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Notes

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ABSTRACT

For a given subset $S \subseteq V(G)$ of a graph G, the pair (G, S) is knitted if for every partition of S into non-empty subsets $S_1, S_2, \dots, S_t$, there are disjoint connected subgraphs $C_1, C_2, \dots, C_t$ in G so that $S_i \subseteq C_i$. A graph G is ℓ-knitted if (G, S) is knitted for all $S \subseteq V(G)$ with $|S| = \ell$. In this paper, we prove that every 9ℓ-connected graph is ℓ-knitted.

Hadwiger's Conjecture states that every k-chromatic graph contains a K_k-minor. We use the above result to prove that the connectivity of minimal counterexamples to Hadwiger's Conjecture is at least $k/9$, which was proved to be at least $2k/27$ in Kawarabayashi (2007) [4].

1. Introduction

One of the most interesting problems in graph theory is Hadwiger's Conjecture, which states that *every k-chromatic graph has a K_k-minor*, where a graph H is a minor of a graph G if H can be obtained from a subgraph of G by contracting edges.

It is known that Hadwiger's Conjecture holds for $k \leqslant 6$. Wagner [11] in 1937 proved that the case $k = 5$ is equivalent to Four Color Theorem. About 60 years later, Robertson, Seymour and Thomas [8] proved that the case $k = 6$ is also equivalent to the Four Color Theorem. In their proof, minimal

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counterexamples, which are also called "contraction-critical non-complete graphs", play an important role. Kawarabayashi and Toft [5] showed that 7-chromatic graphs contain a K_7-minor or a $K_{4,4}$-minor, in which the connectivity property of minimal counterexamples are, again, really important.

Many researchers have considered the connectivity property of contraction-critical graphs. Dirac [2] proved that every k-contraction-critical graph is 5-connected for $k \geqslant 5$, and Mader [7] extended 5-connectivity to the deep result that every k-contraction-critical graph is 7-connected for $k \geqslant 7$ and every 6-contraction-critical graph is 6-connected. Toft [10] proved that k-contraction-critical graphs are k-edge-connected. Kawarabayashi [4] proved the first general result on the vertex connectivity of minimal counterexamples to Hadwiger's Conjecture.

Theorem 1. *(See Kawarabayashi [4].) For all positive integers k, every minimal (with respect to the minor relation) k-chromatic counterexample to Hadwiger's Conjecture is* $\lceil \frac{2k}{27} \rceil$*-connected.*

In the proof of the above theorem, the main tool used was so-called k-linked graphs. A graph G is *k-linked* if for every $2k$ distinct vertices $u_1, v_1, u_2, v_2, \dots, u_k, v_k$ in G, there are k disjoint paths $P_1, P_2, \dots, P_k$ such that P_i connects u_i and v_i. k-linked graphs are very well-studied and play a very important role in the study of graph structures.

In this paper, we improve the result in Theorem 1, by studying a notion called "knitted graph" introduced by Bollobás and Thomason [1].

For $1 \leqslant m \leqslant k \leqslant |V(G)|$, a graph is *$(k, m)$-knit* if whenever S is a set of k vertices of G and $S_1, \dots, S_t$ is a partition of S into $t \geqslant m$ non-empty parts, G contains vertex-disjoint connected subgraphs $C_1, \dots, C_t$ such that $S_i \subseteq V(C_i)$, $1 \leqslant i \leqslant t$. Clearly, a $(2k, k)$-knit graph is k-linked. In [1], Bollobás and Thomason proved that *if a k-connected graph G contains a minor H, where H is a graph with minimum degree at least $0.5(|H| + \lfloor 5k/2 \rfloor - 2 - m)$, then G is (k, m)-knit.* They used this result to show that $22k$-connected graphs are k-linked, which is the first linear upper bound of connectivity for a graph to be k-linked.

We consider a slightly more general notion than (k, m)-knit. For a set $S \subseteq V(G)$ of a graph G, the pair (G, S) *is knitted* if for every partition of S into non-empty subsets $S_1, S_2, \dots, S_t$, there are disjoint connected subgraphs $C_1, C_2, \dots, C_t$ in G so that $S_i \subseteq C_i$. A graph G is ℓ-knitted if (G, S) is knitted for all $S \subseteq V(G)$ with $|S| = \ell$. It is clear that an ℓ-knitted graph is (ℓ, m)-knit for all $m \leqslant \ell$.

