A Survey of Methods to Determine Quantum Symmetry of Graphs

Samantha Phillips

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A survey of methods to determine quantum symmetry of graphs

A thesis submitted in partial fulfillment of the requirement
for the degree of Bachelor of Science in Mathematics from
William & Mary

by

Samantha Phillips

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Williamsburg, VA
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A Survey of Methods to Determine Quantum Symmetry of Graphs

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May 12, 2021
Abstract

We introduce the theory of quantum symmetry of a graph by starting with quantum permutation groups and classical automorphism groups. We study graphs with and without quantum symmetry to provide a comprehensive view of current techniques used to determine whether a graph has quantum symmetry. Methods provided include specific tools to show commutativity of generators of algebras of quantum automorphism groups of distance-transitive graphs; a theorem that describes why nontrivial, disjoint automorphisms in the automorphism group implies quantum symmetry; and a planar algebra approach to studying symmetry.
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Chapter 1

Introduction

Quantum permutation groups were first conceptualized by Wang in 1998 \cite{20}. He discovered that the set $X = \{1, 2, \ldots, n\}$ has quantum permutation group $S_n^+$ such that $S_n \subseteq S_n^+$, where $S_n$ is the classical permutation group of $X$. An automorphism of a graph $\Gamma$ is a bijective map $\sigma : V \to V$, where $V$ is the set of vertices, such that $\sigma$ preserves adjacency and nonadjacency between vertices. We will describe why automorphisms of a graph are a subgroup of permutations in Chapter 3. The set of automorphisms form a group under composition denoted $G_{aut}(\Gamma)$. Then the algebra $C(G_{aut}(\Gamma))$ of functions on $G_{aut}(\Gamma)$ is commutative. Similarly, we can describe the algebra $C(G_{aut}^+(\Gamma))$ of functions on the quantum automorphism group $G_{aut}^+(\Gamma)$, originally defined by Bichon in 2003 and followed by a definition by Banica in 2005 that we will use in this thesis \cite{6, 2}. However, the algebra associated with quantum automorphisms is not necessarily commutative. The natural question follows: when do the quantum automorphism group and classical automorphism group coincide for a graph? This is directly connected to if a graph has quantum symmetry, as described in this thesis. We say a graph does not have quantum symmetry if the quantum and classical automorphism groups are equal, and does have quantum symmetry otherwise. Additionally, this field has been connected to planar algebras and subfactors.
Recently, topics related to quantum automorphism groups have found applications in certain quantum information science problems, such as the quantum isomorphism game [12].

To begin, we will describe the contents of my thesis. In Chapter 2, we provide background about classical and quantum permutation groups. In addition, we include examples of permutation groups with and without quantum permutations. These are the building blocks to understanding quantum symmetry technically. This is connected to the following section, Chapter 3, in which classical and quantum automorphism groups of graphs are explained. We will explore C*-algebras and the process to find quantum automorphisms. Then we will define quantum symmetry and provide a series of examples of graphs that do or do not have quantum symmetry in Chapter 4. Moreover, in Chapter 5, we describe a planar algebra approach to determine if a given graph has quantum symmetry. There is also a proof that the 5-cycle, \( C_5 \), does not have quantum symmetry, which is an example provided in Section 4.2. Finally, we will discuss future directions of this research and conclude the report in the final chapter.

To understand quantum symmetry, it is necessary to define the following basic concepts. These can be referred to as the topics come up in the thesis.

**Definition 1.1.** [9] Group

A group \( G \) is a set with binary operation, denoted here by multiplication, such that the following are satisfied:

(i) **Associativity:** the operation is associative, meaning \((ab)c = a(bc)\) for all \( a, b, c \in G \);

(ii) **Identity:** there exists an element \( e \) (called the identity) in \( G \) such that \( ae = ea = a \) for all \( a \in G \);

(iii) **Inverses:** for each element \( a \) in \( G \), there is an element \( b \in G \) (called the inverse
of a) such that \( ab = ba = e \).

An example of a group under multiplication is the set of nonzero real numbers, often denoted \( \mathbb{R} \setminus \{0\} \). The set of complex numbers \( \mathbb{C} \) is a group under addition. Additionally, a set containing only the identity is a group, called the trivial group.

It is also helpful to be clear about what a graph is here.

**Definition 1.2.** A graph \( \Gamma \) is defined to be \( \Gamma = (V, E) \), where \( V \) represents the set of vertices and \( E \) denotes the set of edges, which are unordered pairs of vertices. Under our definition, there cannot be an edge between a vertex and itself and there cannot be multiple edges between two vertices.

There are important concepts related to graphs that are used here. Consider a pair of vertices \( v_1, v_2 \in V \). Then \( v_1 \) and \( v_2 \) are *adjacent*, or referred to as neighbors, if there exists an edge between them and *nonadjacent* if there does not exist an edge between them. In Figure 1.1, vertices 1, 2, and 3 are adjacent. If \( v_1 = v_2 \), then the pair of vertices are neither adjacent nor nonadjacent. A *path* is defined as an ordered tuple of vertices such that every vertex is adjacent to the one before it, so there exists an edge between \( \{v_i, v_{i+1}\} \), and each \( v_i \) is distinct. The *distance* between two vertices is the shortest path, or smallest number of edges, required to get from \( v_1 \) to \( v_2 \). An *n-cycle* is a path that starts at a specified vertex and ends at that same vertex, so in addition to there being an edge between \( \{v_i, v_{i+1}\} \) for \( 1 \leq i \leq n - 1 \), there is also an edge between \( v_1 \) and \( v_n \). Finally, the *girth* of a graph is the length of the smallest cycle of the graph. Figure 1.1 is an example of a graph with a girth of 3.

![Figure 1.1: 3-cycle](image)

A \((v, k, \lambda, \mu)\)-*strongly regular graph* is a graph with \( v \) vertices where every pair of
adjacent vertices has $\lambda$ common neighbors, every pair of nonadjacent vertices has $\mu$ common neighbors, and every vertex has $k$ neighbors. A distance-transitive graph is a graph with the additional property that if vertices $a$ and $b$ are distance $i$ and vertices $c$ and $d$ are distance $i$ as well, then there exists an automorphism of the graph that sends $a$ to $c$ and $b$ to $d$.

**Definition 1.3.** Define a $*$-homomorphism to be a structure preserving mapping $\phi : A \to B$ such that the following holds:

- $\phi(a^*) = \phi(a)^*$;
- $\phi(a + b) = \phi(a) + \phi(b)$;
- $\phi(\lambda a) = \lambda \phi(a)$;
- $\phi(ab) = \phi(a)\phi(b)$

for $a, b \in A$ and $\lambda \in \mathbb{C}$.

**Definition 1.4.** [8, Theorem 2.6.1] C*-algebra

A C*-algebra $A$ is a (non-empty) set with the following operations:

(i) addition, which is commutative and associative;

(ii) multiplication, which is associative;

(iii) multiplication by scalars $\lambda \in \mathbb{C}$;

(iv) an involution for each $a \in A$ denoted $a^*$ (so, $a^{**} = (a^*)^* = a$ for all $a \in A$).

Both types of multiplication distribute over addition. For $a, b \in A$, we have $(ab)^* = b^*a^*$, $(a + b)^* = a^* + b^*$, and $(\lambda a)^* = \overline{\lambda}a^*$. A Banach algebra is complete in the metric
associated with its norm. That is,

\[
\|\lambda a\| = |\lambda| \|a\|, \\
\|a + b\| \leq \|a\| + \|b\|,
\]

for all \(a, b \in A\) and \(\lambda \in \mathbb{C}\). We will explore how the norm is found as we define the necessary background. Finally, for all \(a \in A\), we have \(\|a^*a\| = \|a\|^2\).

**Definition 1.5.** [8, 13] A *complex Hilbert space* is a vector space with a complex inner product and norm. Let \(\mathcal{H}\) be a complex Hilbert space with inner product denoted \(\langle \cdot, \cdot \rangle\). The collection of linear bounded operators on \(\mathcal{H}\), denoted \(\mathcal{B}(\mathcal{H})\), is a C*-algebra. The multiplication is given by composition of operators. The * is the involution, defined by the equation \(\langle a^*\xi, \eta \rangle = \langle \xi, a\eta \rangle\) for all operators on \(\mathcal{H}\) and \(\xi\) and \(\eta\) in \(\mathcal{H}\). Finally, the norm is given by

\[
\|a\| = \sup\{\|a\xi\| : \xi \in \mathcal{H}, \|\xi\| \leq 1\},
\]

for any \(a \in \mathcal{B}(\mathcal{H})\).

In fact, every C*-algebra is an algebra of operators on a Hilbert space [13]. For our purposes, the involution * is simply acting like the conjugate transpose operation on complex matrices.

There are essential definitions to know before we proceed:

**Definition 1.6.** [4] Let \(A\) be a C*-algebra.

(i) A projection is an element \(p \in A\) satisfying \(p^2 = p = p^*\).

(ii) Two projections \(p, q \in A\) are called orthogonal when \(pq = 0\).

(iii) A partition of unity is a set of orthogonal projections that sum to 1.
Chapter 2

Quantum Permutation Groups

In this chapter we will describe the liberation process that takes us from a classical permutation group to a quantum permutation group. We characterize classical permutation groups with a Woronowicz algebra, a specific way of defining matrix quantum groups.

2.1 Permutations as Matrices

A permutation of a set $X$ is a bijection $\sigma : X \rightarrow X$. The permutations of the set $X = \{1, 2, ..., n\}$ form the symmetric group under composition, denoted $S_n$, of degree $n$. We will describe how to map $S_n$ to the matrix group with complex coefficients $M_n(\mathbb{C})$. We can identify each permutation $\sigma \in S_n$ with a permutation matrix of size $n \times n$:

$$\bar{\sigma} := \begin{bmatrix} e_{\sigma(1)} & \cdots & e_{\sigma(k)} & \cdots & e_{\sigma(n)} \end{bmatrix}$$
where \( \{ e_1, ..., e_n \} \) denotes the canonical basis of \( \mathbb{C}^n \). Here we can view

\[
e_i = \begin{bmatrix}
0 \\
\vdots \\
0 \\
1 \\
0 \\
\vdots \\
0
\end{bmatrix},
\]

where the 1 is in the \( i \)th position.

For example, we know the permutations (12) and (132) are in \( S_3 \). Note that we are using cyclic notation, meaning (12) denotes a mapping that sends 1 to 2 and 2 to 1 with 3 fixed and (132) denotes a mapping that sends 1 to 3, 3 to 2, and 2 to 1. Fixed points are omitted when the permutation is written in cyclic notation. The canonical basis of \( \mathbb{C}^3 \) is

\[
\begin{bmatrix}
e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Then the permutations (12) and (132) in \( S_3 \) are represented by

\[
\tilde{(12)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \tilde{(132)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\]

A matrix’s conjugate transpose is calculated by taking the transpose and then the complex conjugate of each entry. The involution * is an operation such that \( a^{**} = (a^*)^* = a \) with \( (a + b)^* = a^* + b^* \), \( (ab)^* = b^*a^* \), and \( (\lambda a)^* = \overline{\lambda}a^* \) where
Let \( \lambda \in \mathbb{C} \). Recall from Chapter 1 that the involution acts like the conjugate transpose operation on complex matrices for our purposes. A unitary matrix has the property that its conjugate transpose is equal to its inverse. The unitary group \( U(n) \) is the group of unitary \( n \times n \) complex matrices under matrix multiplication, such that

\[
U(n) = \{ M \in M_n(\mathbb{C}) | M^*M = I_n \}
\]

where \( M_n(\mathbb{C}) \) denotes the set of \( n \times n \) matrices with complex coefficients. We can see that every permutation matrix \( \tilde{\sigma} \) is unitary, so

\[
\tilde{\sigma}^* \tilde{\sigma} = I_n
\]

where \( I_n \) denotes the \( n \times n \) identity matrix. This is further justified because each column \( e_{\sigma(1)}, e_{\sigma(2)}, ..., e_{\sigma(n)} \) of a permutation matrix contains one “1” and the rest 0’s. Then \( e_{\sigma(i)}^* e_{\sigma(j)} = 1 \) if \( \sigma(i) = \sigma(j) \) and 0 otherwise. Additionally, each column (and row) vector is a unit vector so \( \tilde{\sigma} \) has an orthogonal basis and it follows that \( \tilde{\sigma}^* \tilde{\sigma} = I_n \).

**Proposition 2.1.** The map \( \phi : S_n \rightarrow U(n) \) sending \( \sigma \rightarrow \tilde{\sigma} \) is an injective group homomorphism with the operation composition.

**Proof.** Let \( \sigma, \tau \in S_n \). Since \( \tilde{\sigma} \) is defined by how \( \sigma \) acts on the set of column vectors \( \{ e_1, ..., e_n \} \) of the canonical basis of \( \mathbb{C}^n \), we can show that \( \phi \) is an injective group homomorphism by considering the map \( \phi(\sigma) : e_i \rightarrow e_{\sigma(i)} \).

First, we will show \( \phi \) is a group homomorphism. Note that \( \phi \) is a group homomorphism if \( \phi(\sigma \tau) = \phi(\sigma)\phi(\tau) \) for all \( \sigma, \tau \in S_n \). Thus, we can check that

\[
\phi(\sigma \tau)(e_i) = \phi(\sigma)(\phi(\tau)(e_i)) \quad \text{for} \quad 1 \leq i \leq n.
\]

Indeed, \( \phi(\sigma \tau)(e_i) = e_{\sigma \tau(i)} = e_{\sigma(\tau(e_i))} \) and \( \phi(\sigma)(\phi(\tau)(e_i)) = \phi(\sigma)e_{\tau(i)} = e_{\sigma(\tau(e_i))} \). Thus, \( \phi(\sigma \tau)(e_i) = e_{\sigma(\tau(e_i))} = \phi(\sigma)(\phi(\tau)(e_i)) \).

