. Since the distance of any two vertices of degree two is at least five, and the number of edges which have one end in Q and the other end in $G - Q$ is at most three, $G - Q$ contains no 5-cycle as a component. Hence, we can apply induction to $G - Q$ to obtain a superlinear L-coloring f of $G - Q$. Now, we shall extend f to a superlinear L-coloring of G.

Note that we ignore $f(u_3)$ in the following sentence if u_3 is not defined. Define $f(v_1)$ to be a color in $L(v_1) - \{f(u_1), f(u_2), f(u_3)\}$, $f(v_2)$ to be a color in $L(v_2) - \{f(v_1), f(u_2), f(u_3)\}$, and $f(v_3)$ to be a color in $L(v_3) - \{f(v_1), f(v_2), f(u_3)\}$, and then define $f(x_i)$ to be a color in $L(x_i) - \{f(v_i), f(u_i)\}$ for those i such that x_i are defined. It is clear that the neighbors of any vertex of degree two receive different colors, so no 2-colored cycles pass through a vertex of degree two. If there is a 2-colored cycle, then it must contain the path $u_i v_i v_j u_j$ for some $i < j$, and u_i is adjacent to v_i, and u_j is adjacent to v_j, but that is impossible since $f(v_i) \neq f(u_j)$. Therefore, there are no 2-colored cycles. In addition, as the neighbors of u_i (other than v_i) have different colors, it is clear that no vertex is adjacent to three vertices that have the same color, so f is a superlinear L-coloring of G. $\square$

Lemma 10. *No 4-cycle in G contains a vertex of degree two.*

Proof. Suppose that G contains a 4-cycle $Q = v_1v_2v_3v_4v_1$ with $\deg(v_1) = 2$. Let u_i be the neighbor of v_i outside C for $2 \leq i \leq 4$. By Lemma 8, $\deg(v_i) = \deg(u_i) = 3$ for $2 \leq i \leq 4$, and $G - Q$ does not contain $K_{3,3}$ or C_5 as a component. Furthermore, if some vertex outside Q is adjacent to two vertices of Q, then $u_2 = u_4$ is the only such vertex by Lemma 9. By induction, $G - Q$ has a superlinear L-coloring f. Define $f(v_3)$ to be a color in $L(v_3) - \{f(u_2), f(u_3), f(u_4)\}$, $f(v_2)$ to be a color in $L(v_2) - \{f(u_2), f(v_3)\}$ but different from the f-value of a colored neighbor of u_2, and define $f(v_4)$ to be a color in $L(v_4) - \{f(v_2), f(v_3), f(u_4)\}$, $f(v_1)$ to be a color in $L(v_1) - \{f(v_2), f(v_3), f(v_4)\}$. Clearly, f is a superlinear L-coloring of G, a contradiction. □

Lemma 11. *No vertex of degree two in G is adjacent to a 4-cycle.*

Proof. Suppose that there exist a 4-cycle $C = v_1v_2v_3v_4v_1$ in G and a vertex v of degree two adjacent to v_1. Let u_1 be the neighbor of v other than v_1, and u_i the neighbor of v_i outside C for $2 \leq i \leq 4$. Let Q be the subgraph of G induced by $\{v, v_i : 1 \leq i \leq 4\}$. Note that $v \neq u_3$ by Lemma 10. By Lemma 8, $\deg(u_i) = 3$ for $1 \leq i \leq 4$, and $G - Q$ contains no component isomorphic to $K_{3,3}$ or C_5. Furthermore, Lemmas 9 and 10 ensure that $\{u_1, u_3\} \cap \{u_2, u_4\} = \emptyset$. By induction, $G - Q$ has a superlinear L-coloring f. Define $f(v_3)$ to be a color in $L(v_3) - \{f(u_2), f(u_3), f(u_4)\}$, $f(v_2)$ to be a color in $L(v_2) - \{f(u_2), f(v_3)\}$ but different from the f-value of a neighbor of u_2, $f(v_4)$ to be a color in $L(v_4) - \{f(v_2), f(v_3), f(u_4)\}$, $f(v_1)$ to be a color in $L(v_1) - \{f(u_1), f(v_2), f(v_4)\}$, and define $f(v)$ to be a color in $L(v) - \{f(u_1), f(v_1)\}$ but different from the f-value of a neighbor of u_1. Clearly, f is a superlinear L-coloring of G, a contradiction. □