In this paper, we give a connectivity condition for a graph to be ℓ-knitted.

Definition 1. The pair (A, B) is a *separation* of G if $V(G) = A \cup B$ and there is no edge between $A - B$ and $B - A$. The *order* of a separation (A, B) is $|A \cap B|$. If $S \subseteq A$, then we say that (A, B) is a separation of (G, S).

We shall prove the following theorem.

Theorem 2. *Let k and ℓ be positive integers and $S \subseteq V(G)$ with $|S| = \ell < k/9$. If there is no separation of (G, S) of size less than ℓ, and every vertex in $G - S$ has degree at least $k - 1$, then (G, S) is knitted.*

The theorem we will prove, Theorem 7, on edge-density in Section 3 is actually stronger than Theorem 2.

We are now ready to state and prove our result on connectivity of minimal counterexamples to Hadwiger's Conjecture.

Theorem 3. *For all positive integer k, every k-chromatic minimal (with respect to the minor relation) counterexample to Hadwiger's Conjecture is* $\lceil \frac{k}{9} \rceil$*-connected.*

Proof. Assume by contradiction that the statement fails. Then we have a minimal k-chromatic graph G that has no K_k-minor and is not $k/9$-connected. Take a minimum cutset S. Then $|S| < k/9$. Let A_1 be a component of $G - S$ and $A_2 = G - S - A_1$. Then both $G[A_1 \cup S]$ and $G[A_2 \cup S]$ have the chromatic number at most $k - 1$.

Let S_1 be a maximum independent set in $G[S]$, and let S_i be a maximum independent set in $G[S - \bigcup_{j=1}^{i-1} S_j]$ for $i \geqslant 2$. Let $v_1, v_2, \ldots, v_{|S|}$ be the set of vertices in S such that $v_1, \ldots, v_{|S_1|} \in S_1$, $v_{|S_1|+1}, \ldots, v_{|S_1|+|S_2|} \in S_2$, and so on. Observe that if we contract each of the subgraph induced by S_i into one vertex, then the resulting graph in S is a clique.

Note that the minimum degree of G is at least $k-1$, thus each vertex in A_p has at least $k-1$ neighbors in $A_p \cup S$ for $p \in \{1, 2\}$. Note also that a separation in $(A_1 \cup S, S)$ or $(A_2 \cup S, S)$ is a separation in (G, S), thus $(A_1 \cup S, S)$ and $(A_2 \cup S, S)$ have no separation of size less than ℓ. By Theorem 2, both $(A_1 \cup S, S)$ and $(A_2 \cup S, S)$ are knitted. So there are disjoint connected subgraphs $C_i \subseteq A_1 \cup S$'s and $D_i \subseteq A_2 \cup S$ so that $S_i \subseteq C_i$ and $S_i \subseteq D_i$. Hence we can contract $A_1 \cup S$ into $S_1, S_2, \ldots$ such that the resulting graph on S is complete. Let G_1 be the resulting graph plus A_2. Similarly, we can also contract $A_2 \cup S$ into $S_1, S_2, \ldots$ such that the resulting graph on S is complete (let G_2 be the resulting graph plus A_1).

Then $\chi(G_1), \chi(G_2) \leqslant k-1$ by minimality of G. But clearly we can combine the colorings of G_1 and G_2 to the whole graph G using at most $k-1$ colors. This is a contradiction. This completes the proof of the theorem. □

The rest of the paper is to prove Theorem 2. We will do this in two steps: in the first step (Section 3), we will show a graphs under study either is knitted or has a dense subgraph; in the second step (Section 2), we find a knitted subgraph in the dense subgraph. Note that this approach is very much similar to the one used by Thomas and Wollan [9].

2. Dense graphs are knitted

In this section, we study when a small dense graph contains a knitted subgraph. This is needed in our proof of Theorem 2 in Section 3.

To show a small dense graph is k-knitted, we use a result by Faudree et al. [3] on k-ordered graphs, where a graph is k-ordered if for every k vertices of given order, there is a cycle containing the k vertices of the given order. It is clear that a k-ordered graph is k-knitted. Throughout the paper, we will use $d(x, H)$ to denote the number of neighbors (degree) of x in subgraph H of G.

