5-2021

# Determining Quantum Symmetry in Graphs Using Planar Algebras

Akshata Pisharody

Determining Quantum Symmetry in Graphs Using Planar Algebras

A thesis submitted in partial fulfillment of the requirement
for the degree of Bachelor of Science in Mathematics from
William & Mary

by

Akshata Pisharody

Accepted for Honors

Pierre Clare

Prof. P. Clare, Director

Eric Swartz

Prof. E. Swartz

Greg Conradi Smith

Prof. G. Conradi Smith

Williamsburg, VA
May 11, 2021

# Determining Quantum Symmetry in Graphs Using Planar Algebras

**Akshata Pisharody**

William & Mary

Department of Mathematics

May 14, 2021

# Abstract

A graph has quantum symmetry if the algebra associated with its quantum automorphism group is non-commutative. We study what quantum symmetry means and outline one specific method for determining whether a graph has quantum symmetry, a method that involves studying planar algebras and manipulating planar tangles. Following the method of [8], we prove that the 5-cycle has no quantum symmetry by showing it has the generating property.

# Contents

# Chapter 1

# Introduction

## 1.1 Background and Notation

We recall some introductory definitions that will be used in the course of this paper.

**Definition 1.1.1.** The **symmetric group of order** $n$ is the group of all permutations of $n$ elements with group operation of composition. Formally, $S_n$ is the set of all bijections of the set $\{1, ..., n\}$ to itself.

*Remark* 1.1.2. Any permutation $\sigma \in S_n$ can be written in cycle notation. For example, the permutation $\sigma$ sending 1 to 3, 3 to 2, 2 to 1, 4 to 5, and 5 to 4 can be written as $(1\,3\,2)(4\,5)$.

**Definition 1.1.3.** The **adjoint** of a bounded linear operator $T$ on a Hilbert space $H$ is another bounded linear operator $T^*$ on $H$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$

**Definition 1.1.4.** The *unitary group* $U(n)$ is the group of all matrices $M \in M_n(\mathbf{C})$ such that $\overline{M}^T M = M\overline{M}^T = I_n$ (where $\overline{M}$ is the matrix such that $\overline{M}_{ij} = \overline{M_{ij}}$). A matrix in $M_n(\mathbf{C})$ is said to be **unitary** if it is in the unitary group.

**Definition 1.1.5.** A **projection** is a linear map $p$ from a vector space $V$ to itself such that $p^2 = p$.

**Definition 1.1.6.** A **partition of unity** is a finite set of projections $P$ such that:

a) The projections are orthogonal, meaning for any $p, q \in P$, $pq = 0$

b) They sum to the identity.4

## 1.2 Permutation Groups

Every permutation in $S_n$ can be associated with a matrix as follows. Let a permutation $\sigma \in S_n$. Let $\{e_1, e_2, ..., e_n\}$ be the canonical basis vectors of $\mathbf{C}^n$.

Then, define the map $\pi : S_n \to M_n(\mathbf{C})$ by

$$\pi(\sigma) = [e_{\sigma(1)} e_{\sigma(2)} ... e_{\sigma(n)}]. \tag{1.1}$$

Below is an example permutation for $\sigma = (132)$

$$\pi(\sigma) = [e_{\sigma(1)} e_{\sigma(2)} e_{\sigma(3)}] = [e_3 e_1 e_2] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

*Remark* 1.2.1. From the example and from the construction using basis vectors, one can see that every entry in a permutation matrix is either a 0 or a 1, and there is exactly one entry that is 1 along every row and column.

Because of this, the sum along any row or column of a permutation matrix is 1.

**Proposition 1.2.2.** *The map $\pi : S_n \to M_n(\mathbf{C})$ is an injective group homomorphism.*

*Proof.* We show $\pi$ is a group homomorphism from $S_n$ (with group operation composition) to $M_n(\mathbf{C})$ (with group operation matrix multiplication). Specifically, we show that $\pi(\sigma_1\sigma_2) = \pi(\sigma_1)\pi(\sigma_2)$.

Let the $n \times n$ matrices $A, B$, and $C$ be defined by $A = \pi(\sigma_1)$, $B = \pi(\sigma_2)$, and $C = AB$. Note that in the matrix $C$, $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ for all $i, j \in \{1, ..., n\}$. From Remark 1.2.1, it is clear that each $a_{ij}$, and $b_{ij}$ are either 1 or 0, so each $c_{ij}$ is also either 1 or 0.

Thus, for $c_{ij} \neq 0$, $a_{ik}b_{kj} = 1$ for some $k \in \{1, ..., n\}$. In fact, there is only one $k$ where this is the case, because from Remark 1.2.1, there is only 1 in each row or column of $A$ and $B$. Thus, for a specific $k$, $1 = a_{ik} = b_{kj}$.

Since $A = \pi(\sigma_1) = [e_{\sigma_1(1)}, ..., e_{\sigma_1(n)}]$, if $a_{ik} = 1$, then $e_{\sigma_1(k)} = e_i$, so $\sigma_1(k) = i$. Likewise, since $B = \pi(\sigma_2) = [e_{\sigma_2(1)}, ..., e_{\sigma_2(n)}]$, if $b_{kj} = 1$, then $e_{\sigma_2(j)} = e_k$, so $\sigma_2(j) = k$.

This means that for any $j \in \{1, ..., n\}$, $(\sigma_1\sigma_2)(j) = \sigma_1(\sigma_2(j)) = \sigma_1(k) = i$. Therefore, $c_{ij} = 1$ if and only if $(\sigma_1\sigma_2)(j) = i$ Now, for the $n \times n$ matrix $M = \pi(\sigma_1\sigma_2) = [e_{(\sigma_1\sigma_2)(1)}, ..., e_{(\sigma_1\sigma_2)(n)}]$, $m_{ij} = 1$ if and only if $(\sigma_1\sigma_2)j = i$. Therefore, $M$ and $C$ are the same matrix.

The injectivity of $\pi$ follows from showing that $ker(\pi) = \{\varepsilon\}$, where $\varepsilon$ is the identity permutation. Thus, we show that $\sigma \in ker(\pi)$ if and only if $\sigma = \varepsilon$.

$$\begin{aligned}
\sigma \in ker(\pi) &\iff \pi(\sigma) = I_n \\
&\iff [e_{\sigma(1)} ... e_{\sigma(n)}] = I_n \\
&\iff \sigma(j) = j \text{ for all } j \in \{1, ..., n\} \\
&\iff \sigma = \varepsilon.
\end{aligned}$$

□

## 1.3 $C^*$-Algebras

**Definition 1.3.1.** A $C^*$ algebra is a normed algebra $A$ over the complex numbers that has all the following properties:

- $A$ is complete with respect to that norm (it is a Banach algebra).
- $A$ has an anti-multiplicative, conjugate-linear involution. That is for every $a, b \in A$, there is an $a^* \in A$ such that $(a^*)^* = a$, $(ab)^* = b^*a^*$, and $(a + \lambda b)^* = a^* + \bar{\lambda}b^*$ (where $\lambda \in \mathbf{C}$).

- For all $a \in A$, the identity $||a^*a|| = ||a||^2$ is satisfied.

*Remark* 1.3.2. $\mathbf{C}$ is a $C^*$-algebra with the involution being the complex conjugate.

*Remark* 1.3.3. $M_n(\mathbf{C})$ is a $C^*$-algebra with the involution of a matrix $M \in M_n(\mathbf{C})$ given by $M^* = \overline{M}^T$.

*Remark* 1.3.4. The space $C(X)$ of continuous, complex valued functions on a compact space $X$ is a $C^*$-algebra. The norm of $f(x) \in C(X)$ in this space is given by the supremum norm $\|f\|_\infty$, and the involution of $f(x)$ is $f(x)^* = \overline{f(x)}$ given by pointwise conjugation.

In fact, every commutative $C^*$ algebra is isomorphic to an algebra of bounded linear operators on a Hilbert space according to a theorem by Gelfand and Naimark (see [6, Theorem 1.4.1]).

**Definition 1.3.5.** A **projection in a $C^*$-algebra** is an element $p \in C^*$-algebra such that $p^2 = p = p^*$.

The terminology comes from orthogonal projections in vector geometry. For a vector space $V$, if $p$ is the map $p : V \to V$ that orthogonally projects $v \in V$ onto a subspace $Y$, then $p^2 = p$ as in Definition 1.1.5. In the geometric case, this can be thought of as projecting a vector onto another vector. If the result after the first projection is then projected again, there is no change. The map $p$ is thus a projection in the sense of Definition 1.3.5 within the algebra of endomorphisms of $V$, which is also a $C^*$-algebra as in 1.3.3

Moreover, from the Gelfand-Naimark Theorem, a projection $p$ in a $C^*$ algebra $A$ can be associated with a bounded linear operator $P$ on some Hilbert space $H$. Then, it is known that $P^*$ (where in this case the star denotes the adjoint of $H$, see Def 1.1.3), is the operator associated with $p^*$. Therefore, since $p^* = p$, we say that $p$ is **self-adjoint** and the $*$ of the involution in $A$ is used to simultaneously refer to the $*$ of the adjoint operator.

# Chapter 2

# Quantum Permutation Groups and Quantum Symmetry

## 2.1 The Classical Permutation Group as a $C^*$-Algebra

Now, we may consider $C(G)$ where $G \subset U(n)$ (See Definition 1.1.4) is a compact Lie group. From work done by Woronowicz [13], it is possible to translate the algebraic properties of $G$ onto $C(G)$.

**Definition 2.1.1.** A **biunitary** matrix is a matrix $u \in M_n(A)$ over a $C^*$-algebra $A$ such that $u^* = u^{-1}$ and $u^T = \bar{u}^{-1}$, where $u^*$ is the involution of $u \in A$ as described in Definition 1.3.1.

**Definition 2.1.2.** A **Woronowicz Algebra** is a $C^*$-algebra $A$ generated by $n^2$ elements $u_{ij}$ $(1 \leq i, j, \leq n)$ such that the matrix $u = [u_{ij}]$ is a biunitary matrix (see Definition 2.1.1) and the maps

a) $\Delta : A \to A \otimes A$ given by $\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}$, the *comultiplication*,

b) $\varepsilon : A \to \mathbf{C}$ given by $\delta(u_{ij})$, the *counit*,

c) $S : A \to A^{op}$ given by $S(u_{ij}) = u_{ji}^*$, the *antipode*, are morphisms.

*Remark* 2.1.3. From Definition 1.1.4 and Remark 1.3.3, if the algebra $A$ is commutative, like $\mathbf{C}$, then a biunitary matrix is the same as a unitary matrix.

For a full explanation of comultiplication, counit, and antipode see [4]. The terminology is reminiscent of the axioms of groups —a group has an associative binary operation, a identity element, and elements are invertible.

For $G \subset U(n)$, each $g \in G$ can be written as

$$g = \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \cdots & \cdots & \cdots \\ g_{n1} & \cdots & g_{nn} \end{bmatrix}.$$

Now, for $1 \leq i, j \leq n$, the $n^2$ functions $u_{ij} : G \to \mathbf{C}$ are given by

$$u_{ij} : \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \cdots & \cdots & \cdots \\ g_{n1} & \cdots & g_{nn} \end{bmatrix} \mapsto g_{ij}.$$

These $u_{ij}$ functions generate $C(G)$ as an algebra (a fact which is explained as it relates to our purposes in [2] as a consequence of the Stone-Weierstrass Theorem). Furthermore, the maps $\Delta$, $\varepsilon$, and $S$ are also defined as the comultiplication, counit, and antipode outlined in Definition 2.1.2, details about which can be seen in [13].