Lemma 12. *No 5-cycle in G contains a vertex of degree two.*

Proof. Suppose that there exist a 5-cycle $Q = v_1v_2v_3v_4v_5v_1$ in G with $\deg(v_1) = 2$. Let u_i be the neighbor of v_i outside Q for $2 \leq i \leq 5$. By Lemma 8, $\deg(v_i) = \deg(u_i) = 3$ for $2 \leq i \leq 5$, and $G - Q$ contains no component isomorphic to $K_{3,3}$ or C_5. Furthermore, Lemmas 9–11 ensure that u_i are pairwise distinct for $2 \leq i \leq 5$. By induction, $G - Q$ has a superlinear L-coloring f. Define $f(v_3)$ to be a color in $L(v_3) - \{f(u_2), f(u_3), f(u_4)\}$, $f(v_4)$ to be a color in $L(v_4) - \{f(v_3), f(u_4), f(u_5)\}$, $f(v_2)$ to be a color in $L(v_2) - \{f(u_2), f(v_3), f(v_4)\}$, $f(v_5)$ to be a color in $L(v_5) - \{f(v_2), f(v_4), f(u_5)\}$, and define $f(v_1)$ to be a color in $L(v_1) - \{f(v_2), f(v_5)\}$. Clearly, f is a superlinear L-coloring of G, a contradiction. □

Lemma 13. *G is cubic.*

Proof. Let $P = v_1v_2v_3v_4$ be a path in G, where the degree of v_4 is two in G. Since the distance between any two vertices of degree two is at least five, v_i and its neighbor that is not in P are of degree three for $1 \leq i \leq 4$. Furthermore, P is an induced path, and the neighbors of each v_i are pairwise distinct by Lemmas 9–12. For each $1 \leq i \leq 4$, let u_i be a neighbor of v_i other than vertices in P, and let w be the neighbor of v_1 other than u_1 and v_2. Note that the number of edges which have one end in P and the other end in $G - P$ is at most five, and the distance of any two vertices of degree two is at least five, so $G - P$ does not contain C_5 as a component when there are at most four edges between P and $G - P$. On the other hand, if there are exactly five edges between P and $G - P$ such that $G - P$ contains a 5-cycle as a component, then G contains exactly 9 vertices, and v_1 together with three vertices of the 5-cycle induce a 4-cycle by Lemma 9, so there is a path P' having v_4 as an end such that P' is disjoint with the 4-cycle, and hence $G - P'$ has no component isomorphic to C_5, and we replace P by P'. Therefore, we may assume that $G - P$ does not contain C_5 as a component. In addition, $G - P$ does not contain $K_{3,3}$ as a component. Hence, we can apply induction to $G - P$ to obtain a superlinear L-coloring f of $G - P$. And we shall extend f to a superlinear L-coloring of G.

Suppose that $f(u_1) \neq f(w)$. Define $f(v_1)$ to be a color in $L(v_1) - \{f(u_1), f(w), f(u_2)\}$, $f(v_2)$ to be a color in $L(v_2) - \{f(v_1), f(u_2), f(u_3)\}$, $f(v_3)$ to be a color in $L(v_3) - \{f(v_2), f(u_3), f(u_4)\}$, and define $f(v_4)$ to be a color in $L(v_4) - \{f(u_3), f(v_3), f(u_4)\}$. Note that neighbors of vertices of degree two receive different colors, so no 2-colored cycle passes v_4. If there is a 2-colored cycle, then it must pass through v_i, v_j, u_j for some $i < j < 4$ and v_k for all k with $i \leq k \leq j$ since $f(u_1) \neq f(w)$, but it is a contradiction since $f(v_{j-1}) \neq f(u_j)$. Therefore, there are no 2-colored cycles. And it is clear that no

vertex is adjacent to three vertices that receive the same color, so f is a superlinear L-coloring and G cannot be a counterexample, a contradiction. So, let $f(u_1) = f(w)$.

