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Composite Gravity in Curved Spacetime

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Composite Gravity in Curved Spacetime

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Abstract

This work presents the development of a quantum theory of gravity motivated by diffeomorphism-invariance and background-independence. A composite graviton state that satisfies the linearized Einstein's field equations has been identified via perturbative expansion about a curved vacuum spacetime. The emergence of this gravitational interaction is discussed, as well as cancellation of tadpoles and treatment of ultraviolet divergences via dimensional regularization. In other words, the formalism of quantum field theory is used to identify a gravitational interaction as an emergent phenomenon rather than as a fundamental aspect of nature. The lattice is proposed as a candidate for a physical regulator, and future directions of research are discussed.

Chapter 1

Introduction

One of the central questions in modern physics is how to formulate a model that accurately predicts the behavior of both the smallest and largest structures in the universe. Quantum field theory (QFT) is the dominant model in the subatomic regime, having predicted the existence of antiparticles, radiative corrections, and the behavior of fundamental particles such as in the electromagnetic interaction. Einstein's theory of general relativity (GR) is our best description of gravitation and makes exquisitely precise predictions concerning cosmology, the motion of celestial bodies, and gravitational waves. Despite the success of these descriptions individually, they appear at face value to contradict. For example, QFT has been unable to produce a prediction for the cosmological constant that agrees with observation, which is an issue known as the cosmological constant problem [1]. Also, it is difficult to define time evolution in QFT when the Hamiltonian in GR vanishes, and this is called the problem of time [2]. Our goal is to present a theory that uses QFT as a starting point and has consequences consistent with GR in the long-distance limit.

The two primary motivations for our theory are diffeomorphism invariance (also known as general covariance) and independence of spacetime background. A diffeomorphism invariant physical law remains unchanged under any continuous coordinate transformation, and a background-independent theory is independent of non-physical fields (called clocks and rods) that determine the scale of space and time intervals [3]. GR has both of these properties by construction, but QFT was developed with implicit assumptions about the geometry of spacetime. In particular, traditional QFT assumes a flat spacetime background, commonly referred to as Minkowski space. As shown in [4], it is possible to construct a diffeomorphism invariant quantum theory of gravity with a Minkowski space background that agrees with GR, up to quantum corrections. Moreover, it was demonstrated in [5] that the same approach holds when the background deviates slightly from Minkowski space. We further generalize these findings in [6], but for our theory to be fully background-independent, we needed to adopt the formalism of QFT in curved spacetime. Chapter 2 contains the foundational theory required for the uninitiated to follow our argument. Chapter 3 outlines our model and how it results in an emergent gravitational interaction. Chapter 4 begins the discussion of performing calculations when the model is put on a lattice. In Chapter 5, implications of the theory and areas where more work is needed are discussed.

Chapter 2

Prerequisite Theory

Before discussing our particular theory, some of the notational and conceptual ideas from GR and QFT must be addressed. For example, we take advantage of Einstein's summation notation throughout our work. Objects with indices, such as vectors, can be written with upper indices (v^μ) or lower indices (v_μ). Repeated indices are summed over unless otherwise specified, and this summation can only occur with one index raised and the other lowered. A matrix is a two-index object, usually with one upper index and one lower index so that the familiar multiplication of matrices A and B:

$$AB = \sum_k A_{ik} B_{kj} = C \tag{2.0.1}$$

becomes

$$A^\mu{}_\lambda B^\lambda{}_\nu = C^\mu{}_\nu \tag{2.0.2}$$

where the sum was carried out implicitly over the index λ. A rank-n tensor is an object with n indices and is a generalization of a matrix that transforms covariantly under general coordinate transformations. That is, under the coordinate transformation $x^\alpha \to x'^\alpha = \Lambda^\alpha{}_\beta \, x^\beta$, the object $T^{\alpha\cdots\beta}{}_{\mu\cdots\nu}$ is a tensor if it transforms as [7]:

$$T^{\alpha\cdots\beta}{}_{\mu\cdots\nu} \rightarrow T'^{\alpha\cdots\beta}{}_{\mu\cdots\nu} = (\Lambda^{\alpha}{}_{\lambda}\cdots\Lambda^{\beta}{}_{\kappa})(\Lambda_{\mu}{}^{\rho}\cdots\Lambda_{\nu}{}^{\sigma})T^{\lambda\cdots\kappa}{}_{\rho\cdots\sigma} \tag{2.0.3}$$

where $\Lambda^{\alpha}{}_{\beta}$ is a real unitary operator. For our purposes, unless otherwise specified, indices run from 0 to $D-1$, where D is the number of spacetime dimensions (usually four). Two common special tensors are the Kronecker delta $\delta^{\mu}{}_{\nu}$ and the Minkowski tensor $\eta_{\mu\nu}$ where

$$\delta^{\mu}{}_{\nu} = \begin{cases} 1 \ , \mu = \nu \\ 0 \ , \mu \neq \nu \end{cases} \qquad \eta_{\mu\nu} = \begin{cases} 1 \ , \mu = \nu = 0 \\ -1 \ , \mu = \nu \neq 0 \\ 0 \ , \mu \neq \nu \end{cases} \tag{2.0.4}$$

and the sign of $\eta_{\mu\nu}$ is according to the standard convention in particle physics, which differs from the traditional standard in general relativity. We also work in natural units:

$$\hbar = c = 1 \tag{2.0.5}$$

where $\hbar$ is the reduced Planck's constant, and c is the speed of light. Finally, the partial derivative $\frac{\partial}{\partial x^{\mu}}$ will be denoted ∂_{μ} and has a lowered index.