This holds for \( \sigma, \tau \in S_n \) and \( 1 \leq i \leq n \) so \( \phi \) is a group homomorphism under composition.

Second, we will prove \( \phi \) is injective. This holds if \( \phi(\sigma) = \phi(\tau) \) implies \( \sigma = \tau \). We have \( I_n \) as the identity matrix in \( U(n) \). The only element in \( S_n \) that is sent to the identity matrix is the identity permutation, denoted \( \text{id} \). The kernel of a homomorphism \( \pi \) from a group \( G \) to a group with the identity \( \text{id} \) is the set \( \{ x \in G | \pi(x) = \text{id} \} \) [9]. So, the kernel of \( \phi \) contains only \( \text{id} \) as \( \{ x \in S_n | \phi(x) = I_n \} = \{ \text{id} \} \).
Therefore since the only element in the kernel of $\phi$ is the identity of $S_n$, we know $\phi$ is a one-to-one, or injective, mapping.

Thus, we have shown $\phi : S_n \rightarrow U(n)$ is an injective group homomorphism. \qed

The permutation matrices act on the canonical basis the same way the corresponding permutation acts on the set $\{1, ..., n\}$. Thus, $S_n$ can be viewed as a subgroup of the unitary group $U(n)$. Henceforth we may identify $S_n$ with the subgroup of permutation matrices in $U(n)$.

### 2.2 Permutation Groups as C*-algebras

Recall the definition of a C*-algebra from the Chapter 1 and definition of the involution in Section 2.1. Let $X$ be a compact space, meaning every open cover of $X$ has a finite subcover. Then

$$C(X) = \{ f : X \rightarrow \mathbb{C} | f \text{ continuous}\}$$

is the algebra of continuous function on $X$ and denotes a commutative C*-algebra with the operations scalar multiplication, pointwise addition, and multiplication [13]. $C(X)$ is unital and commutative. Define

$$C(S_n) = \{ f : S_n \rightarrow \mathbb{C} | f \text{ continuous}\}$$

to be the algebra of continuous functions on $S_n$ with the operations scalar multiplication, pointwise addition, and multiplication.

**Definition 2.2.** [4] Let $A$ be a C*-algebra. A magic unitary is a square matrix $[u_{ij}] = u \in M_n(A)$ with the following conditions

(i) each $u_{ij}$ is a projection: $u_{ij} = u_{ij}^* = u_{ij}^2$;
(ii) each row and column of \( u \) is a partition of unity:
\[
\sum_{i=1}^{n} u_{il} = 1 = \sum_{i=1}^{n} u_{li};
\]

(iii) \( u \) is unitary: \( uu^* = I_n = u^*u \).

The third condition in the definition above indicates the matrix is biunitary as well since a binunitary matrix is a unitary matrix \( u \) whose conjugate transpose \( u^* \) is also unitary. We note that the matrices coming from \( S_n \) described in the previous section are magic unitary matrices. This is an essential element in defining quantum permutations. Suppose \( S_n \) acts in the natural way on the compact set \( X = \{1, 2, ..., n\} \). Then coefficients \( u_{ij} \), defined to be
\[
\begin{bmatrix}
g_{11} & \cdots & g_{1n} \\
\vdots & \ddots & \vdots \\
g_{n1} & \cdots & g_{nn}
\end{bmatrix}
\]
are characteristic functions of the sets \( \{ g \in S_n | g(j) = i \} \) where \( [g_{ij}] = g \) is a permutation matrix [2]. These sets form a partition of \( S_n \). In this case, \( u_{ij} \) selects the \((i, j)\) element in the given permutation matrix \( g = \tilde{\sigma} \). Since every element in a permutation matrix is 0 or 1, we know \( u_{ij} = 1 \) if \( \sigma(i) = j \) and 0 otherwise. Thus, \( C(S_n) \) is generated by these \( u_{ij} \)'s. This generating family is the entries in a biunitary matrix \( u = [u_{ij}] \in M_n(C(S_n)) \) [5].

As a consequence of these definitions, we observe each \( u_{ij} \) is a projection in \( C(S_n) \), \( u_{ij}u_{ik} = \delta_{jk}u_{ij} \) and \( u_{ji}u_{ki} = \delta_{jk}u_{ji} \) for \( 1 \leq i, j \leq n \), and \( \sum_{i=1}^{n} u_{il} = 1 = \sum_{i=1}^{n} u_{li} \) for \( 1 \leq i \leq n \). Here, \( \delta_{jk} = 1 \) if \( j = k \) and \( \delta_{jk} = 0 \) otherwise. Thus, \( u = [u_{ij}] \in M_n(C(S_n)) \) is a magic unitary matrix in \( M_n(C(S_n)) \) by Definition 2.2.

Before concluding our description of \( C(S_n) \), we will define Woronowicz’s matrix quantum groups.
**Definition 2.3.** [21] Matrix Quantum Groups

A matrix quantum group is a unital C*-algebra $\mathcal{A}$ with maps $\Delta$, $\varepsilon$, and $S$ such that

- $\mathcal{A}$ is generated by $n^2$ elements $\{u_{ij} : 1 \leq i, j \leq n\}$
- They form a biunitary matrix $u = [u_{ij}] \in M_n(\mathcal{A})$
- $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : \mathcal{A} \to \mathcal{A} \times \mathcal{A}$
- $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : \mathcal{A} \to \mathbb{C}$
- $S(u_{ij}) = u_{ji}$ defines a morphism $S : \mathcal{A} \to \mathcal{A}^{\text{op}}$

where $\delta$ is the comultiplication of $\mathcal{A}$, $\varepsilon$ is the counit, and $S$ is the antipode.

**Theorem 2.4.** [4] $C(S_n)$ is the universal commutative C*-algebra generated by $n^2$ elements $u_{ij}$, with relations making $[u_{ij}]$ a magic unitary. Moreover, the maps

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$$

$$\varepsilon(u_{ij}) = \delta_{ij}$$

$$S(u_{ij}) = u_{ji}$$

are the comultiplication, counit, and antipode of $C(S_n)$, respectively.

$C(S_n)$ is commutative in the usual sense of pointwise multiplication of functions with values in $\mathbb{C}$. By Woronowicz’s definition, $C(S_n)$ is a matrix quantum group and thus, Woronowicz algebra. In fact, every commutative Woronowicz algebra is of the form $C(G)$, where $G$ is a compact group like $S_n$ [21].

In conclusion, the C*-algebra $C(S_n)$ is the universal commutative C*-algebra generated by $n^2$ elements $\{u_{ij} : 1 \leq i, j \leq n\}$, with relations that make $u = [u_{ij}] \in M_n(C(S_n))$ into a magic unitary matrix. We write

$$C(S_n) = C^*_{\text{comm}}(u_{ij} | u = n \times n \text{ magic unitary})$$
2.3 Liberation Process

Now we will explain the liberation process to reveal quantum permutations. The idea of liberation is to drop the commutativity condition. In other words, $C(S_n^+)$ is defined as the not necessarily commutative C*-algebra satisfying the magic unitary conditions from Definition 2.2. We write

$$C(S_n^+) = C^*_n(u_{ij} | u = n \times n \text{ magic unitary}).$$

We use these algebras to compare $S_n$ and $S_n^+$ by interpreting the surjection $C(S_n^+) \to C(S_n^+)/\langle ab = ba \rangle \simeq C(S_n)$ to imply $S_n \leq S_n^+$, where $\langle ab = ba \rangle = \langle ab - ba | a, b \in C(S_n^+) \rangle$ is the ideal generated by commutators of $C(S_n^+)$. Then $S_n^+$ is the formal dual of $C(S_n^+)$.

The surjection $C(S_n^+) \to C(S_n^+)/\langle ab = ba \rangle \simeq C(S_n)$ is thought of as an injection $S_n \to S_n^+$, so $S_n \leq S_n^+$. In this sense, it “exists” and can be considered a “group-like” object, but it is not necessarily a group. The corresponding compact quantum group $S_n^+$ consists of “quantum permutations.” To summarize, we have the following C*-algebras:

$$C(S_n) = C^*_n(u_{ij}, 1 \leq i, j \leq n | u \text{ is an } n \times n \text{ magic unitary})$$

$$C(S_n^+) = C^*(u_{ij}, 1 \leq i, j \leq n | u \text{ is an } n \times n \text{ magic unitary})$$

Therefore, $C(S_n) \simeq C(S_n^+)$ if and only if $C(S_n^+)$ is commutative.

2.4 Examples

In this section, we will discuss quantum permutations of $\{1, ..., n\}$ for $n = 1, 2, 3,$ and 4.

First, we look at $S_1$. By construction, $S_1$ contains only one element, the identity which we will denote $e$. This forces $C(S_1^+)$ to be commutative as well, as it is generated by a single element. Thus, when we relax the commutativity requirement
\(C(S_1) \simeq C(S_1^+)\). So, we can write \(S_1 = S_1^+\) and \(S_1^+\) does not contain any quantum permutations.

Now consider \(S_2\). Any \(2 \times 2\) magic unitary matrix is of the form

\[
\begin{bmatrix}
  p & 1 - p \\
  1 - p & p
\end{bmatrix}
\]

where \(p\) and \(1 - p\) are projections. Then \(p\) and \(1 - p\) automatically commute \((p(1 - p) = p - p^2 = p - p = 0)\), and so \(C(S_2^+)\) is commutative. Therefore, \(C(S_2) \simeq C(S_2^+)\). Again, \(S_2^+\) does not contain any quantum permutations and \(S_2 = S_2^+\).

Before discussing \(C(S_3^+)\) we prove the following lemma.

**Lemma 2.5.** [16] Let \(u_{ij}, 1 \leq i, j \leq n\), be generators of \(C(S_n^+)\). If we have \(u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}\), then \(u_{ij}\) and \(u_{kl}\) commute.

**Proof.** We apply the involution \(*\) on \(u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}\) to get \(u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij} = (u_{ij}u_{kl}u_{ij})^* = (u_{ij}u_{kl})^* = u_{kl}^*u_{ij}^* = u_{kl}u_{ij}\). □

**Proposition 2.6.** [12] \(C(S_3^+)\) is commutative.

**Proof.** Consider \(S_3\). Suppose \(u = [u_{ij}]_{i,j \in [3]}\) is a magic unitary matrix. By the definition of a magic unitary, all pairs \(u_{ij}\) and \(u_{lk}\) in the same row \((i = l)\) and/or column \((j = k)\) commute. So we only need to show that \(u_{ij}\) and \(u_{lk}\) commute for \(i \neq l\) and \(j \neq k\). Since we can permute the rows and columns of a magic unitary independently and always have a magic unitary, it suffices to show that \(u_{11}u_{22} = u_{22}u_{11}\).

We begin with \(u_{11}u_{22}\). We will show that \(u_{11}u_{22} = u_{11}u_{22}u_{11} + u_{12}u_{22}u_{13}\). Because each column of a magic unitary sums to 1, we can write \(u_{11}u_{22} = u_{11}u_{22}(u_{11} + u_{12} + u_{13})\). Then we distribute to get \(u_{11}u_{22} = u_{11}u_{22}u_{11} + u_{11}u_{22}u_{12} + u_{11}u_{22}u_{13}\). Because \(u_{22}u_{12} = 0\), we finally get \(u_{11}u_{22} = u_{11}u_{22}u_{11} + u_{11}u_{22}u_{13}\).
Now we will examine $u_{11}u_{22}u_{13}$. Knowing that each row of a magic unitary sums to 1, we can write $u_{22}$ as $1-u_{21}-u_{23}$. Thus, we can write $u_{11}u_{22}u_{13} = u_{11}(1-u_{21}-u_{23})u_{13}$. Distributing, we get $u_{11}u_{22}u_{13} = u_{11}u_{13} - u_{11}u_{21}u_{13} - u_{11}u_{23}u_{13}$. Recall the property that $u_{ij}u_{ik} = \delta_{jk}u_{ij}$ and $u_{ji}u_{ki} = \delta_{jk}u_{ji}$, where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ otherwise. We then can state $u_{11}u_{21} = 0$ and $u_{23}u_{13} = 0$. Then the second and third terms are 0 so we have $u_{11}u_{22}u_{13} = u_{11}u_{13}$. By the property above, $u_{11}u_{13} = 0$ as well. Therefore, $u_{11}u_{22}u_{13} = 0$.

Thus, we have shown that $u_{11}u_{22} = u_{11}u_{22}u_{11}$. By Lemma 2.5, we know $u_{11}$ and $u_{22}$ commute. We have proven $u_{ij}$ and $u_{ik}$ commute for $i \neq l$ and $j \neq k$. Thus, $C(S_3^+)$ is commutative and $C(S_3) \simeq C(S_3^+)$, implying that $S_3^+$ does not contain any quantum permutations.