Define $f(v_1)$ to be a color in $L(v_1) - \{f(z) : z \in N[w] - \{v_1\}\}$. If $f(v_1) = f(u_2)$, then define $f(v_3)$ to be a color in $L(v_3) - \{f(u_2), f(u_3), f(u_4)\}$, $f(v_2)$ to be a color in $L(v_2) - \{f(u_1), f(u_2), f(v_3)\}$, and $f(v_4)$ to be a color in $L(v_4) - \{f(u_3), f(v_3), f(u_4)\}$. If $f(v_1) \neq f(u_2)$, then define $f(v_2)$ to be a color in $L(v_2) - \{f(u_1), f(v_1), f(u_2)\}$, and further define $f(v_3)$ to be a color in $L(v_3) - \{f(u_2), f(v_2), f(u_4)\}$ $(L(v_3) - \{f(v_2), f(u_3), f(u_4)\}$, respectively) when $f(v_2) = f(u_3)$ $(f(v_2) \neq f(u_3)$, respectively), and define $f(v_4)$ to be a color in $L(v_4) - \{f(u_3), f(v_3), f(u_4)\}$. It is clear that neighbors of a vertex of degree two receive different colors. Also, if a 2-colored cycle exists, then it must pass through u_i, v_i, v_j, u_j for some $i < j < 4$ and v_k for $i \leq k \leq j$, but it is impossible since $f(v_2) \neq f(u_1)$ and either $f(v_3) \neq f(u_2)$ or $f(u_3) \neq f(v_2)$. And no vertex is adjacent to three vertices that receive the same color. Hence, f is a superlinear L-coloring of G, contradicting that G is a counterexample. This completes the proof. □

Lemma 14. *G does not contain $K_{2,3}$ as a subgraph.*

Proof. Let H be a subgraph of G isomorphic to $K_{2,3}$. In fact, H is an induced subgraph by Lemma 9. Let $V(H) = \{v_1, v_2, u_1, u_2, u_3\}$ and $E(H) = \{v_i u_j : i = 1, 2, j = 1, 2, 3\}$. Note that v_1 and v_2 are all the common neighbors of u_1, u_2, u_3, since G does not contain $K_{3,3}$ as a component.

Suppose that u_1 and u_2 have a common neighbor v_3 other than v_1 and v_2. Note that $G - (H \cup \{v_3\})$ does not contain $K_{3,3}$ or C_5 as a component by Lemma 8. So we can apply induction to $G - (H \cup \{v_3\})$ to obtain a superlinear L-coloring f of $G - (H \cup \{v_3\})$, and we shall extend f to G. Let x be the neighbor of u_3 outside H, and y the neighbor of v_3 outside H. Define $f(u_3)$ to be a color in $L(u_3) - \{f(v) : v \in N[x] - \{u_3\}\}$, $f(v_3)$ to be a color in $L(v_3) - \{f(v) : v \in N[y] - \{v_3\}\}$, $f(u_2)$ to be a color in $L(u_2) - \{f(u_3), f(v_3), f(y)\}$, $f(v_2)$ to be a color in $L(v_2) - \{f(u_2), f(u_3), f(x)\}$, $f(v_1)$ to be a color in $L(v_1) - \{f(u_2), f(v_2), f(u_3)\}$, and $f(u_1)$ to be a color in $L(u_1) - \{f(v_1), f(v_2), f(v_3)\}$ $(L(u_1) - \{f(v_1), f(v_2), f(v_3), f(u_2)\}$, respectively) when $f(v_1), f(v_2), f(v_3)$ are pairwise distinct (when $f(v_1), f(v_2), f(v_3)$ are not pairwise distinct, respectively). By the choices of colors, we see that no vertex is adjacent to three vertices of the same color and there are no two-colored cycles, so G is not a counterexample. Hence, we may assume that v_1 and v_2 are the only common neighbors of any two of u_1, u_2, u_3 by symmetry. In other words, it is impossible to add an edge to G to make G contain $K_{3,3}$ as a subgraph.

Let a, b, c be the three vertices in $G - H$ adjacent to u_1, u_2, u_3 in G, respectively. Note that $G - H$ does not contain 5-cycles and $K_{3,3}$ as components, so we can apply induction to $G - H$ and obtain a superlinear L-coloring f of $G - H$. Now we extend this coloring to G.