2.1 General Relativity

The important concepts from GR that are needed to understand this project are the metric tensor, spacetime curvature, the energy-momentum tensor, and Einstein's field equations. The metric tensor $g_{\mu\nu}$ contains all needed information about the geometry of spacetime. The inverse metric $g^{\mu\nu}$ is defined so that $g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}{}_{\nu}$. The raising and lowering of indices on tensors is defined via the metric tensor. For example, if $A^{\alpha}{}_{\beta}$ is a tensor:

$$A_{\alpha\beta} = g_{\alpha\lambda} A^{\lambda}{}_{\beta} \qquad\qquad A^{\alpha\beta} = g^{\beta\lambda} A^{\alpha}{}_{\lambda} \tag{2.1.1}$$

The generally covariant trace of a two-index object $B_{\mu\nu}$ is $B^{\mu}{}_{\mu} = g^{\mu\nu} B_{\mu\nu}$, keeping in mind that repeated indices are summed over.

The metric tensor is far more than notational convenience. The infinitesimal affine parameter ds is given by

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{2.1.2}$$

where the components of x^{μ} are coordinates, and $g_{\mu\nu}$ is in general a function of the coordinates. The spacetime interval ds is preserved under general coordinate transformations, and for a massive object, it is interpreted as the proper time. Therefore, the metric tensor defines the meaning of distances in spacetime. In special relativity, $g_{\mu\nu} = \eta_{\mu\nu}$. This special spacetime is what we call flat space or Minkowski space. A curved spacetime is defined by a different $g_{\mu\nu}$, and the curvature of a spacetime is described by the four-index curvature tensor $R^{\mu}{}_{\nu\alpha\beta}$. The curvature tensor is a function of the metric tensor, and also useful to us are the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R defined as

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \qquad\qquad R = R^{\mu}{}_{\mu} \tag{2.1.3}$$

The Ricci tensor and Ricci scalar are important to this discussion because they appear in Einstein's field equations.

The next piece of this story is the energy-momentum tensor $T^{\mu\nu}$. As the name implies, this object contains information about the density distribution of energy and momentum in space. Notably, the Hamiltonian is given by the integral over all space of T^{00}, where the zero-component of an object with indices is the time component, and the other components are space components. With these objects defined, we can write down Einstein's field equations (in our convention):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{(M_{\mathrm{P}})^2}(g_{\mu\nu}\Lambda + T_{\mu\nu}) \tag{2.1.4}$$

where Λ is the cosmological constant governing the expansion of the universe, and M_{P} is the Planck mass, which determines the strength of the gravitational interaction. Note that the left-hand side of the equation is entirely a function of $g_{\mu\nu}$, so it depends only on the geometry of spacetime. The right-hand side is a function of $g_{\mu\nu}$ and $T_{\mu\nu}$, so it depends on geometry and the distribution of energy and momentum of objects in spacetime. The conceptual insight of this equation is this direct relationship between the geometry of spacetime and the dynamics of objects.

We are interested in using Einstein's field equations when $T_{\mu\nu} = 0$ (that is, in the vacuum) and when the metric can be written in the form $g_{\mu\nu} = G_{\mu\nu} + h_{\mu\nu}$ where $G_{\mu\nu}$ is some background spacetime metric that does not contribute to dynamics, and $h_{\mu\nu}$ is a perturbation of the background that is small relative to $G_{\mu\nu}$. In this regime, indices are raised and lowered with the background metric. A flat space background corresponds to $G_{\mu\nu} = \eta_{\mu\nu}$, which is a common choice, but recall that our goal is background-independence. In GR, $h_{\mu\nu}$ is traditionally interpreted to be a propagating gravitational wave. When Einstein's field equations are written to linear order $h_{\mu\nu}$, they are called the linearized Einstein's field equations, and these are how we check for consistency between our theory and GR.

2.2 Quantum Field Theory

The important concepts from QFT to address before discussing our theory are correlation functions in the functional integral formalism, the free scalar theory, the effective action, and regularization. In moving from non-relativistic quantum mechanics to QFT, wave functions are replaced by field operators. Our model is constructed of scalar fields ϕ, which follow Bose-Einstein statistics. The state $|0\rangle$ represents the vacuum. The expression $\phi(x_1)|0\rangle$ represents the creation of a particle from the vacuum by the field ϕ at the location in spacetime identified by the four-vector x_1. The expression $\langle 0|\phi(x_2)$ represents the annihilation of a particle at x_2. The expression

$$G_F(x_1, x_2) = \langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle = \begin{cases} \langle 0|\phi(x_1)\phi(x_2)|0\rangle \ , {x_1}^0 > {x_2}^0 \\ \langle 0|\phi(x_2)\phi(x_1)|0\rangle \ , {x_2}^0 > {x_1}^0 \end{cases} \tag{2.2.1}$$

represents a particle being created in one location, propagating to the other location, and being annihilated. The time-ordering operator T ensures that the particle is created at the earlier time and annihilated at the later time. The symbol G_F is called the Feynman propagator, and its definition may need to be adjusted depending on the conventions of one's particular formalism (for example, including an extra factor of i). This time-ordered product of fields (which is also a two-point correlation function of the fields $\phi(x_1)$ and $\phi(x_2)$ in the free theory) is a sort of expectation value, and indeed one can derive scattering amplitudes from such objects, but it should be thought of more abstractly as one of the building blocks for analysis in QFT.

In classical mechanics, one derives equations of motion from the Lagrangian using the principle of variation of the action. In QFT, the action also plays a fundamental role. The action S is the integral over all spacetime of the Lagrangian density $\mathcal{L}$:

$$S = \int d^4x \sqrt{|g|}\ \mathcal{L} \tag{2.2.2}$$

where $g = \det(g_{\mu\nu})$. The d^4x indicates that the integral is over all spacetime. By contrast, an integral over only space would have a d^3x. The factor of $\sqrt{|g|}$ is present because the coordinates are general. For a familiar example, in flat Euclidean space spherical coordinates, $d^3x\sqrt{|g|} = dr\, d\theta\, d\phi\; r^2 \sin\theta$.