Additionally, the matrix

$$u = \begin{bmatrix} u_{11} & \dots & u_{1n} \\ \dots & \dots & \dots \\ u_{n1} & \dots & u_{nn} \end{bmatrix}$$

is biunitary.

Ergo, $C(G)$ is a Woronowicz algebra. In fact, this explanation, along with Theorem 3.1

in [4], allows us to state the remark below.

*Remark* 2.1.4. If $G$ is a finite group, $C(G)$ is a commutative Woronowicz algebra

If we consider $\pi_n = \{\pi(\sigma) \mid \sigma \in S_n\}$, the group of permutation matrices associated with $S_n$, it is possible to see that $\pi_n \subset U(n)$. However, as notation shorthand we may consider $S_n \subset U(n)$ remembering to consider each permutation in $S_n$ as its permutation matrix using the map $\pi$ described in Equation 1.1.

From Remark 2.1.4, we are able to say that $C(S_n)$ is a commutative Woronowicz algebra, and following [12], we may define $C(S_n)$ as using the Woronowicz algebra framework. Specifically, each of the generators $u_{ij}$ are given by [4]

$$u_{ij} = \chi\{\sigma \in S_n \mid \sigma(j) = i\}, \tag{2.1}$$

This can be seen because for a permutation $\sigma \in S_n$, the coefficient in the $ij^{th}$ entry of the matrix $g = \pi(\sigma)$ is 1 if $\sigma(j) = i$ and zero otherwise. These $u_{ij}$'s generate the $C^*$-algebra $C(S_n)$ by Remark 2.1.4 and Definition 2.1.2.

**Proposition 2.1.5.** *Each of the $u_{ij}$'s satisfy the following properties [12]:*

*a)* $u_{ij}^2 = u_{ij} = u_{ij}^*$, *for all* $i, j \in \{1, ..., n\}$. *Namely all of the $u_{ij}$'s are projections (see Definition 1.3.5).*

*b)* $\sum_{j=1}^{n} u_{ij} = 1$, *for all* $i \in \{1, ..., n\}$

*c)* $\sum_{i=1}^{n} u_{ij} = 1$, *for all* $j \in \{1, ..., n\}$

*Proof.* a) Each of the $u_{ij}$'s are taken from entries in matrices in $\pi_n$, which from Remark 1.2.1 means each $u_{ij}$ takes either 0 or 1 as a value, and since both $0^2 = 0 = 0^*$ and $1^2 = 1 = 1^*$, all the $u_{ij}$'s are projections in $C(S_n)$.

b) Fix $i \in \{1, ..., n\}$. Since all of the $u_{ij}$'s take either 0 or 1 as a value, let $u_{ij} = 1$. Then, from 2.1, for $\sigma \in S_n$, $\sigma(j) = i$. Then, for all $j' \neq j$, $\sigma(j') \neq i$ because $\sigma$ is a permutation, meaning $u_{ij'} = 0$. Thus, $\sum_{j=1}^{n} u_{ij} = 1$.

c) This follows from the proof of b) by fixing $j \in \{1, ..., n\}$ and summing $\sum_{i=1}^{n} u_{ij}$.

□

Moreover, in $C(S_n)$, the $u_{ij}$'s are projections with each row and column of the matrix $u = [u_{ij}]$ summing to 1, so $u$ is a magic unitary.

**Definition 2.1.6.** [4, Definition 5.1] For a $C^*$-algebra $A$, a matrix $u \in M_n(A)$ is a **magic unitary** if all entries of $u$ are projections, along each row and column the projections are orthogonal (See Definition 1.1.6), and each row and column sums up to 1.

It is also said that a magic unitary matrix is a matrix such that each row and column forms **a partition of unity** (See Definition 1.1.6).

The word "magic" in this definition comes from a tangential reference to magic squares where each row, column, and the main diagonals of a square of numbers is the same.

In fact, $C(S_n)$ is the unique commutative C* algebra with generators $u_{ij}$ forming a magic unitary matrix [4], which yields the following observation.

*Remark* 2.1.7. $C(S_n)$ is the universal commutative $C^*$-algebra generated by $n^2$ elements

$$\{u_{ij} \mid i, j \in \{1, ..., n\}\}$$

such that that the matrix $u = [u_{ij}] \in M_n(C(S_n))$ is a magic unitary.

## 2.2 Quantum Permutation Groups

From Definition 2.1.7, the algebra $C(S_n)$ is *commutative*, however we may remove the commutativity condition in a process called **liberation** to study a new algebra [12].

**Definition 2.2.1.** $C(S_n^+)$ is the universal $C^*$-algebra generated by $n^2$ elements

$$\{u_{ij} \mid i, j \in \{1, ..., n\}\}$$

such that that the matrix $u = [u_{ij}] \in M_n(\mathbf{C})$ is a magic unitary.

It is clear to see that $C(S_n^+)$ has the same properties as $C(S_n)$ however the latter has a further requirement of commutativity. Thus, from the universal property, it follows that

$$C(S_n) \cong C(S_n^+)/\langle ab = ba \rangle.$$

We may think of the surjection from $C(S_n^+)$ to $C(S_n)$ defined by

$$C(S_n^+) \to C(S_n^+)/\langle ab = ba \rangle \cong C(S_n)$$

as the restriction of functions from $S_n^+$ to $S_n$. Although there is no obvious group or object known as $S_n^+$ made up of elements, we may think of $S_n^+$ as the **quantum permutation group** with $S_n \subseteq S_n^+$ using the restriction of functions above.

If $C(S_n^+)$ is commutative, then

$$C(S_n^+)/\langle ab = ba \rangle \cong C(S_n^+)$$

and the surjection above is an isomorphism, so

$$C(S_n) \cong C(S_n^+).$$

In this instance, since their respective algebras are isomorphic, we may write $S_n = S_n^+$ while keeping in mind all caveats about the latter group not being a classical object.

*Remark* 2.2.2. [12]

For $n = 1, 2, 3$, $C(S_n^+) = C(S_n)$, so the quantum permutation group is equivalent to the classical one. For $n \geq 4$, $C(S_n^+)$ is non-commutative, meaning the quantum permutation group $S_n^+$ is a different object than the classical $S_n$.

For $n = 1$, this is clear because there is only one generator which commutes with itself.

For $n = 2$, any $2 \times 2$ magic unitary matrix of generators has the form [4]

$$\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}.$$

Clearly $p$ commutes with itself. Also, $p$ commutes with $1-p$ since $p(1-p) = 0 = (1-p)(p)$ because entries in the same row and column are orthogonal projections.

When $n = 3$, $C(S_3^+)$ is commutative, so $S_3^+ = S_3$, and a description for this result is shown in [4, Theorem 6.2].

When $n = 4$, the generators of $C(S_4^+)$ can be put into the matrix below where $p, q$ are projections.

$$\begin{bmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{bmatrix}$$

Therefore, the algebra formed by these generators does not need to be necessarily commutative, so $S_4^+ \neq S_4$

## 2.3 Automorphism groups

Let $G$ be a graph on $n$ vertices (we consider a general case where edges are not directed and there are no multiple edges between vertices or loops). A function is an automorphism on $G$ if it is a bijection from the vertex set of $G$ to itself that preserves adjacency and non-adjacency of edges.

**Definition 2.3.1.** Let $V(G)$ be the set of vertices of an $n$-vertex graph $G$. A function $\alpha : V(G) \to V(G)$ is **graph automorphism** if $\alpha$ is a bijection and $\alpha(u)$ is adjacent to $\alpha(v)$ if and only if $u$ is adjacent to $v$ for all $u, v \in V(G)$

Intuitively, if the vertices of $G$ are labelled from $\{1, ..., n\}$, a graph automorphism is a re-labelling of those vertices that preserves the same connections after the re-ordering. In 2.1, each vertex is connected with the same labelled vertices before and after applying the example graph automorphism.

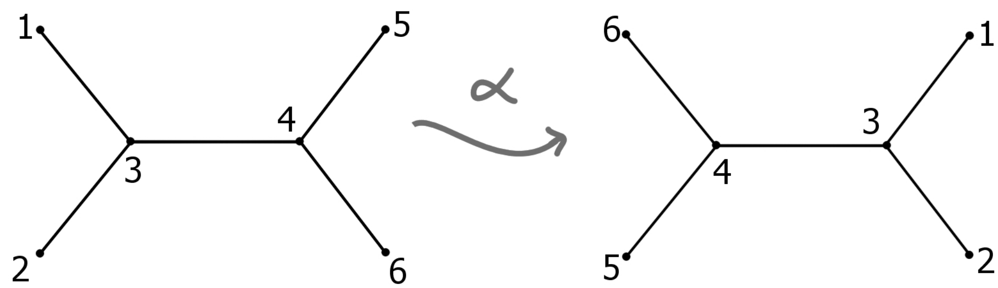



Figure 2.1: $\alpha = (1\,6\,2\,5)(3\,4)$ (in cycle notation)

Each graph automorphism on an $n$-vertex graph defines an element in $S_n$. In fact, the set of all of a graph's automorphisms is a subgroup of $S_n$.

**Definition 2.3.2.** The **automorphism group** of an $n$-vertex graph $G$, $\mathrm{Aut}(G)$, is the group of all graph automorphisms of $G$.

From Section 1.2, each permutation can be thought of as a matrix, which means each automorphism on an $n$ vertex graph is associated with a matrix.

Moreover, the automorphism group of a graph can be shown to be the set of matrices with a specific property.

**Definition 2.3.3.** For an $n$-vertex simple graph $G$ with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$, the **adjacency matrix of** $G$ is an $n \times n$ matrix $A$ such that $a_{ij} = 1$ only when $v_i, v_j$ is an edge in $G$ and $a_{ij} = 0$ otherwise.

*Remark* 2.3.4. For a graph on $n$ vertices with adjacency matrix $A$,

$$\mathrm{Aut}(G) = \{\sigma \in S_n \mid \pi(\sigma)A = A\pi(\sigma)\},$$

where $\pi$ is the map in Equation 1.1.

do we need this part?

Recall that in Section 1.2, we associated every permutation in $S_n$ with a permutation matrix, so for ease of notation, we may write $\sigma A = A\sigma$ and omit the $\pi$ map.

## 2.4 Quantum Automorphism Groups

Since $\mathrm{Aut}(G) \subseteq S_n$, we may use the same framework of $C^*$-algebras as shown in the earlier section to analyze $C(\mathrm{Aut}(G))$. From Remark 2.3.4, we can consider the $C^*$-algebra $C(\mathrm{Aut}(G))$ as the the elements in $C(S_n)$ that commute with the adjacency matrix of $G$. The definitions below follow from Banica [1].

**Definition 2.4.1.** $C(\mathrm{Aut}(G))$ is the universal commutative $C^*$-algebra generated by $n^2$ elements

$$\{u_{ij} \mid i, j \in \{1, ..., n\}\}$$

such that that the matrix $u = [u_{ij}] \in M_n(\mathbf{C})$ is a magic unitary and

$$uA = Au.$$

Note that Definition 2.4.1 only differs from Definition 2.1.7 by the final condition requiring that the matrix $u$ of generators commutes with the graph $G$'s adjacency matrix.