We define $f(u_1), f(u_2)$ so that $f(u_1)$ is different from $f(a)$ and the colors of the two other neighbors of a, and $f(u_2)$ is different from $f(u_1), f(b)$. Now define $f(v_1)$ so that $f(v_1) \in L(v_1) - \{f(u_1), f(u_2), f(c)\}$ and define $f(v_2)$ so that $f(v_2) \in L(v_2) - \{f(v_1), f(u_1), f(u_2)\}$. Finally we define $f(u_3)$ so that $f(u_3) \in L(u_3) - \{f(c), f(v_1), f(v_2)\}$ and if possible, $f(u_3) \neq f(u_2)$.

By induction, the two neighbors of c other than u_3 get different colors, c is not adjacent to three vertices of the same color. Since $f(v_1) \neq f(c)$, u_3 is not adjacent to three vertices of the same color. Since u_1, u_2, v_1, v_2 have distinct colors, none of them is adjacent to three vertices of the same color. By the choices of $f(u_1)$ and $f(u_2)$, none of a and b is adjacent to three vertices of the same color.

Since u_1, u_2, v_1, v_2 have distinct colors, no 2-colored cycle is in H. Because of the choice of $f(u_1)$, no 2-colored cycle contains a and u_1. By the choice of $f(u_3)$, either $f(u_3) \neq f(u_2)$, or $f(c) = f(u_2) \neq f(v_2)$ (and $f(v_1) \neq f(c)$), so no 2-colored cycle contains b and c.

So the extension gives a superlinear L-coloring of G. □

Now we prove a lemma about list-coloring.

Lemma 15. *Let C be a cycle and $L' = \{L'(v) : v \in V(G)\}$ a family of lists such that $|L'(v)| \geq 2$ for every vertex v of C. If either $|L'(w)| \geq 3$ or $L'(u) \neq L'(v)$ for some vertices u, v, w of C, then C has a linear L'-coloring.*

Proof. Let $C = v_1 v_2 \cdots v_k v_1$ and either $|L'(v_k)| \geq 3$ or $L'(v_1) \neq L'(v_k)$. Define an L'-coloring f on C by letting $f(v_1)$ be a color in $L'(v_1) - L'(v_k)$ if possible, and define $f(v_i)$ to be a color in $L'(v_i) - \{f(v_{i-1})\}$

for $2 \leq i \leq k - 1$, and define $f(v_k)$ to be a color in $L'(v_k) - \{f(v_1), f(v_{k-1})\}$. Notice that f is a proper L'-coloring since $L'(v_k) - \{f(v_1), f(v_{k-1})\}$ is not empty by the choice of $f(v_1)$. If C is 2-colored, then $f(v_1) = f(v_{k-1})$, and we can redefine $f(v_k)$ to be a color in $L'(v_k) - \{f(v_1) = f(v_{k-1}), f(v_{k-2})\}$ to make f be a linear L'-coloring in this case. $\quad \square$

Lemma 16. *If C is an induced cycle in G such that no vertex is adjacent to at least two vertices in C, then $G - C$ contains C_5 as a component.* $\quad \square$

Proof. Let $C = v_0 v_1 \cdots v_{k-1} v_0$ and u_i be the neighbor of v_i other than vertices in C for $0 \leq i \leq k - 1$. Suppose that $G - C$ does not contain C_5 as a component, so we can apply induction to $G - C$ to obtain a superlinear L-coloring f of $G - C$. Define $L^+(v_i) = L(v_i) - \{f(u_i), f(u_{i+1})\}$ and $L^-(v_i) = L(v_i) - \{f(u_i), f(u_{i-1})\}$ for $0 \leq i \leq k - 1$, where the indices are computed modulo k. Note that if C has a proper L^+-coloring or a proper L^--coloring, then this coloring together with f gives a proper L-coloring of G such that no vertex is adjacent to three vertices of the same color, and the only possible 2-colored cycle is C since no 2-colored path contains some vertices in C as internal vertices. As a result, f can be extended to a superlinear L-coloring if C has a linear L^+-coloring or a linear L^--coloring.