In our theory, we use the functional integral formalism, which involves integrals over all configurations of the fields. An important object in this regime is the vacuum-to-vacuum expectation value $\langle 0|0\rangle$, given by

$$Z = \langle 0|0\rangle = \int \mathcal{D}\phi\; e^{iS[\phi]} \tag{2.2.3}$$

when the action is a function of the fields ϕ. This object plays the role of a partition function in analogy to statistical mechanics. The mathematical interpretation of the functional integral will be discussed in more detail as it becomes relevant. In this formalism, the correlation function of the product of objects $A^{\mu\nu}$ and $B^{\alpha\beta}$ (which could in general alternatively have any number and configuration of indices) can be written

$$\langle A^{\mu\nu} B^{\alpha\beta}\rangle = \frac{\int \mathcal{D}\phi\; e^{iS[\phi]}\; A^{\mu\nu} B^{\alpha\beta}}{\langle 0|0\rangle} \tag{2.2.4}$$

This notation is particularly useful when we vary the effective action (soon to be defined) with respect to the metric in order to obtain expressions for correlation functions of the energy-momentum tensors of the fields.

Before defining the effective action, we must discuss the free scalar theory, which describes scalar fields ϕ that do not interact. The free theory action in 4-dimensional Minkowski space is given by

$$S_{\text{free}} = \int d^4x\; \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{m^2}{2}\phi^2 \tag{2.2.5}$$

where m is the mass of the field excitations. Through analysis similar to using the Euler-Lagrange equations in classical mechanics, one can derive the Klein-Gordon equation for free scalar fields, effectively their equation of motion:

$$(\Box + m^2)\phi = 0 \tag{2.2.6}$$

where the d'Alembertian operator $\Box$ is $\partial^\mu \partial_\mu$. In our theory, the free action is stated more generally in terms of the metric and number of dimensions for reasons that will become clear, but now we can define the effective action in the free theory W_{eff} as satisfying

$$e^{iW_{\text{eff}}} = \int \mathcal{D}\phi \; e^{iS_{\text{free}}} \tag{2.2.7}$$

in the functional integral formalism.

What remains is actually calculating these quantities. The challenge is that many of the integrals in QFT are formally divergent. However, they are still physically meaningful, and the process of making the infinities finite is called regularization. We introduce a mathematical maneuver, called a regulator, that separates the divergent parts of a calculation from the finite parts. There are different ways of accomplishing this, but we mainly use dimensional regularization in our approach. We express integrals as functions of the number of spacetime dimensions D, make sure the divergences cancel by fine-tuning the parameters of the theory, then take the limit $D \to 4$ in the end. Dimensional regularization is not typically considered physical, but our theory relies on the existence of a physical regulator, so we are using dimensional regularization as a proxy. We are also interested in the lattice as a candidate for physical regulator, as discussed in Chapter 4. The other important tool for performing calculations is Feynman diagrams, but these will be discussed as they appear.

Chapter 3

The Emergence of Composite Gravity

In our theory, gravitation is not a fundamental aspect of nature, but rather a phenomenon that emerges at long distances from QFT. This idea comes from an observation by the physicist Andrei Sakharov in 1967 that the effective action of a quantum field theory is guaranteed to contain terms that resemble the Lagrangian for GR [8]. More specifically, if we do not quantize gravity *a priori* but rather leave it semiclassical (*i.e.* Einstein's equations couple to the energy-momentum tensors of the fields in the theory), then the effective action in QFT is guaranteed to contain the terms [9]:

$$\int d^4x \sqrt{|g|} \left(c_0 + c_1 R + c_2(\text{``}R^2\text{''}) + \dots \right) \tag{3.0.1}$$

where the factors c_k are constants, and the expression "R^2" refers to terms second order in curvature. When comparing this to the action for gravity, called the Einstein-Hilbert action:

$$\int d^4x \sqrt{|g|} \left(\Lambda + \frac{R}{16\pi G_N} + K(\text{``}R^2\text{''}) + \mathcal{L}_{\text{matter}} \right) \tag{3.0.2}$$

where K is some constant, it appears as though all of GR could emerge from QFT without the metric being quantized from the outset. This observation is by no means conclusive on its own, which is why we needed to work through the subtleties. Sakharov called this phenomenon induced gravity, though it has also been referred to as emergent gravity. We are calling it composite gravity because we can identify the composite graviton state that mediates the interaction. The term composite means that the graviton is not fundamental, but rather constructed from other states.

3.1 Regularizing the Model

Our model is given by the action

$$S = \int d^D x \left(\frac{D/2-1}{V(\phi^a)}\right)^{D/2-1} \sqrt{\left|\det\left(\sum_{a=1}^{N} \partial_\mu \phi^a \partial_\nu \phi^a\right)\right|} \tag{3.1.1}$$

in D spacetime dimensions for a large number N of scalar fields ϕ, where

$$V(\phi^a) = V_0 + \Delta V_{ct} + \Lambda + \sum_{a=1}^{N} \frac{1}{2} m^2 \phi^a \phi^a \tag{3.1.2}$$

with constants V_0, Λ, and ΔV_{ct} tuned to cancel divergences in the theory. The action can be rewritten as

$$S = \int d^D x \sqrt{|g|} \left[\frac{1}{2} g^{\mu\nu} \left(\sum_{a=1}^{N} \partial_\mu \phi^a \partial_\nu \phi^a\right) - V(\phi^a)\right] \tag{3.1.3}$$

when the metric $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \frac{D/2-1}{V(\phi^a)} \left(\sum_{M,N=1}^{D} \partial_\mu X^M \partial_\nu X^N G_{MN} - \sum_{M,N=1}^{D} \partial_\mu X^M \partial_\nu X^N G_{MN} + \sum_{a=1}^{N} \partial_\mu \phi^a \partial_\nu \phi^a \right) \tag{3.1.4}$$

where G_{MN} is an auxiliary background metric that depends on the clock and rod fields X^M. Note that the terms involving G_{MN} are identically cancelled in this expression, which preserves the diffeomorphism invariance of the theory because the clock and rod fields do not contribute to the action. One of these terms allows us to organize perturbation theory about a vacuum spacetime metric, and the other cancels divergences. The generality of G_{MN} gives us background-independence. By contrast, previous works set G_{MN} to η_{MN} or a small perturbation of η_{MN}. We choose the basis for X^M to satisfy

