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Bifurcation from a degenerate simple eigenvalue

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Abstract

It is proved that a symmetry-breaking bifurcation occurs at a simple eigenvalue despite the usual transversality condition fails, and this bifurcation from a degenerate simple eigenvalue result complements the classical one with the transversality condition. The new result is applied to an imperfect pitchfork bifurcation, in which a forward transcritical bifurcation changes to a backward one when the perturbation parameter changes. Several applications in ecological and genetics models are shown.

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1. Introduction

In this paper we revisit the bifurcation problem of the nonlinear equation

\[ F(\lambda, u) = 0, \]

(1.1)
where $F \in C^p(\mathbb{R} \times X, Y)$, $p \geq 1$, is a nonlinear differentiable mapping, and $X$, $Y$ are Banach spaces.

If Eq. (1.1) has a trivial solution $u = u_0$ for any $\lambda \in \mathbb{R}$, then a necessary condition for a bifurcation point $(\lambda_0, u_0)$ is that the linearization of $F$ with respect to $u$ at $\lambda = \lambda_0$ is not invertible. Indeed let $F_u(\lambda_0, u_0)$ be the Fréchet derivative of $F(\lambda, u)$ in $u$ at $(\lambda_0, u_0)$, then the null space $N(F_u(\lambda_0, u_0))$ contains a non-zero element if $(\lambda_0, u_0)$ is a bifurcation point. A well-known sufficient condition for bifurcation is that zero is a simple eigenvalue of $F_u(\lambda_0, u_0)$, and the zero eigenvalue moves across $\lambda = \lambda_0$ “transversally”. To be more precise, 0 is a simple eigenvalue of $F_u(\lambda_0, u_0)$ if the following assumption is satisfied:

\[(F_1) \quad \dim N(F_u(\lambda_0, u_0)) = \text{codim } R(F_u(\lambda_0, u_0)) = 1, \quad \text{and } N(F_u(\lambda_0, u_0)) = \text{span}\{u_0\},\]

where $N(F_u)$ and $R(F_u)$ are the null space and the range of linear operator $F_u = F_u(\lambda_0, u_0)$. Crandall and Rabinowitz [6] prove the following celebrated “bifurcation from a simple eigenvalue” theorem (see [6, Theorem 1.7]):

**Theorem 1.1.** Let $U$ be a neighborhood of $(\lambda_0, u_0)$ in $\mathbb{R} \times X$, and $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. Assume that the partial derivatives $F_u$, $F_\lambda$ and $F_{\lambda u}$ exist and are continuous in $U$; at $(\lambda_0, u_0)$, $F$ satisfies (F1) and

\[(F_3) \quad F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0)).\]

Let $Z$ be any complement of $\text{span}\{u_0\}$ in $X$. Then the solution set of $F(\lambda, u) = 0$ near $(\lambda_0, u_0)$ consists precisely of the curves $u = u_0$ and $\Sigma = \{u(t): s \in I \equiv (-\epsilon, \epsilon)\} = \{\lambda(t), u(t)\}$, where $\lambda : I \to \mathbb{R}, z : I \to Z$ are continuous functions such that $u(t) = u_0 + sw_0 + sz(t), \lambda(0) = \lambda_0, z(0) = 0$.

The bifurcation from a simple eigenvalue theorem has been one of the fundamental tools in showing the occurrence of symmetry-breaking spatial patterns in many nonlinear problems. Some novel applications in nonlinear partial differential equation models include the existence of steady periodic water waves [5,30,43], free boundary problem in tumor models and cell growth [14–16], and the existence of nonconstant stationary patterns in spatial ecological models [9,10,20,44,45]. Other applications are also found in nonlinear matrix population models [8], and nonlinear ordinary differential equation population models [21]. In such a bifurcation, a curve of non-trivial solutions emanates from the line of trivial ones. It is also important to determine the direction of the bifurcating curve and the stability of the bifurcating solutions. If in addition, $F$ is $C^2$ in $u$, then the bifurcating curve $\Sigma = \{\lambda(s), u(s)\}$ in Theorem 1.1 is differentiable. If $\lambda'(0) \neq 0$, then a transcritical bifurcation occurs near $(\lambda_0, u_0)$ (see Fig. 1 left panel); and if $\lambda'(0) = 0$, and $F \in C^3$, $\lambda''(0) \neq 0$, then a pitchfork bifurcation occurs near $(\lambda_0, u_0)$ (see Fig. 1 right panel).

The transversality condition (F3) holds in generic situations. But there are important exceptions for which (F3) fails. In this paper, we consider a degenerate bifurcation scenario in which (F1) is satisfied but (F3) is not satisfied. In this case, we prove that, under some higher order transversality conditions on $F$, the local solution set of (1.1) near the bifurcation point $(\lambda_0, u_0)$ consists of the line of trivial solutions, and two other solution curves; each of these two curves could be similar to the one in transcritical or pitchfork bifurcation, or in a degenerate case, identical to the curve of trivial solutions. To compare our results with Theorem 1.1, we use some
Fig. 1. Bifurcation diagrams. (Left): $F(\lambda, u) = \lambda u - u^2$; (Right): $F(\lambda, u) = \lambda u - u^3$.

Fig. 2. Bifurcation diagrams. (Left): $F(\lambda, u) = u(\lambda^2 - u^2)$; (Right): $F(\lambda, u) = u(u - \lambda)(u - 2\lambda)$.

normal form mappings $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to illustrate our results. The normal forms of transcritical and pitchfork bifurcations shown in Theorem 1.1 are (always assuming $(\lambda_0, u_0) = (0, 0)$, see Fig. 1)

$$F_1(\lambda, u) = \lambda u - u^2, \quad \text{and} \quad F_2(\lambda, u) = \lambda u - u^3.$$  \hspace{1cm} (1.2)

The new bifurcation theorem which we prove here (see Figs. 2 and 3) shows normal forms of

$$F_3(\lambda, u) = u(\lambda - u)(\lambda + au),$$

and

$$F_4(\lambda, u) = u(\lambda - u)(\lambda - au), \quad a > 0(\neq 1);$$  \hspace{1cm} (1.3)

and also the degenerate cases

$$F_5(\lambda, u) = \lambda u^2 - u^3, \quad \text{and} \quad F_6(\lambda, u) = \lambda u^2 - u^4.$$  \hspace{1cm} (1.4)
Fig. 3. Bifurcation diagrams. (Left): $F(\lambda, u) = \lambda u^2 - u^3$; (Right): $F(\lambda, u) = \lambda u^2 - u^4$.

Fig. 4. Imperfect bifurcation not preserving the trivial solutions: bifurcation diagrams of $F(\varepsilon, \lambda, u) = \lambda u - u^3 - \varepsilon = 0$. (Left: $\varepsilon < 0$; Middle: $\varepsilon = 0$; Right: $\varepsilon > 0$.)

Fig. 5. Imperfect bifurcation preserving the trivial solutions: bifurcation diagrams of $F(\varepsilon, \lambda, u) = \lambda u - u^3 - \varepsilon u^2 = 0$. (Left: $\varepsilon < 0$; Middle: $\varepsilon = 0$; Right: $\varepsilon > 0$.)

The motivation of studying such degenerate bifurcations is to consider some imperfect bifurcations arising from applications. A typical perturbation to a pitchfork bifurcation destroys the original symmetry, then the trivial solutions cannot be preserved (see Fig. 4). But in many application problems, the trivial solutions are preserved under the perturbation, but the perturbed problem may have a different bifurcation structure. Typically a forward (supercritical) bifurcation which produces a stable non-trivial equilibria becomes a backward (subcritical) bifurcation one in which the bifurcating solutions are unstable (see Fig. 5). This can be best demonstrated by the backward bifurcations in epidemics models [11,18], and catastrophic shifts in ecosystems.
such as deserts, lakes, and coral reefs [34,35]. We show that this phenomenon (as shown in Fig. 5) can be put under the framework of a perturbed bifurcation problem

\[ F(\varepsilon, \lambda, u) = 0, \quad (1.5) \]

where \( \varepsilon \) is a perturbation parameter, and the slight change of \( \varepsilon \)-value causes variation of the bifurcation diagram in \((\lambda, u)\)-space. Such imperfect bifurcations have been considered in [23,38], and this study is partially a sequel to these previous ones as we analyze the new bifurcation scenario as in Fig. 5 by applying the new bifurcation from a degenerate simple eigenvalue theorem mentioned above. Some other related recent studies of imperfect bifurcations can be found in [24,32]. The question of whether a perturbation would preserve or destroy the trivial solutions was also considered in [37], but with a different approach.

As remarked in [23], Lyapunov–Schmidt reduction is an important method to reduce an infinite-dimensional problem to a finite-dimensional one, and the theory of singularities of differentiable maps and catastrophe theory are useful in the qualitative studies of such finite-dimensional problems. In particular, the imperfect bifurcation of (Lyapunov–Schmidt) reduced maps has been considered in Golubitsky and Schaeffer [17]. The approach given here (as well as in the one in [23]) directly deals with the original infinite-dimensional problems, and the conditions are based on various partial derivatives of the nonlinear maps on Banach spaces but not derivatives of reduced finite-dimensional maps. This follows the approach in Crandall and Rabinowitz [6,7], which has been widely utilized in applications mentioned above.

In Section 2, we prove the bifurcation from a degenerate simple eigenvalue theorem, and in Section 3 we prove the related stability results. In Section 4 we apply the bifurcation from a degenerate simple eigenvalue theorem to the imperfect bifurcation problem to obtain the precise local bifurcation diagrams near \( \varepsilon = \varepsilon_0 \). In Section 5, we demonstrate the applications to several imperfect bifurcation problems from mathematical biology.

We use the same labeling of conditions such as \((F1)\), \((F2)\) on \( F \) as in our previous work [38,23], and we use the convention that \((Ff)\) stands for the negation of \((Fi)\) for \( i \in \mathbb{N} \). In the paper, we use \( \| \cdot \| \) as the norm of Banach space \( X \), \( \langle \cdot, \cdot \rangle \) as the duality pair of a Banach space \( X \) and its dual space \( X^* \). For a linear operator \( L \), we use \( N(L) \) as the null space of \( L \) and \( R(L) \) as the range space of \( L \), and we use \( L[w] \) to denote the image of \( w \) under the linear mapping \( L \).

For a multilinear operator \( L \), we use \( L[w_1, w_2, \ldots, w_k] \) to denote the image of \( (w_1, w_2, \ldots, w_k) \) under \( L \), and when \( w_1 = w_2 = \cdots = w_k \), we use \( L[w_1]^k \) instead of \( L[w_1, w_1, \ldots, w_1] \). For a nonlinear operator \( F \), we use \( F_u \) as the partial derivative of \( F \) with respect to argument \( u \).

### 2. Bifurcation from a degenerate simple eigenvalue

We assume that \( F \) satisfies \((F1)\) at \((\lambda_0, u_0)\), then we have decompositions of \( X \) and \( Y: X = N(F_u(\lambda_0, u_0)) \oplus Z \) and \( Y = R(F_u(\lambda_0, u_0)) \oplus Y_1 \), where \( Z \) is a complement of \( N(F_u(\lambda_0, u_0)) \) in \( X \), and \( Y_1 \) is a complement of \( R(F_u(\lambda_0, u_0)) \). In particular, \( F_u(\lambda_0, u_0) \) is an isomorphism. Since \( R(F_u(\lambda_0, u_0)) \) is codimension one, then there exists \( l \in Y^* \) such that \( R(F_u(\lambda_0, u_0)) = \{ v \in Y : \langle l, v \rangle = 0 \} \).

We comment that in Theorem 1.1, if \( F \) is \( C^2 \) near \((\lambda_0, u_0)\), then the curve of non-trivial solutions is differentiable, and one has the formula for the bifurcation direction:

\[ \lambda'(0) = - \frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0] \rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0] \rangle}. \quad (2.1) \]
If $F$ satisfies

\[(F4) \quad F_{uu}(\lambda_0, u_0)[w_0]^2 \neq R(F_u(\lambda_0, u_0)),\]

then we have $\lambda'(0) \neq 0$, and a transcritical bifurcation occurs. If $F$ satisfies $(F4')$ and $F \in C^3$, then $\lambda'(0) = 0$ and

$$\lambda''(0) = -\frac{\langle l, F_{uuu}(\lambda_0, u_0)[w_0]^3 + 3\langle l, F_{uu}(\lambda_0, u_0)[w_0, v_2]\rangle}{3\langle l, F_{uu}(\lambda_0, u_0)[w_0]\rangle}, \quad (2.2)$$

where $v_2$ satisfies $F_{uu}(\lambda_0, u_0)[w_0]^2 + F_u(\lambda_0, u_0)[v_2] = 0$. A pitchfork bifurcation typically satisfies $\lambda''(0) \neq 0$.

We recall two important lemmas from our previous work [23]. First is the well-known Lyapunov–Schmidt reduction under the condition $(F1)$ which is standard from any textbook (see for example [4,29]).

**Lemma 2.1.** Suppose that $F : \mathbb{R} \times X \to Y$ is a $C^p$ ($p \geq 1$) mapping such that $F(\lambda_0, u_0) = 0$, and $F$ satisfies $(F1)$ at $(\lambda_0, u_0)$. Then the equation $F(\lambda, u) = 0$ for $(\lambda, u)$ near $(\lambda_0, u_0)$ can be reduced to

$$\langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0,$$

where $t \in (-\delta, \delta)$, $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$ where $\delta$ is a small constant, $l \in Y^*$ such that $\langle l, v \rangle = 0$ if and only if $v \in R(F_u(\lambda_0, u_0))$, and $g$ is a $C^p$ function into $Z$ such that $g(\lambda_0, 0) = 0$ and $Z$ is a complement of $N(F_u(\lambda_0, u_0))$ in $X$.

Next we recall the following lemma (see [23, Lemma 2.5]) which describes the structure of zero-set of a function defined near a critical point in $\mathbb{R}^2$.