The inclusion $\mathrm{Aut}(G) \leq S_n$ can be seen by the restriction from $C(S_n)$ to $C(\mathrm{Aut}(G))$

$$C(S_n) \to C(S_n)/\langle uA = Au \rangle \cong C(\mathrm{Aut}(G)).$$

Then, the same liberation process can be applied to the commutative $C^*$-algebra $C(\mathrm{Aut}(G))$ to form a new, not-necessarily commutative algebra [1]. Recall from Definition 2.2.1 that $C(S_n^+)$ was defined as the universal $C^*$-algebra generated by $n^2$ elements $u_{ij}$ that form a magic unitary matrix.

**Definition 2.4.2.** For an $n$-vertex graph with adjacency matrix $A$, define

$$C(\mathrm{Aut}(G)^+) = C(S_n^+)/\langle uA = Au\rangle$$

where $u$ is the matrix formed by the generators of $C(S_n^+)$ as outlined in Definition 2.2.1.

Note that Definition 2.4.2 means that $C(\mathrm{Aut}(G)^+)$ is the algebra $C(S_n^+)$ with a further requirement that the matrix $u$ of generators $u_{ij}$ commutes with the adjacency matrix of $G$. Also, from Definition 2.2.1, $C(S_n^+)$ is formed by liberation of $C(S_n)$, meaning we let go of the requirement for commutativity. Therefore, imposing a commutativity condition on $C(\mathrm{Aut}(G)^+)$ would once again yield a commutative, universal $C^*$-algebra with generators $u_{ij}$ forming a magic unitary matrix $u$ commuting with the adjacency matrix of $G$. Condensing this into symbols, we have

$$C(\mathrm{Aut}(G)) \cong C(\mathrm{Aut}(G)^+)/\langle ab = ba\rangle$$

for $a, b \in C(\mathrm{Aut}(G)^+)$.

This allows us to consider the surjection from $C(\mathrm{Aut}(G)^+)$ to $C(\mathrm{Aut}(G))$ as the restriction of functions

$$C(\mathrm{Aut}(G)^+) \to C(\mathrm{Aut}(G)^+)/\langle ab = ba\rangle \cong C(\mathrm{Aut}(G))$$

meaning we think of $\mathrm{Aut}(G) \subset \mathrm{Aut}(G)^+$. Again like with $S_n^+$, $\mathrm{Aut}(G)^+$ is not necessarily a group with elements or permutations in it, but we call it the **quantum automorphism group** of G and we may use the algebra $C(\mathrm{Aut}(G)^+)$ to study it.

It is important to note at this stage that there are two different types of commutativity present in these definitions. The first is commutativity in a $C^*$-algebra (or in another space in the traditional sense) where $ab = ba$ for all $a, b$ in the space. This is the requirement we loosen in the liberation process to define the quantum groups $S_n^+$ and $\mathrm{Aut}(G)^+$ (see Definitions 2.2.1 and 2.4.2). The second is a commutation relation where the matrix $u$ made up of the generators $u_{ij}$ commutes with the adjacency matrix $A$ of a graph $G$. Specifically, since the automorphism group of a graph $G$ is tied to commutativity with the graph's adjacency matrix (see Remark 2.3.4), we use the condition $uA = Au$ to delineate $C(\mathrm{Aut}(G))$ from $C(S_n)$ (see Definition 2.4.1), and $C(\mathrm{Aut}(G)^+)$ from $C(S_n^+)$ (see Definition 2.4.2).

Like outlined in Section 2.2 with $C(S_n^+)$, if $C(\mathrm{Aut}(G)^+)$ is a commutative algebra, then

$$C(\mathrm{Aut}(G)^+)/\langle ab = ba\rangle \cong C(\mathrm{Aut}(G)^+),$$

which means that the restriction of functions from $C(\mathrm{Aut}(G)^+)$ to $C(\mathrm{Aut}(G))$ is an isomorphism and

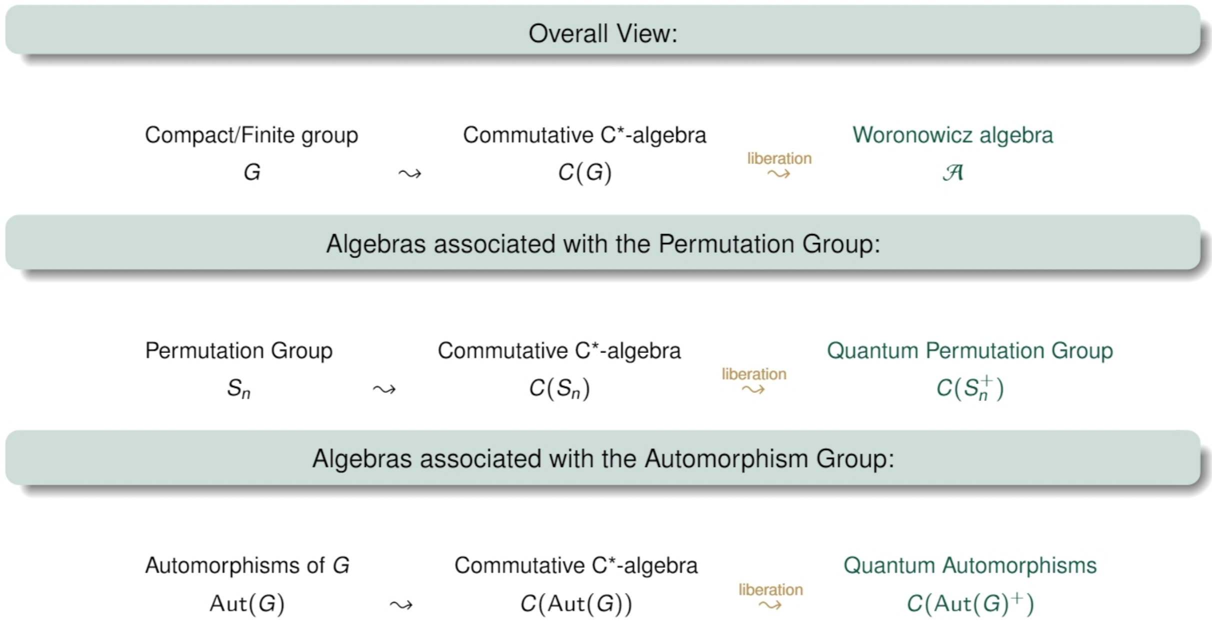
$$C(\mathrm{Aut}(G)) \cong C(\mathrm{Aut}(G)^+).$$

In this case, we then write $\mathrm{Aut}(G) = \mathrm{Aut}(G)^+$. The implications and analysis of this are discussed in the next section.

As stated in Remark 2.2.2, the quantum permutation groups $S_1^+$, $S_2^+$, and $S_3^+$ are equivalent to their classical counterparts. This means that $C(S_n^+)$ is a commutative algebra for $n = 1, 2, 3$, so then $C(\mathrm{Aut}(G)^+)$ is a commutative algebra for graphs on $1, 2$ or 3 vertices. Thus, the quantum automorphism groups for graphs on $1, 2$, and 3 vertices are equivalent to their classical automorphism groups, so these graphs have no quantum symmetry. Quantum symmetry in graphs begins with graphs with 4 or more vertices, for example, $K_4$, the complete graph on 4 vertices has quantum symmetry [3].

## 2.5 Quantum Symmetry

At this point, the diagram below summarizes the relationship between $C^*$-algebras and groups mentioned thus far.




Although the groups $S_n^+$ and $\mathrm{Aut}(G)^+$ do not have clear elements or structure that we know of, they can be studied by looking at the algebras $C(S_n^+)$ and $C(\mathrm{Aut}(G)^+)$. These spaces are are $C^*$-algebras clearly outlined by definitions (2.1.7 and 2.4.2 respectively) based on the Woronowicz algebra structure of $C(S_n)$ and $C(\mathrm{Aut}(G))$, but by liberating each of them from the commutativity requirement.

Overall, the idea of determining whether or not a graph has quantum symmetry involves studying $C(\mathrm{Aut}(G)^+)$ for that graph. If this algebra is commutative, it is isomorphic to $C(\mathrm{Aut}(G))$, meaning we say $\mathrm{Aut}(G) = \mathrm{Aut}(G)^+$ and thus the graph **has no quantum symmetry**. However, if there are elements in $C(\mathrm{Aut}(G)^+)$ that do not commute, then the restriction of functions from $C(\mathrm{Aut}(G)^+)$ to $C(\mathrm{Aut}(G))$ is in fact surjective but not bijective, and we can say $\mathrm{Aut}(G) \subset \mathrm{Aut}(G)^+$. There are no concrete "quantum automorphisms," but since the algebra $C(\mathrm{Aut}(G)^+)$ has non-commutative elements in it whereas $C(\mathrm{Aut}(G))$ does not, we consider $\mathrm{Aut}(G)^+$ to be in a sense "bigger" than $\mathrm{Aut}(G)$, and thus we say the graph **has quantum symmetry.**

It is possible to directly manipulate the generators $u_{ij}$ for a specific $C(\mathrm{Aut}(G)^+)$ and deduce whether or not the algebra is commutative. Many tools and results for doing so are congregated in [9], and some of those methods are applied in [11] to show that the Petersen graph does not have quantum symmetry.

However, the more complicated a graph is, the more opaque $C(\mathrm{Aut}(G)^+)$ becomes to directly manipulate. There are however other methods to analyze the quantum symmetry in graphs, one of which will be explored in the next section of this paper.

# Chapter 3

# Planar Algebras

## 3.1 Planar Algebras and Planar Tangles

Planar algebras were introduced by Jones [7] to study subfactors in the theory of von Neumann algebras. In general, a **planar algebra** is a graded vector space

$$V = \bigcup_{n \in \mathbf{Z}^+ \cup \{+,-\}} V_n$$

over a field $K$ and a set of specific maps between tensor products of $V_k$'s for $k \in \mathbf{Z}^+ \cup \{+,-\}$.

These specific maps between tensor products are a generalization of multiplication in regular algebras and they are defined by other mathematical objects, planar tangles [7]. The presentation in this section is directly lifted from Curtin's description [5], but there are many ways to depict planar tangles diagrammatically depending on ease of notation or what they are used to study [7, 5]. We first give a general description for how planar tangles are used to construct these maps.

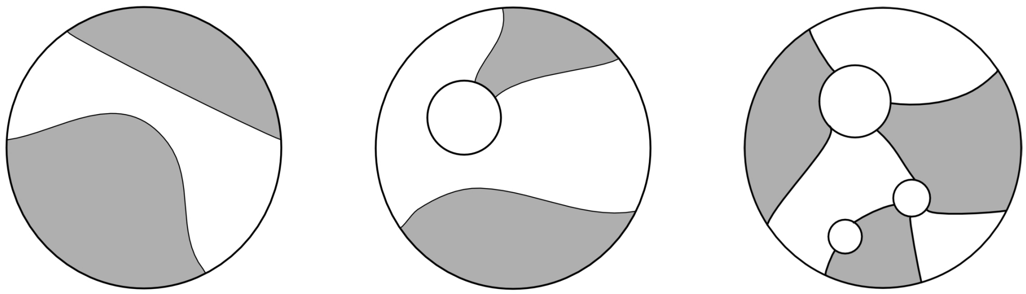


Figure 3.1: Some simple tangles with varying number of strings and inner disks

Let $D_0$ be a disk and $D_1, D_2, ..., D_n$ disjoint disks in the interior of $D_0$. Furthermore, let each disk have an even number of points on its boundary such that there are simple, smooth curves called "strings" in $D_0 / \cup_{i=1}^{n} D_i$ connecting all the boundary points, so disk $D_i$ has $2k_i$ boundary points. Now, each of the regions made by the boundary of $D_0$ and the curves can be colored black or white so that no two adjacent regions are the same color.