$$X^M = \sqrt{\frac{V_0 + \Lambda}{D/2 - 1}} \delta_\mu{}^M x^\mu \tag{3.1.5}$$

and we define the vacuum spacetime metric

$$G_{\mu\nu} \equiv \frac{D/2 - 1}{V_0 + \Lambda} \sum_{M,N=1}^{D} \partial_\mu X^M \partial_\nu X^N G_{MN} \tag{3.1.6}$$

with the clock and rod fields serving as a coordinate basis for $G_{\mu\nu}$. The partition function (in accordance with Eq. (2.2.3)) is

$$Z = \int \mathcal{D}\phi^a \int \mathcal{D}X^M e^{iS} \delta\left(X^M - \sqrt{\frac{V_0 + \Lambda}{D/2 - 1}} \delta_\mu{}^M x^\mu \right) \tag{3.1.7}$$

so that the regularized $\langle g_{\mu\nu} \rangle$ is $G_{MN} \delta_\mu{}^M \delta_\nu{}^N$.

The next step is tuning the parameters of the theory to cancel divergences. These poles occur because of correlation functions of products of operators that are evaluated at the same spacetime point $\langle \phi(x)\phi(x) \rangle$ and $\langle \partial_\mu \phi(x) \partial_\nu \phi(x) \rangle$. These correlation functions, if left unattended, would lead to instability of the vacuum (creation of particles from nothing) from so-called tadpole diagrams. We are aided in calculation by the effective action in the free theory W_{eff}, as defined in Eq. (2.2.7). In our case, the free theory action is given by

$$S_{\text{free}} = \int d^D x \sqrt{|g|} \left[\frac{1}{2} g^{\mu\nu} \left(\sum_{a=1}^{N} \partial_\mu \phi^a \partial_\nu \phi^a \right) - \frac{m^2}{2} \sum_{a=1}^{N} \phi^a \phi^a \right] \tag{3.1.8}$$

which deviates from Eq. (2.2.5) to allow for a different number of spacetime dimensions D, a general metric, and sums over fields. We use the free field action in this case because it is the energy-momentum tensor of the free fields:

$$\mathcal{T}_{\mu\nu} = \sum_{a=1}^{N} \left(\partial_\mu \phi^a \partial_\nu \phi^a - \frac{1}{2} g_{\mu\nu} \left(\partial^\alpha \phi^a \partial_\alpha \phi^a - m^2 \phi^a \phi^a \right) \right) \tag{3.1.9}$$

that couples to the metric in the gravitational interaction. The effective action is dominated by divergent terms W_{div}, and using curved spacetime QFT with dimensional regularization, we confirmed that

$$W_{\text{div}} = N \int d^D x \sqrt{|g|} \left\{ -\frac{1}{(4\pi)^{D/2}} \frac{1}{D-4} \left[\frac{4m^D}{D(D-2)} - \frac{m^{D-2} R}{3(D-2)} + \mathcal{O}(R^2) \right] \right\} \tag{3.1.10}$$

by taking advantage of the fact that the propagator G_F can be expanded as a series in R, in agreement with [10, 11]. In [6], we have a more detailed discussion of the $\mathcal{O}(R^2)$ terms, but we dropped these in the results for simplicity of presentation.

We can compute correlation functions by varying the effective action with respect to various parameters. We found that the tadpoles are given by

$$\begin{aligned} \langle \phi(x)\phi(x) \rangle &= -\frac{2}{\sqrt{|g|}} \frac{\delta W_{\text{eff}}}{\delta(m^2)} = \frac{1}{m^2} \langle \mathcal{T}^\mu_{\ \mu} \rangle \\ &= \frac{1}{(4\pi)^{D/2}} \frac{N}{D-4} \frac{4m^{D-4}}{(D-2)} \left[m^2 - \frac{D-2}{12} R \right] \end{aligned} \tag{3.1.11}$$

$$\begin{aligned} \langle \partial_\mu \phi(x) \partial_\nu \phi(x) \rangle &= \langle \mathcal{T}_{\mu\nu} \rangle = \frac{2}{\sqrt{|g|}} \frac{\delta W^{(2)}_{\text{div}}}{\delta g^{\mu\nu}} \\ &= \frac{1}{(4\pi)^{D/2}} \frac{N}{D-4} \frac{4m^{D-2}}{D(D-2)} \left[m^2 g_{\mu\nu} + \frac{D}{6} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \right] \end{aligned} \tag{3.1.12}$$

The $\langle \partial_\mu \phi(x) \partial_\nu \phi(x) \rangle$ tadpole only appears in the action within the expression

$$\langle \partial_\mu \phi(x) \partial_\nu \phi(x) \rangle - \frac{2}{D-2}(V_0 + \Lambda) G_{\mu\nu} \tag{3.1.13}$$

and therefore is canceled if this expression vanishes. If we fix

$$V_0 = \frac{1}{(4\pi)^{D/2}} \frac{N}{D-4} \frac{2}{D} m^D. \tag{3.1.14}$$

Then we are left with

$$\frac{1}{(4\pi)^{D/2}} \frac{N}{D-4} \frac{m^{D-2}}{3} \left(R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R \right) = \Lambda G_{\mu\nu}, \tag{3.1.15}$$

as the condition to cancel the tadpole. This is Einstein's equations in the vacuum as in Eq. (2.1.4), which gives us an expression for the Planck mass

$$(M_P)^{D-2} = \frac{1}{(4\pi)^{D/2}} \frac{N}{D-4} \frac{m^{D-2}}{3} = \frac{V_0 D}{6m^2}. \tag{3.1.16}$$

Likewise, $\frac{1}{2} m^2 \langle \phi(x) \phi(x) \rangle$ appears with ΔV_{ct}, so this tadpole is canceled by fixing

$$\Delta V_{ct} = -\frac{1}{2} m^2 \langle \phi(x) \phi(x) \rangle = -\frac{D}{D-2}(V_0 + \Lambda). \tag{3.1.17}$$

With these parameters defined and divergences regularized, we have a self-consistent theory.