**Lemma 2.2.** Suppose that $(x_0, y_0) \in \mathbb{R}^2$ and $U$ is a neighborhood of $(x_0, y_0)$. Assume that $f : U \to \mathbb{R}$ is a $C^p$ function for $p \geq 2$, $f(x_0, y_0) = 0$, $\nabla f(x_0, y_0) = 0$, and the Hessian matrix $H = \nabla^2 f(x_0, y_0)$ is non-degenerate. Then

1. If $H$ is definite (i.e. $\det(H) > 0$), then $(x_0, y_0)$ is the unique zero point of $f(x, y) = 0$ near $(x_0, y_0)$;
2. If $H$ is indefinite (i.e. $\det(H) < 0$), then there exist two $C^{p-1}$ curves $(x_1(t), y_1(t))$, $i = 1, 2$, $t \in (-\delta, \delta)$, such that the solution set of $f(x, y) = 0$ consists of exactly the two curves near $(x_0, y_0)$, $(x_1(0), y_1(0)) = (x_0, y_0)$. Moreover $t$ can be rescaled and indices can be rearranged so that $(x_1'(0), y_1'(0))$ and $(x_2'(0), y_2'(0))$ are the two linear independent solutions of

$$f_{xx}(x_0, y_0)\eta^2 + 2f_{xy}(x_0, y_0)\eta\tau + f_{yy}(x_0, y_0)\tau^2 = 0. \quad (2.3)$$

Now we are ready to state our main result on the bifurcation from a degenerate simple eigenvalue.

**Theorem 2.3.** Let $U$ be a neighborhood of $(\lambda_0, u_0)$ in $\mathbb{R} \times X$, and let $F \in C^3(U, Y)$. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. At $(\lambda_0, u_0)$, $F$ satisfies $(F1)$,
Assume that the matrix $X$ and $v$ are definite, i.e., $\langle l, v \rangle > 0$, hence $g(\lambda, 0) = 0$ and $g_{22}(\lambda, 0) = 0$. It is easy to calculate that $u_0$ is a trivial solution for all $\lambda$ near $\lambda_0$, that is, $F(\lambda, u_0) = 0$.

Proof. We denote the projection from $Y$ into $R(F_u(\lambda_0, u_0))$ by $Q$. Then the function $g(\lambda, t)$ in Lemma 2.1 is obtained from (see [23])

$$f_i(\lambda, t) \equiv Q \circ F(\lambda, u_0 + tw_0 + g(\lambda, t)) = 0.$$

Since $u_0$ is a trivial solution for all $\lambda$ near $\lambda_0$, that is, $F(\lambda, u_0) = 0$, then by Lemma 2.1 we have $g(\lambda, 0) \equiv 0$, hence $g_1(\lambda, 0) = 0$ and $g_{22}(\lambda, 0) = 0$. It is easy to calculate that

\begin{align*}
(F_3') & \quad F_{\lambda u}(\lambda_0, u_0)[w_0] \in R(F_u(\lambda_0, u_0)), \text{ and} \\
(F_4') & \quad F_{uu}(\lambda_0, u_0)[w_0]^2 \in R(F_u(\lambda_0, u_0)).
\end{align*}

Let $X = N(F_u(\lambda_0, u_0)) \oplus Z$ be a fixed splitting of $X$, and let $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y: \langle l, v \rangle = 0\}$. Denote by $v_1 \in Z$ the unique solution of

$$F_{\lambda u}(\lambda_0, u_0)[w_0] + F_u(\lambda_0, u_0)[v] = 0,$$

and $v_2 \in Z$ the unique solution of

$$F_{uu}(\lambda_0, u_0)[w_0]^2 + F_u(\lambda_0, u_0)[v] = 0.$$

We assume that the matrix (all derivatives are evaluated at $(\lambda_0, u_0)$)

$$H = H(\lambda_0, u_0) = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}$$

is non-degenerate, i.e., $\det(H) \neq 0$, where $H_{ij}$ is given by

\begin{align*}
H_{11} &= \langle l, F_{\lambda u}[w_0] + 2F_u[v_1] \rangle, \\
H_{12} &= \frac{1}{2} \langle l, F_{u u}[w_0]^2 + F_{\lambda u}[v_2] + 2F_u[w_0, v_1] \rangle, \\
H_{22} &= \frac{1}{3} \langle l, F_{u u}[w_0]^3 + 3F_u[w_0, v_2] \rangle.
\end{align*}

1. If $H$ is definite, i.e. $\det(H) > 0$, then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is the line $\{(\lambda, u_0)\}$.

2. If $H$ is indefinite, i.e. $\det(H) < 0$, then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is the union of $C^1$ curves intersecting at $(\lambda_0, u_0)$, including the line of trivial solutions $\Gamma_0 = \{(\lambda, u_0)\}$ and two other curves $\Gamma_i = \{(\lambda_i(s), u_i(s)) : |s| < \delta\}$ $(i = 1, 2)$ for some $\delta > 0$, with

$\lambda_i(s) = \lambda_0 + \mu_i s + s \theta_i(s)$, \hspace{1cm} u_i(s) = u_0 + \eta_i s w_0 + s v_i(s),$

where $(\mu_1, \eta_1)$ and $(\mu_2, \eta_2)$ are non-zero linear independent solutions of the equation

$$H_{11} \mu^2 + 2H_{12} \mu \eta + H_{22} \eta^2 = 0,$$

and $\theta_i(0) = \theta'_i(0) = 0$, $v_i(0) \in Z$, and $v_i(0) = v'_i(0) = 0$, $i = 1, 2$. 

\[\theta_i(0) = \theta'_i(0) = 0, \quad v_i(0) \in Z, \quad \text{and} \quad v_i(0) = v'_i(0) = 0, \quad i = 1, 2.\]
\[
\frac{\partial f_1}{\partial t}(\lambda_0, 0) = Q \circ F_u(\lambda_0, u_0)[w_0 + g_t(\lambda_0, 0)] = F_u(\lambda_0, u_0)[g_t(\lambda_0, 0)] = 0,
\]
thus \(g_t(\lambda_0, 0) = 0\) from \(F_u(\lambda_0, u_0)[w_0] = 0\), \(F_u(\lambda_0, u_0)|_Z : Z \to R(F_u(\lambda_0, u_0))\) is an isomorphism and \(g_t(\lambda_0, 0) \in Z\). Next we calculate the second derivatives of \(f_1\):

\[
\frac{\partial^2 f_1}{\partial \lambda \partial t}(\lambda_0, 0) = Q \circ (F_{\lambda u}(\lambda_0, u_0)[w_0 + g_t(\lambda_0, 0)] + F_{uu}(\lambda_0, u_0)[w_0 + g_t(\lambda_0, 0), g_\lambda(\lambda_0, 0)])
\]

\[
= F_{\lambda u}(\lambda_0, u_0)[g_{\lambda t}(\lambda_0, 0)]
\]

\[
= F_{\lambda u}(\lambda_0, u_0)[w_0] + F_u(\lambda_0, u_0)[g_{\lambda t}(\lambda_0, 0)] = 0,
\]
thus \(g_{\lambda t}(\lambda_0, 0) = v_1\) from (F3'), where \(v_1\) is defined as in (2.4); and

\[
\frac{\partial^2 f_1}{\partial t^2}(\lambda_0, 0) = Q \circ (F_{uu}(\lambda_0, u_0)[w_0 + g_t(\lambda_0, 0)]^2 + F_u(\lambda_0, u_0)[g_{tt}(\lambda_0, 0)])
\]

\[
= F_{uu}(\lambda_0, u_0)[w_0] + F_u(\lambda_0, u_0)[g_{tt}(\lambda_0, 0)] = 0,
\]
thus \(g_{tt}(\lambda_0, 0) = v_2\) from (F4') where \(v_2\) is defined as in (2.5).

We define the bifurcation function

\[
f(\lambda, t) = \langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t)) \rangle.
\] (2.12)

From the assumptions, \(f\) is \(C^3\) in \(U\). Since \(g(\lambda, 0) \equiv 0\), then \(f(\lambda, 0) \equiv 0\). To prove the statement in Theorem 2.3, we apply Lemma 2.2 to

\[
h(\lambda, t) = \begin{cases} \frac{1}{t} f(\lambda, t), & \text{if } t \neq 0, \\ f(\lambda, 0), & \text{if } t = 0. \end{cases}
\] (2.13)

First we verify that \(h(\lambda, t)\) is \(C^2\) at \(t = 0\). By the definition,

\[
h_t(\lambda, 0) = \lim_{t \to 0} \frac{1}{t} \left( h(\lambda, t) - h(\lambda, 0) \right) = \lim_{t \to 0} \frac{1}{t} \left( \frac{1}{t} f(\lambda, t) - f_t(\lambda, 0) \right)
\]

\[
= \lim_{t \to 0} \frac{1}{t^2} \left( f(\lambda, t) - f(\lambda, 0) - f_t(\lambda, 0)t \right) = \frac{1}{2} f_{tt}(\lambda, 0),
\]

\[
h_{\lambda t}(\lambda, 0) = \lim_{t \to 0} \frac{1}{t} \left( h_{\lambda}(\lambda, t) - h_{\lambda}(\lambda, 0) \right) = \lim_{t \to 0} \frac{1}{t} \left( \frac{1}{t} f_{\lambda}(\lambda, t) - f_{\lambda t}(\lambda, 0) \right)
\]

\[
= \lim_{t \to 0} \frac{1}{t^2} \left( f_{\lambda}(\lambda, t) - tf_{\lambda t}(\lambda, 0) \right) = \frac{1}{2} f_{\lambda tt}(\lambda, 0),
\]

\[
h_{\lambda}(\lambda, 0) = f_{\lambda t}(\lambda, 0), \quad h_{\lambda \lambda}(\lambda, 0) = f_{\lambda \lambda t}(\lambda, 0),
\]

and
\[ h_{tt}(\lambda, 0) = \lim_{t \to 0} \frac{1}{t} \left( h_{t}(\lambda, t) - h_{t}(\lambda, 0) \right) = \lim_{t \to 0} \frac{1}{t} \left( -\frac{1}{t^2} f(\lambda, t) + \frac{1}{t} f_t(\lambda, t) - \frac{1}{2} f_{tt}(\lambda, 0) \right) \]
\[ = \lim_{t \to 0} \frac{1}{t^3} \left[ -f(\lambda, t) + t f_t(\lambda, t) - \frac{1}{2} f_{tt}(\lambda, 0)t^2 \right] = \frac{1}{3} f_{ttt}(\lambda, 0). \]

Hence \( h_t \) and \( h_{tt} \) exist when \( t = 0 \). Moreover
\[ h_{tt}(\lambda, 0) - h_{tt}(\lambda, 0) = 2t \left( f(\lambda, t) - tf_t(\lambda, t) + \frac{t^2}{2} f_{tt}(\lambda, t) - \frac{t^3}{6} f_{ttt}(\lambda, 0) \right) = o(t), \]
and
\[ h_{\lambda \lambda}(\lambda, t) - h_{\lambda \lambda}(\lambda, 0) = \frac{1}{t} f_{\lambda \lambda}(\lambda, t) - f_{\lambda \lambda t}(\lambda, 0) \]
\[ = \frac{1}{t} \left[ f_{\lambda \lambda}(\lambda, t) - f_{\lambda \lambda}(\lambda, 0) - f_{\lambda \lambda t}(\lambda, 0)t \right] = o(t), \]
\[ h_{\lambda t}(\lambda, t) - h_{\lambda t}(\lambda, 0) = -\frac{1}{t^2} f_{\lambda}(\lambda, t) + \frac{1}{t} f_{\lambda t}(\lambda, t) - \frac{1}{2} f_{\lambda \lambda t}(\lambda, 0) \]
\[ = -\frac{1}{t^2} \left[ f_{\lambda}(\lambda, t) - tf_{\lambda t}(\lambda, t) + \frac{1}{2} f_{\lambda \lambda t}(\lambda, 0)t^2 \right] = o(t), \]
thus \( h \in C^2 \) at \( t = 0 \) in \( U \).

Next we claim that \( h \) defined above satisfies the conditions in Lemma 2.2, that is, \( h(\lambda_0, 0) = 0 \), \( \nabla h(\lambda_0, 0) = (h_\lambda(\lambda_0, 0), h_t(\lambda_0, 0)) = 0 \), and the Hessian matrix \( \text{Hess}(h) \) is non-degenerate. Indeed from the information of the partial derivatives of \( g \) at \( (\lambda_0, 0) \), we have
\[ h(\lambda_0, 0) = f_t(\lambda_0, 0) = \langle l, F_u(\lambda_0, u_0)[w_0 + g_t(\lambda_0, 0)] \rangle = 0, \]
\[ h_\lambda(\lambda_0, 0) = f_{\lambda t}(\lambda_0, 0) \]
\[ = \langle l, F_{\lambda u}(\lambda_0, u_0)[w_0 + g_t(\lambda_0, 0)] + F_{uu}(\lambda_0, u_0)[g_\lambda(\lambda_0, 0), w_0 + g_t(\lambda_0, 0)] \rangle \]
\[ + \langle l, F_u(\lambda_0, u_0)[g_{\lambda t}(\lambda_0, 0)] \rangle \]
\[ = \langle l, F_{\lambda u}(\lambda_0, u_0)[w_0] \rangle, \]
and
\[ h_t(\lambda_0, 0) = \frac{1}{2} f_{tt}(\lambda_0, 0) \]
\[ = \frac{1}{2} \langle l, F_{uu}(\lambda_0, u_0)[w_0 + g_t(\lambda_0, 0)]^2 + F_u(\lambda_0, u_0)[g_{tt}(\lambda_0, 0)] \rangle \]
\[ = \frac{1}{2} \langle l, F_{uu}(\lambda_0, u_0)[w_0]^2 \rangle. \]

For the Hessian matrix \( \text{Hess}(h) = (h_{\lambda \lambda}, h_{\lambda t}, h_{tt}) \), we evaluate each entry, with the partial derivatives of \( F \) being always evaluated at \( (\lambda_0, u_0) \), and the partial derivatives of \( g \) being evaluated at \( (\lambda_0, 0) \). First
\[ h_{\lambda, \lambda}(\lambda_0, 0) = f_{\lambda, \lambda}(\lambda_0, 0) \]
\[ = \langle l, F_{\lambda, \lambda}[w_0 + g_t] + 2F_{\lambda, uu}[g_{\lambda}, w_0 + g_t] + 2F_{\lambda, u}[g_{\lambda, \lambda}, g_{\lambda}] \]
\[ + F_{u}[g_{\lambda, \lambda}] + F_{uu}[g_{\lambda}, g_{\lambda}, w_0 + g_t] \]
\[ = \langle l, F_{\lambda, \lambda}[w_0] + 2F_{\lambda, u}[v_1] \rangle. \]

Next we have
\[ h_{\lambda, t}(\lambda_0, 0) = \frac{1}{2} f_{\lambda, t}(\lambda_0, 0) \]
\[ = \frac{1}{2} \langle l, F_{\lambda, uu}[w_0 + g_t]^2 + F_{\lambda, u}[g_{tt}] + 2F_{uu}[g_{\lambda, t}, w_0 + g_t] + F_{u}[g_{\lambda, tt}] \]
\[ + F_{uu}[g_{\lambda}, w_0 + g_t, w_0 + g_t] \]
\[ = \frac{1}{2} \langle l, F_{\lambda, uu}[w_0]^2 + F_{\lambda, u}[g_{tt}] + 2F_{uu}[g_{\lambda, t}, w_0] \rangle \]
\[ = \frac{1}{2} \langle l, F_{\lambda, uu}[w_0]^2 + F_{\lambda, u}[v_2] + 2F_{uu}[v_1, w_0] \rangle; \]
and finally,
\[ h_{tt}(\lambda_0, 0) = \frac{1}{3} f_{tt}(\lambda_0, 0) \]
\[ = \frac{1}{3} \langle l, F_{uuu}[w_0 + g_t] + 3F_{uu}[g_{tt}, w_0 + g_t] + F_{u}[g_{ttt}] \rangle \]
\[ = \frac{1}{3} \langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_2] \rangle. \]

Therefore from Lemma 2.2, we conclude that the solution set of \( h(\lambda, t) = 0 \) near \( (\lambda, t) = (\lambda_0, 0) \) is a pair of intersecting curves if the matrix in (2.6) is indefinite, or is a single point if it is definite. Thus the solution set of \( F(\lambda, u) = 0 \) near \( (\lambda_0, u_0) \) is exactly the union of pair of intersecting curves which solve \( h(\lambda, t) = 0 \) and the line of trivial solutions.