The diagrams above are not labelled; they are overall pictures of what tangles look like. However specific labelling conventions are used in order to make tangles useful for defining maps in a planar algebra. For a specific example, see Figure 3.2.

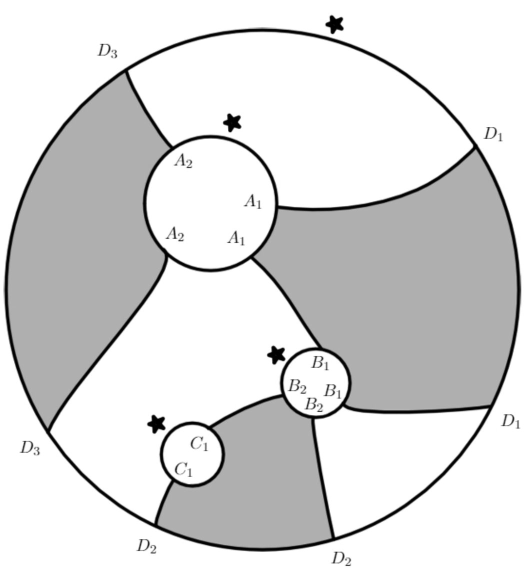



Figure 3.2: Labelled version of the rightmost tangle from Figure 3.1

Each of the boundary points divide the disks into intervals, and one of these intervals on each disk that is touching a white region should be marked as "privileged", denoted above with a $\star$. If the entirety of a disk is one interval, it is privileged only if it touches a white region. Additionally, the boundary points of each disk are numerically labelled clockwise starting at the privileged interval, but with duplicated indices. Note that the process of selecting privileged intervals makes the coloring redundant [5].

**Definition 3.1.1.** For an outer disk $D_0$ the isotopy class of such diagrams with the same disks, strings, colors, and labelled intervals is called a **planar $k_0$-tangle** or a **$k_0$-tangle**. Additionally, a diagram with outer disk $D_0$, inner disks $D_1, ..., D_n$, and each $D_i$ having $2k_i$ boundary points for $i \in \{0, ..., n\}$ can also be called a $(k_1, ..., k_n, k_0)$-tangle. [1].

These planar tangles can be naturally composed within each other if the boundary points on one planar tangle's inner disk are matched with the boundary points of another's outer disk, the strings are connected, and the coloring is kept the same (see Figure 3.3 for an example). The collection of all planar tangles along with this composition operation is known as the **planar operad**.

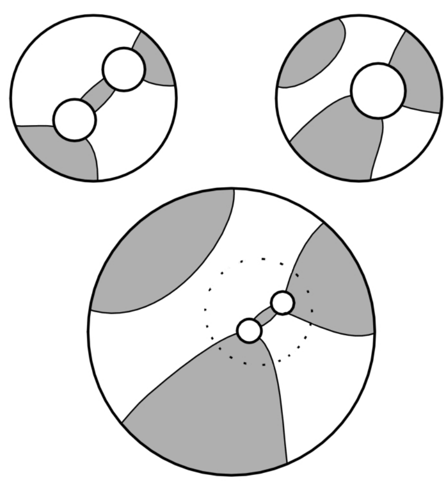


Figure 3.3: Composition of two tangles. The one on the left can be put inside the inner disk of the one on the right

For ease of reference, this paper will shorten the term "planar tangle" to "tangle." It should be noted however that diagrams that are not necessarily planar but still constructed similarly to planar tangles are also called tangles.

There are different types of planar algebras coming from different structures [5, 7], but for this instance we are interested in the planar algebra of functions on a finite set.

**Definition 3.1.2.** [5, Section 3.1] Let $X$ be a finite set and for all $i > 0$, let $P_i(X)$ be the vector space of all complex linear functionals on $X^i$, and let $P_+(X)$ and $P_-(X)$ be identified with $\mathbf{C}$. Then

$$P(X) = \bigcup_{i \in \mathbf{Z}^+ \cup \{+,-\}} P_i(X)$$

is the **spin planar algebra** [5].

Planar tangles define maps between tensor products of $P_i(X)$, and the formalism that follows from [1] informs us how.

**Definition 3.1.3.** [1, Definition 5.2] Every $(k_1, ..., k_n, k_0)$-tangle $T$ defines a map

$$Z_T : \bigotimes_{i=1}^{n} P_{k_i}(X) \to P_n(X)$$

by

$$Z_T(x_1 \otimes ... \otimes x_n) = \beta^{c(T)} \sum_y (x_1, ..., x_n, y)^T y$$

where the sum is over all $n$-fold tensor products of Dirac masses, $\beta$ is the number of elements in $X$, and $c(T)$ is the number of closed circles in $T$, and $(x_1, ..., x_n, y)^T = 1$ If all strings in $T$ are connected to equal indices and 0 otherwise.

Using the example the tangle $T$ below, we will show how this map is defined. Let us say the upper inner disk is $D_1$ and the lower inner disk is $D_2$, and let the boundary points be labelled as shown in Figure 3.4.

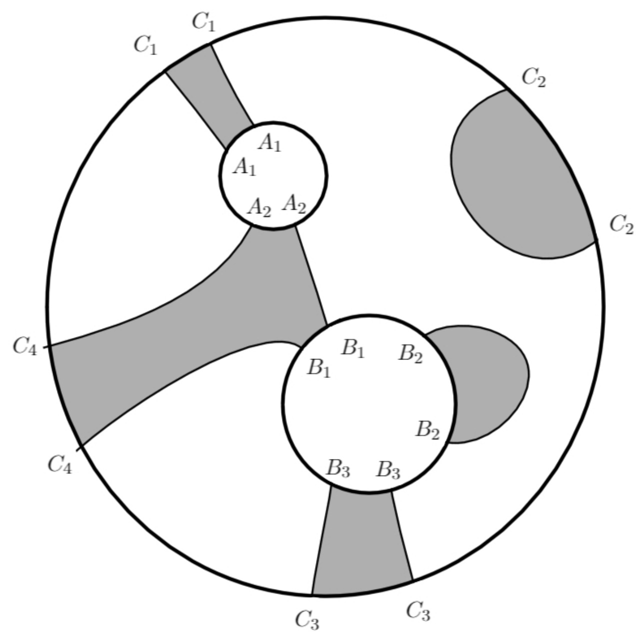



Figure 3.4: The planar tangle $S$

From Definition 3.1.1 it is clear that $S$ is a $(2,3,4)$-tangle, so it gives a map $Z_S$ from $P_2(X) \otimes P_3(X)$ to $P_4(X)$. For $x_1 \in P_2(X)$, $x_2 \in P_3(X)$, and $y \in P_4(X)$, let

$$x_1 = \delta_{A_1} \otimes \delta_{A_2}$$

$$x_2 = \delta_{B_1} \otimes \delta_{B_2} \otimes \delta_{B_3}$$

$$y = \delta_{C_1} \otimes \delta_{C_2} \otimes \delta_{C_3} \otimes \delta_{C_4}$$

where $\delta$ represents the Dirac measure or Dirac mass. Dirac masses form a basis for $\mathbf{C}(X)$ on a finite set $X$ [1]. Therefore $x_1 \in P_2(X)$, can be written as a 2-fold tensor product of Dirac masses on elements in $X^2$. Similarly, $x_2 \in P_3(X)$, and $y \in P_4(X)$ can be written as a 3-fold, and 4-fold product of Dirac masses, respectively.

From the Definition 3.1.3, and since there are zero closed circles in $S$, the map associated with it is given by

$$Z_S(x_1 \otimes x_2) = \sum_y (x_1, ..., x_n, y)^S y \tag{3.1}$$

and it is nonzero only when $(x_1, ..., x_n, y)^S = 1$, meaning all strings in $S$ must join equal indices. Thus, if $A_1$ and $A_2$ are fixed, $B_1 = A_2$ and $B_2 = B_2$ (meaning there are no restrictions on $B_2$). Furthermore, since the sum is over all $y \in P_4(X)$, there are only certain indices of the Dirac masses in $y$ that make $(x_1, ..., x_n, y)^T$ nonzero. Namely $C_1 = A_1$, $C_3 = B_3$, and $C_4 = B_1 = A_2$ (along with $C_2 = C_2$).

Thus, the only nonzero map is of the form

$$Z_S(x_1 \otimes x_2) = \sum_y (x_1, ..., x_n, y)^S y$$

where

$$x_1 = \delta_{A_1} \otimes \delta_{A_2}$$

$$x_2 = \delta_{A_2} \otimes \delta_{B_2} \otimes \delta_{B_3}$$

and the sum is over all $y$ of the form

$$y = \delta_{A_1} \otimes \delta_{C_2} \otimes \delta_{B_3} \otimes \delta_{A_2}$$

Therefore, the tangle $S$ gives a map $Z_S : P_2(X) \otimes P_3(X) \to P_4(X)$. This map can be thought of as a way to do multiplication in the spin-planar algebra. Often, for ease of notation, if a tangle $T$ gives a map $Z_T(x_1 \otimes ... \otimes x_n)$, we write the function as $T(x_1 \otimes ... \otimes x_n) = Z_T(x_1 \otimes ... \otimes x_n)$.

Now we define a specific type of spin-planar algebra that we will use for the remainder of this paper.

**Definition 3.1.4.** [10, Definition 6.1.2][5]

Let $P(X)$ be the spin planar algebra associated with some finite set $X = \{e_1, ..., e_d\}$

and let $G \subseteq S_d$ be a group. Then, $G$ has a natural action on $\alpha : X^k \times G \to X^k$ where $((e_1, ...e_k), g) \mapsto (e_{g(1)}, ...e_{g(k)})$ for $g \in G$. This action can be extended to

$$\alpha^{\otimes k} : X^{\otimes k} \times G \to X^{\otimes k}$$

$$(e_{i_1} \otimes ... \otimes e_{i_k}, g) \mapsto e_{g(i_1)} \otimes ... \otimes e_{g(i_k)}.$$

This induces an action of $G$ on $P(X)$. Thus, the **group-action planar algebra** $\mathcal{P}^G$ is the fixed-point algebra of the group action, meaning

$$\mathcal{P}_k^G = \left\{ x \in P(X) \mid \alpha^{\otimes k}(x, g) = x \text{ for all } g \in G \right\}.$$

and

$$\mathcal{P}^G = \bigcup_{i \in \mathbf{Z}^+ \cup \{+,-\}} \mathcal{P}_i^G$$

(with $\mathcal{P}_+^G$ and $\mathcal{P}_-^G$ associated with $\mathbf{C}$ as in Definition 3.1.2).

## 3.2 Other Drawings

As noted in Definition 3.1.1, planar tangles can be depicted in many ways up to isotopy, meaning any drawing of them can be deformed or stretched into equivalent ones.

Since all of the planar tangles necessary for the scope of this paper have at most one inner disk, it is useful to consider one of the disks as the input disk and the other as the output disk. This helps intuitively consider compositions of tangles as seen in Figure 3.3. The rest of this paper will use a formalism for drawing that is adapted from [1] and [8] that allows visualization of this fact using vertically stacked tangles instead of concentric ones.