3.2 Identifying the Gravitational Interaction

We now write the full metric as a perturbation of the vacuum: $g_{\mu\nu} = G_{\mu\nu} + h_{\mu\nu}$. This forces the energy-momentum tensor that appears in Einstein's equations to vanish, as in Eq. (3.1.15). The action to $\mathcal{O}(h^2)$ becomes

$$S = \int d^D x \sqrt{|G|} \left(\frac{V_0 + \Delta V_{\rm ct} + \Lambda}{D/2-1} + \frac{1}{2} \sum_{a=1}^{N} \left[\partial^\mu \phi^a \partial_\mu \phi^a - m^2 \phi^a \phi^a \right] - \frac{1}{2} h^{\mu\nu} \mathcal{T}_{\mu\nu} \right.$$
$$+ \frac{1}{4} \left(h^{\mu\alpha} h^{\nu\beta} - \frac{1}{2} h^{\mu\nu} h^{\alpha\beta} \right) \sum_{a=1}^{N} \left[G_{\mu\nu} \partial_\alpha \phi^a \partial_\beta \phi^a + G_{\alpha\beta} \partial_\mu \phi^a \partial_\nu \phi^a \right.$$
$$\left. \left. - \frac{1}{2} G_{\mu\nu} G_{\alpha\beta} \left(\partial^\gamma \phi^a \partial_\gamma \phi^a - m^2 \phi^a \phi^a - \frac{2}{N} \frac{V_0 + \Lambda}{D/2-1} \right) \right] + \mathcal{O}(h^3) \right) \qquad (3.2.1)$$

where in the definition of $\mathcal{T}_{\mu\nu}$ from Eq. (3.1.9), the metric is replaced by the vacuum metric $G_{\mu\nu}$, and $G = \det(G_{\mu\nu})$. Note that the action contains products of $h^{\mu\nu}$ with $\mathcal{T}_{\mu\nu}$, $\partial_\mu \phi \, \partial_\nu \phi$, and ϕ^2. Using the definitions of $\mathcal{T}_{\mu\nu}$ and $h_{\mu\nu}$, these interaction terms can be rewritten as products of $\mathcal{T}_{\mu\nu}$'s and ϕ's with $\mathcal{T}_{\mu\nu}$'s to leading order in $1/N$ and $1/(V_0 + \Lambda)$.

Our strategy for analyzing the interaction is to study two-into-two scattering of fields ϕ that couple to $\mathcal{T}_{\mu\nu}$ using Feynman diagrams. To do this, we need the four-point correlation function

$$\langle 0 | T \left\{ \phi^a(x_1) \phi^a(x_2) \phi^c(x_3) \phi^c(x_4) e^{i S_{\rm int}} \right\} | 0 \rangle \qquad (3.2.2)$$

where $S_{\rm int}$ is the interaction terms of the action. The purpose of this exercise is look for consistency between the equation of motion for the fields and the equation of motion for linearized gravity in curved spacetime. We write this four-point correlation function in the form

$$\int d^D y_1 d^D y_2 \sqrt{|G(y_1) G(y_2)|} E_{\mu\nu}(x_1, x_2, y_1) i A^{\mu\nu\alpha\beta}(y_1, y_2) E_{\alpha\beta}(x_3, x_4, y_2) \qquad (3.2.3)$$

where $E_{\mu\nu}(x_1, x_2, y_1)$ can be written as $\langle \phi^a(x_1) \phi^a(x_2) \mathcal{T}_{\mu\nu}(y_1) \rangle$. The factor $iA^{\mu\nu\alpha\beta}$ acts as a Green's function for the equation of motion with the energy-momentum tensors as a source.

The contributing Feynman diagrams to leading order in $1/N$ are shown in Fig. 3.1 (found in [6]). The shaded blob represents the full interaction, which is the infinite sum of loop diagrams shown in Fig. 3.1a. The diagram with no loops is called the tree-level diagram. The sum is similar to a geometric series, and we can analogously use a trick to calculate it explicitly. As shown in Fig. 3.1b, we can use a recursion relation to state the full sum as the tree-level diagram plus the full interaction connected to one loop, which is called the kernel. This way, we need only evaluate the tree-level contribution and the kernel in order to describe the full interaction. The recursion relation is

$$A^{\mu\nu\alpha\beta} = A_0^{\mu\nu\alpha\beta} + K^{\mu\nu}{}_{\lambda\kappa} A^{\lambda\kappa\alpha\beta} \tag{3.2.4}$$

where $A_0^{\mu\nu\alpha\beta}$ is the tree-level amplitude, and $K^{\mu\nu}{}_{\lambda\kappa}$ is the kernel.

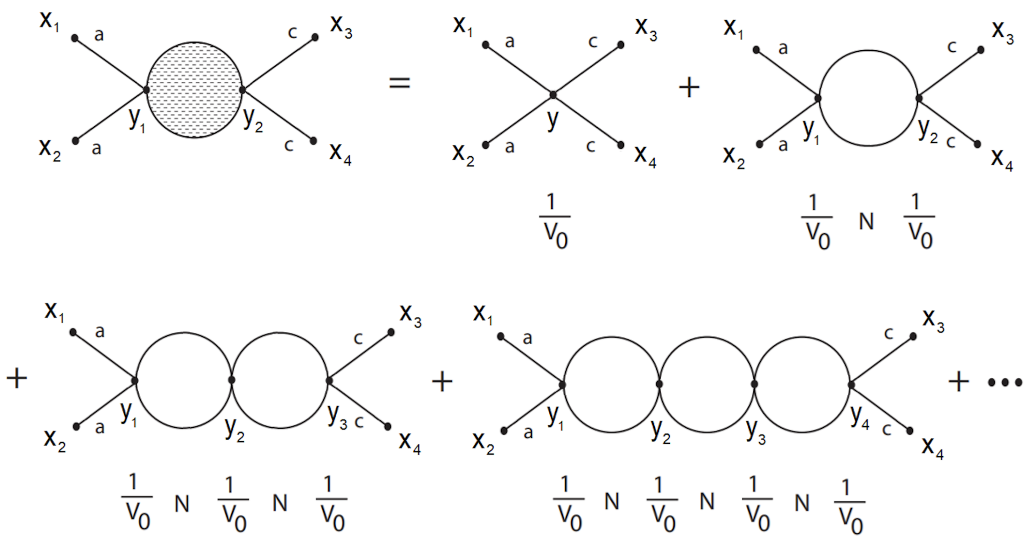
We found that the tree-level amplitude is