Theorem 4. *(See Faudree et al. [3].) For every graph G with order $n \geqslant 2\ell \geqslant 2$, if $d(x, G) + d(y, G) \geqslant n + \frac{3\ell-9}{2}$ for every pair of non-adjacent vertices x and y, then G is ℓ-ordered.*

Note that for $n \geqslant 5\ell$, Kostochka and Yu [6] showed that a graph G with minimum degree at least $\frac{n+\ell}{2} - 1$ is ℓ-ordered. Since we do not know if the minimum degree condition still holds for $n < 5\ell$, we are unable to use this less demanding degree conditions in our proof.

Theorem 5. *Let $\alpha \geqslant 4.5$. A graph H with minimum degree $\delta(H) \geqslant \alpha\ell + 1$ and $|V(H)| \leqslant 2\alpha\ell$ contains an ℓ-knitted subgraph.*

Proof of Theorem 5. Assume by contradiction that H is not ℓ-knitted. Then there is a subset $S \subseteq V(H)$ with $|S| = \ell$, and a partition $S = \bigcup_{i=1}^{t} S_i$ such that we cannot find disjoint connected subgraphs containing S_i's.

We consider partial (ℓ, t)-knit $C = \bigcup_{i=1}^{t} C_i$, which is a subgraph of G in which $S_i \subseteq C_i$ but C_is are not necessarily connected.

An optimal (ℓ, t)-knit $C = \bigcup_{i=1}^{t} C_i$ is a partial (ℓ, t)-knit such that

(a) $|C| \leqslant \alpha\ell$;
(b) the number of components of C is minimized; and
(c) subject to (a) and (b), $|C|$ is minimized.

We observe that the components in C containing exactly one vertex in S consist of one vertex, and a component with two vertices in S is a path.

We may assume that $S_1 \subseteq C_1$, but C_1 is not connected. Then there exists $x, y \in S_1$ such that x and y belong to different components of C_1. Note that $H - C \neq \emptyset$, since $d(x, H - C) = d(x) - |C| \geqslant (\alpha\ell + 1) - \alpha\ell = 1$.

Now we show that for every $u \in H - C$ and for every component P in C with $|V(P) \cap S| \geqslant 2$, $d(u, P) \leqslant |V(P) \cap S| + 1$. We actually will give the following more general statement, which might be of independent interest.

Lemma 1. *Let W be a graph. Let S' be a subset of $V(W)$ with $|S'| \geqslant 2$, and let F be subtree of W such that $F \supseteq S'$ and all leaves of F belong to S'. Let $u \in W - F$, and suppose that $d(u, F) \geqslant |S'| + 2$. Then $W[V(F) \cup \{u\}]$ contains a subtree F_0 with $u \in F_0$ such that $|F_0| < |F|$, $F_0 \supseteq S'$ and all leaves of F_0 belong to S'.*

Proof. Let $k = |S'|$. When $k = 2$, F is a path with both leaves in S', then since $d(u, F) \geqslant 4$, we can replace a segment of F by u to get a smaller subtree F_0 so that the leaves of F_0 belong to S'. So let $k \geqslant 3$.

Now we use induction on $|F|$. Note that F has at least two leaves, and let $u_1, u_2 \in S$ be two of them. For $i = 1, 2$, let P_i be maximal paths such that $u_i \in P_i$ and the subtree $F - V(P_i)$ contains $S' - \{u_i\}$. Note that $P_1 \cap P_2 = \emptyset$. For each i, let x_i be the vertex in $F - P_i$ which is adjacent (in F) to an endpoint of P_i.

Let $i = 1$ or 2. First assume $d(u, P_i) = 0$. Then by the induction assumption, $W[V(F - P_i) \cup \{u\}]$ contains a subtree F' with $u \in F'$ such that $|F'| < |F - P_i|$, $F' \supseteq (S' - \{u'\}) \cup \{x_i\}$ and all leaves of F' belong to $(S' - \{u_i\}) \cup \{x_i\}$. Adding P_i to F', we obtain a desired tree. Next assume $d(u, P_i) = 1$. Then by the induction assumption, $W[V(F - P_i) \cup \{u\}]$ contains a subtree F' with $u \in F'$ such that $|F'| < |F - P_i|$, $F' \supseteq S' - \{u_i\}$ and all leaves of F' belong to $S' - \{u_i\}$. Adding P_i to F', we obtain a desired tree. Thus we may assume $d(u, P_i) \geqslant 2$ for each $i = 1, 2$. Let $P_i = u_i P_i v_i v_i' P_i x_i'$ so that x_i' is adjacent to x_i and v_i is the only neighbor of u on $u_i P_i v_i$. Then $|V(v_i' P_i x_i')| \geqslant 1$. Now $F_0 = (F - \bigcup_{i=1}^{2} V(v_i' P_i x_i')) \cup \{u\}$ is a subtree (note that $k \geqslant 3$, so F_0 is connected) with desired properties. □