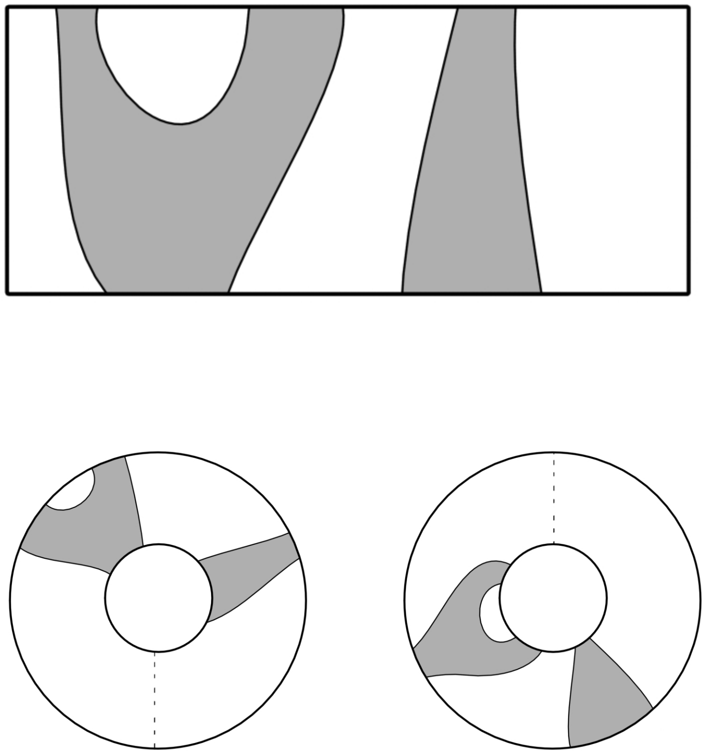


Figure 3.5: The tangle on top can be deformed to the ones on the bottom.

Planar tangles will be depicted as a rectangle with the boundary points at the top being the "input boundary" and the ones at the bottom being the "output boundary". Note that by wrapping the edge of the rectangle around and connecting it, as seen in Figure 3.5, this diagram can be transformed into circular planar tangles that were shown in Section 3.

This means it is intuitive to consider compositions of planar tangles as stacked rectangular diagrams with an input at the top and an output at the bottom, as in Figure 3.6.

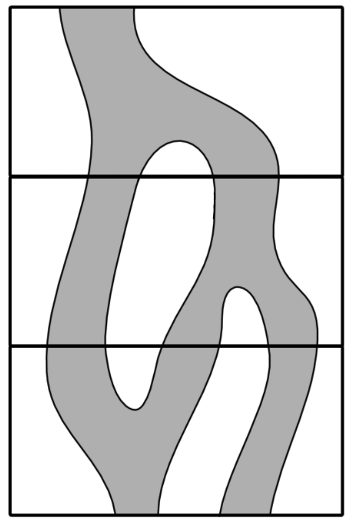


Figure 3.6: Three tangles stacked on top of each other. If the tangles are labelled from top to bottom as A, B, and C, the stacked diagram above depicts the map $Z_C(Z_B((Z_A)))$.

Also note that by wrapping in the other direction, the input and output disk can be switched, as seen in the bottom of Figure 3.5. In fact, there is no clear distinction between what the "input" and "output" are, however for intuition and consistency, we will use **input boundary** to refer to the top of the rectangle and **output boundary** to refer to the bottom of the rectangle.

## 3.3 The Generating Property

In order to relate planar algebras and planar tangles to quantum symmetry, we go over some terms and results from [10].

**Definition 3.3.1.** [10, Definition 6.1.3]

The planar algebra $P$ is said to be generated by a set $S$ if for every $x \in P$, there exists a planar tangle $T$ such that $T(s_1, ..., s_k)) = x$ for some $s_i \in S$.

**Definition 3.3.2.** For a finite graph $G$ with vertex set $X$ and automorphism group $\mathrm{Aut}(G)$, let $\mathcal{P} = \mathcal{P}^{Aut(G)}$ which is the group action planar algebra denoted in 3.1.4.

**Definition 3.3.3.** Let $\mathcal{A}^{\mathrm{Aut}(G)}$ be the algebra generated by $P_2^{\mathrm{Aut}(G)}$. In [10], $P_2^{\mathrm{Aut}(G)}$ is called the **2-box space**, so $A^{\mathrm{Aut}(G)}$ is the algebra generated by the 2-box space.

**Definition 3.3.4.** [10, Definition 6.1.5] A finite graph $G$ has the **generating property** if $\mathcal{P}$ is generated by its 2-box space, meaning $\mathcal{A}^{\mathrm{Aut}(G)} = \mathcal{P}$.

For the graphs we will consider, it is known [10, Corollary 6.2.4] that the generating property exists for a graph if and only if that graph has no quantum symmetry. This is based on associating other algebraic objects known as orbital algebras to these graphs, but this will not be considered in this paper. In this instance, a key simplification can be made. Recall the definition of a distance-transitive graph.

**Definition 3.3.5.** A graph $G$ is **distance-transitive** if for any two vertices $u, v \in V(G)$ distance $a$ apart and any other pair of vertices $z, w \in V(G)$ also distance $a$ apart, there exists a $\sigma \in \mathrm{Aut}(G)$ such that $\sigma(u) = z$ and $\sigma(v) = w$.

Intuitively, this means that for any pair of vertices at a certain distance apart, there exists a graph automorphism that maps these two vertices to any other two vertices that same distance apart. The graphs we will look at will all be distance-transitive, which means we may use [10, Example 2.4.13] to say the following:

*Remark* 3.3.6. [10, Example 6.3.1] For a distance-transitive graph $G$, $G$ has the generating property if and only if it has no quantum symmetry.

This means Remark 3.3.6 and Definition 3.3.4 can be used to say the following.

*Remark* 3.3.7. For a distance-transitive graph $G$, if $\mathcal{P}$ is generated by its 2-box space (meaning $\mathcal{A}^{\mathrm{Aut}(G)} = \mathcal{P}$), then the graph has no quantum symmetry.

This is a method of showing a graph does or does not have quantum symmetry that does not involve directly manipulating the $u_{ij}$ generators of the algebra associated with the quantum automorphism group. This method was used to show that the distance-transitive graph known as the Higman-Sims graph $HS$ has quantum symmetry [10, Theorem 6.3.3].

The paper showed that the dimension of $\mathcal{A}^{\mathrm{Aut}(HS)}$ is not equal to the dimension of $\mathcal{P}^{\mathrm{Aut}(HS)}$, so the algebras are not equal and thus the graph does not have the generating property.

Furthermore, a different proof has been given that the Petersen graph $PS$ has no quantum symmetry by showing that $\mathcal{P}^{\mathrm{Aut}(PS)}$ is generated by its 2-box space [8, Theorem 3.1.9]. This proof was used in service of studying subfactor algebras, but the method used can be extended to look at other distance-transitive graphs and judge their quantum symmetry or lack thereof.

## 3.4 The 5-Cycle

In this section we will adapt the method used to prove that the 5-cycle, or $C_5$, has no quantum symmetry. We modify the method used in [8, Theorem 3.1.9] but do it for a different example.

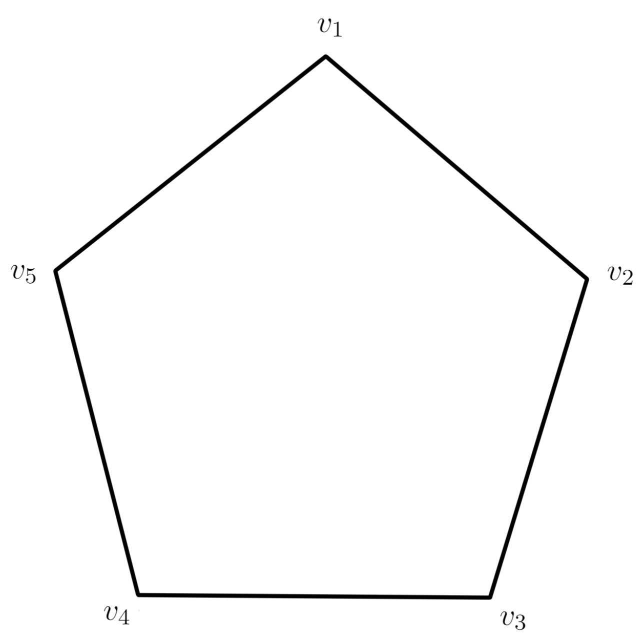



Figure 3.7: A drawing of $C_5$

Let the vertex set of $C_5$ be $V = \{v_1, v_2, v_3, v_4, v_5\}$. For any $i, j \in \{1, 2, 3, 4, 5\}$, $v_iv_j$ is an edge if and only if $i = j \pm 1 \pmod 5$.

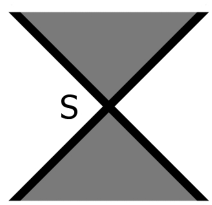
It is known that the automorphism group of $C_5$ is $D_5 \subseteq S_5$, the dihedral group of order 10, and this group acts on $V$ in the way defined below for a permutation $\sigma \in D_5$.

$$\sigma(v_i) = v_{\sigma(i)} \text{for any } v_i \in V.$$

Therefore, we let $\mathcal{P}^D$ be the group action planar algebra that is the fixed-point algebra of this group action as defined in 3.1.4.

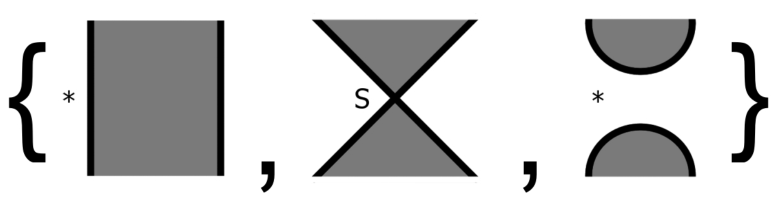
**Definition 3.4.1.** Let $S$ be $\chi_{[u \otimes v]}$ where $u, v \in V$ and $uv$ is an edge of $C_5$. This is the characteristic function of the orbit of $u \otimes v$ under the automorphism group of $C_5$. $S \in \mathcal{P}_2^D$ can be depicted by the following drawing

$$S = \chi_{[u \otimes v]} =$$




We note that an element in $P_k^D$ is the characteristic function of the orbit of a $k$-fold tensor product of vertices in $C_5$. This means an element can be depicted as a tangle with $2k$-boundary points on the top and bottom each, with the filled in regions representing the vertices in the tensor product and the structure of the tangle depicting their relationship. This is what we have done in Definition 3.4.1 to show the tangle $S$.

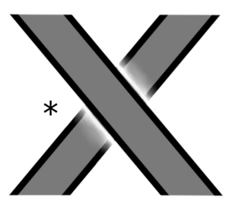
**Proposition 3.4.2.** *[8, Prop 3.1.6] The 2-box space $\mathcal{P}_{2,+}^D$ has the following basis*



*Proof.* Let $v, u, w \in V$ such that $u \neq v \neq w$, $uv$ is an edge of the 5-cycle, and $vw$ is not an edge of the 5-cycle. Now note that $\mathcal{P}^D_{2,+}$ has a basis of $\left\{\chi_{[v\otimes v]}, \chi_{[v\otimes u]}, \chi_{[v\otimes w]}\right\}$. This is because there are only three possibilities for the orbits of a 2-fold tensor product of vertices of $C_5$. The two vertices are either the same, unequal and adjacent to each other, or unequal and non-adjacent to each other.

The first drawing, in the image of Prop 3.4.2 denotes a characteristic function of an orbit with two of the same vertices, so $\chi_{[v\otimes v]}$. The second drawing denotes a characteristic function of an orbit with two different vertices related by an edge, so it is $\chi_{[v\otimes u]} = S$. Finally, the third drawing denotes any pair of vertices. Thus, $\mathcal{P}^D_{2,+}$ can be generated by the three elements in the image. □

**Definition 3.4.3.** [8, Def 3.1.7] The element $R \in P_{4,+}$ is equal to $\chi_{[u\otimes v\otimes v\otimes u]}$ where $u, v \in V$ are arbitrary. A drawing of $R$ is given below.