For the case of two intersecting curves, we denote the two curves by \( (\lambda_i(s), u_i(s)) = (\lambda_i(s), u_0 + t_i(s)w_0 + g(\lambda_i(s), t_i(s))) \), with \( i = 1, 2 \). Then
\[ F(\lambda_i(s), u_0 + t_i(s)w_0 + g(\lambda_i(s), t_i(s))) = 0. \]  
(2.14)

From Lemma 2.2, the vectors \( v_i = (\lambda_i'(0), t_i'(0)) \) are the solutions of \( v^T H v = 0 \), which are the solutions \( (\mu, \eta) \) of (2.10). \( \square \)

Apparently the more interesting case in Theorem 2.3 is when \( H_0 \) is indefinite, i.e. \( \det(H_0) < 0 \), thus the following remark is only for that case.
Remark 2.4.

1. If $H_{11} \neq 0$, or equivalently,

$$F_{\lambda \lambda u}(\lambda_0, u_0)[w_0] + 2F_{\lambda u}(\lambda_0, u_0)[v_1] \notin R(F_u(\lambda_0, u_0)),$$

(2.15)

then $\eta_i \neq 0$ for $i = 1, 2$. In that case, both $\Gamma_1$ and $\Gamma_2$ are transversal to $\Gamma_0$. If $\mu_i \neq 0$ also holds for both $i = 1$ and $i = 2$, then (1.1) has exactly three solutions locally for any $\lambda \neq \lambda_0$ (see Fig. 2).

2. If $F_{\lambda u}(\lambda_0, u_0)[w_0] = 0$, then $v_1 = 0$, and similarly if $F_{uu}(\lambda_0, u_0)[w_0]^2 = 0$, then $v_2 = 0$. If $v_1 = v_2 = 0$, then the matrix $H_0$ in Theorem 2.3 is simplified to

$$H_0 = H_1(\lambda_0, u_0) \equiv \left( \begin{array}{c}
\langle l, F_{\lambda \lambda u}(\lambda_0, u_0)[w_0] \rangle \\
\frac{1}{2} \langle l, F_{\lambda uu}(\lambda_0, u_0)[w_0]^2 \rangle \\
\frac{1}{3} \langle l, F_{uuu}(\lambda_0, u_0)[w_0]^3 \rangle
\end{array} \right),$$

(2.16)

and (2.10) becomes

$$\langle l, F_{\lambda \lambda u}(\lambda_0, u_0)[w_0] \rangle \mu^2 + \langle l, F_{\lambda uu}(\lambda_0, u_0)[w_0]^2 \rangle \mu \eta + \frac{1}{3} \langle l, F_{uuu}(\lambda_0, u_0)[w_0]^3 \rangle \eta^2 = 0.$$

(2.17)

The local solution set of (1.1) near $(\lambda_0, u_0)$ described in Theorem 2.3 could be the union of three distinct curves $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$. But it is possible that one of $\Gamma_1$ or $\Gamma_2$ is identical to $\Gamma_0$, see for example, the mappings defined in (1.4) (Fig. 3). Indeed we have the following corollary:

**Corollary 2.5.** Assume the conditions in Theorem 2.3 are satisfied, and in addition we assume that

$$R(F_u(\lambda, u_0)) \subseteq R(F_u(\lambda_0, u_0)) \quad \text{for } \lambda \text{ near } \lambda_0,$$

(2.18)

then $\Gamma_1$ is identical to $\Gamma_0$, and $\Gamma_2$ is in form of $(\lambda_2(s), u_2(s)) = (\lambda_0 + \mu_2 s + s \theta_2(s), u_0 + s w_0 + s v_2(s))$, where

$$\mu_2 = -\frac{H_{22}}{2H_{12}},$$

(2.19)

which determines the bifurcation direction: if $\mu_2 \neq 0$, then a transcritical bifurcation occurs, and on either side of $\lambda = \lambda_0$, (1.1) has exactly two solutions locally; and if $\mu_2 = 0$ but a higher order non-degeneracy condition is satisfied, then a pitchfork bifurcation occurs.

**Proof.** From (2.1.3) and (2.1.8), we have

$$h(\lambda, 0) = f_t(\lambda, 0) = \langle l, F_u(\lambda, u_0)[w_0 + g_t(\lambda, 0)] \rangle = 0.$$

Hence one of solution curves of $h(\lambda, t) = 0$ is still given by $t = 0$, while the other solution curve for $h(\lambda, t) = 0$ is non-trivial. Moreover from (2.7) and (2.8),
Fig. 6. Global bifurcation diagrams. (Left): $F(\lambda, u) = u(\lambda^2 - u^2 + u^4)$; (Right): $F(\lambda, u) = u(\lambda^4 - \lambda^2 + u^2)$.

\[ H_{11} = \langle l, F_{\lambda\lambda u}[w_0] + 2F_{\lambda u}[v_1] \rangle = 0, \]
\[ H_{12} = \frac{1}{2} \langle l, F_{\lambda uu}[w_0]^2 + 2F_{uu}[w_0, v_1] \rangle, \]

thus we can assume that $(\mu_1, \eta_1) = (1, 0)$, thus $\Gamma_1$ is identical to $\Gamma_0$ and $(\mu_2, \eta_2) = (-\frac{H_{22}}{2H_{12}}, 1)$. □

The condition (2.18) implies that $N(F_u(\lambda, u_0)) \neq \{0\}$ for $\lambda$ near $\lambda_0$, since the Fredholm index of $F_u(\lambda, u)$ is locally a constant. This can be satisfied if $w_0 \in N(F_u(\lambda, u_0))$ for $\lambda$ near $\lambda_0$ and $R(F_u(\lambda, u_0)) = R(F_u(\lambda_0, u_0))$. In this case, (F3′) is satisfied for all nearby $\lambda$ and $v_1 = 0$. Hence a stronger degeneracy occurs here as each $(\lambda, u_0)$ is a degenerate point for $F(\lambda, u) = 0$. However only at $(\lambda_0, u_0)$, a bifurcation of non-trivial solutions occurs as (F4′) is satisfied at $(\lambda_0, u_0)$, and it is not satisfied for other $(\lambda, u_0)$ since $\det(H) = -H_{12}^2 \neq 0$. This is demonstrated by the examples in Fig. 3.

We end this section with two more one-dimensional examples to show the global nature of the bifurcation branches $\Gamma_1$ and $\Gamma_2$ obtained in Theorem 2.3. For the classical bifurcation from simple eigenvalue case, Shi and Wang [41] showed that the connected component of the set of non-trivial solutions of (1.1) containing the curve emanating from $(\lambda_0, u_0)$ as in Theorem 1.1 is either unbounded or it connects to another $(\lambda_*, u_0)$ which is another bifurcation point. This result extends the earlier one by Rabinowitz [33] which assumes $X = Y$ and the operators are compact ones. The bifurcation branches $\Gamma_1$ and $\Gamma_2$ in Theorem 2.3 can be unbounded in $\mathbb{R} \times X$ (see the examples in (1.3) and (1.4)), or they can connect to another bifurcation point (see Fig. 6 left panel), or each of $\Gamma_1$ and $\Gamma_2$ is bounded (see Fig. 6 right panel). It is easy to verify that the bifurcation at $(\lambda, u) = (0, 0)$ in both diagrams of Fig. 6 satisfies the conditions in Theorem 2.3. Note that the latter alternative is not possible for the classical bifurcation from simple eigenvalue case.

3. Stability

In this section, we consider the stability of the bifurcating solutions on $\Gamma_i$, $i = 0, 1, 2$ obtained in Theorem 2.3. At the bifurcation point $(\lambda_0, u_0)$, 0 is an eigenvalue of $F_u(\lambda_0, u_0)$. We are interested in the perturbation of this zero eigenvalue for solution $(\lambda, u)$ of (1.1) near the bifurcation
point. First we recall the following definition of $K$-simple eigenvalue and a fundamental result due to Crandall and Rabinowitz [7]. Here we denote by $B(X, Y)$ the set of bounded linear maps from $X$ into $Y$.

**Definition 3.1.** (See [7, Definition 1.2].) Let $T, K \in B(X, Y)$. We say that $\mu \in \mathbb{R}$ is a $K$-simple eigenvalue of $T$, if

$$\dim N(T - \mu K) = \text{codim } R(T - \mu K) = 1, \quad N(T - \mu K) = \text{span}\{w_0\},$$

and

$$K[\{w_0\}] \notin R(T - \mu K).$$

**Lemma 3.2.** (See [7, Lemma 1.3].) Suppose that $T_0, K \in B(X, Y)$ and $\mu_0$ is a $K$-simple eigenvalue of $T_0$. Then there exists $\delta > 0$ such that if $T \in B(X, Y)$ and $\|T - T_0\| < \delta$, then there exists a unique $\mu(T) \in \mathbb{R}$ satisfying $\|\mu(T) - \mu_0\| < \delta$ such that $N(T - \mu(T)K) \neq \emptyset$ and $\mu(T)$ is a $K$-simple eigenvalue of $T$. Moreover if $N(T_0 - \mu_0 K) = \text{span}\{w_0\}$ and $Z$ is a complement of $\text{span}\{w_0\}$ in $X$, then there exists a unique $w(T) \in X$ such that $N(T - \mu(T)K) = \text{span}\{w(T)\}$, $w(T) - w_0 \in Z$ and the map $T \mapsto (\mu(T), w(T))$ is analytic.

By using Lemma 3.2 in the same way as in [7, Corollary 1.13], we assume that $X \subset Y$, $K = i : X \to Y$ is the inclusion mapping $i(x) = x$ for $x \in X$, and $i$ is continuous. Then we have the following result for the linearized equation for the bifurcating solutions in Theorem 2.3.

**Proposition 3.3.** Let $X, Y, U, F, Z, \lambda_0, w_0, v_1$ and $v_2$ be the same as in Theorem 2.3, and let all assumptions in Theorem 2.3 on $F$ be satisfied. In addition we assume that $X \subset Y$, and the inclusion mapping $i : X \to Y$ is continuous. Let $(\lambda_i(s), u_i(s))$ $(i = 1, 2)$ be the solution curves in Theorem 2.3. Then there exist $\varepsilon > 0$, $C^2$ functions $\gamma : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to \mathbb{R}$, $\sigma_i : (-\varepsilon, \varepsilon) \to \mathbb{R}$, $v : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to X$, $w_i : (-\varepsilon, \varepsilon) \to X$ such that

$$F_u(\lambda, u_0)[v(\lambda)] = \gamma(\lambda)v(\lambda) \quad \text{for } \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon), \quad (3.1)$$

$$F_u(\lambda_i(s), u_i(s))[w_i(s)] = \sigma_i(s)w_i(s) \quad \text{for } s \in (-\varepsilon, \varepsilon), \quad (3.2)$$

where $\gamma(\lambda_0) = \sigma_i(0) = 0$, $v(\lambda_0) = w_i(0) = w_0$, and $v(\lambda) - w_0 \in X$, $w_i(s) - w_0 \in Z$.

The signs of $\gamma(\lambda)$ and $\sigma_i(s)$ determine the stability of the bifurcating solutions. In [7], the stability of bifurcating solutions obtained in Theorem 1.1 was considered. Here we consider the stability of bifurcating solutions obtained in Theorem 2.3.

**Theorem 3.4.** Let the assumptions of Proposition 3.3 hold and let $\gamma, \sigma_i$ be the functions provided by Proposition 3.3. In addition, we assume that

$$w_0 \notin R(F_u(\lambda_0, u_0)), \quad \text{where } w_0(\neq 0) \in N(F_u(\lambda_0, u_0)). \quad (3.3)$$

Then:
1. \( \gamma'(\lambda_0) = 0 \), and
\[
\gamma''(\lambda_0) = \frac{H_{11}}{\langle l, w_0 \rangle}.
\]  
(3.4)

If \( H_{11} = 0 \) and we assume that \( F \in C^4 \) near \((\lambda_0, u_0)\), then we have \( \gamma''(\lambda_0) = 0 \) and
\[
\gamma'''(\lambda_0) = -\frac{\langle l, F_{\lambda u}(\lambda_0, u_0)v(\lambda_0) \rangle + 3F_{\lambda u}(\lambda_0, u_0)v_1 + 3F_{u}(\lambda_0, u_0)v_3}{\langle l, w_0 \rangle},
\]
where \( v_3 \in Z \) is the unique solution of
\[
F_{\lambda u}(\lambda_0, u_0)v_0 + 2F_{\lambda u}(\lambda_0, u_0)v_1 + F_u(\lambda_0, u_0)v_3 = 0.
\]  
(3.6)

2. \( \sigma'_i(0) = 0 \) and
\[
\sigma''_i(0) = \frac{H_{22}\eta_i^2 - H_{11}\mu_i^2}{\langle l, w_0 \rangle},
\]
where \( H_{11}, H_{22}, \mu_i, \eta_i \) \((i = 1, 2)\) are defined in Theorem 2.3.