Finally, we will look at $S_4$. Recall Definition 1.3 of a *-homomorphism. Note $Sym(\{1, 2\})$ and $Sym(\{3, 4\})$ are the set of all permutations on $\{1, 2\}$ and $\{3, 4\}$, respectively. Then $Sym(\{1, 2\}) \times Sym(\{3, 4\}) \leq S_4$, so we have two disjoint $S_2$ groups in $S_4$. This gives us the surjective *-homomorphism $\phi : C(S_4^+) \to C^*(p, q|p = p^* = p^2, q = q^* = q^2)$ by

$$u \to \begin{bmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{bmatrix}$$

where projections $p$ and $q$ are noncommutative [19]. Observe that this matrix is a magic unitary, so $C(S_4^+)$ has a noncommutative quotient, and $C(S_4^+)$ is not commutative as well. Therefore, $C(S_4^+)$ is not isomorphic to $C(S_4)$. Then $S_4^+$ consists of quantum permutations in addition to classical permutations.

More generally, it know that for $n \geq 4$, $C(S_n^+)$ is not commutative and is infinite dimensional [3]. Thus, $C(S_n) \not\simeq C(S_n^+)$. In fact, quantum permutations start existing
at $n = 4$. 
Chapter 3

Quantum Automorphisms Groups

3.1 Automorphism Groups of Graphs

We will first explain what a graph automorphism is in the classical sense. We begin with a definition. Every graph \( \Gamma = (V,E) \), as defined in Chapter 1, has a group of bijections from \( V \rightarrow V \) that is identified with \( S_n \) where \( n \) denotes the number of distinct element in \( V \).

**Definition 3.1.** [16] Automorphism group of a finite graph

Given a graph \( \Gamma = (V,E) \), we can consider a subgroup of the group of bijections of the graph such that the mappings preserve adjacency and nonadjacency between vertices. This subgroup is referred to as the automorphism group, denoted \( G_{\text{aut}}(\Gamma) \).

Figure 3.1 shows a graph \( \Gamma \) containing three vertices, denoted 1, 2 and 3, and three edges. The transposition \( \sigma = (23) \) is an automorphism of \( \Gamma \).

![Figure 3.1: 3-cycle (a)](image)
Notice the orientation of the labels 2 and 3 of Figure 3.2. The graph \( \Gamma \) has essentially been flipped across the vertical axis by the automorphism \( \sigma \). There is still an edge between vertices 1 and 2, 2 and 3, and 1 and 3, as required.

In the example above,

\[
G_{aut}(\Gamma) = \{ e, (23), (13), (12), (123), (132) \} = S_3,
\]

where \( e \) is the identity element.

**Definition 3.2. Adjacency matrix**

Define the *adjacency matrix* \( \varepsilon \) of a graph \( \Gamma \) with \( n \) vertices to be the \( n \times n \) matrix where the \((i,j)\) position is equal to 1 if there exists an edge between vertices \( i \) and \( j \). Otherwise, the \((i,j)\) position of \( \varepsilon \) is equal to 0.

Since we have established there is an injective group homomorphism from \( S_n \) to \( U(n) \) in Section 2.1, we identify an injective group homomorphism from \( G_{aut}(\Gamma) \) to \( U(n) \). We can characterize the image of \( G_{aut}(\Gamma) \) in \( U(n) \) as follows.

**Theorem 3.3.** Permutation matrices of automorphisms of a graph commute with the adjacency matrix.

*Proof.* Let \( \bar{\sigma} \) be the automorphism \( \sigma \) represented by a matrix in \( U(n) \) of a graph \( \Gamma \) with \( n \) vertices and \( \varepsilon \) be the adjacency matrix of \( \Gamma \). We will prove \( \bar{\sigma} \varepsilon = \varepsilon \bar{\sigma} \). Recall that \( e_i \) is the \( i \)th vector in the canonical basis. We will verify that \( \bar{\sigma} \varepsilon (e_i) = \varepsilon \bar{\sigma}(e_i) \) for \( i \in \{1,..,n\} \). Define \( i \sim j \) to mean vertex \( i \) and \( j \) are related by an edge. Then the corresponding \((i,j)\) element of the adjacency matrix would be 1. We have the
following maps: $\bar{\sigma} : e_i \rightarrow e_{\bar{\sigma}(i)}$ and $\varepsilon : e_i \rightarrow \sum_{i \sim j} e_j$. Then the composition of functions gives us

$$\varepsilon \circ \bar{\sigma} : e_i \rightarrow \sum_{\bar{\sigma}(i) \sim j} e_j$$

and

$$\bar{\sigma} \circ \varepsilon : e_i \rightarrow \sum_{i \sim j} e_{\bar{\sigma}(j)}$$

Since $\bar{\sigma} \in G_{\text{aut}}(\Gamma)$, we know $\bar{\sigma}(i) \sim j$ if and only if $j = \bar{\sigma}(k)$ with $i \sim k$. Therefore,

$$\varepsilon \circ \bar{\sigma}(e_i) = \sum_{i \sim k} e_{\bar{\sigma}(k)} = \bar{\sigma} \circ \varepsilon(e_i)$$

and so, $\varepsilon$ and $\bar{\sigma}$ commute on the basis $\{e_1, ..., e_n\}$. Hence, $\varepsilon \bar{\sigma} = \bar{\sigma} \varepsilon$ and we are done. 

So, $G_{\text{aut}}(\Gamma) = \{\sigma \in S_n | \bar{\sigma} \varepsilon = \varepsilon \bar{\sigma}\} \leq S_n$, meaning $G_{\text{aut}}(\Gamma)$ is a subgroup of $S_n$ [16]. The main result here is that $G_{\text{aut}}(\Gamma)$ is a subgroup of $S_n$ where elements have permutation matrices that commute with $\varepsilon$. This implies we are able to represent automorphisms as matrices as well.

### 3.2 Automorphism Groups as C*-algebras

Now we will characterize the functions from $G_{\text{aut}}(\Gamma)$ to $C(G_{\text{aut}}(\Gamma))$. Consider a graph $\Gamma$ with $n$ vertices. Recall the universal commutative C*-algebra $C(S_n)$ from Section 2.2:

$$C(S_n) = C^*_{\text{comm}}(u_{ij}, 1 \leq i, j \leq n | u \text{ is an } n \times n \text{ magic unitary})$$

Similarly, $C(G_{\text{aut}}(\Gamma))$ is the universal commutative C*-algebra:

$$C(G_{\text{aut}}(\Gamma)) = C^*_{\text{comm}}(u_{ij} | u \text{ is an } n \times n \text{ magic unitary, } u\varepsilon = \varepsilon u)$$

Notice the additional requirement of commutativity with the adjacency matrix. As $G_{\text{aut}}(\Gamma)$ is a subgroup of $S_n$, we reflect this by a surjection from $C(S_n) \rightarrow C(G_{\text{aut}}(\Gamma)) \simeq$
\[C(S_n)/\langle \varepsilon u = u\varepsilon \rangle. \] So, \(C(G_{\text{aut}}(\Gamma))\) is isomorphic to \(C(S_n)\) quotiented by the ideal \(\langle \varepsilon u = u\varepsilon \rangle\). The quotient of \(C(S_n), C(S_n)/\langle \varepsilon u = u\varepsilon \rangle\), is the set of permutations that preserve adjacency which is what we require for permutations contained in \(G_{\text{aut}}(\Gamma)\). A proof that this holds can be found in [1, Proposition 9.1].

3.3 Liberation Process

In this section, we will explain the liberation process to reveal quantum automorphisms by, again, dropping the commutativity condition. We can define quantum automorphism groups, denoted \(G_{\text{aut}}^{+}(\Gamma)\), in terms of quantum permutation groups and classical automorphism groups.

First, recall the not necessarily commutative C*-algebra

\[C(S_n^+) = C^*(u_{ij}, 1 \leq i, j \leq n | u \text{ is an } n \times n \text{ magic unitary})\]

from Section 2.3. From Section 3.2, we established that \(C(G_{\text{aut}}(\Gamma)) \simeq C(S_n)/\langle \varepsilon u = u\varepsilon \rangle\). Similarly,

\[C(G_{\text{aut}}^{+}(\Gamma)) \simeq C(S_n^+)/\langle \varepsilon u = u\varepsilon \rangle\]

since \(C(G_{\text{aut}}^{+}(\Gamma))\) requires \(u\) with coefficients in \(C(G_{\text{aut}}^{+}(\Gamma))\) commute with the adjacency matrix \(\varepsilon\), while \(C(S_n^+)\) does not.

Second, we know that \(C(G_{\text{aut}}(\Gamma))\) has a commutativity condition like \(C(S_n)\) because \(G_{\text{aut}}(\Gamma) \subseteq S_n\). To examine \(C(G_{\text{aut}}^{+}(\Gamma))\), we can remove this commutativity condition. So,

\[C(G_{\text{aut}}^{+}(\Gamma))/\langle ab = ba \rangle \simeq C(G_{\text{aut}}(\Gamma))\]

In fact, we think of this surjection \(C(G_{\text{aut}}^{+}(\Gamma)) \rightarrow C(G_{\text{aut}}^{+}(\Gamma))/\langle ab = ba \rangle \simeq C(G_{\text{aut}}(\Gamma))\) as a restriction of functions from \(C(G_{\text{aut}}^{+}(\Gamma))\) to \(C(G_{\text{aut}}(\Gamma))\). Thus, we consider \(G_{\text{aut}}(\Gamma) \leq G_{\text{aut}}^{+}(\Gamma)\), meaning \(G_{\text{aut}}(\Gamma)\) is thought of as a subgroup of \(G_{\text{aut}}^{+}(\Gamma)\).
Our characterization from the liberation process above leads us to the following definition from Banica:

**Definition 3.4.** [2] Quantum Automorphism Group $G^+_{\text{aut}}(\Gamma)$

Let $\Gamma = (V, E)$ be a finite graph with $n$ vertices and adjacency matrix $\varepsilon \in M_n(\{0, 1\})$. The quantum automorphism group $G^+_{\text{aut}}(\Gamma)$ is the compact matrix quantum group $(\mathbb{C}(G^+_{\text{aut}}(\Gamma)), u)$, where $\mathbb{C}(G^+_{\text{aut}}(\Gamma))$ is the universal C*-algebra with generators $u_{ij}, 1 \leq i, j \leq n$ and relations

(i) $u_{ij} = u_{ij}^*, u_{ij}u_{ij} = \delta_{jk}u_{ij}, u_{ji}u_{ki} = \delta_{jk}u_{ji}, 1 \leq i, j, k \leq n$;

(ii) $\sum_{l=1}^n u_{il} = 1 = \sum_{l=1}^n u_{li}, 1 \leq i \leq n$;

(iii) $u\varepsilon = \varepsilon u$. 


Chapter 4

Quantum Symmetry of Graphs

4.1 Definition

We are now prepared to define quantum symmetry of graphs. Although there are a multitude of ways to determine if a given graph has quantum symmetry, we will describe this quality using quantum automorphism groups. We have two cases: either $C(G_{\text{aut}}^+(\Gamma))$ is commutative or $C(G_{\text{aut}}^+(\Gamma))$ is not commutative.

First, suppose $C(G_{\text{aut}}^+(\Gamma))$ is commutative. Then

$$C(G_{\text{aut}}^+(\Gamma))/\langle ab = ba \rangle = C(G_{\text{aut}}^+(\Gamma)) \simeq C(G_{\text{aut}}(\Gamma)).$$

Thus, $C(G_{\text{aut}}^+(\Gamma))$ is the same as $C(G_{\text{aut}}(\Gamma))$, and we say the graph $\Gamma$ has no quantum symmetry with $G_{\text{aut}}^+(\Gamma) = G_{\text{aut}}(\Gamma)$.

Second, suppose $C(G_{\text{aut}}^+(\Gamma))$ is not commutative. Then we know the mapping $C(G_{\text{aut}}^+(\Gamma)) \rightarrow C(G_{\text{aut}}^+(\Gamma))/\langle ab = ba \rangle \simeq C(G_{\text{aut}}(\Gamma))$ is strictly a surjection, not a bijection. Similar to how we understood quantum permutations, we now let the “extra elements” in $G_{\text{aut}}^+(\Gamma)$ correspond to quantum automorphisms. So, we say the graph $\Gamma$ has quantum symmetry with $G_{\text{aut}}(\Gamma) \subset G_{\text{aut}}^+(\Gamma)$. 
4.2 Examples of graphs without quantum symmetry

In this section, we will provide a few examples of graphs that do not have quantum symmetry. First, consider the graphs with one, two, and three vertices in Figure 4.1. As described in Section 2.4, when \( n < 4 \), \( C(S_n) \simeq C(S_n^+) \) and so \( C(S_n^+) \) does not contain any quantum permutations. It follows that \( C(G_{\text{aut}}(\Gamma)) = C(G_{\text{aut}}^+(\Gamma)) \) because, by definition, \( C(G_{\text{aut}}^+(\Gamma)) \simeq C(S_n^+)/\langle \varepsilon u = u \varepsilon \rangle = C(S_n)/\langle \varepsilon u = u \varepsilon \rangle \simeq C(G_{\text{aut}}(\Gamma)) \).

Therefore, \( G_{\text{aut}}^+(\Gamma) \) does not contain any “quantum automorphisms” when \( \Gamma \) denotes the graphs above with 1, 2, and 3 vertices. The four graphs above do not have quantum symmetry.

