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## B-Meson Decay Constants from Improved Lattice Nonrelativistic QCD with Physical u,d,s, and c Quarks

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**B-Meson Decay Constants from Improved Lattice  
Nonrelativistic QCD with Physical u, d, s, and c Quarks**  
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TABLE II: Parameters used for the valence quarks.  $am_b$  is the bare  $b$  quark mass in lattice units,  $u_{0L}$  is the Landau link value used for tadpole-improvement, and  $am_l^{\text{val}}$ ,  $am_s^{\text{val}}$  are the HISQ light and strange quark masses.

Set	$am_b$	$u_{0L}$	$am_l^{\text{val}}$	$am_s^{\text{val}}$
1	3.297	0.8195	0.013	0.0641
2	3.263	0.82015	0.0064	0.0636
3	3.25	0.819467	0.00235	0.0628
4	2.66	0.834	0.01044	0.0522
5	2.62	0.8349	0.00507	0.0505
6	2.62	0.834083	0.00184	0.0507
7	1.91	0.8525	0.0074	0.0364
8	1.89	0.851805	0.0012	0.0360

slight discrepancy. The  $s$  quark is tuned using the  $\eta_s$  meson ( $M_{\eta_s} = 0.6893(12)$  GeV [3]). Values very close to the sea  $s$  masses are found, meaning that partial quenching effects will be small.

To improve the statistical precision of the correlators, we take  $U(1)$  random noise sources for the valence quarks using the methods developed in [13]. Along with the point source required for the matrix element, we include gaussian smearing functions for the  $b$  quark source with two different widths. We include 16 time sources with  $b$  quarks propagating both forward and backward in time on each configuration. We checked the statistical independence of results using a blocked autocorrelation function [3]. Even on the finer physical point ensembles, the correlations are very small between adjacent configurations and the integrated autocorrelation time is consistent with one.

The decay constant is defined from  $\langle 0|A_0|B_q\rangle_{\text{QCD}} = M_{B_q} f_{B_q}$ , but the quantity that we extract directly from the amplitude of our correlator fits is  $\Phi_{B_q} = \sqrt{M_{B_q}} f_{B_q}$ ; we convert to  $f_{B_q}$  at the end. For NRQCD, the full QCD matrix element is constructed from effective theory currents arranged in powers of  $1/m_b$ . For  $A_0$  we consider the following currents, made from heavy quark  $\Psi_Q$  and light quark fields  $\Psi_q$ :

$$J_0^{(0)} = \bar{\Psi}_q \gamma_5 \gamma_0 \Psi_Q \quad (1)$$

$$J_0^{(1)} = \frac{-1}{2m_b} \bar{\Psi}_q \gamma_5 \gamma_0 \gamma \cdot \nabla \Psi_Q \quad (2)$$

$$J_0^{(2)} = \frac{-1}{2m_b} \bar{\Psi}_q \gamma \cdot \overleftarrow{\nabla} \gamma_5 \gamma_0 \Psi_Q. \quad (3)$$

These currents are related to the full QCD current through  $\mathcal{O}(\alpha_s, \alpha_s \Lambda_{\text{QCD}}/m_b)$  by

$$\langle A_0 \rangle = (1 + \alpha_s z_0) \left( \langle J_0^{(0)} \rangle \right) + (1 + \alpha_s z_1) \langle J_0^{(1)} \rangle + \alpha_s z_2 \langle J_0^{(2)} \rangle \quad (4)$$

One-loop coefficients were calculated in [15]. Here we re-order the perturbation series to make the process of renormalisation clearer. The  $z_i$  depend on  $am_b$  and are

TABLE III: Coefficients for the perturbative matching of the axial vector current (Eq. 4).  $z_0 = \rho_0 - \zeta_{10}$ ,  $z_1 = \rho_1 - z_0$ ,  $z_2 = \rho_2$  from [15].

Set	$z_0$	$z_1$	$z_2$
1	0.024(2)	0.024(3)	-1.108(4)
2	0.022(2)	0.024(3)	-1.083(4)
3	0.022(1)	0.024(2)	-1.074(4)
4	0.006(2)	0.007(3)	-0.698(4)
5	0.001(2)	0.007(3)	-0.690(4)
6	0.001(2)	0.007(2)	-0.690(4)
7	-0.007(2)	-0.031(4)	-0.325(4)
8	-0.007(2)	-0.031(4)	-0.318(4)

TABLE IV: Raw lattice amplitudes for  $B_s$  and  $B$  from each ensemble, errors are from statistics/fitting only.  $a^{3/2}\Phi_q^{(0)}$  and  $a^{3/2}\Phi_q^{(1)}$  are the leading amplitude and  $1/m_b$  correction.