Let δ^* be the minimum degree of $H - C$. We have the following

Lemma 2. $\delta^* \geqslant (\alpha - 1.5)\ell$.

Proof. For every $u \in H - C$,

$$d(u, H - C) = d(u, H) - d(u, C) \geqslant \delta(H) - d(u, C) \geqslant \alpha\ell + 1 - d(u, C).$$

So we just need to prove that $d(u, C) \leqslant 1.5\ell$ for every $u \in H - C$.

Let P_j, $1 \leqslant j \leqslant c_i$, be the components of C_i in which u has neighbors. If $|P_j \cap S| \geqslant 2$, then by Lemma 1 we have $d(u, P_j) \leqslant |P_j \cap S| + 1 \leqslant 3|P_j \cap S|/2$ and, if $|P_j \cap S| = 1$ then $|P_j| = 1$, and hence $d(u, P_j) = |P_j \cap S| \leqslant 3|P_j \cap S|/2$, which implies

$$d(u, C_i) = \sum_{j=1}^{c_i} d(u, P_j) \leqslant \sum_{j=1}^{c_i} 3|P_j \cap S|/2 \leqslant 3|C_i \cap S|/2.$$

Therefore $d(u, C) = \sum_{C_i} d(u, C_i) \leqslant 1.5|S| = 1.5\ell$, and the lemma is proven. □

Lemma 3. *The subgraph $H - C$ is connected.*

Proof. Let $H_1, \dots, H_p$ with $p \geqslant 1$ be the components of $H - C$. Then H_i is not ℓ-knitted, thus not ℓ-ordered. So by Theorem 4, $2\delta^* < |H_i| + \frac{3\ell - 9}{2}$. Therefore we have

$$|H_i| > (2\alpha - 4.5)\ell + 4.5.$$

If $p \geqslant 2$, then $|H| \geqslant |C| + |H_1| + |H_2| > \ell + 2(2\alpha - 4.5)\ell + 9$, that is, $2\alpha\ell > (4\alpha - 8)\ell + 9$. So $(8 - 2\alpha)\ell > 9$, a contradiction to $\alpha \geqslant 4$. □

Lemma 4. $|C| \leqslant \alpha\ell - 5$.

Proof. For otherwise, $|H-C| \leqslant 2\alpha\ell - |C| \leqslant 2\alpha\ell - (\alpha\ell - 4) = \alpha\ell + 4$. Then $2\delta^* - (|H-C| + \frac{3\ell-9}{2}) \geqslant (2\alpha - 3)\ell - (\alpha\ell + 4) - \frac{3\ell-9}{2} = (\alpha - 4.5)\ell + 0.5 > 0$. By Theorem 4, $H - C$ is ℓ-ordered, thus ℓ-knitted, a contradiction. □

Let $A = N(x) \cap (H - C)$ and $B = N(y) \cap (H - C)$. Furthermore, let $A' = N(A) \cap (H - C) - A$ and $B' = N(B) \cap (H - C) - B$. Let $D = (H - C) - (A \cup A' \cup B \cup B')$. Then there is no path of length at most 6 from x to y through $A \cup A' \cup D \cup B' \cup B$, for otherwise, we may get C' by adding this path to C. Note that C' has less components than C, and $|C'| \leqslant |C| + 5 \leqslant (\alpha\ell - 5) + 5 = \alpha\ell$, a contradiction to the assumption that C is optimal.

Take $u \in D - N(A')$, then u has no neighbors in $A' \cup A$. Take $v \in A$, then every pair of u, v, y has no common neighbors in $H - C$. Thus $|H| \geqslant d(y) + d(u, H - C) + d(v, H - C) \geqslant \delta(H) + 2\delta^* > \alpha\ell + (2\alpha - 3)\ell = (3\alpha - 3)\ell$, and it follows that $2\alpha\ell > (3\alpha - 3)\ell$, or $\alpha < 3$, a contradiction.