Before continuing, recall that by [16] and our definition of \( C(G_{\text{aut}}^+(\Gamma)) \) in Section 3.3, we know \( C(G_{\text{aut}}^+(\Gamma)) \) is a C*-algebra generated by \( u_{ij} \)'s with the following properties:

(i) \( u_{ij} = u_{ij}^* \), \( u_{ij} u_{ik} = \delta_{jk} u_{ij} \), \( u_{ji} u_{ki} = \delta_{jk} u_{ji} \), \( 1 \leq i, j, k \leq n \);

(ii) \( \sum_{l=1}^{n} u_{il} = 1 = \sum_{l=1}^{n} u_{li} \), \( 1 \leq i \leq n \);

(iii) \( u_{ji} u_{ik} = u_{ik} u_{ji} = 0 \), \( (i, k) \notin E \), \( (j, l) \in E \);

(iv) \( u_{ij} u_{kl} = u_{kl} u_{ij} = 0 \), \( (i, k) \notin E \), \( (j, l) \in E \).

I will reference (i), (ii), (iii), (iv) to refer to the properties of \( u_{ij} \)'s. Second, we will examine graphs with 4 or more vertices. Figure 4.2 is the cycle with 5 vertices, \( C_5 \). It does not have quantum symmetry. There are multiple ways to prove this. One way to
show $C_5$ does not have quantum symmetry is using a theorem that strongly regular
graphs of girth 5 do not have quantum symmetry [17]. We know $C_5$ is a strongly
regular graph because every pair of adjacent vertices has 0 common neighbors, every
pair of nonadjacent vertices has 1 common neighbor, and all vertices have exactly
two neighbors. I will prove that because of this property, $C_5$ does not have quantum
symmetry. First, I will prove a technical lemma.

**Lemma 4.1.** [17] Let $\Gamma$ be a finite, undirected graph and let $(u_{ij})$, $1 \leq i, j \leq n$, be
the generators of $C(G_{\text{aut}}(\Gamma))$. If we have $d(i, k) \neq d(j, l)$, then $u_{ij}u_{kl} = 0$.

**Proof.** Let $m := d(i, k)$ and $n := d(j, l)$. Without loss of generality, let $m < n$. First,
let $m = 0$. Then $i = k$ and $u_{ij}u_{kl} = u_{ij}u_{il} = 0$ by (i) since $d(j, l) = n > 0$ and $j \neq l$.
Thus, $u_{ij}u_{kl} = u_{kl}u_{ij}$ when $d(i, k) = 0$. Second, let $m = 1$. Then $i$ and $k$ must be
adjacent. Since $d(j, l) = n > 1$, we know $j$ and $l$ are nonadjacent. By (iii), we have
$u_{ij}u_{kl} = u_{kl}u_{ij} = 0$. Third, suppose $m \geq 2$. Then the distance between $i$ and $k$ is
greater than or equal to 2 and so, $i$ and $k$ are nonadjacent. Let the vertices on the
shortest path from $i$ to $k$ be $a_1, a_2, ..., a_{m-1}$.

$$u_{ij}u_{kl} = u_{ij} \left( \sum_{b_1} u_{a_1b_1} \right) \left( \sum_{b_2} u_{a_2b_2} \right) \cdots \left( \sum_{b_{m-1}} u_{a_{m-1}b_{m-1}} \right) u_{kl} \quad (\text{ii})$$

$$= \sum_{b_1, ..., b_{m-1}} u_{ij}u_{a_1b_1}u_{a_2b_2}...u_{a_{m-1}b_{m-1}}u_{kl}$$

However, since $d(j, l) > m$, there does not exist a path a length $m$ between $j$ and $l$.
Thus, for all $b_0 := j, b_1, ..., b_{m-1}, b_m := l$, there must exist at least two vertices $b_r, b_{r+1}$
such that $(b_r, b_{r+1}) \notin E$. Since $a_r, a_{r+1} \in E$ for all $r \in \{0, ..., m-1\}$, we know by (iii)
that \( u_{a,b} u_{a_{r+1} b_{r+1}} = 0 \). Therefore,

\[
\begin{align*}
  u_{ij} u_{kl} &= \sum_{b_1, \ldots, b_{m-1}} u_{ij} u_{a_1 b_1} u_{a_2 b_2} \cdots u_{a_{m-1} b_{m-1}} u_{kl} \\
  &= \sum_{b_1, \ldots, b_{m-1}} u_{ij} 0 u_{kl} \\
  &= 0
\end{align*}
\]

So, we have shown that for \( m \geq 2 \), \( u_{ij} u_{kl} = 0 = u_{kl} u_{ij} \). Thus, whenever \( d(i, k) \neq d(j, l) \), we know \( u_{ij} \) and \( u_{kl} \) commute.

\begin{center}
\textbf{Theorem 4.2.} [17] The strongly regular graph \( C_5 \) with girth five has no quantum symmetry.
\end{center}

\begin{proof}
Let \( d(i, k) \) denote the distance between vertex \( i \) and vertex \( k \), which is the length of the shortest path between \( i \) and \( k \) measured in edges. In order to show \( C_5 \) does not have quantum symmetry, we must prove that the \( u_{ij} \)’s that generate \( C(G^+_{\text{aut}}(C_5)) \) commute. We need to prove that \( u_{ij} \) and \( u_{kl} \) commute in three cases: (1) \( i = k, j = l \), (2) \( d(i, k) = d(j, l) = 1 \), and (3) \( d(i, k) = d(j, l) = 2 \). These are the only possible cases because \( d(i, k) \leq 2 \) for all vertices \( i, k \) since \( C_5 \) is a cycle of length 5. Additionally, we do not need to consider when \( d(i, k) \neq d(j, l) \) because of the Lemma 4.1.

First, consider \( u_{ij} \) and \( u_{kl} \) where \( i = k, j = l \). Then \( u_{ij} u_{kl} = u_{ij} u_{ij} = \delta_{jj} u_{ij} = u_{ij} = u_{ij} u_{ij} = u_{kl} u_{ij} \) by (i). Therefore, given \( i = k, j = l \), we know \( u_{ij} \) and \( u_{kl} \) commute.

Second, we will examine \( u_{ij} \) and \( u_{kl} \) when \( d(i, k) = d(j, l) = 1 \). Then there must exist an edge between \( i \) and \( k \), as well as between \( j \) and \( l \). We write \( \{i, k\} \in E \) and \( \{j, l\} \in E \). By (ii), we have \( \sum_{s=1}^{5} u_{is} = 1 \) since there are 5 vertices in \( C_5 \). Note that by (iii), \( u_{kl} u_{is} = 0 \) if \( \{l, s\} \notin E \) since we know \( \{i, k\} \in E \) by our set-up. Therefore,
we have
\[ u_{ij}u_{kl} = u_{ij}u_{kl} \left( \sum_{s:(l,s) \in E} u_{is} \right) \]

Suppose \( s \neq j \). Since \( \{l, s\} \in E \) and \( \{j, l\} \in E \), the only common neighbor of \( j \) and \( s \) is \( l \). Otherwise we would get a quadrangle, or 4-cycle, that is a contradiction to our assumption that \( C_5 \) has a girth of 5. Therefore, \( u_{ij}u_{kl} \left( \sum_{s:(l,s) \in E} u_{is} \right) = u_{ij}u_{kl}u_{is} \) for the \( s \) described above.

Then no other vertex of \( C_5 \) can be adjacent to both \( j \) and \( s \), i.e., for \( a \neq l \), \( \{a, s\} \notin E \) or \( \{a, j\} \notin E \). By (iii), either \( u_{ka}u_{is} = 0 \) or \( u_{ij}u_{ka} = 0 \) since \( \{i, k\} \in E \). Next, we know by (ii), \( \sum_{a=1}^{5} u_{is} = 1 \). Thus, we can write
\[
\sum_{a=1}^{5} u_{ka}u_{is} = u_{ij}u_{i} = u_{ij}u_{is}
\]

So, \( u_{ij}u_{is} = \delta_{js}u_{ij} = 0 \) since \( j \neq s \).

Therefore, we know \( s = j \) and

\[
u_{ij}u_{kl} = u_{ij}u_{kl} \left( \sum_{s:(l,s) \in E} u_{is} \right) = u_{ij}u_{kl}u_{ij}\]

Finally, we have \( u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij} \). By Lemma 2.5, this implies \( u_{ij} \) and \( u_{kl} \) commute. Therefore, our second case holds and \( u_{ij}u_{kl} \) commutes for \( d(i, k) = d(j, l) = 1 \).

The third case requires we prove \( u_{ij} \) and \( u_{kl} \) commute for \( d(i, k) = d(j, l) = 2 \). We will do this by considering the complement of \( C_5 \), denoted \( C_5^c \). We can find \( C_5^c \) by switching non-edges and edges. Let \( C_5 \) be labelled as the graph in Figure 4.3. Consistent with our ordering of \( C_5 \), we get \( C_5^c \) in Figure 4.4. Then any automorphism of \( C_5 \) is also an automorphism of \( C_5^c \) since automorphisms preserve adjacent and non-adjacent relationships between vertices and \( C_5^c \) just has edges and non-edges flipped. Notice that if \( d(i, j) = 2 \) in \( C_5 \), \( d(i, j) = 1 \) in \( C_5^c \). So, we can consider the \( u_{ij} \)'s that generate \( C(G^{+}_{aut}(C_5)) = C(G^{+}_{aut}(C_5^c)) \). However, we need only look at \( u_{ij} \) and \( u_{kl} \) such
that $d(i,k) = d(j,l) = 1$ for $i,j,k,l \in C^e_5$. In an analogous proof to the one for the second case, we can show that $u_{ij}$ and $u_{kl}$ commute here.

Therefore, we have shown that $u_{ij}$ and $u_{kl}$ commute in all three cases. Because $C(G^{+}_{\text{aut}}(C_5))$ is commutative, we know that $C_5$ does not have quantum symmetry and we are done.

Second, there is a planar algebra approach to prove $C_5$ does not have quantum symmetry that will be explored in the Chapter 5.

Lastly in this section, we will take a look at the Petersen graph. This graph is well known in graph theory because of its unique properties. We will describe the proof that shows the Petersen graph $P$ does not have quantum symmetry. Recall the following relevant properties of the Petersen graph. The Petersen graph is a strongly regular graph where adjacent vertices have 0 common neighbors, nonadjacent vertices have 1 common neighbor, and each vertex has a total of 3 neighbors. We will use the following labeling for the vertices of $P$: each vertex is denoted as an unordered pair

![Figure 4.5: Petersen graph](image)
\{a, b\} where \(a, b \in \{1, 2, 3, 4, 5\}\) and \(a \neq b\). Then there is an edge between two vertices if \(\{a, b\} \cap \{c, d\} = \emptyset\), meaning the intersection of the unordered pairs is empty.

Before proceeding, we define the following for an action of a group \(G\) on the set \(V\):

We say that \(G\) acts on a set \(V\) if there exists a function \(f : V \times G \to V\) sending \((v, g) \to v^g\) for each \(g \in G\) and \(v \in V\) that satisfies:

- If \(a \in V\) and \(g, h \in G\), then \(a^{(gh)} = (a^g)^h\),
- If \(e\) is the identity of \(G\), then \(a^e = a\) for all \(a \in V\).

Note that this is right action notation, so we represent \(g \in G\) acting on \(v \in V\) by \(v^g\). However, we also use the notation \(g(v)\) when convenient in this thesis to denote a group action. It will be labeled as needed.

The orbit of a vertex \(v \in V\) is the subset of \(V\) such that \(v^G := \{v^g | g \in G\} \subseteq V\).

The stabilizer of a vertex \(v \in V\) is the subgroup of \(G\) such that \(G_v := \{g \in G | v^g = v\} \subseteq G\). The Orbit-Stabilizer Theorem states that if a finite group \(G\) acts on a set \(V\), then for every \(v \in V\), we have \(|G| = |v^G| \cdot |G_v|\). A final definition:

**Definition 4.3.** A graph \(\Gamma\) has rank 3 if \(G_{\text{aut}}(\Gamma)\) has exactly three orbits on \(V \times V\) where \(G_{\text{aut}}(\Gamma)\) acts on \(V \times V\) by \((v_1, v_2)^g = (v_1^g, v_2^g)\) with \((v_1, v_2) \in V \times V\) and \(g \in G_{\text{aut}}(\Gamma)\). Let \(v_1, v_2 \in V\). Then the three orbits are \((v_1, v_1)^G\); \((v_1, v_2)^G\) where \(v_1\) and \(v_2\) are adjacent; and \((v_1, v_2)^G\) where \(v_1\) and \(v_2\) are nonadjacent.

The ideas in the next proof are from [11].

**Lemma 4.4.** The automorphism group of Petersen Graph \(P\) is \(S_5\).

**Proof.** First, note that the order of \(S_5\), denoted \(|S_5|\), is \(5! = 120\). Since each vertex is represented by a subset of size 2 of the set \(\{1, 2, 3, 4, 5\}\), we consider permutations of the set \(\{1, 2, 3, 4, 5\}\), written \(S_5\). First, we will show that \(S_5\) is a subgroup of \(G_{\text{aut}}(P)\). By construction, there is an edge between two vertices if \(\{a, b\} \cap \{c, d\} = \emptyset\).
Since $\sigma \in S_5$ is a bijection, we know $\{a, b\} \cap \{c, d\} = \emptyset$ if and only if $\{\sigma(a), \sigma(b)\} \cap \{\sigma(c), \sigma(d)\} = \emptyset$. Therefore, for $\sigma \in S_5$, $\sigma$ is an automorphism of $P$. Moreover, if $\sigma$ is not the identity of $S_5$, then $\sigma$ does not fix all vertices of $P$. So, we have $S_5 \leq G_{\text{aut}}(P)$.