Suppose that C does not have a linear L^+-coloring nor a linear L^--coloring. By Lemma 15, $L^+(v_i) = L^+(v_{i+1})$, $L^-(v_i) = L^-(v_{i+1})$, $|L^+(v_i)| = |L^-(v_i)| = 2$, for every $0 \leq i \leq k - 1$. So $f(u_{i-1}) \neq f(u_i) \neq f(u_{i+1})$ for all $0 \leq i \leq k - 1$, for otherwise $L^+(v_i)$ or $L^-(v_i)$ has size at least three. Furthermore, $L(v_i) = L^+(v_i) \cup \{f(u_i), f(u_{i+1})\}$ and $L(v_{i+1}) = L^+(v_{i+1}) \cup \{f(u_{i+1}), f(u_{i+2})\}$, so $L^-(v_i) = L^+(v_i) \cup \{f(u_{i+1})\} - \{f(u_{i-1})\}$ and $L^-(v_{i+1}) = L^+(v_{i+1}) \cup \{f(u_{i+2})\} - \{f(u_i)\}$ for every $0 \leq i \leq k - 1$. Since $L^-(v_i) = L^-(v_{i+1})$ and $f(u_{i-1}) \neq f(u_i) \neq f(u_{i+1})$, $f(u_{i+1}) = f(u_{i-1})$ for every $0 \leq i \leq k - 1$. If C is odd, then $f(u_1) = f(u_2)$, a contradiction. So C is an even cycle. In addition, every even cycle is 2-choosable, so C must be 2-colored by every proper L^+(and L^-)-coloring of C. Without loss of generality, we may assume that C is colored by color 1 and 2, and $L^+(v_i) = L^-(v_i) = \{1, 2\}$. So $L(v_i) = \{1, 2, f(u_0), f(u_1)\}$ for every $0 \leq i \leq k - 1$.

For any $0 \leq i \leq k - 1$, we can redefine $f(v_i)$ by a color in $L(v_i) - \{f(u_i), f(v_{i-1}) = f(v_{i+1})\}$ and different from the current color, and this alteration will make f be a superlinear L-coloring unless the subpath $x_{i-1} u_{i-1} v_{i-1} v_i v_{i+1} u_{i+1} x_{i+1}$ is contained in a 2-colored cycle, where x_{i-1} and x_{i+1} are neighbors of u_{i-1} and u_{i+1} other than vertices in C, respectively. Hence, one neighbor of u_{i-1} and one neighbor of u_{i+1} other than vertices in C has the same f-value as v_{i-1}, for every $0 \leq i \leq k - 1$. Similarly, we can swap color 1 and 2 on C before we change $f(v_i)$, so one neighbors of u_{i-1} has f-value 1 and one neighbor of u_{i-1} has f-value 2 for every $0 \leq i \leq k - 1$. Consequently, let g be the L-coloring of G such that $g(v) = f(v)$ for $v \in G - C$, and $g(v_0) = f(u_1), g(v_1) = f(u_0), g(v_{2j}) = 1$ and $g(v_{2j+1}) = 2$ for $1 \leq j \leq k/2 - 1$. It is easy to see that g is a superlinear L-coloring of G. $\quad \square$

We are now ready to finish the proof of the theorem.

Proof of Theorem 2. Let G be a minimum counterexample and $C = v_1 v_2 \cdots v_k$ be a shortest cycle in G, so the length of C is at least four by Lemma 9. Suppose that there is a vertex v adjacent to at least two vertices in C. If the shortest cycles have length five or more, then we have a shorter cycle by replacing a segment of C by v. So C has length four, but then $C \cup \{v\}$ induces a $K_{2,3}$, contradicting Lemma 14. Therefore, there is no vertex adjacent to at least two vertices in C, so $G - C$ has a 5-cycle as a component by Lemma 16. On the other hand, the shortness of C implies that C has length at most five, so G can be partitioned into two 5-cycles and G has girth five since G is cubic by Lemma 13. In other words, G is the Petersen graph.