$$A_0^{\mu\nu\alpha\beta} = -\frac{1}{4(V_0+\Lambda)}\left[(D/2-1)\left(G^{\nu\alpha}G^{\mu\beta} + G^{\mu\alpha}G^{\nu\beta}\right) - G^{\mu\nu}G^{\alpha\beta}\right] \tag{3.2.5}$$

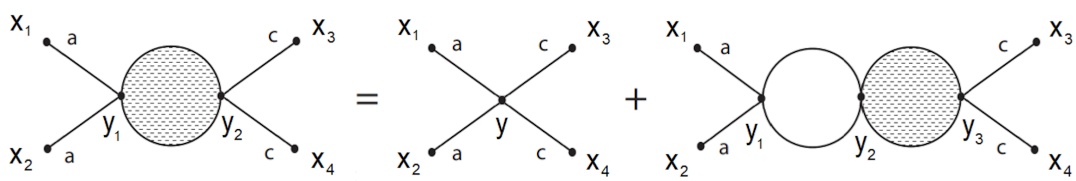
and the kernel is

$$\begin{aligned} K^{\mu\nu}{}_{\lambda\kappa}(y_1, y_2) = \Bigg[& \frac{V_0}{(V_0+\Lambda)}\frac{1}{2}(\delta^{\mu}{}_{\lambda}\delta^{\nu}{}_{\kappa} + \delta^{\nu}{}_{\lambda}\delta^{\mu}{}_{\kappa}) \\ & + iA_0^{\mu\nu\rho\sigma}\frac{4M_{\mathrm{P}}^{D-2}}{(D-2)}D_{\rho\sigma\lambda\kappa}(y_1)\Bigg]\delta^{(D)}(y_1 - y_2) \end{aligned} \tag{3.2.6}$$

where the four-index $D_{\mu\nu\alpha\beta}$ is the linearized gravitational wave operator in curved space, given by



(a) These diagrams represent the full geometric series of loops.



(b) These diagrams are equivalent, requiring calculation of only the tree-level diagram and the one loop (*i.e.* the kernel) connected to the shaded blob on the right-hand side.

Figure 3.1: These are the diagrams that contribute to four-point correlation function stated in Eq. (3.2.2) to leading order in $1/N$. The shaded blob is the full interaction.

$$D_{\mu\nu\alpha\beta} \equiv \frac{1}{2}\Big(G_{\mu\alpha}G_{\nu\beta}\Box - \frac{1}{2}G_{\mu\nu}G_{\alpha\beta}\Box + R_{\mu\alpha}G_{\nu\beta} + R_{\nu\alpha}G_{\mu\beta} \\ -2G_{\mu\nu}R_{\alpha\beta} + 2R_{\mu\nu\alpha\beta} - RG_{\mu\alpha}G_{\nu\beta} + \frac{1}{2}RG_{\mu\nu}G_{\alpha\beta}\Big) \tag{3.2.7}$$

when $D = 4$. In performing these calculations, we again took advantage of the effective action, using

$$\frac{\sqrt{G(x)G(y)}}{4}\langle \mathcal{T}_{\mu\nu}(x)\mathcal{T}_{\alpha\beta}(y)\rangle = -\frac{\delta^2 W_{\text{eff}}}{\delta h^{\mu\nu}(x)\delta h^{\alpha\beta}(y)}. \tag{3.2.8}$$

We found that the Green's function $A^{\mu\nu\alpha\beta}$ satisfies the recursion relation in Eq. (3.2.4) if it solves the equation

$$\left[\frac{4\Lambda}{(D-2)}\left(G_{\rho\lambda}G_{\sigma\kappa}-\frac{1}{2}G_{\rho\sigma}G_{\lambda\kappa}\right)-\frac{4M_{\mathrm{P}}}{(D-2)}D_{\rho\sigma\lambda\kappa}(y_1)\right]A^{\lambda\kappa\alpha\beta}(y_1,y_2)$$
$$=\frac{1}{\sqrt{|G(y_1)|}}\frac{1}{2}(\delta^{\alpha}{}_{\rho}\delta^{\beta}{}_{\sigma}+\delta^{\beta}{}_{\rho}\delta^{\alpha}{}_{\sigma})\delta^{(D)}(y_2-y_1), \tag{3.2.9}$$

which is the linearized Einstein's field equations in vacuum with a curved background $G_{\mu\nu}$ with a delta-function source. Therefore, the equation of motion for the fields ϕ is as predicted by GR. This result confirms that an emergent composite gravitational interaction between scalar fields can be identified from QFT in curved spacetime.

Chapter 4

Putting the Model on a Lattice

So far, our analysis has relied on dimensional regularization to evaluate integrals that diverge in the large momentum regime (known as ultraviolet divergences). However, this mathematical trick of considering the number of dimensions as slightly different from four is not considered physical. A complete quantum field theory must have a candidate for a physical regulator, so we began exploring how our model behaves on the lattice. We consider all spacetime points as lying on a four-dimensional mesh of discrete lattice points, and each point is connected to two other lattice points in each direction. There is a finite (presumably small on some scale) spacing a between lattice points, and though a could in general vary with location on the lattice, we will consider it as constant for simplicity. The lattice serves as a regulator because integrals in momentum space that diverge as momentum goes to infinity are cut off at a large momentum proportional to $1/a$ [12]. Results can be compared to the continuum formulation by taking the limit $a \to 0$.