The element $R$ can be thought of as something that switches two vertices as seen in the drawing. Specifically, if $x = \chi_{[a\otimes b]}$, then $R(x) = \chi_{[b\otimes a]}$. This means we can write $R$ as the characteristic function of a 4-fold tensor product of vertices in $C_5$ where the second two terms in the tensor product are the reverse of the first two.

Along with the element $S$ from Definition 3.4.1, which connects two adjacent vertices, we may show a useful statement about $\mathcal{P}^D$.

**Theorem 3.4.4.** *[8, Theorem 3.1.8]*

*$\mathcal{P}^D$ is generated by $\{S, R\}$.*

*Proof.* Let $A$ be the algebra generated by $\{S, R\}$. We show that if the characteristic function of any arbitrary orbit in $\mathcal{P}^D$ can be generated by elements in $A$. We will build up to this by using other intermediary tangles in $A$.

First, we will use the 5-cycle to create the element $X$ in $A$. To do this, consider the labelled drawing below of the 5-cycle with all vertices put on an outer circle with a $*$ on the left most interval

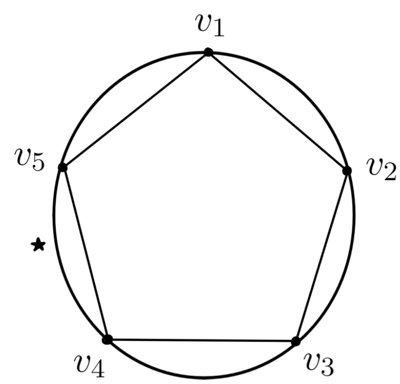



Figure 3.8

We can create an element in $\mathcal{P}_{5,+}$ by replacing every vertex in the diagram above with the shape at the top of Figure 3.9, since each vertex is of degree 2. Also, if there is an edge between two vertices, it will be replaced by the element $S$.

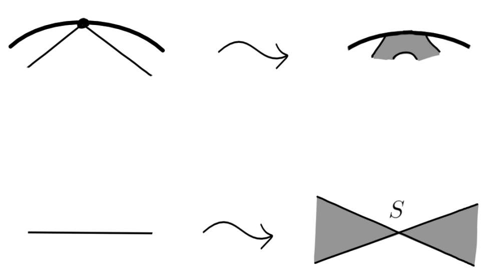



Figure 3.9

As a side-note, in Ren's proof for the Petersen graph in [8], overlapping vertices were replaced by the element $R$, but since $C_5$ is planar, that step is not necessary in this instance.

This gives us the diagram below, an element $X$ in $\mathcal{P}_{5,+}$

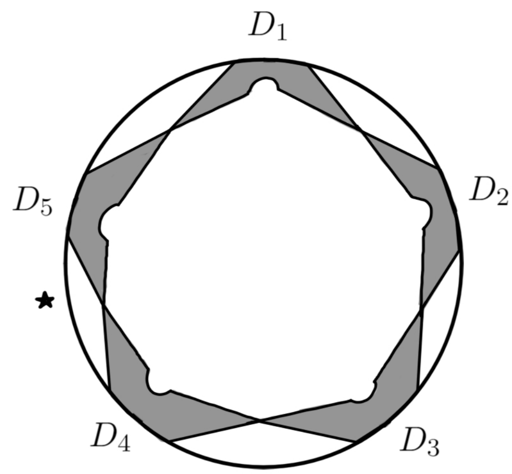



Figure 3.10: The Tangle $X$

The filled-in regions can be labelled clockwise as $D_1, ..., D_5$. Furthermore, let $\sigma$ be a function assigning each filled-in region to a unique vertex in $V$. Therefore, the map

$$X(\sigma(D_1) \otimes \sigma(D_2) \otimes \sigma(D_3) \otimes \sigma(D_4) \otimes \sigma(D_5))$$

is nonzero if and only if $\sigma(D_i) = \sigma(D_{j \pm 1 \pmod 5})$ for any $v_i v_j$ an edge of the 5-cycle. For the bijection $\sigma(D_i) = v_i$, this will always be the case.

Therefore, we may say that

$$X = \chi_{[v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5]} \in A$$

where $v_i \in V$ and $v_i \neq v_j$ for $i \neq j$.

We may say that $X \in A$ because $X$ was constructed using only the element $S$. Now, before continuing with the main proof, some additional tangles must be constructed.

Next, we will define the following tangle $\mathcal{O}_j^m \in A$ with $2m$ input and $2m$ output boundary points.

**Definition 3.4.5.** The $\mathcal{O}_j^m$ tangle has $2m$ input boundary points and $2m$ output boundary points, where the $j^{th}$ and $j+1^{st}$ regions are input into the tangle $R$. A drawing is given in

Figure 3.11.

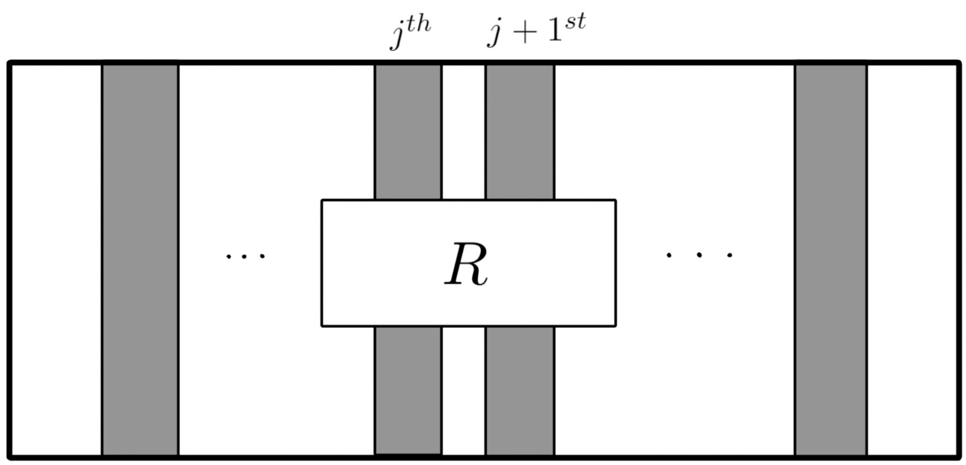



Figure 3.11: A general $\mathcal{O}_j^m$ tangle

From the construction of $R$ (See Def 3.4.3), we may see that applying this tangle to the characteristic function of the orbit of an $m$-fold tensor product of vertices yields almost the same product, but with the $j^{\text{th}}$ and $j+1^{\text{st}}$ entries transposed for $j \in \{1, ...m\}$ (if $j = m$ then it switches the first and last entry, so in effect $\mathcal{O}_j^m$ transposes the $j^{th}$ and $j+1 \pmod{m}^{st}$).

$$\mathcal{O}_j^m(\chi_{[v_1 \otimes v_2 \otimes \ldots v_j \otimes v_{j+1} \ldots \otimes v_m]}) = \chi_{[v_1 \otimes v_2 \otimes \ldots v_{j+1} \otimes v_j \ldots \otimes v_m]} \tag{3.2}$$

Furthermore, multiple $\mathcal{O}_j^m$ tangles can be stacked in order to permute the entries in the characteristic orbit of any $m$-fold tensor product. This means we have an action of $S_m$ on $P_{m,+}$. We may label $\mathcal{O}_\sigma^m$ be the tangle that permutes the entries in the characteristic function of the orbit of an $m$-fold tensor product using $\sigma \in S_m$. The effect of the tangle is summarized in the equation below.

$$\mathcal{O}_\sigma^m(\chi_{[v_1 \otimes v_2 \ldots \otimes v_m]}) = \chi_{[v_{\sigma(1)} \otimes v_{\sigma(2)} \ldots \otimes v_{\sigma(m)}]} \tag{3.3}$$

Now we will define another tangle $\mathcal{C}_j^m \in A$.

**Definition 3.4.6.** The $\mathcal{C}_j^m$ tangle has $2m$ input boundary points and $2(m-1)$ output boundary points, where the $j^{th}$ and $j+1^{st}$ regions are condensed into one region. A drawing is given in Figure 3.12.

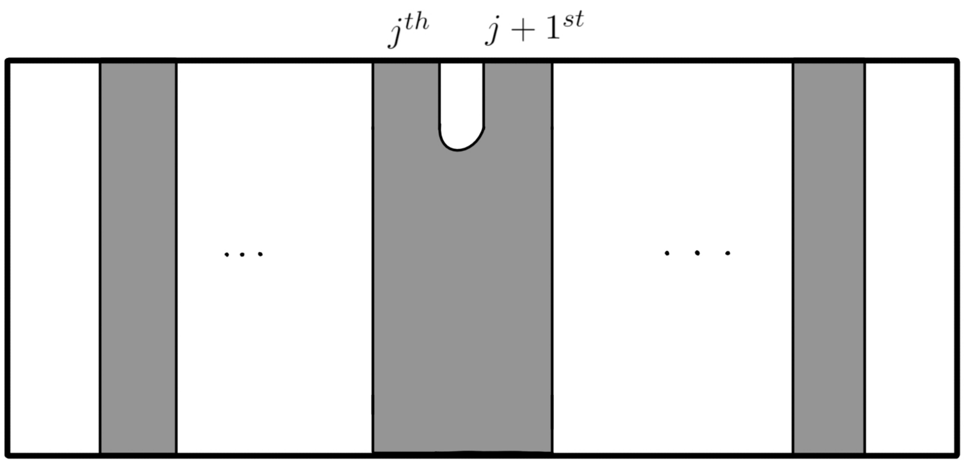



Figure 3.12: A general $\mathcal{C}_j^m$ tangle

Applying $\mathcal{C}_j^m$ to a characteristic function of the orbit of an $m$-fold tensor product shortens the tensor product into an $m-1$-fold one by condensing the $j^{th}$ and $j+1^{st}$ entries. In effect the function associated with this tangle is only nonzero when the $j^{th}$ and $j+1^{st}$ entries in the $m$-fold tensor product are the same, and the output is an $m-1$-fold tensor product where the duplicates are condensed into one entry.

$$\mathcal{C}_j^m(\chi_{[v_1\otimes v_2\otimes \ldots v_j \otimes v_{j+1}\ldots \otimes v_m]}) = \chi_{[v_1\otimes v_2\otimes \ldots v_j\otimes v_{j+2}\ldots\otimes v_m]} \tag{3.4}$$

Now, we show the result, that $\mathcal{P}^D$ is generated by $\{S, R\}$. Let

$$Q = \chi_{[v_{j_1}\otimes\ldots\otimes v_{j_m}]}$$

be the characteristic function of an arbitrary orbit, so $Q \in \mathcal{P}^D$ is arbitrary. Note that $j_1, \ldots j_m$ can take any values in $\{1, 2, 3, 4, 5\}$ at any order. For an example that can be followed for the rest of this proof, consider $E = \chi_{[v_4 \otimes v_3 \otimes v_2 \otimes v_2 \otimes v_5 \otimes v_4]}$.

First we will take $j_1, ...j_m$ and reorder them such that the indices are increasing in order. This can be done using some permutation $\alpha \in S_m$, which means we may use the tangle $\mathcal{O}_\alpha^m \in A$. Thus,

$$\mathcal{O}_\alpha^m(\chi_{[v_{j_1} \otimes ... \otimes v_{j_m}]}) = (\chi_{[v_{i_1} \otimes ... \otimes v_{i_m}]})$$

where $i_k = \alpha(j_k)$.