Set	$a^{3/2}\Phi_s^{(0)}$	$a^{3/2}\Phi_s^{(1)}$	$a^{3/2}\Phi^{(0)}$	$a^{3/2}\Phi^{(1)}$
1	0.3720(10)	-0.0300(3)	0.3220(19)	-0.0260(3)
2	0.3644(6)	-0.0291(3)	0.3093(11)	-0.0257(8)
3	0.3621(16)	-0.0288(2)	0.2986(17)	-0.0237(4)
4	0.2733(4)	-0.0234(2)	0.2373(9)	-0.0197(4)
5	0.2679(3)	-0.0234(1)	0.2272(7)	-0.0197(3)
6	0.2653(2)	-0.0229(1)	0.2193(8)	-0.0194(3)
7	0.1747(3)	-0.0170(1)	0.1525(8)	-0.0146(6)
8	0.1694(3)	-0.0167(0)	0.1386(5)	-0.0136(1)

given in Table III for the range of masses needed here. We see that the one-loop renormalisation of the tree-level current,  $J_0^{(0)} + J_0^{(1)}$ , is tiny [32].  $z_0$  includes the effect of mixing between  $J_0^{(0)}$  and  $J_1^{(1)}$  at one-loop. We evaluate the renormalisation of Eq. 4 using  $\alpha_s$  in the V-scheme at scale  $q = 2/a$ . Values for  $\alpha_s$  are obtained by running down from  $\alpha_s^{\overline{\text{MS}}}(M_Z) = 0.1184$  [16] and range from 0.285 to 0.314.

### III. RESULTS

We fit heavy-light meson correlators with both  $J_0^{(0)}$  and  $J_0^{(1)}$  operators at the sink simultaneously using a multi-exponential Bayesian fitting procedure [17]. The  $B$  and  $B_s$  are fit separately; priors used in the fit are

TABLE V: Raw lattice energies from each ensemble, errors are from statistics/fitting only.  $aM_\pi$  are the pion masses used in the chiral fits,  $aE(B_s)$  and  $aE(B)$  are the energies of the  $B_s$  and  $B$  meson. Results on sets 3, 6 and 8 are new, others are given in [11].

Set	$aM_\pi$	$aE(B_s)$	$aE(B)$
3	0.10171(4)	0.6067(7)	0.5439(12)
6	0.08154(2)	0.5158(1)	0.4649(6)
8	0.05718(1)	0.4025(2)	0.3638(5)

described in [11]. The amplitudes and energies from the fits are given in Tables IV and V.  $a^{3/2}\Phi_q^{(0)}$  is the matrix element of the leading current  $J_0^{(0)}$  and  $a^{3/2}\Phi_q^{(1)}$  that of  $J_0^{(1)}$  and  $J_0^{(2)}$ , whose matrix elements are equal at zero meson momentum. Notice that the statistical errors in  $\Phi$  do not increase on the physical point lattices, because they have such large volumes.

We take two approaches to the analysis. The first is to perform a simultaneous chiral fit to all our results for  $\Phi$ ,  $\Phi_s$ ,  $\Phi_s/\Phi$  and  $M_{B_s} - M_B$  using  $SU(2)$  chiral perturbation theory. The second is to study only the physical  $u/d$  mass results as a function of lattice spacing.

For the chiral analysis we use the same formula and priors for  $M_{B_s} - M_B$  as in [11]. Pion masses used in the fits are listed in Table V and the chiral logarithms,  $l(M_\pi^2)$ , include the finite volume corrections computed in [18] which have negligible effect on the fit. For the decay constants the chiral formulas, including analytic terms up to  $M_\pi^2$  and the leading logarithmic behaviour, are (see e.g. [19]):

$$\Phi_s = \Phi_{s0}(1.0 + b_s M_\pi^2/\Lambda_\chi^2) \quad (5)$$

$$\Phi = \Phi_0 \left( 1.0 + b_l \frac{M_\pi^2}{\Lambda_\chi^2} + \frac{1 + 3g^2}{2\Lambda_\chi^2} \left( -\frac{3}{2} l(M_\pi^2) \right) \right) \quad (6)$$