**Proof.** Differentiating (3.1) with respect to \( \lambda \), we obtain that
\[
F_{\lambda u}(\lambda, u_0)[v'(\lambda)] + F_u(\lambda, u_0)[v(\lambda)] = \gamma'(\lambda)v(\lambda) + \gamma(\lambda)v'(\lambda),
\]  
(3.8)

and evaluating at \( \lambda = \lambda_0 \), we have
\[
F_{\lambda u}(\lambda_0, u_0)[v_0] + F_u(\lambda_0, u_0)[v'(\lambda_0)] = \gamma'(\lambda_0)v_0.
\]

Then the assumption \((F3')\) implies that \( \gamma'(\lambda_0) = 0 \) and \( v'(\lambda_0) = v_1 \) from (2.4). We differentiate (3.8) again, and we obtain that
\[
F_{\lambda u}(\lambda, u_0)[v(\lambda)] + 2F_{\lambda u}(\lambda, u_0)[v'(\lambda)] + F_u(\lambda, u_0)[v''(\lambda)]
\]
\[
= \gamma''(\lambda)v(\lambda) + 2\gamma'(\lambda)v'(\lambda) + \gamma(\lambda)v''(\lambda).
\]  
(3.9)

Setting \( \lambda = \lambda_0 \) in (3.9), we have
\[
F_{\lambda u}(\lambda_0, u_0)[v_0] + 2F_{\lambda u}(\lambda_0, u_0)v_1 + F_u(\lambda_0, u_0)[v''(\lambda_0)] = \gamma''(\lambda_0)v_0,
\]  
(3.10)

and by applying \( l \in Y^* \) to (3.10), we obtain (3.4) from (3.3). If \( F \) satisfies (2.15), then \( H_{11} \neq 0 \) and \( \gamma''(\lambda_0) \neq 0 \). If \( H_{11} = 0 \), then \( \gamma''(\lambda_0) = 0 \), and
\[
F_{\lambda u}(\lambda_0, u_0)[v_0] + 2F_{\lambda u}(\lambda_0, u_0)v_1 + F_u(\lambda_0, u_0)[v''(\lambda_0)] = 0.
\]

We have \( v''(\lambda_0) = v_3 \), where \( v_3 \) is defined by (3.6). We differentiate (3.9) again, and we obtain
\[ F_{\lambda,\lambda,u}(\lambda, u_0)[v(\lambda)] + 3 F_{\lambda,\lambda,u}(\lambda, u_0)[v'(\lambda)] + 3 F_{\lambda,u}(\lambda, u_0)[v''(\lambda)] + F_u(\lambda, u_0)[v'''(\lambda)] = \gamma'''(\lambda)v(\lambda) + 3\gamma''(\lambda)v'(\lambda) + 3\gamma'(\lambda)v''(\lambda) + \gamma(\lambda)v'''(\lambda). \]

By setting \( \lambda = \lambda_0 \), we have
\[ \gamma'''(\lambda_0)w_0 = F_{\lambda,\lambda,\lambda,u}(\lambda_0, u_0)[w_0] + 3 F_{\lambda,\lambda,u}(\lambda_0, u_0)[v_1] + 3 F_{\lambda,u}(\lambda_0, u_0)[v_3] + F_u(\lambda_0, u_0)[v'''(\lambda_0)]. \] (3.11)

Thus by applying \( l \in Y^* \) to (3.11), we obtain (3.5).

On the other hand, we differentiate \( F(\lambda_i(s), u_i(s)) = 0 \) twice to obtain
\[ F_{\lambda\lambda}[\lambda_i'(s)]^2 + 2 F_{\lambda,u}[\lambda_i'(s)\lambda_i(s) + F_{\lambda}\lambda_i''(s) + F_{uu}[u_i'(s)]^2 + F_u[u_i''(s)] = 0. \] (3.12)

By setting \( s = 0 \) in (3.12), we get
\[ u_i''(0) = 2\mu_i\eta_i v_1 + \eta_i^2 v_2. \] (3.13)

Similarly by differentiating (3.2), we obtain
\[ F_{\lambda,u}[w_i(s)]\lambda_i'(s) + F_{uu}[w_i(s), u_i'(s)] + F_u[w_i'(s)] = \sigma_i'(s)w_i(s) + \sigma_i(s)w_i'(s). \] (3.14)

By setting \( s = 0 \) in (3.14), we get
\[ \mu_i F_{\lambda,u}[w_0] + \eta_i F_{uu}[w_0]^2 + F_u[w_i'(0)] = \sigma_i'(0)w_0. \] (3.15)

Hence by applying \( l \) to (3.15), we obtain \( \sigma_i'(0) = 0 \) and
\[ u_i'(0) = \mu_i v_1 + \eta_i v_2. \] (3.16)

We differentiate (3.14) again, and we have
\[ F_{\lambda,\lambda,u}[w_i(s)][\lambda_i'(s)]^2 + 2 F_{\lambda,u}[w_i(s), u_i'(s)]\lambda_i'(s) + F_{uuu}[w_i(s), u_i'(s), u_i'(s)] + 2 F_{\lambda,u}[w_i(s)]\lambda_i''(s) + 2 F_{uuu}[w_i'(s), u_i'(s)] \]
\[ + F_{uuu}[w_i(s), u_i'(s)] + F_u[w_i''(s)] = \sigma_i''(s)w_i(s) + 2\sigma_i'(s)w_i'(s) + \sigma_i(s)w_i''(s). \] (3.17)

By setting \( s = 0 \) in (3.17) and using (3.13) and (3.16), we get
\[ \sigma_i''(0)w_0 = \mu_i^2 F_{\lambda,\lambda,u}[w_0] + 2\mu_i\eta_i F_{\lambda,u}[w_0]^2 + \eta_i^2 F_{uuu}[w_0]^3 + 2\mu_i F_{\lambda,u}[\mu_i v_1 + \eta_i v_2] + F_{\lambda,u}[w_0]\lambda_i''(0) + 2\eta_i F_{uuu}[\mu_i v_1 + \eta_i v_2, w_0] + F_{uuu}[w_0, 2\mu_i\eta_i v_1 + \eta_i^2 v_2] + F_u[w_i''(0)]. \] (3.18)
Thus by applying $l$ to (3.18), we obtain
\[ H_{11} \mu_i^2 + 4H_{12} \mu_i \eta_i + 3H_{22} \eta_i^2 = \sigma''_i(0) \langle l, w_0 \rangle, \]  
which implies (3.7) by using (2.10) and (3.3). \[ \Box \]

**Remark 3.5.**

1. If $F$ satisfies (2.15), then we have $\gamma'(0) = 0$ and $\gamma''(0) \neq 0$ from (3.4), thus the trivial solution on $\Gamma_0$ in Theorem 2.3 has the same stability near the bifurcation point. Similarly if $H_{22} \eta_i^2 \neq H_{11} \mu_i^2$, we have $\sigma'_i(0) = 0$ and $\sigma''_i(0) \neq 0$ from (3.7), thus the non-trivial solutions on $\Gamma_i$ ($i = 1, 2$) in Theorem 2.3 also have the same stability before and after the bifurcation point. This shows that in general, the bifurcation from a degenerate simple eigenvalue does not cause an exchange of stability as in the non-degenerate simple eigenvalue case [7].

2. When $\Gamma_1$ and $\Gamma_2$ are both distinctive from $\Gamma_0$, the stability of non-trivial solutions on $\Gamma_1$ and $\Gamma_2$ depends on the sign of $H_{11} \cdot H_{22}$. If $H_{11} \cdot H_{22} < 0$, then from Theorems 2.3 and 3.4, $\sigma_1(s) = \sigma_2(s)$ for small $s$ such that $|s| \neq 0$. In this case, the stability of solutions on one of $\Gamma_1$ or $\Gamma_2$ have the same stability as the ones on $\Gamma_0$ (see Fig. 2 left panel). But if $H_{11} \cdot H_{22} > 0$, then the solutions on both of $\Gamma_1$ and $\Gamma_2$ have the same stability as the ones on $\Gamma_0$ (see Fig. 2 right panel).

3. From Remark 2.4, if $R(F_u(\lambda, u_0)) \subseteq R(F_u(\lambda_0, u_0))$ for $\lambda$ near $\lambda_0$ and $F_{\lambda uu}(\lambda_0, u_0)[w_0]^2 \notin R(F_u(\lambda_0, u_0))$, then one can assume that $(\mu_1, \eta_1) = (1, 0)$ thus $\Gamma_1$ is identical to $\Gamma_0$, and $\Gamma_2$ is distinctive from $\Gamma_0$. In this case we have $H_{11} = 0$ so $\gamma''(0) = \sigma''_1(0) = 0$, $\gamma'''(0)$ is given by (3.5), and $\sigma''_2(0) = \frac{H_{22}}{\langle l, w_0 \rangle}$ (see Fig. 3). In this case, if $\gamma'''(0) \neq 0$ and $\sigma''_2(0) \neq 0$, then the stability of the non-trivial solutions does not change across the bifurcation point, and all the trivial solutions are always degenerate.

4. **Perturbation problem**

In this section, we shall consider a nonlinear equation with two parameters $\varepsilon$ and $\lambda$:
\[ F(\varepsilon, \lambda, u) = 0, \]  
where $F \in C^1(M, Y)$, $M \equiv \mathbb{R} \times \mathbb{R} \times X$, and $X, Y$ are Banach spaces. Here we consider the variation of bifurcation diagrams in $(\lambda, u)$-space when the value of an additional parameter $\varepsilon$ changes, following the consideration in our previous work [23,38]. To consider the original equation and its linearization together, we define an augmented operator
\[ G(\varepsilon, \lambda, u, w) = \begin{pmatrix} F(\varepsilon, \lambda, u) \\ F_u(\varepsilon, \lambda, u)[w] \end{pmatrix}. \]  
We consider the solutions $(\varepsilon_0, \lambda_0, u_0, w_0)$ of $G(\varepsilon, \lambda, u, w) = 0$. For $(\varepsilon_0, \lambda_0, u_0) \in M$ and $w_0 \in X_1 = \{ x \in X : \| x \| = 1 \}$, by using Hahn–Banach Theorem (see [38, Lemma 7.1]), there exists a closed subspace $X_3$ of $X$ with codimension 1 such that $X = L(w_0) \oplus X_3$, where $L(w_0) =$
verify all the assumptions in Theorem 2.3. We define the linearized operator $X$ and transcritical bifurcation (see Theorem 4.3 below and Fig. 5). In the following new result for the perturbed problem (4.1), the original transcritical or pitchfork bifurcation is not preserved by the perturbation (see Figs. 1–4 in [23] or Fig. 4 in this paper). In the following new result for the perturbed problem (4.1), the trivial solutions are preserved by the perturbation, but a pitchfork bifurcation is perturbed into a transcritical bifurcation (see Theorem 4.3 below and Fig. 5).

In the following we will still use the conditions (F1) on $F$ defined in previous sections and in [23, 38], but we shall understand that the variables are $(\varepsilon_0, \lambda_0, u_0)$ instead of $(\lambda_0, u_0)$ in all these conditions. In the following theorem, we consider a situation that a classical pitchfork bifurcation occurs and there is a unique degenerate trivial solution when $\varepsilon = \varepsilon_0$, then for a perturbed problem with $\varepsilon$ near $\varepsilon_0$, the degenerate trivial solution persists but some new degenerate solutions emerge for $\varepsilon \neq \varepsilon_0$.

**Theorem 4.1.** Let $F \in C^3(M, Y)$, and let there be $T_0 = (\varepsilon_0, \lambda_0, u_0, w_0) \in M_1$ such that $G(T_0) = (0, 0)$. Assume that there exists $\delta_0 > 0$ such that

$$F(\varepsilon, \lambda, u_0) = 0, \quad F_u(\varepsilon, \lambda_0, u_0)[w_0] = 0, \quad \text{for } |\varepsilon - \varepsilon_0| < \delta_0, \ |\lambda - \lambda_0| < \delta_0. \quad (4.3)$$

Suppose that $F$ satisfies (F1), (F3), (F4') at $T_0$, and

$$F_{uuu}(\varepsilon_0, \lambda_0, u_0)[w_0]^2 + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_2] \notin R(F_u(\varepsilon_0, \lambda_0, u_0)), \quad (4.4)$$

where $v_2 \in X_3$ is the unique solution of

$$F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0]^2 + F_u(\varepsilon_0, \lambda_0, u_0)[v] = 0. \quad (4.5)$$

Then the solution set of $G(\varepsilon, \lambda, u, w) = (0, 0)$ near $T_0$ is the union of $C^1$ curves intersecting at $T_0$ including the line of trivial solutions $\Sigma_0 = \{(\varepsilon, \lambda_0, u_0, w_0) : |\varepsilon - \varepsilon_0| < \delta_1\}$ and two other curves $\Sigma_i = \{T_{is} = (\varepsilon_i(s), \lambda_i(s), u_i(s), w_i(s)) : |s| < \delta_2\}$, for some $\delta_1, \delta_2 > 0$, where $\varepsilon_i(s) = \varepsilon_0 + \mu_is + s\eta_i, u_i(s) = u_0 + s\eta_i, w_i(s) = u_0 + s\eta_iv_2 + s\zeta_i(s), w_i(s) = u_0 + s\eta_i, v_2 + s\zeta_i(s), \lambda_i(s) = \lambda_0 + s\eta_i, u_i(s) = u_0 + s\eta_i, w_0 + s\zeta_i(s), w_i(s) = u_0 + s\eta_i, v_2 + s\zeta_i(s), \zeta_i(0) = \zeta_{i1}(0) = \zeta_{i2}(0) = \zeta_{i3}(0) = 0, \ i = 1, 2, (\mu_1, \eta_1) = (1, 0) \text{ and}$

$$\left(\mu_2, \eta_2\right) = \left(\frac{-2[\lambda_0, F_{uuu}(w_0)^3 + 3F_{uu}(w_0, v_2)]}{3[\lambda_0, F_{uuu}(w_0)^2 + F_{uu}(v_2)]}, 1\right). \quad (4.6)$$

**Proof.** We apply Theorem 2.3 to the equation $G(\varepsilon, \lambda, u, w) = (0, 0)$ with $\varepsilon$ as parameter, and we verify all the assumptions in Theorem 2.3. We define the linearized operator $K : \mathbb{R} \times X \times Y \times Y \rightarrow Y \times Y$ of $G$ by
\[ K[\tau, v, \psi] = G_{(\lambda, u, w)}(T_0)[\tau, v, \psi] = \begin{pmatrix} \tau F_\lambda(\varepsilon_0, \lambda_0, u_0) + F_u(\varepsilon_0, \lambda_0, u_0)[v] \\ \tau F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] \end{pmatrix}. \] (4.7)

We prove it in several steps, and we recall that \( l \in Y^* \) satisfies \( N(l) = R(F_u(\varepsilon_0, \lambda_0, u_0)) \). In the following, for the simplicity of notations, we denote \( U = (\lambda, u, w) \) and \( U_0 = (\lambda_0, u_0, w_0) \).