3. Proof of Theorem 2

We first introduce some notations.

Definition 2. A separation (A, B) of (G, S) is rigid if $(G[B], A \cap B)$ is knitted.

For a set $H \subseteq V(G)$, let $\rho(H)$ be the number of edges with at least one endpoint in H.

Definition 3. Let G be a graph and $S \subseteq V(G)$, and $\alpha > 1$ be a real number. The pair (G, S) is $\alpha\ell$-*massed* if

(i) $\rho(V(G) - S) > \alpha\ell|V(G) - S| - 1$, and
(ii) every separation (A, B) of (G, S) of order at most $|S| - 1$ satisfies $\rho(B - A) \leqslant \alpha\ell|B - A|$.

Definition 4. Let G be a graph and $S \subseteq V(G)$, and let $\alpha > 1$ be a real number. The pair (G, S) is (α, ℓ)-minimal if

1. (G, S) is $\alpha\ell$-massed,
2. $|S| \leqslant \ell$ and (G, S) is not knitted,
3. subject to above two, $|V(G)|$ is minimum,
4. subject to above three, $\rho(G - S)$ is minimum, and
5. subject to above four, the number of edges of G with both ends in S is maximum.

Theorem 6. *Let $\ell \geqslant 1$ be an integer and $\alpha \geqslant 2$ be a real number. Let G be a graph and $S \subseteq V(G)$ such that (G, S) is (α, ℓ)-minimal. Then G has no rigid separation of order at most $|S|$, and G has a subgraph H with $|V(H)| \leqslant 2\alpha\ell$ and minimum degree at least $\alpha\ell + 1$.*

With Theorem 6 and Theorem 5, we can actually obtain the following result, which is a little stronger than Theorem 2.

Theorem 7. *Let ℓ be an integer. Let G be a graph and $S \subseteq V(G)$ be an ℓ-subset such that (G, S) is $(4.5, \ell)$-massed. Then (G, S) is knitted.*

Proof. Suppose that some $(4.5, \ell)$-massed graph is not knitted and take such a graph G so that (G, S) is $(4.5, \ell)$-minimal. By Theorems 6 and 5, the graph G has no rigid separation of order at most ℓ and has an ℓ-knitted subgraph K.

If there are $|S| = \ell$ disjoint paths from S to K (we may suppose that each path uses one vertex in K), then for every partition of S, there is a corresponding partition of the endpoints of the paths in K; since K is knitted, there are disjoint connected subgraphs in K containing the parts of the endpoints, thus we have disjoint connected subgraph containing the parts of S.

If there is no $|S|$ disjoint paths from S to K, then there is separation (A, B) with $S \subseteq A$, $K \subseteq B$ of order at most $\ell - 1$. We may assume (A, B) is a separation with smallest order. Then there are $|A \cap B|$ disjoint paths from $A \cap B$ to K. Similar to the above, for every partition of $A \cap B$, we have disjoint connected subgraph containing the parts of $A \cap B$. So $G[B, A \cap B]$ is knitted, that is, (A, B) is a rigid separation of order at most $\ell - 1$, a contradiction. □

Proof of Theorem 6. We prove this theorem in the following three claims.

Claim 1. *G has no rigid separation of order at most $|S|$.*

Proof. For otherwise, take a rigid separation (A, B) with minimum A.

We first assume that $|A \cap B| < |S|$. Let $G^*[A]$ be the resulting graph from $G[A]$ by adding all missing edges in $A \cap B$. Consider $(G^*[A], S)$. If it also satisfies both (i) and (ii), then $(G^*[A], S)$ is knitted, and a knit in $G^*[A]$ can be easily converted into a knit in G since (A, B) is a rigid separation. Since G is $\alpha\ell$-massed, $\rho(B - A) \leqslant \alpha\ell|B - A|$, hence $\rho(A - S) > \alpha\ell|A - S| - (\alpha - 0.5)\ell^2$. So it satisfies (i), and thus does not satisfy (ii).