The coefficients of the analytic terms  $b_s, b_l$  are given priors 0.0(1.0),  $g$  has prior 0.5(5) and  $\Phi_0, \Phi_{s0}$  have 0.5(5). To allow for discretisation errors each fit formula is multiplied by  $(1.0 + d_1(\Lambda a)^2 + d_2(\Lambda a)^4)$ , with  $\Lambda = 0.4$  GeV. We expect discretisation effects to be very similar for  $\Phi$  and  $\Phi_s$  and so we take the  $d_i$  to be the same, but differing from the  $d_i$  used in the  $M_{B_s} - M_B$  fit. Since all actions used here are accurate through  $a^2$  at tree-level, the prior on  $d_1$  is taken to be 0.0(3) whereas  $d_2$  is 0.0(1.0). The  $d_i$  are allowed to have mild  $m_b$  dependence as in [11]. The ratio  $\Phi_s/\Phi$  is allowed additional light quark mass dependent discretisation errors that could arise, for example, from staggered taste-splittings. For comparison, we have fit the results using  $SU(2)$  heavy meson staggered chiral perturbation theory [20, 21] which changes the results by less than 1-sigma. We have tested that the fit is stable with respect to changes to the priors for  $g, b_l, b_s, d_i$  and adding/removing discretisation corrections.

Error %	$\Phi_{B_s}/\Phi_B$	$M_{B_s} - M_B$	$\Phi_{B_s}$	$\Phi_B$
EM:	0.0	1.2	0.0	0.0
$a$ dependence:	0.01	0.9	0.9	0.9
chiral:	0.01	0.2	0.04	0.04
$g$ :	0.01	0.1	0.0	0.01
stat/scale:	0.30	1.2	0.7	0.7
operator:	0.0	0.0	1.3	1.3
relativistic:	0.5	0.5	1.0	1.0
total:	0.6	2.0	2.0	2.0

TABLE VI: Full error budget from the chiral fit as a percentage of the final answer.

The results of the decay constant chiral fits are plot-

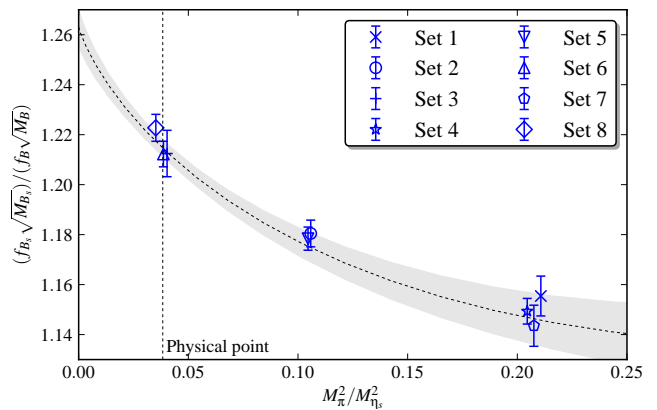


FIG. 1: Fit to the decay constant ratio  $\Phi_{B_s}/\Phi_B$ . The fit result is shown in grey and errors include statistics, and chiral/continuum fitting.

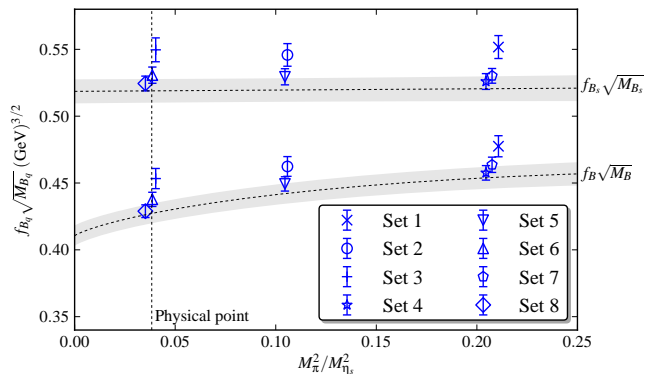


FIG. 2: Fit to the decay constants  $\Phi_{B_s}$  and  $\Phi_B$ . Errors on the data points include statistics/scale only. The fit error, in grey, includes chiral/continuum fitting and perturbative errors.

ted in Figs. 1 and 2. Extrapolating to the physical point appropriate to  $m_l = (m_u + m_d)/2$  in the absence of electromagnetism, i.e.  $M_\pi = M_{\pi^0}$ , we find  $\Phi_{B_s} = 0.519(10)$  GeV $^{3/2}$ ,  $\Phi_B = 0.427(9)$  GeV $^{3/2}$ ,  $\Phi_{B_s}/\Phi_B = 1.215(7)$ . For  $M_{B_s} - M_B$  we obtain 86(1) MeV, in agreement with the result of [11].