Second, we will show that $A = G_{\text{aut}}(P)$ does not contain any automorphisms other than $\sigma \in S_5$. Let $A_0$ be the set of automorphisms that fix the vertex $v_0 = \{1, 2\} \in V$, meaning $A_{v_0} = A_0$. Let $v^A_0$ be the orbit of $v_0$. Under the action of $S_5$, $v^A_0 = V$ since $v_0$ can be sent to any other vertex of $P$ and so, $|v^A_0| = 10 = |V|$. By the Orbit-Stabilizer Theorem, we have $|G_{\text{aut}}(P)| = 10 \cdot |A_0|$. Now let $v_1 = \{3, 4\}$, which is adjacent to $v_0$ by construction, and $A_{0,1}$ be the stabilizer of $v_1$ contained in $A_0$. Under the action of $A_0$, $v_1$ can be mapped to any of the other three neighbors of $v_0$. So, we have $|A_0| = 3 \cdot |A_{0,1}|$. Thus, we now obtain by the Orbit-Stabilizer Theorem that $|G_{\text{aut}}(P)| = 30 |A_{0,1}|$. Let $v_2$ be another one of the three neighbors of $v_0$. There exists a permutation, $(34)$, that fixes $v_0$ and $v_1$ and interchanges the other two neighbors of $v_0$, denoted $v_2$ and $v_3$. Let $A_{0,1,2}$ be the stabilizer of $v_2$ in $A_{0,1}$. Fixing $v_2$ fixes the third neighbor as well. It follows that $|A_{0,1}| = 2 |A_{0,1,2}|$. Therefore, again using the Orbit-Stabilizer Theorem, $|G_{\text{aut}}(P)| = 60 |A_{0,1,2}|$. Now we have $v_0$ and all three neighbors of $v_0$, $v_1$, $v_2$, $v_3$, fixed. So, we have two options for an automorphism: either the identity or $(12)$, meaning 1 and 2 are sent to each other. Let $A_{0,1,2,4}$ be the set of automorphisms that fix $v_4$ in $A_{0,1,2}$ where $v_4$ is a neighbor of $v_1$ and not $v_0$. Since $P$ is strongly regular, we know there are 2 options for $v_4$. So, $|A_{0,1,2}| = 2 |A_{0,1,2,4}|$. With this, all vertices of the graph are fixed so $|A_{0,1,2,4}| = 1$. Therefore, $|G_{\text{aut}}(P)| = 120 |A_{0,1,2,4}| = 120 = |S_5|$. The only automorphisms in $|G_{\text{aut}}(P)|$ must be in $S_5$ since we have shown they have the same order. Therefore, $G_{\text{aut}}(P) = S_5$.  

\begin{theorem}
The Petersen Graph $P$ has no quantum symmetry, that is,

$$G^+_{\text{aut}}(P) = G_{\text{aut}}(P) = S_5$$

\end{theorem}
Proof. By the previous lemma, we know $G_{\text{aut}}(P) = S_5$. It remains to show that $G^+_{\text{aut}}(P) = G_{\text{aut}}(P) = S_5$. Let $(u_{ij})_{1 \leq i, j \leq n}$ be the generators of $G^+_{\text{aut}}(P)$. I will reference (i), (ii), (iii), (iv) to refer to the properties of $(u_{ij})$’s. In order to prove $G^+_{\text{aut}}(P) = G_{\text{aut}}(P)$, we must show that $u_{ij}u_{kl} = u_{kl}u_{ij}$, namely each $u_{ij}$ commutes with each $u_{kl}$. We need to prove that $u_{ij}$ and $u_{kl}$ commute in three cases: (1) $i = k$, $j = l$, (2) $\{i, k\} \in E$, $\{j, l\} \in E$, and (3) $\{i, k\} \notin E$, $\{j, l\} \notin E$.

Because $P$ is a rank 3 graph by Definition 4.3, we only need to consider when vertices are the same, adjacent, or nonadjacent. Additionally, we do not need to consider when $d(i, k) \neq d(j, l)$ because of Lemma 4.1.

First, consider $u_{ij}$ and $u_{kl}$ where $i = k$, $j = l$. Then $u_{ij}u_{kl} = u_{ij}u_{ij} = \delta_{jj}u_{ij} = u_{ij} = u_{ij}u_{ij} = u_{kl}u_{ij}$ by (i). Therefore, given $i = k$, $j = l$, we know $u_{ij}$ and $u_{kl}$ commute.

Now we will look at the second case: $\{i, k\} \in E$, $\{j, l\} \in E$. By (ii), $\sum_{s=1}^{10} u_{is} = 1$ since $P$ has 10 vertices. In fact, if we multiply $u_{kl} \cdot 1 = u_{kl} \sum_{s=1}^{10} u_{is}$, we only need to consider $s \in V$ such that $\{l, s\} \in E$ by (iii). Thus, we have

$$u_{ij}u_{kl} = u_{ij}u_{kl} \left( \sum_{s \in \{l, s\} \in E} u_{is} \right).$$

So, we have $s$ with $\{l, s\} \in E$. Consider $s \neq j$. By our set-up, we know $\{j, l\} \in E$. We can conclude $\{j, s\} \notin E$, as any adjacent vertices, such as $l$ and $j$, do not have any common neighbors. In fact, since $\{j, s\} \notin E$, we know the only common neighbor of $j$ and $s$ is $l$. Then any other vertex in $V$, say $a$, is not adjacent to both $s$ and $j$, i.e., $\{a, s\} \notin E$ or $\{a, j\} \notin E$. Then (iii) implies $u_{ij}u_{ka} = 0$ or $u_{ka}u_{is} = 0$ for $a \neq l$. Again using (ii), we can write

$$u_{ij}u_{kl}u_{is} = u_{ij} \left( \sum_{a=1}^{10} u_{ka} \right) u_{is} = u_{ij}u_{is}$$
since any \( a \neq l \) will lead to \( u_{ij}u_{ka}u_{is} = 0 \). Additionally, by (i), since \( j \neq s \), we have \( u_{ij}u_{is} = 0 \). Therefore, we only need to consider \( s = j \) and obtain

\[
    u_{ij}u_{kl} = u_{ij}u_{kl} \left( \sum_{s, \{l,s\} \in E} u_{is} \right) = u_{ij}u_{kl}u_{ij}
\]

The final step is using Lemma 2.5 to get \( u_{ij}u_{kl} = u_{kl}u_{ij} \) for \( \{i, k\} \in E, \{j, l\} \in E \).

The second case holds.

Finally, we will examine the third case: \( \{i, k\} \notin E, \{j, l\} \notin E \). First, let \( \{i, k\} \notin E, \{j, l\} \notin E, i \neq k, j = l \). Then \( u_{ij}u_{kl} = u_{ij}u_{kj} = \delta_{ik}u_{ij} = 0 = u_{kj}u_{ij} \) by (ii) and we are done.

Now let \( \{i, k\} \notin E, \{j, l\} \notin E, i = k, j \neq l \). Then \( u_{ij}u_{kl} = u_{ij}u_{il} = \delta_{jl}u_{ij} = 0 = u_{kj}u_{ij} \) by (ii) and we are done.

Finally, let \( \{i, k\} \notin E, \{j, l\} \notin E, i \neq k, j \neq l \). We know that a property of the Petersen graph is that every pair of nonadjacent vertices has exactly 1 common neighbor. Thus, there exists exactly one \( s \in E \) such that \( \{i, s\} \in E, \{k, s\} \in E \) and exactly one \( t \in E \) such that \( \{j, t\} \in E, \{l, t\} \in E \).

Step 1: The first step is to show that \( u_{ij}u_{kl} = u_{ij}u_{st}u_{kl} \). Since \( t \) is the only common neighbor of \( j \) and \( l \), we know \( \{j, b\} \notin E \) or \( \{b, l\} \notin E \) for all vertices \( b \neq t \). Therefore, since \( \{i, s\} \in E \) and \( \{k, s\} \in E \), by (iii), we have \( u_{ij}u_{sb} = 0 \) or \( u_{sb}u_{kl} = 0 \) for all \( b \neq t \).

By (ii), we get

\[
    u_{ij}u_{kl} = u_{ij} \left( \sum_{b=1}^{n} u_{sb} \right) u_{kl} = u_{ij}u_{st}u_{kl}
\]

Step 2: Second, we will show that \( u_{ij}u_{kl} = u_{ij}u_{st}u_{kl}(u_{ij} + u_{il} + u_{iq}) \), where \( q \) is the third neighbor of \( t \). Given \( u_{ij}u_{kl} = u_{ij}u_{st}u_{kl} \) from immediately above and \( u_{st}u_{kl} = u_{kl}u_{st} \) by Case 2 proved previously, we have \( u_{ij}u_{kl} = u_{ij}u_{st}u_{kl} = u_{ij}u_{kl}u_{st} \). Relations (ii) and
(iii) imply

\[ u_{ij}u_{kl} = u_{ij}u_{kl}u_{st} \left( \sum_{p; \{t,p\} \in E} u_{ip} \right) \]

Recall that each vertex of the Petersen graph has 3 neighbors. Since we know \( j \) and \( l \) are neighbors of \( t \), we have

\[ u_{ij}u_{kl} = u_{ij}u_{kl}u_{st}(u_{ij} + u_{il} + u_{iq}) = u_{ij}u_{st}u_{kl}(u_{ij} + u_{il} + u_{iq}) \]

where \( q \) is the third neighbor of \( t \).

Step 3: The third step is to prove \( u_{ij}u_{st}u_{kl}u_{il} = 0 \) and \( u_{ij}u_{st}u_{kl}u_{iq} = 0 \). First, by (i), since \( k \neq i \), we have \( u_{kl}u_{il} = \delta_{kl}u_{il} = 0 \), as \( k \neq l \). Thus, we have \( u_{ij}u_{st}u_{kl}u_{il} = 0 \).

Now consider \( u_{ij}u_{st}u_{kl}u_{iq} = 0 \). By (i), we have \( u_{ij}u_{st}u_{kq}u_{iq} = 0 \) since \( k \neq i \). Also by (i), we know \( u_{ij}u_{kq}u_{st}u_{iq} = 0 \) since \( k \neq i \). Using Case 2, we can swap \( u_{kq}u_{st} = u_{st}u_{kj} \), as \( \{s,k\} \in E \) and \( \{t,j\} \in E \) by our set-up. Thus, we have \( u_{ij}u_{kq}u_{st}u_{iq} = u_{ij}u_{st}u_{kq}u_{iq} = 0 \). And so, we can plug in

\[
\begin{align*}
    u_{ij}u_{st}u_{kl}u_{iq} &= u_{ij}u_{st}(u_{kl} + u_{kj} + u_{kq})u_{iq} & u_{ij}u_{st}u_{kq}u_{iq} &= 0, \quad u_{ij}u_{st}u_{kj}u_{iq} \\
    &= u_{ij}u_{st} \left( \sum_{a; (t,a) \in E} u_{ka} \right) u_{iq} & \text{neighbors of } t \text{ are } l, j, \text{ and } q \\
    &= u_{ij}u_{st} \left( \sum_{a=1}^{n} u_{ka} \right) u_{iq} & \text{(ii)} \\
    &= u_{ij}u_{st}u_{iq} & \text{(ii)}
\end{align*}
\]

We know \( u_{ij}, u_{st} \) commute by Case 2 since \( \{t, j\} \in E \), \( \{i, s\} \in E \). We showed above that \( u_{ij}u_{st}u_{kl}u_{iq} = u_{ij}u_{st}u_{iq} \). Then \( u_{ij}u_{st}u_{iq} = u_{st}u_{ij}u_{iq} \) by Case 2. Finally, we have \( u_{st}u_{ij}u_{iq} = 0 \) by (i) since \( q \neq j \).

Finally, by Step 2 we have \( u_{ij}u_{kl} = u_{ij}u_{st}u_{kl}(u_{ij} + u_{il} + u_{iq}) = u_{ij}u_{st}u_{kl}u_{ij} + u_{ij}u_{st}u_{kl}u_{il} + u_{ij}u_{st}u_{kl}u_{iq} \). By Step 3, we know \( u_{ij}u_{st}u_{kl}u_{il} = 0 \) and \( u_{ij}u_{st}u_{kl}u_{iq} = 0 \),
so we have $u_{ij}u_{kl} = u_{ij}u_{st}u_{kl}u_{ij} + 0 + 0 = u_{ij}u_{st}u_{kl}u_{ij}$. Using Step 1, we get $u_{ij}u_{kl} = u_{ij}u_{kl}u_{ij}$. By Lemma 2.5, we know this implies $u_{ij}$ and $u_{kl}$ commute. Therefore, we finally get that $u_{ij}u_{kl} = u_{kl}u_{ij}$ when $\{i, k\} \notin E$, $\{j, l\} \notin E$, $i \neq k$, $j \neq l$ and our fourth case holds.

Since $u_{ij}$ and $u_{kl}$ commute for all $i, j, k, l \in V$, we have shown that $G^+_{\text{aut}}(P) = G_{\text{aut}}(P)$. Therefore, the Petersen graph $P$ does not have quantum symmetry by definition.

4.3 Examples of graphs with quantum symmetry

**Definition 4.6.** [19] Disjoint automorphisms

Let $V = \{1, \ldots, r\}$ be the set of vertices of a graph $\Gamma$. We say that two automorphisms $\sigma : V \to V$ and $\tau : V \to V$ are disjoint, if $\sigma(i) \neq i$ implies $\tau(i) = i$ and vice versa, for all $i \in V$. Here we are using left action notation to denote the action of $\sigma$ and $\tau$ on elements in $V$.