1. Suppose that \((\tau, v, \psi) \in N(K)\) and \((\tau, v, \psi) \neq (0, 0, 0)\). Notice that \( F_\lambda(\varepsilon_0, \lambda_0, u_0) = 0 \) from (4.3), then the first equation of \( K[\tau, v, \psi] = (0, 0) \) is reduced to \( F_u(\varepsilon_0, \lambda_0, u_0)[v] = 0 \). Hence from (F1), we have \( v = kw_0 \) for \( k \in \mathbb{R} \) and

\[ \tau F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[k w_0, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] = 0. \] (4.9)

Applying \( l \) to (4.9), we obtain \( \tau = 0 \) from (F3) and (F4'). Thus \( \psi = kv_2 \), where \( v_2 \) is uniquely determined by (4.5). Therefore \( N(K) = \text{span}\{W_0 = (0, w_0, v_2)\} \).

2. \( \dim N(K) = 1 \). Let \((h, g) \in R(K)\), and let \((\tau, v, \psi) \in \mathbb{R} \times X \times X_3 \) satisfy

\[ \tau F_\lambda(\varepsilon_0, \lambda_0, u_0) + F_u(\varepsilon_0, \lambda_0, u_0)[v] = h, \] (4.10)

\[ \tau F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] + F_{uu}(\varepsilon_0, \lambda_0, u_0)[v, w_0] + F_u(\varepsilon_0, \lambda_0, u_0)[\psi] = g. \] (4.11)

Applying \( l \) to (4.10), we get \( \langle l, h \rangle = 0 \), hence \( h \in R(F_u(\varepsilon_0, \lambda_0, u_0)) \), and \( R(K) \subseteq R(F_u) \times Y \). Conversely, for any \((h, g) \in R(F_u) \times Y\), there exists a unique \( v_3 \in X_3 \) such that \( F_u(\varepsilon_0, \lambda_0, u_0)[v_3] = h \), then \( v_3 \) solves (4.10). Substituting \( v = v_3 \) into (4.11), and applying \( l \), we obtain

\[ \tau \langle l, F_{\lambda u}[w_0] \rangle + \langle l, F_{uu}[w_0, v_3] \rangle = \langle l, g \rangle. \] (4.12)

Then from (F3) there exists a unique \( \tau \) so that (4.12) holds for any \( g \in Y \). With such choice of \((\tau, v, \psi) \) in (4.11) it is uniquely solvable in \( X_3 \). Therefore this \((\tau, v_3, \psi) \) is a pre-image of \((h, g)\), which implies that \( R(K) = R(F_u(\varepsilon_0, \lambda_0, u_0)) \times Y \), and \( \text{codim} R(K) = 1 \). This proves that \( G \) satisfies the condition (F1) at \((\varepsilon_0, U_0)\).

3. \( G \) satisfies (F3'), that is \( G_{eU}(T_0)[W_0] \in R(K) \). From (4.3), we have

\[ G_{eU}(T_0)[W_0] = \begin{pmatrix} F_{\varepsilon u}(\varepsilon_0, \lambda_0, u_0)[w_0] \\ F_{\varepsilon uu}(\varepsilon_0, \lambda_0, u_0)[w_0]^2 + F_{\varepsilon uu}(\varepsilon_0, \lambda_0, u_0)[v_2] \end{pmatrix} = \begin{pmatrix} 0 \\ F_{\varepsilon uu}(\varepsilon_0, \lambda_0, u_0)[w_0]^2 + F_{\varepsilon uu}(\varepsilon_0, \lambda_0, u_0)[v_2] \end{pmatrix}, \]

which belongs to \( R(K) \) from part (3). Eq. (2.4) now becomes \( G_{eU}[W_0] + K[\tau, v, \psi] = 0 \), which is
Looking for a solution \((\tau, v, \psi)\) \(\in Z_1 \equiv \mathbb{R} \times X \times X_3\), first we know that a solution of (4.13) is given by \(v = kw_0\) for \(k \in \mathbb{R}\), and we choose \(k = 1\) here. Next we apply \(l\) to (4.14) and we obtain

\[
\tau = \tau_1 \equiv -\frac{\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0]^2 + F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0]v_2 \rangle}{\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0] \rangle},
\]

and subsequently \(\psi\) can be uniquely determined with \(v = w_0\) and \(\tau_1\) given by (4.15). We denote this solution to be \(V_1 = (\tau_1, w_0, \psi_1)\).

(5) \(G\) satisfies (F4') at \((\varepsilon_0, U_0)\), that is \(G_{UU}(T_0)[W_0]^2 \in R(K)\). Notice that

\[
G_{UU}(T_0)[W_0]^2 = \left( F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0]^3 + 2F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, v_2] \right)
\]

thus (F4') is satisfied for \(G\) as \(F\) satisfies (F4') at \(T_0\). Eq. (2.5) now becomes \(G_{UU}[W_0]^2 + K[\tau, v, \psi] = 0\), which is

\[
F_{uu}[w_0]^2 + F_u[v] = 0,
\]

\[
F_{uu}[w_0]^3 + 2F_{uu}[w_0, v_2] + \tau F_{uu}[w_0] + F_u[v, w_0] + F_u[\psi] = 0.
\]

We look for a solution \((\tau, v, \psi)\) \(\in Z_1 \equiv \mathbb{R} \times X \times X_3\). Then a solution of (4.16) is given by \(v = v_2\) from (F4'). We apply \(l\) to (4.17) and we obtain

\[
\tau = \tau_2 \equiv -\frac{\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0]^3 + 3F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, v_2] \rangle}{\langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0]^2 \rangle}.
\]

Then \(\psi\) can also be uniquely determined. We denote this solution to be \(V_2 = (\tau_2, v_2, \psi_2)\).

(6) We prove that \(\det(\tilde{H}) < 0\), where the matrix \(\tilde{H}\) is given by

\[
\tilde{H} = \tilde{H}(\varepsilon_0, U_0) = \begin{pmatrix}
\tilde{H}_{11} & \tilde{H}_{12} \\
\tilde{H}_{12} & \tilde{H}_{22}
\end{pmatrix}
\]

and \(\tilde{H}_{ij}\) are given by

\[
\tilde{H}_{11} = \langle l_1, G_{eeU}[W_0] + 2G_{eU}[V_1] \rangle,
\]

\[
\tilde{H}_{12} = \frac{1}{2}\langle l_1, G_{eUU}[W_0] + G_{eU}[V_2] + 2G_{UU}[W_0, V_1] \rangle,
\]

\[
\tilde{H}_{22} = \frac{1}{3}\langle l_1, G_{UUU}[W_0]^3 + 3G_{UU}[W_0, V_2] \rangle.
\]

Here \(l_1 \in (Y \times Y)^*\) is defined by
\[\{l_1, (y_1, y_2)\} = (l, y_1),\]  

where \(l \in Y^*\) is defined previously so that \((l, y) = 0\) if and only if \(y \in R(F_u(\epsilon_0, \lambda_0, u_0))\).

From (4.3), (4.20), we have

\[
\begin{align*}
\tilde{H}_{11} &= \langle l, F_{\tilde{e}_U}[\tilde{W}_0] + 2F_{\tilde{e}_U}[\tilde{V}_1] \rangle \\
&= \langle l, F_{\tilde{e}u}[w_0] + 2F_{\tilde{e}u}[w_0] \rangle = 0.
\end{align*}
\]

On the other hand, from (4.4), (4.15), (4.21) and \((F^4)\), we have

\[
\begin{align*}
\tilde{H}_{12} &= \frac{1}{2}\langle l, F_{\tilde{e}_U}[\tilde{W}_0]^2 + F_{\tilde{e}_U}[\tilde{V}_2] + 2F_{\tilde{e}_U}[\tilde{W}_0, \tilde{V}_1] \rangle \\
&= \frac{1}{2}\langle l, F_{\tilde{e}uu}[w_0]^2 + \tau_1 F_{\tilde{e}u}[\tilde{u}][w_0] + 2F_{\tilde{e}u}[w_0]^2 \rangle \\
&= \frac{1}{2}\langle l, F_{\tilde{e}uu}[w_0]^2 + F_{\tilde{e}u}[v_2] + 2\tau_1 F_{\tilde{e}u}[w_0] \rangle \\
&= -\frac{1}{2}\langle l, F_{\tilde{e}uu}[w_0]^2 + F_{\tilde{e}u}[v_2] \rangle \neq 0.
\end{align*}
\]

And from (4.22) and (4.18), we have

\[
\begin{align*}
\tilde{H}_{22} &= \frac{1}{3}\langle l, F_{\tilde{e}_U}[\tilde{W}_0]^3 + 3F_{\tilde{e}_U}[\tilde{W}_0, \tilde{V}_2] \rangle \\
&= \frac{1}{3}\langle l, F_{\tilde{u}uu}[w_0]^3 + 3F_{\tilde{u}u}[w_0] + 3F_{\tilde{u}u}[w_0, v_2] \rangle \\
&= -\frac{2}{3}\langle l, F_{\tilde{u}uu}[w_0]^3 + 3F_{\tilde{u}u}[w_0, v_2] \rangle
\end{align*}
\]

where \(\tilde{U} = (\lambda, u), \tilde{W}_0 = (0, w_0), \tilde{V}_1 = (\tau_1, w_0),\) and \(\tilde{V}_2 = (\tau_2, v_2)\).

From (4.19), (4.20) and (4.25), we obtain

\[
\det(\tilde{H}) = -\tilde{H}_{12} = -\frac{1}{4}\langle l, F_{\tilde{e}uu}[w_0]^2 + F_{\tilde{e}u}[v_2] \rangle^2 < 0.
\]

Now we can apply Theorem 2.3 to \(G(\epsilon, U) = (0, 0)\). Then the solution set of \(G(\epsilon, U) = (0, 0)\) near \(T_0\) is the union of \(C^1\) curves intersecting at \(T_0\) including the line of trivial solutions \(\Sigma_0 = \{(\epsilon, U_0): |\epsilon - \epsilon_0| < \delta_1\}\) and two other curves \(\Sigma_i = \{T_{i\delta} = (\epsilon_i(s), U_i(s))$: \(|s| < \delta_2\}, \) for some \(\delta_1, \delta_2 > 0\), where \(\epsilon_i(s) = \epsilon_0 + \mu_is + s\eta(s), U_i(s) = U_0 + s\eta_iW_0 + sZ_i(s),\) where \(Z_i(0) = Z_i(0) = 0, i = 1, 2, \) and \((\mu_1, \eta_1)\) and \((\mu_2, \eta_2)\) satisfy

\[
2\tilde{H}_{12}\mu + \tilde{H}_{22}\eta^2 = 0.
\]

We can choose \((\mu_1, \eta_1) = (1, 0)\) and \((\mu_2, \eta_2) = (-\tilde{H}_{22}/(2\tilde{H}_{12}), 1)\). This completes the proof. \(\square\)

Following Corollary 2.5, we show the following special case of Theorem 4.1:
Corollary 4.2. Assume the conditions in Theorem 4.1 are satisfied, and in addition we assume that

\[ R(F_u(\varepsilon, \lambda_0, u_0)) \subseteq R(F_u(\varepsilon_0, \lambda_0, u_0)), \quad \text{for } |\varepsilon - \varepsilon_0| < \delta_0. \]  

(4.29)

Then the solution set of \( G(\varepsilon, \lambda, u, w) = (0, 0) \) near \( T_0 \) consists precisely of the curves \( \Sigma_0 = \{ (\varepsilon, \lambda_0, w_0, 0) : |\varepsilon - \varepsilon_0| < \delta_1 \} \) and \( \Sigma_2 = \{ T_s = (\varepsilon(s), \lambda_2(s), u_2(s), w_2(s)) : |s| < \delta_2 \} \), for some \( \delta_1, \delta_2 > 0 \), where \( \varepsilon_2(s) = \varepsilon_0 + \alpha s + s\zeta_0(s), \lambda_2(s) = \lambda_0 + s\zeta_2_1(s), u_2(s) = u_0 + sw_0 + s\zeta_2_2(s), w_2(s) = w_0 + sv_2 + s\zeta_2_3(s) \).

Proof. We apply Corollary 2.5 by showing (2.18) holds, that is

\[ \langle l, F_{uuu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_2] \rangle = -2(\varepsilon, \lambda, u, w)[w_0]^3 + 3F_{uu}(\varepsilon_0, \lambda_0, u_0)[w_0, w_2] \].