Let L be a family of lists of size four. Note that if $G - C$ has a superlinear L-coloring f, then we can extend f to a superlinear L-coloring of G by Lemma 16. On the other hand, $G - C$ does not have a superlinear L-coloring only if all lists $L(v)$ of vertices v in $G - C$ are the same. This implies that for every 5-cycle C' in G, every vertex in C' has the same list of colors. Hence, every vertex in the Petersen graph has the same list of colors, say $\{1, 2, 3, 4\}$.

We denote the vertex-set of the Petersen graph by $\{u_i, v_i : 1 \leq i \leq 5\}$ such that u_i is adjacent to v_i for $1 \leq i \leq 5$, and $u_1 u_2 \cdots u_5 u_1$ and $v_1 v_3 v_5 v_2 v_4 v_1$ are 5-cycles. Define $f(u_1) = f(u_3) = f(u_5) = 1$, $f(u_2) = f(u_4) = 2, f(v_1) = f(v_2) = f(u_5) = 3$, and $f(v_3) = f(v_4) = 4$. Clearly, f is a superlinear L-coloring of the Petersen graph, so G is not a counterexample, a contradiction. $\quad \square$

3. Linear-time algorithm

In this section, we shall give a linear-time algorithm to find a superlinear L-coloring whenever a family L of lists is given.

For every vertex v of a graph G and for every positive integer k, denote by $N_k(v)$ the set $\{u \in V(G) : 0 < d(u, v) \leq k\}$, where $d(u, v)$ is the distance between u and v, and define $N_k[v] = N_k(v) \cup \{v\}$.

First, we introduce a subroutine to deal with the noncubic case.

Subroutine 1

Input: (G, L, T), where G is a subcubic graph whose every component has minimum degree at most two, $L = \{L(v) : v \in V(G)\}$ is a family of lists of size 4, and T is a list that consists of all vertices of degree at most two in G. Every component of G has minimum degree at most two and does not contain $K_{3,3}$ or C_5 as a component.

Output: A superlinear L-coloring of G.

Running time: $O(|V(G)|)$.

Description: Pick a vertex v from T. Let H be the component of G containing v. Note that v is of degree at most two in G. If the order of H is at most 10, then output the superlinear L-coloring of G by combining a superlinear L-coloring of H found by brute force and the superlinear L-coloring obtained by executing the subroutine with input $(G - H, L, T - V(H))$.

If $\deg(v) \leq 1$, then put the neighbor of v in G to T. Output the superlinear L-coloring of G obtained by extending the superlinear of $G - v$ obtained by executing the subroutine with input $(G - v, L, T - \{v\})$. Note that $G - \{v\}$ does not contain C_5 as a component when $\deg(v) = 1$, since $H - \{v\}$ has order at least 9 and G has no component isomorphic to C_5. So we may assume that every vertex in $N_{12}[v]$ has degree at least two.

If there are two adjacent degree two vertices u, w in $N_{12}[v]$, then let H' be the maximal connected subgraph of H containing u, w induced by degree at most two vertices in H. Note that H' has maximum degree at most two, so H' is a cycle or a path. If H' is a cycle, then $H = H'$, and return the superlinear L-coloring obtained by combining a superlinear L-coloring of H' and the superlinear L-coloring obtained by applying the subroutine with input $(G - H', L, T - V(H'))$. If H' is a path, then let G' be the subgraph obtained from G by deleting H' and all components of $G - H'$ isomorphic to C_5, and let T' be the minimum superset of $T \cap V(G')$ containing $N(H') \cap V(G')$. Apply the subroutine with input (G', L, T') to obtain a superlinear L-coloring of G', and then extend the coloring to a superlinear L-coloring of G as in the proof of Lemma 4. Hence, we assume that no two adjacent degree two vertices u, w in $N_{12}[v]$.

Then, we can find a subgraph Q in $N_6[v]$ defined in the proof of Lemmas 5–7 and 9–12 or 13 such that $G - Q$ does not contain a $K_{3,3}$ or C_5 as a component. Put neighbors of Q in G into T, and remove $V(Q) \cap T$ from T. Output the superlinear L-coloring of G obtained from the superlinear coloring of $G - Q$ obtained by applying the subroutine with input $(G - Q, L, T)$ by extending it as in the proof of the corresponding lemma.