**Theorem 4.7.** [19] For any finite graph $\Gamma$, if there exists two nontrivial, disjoint automorphisms $\sigma, \tau \in G_{\text{aut}}(\Gamma)$, where the order of $\sigma$ and $\tau$ are $n$ and $m$, respectively, $\Gamma$ has quantum symmetry.

**Proof.** Let $\sigma, \tau \in G_{\text{aut}}(\Gamma)$ be of order $n$ and $m$, respectively, where $\sigma$ and $\tau$ are disjoint and nontrivial. Recall the definition of a *-homomorphism in Definition 1.3. We want to show there is a surjective *-homomorphism $\phi : C(G^+_{\text{aut}}(\Gamma)) \to A$ where $A$ is defined as

$$A := C^* \left( p_1, \ldots, p_n, q_1, \ldots, q_m \middle| p_k = p_k^*, q_l = q_l^*, \sum_{k=1}^n p_k = 1 = \sum_{l=1}^m q_l \right) \simeq C^*(\mathbb{Z}_n * \mathbb{Z}_m)$$

So, $A \simeq C^*(\mathbb{Z}_n * \mathbb{Z}_m)$ is the C*-algebra of a noncommutative group with * denoting the free product here. We will define the surjective *-homomorphism $\phi : C(G^+_{\text{aut}}(\Gamma)) \to A$
by its images on the generators of $C(G_{\text{aut}}^+(\Gamma))$. This implies $C(G_{\text{aut}}^+(\Gamma))$ is noncommutative as the $p_k$ and $q_l$ are not necessarily commutative. Suppose $\Gamma$ has $r$ vertices.

Define the matrix

$$u' := \sum_{l=1}^{m} \tau^l \otimes q_l + \sum_{k=1}^{n} \sigma^k \otimes p_k - id_{M_r(A)} \in M_r(A)$$

where $M_r(A)$ denotes the set of $r \times r$ matrices with coefficients in $A$ and $\tau^l, \sigma^k$ represent the permutation matrices corresponding to $\tau^l, \sigma^k \in S_n$. The identity of $M_r(A)$ is $I_r$, the matrix with 1’s on the diagonal and 0’s elsewhere.

Now define

$$u'_{ij} := \sum_{l=1}^{m} \delta_{j \tau^l(i)} \otimes q_l + \sum_{k=1}^{n} \delta_{j \sigma^k(i)} \otimes p_k - \delta_{ij} \in A$$

where $\delta_{ij}$ is the indicator function: $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. So, $\delta_{j \tau^l(i)} = 1$ if $j = \tau^l(i)$ and 0 otherwise and $\delta_{j \sigma^k(i)} = 1$ if $j = \sigma^k(i)$ and 0 otherwise.

Now we need to show that $u'$ fulfills the relations of $u \in M_r(A)$, the fundamental representation of $G_{\text{aut}}^+(\Gamma)$ from Definition 3.4. By our set-up, $\tau^l, \sigma^k$ are disjoint automorphisms in $G_{\text{aut}}(\Gamma)$. By definition, $\tau^l \varepsilon = \varepsilon \tau^l$ and $\sigma^k \varepsilon = \varepsilon \sigma^k$ for all $1 \leq l \leq m$ and $1 \leq k \leq n$. We will first prove $u' (\varepsilon \otimes 1) = (\varepsilon \otimes 1) u'$:

$$u' (\varepsilon \otimes 1) = \left( \sum_{l=1}^{m} \tau^l \otimes q_l + \sum_{k=1}^{n} \sigma^k \otimes p_k - id_{M_r(A)} \right) (\varepsilon \otimes 1) = \sum_{l=1}^{m} \tau^l \varepsilon \otimes q_l + \sum_{k=1}^{n} \sigma^k \varepsilon \otimes p_k - (\varepsilon \otimes 1) = \sum_{l=1}^{m} \varepsilon \tau^l \otimes q_l + \sum_{k=1}^{n} \varepsilon \sigma^k \otimes p_k - (\varepsilon \otimes 1) = (\varepsilon \otimes 1) \left( \sum_{l=1}^{m} \tau^l \otimes q_l + \sum_{k=1}^{n} \sigma^k \otimes p_k - id_{M_r(A)} \right) = (\varepsilon \otimes 1) u'.$$

So, we have shown that the $u' (\varepsilon \otimes 1) = (\varepsilon \otimes 1) u'$ holds. Now we will examine $\sum_{i=1}^{r} u'_{ij}$. 
We want this sum to equal $1 \otimes 1$, and indeed it does:

$$\sum_{i=1}^{r} u'_{ij} = \sum_{i=1}^{r} \left( \sum_{l=1}^{m} \delta_{j\tau^{l}(i)} \otimes q_{l} + \sum_{k=1}^{n} \delta_{j\sigma^{k}(i)} \otimes p_{k} - \delta_{ij} \right)$$

$$= \sum_{i=1}^{r} \sum_{l=1}^{m} \delta_{j\tau^{l}(i)} \otimes \sum_{l=1}^{r} q_{l} + \sum_{i=1}^{r} \sum_{k=1}^{n} \delta_{j\sigma^{k}(i)} \otimes \sum_{k=1}^{n} p_{k} - \sum_{i=1}^{r} \delta_{ij}$$

$$= 1 \otimes \sum_{l=1}^{m} q_{l} + 1 \otimes \sum_{k=1}^{n} p_{k} - 1 \otimes 1$$

$$= 1 \otimes 1 + 1 \otimes 1 - 1 \otimes 1$$

$$= 1 \otimes 1.$$

Similarly, we can calculate

$$\sum_{i=1}^{r} u'_{ji} = \sum_{i=1}^{r} \left( \sum_{l=1}^{m} \delta_{j\tau^{l}(j)} \otimes q_{l} + \sum_{k=1}^{n} \delta_{j\sigma^{k}(j)} \otimes p_{k} - \delta_{ji} \right)$$

$$= \sum_{i=1}^{r} \sum_{l=1}^{m} \delta_{j\tau^{l}(j)} \otimes \sum_{l=1}^{r} q_{l} + \sum_{i=1}^{r} \sum_{k=1}^{n} \delta_{j\sigma^{k}(j)} \otimes \sum_{k=1}^{n} p_{k} - \sum_{i=1}^{r} \delta_{ji}$$

$$= 1 \otimes \sum_{l=1}^{m} q_{l} + 1 \otimes \sum_{k=1}^{n} p_{k} - 1 \otimes 1$$

$$= 1 \otimes 1 + 1 \otimes 1 - 1 \otimes 1$$

$$= 1 \otimes 1.$$

Thus, $\sum_{i=1}^{r} u'_{ij} = 1 \otimes 1 = \sum_{j=1}^{r} u'_{ij}$. Since $\sigma$ and $\tau$ are disjoint, we have

$$u'_{ij} = \sum_{l=1}^{m} \delta_{j\tau^{l}(i)} \otimes q_{l} + \sum_{k=1}^{n} \delta_{j\sigma^{k}(i)} \otimes p_{k} - \delta_{ij} = \begin{cases} 
\sum_{k \in N_{ij}} p_{k}, & \text{if } \sigma(i) \neq i \\
\sum_{l \in M_{ij}} q_{l}, & \text{if } \tau(i) \neq i \\
\delta_{ij} & \text{otherwise,}
\end{cases} \tag{4.1}$$

where $N_{ij} = \{ k \in \{1, \ldots, n\}, \sigma^{k}(i) = j \}$ and $M_{ij} = \{ l \in \{1, \ldots, n\}, \tau^{l}(i) = j \}$.

Therefore, in any case, $u'_{ij}$ is a projection. By the universal property, we have a
*-homomorphism $\phi : C(G^+_{\text{aut}}(\Gamma)) \to A$ by sending $u \to u'$.

We now need to show the *-homomorphism $\phi$ is a surjection. First consider $\sigma$. We know $|\sigma| = n$. We can write $\sigma$ as a product of disjoint cycles, as we can with any permutation. Then there exists $s_1, \ldots, s_a \in V$ such that for all $k_1 \neq k_2$ where $k_1, k_2 \in \{1, \ldots, n\}$, we have

$$(\sigma^{k_1}(s_1), \ldots, \sigma^{k_1}(s_a)) \neq (\sigma^{k_2}(s_1), \ldots, \sigma^{k_2}(s_a)).$$

Similarly, there exists $t_1, \ldots, t_b \in V$ such that for $l_1 \neq l_2$ where $l_1, l_2 \in \{1, \ldots, m\}$,

$$(\tau^{l_1}(t_1), \ldots, \tau^{l_1}(s_b)) \neq (\tau^{l_2}(t_1), \ldots, \tau^{l_2}(t_b)).$$

Thus, we have

$$
\phi(u_{s_1\sigma^{k}(s_1)} \cdots u_{s_a\sigma^{k}(s_a)}) = u'_{s_1\sigma^{k}(s_1)} \cdots u'_{s_a\sigma^{k}(s_a)} = p_k
$$

$$
\phi(u_{t_1\tau^{l}(t_1)} \cdots u_{t_b\tau^{l}(t_b)}) = u'_{t_1\tau^{l}(t_1)} \cdots u'_{t_b\tau^{l}(t_b)} = q_l
$$

for all $k \in \{1, \ldots, n\}$ and $l \in \{1, \ldots, m\}$. Since $A$ is generated by $p_k$ and $q_l$, we know every point in $A$ will be attained by $\phi$ of an element in $C(G^+_{\text{aut}}(\Gamma))$. Therefore, $\phi$ is a surjection by construction.

Now we have shown that there exists a surjective *-homomorphism $\phi : C(G^+_{\text{aut}}(\Gamma)) \to A$. Since $A$ is noncommutative, we know $C(G^+_{\text{aut}}(\Gamma))$ is as well. Therefore, $\Gamma$ has quantum symmetry by definition.

Figure 4.6 is the $K_5$ graph, the complete graph on 5 vertices. Notice that there exists an edge between every pair of vertices of the $K_5$ graph. Thus, any permutation in $S_5$, which acts on the set $\{1, 2, 3, 4, 5\}$, will preserve adjacency between vertices and so $G_{\text{aut}}(K_5) = S_5$. Theorem 4.7 applies here because $S_5$ contains nontrivial disjoint automorphisms with orders greater than or equal to 2. For example, $(12)$ and $(34)$.
are disjoint and both contained in $S_5$. The following section describes a script that returns the set of disjoint automorphisms of a given graph. $S_5$ contains 35 disjoint pairs, calculated in the GAP script. Additionally, since $K_5$ has 5 vertices, any disjoint automorphisms are either both of order 2, or one is of order 2 and the other is of order 3. We calculate that there are \( \binom{\frac{5}{2}}{2} \) = 15 pairs of disjoint 2-cycles. Then there are \( \binom{\frac{3}{2}}{2} \) = 20 disjoint pairs containing a 2-cycle and a 3-cycle. Thus, there are 15 + 20 = 35 pairs of disjoint automorphisms. Therefore, we know $K_5$ has quantum symmetry.

### 4.4 GAP Script

I wrote a script using GAP - Groups, Algorithms, and Programming - a System for Computational Discrete Algebra to determine if a given graph contains disjoint automorphisms [15]. The method works by systematically checking if there exists an intersection between the points moved by pairs of automorphisms of the graph. We exclude checking the trivial automorphism that contains only the identity because it fixes all points, meaning the points it moves (none) won’t intersect with any other permutation. Since the trivial automorphism only contains 1 element, it does not apply in the theorem above. Additionally, we do not need to check any automorphisms that move $n-1$ or $n$ vertices, where $n$ is the number of vertices (for $K_5$, automorphisms of size 4 or 5) because these are guaranteed to intersect with any other automorphism in the automorphism group. Finally, the program returns a list of any found pairs of disjoint automorphisms.
Chapter 5

Planar Algebra Approach

We will prove that the $C_5$ graph does not have quantum symmetry using a planar algebra approach based on a section in Yunxiang Ren’s thesis [14]. To begin, we will provide background about planar algebras and how it relates to quantum symmetry of graphs.

5.1 Background

Definition 5.1. [10, 7] Planar Tangle

A planar $k$-tangle is a collection of $n + 1$ disjoint discs, consisting of the unit disc $D_0$ in $\mathbb{C}$ with $n$ (possibly empty) disjoint discs, which are joined by disjoint smooth curves, called strings. Each disc $D_i$, $0 \leq i \leq n$, will have an even number $2k_i \geq 0$ of marked points on its boundary (with $k = k_0$). These strings may form closed loops which touch no disc. There is a coloring of each region created by the discs and strings. The points on the boundary of each disc where a string touches are denoted as “marked.” These marked points divide the boundary of each disc into intervals.

Figure 5.1 is an example of a planar 4-tangle [10]. We know it is a 4-tangle because there are 4 shaded regions touching the boundary of $D_0$ and $k = k_0$ by
Definition 5.1. The strings lie in the area within the unit disc $D_0$ that complements the inner discs. As in, the discs are not included in the coloring of regions created by strings. Notice how the light and dark regions are adjacent, no light region is next to another light region. Each $\$\$ denotes the chosen light region whose closure meets that disc. The location of dark and light regions, marked points, strings and discs all provide information about the function of the tangle. The operation between tangles is composition.

Definition 5.2. [18] Planar Algebra

A planar algebra $\mathcal{P}$ is a collection of finite-dimensional vector spaces $(\mathcal{P})_{n \in \mathbb{N} \cup \{-, +\}}$ such that each planar tangle $T$ of degree $k$ with $n$ internal disks $D_1, \ldots, D_n$ of degree $k_1, \ldots, k_n$ respectively, yields a multilinear map

$$Z_T : \bigotimes_{1 \leq i \leq n} \mathcal{P}_{k_i} \to \mathcal{P}_k$$

The composition of those maps has to be compatible with the composition of planar tangles.