Figs 3 and 4 show the results of fitting  $M_{B_s} - M_B$  and decay constants from the physical point ensembles only, and allowing only the mass dependent discretisation terms above. The results are  $\Phi_{B_s} = 0.521(8)$  GeV $^{3/2}$ ,  $\Phi_B = 0.428(7)$  GeV $^{3/2}$ ,  $\Phi_{B_s}/\Phi_B = 1.216(7)$  and  $M_{B_s} - M_B = 87(1)$  MeV. Results and errors agree well between the two methods and we take the central values from the chiral fit as this allows us to interpolate to the correct pion mass.

Our error budget is given in Table VI. The errors that are estimated directly from the chiral/continuum fit are those from statistics, the lattice spacing and  $g$  and other chiral fit parameters. The two remaining sources of error in the decay constant are missing higher order corrections in the operator matching and relativistic corrections to

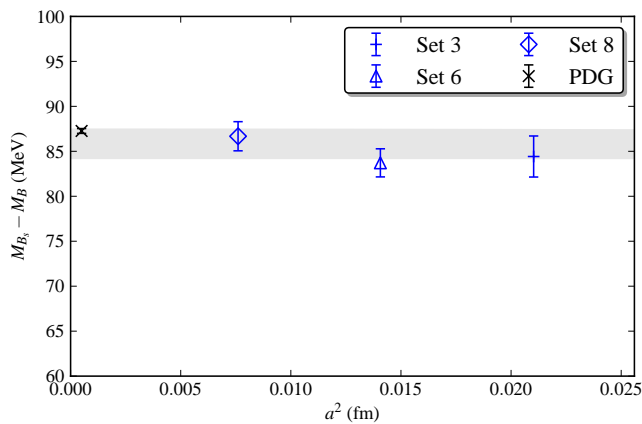


FIG. 3: Fit to the mass difference  $M_{B_s} - M_B$  on the three physical point ensembles only. Errors on data points include statistics and scale, the fit error is shown in grey. An electromagnetic correction of  $-1(1)$  MeV has been applied to the lattice results and the fit to allow comparison with experiment.

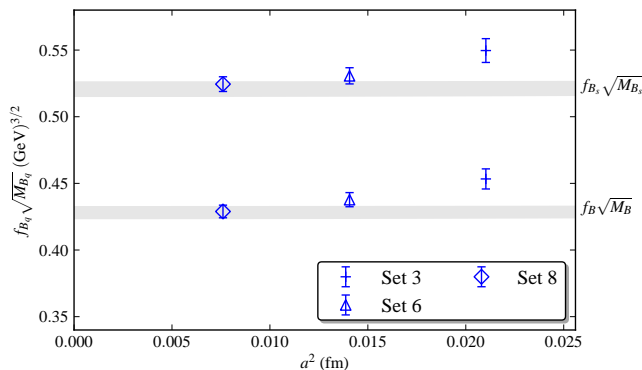


FIG. 4: Fit to the decay constants  $\Phi_{B_s}$  and  $\Phi_B$  on the three physical point ensembles only. Errors on the data points include statistics/scale only. The fit error includes chiral/continuum fitting and perturbative errors.

the current. We estimate the operator matching error by allowing in our fits for an  $am_b$ -dependent  $\alpha_s^2$  correction to the renormalisation in Eq. 4 with prior on the coefficient of  $0.0(2)$  i.e. ten times the size of the one-loop correction,  $z_0$ . This error cancels in the ratio  $f_{B_s}/f_B$ . We also allow for  $\alpha_s^2$  corrections multiplying  $J_0^{(1,2)}$  with coefficient  $0.0(1.0)$ . The matrix element of  $J_0^{(1)}$  is about 10% of  $J_0^{(0)}$  from Table IV. Missing current corrections at the next order in  $1/m_b$  will be of size  $(\Lambda_{\text{QCD}}/m_b)^2 \simeq 0.01$  which we take as an error. Finally, we estimated in [11] that to correct for missing electromagnetic effects,  $M_{B_s} - M_B$  should be shifted by  $-1(1)$  MeV.