End of description

The correctness of the Subroutine 1 is clear. Note that it is not hard to implement Subroutine 1 such that putting any vertex in G into T or removing any vertex from T can be done in constant time. To see the Subroutine 1 runs in time $O(|V(G)|)$, since G has maximum degree at most three, it is sufficient to show that a superlinear L-coloring of a cycle C with length other than five can be found in time $O(|V(C)|)$. In fact, Skulrattanakulchai [16] proved that a L-coloring of C^2 can be found in $O(|V(C)|)$, and this coloring is a superlinear L-coloring of C.

Second, we introduce a subroutine to find a certain subgraph of a connected cubic graph.

Subroutine 2

Input: A connected cubic graph G of order at least 11.

Output: An induced subgraph H, where H is isomorphic to K_3 or $K_{2,3}$, or H is a cycle such that no vertex outside H is adjacent to at least two vertices in H.

Running time: $O(|V(G)|)$.

Description: Use brute force to find a triangle or a $K_{2,3}$ in G. If we find a triangle or a $K_{2,3}$, then return it; otherwise, G is triangle-free and $K_{2,3}$-free, and we pick a cycle C in G.

Let $C = v_0 v_1 \cdots v_{k-1} v_0$ and let u_i be the neighbor of v_i other than v_{i-1} and v_{i+1}, where the indices are computed modulo k. For $i = 0$ to $k - 1$, consecutively see neighbors of u_i. Whenever some u_i is adjacent to at least two vertices in C, since G is triangle-free and $K_{2,3}$-free, we can replace C by 4-cycle or a shorter cycle obtained from u_i and the path in C containing v_{i+1} whose ends are in $N(u_i) \cap V(C)$. So we can repeat this process until no u_i is adjacent to at least two vertices in C.

End of description

The correctness of Subroutine 2 is clear. We shall show that Subroutine 2 runs in time $O(|V(G)|)$. Since G has maximum degree three, it takes time $O(|V(G)|)$ to use brute force to find a triangle or a $K_{2,3}$. Furthermore, finding a desired cycle mentioned in Subroutine 2 runs in linear-time since every vertex is visited by at most a bounded number of times. So Subroutine 2 runs in linear-time.

Now, we give an algorithm to find a superlinear L-coloring of a connected subcubic graph G.

Algorithm for finding a superlinear L-coloring

Input: A connected subcubic graph G which is not $K_{3,3}$ or C_5, and a family $L = \{L(v) : v \in V(G)\}$ of lists of size 4.

Output: A superlinear L-coloring of G.

Running time: $O(|V(G)|)$.

Description: If the order of G is at most 10, then return a superlinear L-coloring of G found by brute force. So we assume that G has order at least 11.

Use breadth-first-search to check whether G is cubic or not, and obtain the set T consisting all vertices of degree at most two in G. If G is not cubic, then return the L-coloring obtained by applying Subroutine 1 with input (G, L, T). So we assume that G is cubic.

Apply Subroutine 2 to obtain an induced subgraph H, where H isomorphic to K_3 or $K_{2,3}$, or H is a cycle such that no vertex outside H is adjacent to at least two vertices in H. Since G is cubic, $G - H$ contains no $K_{3,3}$ or C_5 as a component if H is isomorphic to K_3 or $K_{2,3}$. If $G - H$ contains C_5 as a component, then H is a cycle of length at least 6 as G has at least 11 vertices, and we replace H by this 5-cycle in this case. Hence, $G - H$ contains no C_5 or $K_{3,3}$ as a component.

If H is a triangle or a $K_{2,3}$, respectively, then define Q as in the proof of Lemma 9 or 14, respectively; if H is a cycle other than a triangle, then define Q to be H. Let T' be the set of vertices of degree two in $G - Q$. Note that T' is a subset of $N(H)$, and $G - Q$ does not contain C_5 or $K_{3,3}$ as a component. Return the superlinear L-coloring of G obtained by extending the superlinear L-coloring obtained from applying Subroutine 1 with input $(G - Q, L, T')$ as the proof in Lemmas 9, 14 and 16.

End of the description

The correctness of the algorithm is clear. Note that we only check whether a graph is cubic or not once, so the algorithm runs in linear-time as Subroutines 1 and 2 do.

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