Definition 5.3. [18] Spin Planar Algebra

A spin planar algebra $\mathcal{P}$ of degree $k \in \mathbb{N}$ is the vector space of all linear functionals $f : W^k \to \mathbb{C}$ where $W$ is a finite dimensional vector space. A functional $f : X \to \mathbb{C}$ is an operator from a vector space $X$ into the field $\mathbb{C}$. Because $f$ is a linear functional, we know $f(a + b) = f(a) + f(b)$ and $f(ca) = cf(a)$ where $a, b \in W^k$ and $c \in \mathbb{C}$. Also,
$W^\otimes k$ denotes a $k$-tangle of an element in $W$. We can relate $W^\otimes k$ and $f$ by

$$f : W^k \rightarrow \mathbb{C} \leftrightarrow \sum_{i_1, \ldots, i_k} f(e_{i_1}, \ldots, e_{i_k})e_{i_1} \otimes \ldots \otimes e_{i_k} \in W^\otimes k$$

where $e_{i_1}, \ldots, e_{i_k}$ denotes the basis of $W$.

Note that a planar algebra $P$ is generated by a set of elements $X$ if for every element $a \in P_n$, there exists a planar tangle $T$ such that $Z_T(x_1, \ldots, x_k) = a$ for some $x_i \in X$. [18]

**Definition 5.4.** [18] Group-action Planar Algebra

Let $W$ be an $n$-dimensional vector space with basis $\{e_1, \ldots, e_n\}$. Let $P$ be the associated spin planar algebra, as defined above. Assume $G$ is a subgroup of the symmetric group $S_n$. Then $G$ has the natural action $\alpha : W \times G \rightarrow W$, where $\alpha$ maps $(e_i, g)$, with $e_i \in W$ and $g \in G$, to $e_{ig}$. This can be extended to get $\alpha^\otimes k : W^\otimes k \times G \rightarrow W^\otimes k$ where $\alpha^\otimes k$ maps $(e_{i_1} \otimes \ldots \otimes e_{i_k}, g) \rightarrow e_{i_1g} \otimes \ldots \otimes e_{i_kg}$. So, the group-action planar algebra $P^G$ is the fixed point algebra of the group action:

$$P^G_n = \{ s \in P_n : \alpha^\otimes n(x, g) = x \text{ for all } g \in G \}$$

**Definition 5.5.** [14] A (shaded) planar algebra is a family of vector spaces $P_{n,\pm}$, $n \in \mathbb{N}$, with multilinear maps determined by planar tangles:

$$Z_T : \otimes_{D \in D_T} P_{\partial D} \rightarrow P_{\partial D_T}$$

where $D_T$ is the set of input disks of $T$ and $D_T$ is the output disk. For a disc $D$, $P_{\partial D}$ is $P_{n,+}$ where $n$ is the half of number of the boundary points of $D$ and the $\$ is in a unshaded region and $P_{n,-}$ where $n$ is the half of number of the boundary points of $D$ and the $\$ is in a shaded region.

**Definition 5.6.** Generating Property [18]
Let $\Gamma$ be a finite graph. We say $\Gamma$ has the generating property if the group-action planar algebra $\mathcal{P}^{G_{\text{aut}}(\Gamma)}$, where the acting group is the automorphism group $G_{\text{aut}}(\Gamma)$ of $\Gamma$, is generated by $\mathcal{P}_2^{G_{\text{aut}}(\Gamma)}$. This is saying that the entire group-action planar algebra $\mathcal{P}^{G_{\text{aut}}(\Gamma)}$ is equal to the planar subalgebra generated by the 2-box space $\mathcal{P}_2^{G_{\text{aut}}(\Gamma)}$, a group action planar algebra as defined in Definition 5.4.

In the following section we will tie together the generating property and quantum symmetry of graphs.

### 5.2 Coherent Configurations and Orbital Algebras

First, we have the following definition:

**Definition 5.7.** [12] Coherent Configurations

Given a set $\Omega$, a *coherent configuration* is a partition $\mathcal{R} = \{R_i : i \in I\}$ of $\Omega^2 := \Omega \times \Omega$ into relations satisfying:

(i) Diagonal Relations: there is a subset $D \subseteq I$ of the index set such that $\{R_d : d \in D\}$ is a partition of the diagonal $\{(\alpha, \alpha) : \alpha \in \Omega\}$.

(ii) Converse: for each relation $R_i$, its converse $\{(\beta, \alpha) : (\alpha, \beta) \in R_i\}$ is also a relation, say $R_i' \in \mathcal{R}$. (Note: $R_i' = R_i$ is allowed.)

(iii) Intersection Numbers: for all $i, j, k \in I$ and any $(\alpha, \beta) \in R_k$, the number of $\gamma \in \Omega$ such that $(\alpha, \gamma) \in R_i$ and $(\gamma, \beta) \in R_j$ is a constant $p_{ij}^k$ that does not depend on $\alpha$ or $\beta$.

**Definition 5.8.** Characteristic Matrices

Given any coherent configuration $\mathcal{R} = \{\mathcal{R}_i : i \in I\}$ on $\Omega$, one can construct a matrix $A_i$ for each $i \in I$:

$$(A_i)_{\alpha \beta} := \begin{cases} 1, & \text{if } (\alpha, \beta) \in \mathcal{R}_i \\ 0, & \text{otherwise} \end{cases}$$
Then \( \{ A_i : i \in I \} \) is the set of characteristic matrices of the coherent configuration.

**Definition 5.9.** Coherent Algebra

We construct an algebra as follows: the linear span of the \( \{ A_i : i \in I \} \), i.e., the set of all matrices

\[
\sum_{i \in I} c_i A_i, (c_i \in \mathbb{C})
\]

is the self-adjoint, unital algebra containing the all-ones matrix that is closed under the entrywise product. The algebra \( \mathcal{A}(\mathcal{R}) \) is called a coherent algebra. Note that an element \( a \) is self-adjoint if \( a^* = a \).

Since any coherent algebra is closed under the entrywise product by construction, there is a unique basis of 0,1-matrices which define a partition of \( \Omega \times \Omega \). Thus, coherent configurations coincide with coherent algebras and these two terms are really representing the same idea [12].

**Definition 5.10.** [12] Consider a graph \( \Gamma \) where \( V \) is the set of vertices of the graph and \( G = G_{\text{aut}}(\Gamma) \). Suppose \( a, b \in V \). We have a natural action on pairs in \( V \times V \):

\[
(a, b)^g := (a^g, b^g).
\]

The orbit \( (a, b)^G \) under this action on pairs of vertices is called an **orbital**.

**Definition 5.11.** [12] Let \( V \) denote the set of vertices of the graph \( \Gamma \) and \( a, b, c, d \in V \). Suppose \( u_{ij} \)'s generate \( C(G_{\text{aut}}^+(\Gamma)) \). Define the equivalence relation \( \sim_{2} \) on \( V \times V \) by \( (a, b) \sim_{2} (c, d) \) if \( u_{ac}u_{bd} \neq 0 \).

Then given a quantum automorphism group \( G_{\text{aut}}^+(\Gamma) \), the **quantum orbitals** of \( G_{\text{aut}}^+(\Gamma) \) are the equivalence classes of the relation \( \sim_{2} \) defined above [12].

**Definition 5.12.** [18] Let \( \Gamma = (V, E) \), where \( V \) is the set of vertices and \( E \) is the set of edges, be a graph. We associate the following three coherent algebras for the graph.
(i) The coherent algebra of $\Gamma$, $\mathcal{CA}(\Gamma)$, is the smallest coherent algebra containing the adjacency matrix $\varepsilon$.

(ii) The orbitals of $G_{aut}(\Gamma)$ on $V$ form a coherent configuration. The corresponding coherent algebra is called the orbital algebra $O(\Gamma)$.

(iii) The quantum orbitals of $G_{aut}^+(\Gamma)$ on $V$ form a coherent configuration. The corresponding coherent algebra is called the quantum orbital algebra $QO(\Gamma)$.

As we established in Section 3.3, we consider $G_{aut}(\Gamma) \leq G_{aut}^+(\Gamma)$. Additionally, for $\sigma \in G_{aut}(\Gamma)$, we have

$$u_{ij}(\sigma) = \begin{cases} 1, & \text{if } \sigma(i) = j \\ 0, & \text{otherwise} \end{cases}$$

Then if $\sigma(i) = j$ and $\sigma(k) = l$, we obtain $(u_{ij}u_{kl})(\sigma) = u_{ij}(\sigma)u_{kl}(\sigma) = 1 \cdot 1 = 1 \neq 0$. So, in this case, $u_{ij}u_{kl} \neq 0$. Thus, if $(i,j)$ and $(k,l)$ are in the same classical orbital of $\Gamma$, we can conclude they are also in the same quantum orbital. Additionally, the quantum orbital algebra must contain the adjacency matrix of $\Gamma$ since $u_{ij}u_{kl} \neq 0$ is not possible if one of $(i,j)$ and $(k,l)$ are adjacent and the other is nonadjacent. Therefore, the quantum orbital algebra must contain the coherent algebra of $\Gamma$. It follows that $\mathcal{CA}(\Gamma) \subseteq QO(\Gamma) \subseteq O(\Gamma)$ [12].

**Proposition 5.13.** [18] Let $\Gamma$ be a graph with $O(\Gamma) = QO(\Gamma)$. Then $\Gamma$ has the generating property if and only if $\Gamma$ has no quantum symmetry.

**Proposition 5.14.** [18] Let $\Gamma$ be a graph with $O(\Gamma) = QO(\Gamma)$. If $G_{aut}(\Gamma)$ contains disjoint automorphisms, then $\Gamma$ does not have the generating property. Thus $\Gamma$ does not have quantum symmetry.

The previous proposition follows from the method proved in Section 4.3 that shows a graph does not have quantum symmetry if it has disjoint automorphisms.
It is significant that we can draw a conclusion about quantum symmetry of a graph without directly studying the automorphism group of the given graph.

A class of graphs with $O(\Gamma) = QO(\Gamma)$ is distance-transitive graphs, defined in Chapter 1. This is known because the coherent algebra $CA(\Gamma)$ of a distance-transitive graph $\Gamma$ of diameter $d$ is $(d + 1)$-dimensional. Additionally, the orbital algebra $O(\Gamma)$ of a distance-transitive graph $\Gamma$ of diameter $d$ is $(d + 1)$-dimensional. This implies $QO(\Gamma)$ is also $(d + 1)$-dimensional since $CA(\Gamma) \subseteq QO(\Gamma) \subseteq O(\Gamma)$ [18].

We then have

**Proposition 5.15.** [18] Let $\Gamma$ be a distance-transitive graph. Then $\Gamma$ has the generating property if and only if it has no quantum symmetry.

### 5.3 $C_5$ does not have quantum symmetry

We will consider the 5-cycle $C_5$ in Figure 5.2.

![Figure 5.2: $C_5$ graph](image)

**Definition 5.16.** A basis of the 2-box space for $C_5$, $P_{2,+}$, is shown in Figure 5.3, where

![Figure 5.3: 2-box space basis for $C_5$](image)

the first drawing denotes the identity, the second denotes the matrix of all ones, and
the third represents the adjacency matrix in the following way. They are analogous to the characteristic functions of orbits of 2-tensors of vertices, which correspond to the identity, all ones matrix, and adjacency matrix. Additionally, $S$ is the characteristic function of the orbit $[u \otimes v]$ such that $\{u, v\}$ is an edge of the graph $C_5$. Consider the first diagram. Suppose we input $u$ on top and $v$ on the bottom. Then $u = v$ and we have the identity. Now consider the second diagram. If we input $u$ in the blue area and $v$ in the pink area, they can have any relation. Finally, look at the third diagram. Inputting $u$ in the blue area and $v$ in the pink area of the diagram, we see $u$ and $v$ are adjacent.

**Definition 5.17.** The element $R = \chi_{[u \otimes v \otimes u \otimes v]} \in \mathcal{P}_{4,+}$ is represented by the tangle in Figure 5.4. $R$ is a symmetric braiding viewed as a linear map from $V \otimes V \rightarrow V \otimes V$.

![Figure 5.4: R](image)

The generator flips vertices $v$ and $v$, i.e., $R$ sends $(u, v)$ to $(v, u)$. We will input $u$ and $v$ on top of the diagram. The area the the first and third slot is connected, as well as the second and fourth. So, if the input is $u$ and $v$, then the output is $v$ and $u$.

**Theorem 5.18.** The planar algebra $\mathcal{P}$ of $C_5$ is generated by $\{S, R\}$.

*Proof.* Note this proof is based on a proof in [14]. Let $\mathcal{A}_*$ be the planar subalgebra generated by $\{S, R\}$. Recall that $S$ is the characteristic function of the orbit of two adjacent vertices and $R$ flips the position of the two input vertices. We will show that these two operators generate the entire planar algebra $\mathcal{P}$.