For $E = \chi_{[v_4 \otimes v_3 \otimes v_2 \otimes v_2 \otimes v_5 \otimes v_4]}$ and $\alpha$ being $(3\,1\,4\,2)(5\,6)$.

$$\mathcal{O}_\alpha^6(E) = \chi_{[v_2 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_4 \otimes v_5]} \tag{3.5}$$

Since, for an arbitrary $Q$, the order of the entries in the tensor product can be changed into an increasing order using a tangle in $A$, we only need to show that any $Q = \chi_{[v_{j_1} \otimes ... \otimes v_{j_m}]}$ with $j_1 \leq j_2 \leq ... \leq j_m$ is in $A$. We continue on by assuming $j_1...j_m$ are ordered as such.

Now, note that it is possible for duplicate entries to exist in the tensor product. Say for example $j_1 = j_2 = ... = j_{k_1} = t_1$; $j_{k_1+1} = j_{k_1+2} = ... = j_{k_2} = t_2$;..., $j_{k_{l-1}+1} = ... = j_m = t_l$ and $t_1 < t_2 < ... < t_l$ by the re-ordering that was just done.

Now we may use $\mathcal{C}_i^m \in A$ to condense any two adjacent indices $i$ and $i+1$ that are the same index. Multiple $\mathcal{C}$ tangles can be stacked to condense all the $j_1 = j_2 = ... = j_{k+1}$ into $t_1$, the $j_{k_1+1} = j_{k_1+2} = ... = j_{k_2}$ into $t_2$, and so on until

$$\mathcal{CC}(\chi_{[v_{j_1} \otimes ... \otimes v_{j_m}]}) = \chi_{[v_{t_1} \otimes ... \otimes v_{t_l}]} \in A$$

where $\mathcal{CC}$ is the tangle formed by stacking each of the previous $\mathcal{C}$ tangles and $t_1 < t_2 < ... < t_l$.

At this point, the only thing we need to show is that for arbitrary $t_1 < t_2 < ... < t_l$ with $t_i \in \{1, 2, 3, 4, 5\}$, $\chi_{[v_{t_1} \otimes ... \otimes v_{t_l}]}$ is in $A$.

For the example $E$, this amounts to showing

$$\chi_{[v_2 \otimes v_3 \otimes v_4 \otimes v_5]} \in A \tag{3.6}$$

Now we introduce the tangle $L_{t_1,...,t_l} \in A$. It is difficult to draw a general $L$ tangle since it is dependent on $t_1, ..., t_l$, so the example tangle for $E$ is shown in Figure 3.13 and the process for constructing it will be described.

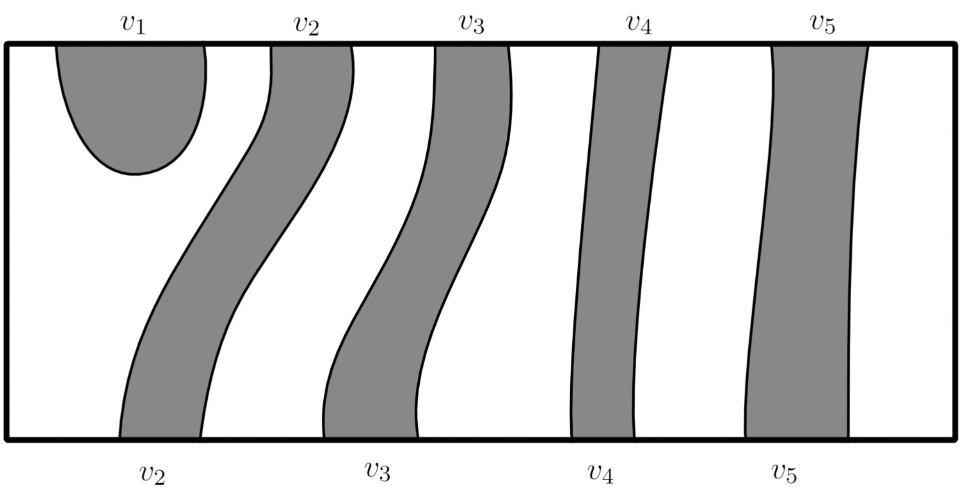



Figure 3.13: $L_{2,3,4,5}$ tangle for the example $E$.

The general process involves drawing a tangle where the characteristic function of the orbit of $v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5$ is the input and the characteristic function of the orbit of $v_{t_1} \otimes ... \otimes v_{t_l}$ is the output. This can be done by drawing a tangle with 10 input boundary points and the $2l$ output boundary points This amounts to a map going from $P_{5,+}$ to $P_{l,+}$ according to Definition 3.1.3.

So for $E$, we draw a box with 10 input boundary points and 8 output boundary points. Then, the input boundary points 3 and 4 are connected to the output boundary points 1 and 2, the input boundary points 5 and 6 are connected to the output boundary points 3 and 4, the input boundary points 7 and 8 are connected to the output boundary points 5 and 6, and the input boundary points 9 and 10 are connected to the output boundary points 7 and

8. Finally, input boundary points 1 and 2 are connected to each other, which can be thought of as removing $v_1$. This is because $v_1$ does not appear in the tensor product $\chi_{[v_2 \otimes v_3 \otimes v_4 \otimes v_5]}$

Thus, we have that

$$L_{2,3,4,5}(\chi_{[v_1 \otimes \ldots \otimes v_5]}) = \chi_{[v_2 \otimes v_3 \otimes v_4 \otimes v_5]}.$$

A general $L_{t_1,\ldots,t_l}$ tangle can be drawn by drawing an aforementioned box with 10 input boundary points labelled $i_1, i_2, \ldots, i_{10}$ and $2l$ output boundary points labelled from left to right as $o_1, o_2, \ldots, o_{2l}$. The input boundary points can be associated with the tensor product of $v_1 \otimes \ldots \otimes v_5$, by associating each $v_k$ with the shaded region between points $i_{2k}$ and $i_{2k-1}$. Likewise, the output boundary points can be associated with the tensor product of $v_{t_1} \otimes \ldots \otimes v_{t_1}$ by associating $v_{t_k}$ with the shaded region between points $o_{2k}$ and $o_{2k-1}$. Then, for each $k \in \{1, \ldots, 5\}$ when $v_k$ on the input boundary is equal to some $v_{t_p}$ on the output boundary, those corresponding regions are connected by joining a string between $i_{2k}$ and $o_{2p}$ and a string between $i_{2k-1}$ and $o_{2p-1}$. Then, for any $k \in \{1, \ldots, 5\}$ where $v_k$ on the input boundary is not equal to any $v_{t_p}$ on the output boundary, $i_{2k}$ is connected to $i_{2k-1}$. In effect, what this does is preserve the indices in the tensor product $v_1 \otimes \ldots \otimes v_5$ that appear in the tensor product $v_{t_1} \otimes \ldots \otimes v_{t_1}$, and remove any that do not appear in the latter product.

With all this described, the goal was to show that for arbitrary $t_1 < t_2 < \ldots < t_l$ with $t_i \in \{1, 2, 3, 4, 5\}$, $\chi_{[v_{t_1} \otimes \ldots \otimes v_{t_l}]}$ is in $A$. Using the corresponding $L_{t_1,\ldots,t_l}$ tangle,

$$L_{t_1,\ldots,t_l}(\chi_{[v_1 \otimes \ldots \otimes v_5]}) = \chi_{[v_{t_1} \otimes \ldots \otimes v_{t_l}]}.$$

Since we showed earlier that $X = \chi_{[v_1 \otimes \ldots \otimes v_5]} \in A$, we have shown that $\chi_{[v_{t_1} \otimes \ldots \otimes v_{t_l}]} \in A$. Therefore, any arbitrary $Q \in \mathcal{P}^D$ is in $A$. $\square$

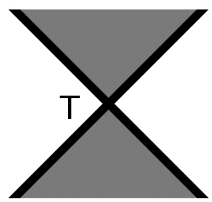
Now, Theorem 3.4.4 is useful because it shows that $\mathcal{P}^D$ is generated by $A$, so in order to show the 5-cycle has the generating property, it just remains to show that $A$ is generated by the 2-box space of $\mathcal{P}^D$. Since $S \in \mathcal{P}_{2,+}$ already (see Proposition 3.4.2), all we must show is that $R$ is generated by $\mathcal{P}_{2,+}$. This will be done by following the method of Ren in [8, Theorem 3.1.9].

**Theorem 3.4.7.** *$\mathcal{P}^D$ is generated by its 2-box space.*

*Proof.* Let $\mathcal{U}$ be the algebra generated by $\mathcal{P}_{2,+}$. We show that $R \in \mathcal{U}$.

Let $a, b, u, v, w$ be vertices of $C_5$ such that $u \neq v \neq w$, $uv$ is an edge of the 5-cycle and $vw$ is not an edge of the $C_5$. Now, we define an element $T \in \mathcal{P}^D_{2,+}$.

**Definition 3.4.8.** Let $T = \chi_{[u \otimes w]}$ such that $u \neq w$ and $uw$ is not an edge of the 5-cycle.




Now, consider $R$ from Definition 3.4.3, which can be considered as the characteristic orbit of a 4-fold tensor product where the last two entries are the first two but in reverse order. Thus, we let $R = \chi_{[a \otimes b \otimes b \otimes a]}$ for any two vertices $a, b$ of $C_5$.

From Proposition 3.4.2, we know that $\chi_{[a \otimes b]} = \chi_{[u \otimes u]} + \chi_{[u \otimes v]} + \chi_{[u \otimes w]}$ where $a, b$ are any vertices in $V$. Again this is intuitive for $C_5$ because any two vertices are either equal, unequal but adjacent, or unequal and nonadjacent. Therefore, for any two vertices $a, b$ whose order is switched by $R$, we have the following equation.

$$R = \chi_{[a \otimes b \otimes b \otimes a]} = \chi_{[a \otimes a \otimes a \otimes a]} + \chi_{[u \otimes v \otimes v \otimes u]} + \chi_{[u \otimes w \otimes w \otimes u]} \tag{3.7}$$

Intuitively, for a 4-fold tensor product of vertices where the last two entries are the first two but in reverse order, these two vertices can have only the relations described in the previous paragraph. This yields the sum in Equation 3.7. which is summarized in Figure 3.14.

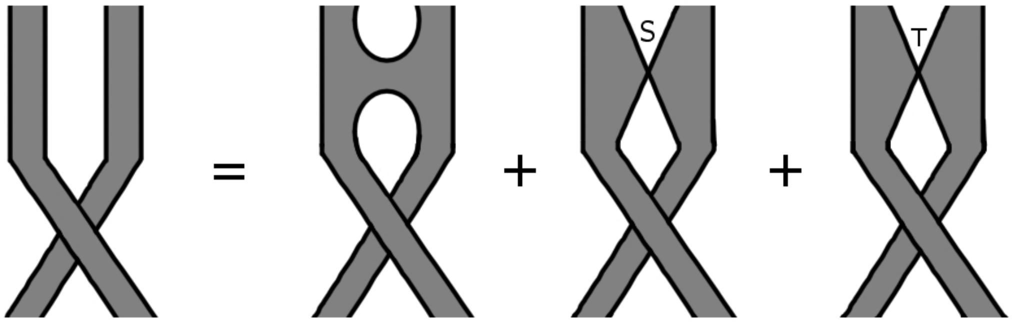



Figure 3.14: $R$ as a sum of three other tangles

All that remains to be shown is that each term on the righthand side of Equation 3.7 is in $\mathcal{U}$. The first term is in $\mathcal{U}$ [1, Example 5.2] due to some known facts about Temperley-Lieb diagrams.