Using the PDG masses  $M_{B_l} = (M_{B^0} + M_{B^\pm})/2 = 5.27942(12)$  GeV and  $M_{B_s} = 5.36668(24)$  GeV [22] to

convert  $\Phi_q$  to  $f_{B_q}$  our final results are:

$$\begin{aligned} f_B &= 0.186(4) \text{ GeV} \\ f_{B_s} &= 0.224(4) \text{ GeV} \\ f_{B_s}/f_B &= 1.205(7) \\ M_{B_s} - M_B &= 85(2) \text{ MeV.} \end{aligned} \quad (7)$$

For the  $B$  meson decay constant we need to distinguish between  $f_{B_d}$  and  $f_{B_u}$ . Since sea quark mass effects are much smaller than valence mass effects we simply do this by extrapolating  $\Phi_{B_s}$  and  $\Phi_B$  to values of  $M_\pi^2$  corresponding to fictitious mesons made purely of  $u$  or  $d$  quarks using  $m_u/m_d = 0.48(10)$  [22]. This gives:

$$\begin{aligned} f_{B_s}/f_{B^+} &= 1.217(8) ; f_{B_s}/f_{B^0} = 1.194(7) \\ f_{B^+} &= 0.184(4) \text{ GeV} ; f_{B^0} = 0.188(4) \text{ GeV} \end{aligned} \quad (8)$$

#### IV. CONCLUSIONS

Our results agree with but improve substantially on two earlier results using nonrelativistic approaches for the  $b$  quark and multiple lattice spacing values on  $N_f = 2 + 1$  ensembles using asqtad sea quarks. These were:  $f_{B_s} = 228(10)$  MeV,  $f_{B_s}/f_B = 1.188(18)$  (NRQCD/HISQ) [14] and  $f_{B_s} = 242.0(9.5)$  MeV and  $f_{B_s}/f_{B^+} = 1.229(26)$  (Fermilab/asqtad) [21]. We also agree well (within the 2% errors) with a previous result for  $f_{B_s}$  of  $225(4)$  MeV obtained using a relativistic (HISQ) approach to  $b$  quarks on very fine  $N_f = 2 + 1$  lattices [23]. Our simultaneous determination of  $M_{B_s} - M_B$  to 2% agrees with experiment ( $87.4(3)$  MeV [22]).

We can determine new lattice ‘world-average’ error-weighted values by combining our results in Eq. 7 with the independent results of [21] and [23] since effects from  $c$  sea quarks, which they do not include, should be negligible [24]. The world averages are then:  $f_{B_s} = 225(3)$  MeV and  $f_{B_s}/f_{B^+} = 1.218(8)$  giving  $f_{B^+} = 185(3)$  MeV.

These allow for significant improvements in predictions for SM rates. For example, updating [25] with the world-average for  $f_{B_s}$  above and our result for  $f_{B^0}$  (Eq. 8) we obtain:

$$\begin{aligned} \text{Br}(B_s \rightarrow \mu^+ \mu^-) &= 3.17 \pm 0.15 \pm 0.09 \times 10^{-9} \\ \text{Br}(B_d \rightarrow \mu^+ \mu^-) &= 1.05 \pm 0.05 \pm 0.05 \times 10^{-10} \end{aligned} \quad (9)$$

where the second error from  $f_{B_q}$  has been halved and is no longer larger than other sources of error such as  $V_{tb}^* V_{tq}$ . Note that this is the flavor-averaged branching fraction at  $t = 0$ ; the time-integrated result would be increased by 10% in the  $B_s$  case (to  $3.47(19) \times 10^{-9}$ ) to allow for the width difference of the two eigenstates [26, 27]. The current experimental results [28] for  $B_s \rightarrow \mu^+ \mu^-$  agree with this prediction.

From the world-average  $f_{B^+}$  above we also obtain the Standard Model rate:

$$\frac{1}{|V_{ub}|^2} \text{Br}(B^+ \rightarrow \tau \nu) = 6.05(20), \quad (10)$$

with 3% accuracy. Calculations of matrix elements for  $B_s/B$  mixing with physical  $u/d$  quarks are now underway.

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- [32] This agrees with expectations from [21, 29] in which the heavy-light renormalisation constant is perturbatively very close to the product of the square roots of the renormalisation of the local temporal vector current for heavy-heavy and light-light. Here the corresponding heavy-heavy current is conserved [30] and the light-light current has a very small renormalisation [31]. This will be discussed further elsewhere.