Derivatives on the lattice can be understood via the limit definition of the derivative. For our purposes, it will be convenient to consider the symmetric derivative

$$\partial_\mu \phi(x) \to \Delta_\mu \phi(x) = \frac{\phi(x+\hat{\mu}) - \phi(x-\hat{\mu})}{2a} \tag{4.0.1}$$

where the $x+\hat{\mu}$ is point one lattice site forward from x in the x^μ direction, and $x-\hat{\mu}$ is one lattice site backward. The distance between spacetime points is finite, so there is no limit $\Delta x \to 0$. The form of the symmetric second derivative on the lattice follows:

$$\Delta_\mu \Delta^\mu \phi(x) = \frac{\phi(x + 2\hat{\mu}) - 2\phi(x) + \phi(x - 2\hat{\mu})}{4a^2} \tag{4.0.2}$$

using the points two sites forward and two sites backward from x. Integrals over all spacetime are replaced with discrete sums over all lattice points, with differential elements again replaced by the lattice spacing.

$$\int d^4x \to \sum_x a^4 \tag{4.0.3}$$

With these tools in place, we can write down our action on the lattice based on Eq. (3.1.1):

$$S = \sum_x a^D \left(\frac{D/2 - 1}{V(\phi^b)} \right)^{D/2-1} \sqrt{\left| \det \left(\sum_{b=1}^{N} \Delta_\mu \phi^b \Delta_\nu \phi^b \right) \right|} \tag{4.0.4}$$

where we stress that the sum over lattice points is the product of sums in four variables. We are free to set $D = 4$ when convenient because we are no longer performing dimensional regularization.

Note that factors of a cancel in the action. The determinant in the action contains $2D$ factors of a in the denominator since $\Delta_\mu \phi^b \Delta_\nu \phi^b$ is a $D \times D$ operator, and if A is an $n \times n$ operator and α a scale factor, then $\det(\alpha A) = \alpha^n \det(A)$. Then, the root determinant has D factors of a in the denominator, which cancel the D factors of a in the numerator of Eq. (4.0.4). This cancellation is due to diffeomorphism invariance and is noteworthy because it suggests (at least naïvely) that the lattice spacing plays no role in dynamics. The lattice spacing does influence the location of the spacetime points used in the definition of the lattice derivative from Eq. (4.0.1), but it will be interesting to see how the lattice results compare to the continuum if there are no explicit factors of a to take to zero.

Recall that we have been attempting to present a background-independent formulation of composite gravity. However, this poses a potential problem for the lattice framework. We currently have no reason to believe that any possible physical metric that solves Einstein's equations will permit a lattice. This question merits further contemplation, but for the present, we take a step back and only consider a flat spacetime background. We know that flat space permits a lattice, so we will perform an analysis in that regime and observe whether or not it is worth generalizing based on its implications.

4.1 Functional Integration on the Lattice

In order to calculate correlation functions in the lattice framework, we must discuss some of the inner workings of functional integration. Recall that in the continuum, n-point correlation functions can be written

$$\langle 0|T\{\phi_1 \cdots \phi_n\}|0\rangle = \frac{\int \mathcal{D}\phi (\phi_1 \cdots \phi_n) e^{iS}}{\int \mathcal{D}\phi \; e^{iS}} \tag{4.1.1}$$

where the integrals are over an infinite number of variables. More precisely,

$$\int \mathcal{D}\phi = \lim_{N\to\infty} \int \prod_{i=1}^{N} d\phi_i \tag{4.1.2}$$

where each ϕ is a function of time, and the limit $N \to \infty$ is the limit as the number of time steps goes to infinity. On the lattice, this limit is omitted until the number of lattice sites is taken to infinity.

In order to evaluate the functional integral, we begin with the free part of the action. After that, the interacting part of $\exp(iS) = \exp(iS_{\text{free}})\exp(iS_{\text{int}})$ can be written as a perturbative expansion in powers of ϕ. On the lattice, with the background metric fixed to $\eta_{\mu\nu}$,

$$S_{\text{free}} = \sum_x a^4 \sum_{b=1}^{N} \left[\frac{1}{2}\Delta_\mu \phi^b \Delta^\mu \phi^b - \frac{1}{2}m^2 \phi^b \phi^b\right] \tag{4.1.3}$$

where perturbations about the background metric have been absorbed into the interacting part of the action. By analogy with integration by parts, we assert that we can rewrite this as

$$\begin{aligned} S_{\text{free}} &= \sum_{b=1}^{N} \sum_x a^4 \left[-\frac{1}{2}\phi^b \left(\Delta_\mu \Delta^\mu + m^2\right) \phi^b\right] \\ &= -\frac{1}{2}a^4 \sum_{b=1}^{N} \left[\Phi_{txyz} \left(\Delta_\mu \Delta^\mu + m^2\right) \delta^{tt'} \delta^{xx'} \delta^{yy'} \delta^{zz'} \Phi_{t'x'y'z'}\right] \\ &= -\frac{1}{2}a^4 \sum_{b=1}^{N} \left[\Phi_{txyz} \mathcal{K}^{tt'xx'yy'zz'} \Phi_{t'x'y'z'}\right] \end{aligned} \tag{4.1.4}$$

where the four-index object Φ_{txyz} contains the ϕ^b's at every lattice site in spacetime, and repeated indices in the last two lines are summed over and run from 1 to the number of lattice sites in each respective direction. The operator $\mathcal{K}$ is referred to as the kernel of the functional integral, which is not to be confused with the previously discussed kernel of the Feynman diagrams. Schematically, our goal is to rewrite the product $\Phi\mathcal{K}\Phi$ as a sum of $\kappa\tilde{\Phi}^2$ terms where κ is a constant, and $\tilde{\Phi}$ is constructed of linear combinations of ϕ^b's. We accomplish this by diagonalizing $\mathcal{K}$. This transformation converts the functional integral into a product of Gaussian integrals, which can be evaluated explicitly.