Let (A', B') be a separation of $G^*[A]$ such that $S \subseteq A'$ and B' is minimal. If $A \cap B \subseteq A'$, then $(A' \cup B, B')$ is a separation in G violating (ii). So $A \cap B \nsubseteq A'$. Since $A \cap B$ forms a cliques, $A \cap B \subseteq B'$. Consider $(G^*[B'], A' \cap B')$. The minimality of B' implies that it satisfies (ii), and $\rho(B' - A') > \alpha\ell|B' - A'| > \alpha\ell|B' - A'| - 1$ means that it satisfies (i) as well. So $(G^*[B'], A' \cap B')$ is knitted. Then $(G^*[B \cup B'], A' \cap B')$ is knitted, which means that $A' \cap B'$ is a rigid separation of (G, S), a contradiction to the minimality of A.

Now assume that $|A \cap B| = |S|$. If there exist $|S|$ disjoint paths from S to $A \cap B$, then the paths together with the rigidity of (A, B) show that (G, S) is knitted, a contradiction. So there is a separation (A'', B'') of $(G[A], S)$ of order less than $|S|$ with $A \cap B \subseteq B''$. Choose such a separation with minimum $|A'' \cap B''|$. Then there are $|A'' \cap B''|$ disjoint paths from $A'' \cap B''$ to $A \cap B$, from the rigidity of (A, B) we have $(A'', B \cup B'')$ is a rigid separation of (G, S) with $|A''| < |A|$, a contradiction to the minimality of A. □

Claim 2. *For every edge uv with $v \notin S$, the vertices u and v have at least $\alpha\ell$ common neighbors.*

Proof. Consider the graph $G' = G/uv$, the resulting graph from G by contradicting the edge uv. If (G', S) is knitted, then (G, S) is knitted. So (G', S) violates (i) or (ii).

If (G', S) violates (i), then

$$\rho\big(G' - S\big) \leqslant \alpha\ell\big|G' - S\big| - 1 = \alpha\ell|G - S| - 1 - \alpha\ell < \rho(G - S) - \alpha\ell.$$

Thus u and v have at least $\alpha\ell$ common neighbors, which gives the difference of sizes of G and G'.

So we may assume that (G', S) violates (ii). Let (A', B') be a separation of G' violating (ii) with B' minimal. By minimality, the pair $(G'[B'], A' \cap B')$ is knitted. So (A', B') is a rigid separation of (G', S) (of order at most $|S| - 1$). Note that the separation induces a separation (A, B) in G. If $\{u, v\} \nsubseteq A \cap B$, then (A, B) is a rigid separation of (G, S) of order at most $|S| - 1$, which a contradiction to Claim 1. So we assume that $u, v \in A \cap B$. Then by minimality of B', $(G[B], A \cap B)$ is $\alpha\ell$-massed thus knitted, so (A, B) is a rigid separation of size at most $|A' \cap B'| + 1 \leqslant |S|$, a contradiction to Claim 1 again. □

Claim 3. *Let δ' be the minimum degree in G among the vertices in $V(G) - S$. Then $\alpha\ell + 1 \leqslant \delta' < 2\alpha\ell$.*

Proof. We only need to prove that $\delta' < 2\alpha\ell$. Take an edge $e = uv$ in G, and consider $G_1 = G - e$. Then G_1 fails (i) or (ii).

If G_1 fails (ii), then $(G-e, S)$ contains a separation (A, B) with $|A \cap B| < |S|$. It follows that $u \in A - B$ and $v \in B - A$, lest (A, B) is a separation in (G, S) violating (ii). Then $|N(u) \cap N(v)| \leqslant |A \cap B| < |S| \leqslant \ell < \alpha\ell$, a contradiction to Claim 2. So G_1 fails (i), that is, $\rho(G-S) \leqslant \alpha\ell|V(G)-S|-1$.

If $\delta' \geqslant 2\alpha\ell$, then

$$2\left(\alpha\ell\left|V(G)-S\right|-1\right) \geqslant 2\rho(G-S) \geqslant \sum_{v \in V(G)-S} deg(v) \geqslant 2\alpha\ell\left|V(G)-S\right|,$$

a contradiction. □

Now let $v \in V(G) - S$ be a vertex with degree δ' in G. Let H be the graph induced by v and its neighbors. Then H has at most $2\alpha\ell$ vertices, and H has minimum degree at least $\alpha\ell + 1$.

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