(4.30)

Thus \( R(G_U(\varepsilon, U_0)) \subseteq R(K) \). And from (4.29), we have \( \langle l, F_u(\varepsilon, \lambda_0, u_0)[v_2] \rangle = 0 \), so we obtain \( \langle l, F_{uu}(\varepsilon_0, \lambda_0, u_0)[v_2] \rangle = 0 \). Now we get (4.30), from (4.6). Hence the conclusions follow from Corollary 2.5. \( \square \)

In Corollary 4.2, we have \( \varepsilon_2(0) = \varepsilon_0, \varepsilon'_2(0) = \alpha, \lambda_2(0) = \lambda_0, \lambda'_2(0) = 0, u'_2(0) = u_0, \) and \( w'_2(0) = v_2 \). To completely determine the turning direction of the curve of degenerate solutions, we calculate \( \lambda''_2(0) \). Let \( \{ T_s = (\varepsilon_2(s), \lambda_2(s), u_2(s), w_2(s)) : s \in (-\delta, \delta) \} \) be a curve of degenerate solutions which we obtain in Corollary 4.2. Differentiating \( G(\varepsilon_2(s), \lambda_2(s), u_2(s), w_2(s)) = 0 \) with respect to \( s \), we obtain (for convenience, we drop all the subscripts 2 in the following)

\[ F\varepsilon''(s) + F_{\lambda}\lambda'(s) + F_u[u'(s)] = 0, \]  

(4.31)

\[ F_{eu}[w(s)]\varepsilon'(s) + F_{euu}[w(s)]\lambda'(s) + F_{uu}[w(s), u'(s)] + F_u[w'(s)] = 0. \]  

(4.32)

Setting \( s = 0 \) in (4.32), we get exactly (4.5). We differentiate (4.31) and (4.32) again, and we have

\[ F\varepsilon''[\varepsilon'(s)]^2 + F_{\lambda}\varepsilon''(s) + F_{\lambda\lambda}[\lambda'(s)]^2 + F_{\lambda\lambda''}(s) + F_{uuu}[u'(s)]^2 \]

(4.33)

\[ + F_{u}u''(s) + 2F_{\lambda\lambda\lambda}[w(s)]\lambda'(s) + 2F_{euu}[u'(s)]\varepsilon'(s) + 2F_{uu}[u'(s)]\lambda'(s) = 0, \]

\[ F_{eeu}[w(s)]\varepsilon''(s)^2 + F_{eu}[w(s)]\varepsilon''(s) + F_{u}[u'(s)]\lambda''(s) \]

\[ + F_{u}[w''(s)] + 2F_{u}[u''(s)]u'(s), w(s)] + F_{uu}[w(s), u'(s)] + F_{eu}[w'(s)] \]

\[ + 2F_{\lambda uu}[u'(s), w(s)]\lambda'(s) + 2F_{\lambda u}[w'(s)]\varepsilon'(s) + 2F_{uu}[u'(s), w(s)]\varepsilon'(s) \]

\[ + 2F_{\lambda uu}[u'(s), w(s)]\lambda'(s) + 2F_{\lambda uu}[w'(s)]\varepsilon'(s) + 2F_{uu}[w'(s)]\lambda'(s) \]

\[ + 2F_{uu}[w'(s), u'(s)] = 0. \]  

(4.34)

Setting \( s = 0 \) in (4.33) and (4.34), we obtain \( u''(0) = v_2 \) and
\[ F_{\lambda u}[w_0]\lambda''(0) + F_{\lambda uu}[w_0]^3 + F_u[w''(0)] + 2\alpha F_{\epsilon uu}[w_0]^2 + 2\alpha F_{\epsilon u}[v_2] + 3F_{uu}[w_0, v_2] = 0 \]

and applying \( l \) to it, we obtain
\[
\lambda''(0) = \lambda''(0) = -\frac{(l, F_{\lambda uu}[w_0]^3 + 3F_{uu}[w_0, v_2])}{(l, F_{\lambda u}[w_0])} = \frac{(l, F_{\lambda uu}[w_0]^3 + 3F_{uu}[w_0, v_2])}{3(l, F_{\lambda u}[w_0])}.
\]

(4.35)

from (4.30). From Theorem 4.1, Corollary 4.2 and above calculations, a one-to-one relation between \( \lambda_2 \) and \( \epsilon \) can be established if \( \epsilon'_{2}(0) = \alpha \neq 0 \) by
\[
\lambda_2(\epsilon) - \lambda_0 \approx \frac{1}{2} \frac{\lambda''_{2}(0)}{\alpha} \left( \frac{\epsilon - \epsilon_0}{\alpha} \right)^2 = \frac{\lambda''_{2}(0)}{2\alpha^2} (\epsilon - \epsilon_0)^2 = \Lambda(\epsilon - \epsilon_0)^2;
\]

(4.36)

where
\[
\Lambda = \frac{\lambda''_{2}(0)}{2\alpha^2} = \frac{3(l, F_{\lambda uu}[w_0]^2)^2}{8(l, F_{\lambda u}[w_0]) \cdot (l, F_{\lambda uu}[w_0]^3 + 3F_{uu}[w_0, v_2])}.
\]

(4.37)

from (4.30) and (4.35).

Now by using the information for the degenerate solutions obtained in Theorem 4.1 and Corollary 4.2, we have the following result for the variation of the bifurcation diagrams of the original equation \( F(\epsilon, \lambda, u) = 0 \) in the \((\lambda, u)\)-space when the parameter \( \epsilon \) is perturbed from \( \epsilon = \epsilon_0 \).

**Theorem 4.3.** Assume that all the conditions in Theorem 4.1 and Corollary 4.2 are satisfied, and \( \{T_{23}\} \) is defined as in Corollary 4.2. For the purpose of fixing an orientation, we also assume that, in addition to (F3) and (4.4),

\[
(l, F_{\lambda u}(\epsilon_0, \lambda_0, u_0)[w_0]) > 0,
\]

(4.38)

\[
(l, F_{\epsilon uu}(\epsilon_0, \lambda_0, u_0)[w_0]^2) < 0,
\]

(4.39)

\[
(l, F_{\lambda uu}(\epsilon_0, \lambda_0, u_0)[w_0]^3 + 3F_{uu}(\epsilon_0, \lambda_0, u_0)[w_0, v_2]) < 0,
\]

(4.40)

where \( v_2 \) is defined by (4.5). Then there exist \( \rho, \eta, \delta_3, \delta_4 > 0 \) such that for \( N = [(\lambda, u) \in \mathbb{R} \times X: |\lambda - \lambda_0| \leq \delta_3, ||u - u_0|| \leq \delta_4, |\epsilon - \epsilon_0| < \rho, \) the set \( F_\epsilon = \{(\lambda, u) \in N: F(\epsilon, \lambda, u) = 0\} \) is the union of two curves \( \Gamma_0 = \{(\lambda, u): |\lambda - \lambda_0| \leq \delta_3 \} \) and \( \Gamma_\epsilon = \{\bar{\lambda}_\epsilon(t), \bar{u}_\epsilon(t)): |t| \leq \eta \} \) with \( \bar{\lambda}_\epsilon(0) = \lambda_0 \). Moreover,

(A) when \( \epsilon = \epsilon_0, \bar{\lambda}_\epsilon'(0) = 0, \bar{\lambda}_\epsilon''(0) > 0, \) and \( \bar{\lambda}_\epsilon(\pm \eta) = \lambda_0 + \delta_3; \)

(B) when \( \epsilon \in (\epsilon_0 - \rho, \epsilon_0), \bar{\lambda}_\epsilon'(0) < 0, \) there exists \( t_+ \in (0, \eta) \) such that \( \bar{\lambda}_\epsilon(t_+) < \lambda_0, \bar{\lambda}_\epsilon'(t_+) = 0, \bar{\lambda}_\epsilon''(t_+) > 0, \bar{\lambda}_\epsilon(\pm \eta) = \lambda_0 + \delta_3, \) and \( (\bar{\lambda}_\epsilon(t_+), \bar{u}_\epsilon(t_+)), (\lambda_0, u_0) \) are the only degenerate solutions on \( \Gamma_\epsilon; \)

(C) when \( \epsilon \in (\epsilon_0, \epsilon_0 + \rho), \bar{\lambda}_\epsilon'(0) > 0, \) there exists \( t_- \in (-\eta, 0) \) such that \( \bar{\lambda}_\epsilon(t_-) < \lambda_0, \bar{\lambda}_\epsilon'(t_-) = 0, \bar{\lambda}_\epsilon''(t_-) > 0, \bar{\lambda}_\epsilon(\pm \eta) = \lambda_0 + \delta_3, \) and \( (\bar{\lambda}_\epsilon(t_-), \bar{u}_\epsilon(t_-)), (\lambda_0, u_0) \) are the only degenerate solutions on \( \Gamma_\epsilon. \)
Proof. Apparently from (4.3), \( \Gamma_0 = \{ (\lambda, u_0) : |\lambda - \lambda_0| \leq \delta_3 \} \) is always a solution curve for any small \( \delta_3 > 0 \) and \( \varepsilon \) near \( \varepsilon_0 \). When \( \varepsilon = \varepsilon_0 \), the conditions (F1) and (F3) imply that a bifurcation of non-trivial solutions occurs at \( (\lambda_0, u_0) \) from Theorem 1.1, and a curve of non-trivial solutions \( \Gamma_{\varepsilon_0} = \{ (\tilde{x}_{\varepsilon_0}(t), \tilde{u}_{\varepsilon_0}(t)) : |t| \leq \eta \} \) exists. Moreover (2.1), (F4') and (F6) imply that \( \tilde{\lambda}'_{\varepsilon_0}(0) = 0 \) and from (2.2), (4.38) and (4.40),

\[
\tilde{\lambda}''_{\varepsilon_0}(0) = - \frac{\langle l, F_{u\varepsilon u}[w_0]^3 + 3F_{u\varepsilon u}[w_0, v_2] \rangle}{3\langle l, F_{\lambda u}[w_0] \rangle} > 0. \tag{4.41}
\]

Thus when \( \varepsilon = \varepsilon_0 \), the bifurcation near \( (\lambda_0, u_0) \) is a supercritical pitchfork one, and there is only one degenerate solution on \( \Gamma_{\varepsilon_0} \) near \( (\lambda_0, u_0) \). Define \( N = \{(\lambda, u) \in \mathbb{R} \times X : |\lambda - \lambda_0| \leq \delta_3, \|u - u_0\| \leq \delta_4 \} \), such that \( F_{\varepsilon_0} \) is the union of \( \Gamma_0 \) and \( \Gamma_{\varepsilon_0} \), and \( \tilde{\lambda}_{\varepsilon_0}(\pm \eta) = \lambda_0 + \delta_3 \), for \( s \in [-\eta, \eta] \), \( \|\tilde{u}_{\varepsilon_0}(t)\| \leq \delta_4/2 \).

From the conditions are satisfied, a bifurcation of the degenerate solutions of \( F = 0 \) occurs near \( (\varepsilon, \lambda, u) = (\varepsilon_0, \lambda_0, u_0) \) as described in Theorem 4.1 and Corollary 4.2. Hence there are two curves of degenerate solutions: the trivial one \( \Gamma_0 \), and \( \tilde{\lambda}_{\varepsilon_0}(\pm \eta) = \lambda_0 + \delta_3 \), for \( s \in [-\eta, \eta] \). Furthermore from (4.30), (4.39) and (4.40), we have

\[
\varepsilon_0'(0) = \alpha = - \frac{2\langle l, F_{u\varepsilon u}(\varepsilon_0, \lambda_0, u_0)[w_0]^3 + 3F_{u\varepsilon u}(\varepsilon_0, \lambda_0, u_0)[w_0, v_2] \rangle}{3\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] \rangle} < 0,
\]

thus there are exactly two degenerate solutions for any \( \varepsilon \in (\varepsilon_0 - \rho, \varepsilon_0 + \rho) \setminus \{\varepsilon_0\} \). Similarly \( \lambda_2'(0) = 0 \), and from (4.35), (4.38) and (4.40), we have

\[
\lambda_2''(0) = \frac{\langle l, F_{u\varepsilon u}(\varepsilon_0, \lambda_0, u_0)[w_0]^3 + 3F_{u\varepsilon u}(\varepsilon_0, \lambda_0, u_0)[w_0, v_2] \rangle}{3\langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] \rangle} < 0,
\]

hence \( \lambda_2(s) < \lambda_0 \) for any \( s \neq 0 \).

We consider the case of \( \varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho) \), and the case of \( \varepsilon \in (\varepsilon_0 - \rho, \varepsilon_0) \) is similar. For such \( \varepsilon \), \( (\lambda_0, u_0) \) is a degenerate solution of \( F \) which satisfies (F1). Indeed, by using [22, p. 235, Theorem 5.17], we have \( \dim N(F_u(\varepsilon, \lambda_0, u_0)) = 1 \) from \( \dim N(F_u(\epsilon_0, \lambda_0, u_0)) \leq \dim N(F_u(\varepsilon_0, \lambda_0, u_0)) = 1 \) and \( N(F_u(\varepsilon, \lambda_0, u_0)) \neq \emptyset \). Furthermore, since Fredholm index is a constant under a small perturbation (again from [22, p. 235, Theorem 5.17]) and \( F \) satisfies (F1) at \( (\varepsilon_0, \lambda_0, u_0) \), we have \( \dim N(F_u(\varepsilon_0, \lambda_0, u_0)) = \text{codim} R(F_u(\varepsilon, \lambda_0, u_0)) = 1 \). Let \( I(\varepsilon) \in Y^* \) be the function such that \( N(I(\varepsilon)) = R(F_u(\varepsilon, \lambda_0, u_0)) \) and \( I(\varepsilon_0) = 1 \). Since \( \langle l, F_{\lambda u}(\varepsilon_0, \lambda_0, u_0)[w_0] \rangle > 0 \), we have \( \langle l(\varepsilon), F_{\lambda u}(\varepsilon, \lambda_0, u_0)[w_0] \rangle > 0 \) for \( \varepsilon \in (\varepsilon_0, \varepsilon_0 + \rho) \) and (F3) is also satisfied at \( (\varepsilon, \lambda_0, u_0) \). Hence all conditions in Theorem 1.1 are satisfied, and a curve of non-trivial solutions \( \Gamma_\varepsilon = \{(\tilde{x}_\varepsilon(t), \tilde{u}_\varepsilon(t)) : |t| \leq \eta \} \) exists such that \( (\tilde{x}_\varepsilon(0), \tilde{u}_\varepsilon(0)) = (\lambda_0, u_0) \), and \( \tilde{\lambda}_\varepsilon'(0) \) is determined by (2.1). Define \( B(\varepsilon) = \langle l(\varepsilon), F_{u\varepsilon u}(\varepsilon, \lambda_0, u_0)[w_0]^2 \rangle \), we have

\[
B'(\varepsilon_0) = \langle l'(\varepsilon_0), F_{u\varepsilon u}[w_0]^2 \rangle + \langle l, F_{uuu}[w_0]^2 \rangle < 0. \tag{4.42}
\]
In (4.42), we obtain \( \langle l' (\varepsilon_0), F_u [v_2] \rangle = - \langle l, F_{\varepsilon u} [v_2] \rangle = 0 \) by differentiating the equation \( \langle l (\varepsilon), F_u (\varepsilon , \lambda_0 , u_0) [v_2] \rangle = 0 \). Thus \( B (\varepsilon) < 0 \) for \( \varepsilon \in (\varepsilon_0 , \varepsilon_0 + \rho) \), and from (2.1), we have

\[
\ddot{\lambda}_e (0) = - \frac{\langle l (\varepsilon), F_{uu} (\varepsilon , \lambda_0 , u_0) [w_0] \rangle^2}{2 \langle l (\varepsilon), F_{\varepsilon u} (\varepsilon , \lambda_0 , u_0) [w_0] \rangle} > 0.
\]

On the other hand, since \( \ddot{\lambda}_e (0) (\pm \eta) = \lambda_0 + \delta_3 \), and for \( s \in (- \eta , \eta) \), \( \parallel \nu (t) \parallel \leq \delta_4 / 2 \). Then by choosing \( \varepsilon \) close enough to \( \varepsilon_0 \), we may assume that \( \ddot{\lambda}_e (\pm \eta) = \lambda_0 + \delta_3 \), and for \( s \in (- \eta , \eta) \), \( \parallel \nu (t) \parallel \leq \delta_4 \). Hence there exists \( t_- \in (- \eta , 0) \) such that \( (\lambda_-, \nu-) = (\ddot{\lambda}_e (t_-), \ddot{\nu} (t_-)) \) is the only degenerate solution on the curve \( \Gamma_e \) other than \( (\lambda_0 , u_0) \).