In order to diagonalize the functional integral kernel, we assume that Φ is a product of functions of the individual coordinates t, x, y, and z by analogy with separation of variables as used when solving partial differential equations. We write $\mathcal{K}$ as a sum of matrix operators acting on Φ.

$$
\begin{aligned}
\mathcal{K}^{tt'xx'yy'zz'} &= \left(\Delta_\mu \Delta^\mu + m^2\right) \delta^{tt'} \delta^{xx'} \delta^{yy'} \delta^{zz'} \\
&= \Delta^{(2)tt'} \delta^{xx'} \delta^{yy'} \delta^{zz'} - \delta^{tt'} \Delta^{(2)xx'} \delta^{yy'} \delta^{zz'} - \delta^{tt'} \delta^{xx'} \Delta^{(2)yy'} \delta^{zz'} \\
&\quad - \delta^{tt'} \delta^{xx'} \delta^{yy'} \Delta^{(2)zz'} + m^2 \delta^{tt'} \delta^{xx'} \delta^{yy'} \delta^{zz'}
\end{aligned}
\tag{4.1.5}
$$

where $\Delta^{(2)}$ is the matrix representing the second derivative on the lattice. If we enforce periodic boundary conditions on the lattice, then from Eq. (4.0.2), it follows that

$$
\Delta^{(2)} = \frac{1}{4a^2} \begin{pmatrix}
-2 & 0 & 1 & & & & 1 & 0 \\
0 & -2 & 0 & 1 & & & 0 & 1 \\
1 & 0 & -2 & 0 & 1 & & & \\
 & 1 & 0 & -2 & 0 & 1 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & 1 & 0 & -2 & 0 & 1 \\
1 & 0 & & & 1 & 0 & -2 & 0 \\
0 & 1 & & & & 1 & 0 & -2
\end{pmatrix}
\tag{4.1.6}
$$

which is a symmetric (also real hermitian) matrix because we chose a symmetric definition of the discretized derivative. If we assume the lattice has the same number n of lattice sites in each direction, Then $\Delta^{(2)}$ is an $n \times n$ matrix for each term in Eq. (4.1.5).

We can diagonalize $\Delta^{(2)}$ using a process similar to one described in [13]. We can write $\Delta^{(2)}$ as $1/(4a^2)(C + C^\dagger - 2)$ where

$$
C = \begin{pmatrix}
0 & 0 & 1 & & & \\
 & 0 & 0 & 1 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & 0 & 0 & 1 \\
1 & 0 & & & 0 & 0 \\
0 & 1 & & & & 0
\end{pmatrix}
\tag{4.1.7}
$$

and $C^\dagger$ is the hermitian conjugate of C. We can easily show that the eigenvalues of C are ω_k^2 where $\omega_k = \exp(2\pi i k/n)$ are called the roots of unity with $k \in \{0, 1, \ldots, n-1\}$.

The eigenvectors are $\bar{\omega}_k = (1, \omega_k, \omega_k^2, \ldots, \omega_k^{n-1})$. Likewise, the eigenvalues of $C^\dagger$ are ω_k^{-2} with the same eigenvectors. The proof:

$$C\bar{\omega}_k = \begin{pmatrix} \omega_k^2 \\ \omega_k^3 \\ \vdots \\ \omega_k^{n-1} \\ 1 \\ \omega_k \end{pmatrix} = \omega_k^2 \begin{pmatrix} 1 \\ \omega_k \\ \vdots \\ \omega_k^{n-3} \\ \omega_k^{n-2} \\ \omega_k^{n-1} \end{pmatrix} \quad \text{and} \quad C^\dagger\bar{\omega}_k = \begin{pmatrix} \omega_k^{n-2} \\ \omega_k^{n-1} \\ 1 \\ \vdots \\ \omega_k^{n-4} \\ \omega_k^{n-3} \end{pmatrix} = \omega_k^{-2} \begin{pmatrix} 1 \\ \omega_k \\ \omega_k^2 \\ \vdots \\ \omega_k^{n-2} \\ \omega_k^{n-1} \end{pmatrix} \tag{4.1.8}$$

can be understood when keeping in mind that $\omega_k^n = 1$. Therefore, the eigenvectors of $\Delta^{(2)}$ are $\bar{\omega}_k$ with eigenvalues

$$\frac{1}{4a^2}(e^{4i\pi k/n} + e^{-4i\pi k/n} - 2) = \frac{1}{2a^2}(\cos(4\pi k/n) - 1) = -\frac{1}{a^2}\sin^2(2\pi k/n) \tag{4.1.9}$$

which are real, as required by the fact that $\Delta^{(2)}$ is hermitian. Following from Eq. (4.1.5), the eigenvalues of $\mathcal{K}$ are $\frac{2}{a^2}\sin^2(2\pi k/n) + m^2$ with the same eigenvectors. With this information, we will be able to diagonalize the full operator $\mathcal{K}$, which will allow us to compute correlation functions on the lattice. These efforts are ongoing.

Chapter 5

Discussion and Conclusion

We have constructed a diffeomorphism invariant and background-independent scalar quantum theory of composite gravity. An emergent gravitational interaction arises at long distances and conforms to the linearized Einstein's field equations in curved spacetime with quantum corrections. The theory pre-supposes the existence of a physical regulator, and we used dimensional regularization as a proxy. We have suggested the lattice as a candidate for a physical regulator and began the discussion on performing calculations in this framework.

Moving forward, the most obvious next step is finishing the lattice calculations, including some discussion of the plausibility of a background-independent lattice. If the lattice is ruled out as a physical regulator, then some other candidate is needed, such as the stochastic picture suggested in [14]. Beyond the regulator, the background-independent model should be generalized to the standard model to include fermion and gauge fields. Even in the scalar theory, another possible future direction of research would be analyzing the effects of higher-order terms in the perturbative expansions. For example, these corrections could have an impact on early-universe cosmology calculations where short-distance scale physics dominates. Also, higher-order terms could reveal a prediction for the value of the cosmological constant. The implications of this quantum theory of gravity merit further investigation.

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