We first draw the $C_5$ graph in the unit disc of $\mathbb{R}^2$ such that the vertices are put on the unit circle with equal distances. The unit circle is divided into five intervals by
these vertices and we put a $ near the leftmost interval as illustrated in the following:

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure5_5.png}
\caption{C$_5$ on a unit circle}
\end{figure}

Next we construct an element in $P_{5,+}$ by the following procedure. The vertices of the $C_5$ graph are put on the unit circle with equal distances. We put two boundary points near the position of each vertex of the $C_5$ graph. Since the degree of each vertex is 2, we apply the operation in the neighborhood of each vertex:

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure5_6.png}
\caption{Operation of each vertex}
\end{figure}

If two vertices are connected by an edge in the $C_5$ graph, we apply the operation in Figure 5.7 to the edge.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure5_7.png}
\caption{Operation to each edge}
\end{figure}

So, Figure 5.8 shows the element $X$ in $P_{5,+}$. By construction, each black region of $X$ that touches the output disc corresponds to a vertex of the $C_5$ graph. Let $D_i$ be the black region which corresponds to the vertex $v_i$ and $\sigma$ be a function from the black regions of $X$ to the set $V$. It follows from the definition of $R$ that $\sigma$ is uniquely determined by the value $\sigma(D_i)$, $1 \leq i \leq 5$. Then $X(\sigma(D_1) \otimes \sigma(D_2) \otimes \sigma(D_3) \otimes \sigma(D_4) \otimes$
\( \sigma(D_5) \) is nonzero if and only if there exists an edge between \((v_i, v_j)\) in \(C_5\). This occurs when \(|i-j| \mod 5 = 1\). Therefore, we have \(X = \chi[v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5] \in \mathcal{A}_*\), where \(v_i \in V\) and \(v_i \neq v_j\) for \(i \neq j\).

Next we will introduce the following annular tangle \(\Phi_j^m\) in \(\mathcal{A}_*\) in Figure 5.9. This tangle is drawn horizontally, rather than as a disk. We can connect each end of the tangle to create a disk if desired, but it has the same meaning drawn this way. It follows that \(\Phi_j^m(\chi[v_1 \otimes v_2 \otimes ... v_j + 1 \otimes v_{j+1} \otimes ... v_m]) = \chi[v_1 \otimes v_2 \otimes ... v_j - 1 \otimes v_{j+1} \otimes ... v_m]\) where the square brackets \([\cdot]\) indicate the orbit of the tensor under the automorphism. Notice that \(v_{ij+1}\) and \(v_{ij}\) are switched. Therefore, we obtain an action of \(S_m\) on \(\mathcal{T}_{m,+}\) induced by the planar tangle \(\Phi_j^m\), \(1 \leq j \leq m\). For a permutation \(\sigma \in S_m\), let \(\Phi^m_\sigma\) be the corresponding planar tangle. For example, say we have \(\chi[v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5]\). Let \(\sigma \in S_5\) be the permutation \((4 5)\). Thus we have \(\chi[v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5] = \Phi_5^0(X) \in \mathcal{A}_*\). This tangle allows us to place the \(v_{ij}\) in any order we want.

Now let \(\chi[v_{k1} \otimes v_{k2} \otimes ... v_{km}]\) be the characteristic function of an arbitrary orbit. Since we have an action of \(S_m\) on \(\mathcal{T}_{m,+}\), one can assume that \(k_1 = k_2 = ... = k_{a_1} = t_1; k_{a_1+1} = k_{a_1+2} = ... = k_{a_2} = t_2; k_{a_2+1} = k_{a_2+2} = ... = k_{a_3} = t_3\) without loss of generality. This means we do not need to consider repeated elements. Instead, we
need only address one copy of each vertex.

Now we consider the following annular tangle $E_j^m$ in $A_\circ$ where the $i$-th boundary point on the input disc is connected to the $(i+1)$-th boundary point to remove the $j$-th vertex. Note that the $j$-th vertex is specified on the tangle while the boundary points that represent this vertex are $i = 2j - 1$ and $i + 1 = 2j$. It follows that $E_j^m(\chi[v_{k_1} \otimes v_{k_2} \otimes v_{k_3} \otimes \ldots \otimes v_{k_m}]) = \chi[v_{k_1} \otimes v_{k_3} \otimes \ldots \otimes v_{k_m}]$. Here $v_{k_2}$ was removed. To show $\chi[v_{k_1} \otimes v_{k_2} \otimes v_{k_3} \otimes \ldots \otimes v_{k_m}] \in A_\circ$, we just need to show that $\chi[v_{k_1} \otimes v_{k_3} \otimes \ldots \otimes v_{k_m}] \in A_\circ$. By repeating this procedure, we only need to show $\chi[v_{t_1} \otimes v_{t_3} \otimes \ldots \otimes v_{t_l}] \in A_\circ$

We are now only considering one copy of each vertex and used $E_j^m$ to remove unnecessary elements.

Let $\Psi_{t_1, t_2, \ldots, t_l}$ be an $(10, 2l)$-tangle such that the inputs $v_1, v_2, v_3, v_4, v_5$ are either connected to the corresponding boundary point of the output disc or connected to itself.

We have an example in Figure 5.11. In this tangle, the input is vertices $v_1, v_2, v_3, v_4, v_5$ and the output is $v_2$ and $v_4$. 
It follows that

\[ \chi[j_{t_1} \otimes j_{t_2} \otimes \ldots \otimes j_{t_l}] = \Psi_{t_1, t_2, \ldots, t_l}(\chi[v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5]) \in \mathcal{A}. \]

So, we have shown that \( \chi[j_{t_1} \otimes j_{t_2} \otimes \ldots \otimes j_{t_l}] \) can be gotten using \( \Psi \), which is contained in \( \mathcal{A} \). Therefore, the planar algebra \( \mathcal{P} \) equals the planar subalgebra \( \mathcal{A} \) which is generated by \( \{S, R\} \). Thus, the planar algebra \( \mathcal{P} \) is generated by \( \{S, R\} \).

To show \( \mathcal{P} \) is generated by its 2-box space, we need to show that \( R \) is in the 2-box space. Let \( \mathcal{Q} \) be the planar subalgebra generated by its 2-box space, \( \mathcal{P}_{2,\pm} \).

**Theorem 5.19.** The subgroup planar algebra \( \mathcal{P} \) for \( C_5 \) is generated by its 2-box space.

**Proof.** Let \( T = \chi[v \otimes u] \), where \( v \neq u \) and \( (v, u) \) is not an edge of the \( C_5 \) graph. Then we know that a pairs of vertices can either be adjacent, nonadjacent, or the same vertex. Thus, we have the equation in Figure 5.12. The tangle on the left hand side denotes all relations between vertices. The first element on the right hand side represents the identity, and second represents adjacent vertices. Then the third tangle on the right hand side denotes nonadjacent vertices.

Then compose these tangles with \( R \) to get the equation in Figure 5.13. We will

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**Figure 5.12:** Equation 1

**Figure 5.13:** Equation 1 composed with \( R \)

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show that every term on the right hand side of Equation 5.13 is in the 2-box space. The characteristic function is generated by summing over the three terms on the right hand side of Equation 5.13: the first denotes when \( v_i = v_j \), the second when \( v_i \) and \( v_j \) are adjacent, and then third represents when \( v_i \) and \( v_j \) are nonadjacent. Note for each of these tangles, the input is two vertices, \( v_i \) and \( v_j \), and the output is these two vertices swapped.

Consider the first element on the right hand side of Equation 5.13. Let \( v_i \) and \( v_j \) be inputs in this tangle, where \( v_i \) and \( v_j \) are denoted by the blue area. Then since the area is connected in the upper part of the tangle, we know \( v_i = v_j \). This implies \( i = j \) as well. Therefore, this tangle is really the identity and must be in the 2-box space.

Now we will prove that the second term belongs to the planar subalgebra \( Q_k \).

We consider the element in Figure 5.14. Let \( [v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_4}] \) be a nonzero state. Therefore, \( (v_{i_j}, v_{i_{j+1}}) \) is an edge in the \( C_5 \) graph for any \( j \in \{1, 2, 3, 4\} \) modulo 4. However, there is no subgraph of the \( C_5 \) graph isomorphic to a square. Therefore, we have either \( v_{i_1} = v_{i_3} \) or \( v_{i_2} = v_{i_4} \). There are three ways we can have a 4-cycle in the \( C_5 \) graph (using the representation of the \( C_5 \) graph from the previous theorem): \( v_1 \rightarrow v_2 \rightarrow v_1 \rightarrow v_2, v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_2, \) and \( v_1 \rightarrow v_2 \rightarrow v_1 \rightarrow v_5 \). So,

\[
\chi[v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_4}] = \chi[v_{i_1} \otimes v_{i_2} \otimes v_{i_1} \otimes v_{i_2}] + \chi[v_{i_1} \otimes v_{i_2} \otimes v_{i_3} \otimes v_{i_2}] + \chi[v_{i_1} \otimes v_{i_2} \otimes v_{i_1} \otimes v_{i_3}]
\]
Similarly, we have the following tangles:

\[
\chi[v_1 \otimes v_2 \otimes v_1 \otimes v_2] + \chi[v_1 \otimes v_2 \otimes v_3 \otimes v_2]
\]

Suppose we input \(v_1\) and \(v_2\) in the above tangle, so \(v_1\) is represented by the orange area and \(v_2\) is represented by blue. Then we see the second and fourth spots must be the same vertex, here \(v_2\). The third spot, denoted by the pink area, must be adjacent to the second (and fourth) spot, so it can be \(v_1\) or \(v_3\).

\[
\chi[v_1 \otimes v_2 \otimes v_1 \otimes v_2] + \chi[v_1 \otimes v_2 \otimes v_1 \otimes v_2]
\]

Suppose we input \(v_1\) and \(v_2\) in the above tangle where \(v_1\) is represented by the blue area and \(v_2\) is denoted by orange. Then we see the first and third spots must be the same vertex, here \(v_1\). The fourth spot, denoted by pink, must be adjacent to the first (and third) spot, so it can be \(v_2\) or \(v_5\). Note the input vertices are arbitrary, the relations between vertices are dictated by the tangle. Recall that we are still using the diagram from the previous proof for the ordering of the vertices.

Then since the second term of the right hand side of Equation 5.13 is equal to \(\chi[v_1 \otimes v_2 \otimes v_1 \otimes v_2]\), we have the equation in Figure 5.15. Thus, we have shown the second term of Equation 5.13 belongs to \(Q_\bullet\).
Finally, we need to show that the third term on the right hand side of Equation 5.13 belongs to the planar subalgebra $Q$. We will use that the second term of Equation 5.13 is in $Q$. We have the element $D$ in Figure 5.16 belonging to $Q$: Let $B_1$ correspond to $v_1$ and $B_2$ correspond to $v_3$. Then the vertex corresponding to $B_3$ must be adjacent to both $v_1$ and $v_3$. The only vertex $B_3$ can correspond to is $v_2$. Then the vertex that corresponds to $B_4$ must be adjacent to $v_2$ and nonadjacent (and not the same as) $v_3$. Similarly, the vertex that corresponds to $B_5$ must be adjacent to $v_2$ and nonadjacent (and not the same as) $v_1$. Therefore, $B_4$ corresponds to $v_1$ and $B_5$ corresponds to $v_3$. Effectively we have shown that the third term in Equation 5.13, that switches two nonadjacent vertices, is in $Q$. Thus, all three of the terms in Equation 5.13 are in $Q$.

From Equation 5.13, we know that the element $R$ belongs to the planar subalgebra $Q$. By the previous theorem, we have the planar algebra $P$ is generated by its 2-box space.

We have now shown that the $C_5$ graph does not have quantum symmetry using a planar algebra approach. These methods have potential to reveal the presence of quantum symmetry in other graphs as well.
Chapter 6

Future Research

There are many potential avenues for research related to quantum symmetry of graphs to go. The first area is studying and generating methods to determine if certain classes of graphs have quantum symmetry. This includes the planar algebra and coherent configuration approaches, as well as constructing and studying the quantum automorphism group. Additionally, there has been a lot of work showing shortcuts for proving quantum symmetry in distance-transitive graphs [17].

The second major area is picking specific graphs of interest and exploring methods already proven. In practice, researchers may have a practical reason for wanting to determine if, say the Schläfli graph, has quantum symmetry. We have reason to believe the planar algebra approach used for $C_5$ can be used in some instances of rank 3, distance-transitive, strongly regular graphs where adjacent and nonadjacent vertices have less than or equal to 2 common neighbors each.

As this field of mathematics is relatively new, with quantum permutations being discovered and published by Wang in 1998 [20], there is much left to explore.
Bibliography


Appendix A

Disjoint automorphisms script

LoadPackage("grape");

# check if there exists a pair of disjoint automorphisms for G
DisjointAutCheck := function(G)
# define local variables
local autJ, autJ-support, autJ-filt, i, l, n, disjoint-pairs;
autJ := AutGroupGraph(G); # automorphism group of G
n := OrderGraph(G); # number of vertices in G
# for each element in autJ, list moved vertices in G
autJ-support := List(AsList(autJ), i -> [i, MovedPoints(i)]);
# remove automorphisms of order 0, n-1, and n
autJ-filt := Filtered(AsList(autJ-support), i -> not (Size(i[2]) in [0, n-1, n]));

l := Length(autJ-filt);
disjoint-pairs := [];
Print("Checking pairs...");
for i in [1..(l-1)] do
    for j in [(i+1)..l] do

if IsEmpty(Intersection(autJ-filt[i][2], autJ-filt[j][2]))
    then Add(disjoint-pairs,[autJ-filt[i], autJ-filt[j]]);
    fi;
    od;
od;
Print("Disjoint pairs of automorphisms found: ", disjoint-pairs);
if IsEmpty(disjoint-pairs) then Print("No pairs of disjoint automorphisms exist.")
if not IsEmpty(disjoint-pairs) then Print("Pairs of disjoint automorphisms exist.")
return [autJ, disjoint-pairs];
end;