We verify that \( (\ddot{\lambda}_e (t_-), \ddot{\nu} (t_-)) \) is a degenerate solution which satisfies the condition of Saddle-node Bifurcation Theorem (see [7, Theorem 3.2]). Again the condition (F1) is satisfied at \( (\ddot{\lambda}_e (t_-), \ddot{\nu} (t_-)) \) since \( (\ddot{\lambda}_e (t_-), \ddot{\nu} (t_-)) \) is a perturbation of \( (\lambda_0 , u_0) \). The only other condition is (F2): \( F_{\dot{\lambda}} (\ddot{\lambda}_e (t_-), \ddot{\nu} (t_-)) \neq R (F_{\ddot{\lambda}} (\ddot{\lambda}_e (t_-), \ddot{\nu} (t_-))) \). To prove that, recall that \( \Sigma_2 = \{ T_s = (\varepsilon_2 (s), \lambda_2 (s), u_2 (s), w_2 (s)) : |s| < \delta_2 \} \) is the curve of non-trivial degenerate solutions of \( F = 0 \). Since we assume \( \varepsilon \in (\varepsilon_0 , \varepsilon_0 + \rho) \), then \( (\lambda_-, \nu-) = (\lambda_2 (s), \nu_2 (s)) \) for some \( s \in (- \delta_2 , 0) \). Define \( A (s) = \langle l (s), F_{\varepsilon_2 (s), \lambda_2 (s)} (s), \nu_2 (s) \rangle \), where \( l (s) \in Y^* \) satisfying \( N (l (s)) = R (F_u (\varepsilon_2 (s), \lambda_2 (s), \nu_2 (s))) \). Then \( A (0) = \langle l, F_{\varepsilon_0 u} (\varepsilon_0 , \lambda_0 , u_0) [w_0] \rangle > 0 \) and \( A (0) = 0 \), so \( A (0) < 0 \) for \( s < 0 \) which implies (F2) holds at \( (\lambda_-, \nu-) \). We also define \( B (s) = \langle l (s), F_{uu} (\varepsilon_2 (s), \lambda_2 (s), u_2 (s)) [w_2 (s)]^2 \rangle \), we have

\[
\dot{B} (0) = \langle l' (0), F_{uu} [w_0]^2 \rangle + \alpha \langle l, F_{\varepsilon uu} [w_0]^2 \rangle + \langle l, F_{uuu} [w_0]^3 \rangle + 2 \langle F_{uu} [v_2], w_0 \rangle \\
= \langle l' (0), F_{uu} [w_0]^2 \rangle - \frac{2}{3} \langle l, F_{uuu} [w_0]^3 \rangle + 3 \langle F_{uu} [w_0, v_2] \rangle + \langle l, F_{uuu} [w_0]^3 \rangle + 2 \langle F_{uu} [v_2], w_0 \rangle \\
= \frac{1}{3} \langle l, F_{uuu} [w_0]^3 \rangle + 3 \langle F_{uu} [w_0, v_2] \rangle < 0,
\]

since \( \varepsilon' (0) = \alpha \), \( \lambda'_2 (0) = 0 \), \( u'_2 (0) = w_0 \), \( w'_2 (0) = v_2 \) and (4.30). In (4.43), we obtain \( \langle l (s), F_{uu} [w_0]^2 \rangle = \langle l, F_{uu} [v_2, w_0] \rangle \) by differentiating \( \langle l (s), F_u (\varepsilon_2 (s), \lambda_2 (s), u_2 (s)) [w_0] \rangle = 0 \) twice and using (4.35). In particular \( B (s_-) > 0 \) from \( B (0) = 0 \). Now from [7, Theorem 3.2], near \( (\lambda_-, u_-) \), the solutions of \( F (\varepsilon , \cdot) = 0 \) form a curve, which is indeed identical to \( \Gamma_e \) defined earlier. Hence \( \ddot{\lambda} (\cdot) = 0 \), \( \dddot{\lambda} (\cdot) = 0 \) and \( \dddot{\lambda} (\pm \eta) = \lambda_0 + \delta_1 \). Here

\[
\dddot{\lambda} (t_-) = - \frac{\langle l (s_-), F_{uu} (\varepsilon, \lambda_-, u_-) [w_-] \rangle}{\langle l (s_-), F_{\varepsilon u} (\varepsilon, \lambda_-, u_-) \rangle} = - \frac{B (s_-)}{A (s_-)} > 0. \tag{4.44}
\]

The results in this section provide an abstract framework for the bifurcations shown in Fig. 5. That is, the bifurcation diagram changes from a “forward” transcritical bifurcation when \( \varepsilon > \varepsilon_0 \), to a pitchfork bifurcation when \( \varepsilon = \varepsilon_0 \), and to a “backward” transcritical bifurcation when \( \varepsilon < \varepsilon_0 \). Here the forward and backward refer to the portion of the solution curve with \( t > 0 \), which often represents the positive solutions.

5. Examples

Reaction–diffusion models have been used to describe various spatiotemporal phenomena in spatial ecology, population genetics. In the ecological models in form
\[ \Delta u + \lambda u f(x, u) = 0, \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial \Omega, \quad (5.1) \]

it was usually assumed that the growth rate per capita \( f(x, u) \) is decreasing in \( u \) due to the crowding effect, and this represents the typical logistic growth. However it has been increasingly recognized that for many species, the function \( f(x, u) \) is increasing for small \( u \) because the population is too small to support a growth, and is decreasing for large \( u \) again due to the competition for the limited resource. The latter growth pattern is termed as Allee effect in ecological studies, see for examples, Cantrell and Cosner [3], Shi and Shivaji [40], Stephens and Sutherland [42]. If \( f(x, 0) < 0 \), then the growth pattern \( f(x, u) \) is of a strong Allee effect; and if \( f(x, 0) \geq 0 \), then it is of a weak Allee effect [40].

**Example 5.1.** In this example, we demonstrate the transition of the bifurcation diagrams of (5.1) when \( f \) is changed from logistic type to weak Allee effect type. To be more specific, we consider the following semilinear elliptic equation

\[ \{ \begin{array}{ll}
\Delta u + \lambda [a(x)u - b(x)u^3 - \varepsilon c(x)u^2] &= 0, & x \in \Omega, \\
u &= 0, & x \in \partial \Omega,
\end{array} \quad (5.2) \]

where \( \lambda \) is a positive parameter, \( \varepsilon \) is a real parameter, \( \Omega \) is a bounded region with smooth boundary in \( \mathbb{R}^n \) for \( n \geq 1 \), \( a(\cdot), b(\cdot), c(\cdot) \in C^\alpha(\overline{\Omega}) \), and there exists an open subset \( \Omega_0 \subseteq \Omega \) such that \( a(x) > 0 \) for any \( x \in \Omega_0 \). It is well-known that the eigenvalue problem

\[ \{ \begin{array}{ll}
\Delta \phi + \lambda a(x)\phi &= 0 & \text{in } \Omega, \\
\phi &= 0 & \text{on } \partial \Omega,
\end{array} \quad (5.3) \]

has a principal eigenvalue \( \lambda_1 > 0 \) such that the corresponding eigenfunction \( \phi_1(x) > 0 \) in \( \Omega \) (see [2]). Moreover \( \lambda_1 \) can be expressed as

\[
\lambda_1 = \inf \left\{ \int_{\Omega} |\nabla \phi(x)|^2 \, dx : \int_{\Omega} a(x)\phi^2(x) \, dx = 1, \, \phi \in H^1_0(\Omega) \right\}.
\]

This apparently implies that \( \int_{\Omega} a(x)\phi_1^2(x) \, dx > 0 \), and here we also assume that

\[
\int_{\Omega} b(x)\phi_1^4(x) \, dx > 0, \quad \text{and} \quad \int_{\Omega} c(x)\phi_1^3(x) \, dx > 0. \quad (5.4)
\]

When \( \varepsilon = 0 \), it is well-known that (5.2) has a unique positive solution and a unique negative solution when \( \lambda > \lambda_1 \), and a pitchfork bifurcation occurs at \( (\lambda_1, 0) \) (see [40]). Here we demonstrate our new abstract theory by analyzing the bifurcation problem with \( \varepsilon \) small.

Define a nonlinear mapping \( F : \mathbb{R} \times \mathbb{R} \times C^2_0(\overline{\Omega}) \to C^\alpha(\overline{\Omega}) \), by

\[
F(\varepsilon, \lambda, u) = \Delta u + \lambda [a(x)u - b(x)u^3 - \varepsilon c(x)u^2]. \quad (5.5)
\]

Then \( u = 0 \) is a trivial solution for any \( \lambda \) and \( \varepsilon \), and \( \lambda = \lambda_1 \) is a bifurcation point for positive and negative solutions of (5.2) for any fixed \( \varepsilon \). It is easy to verify that \( N(F_u(0, \lambda_1, 0)) = \text{span}\{\phi_1\} \), \( R(F_u(0, \lambda_1, 0)) = \{v \in C^\alpha(\overline{\Omega}) : \int_{\Omega} v\phi_1 \, dx = 0\} \); \( F_{\lambda u}(0, \lambda_1, 0)[\phi_1] = a(x)\phi_1 \not\in R(F_u(0, \lambda_1, 0)) \).
that all the degenerate solutions of (5.2) near the corresponding eigenfunction $\lambda_{2294}(\varepsilon, \lambda, u)$.

Theorem 5.1. A global bifurcation result of (5.2).

Indeed by combining with a well-known uniqueness result, we can obtain the following: the bifurcation of positive solutions is a forward one when $\varepsilon > 0$. Indeed by combining with a well-known uniqueness result, we can obtain the following global bifurcation result of (5.2).

Restricting to the positive solutions of (5.2), then near the bifurcation point $(\lambda, u) = (\lambda_1, 0)$, the bifurcation of positive solutions is a forward one when $\varepsilon > 0$, and it is a backward one when $\varepsilon < 0$. Indeed by combining with a well-known uniqueness result, we can obtain the following global bifurcation result of (5.2).

Theorem 5.1. Suppose that $a(\cdot), b(\cdot), c(\cdot) \in C^\omega(\overline{\Omega})$, and there exists an open subset $\Omega_0 \subseteq \Omega$ such that $a(x) > 0$ for any $x \in \Omega_0$. Let $\lambda_1$ be the principal eigenvalue of (5.3), and the corresponding eigenfunction $\phi_1$ satisfies (5.4).

1. There exists $\delta > 0$ such that for $\varepsilon$ satisfying $\varepsilon \in (-\delta, \delta)$, all non-trivial solutions of (5.2) near $(\lambda_1, 0)$ lie on a smooth curve $\Gamma_\varepsilon = \{(\lambda_\varepsilon(t), u_\varepsilon(t, \cdot)): |t| < \varepsilon\}$, such that $\lambda_\varepsilon(0) = \lambda_1$, $u_\varepsilon(0, \cdot) = 0$, $u_\varepsilon(t, x) > 0$ when $t \in (0, \varepsilon)$ and $x \in \Omega$, and $u_\varepsilon(t, x) < 0$ when $t \in (-\varepsilon, 0)$ and $x \in \Omega$. Moreover if $\varepsilon \in (0, \delta)$, then there exists $t_- \in (0, \varepsilon)$ such that $\lambda_\varepsilon'(t_-) = 0$ and $\lambda_\varepsilon''(t_-) > 0$, and $\lambda_\varepsilon'(t - t_-) > 0$ for $t \in (-\varepsilon, \varepsilon) \setminus \{t_-\}$; if $\varepsilon = 0$, then $\lambda_\varepsilon'(0) = 0$, $\lambda_\varepsilon''(0) > 0$, and $\lambda_\varepsilon'(t) > 0$ for $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$; and if $\varepsilon \in (-\delta, 0)$, then there exists $t_+ \in (-\varepsilon, 0)$ such that $\lambda_\varepsilon'(t_+) = 0$ and $\lambda_\varepsilon''(t_+) > 0$, and $\lambda_\varepsilon'(t - t_+) > 0$ for $t \in (-\varepsilon, 0) \setminus \{t_+\}$ (see Fig. 5 for an illustration).