Therefore, now we show that the second term is in $\mathcal{U}$. Consider the following element $H \in \mathcal{U}$.

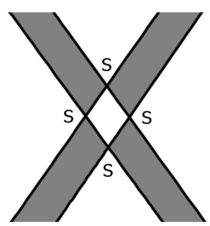



Figure 3.15: $H = \chi_{[v_{i_1} \otimes v_{i_2} \otimes v_{i_4} \otimes v_{i_3}]}$

For $H$ to be nonzero, $v_{i_j} v_{i_{j+1 \pmod 4}}$ is an edge in the 5-cycle for all $j \in \{1, 2, 3, 4\}$. This means each $v_{i_j}$ cannot be unique, since the 5-cycle has no quadrangle, meaning it has no subgraph isomorphic to a square. Thus, either $v_{i_1} = v_{i_3}$ or $v_{i_2} = v_{i_4}$, or both.

Using the labelling of $C_5$ in Figure 3.8, we have

$$H = \chi_{[v_{i_1} \otimes v_{i_2} \otimes v_{i_4} \otimes v_{i_3}]} = \chi_{[v_1 \otimes v_2 \otimes v_5 \otimes v_1]} + \chi_{[v_2 \otimes v_1 \otimes v_1 \otimes v_5]} + \chi_{[v_1 \otimes v_2 \otimes v_2 \otimes v_1]} \tag{3.8}$$

Now consider the elements $I, J \in \mathcal{U}$.

**Definition 3.4.9.** Let $I = \chi_{[v_1 \otimes v_2 \otimes v_2 \otimes v_1]} + \chi_{[v_2 \otimes v_1 \otimes v_1 \otimes v_5]}$

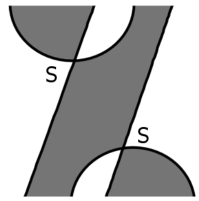



Figure 3.16: I tangle

and $J = \chi_{[v_1 \otimes v_2 \otimes v_2 \otimes v_1]} + \chi_{[v_1 \otimes v_2 \otimes v_5 \otimes v_1]}$

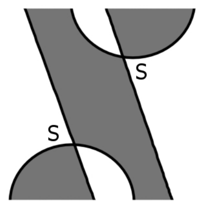



Figure 3.17: J tangle

Since the second term in Equation 3.7 is $\chi_{[v_1 \otimes v_2 \otimes v_2 \otimes v_1]}$, it follows from Definitions 3.4.9 and 3.15 that

$$I + J - H = \chi_{[v_1 \otimes v_2 \otimes v_2 \otimes v_1]}$$

Therefore, the second term in 3.7 is in $\mathcal{U}$.

Finally, all that is left to show is that the third term in 3.7 is in $\mathcal{U}$. Thus, we show that $\chi_{[u \otimes w \otimes w \otimes u]} \in \mathcal{U}$ for $u, w$ unequal and nonadjacent vertices in $V$. To do this, we consider the tangle $F$ shown below in Figure 3.18 shaded regions labelled $B_i$.

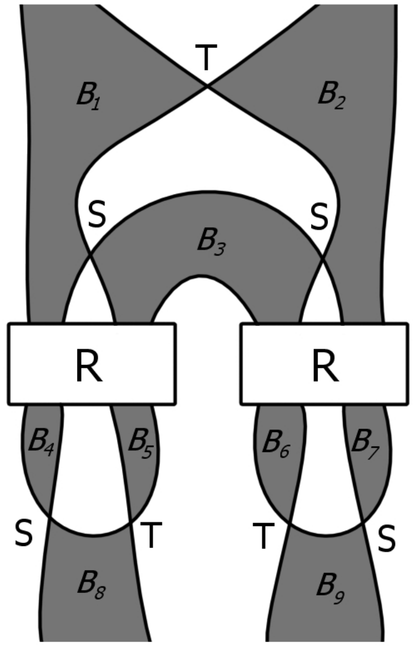



Figure 3.18: The tangle $F$

Note that $F \in \mathcal{U}$ since it is constructed using only the the second term in the equation, which we just showed is in $\mathcal{U}$, and $T$. We show that $F$ is equivalent to the third term in the equation.

Let $F = \chi_{[a \otimes b \otimes c \otimes d]}$ for $a, b, c, d$ vertices of $C_5$. Let $\sigma$ be a function from the shaded regions to $V$ defined by the labelling in Figure 3.18. Since $B_1$ and $B_2$ are connected by $T$, we know $\sigma(B_1)$ is not adjacent to $\sigma(B_2)$. Thus, without loss of generality, let $\sigma(B_1) = v_1$ and $\sigma(B_2) = v_3$. Then, since both $B_1$ and $B_2$ are connected to $B_3$ by $S$, $\sigma(B_3)$ is adjacent to both $v_1$ and $v_3$, so $\sigma(B_3) = v_2$.

Continuing on, we note that $\sigma(B_4) = \sigma(B_3)$, $\sigma(B_5) = \sigma(B_1)$, $\sigma(B_6) = \sigma(B_2)$, and $\sigma(B_7) = \sigma(B_3)$ by the definition of $R$. Therefore, $\sigma(B_4) = \sigma(B_7) = v_2$, $\sigma(B_5) = v_1$, and $\sigma(B_6) = v_3$. Finally, since $B_4$ is connected to $B_8$ by $S$, and $B_5$ is also connected to $B_8$ but by $T$, this means $\sigma(B_8)$ is adjacent to $v_2$ and not adjacent to $v_1$, so $\sigma(B_8) = v_3$. By a similar argument, $B_6$ is connected to $B_9$ by $T$, and $B_7$ is also connected to $B_9$ but by $S$, so $\sigma(B_9)$ is not adjacent to $v_3$ and adjacent to $v_2$, meaning $\sigma(B_9) = v_1$

Therefore, we have shown that

$$F = \chi_{[\sigma(B_1)\otimes(B_2)\otimes\sigma(B_8)\otimes\sigma(B_9)]} = \chi_{[v_1\otimes v_3\otimes v_3\otimes v_1]}$$

which is equivalent to the third term in Equation 3.7.

Thus, all three terms in Equation 3.7 are in $\mathcal{U}$, which means that $R \in \mathcal{U}$. Therefore, $A$ is generated by the 2-box space, and since $\mathcal{P}^D$ is generated $A$, the result is shown. $\square$

Overall, in this section we have shown that the 5-cycle has the generating property, so by Remark 3.3.7, the 5-cycle does not have quantum symmetry.

# Chapter 4

# Conclusions

The previous section that adapted the proof of the Petersen graph in [8] was contingent upon specific properties of the Petersen Graph and the 5-cycle. For any pair of vertices, the two vertices are either at distance 0 from each other (the same), distance 1 from each other (adjacent), or distance 2 from each other (nonadjacent). This means there are only three orbits on pairs under the automorphism group of each respective graph. Therefore we may say that the basis of the 2-box space has the three elements outlined in Proposition 3.4.2.

Therefore, this method can be extended for graphs that are similar to these two examples, namely other distance transitive graphs since Remark 3.3.7 shows us that we may use the properties of the 2-box space to give a result about the quantum symmetry of a graph. Furthermore if a graph is defined in a way that does not match the specific requirements of the proof in Section 3.4, say for example it does contain a quadrangle, it could be possible to adapt the proof by drawing different tangles that help simplify the problem. This would not be a trivial task, but it is another method that can be used to determine the presence or absence quantum symmetry of a graph. Although manipulating generators in a $C^*$-algebra is a viable and useful method to show quantum symmetry, if the algebra is too complex, manipulating planar algebras and tangles could provide a more visual perspective on the problem.

# Acknowledgments

I would like to thank Professor Pierre Clare for his continual support and guidance as my mentor throughout both this project and all other aspects of mathematics. I'd also like to thank Professor Eric Swartz for his indispensable feedback and guidance on this project. Without both of their patient instruction, it would not have been possible to grasp and apply the mathematics present in this project. I would also like to thank my good friend Samantha Phillips who has worked on this material in a parallel fashion and has been both a great confidant and an excellent sounding board to talk about ideas and concepts.

I also would like to thank Professor Gregory Conradi-Smith for his very kind support of my intellectual curiosity and insightful guidance regarding academics as a whole.

I will always be deeply grateful to all of my Professors and teachers who have given me the tools to come into this field and to grow overall as a student.

And finally, I must acknowledge my thanks to all of my friends who endlessly supported me and forced me to take breaks when I was overworked.

# Bibliography


[1] Teodor Banica. "Quantum automorphism groups of homogeneous graphs". In: *Journal of Functional Analysis* 224.2 (2005), pp. 243–280. ISSN: 0022-1236. DOI: https://doi.org/10.1016/j.jfa.2004.11.002. URL: https://www.sciencedirect.com/science/article/pii/S0022123604004070.

[2] Teodor Banica. *Quantum permutations and quantum reflections*. 2020.

[3] Teodor Banica and Julien Bichon. "Quantum automorphism groups of vertex-transitive graphs of order $\leq 11$". In: *J. Algebraic Combin.* 26.1 (2007), pp. 83–105. ISSN: 0925-9899. DOI: 10.1007/s10801-006-0049-9. URL: https://doi.org/10.1007/s10801-006-0049-9.

[4] Teodor Banica, Julien Bichon, and Benoıt Collins. "Quantum permutation groups: a survey". In: *Noncommutative harmonic analysis with applications to probability*. Vol. 78. Banach Center Publ. Polish Acad. Sci. Inst. Math., Warsaw, 2007, pp. 13–34. DOI: 10.4064/bc78-0-1. URL: https://doi.org/10.4064/bc78-0-1.

[5] Brian Curtin. "Some planar algebras related to graphs". In: *Pacific J. Math.* 209.2 (2003), pp. 231–248. ISSN: 0030-8730. DOI: 10.2140/pjm.2003.209.231. URL: https://doi.org/10.2140/pjm.2003.209.231.

[6] Jacques Dixmier. *$C^*$-algebras*. Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977, pp. xiii+492. ISBN: 0-7204-0762-1.

[7] Vaughan F. R. Jones. *Planar algebras, I.* 1999. eprint: arXiv:math/9909027.

[8] Yunxiang Ren. "Skein theory of planar algebras and some applications". PhD thesis. 2017.

[9] Simon Schmidt. "On the quantum symmetry of distance-transitive graphs". In: *Advances in Mathematics* 368 (July 2020), p. 107150. ISSN: 0001-8708. DOI: 10.1016/j.aim.2020.107150. URL: http://dx.doi.org/10.1016/j.aim.2020.107150.

[10] Simon Schmidt. "Quantum automorphism groups of finite graphs". In: (2020).

[11] Simon Schmidt. "The Petersen graph has no quantum symmetry". In: *Bulletin of the London Mathematical Society* 50.3 (Apr. 2018), pp. 395–400. ISSN: 0024-6093. DOI: 10.1112/blms.12154. URL: http://dx.doi.org/10.1112/blms.12154.

[12] Shuzhou Wang. "Quantum symmetry groups of finite spaces". In: *Comm. Math. Phys.* 195.1 (1998), pp. 195–211. ISSN: 0010-3616. DOI: 10.1007/s002200050385. URL: https://doi.org/10.1007/s002200050385.

[13] S. L. Woronowicz. "Compact matrix pseudogroups". In: *Comm. Math. Phys.* 111.4 (1987), pp. 613–665. ISSN: 0010-3616. URL: http://projecteuclid.org/euclid.cmp/1104159726.