2. If in addition, we assume that $a(x) \equiv a > 0$, $b(x) \equiv b > 0$ and $c(x) \equiv c > 0$, then all the positive and negative solutions of (5.2) lie on a smooth curve $\Gamma_\varepsilon = \{(\lambda_\varepsilon(t), u_\varepsilon(t, \cdot)): t \in \mathbb{R}\}$, such that $\lambda_\varepsilon(0) = \lambda_1$, $u_\varepsilon(0, \cdot) = 0$, $u_\varepsilon(t, x) > 0$ when $t > 0$ and $x \in \Omega$, and $u_\varepsilon(t, x) < 0$ when $t < 0$ and $x \in \Omega$. Moreover if $\varepsilon \in (0, \delta)$, then there exists $t_- < 0$ such that $\lambda_\varepsilon'(t_-) = 0$ and $\lambda_\varepsilon''(t_-) > 0$, and $\lambda_\varepsilon'(t - t_-) > 0$ for $t \in \mathbb{R} \setminus \{t_-\}$; if $\varepsilon = 0$, then $\lambda_\varepsilon'(0) = 0$, $\lambda_\varepsilon''(0) > 0$, and $\lambda_\varepsilon'(t) > 0$ for $t \in \mathbb{R} \setminus \{0\}$; and if $\varepsilon \in (-\delta, 0)$, then there exists $t_+ > 0$ such that $\lambda_\varepsilon'(t_+) = 0$ and $\lambda_\varepsilon''(t_+) > 0$, and $\lambda_\varepsilon'(t - t_+) > 0$ for $t \in (-\varepsilon, 0) \setminus \{t_+\}$. 

$$\langle l, F_{uu}(0, \lambda_1, 0)[\phi_1] \rangle = \int_{\Omega} a(x)\phi_1^2(x) dx > 0,$$

$$\langle l, F_{uu}(0, \lambda_1, 0)[\phi_1]^3 \rangle = -6\lambda_1 \int_{\Omega} b(x)\phi_1^4(x) dx < 0,$$

$$\langle l, F_{uu}(0, \lambda_1, 0)[\phi_1]^2 \rangle = -2\lambda_1 \int_{\Omega} c(x)\phi_1^3(x) dx < 0,$$

and the bifurcation diagrams near $\varepsilon = 0$ are exactly as in the ones in Fig. 5.
Proof. The local bifurcation diagrams follow from the calculation above and Theorem 4.3. For the global bifurcation diagrams, we assume that \( a(x) \equiv a > 0 \), \( b(x) \equiv b > 0 \) and \( c(x) \equiv c > 0 \). We first prove the case of \( \varepsilon = 0 \). From [39, Theorem 2.3] (see also [31]), (5.2) has no positive solution if \( \lambda \leq \lambda_1 \), and has exactly one positive solution \( v_\lambda \) if \( \lambda > \lambda_1 \). Moreover, all \( v_\lambda \)'s lie on a smooth curve, \( v_\lambda \) is stable and \( v_\lambda \) is increasing with respect to \( \lambda \). Hence by letting \( \lambda_0(t) = \lambda_1 + (\lambda - \lambda_1) t^2 \), \( u_0(t, \cdot) = v_{\lambda_0(t)}(\cdot) \) for \( t > 0 \), and \( u_0(t, \cdot) = -v_{\lambda_0(-t)}(\cdot) \) for \( t < 0 \), we prove the global bifurcation diagram for \( \varepsilon = 0 \) as the positive (hence also negative) solution of (5.2) is unique for \( \lambda > \lambda_1 \).

For the case of \( \varepsilon \in (-\delta, 0) \), it is easy to see that the negative solution branch satisfies the conditions in [39, Theorem 2.3], hence the negative solution is unique for any \( \lambda > \lambda_1 \). For the positive solutions, the local bifurcation near \((\lambda_1, 0)\) and the saddle-node bifurcation point \((\lambda_\varepsilon(t_+), u_\varepsilon(t_+))\) have been shown above. From the global bifurcation theorem (see [33,41]), the local curve of positive solutions from \((\lambda_1, 0)\) is unbounded, and indeed it can be extended to \( \lambda = \infty \) as all positive solutions are bounded by \( \sqrt{a/b} \). Thus there exists at least a positive solution for any \( \lambda > \lambda_\varepsilon(t_+) \) and they are on the global branch emanating from \((\lambda_1, 0)\). To prove the uniqueness of positive solution for \( \lambda > \lambda_1 + \delta_0 \), we first use [1, Theorem A] to conclude the uniqueness when \( \lambda > M \) for some large \( M \) > 0. For \( \lambda \in [\lambda_1 + \delta_0, M] \), the unique positive solution \( v_\lambda \) of (5.2) with \( \varepsilon = 0 \) is stable, hence a small perturbation for \( \varepsilon \) near 0 is still stable, uniformly for \( \lambda \in [\lambda_1 + \delta_0, M] \). Hence the positive solution for \( \lambda \in [\lambda_1 + \delta_0, M] \) is also unique. The case of \( \varepsilon \in (0, \delta) \) is similar and we omit the details. \( \square \)

We extract the global bifurcation for the positive solutions in the following corollary:

**Corollary 5.2.** Let \( a, b, c > 0 \), and let \( \lambda_1(a) \) be the principal eigenvalue of (5.3) with \( a(x) \equiv a \). Then for the boundary value problem

\[
\begin{cases}
\Delta u + \lambda (au - bu^3 - \varepsilon cu^2) = 0, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
\]

there exists \( \delta > 0 \) such that

1. When \( \varepsilon \in [0, \delta) \), (5.6) has no positive solution when \( \lambda \leq \lambda_1(a) \), it has a unique positive solution \( \tilde{u}_\lambda \) when \( \lambda > \lambda_1(a) \), all \( \tilde{u}_\lambda \) are on a smooth curve \( \Gamma_\varepsilon = \{(\lambda, \tilde{u}_\lambda): \lambda > \lambda_1(a)\} \), and \( \tilde{u}_\lambda \) is stable and it is increasing in \( \lambda \);

2. When \( \varepsilon \in (-\delta, 0) \), there exists \( \lambda_* \in (0, \lambda_1(a)) \) such that (5.6) has no positive solution when \( \lambda \leq \lambda_* \), it has a unique positive solution \( \tilde{u}_\lambda \) when \( \lambda \geq \lambda_1(a) \) and \( \lambda = \lambda_* \), and it exactly two positive solutions \( \tilde{u}_\lambda \) and \( \tilde{u}_\lambda \) such that \( \tilde{u}_\lambda > \tilde{u}_\lambda \) when \( \lambda_* < \lambda < \lambda_1(a) \); all \( \tilde{u}_\lambda \) are on a smooth curve \( \Gamma_\varepsilon = \{(\lambda, \tilde{u}_\lambda): \lambda \geq \lambda_*\} \), and \( \tilde{u}_\lambda \) is stable and it is increasing in \( \lambda \); all \( \tilde{u}_\lambda \) are on a smooth curve \( \Gamma_\varepsilon = \{(\lambda, \tilde{u}_\lambda): \lambda < \lambda < \lambda_1(a)\} \), \( \tilde{u}_\lambda \) is unstable, \( \lim_{\lambda \to \lambda_*^+} \tilde{u}_\lambda = \tilde{u}_\lambda \), and \( \lim_{\lambda \to \lambda_*^-} \tilde{u}_\lambda = \tilde{u}_\lambda \).

From (4.37), we can have an estimate of \( \lambda_* \):

\[
\lambda_* \approx \lambda_1(a) + \Delta \varepsilon^2,
\]
where

\[
\Lambda = - \frac{\lambda_1(a) \int_{\Omega} c(x) \phi_1^3(x) \, dx}{4 \int_{\Omega} a(x) \phi_1^2(x) \, dx \cdot \int_{\Omega} b(x) \phi_1^4(x) \, dx}.
\] (5.8)

**Example 5.2.** Here we revisit an equation from the study of gene frequency under the combined influence of selection and migration (in equilibrium form):

\[
\begin{cases}
\Delta u + \mu s(x)u(1-u)[hu + (1-h)(1-u)] = 0, & x \in \Omega, \\
\frac{\partial u}{\partial n} = 0, & x \in \partial \Omega,
\end{cases}
\] (5.9)

where \( s \in L^\infty(\Omega) \), \( 0 \leq h \leq 1 \), and \( 0 \leq u(x) \leq 1 \). This model was introduced by Fisher [12], and it was considered by Fleming [13], Henry [19, Chapter 10], see also more recent work in [25–28]. Here we only consider the case that \( \int_{\Omega} s(x) \, dx < 0 \) and \( s(x) > 0 \) on a subset \( \Omega_0 \) with positive measure. Clearly \( u = 0 \) and \( u = 1 \) are constant solutions to (5.9), and from [19, Lemma 10.1.2, Lemma 10.1.3], [36], \( \mu_0(h) = \inf \left\{ \int_{\Omega} |\nabla \phi|^2 \, dx : (1-h) \int_{\Omega} s(x) \phi^2 \, dx = 1 \right\} \) is a bifurcation point from \( u = 0 \). To cast Eq. (5.9) into the framework in Section 4, we define

\[
\lambda = \mu(1-h), \quad \varepsilon = 3 - \frac{1}{1-h}.
\] (5.10)

Then (5.9) becomes

\[
\begin{cases}
\Delta u + \lambda s(x)[u - \varepsilon u^2 + (-1 + \varepsilon)u^3] = 0, & x \in \Omega, \\
\frac{\partial u}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\] (5.11)

We consider the bifurcation near \( \varepsilon = \varepsilon_0 = 0 \).

Define a nonlinear mapping \( F : \mathbb{R} \times \mathbb{R} \times C^{2,\alpha}_N(\overline{\Omega}) \to C^\alpha(\overline{\Omega}) \), by

\[
F(\varepsilon, \lambda, u) = \Delta u + \lambda s(x)[u - \varepsilon u^2 + (-1 + \varepsilon)u^3],
\] (5.12)

where \( C^{2,\alpha}_N(\overline{\Omega}) = \{ u \in C^{2,\alpha}(\overline{\Omega}) : \frac{\partial u}{\partial n} = 0 \} \). Let \( \lambda_0 \) be the positive principal eigenvalue of

\[
\begin{cases}
\Delta \phi + \lambda s(x) \phi = 0, & x \in \Omega, \\
\frac{\partial \phi}{\partial n} = 0, & x \in \partial \Omega,
\end{cases}
\] (5.13)

and \( \phi_1 \) is the corresponding positive eigenfunction. Then \( \lambda = \lambda_0 = \frac{1}{3} \mu_0 \) is a bifurcation point for positive and negative solutions of (5.11) for any fixed \( \varepsilon \). One can verify that \( N(F_u(0, \lambda_0, 0)) = \operatorname{span}\{ \phi_1 \} \), \( R(F_u(0, \lambda_0, 0)) = \{ u \in C^\alpha(\overline{\Omega}) : \int_{\Omega} v \phi_1 \, dx = 0 \} \); and \( F_{\lambda,u}(0, \lambda_0, 0)[\phi_1] = s(x)\phi_1 \notin R(F_u(0, \lambda_0, 0)) \), \( F_{uu}(0, \lambda_0, 0)[\phi_1]^2 = 0 \in R(F_u(0, \lambda_0, 0)) \), thus (F1), (F3) and (F4') are satisfied and \( v_2 = 0 \) at \( (\varepsilon, \lambda, u) = (0, \lambda_0, 0) \). The condition (4.3) is also satisfied as \( F_u(\varepsilon, \lambda_0, 0) = F_u(0, \lambda_0, 0) \) hence \( R(F_u(\varepsilon, \lambda_0, 0)) = R(F_u(0, \lambda_0, 0)) \). Furthermore we can compute that
\[ \langle l, F_{\lambda u}(0, \lambda_0, 0)[\phi_1] \rangle = \int_{\Omega} s(x) \phi_1^2(x) \, dx = \frac{1}{\lambda_0} \int_{\Omega} |\nabla \phi_1|^2 \, dx > 0, \]

\[ \langle l, F_{uuu}(0, \lambda_0, 0)[\phi_1]^3 \rangle = -6\lambda_0 \int_{\Omega} s(x) \phi_1^4(x) \, dx = -18 \int_{\Omega} |\nabla \phi_1|^2 \phi_1^2 \, dx < 0, \]

\[ \langle l, F_{e uu}(0, \lambda_0, 0)[\phi_1]^2 \rangle = -2\lambda_0 \int_{\Omega} s(x) \phi_1^3(x) \, dx = -4 \int_{\Omega} |\nabla \phi_1|^2 \phi_1 \, dx < 0, \]

hence (4.38), (4.39) and (4.40) are also satisfied. Then all conditions in Theorem 4.3 are satisfied, and similar to Example 5.1, the bifurcation diagrams in \((\lambda, u)\)-space near \(\varepsilon = 0\) are exactly as in the ones in Fig. 5. In particular, the bifurcation is supercritical when \(\varepsilon \in (0, \delta)\) (or \(h \in (2/3 - \delta_1, 2/3)\)), and it is subcritical when \(\varepsilon \in (-\delta, 0)\) (or \(h \in (2/3, 2/3 + \delta_1)\)).

A similar global bifurcation picture as the ones in Example 5.1 can be obtained for (5.9). As proved in [19], the positive solution is unique for any \(\mu > \mu_0\) for \(h \in [1/3, 2/3]\). For \(h \in (2/3, 2/3 + \delta_1)\), there exists a \(\mu_* < \mu_0\) such that (5.9) has no positive solution when \(\mu < \mu_*\), has exactly one positive solution when \(\mu = \mu_0\) or \(\mu = \mu_*\), and has exactly two positive solutions when \(\mu_* < \mu < \mu_0\). This is based on the perturbation results proved above and the uniqueness of the positive solution for large \(\mu\). The latter uniqueness result can be proved by using the method outlined in [19, p. 319, Exercise 3] and topological degree argument as in [1].

From (4.37), we can have the following estimate of the saddle-node bifurcation point \(\mu_* = \mu_*(h)\) \((h \in (2/3, 2/3 + \delta_1))\):

\[
\mu_*(h) = \frac{\lambda_0}{1 - h} \left[ 1 - \frac{\left( \int_{\Omega} |\nabla \phi_1|^2 \phi_1 \, dx \right)^2}{3 \int_{\Omega} |\nabla \phi_1|^2 \phi_1^2 \, dx \cdot \int_{\Omega} |\nabla \phi_1|^2 \, dx} \left( \frac{2 - 3h}{1 - h} \right)^2 \right].
\]  

(5.14)

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References