2013

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Recommended Citation

Johnson, C. R., Nuckols, J., & Spicer, C. (2013). The implicit construction of multiplicity lists for classes of trees and verification of some conjectures. Linear Algebra and its Applications, 438(5), 1990-2003.

Linear Algebra and its Applications 438 (2013) 1990–2003

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

The implicit construction of multiplicity lists for classes of trees and verification of some conjectures ☆

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ARTICLE INFO

Article history:
Received 14 March 2011
Accepted 7 November 2012
Available online 8 December 2012

Submitted by H. Schneider

AMS classification:
05C50
15A18

Keywords:
Eigenvalues
Trees
Implicit function theorem construction
Multiplicities
Degree conjecture

ABSTRACT

For the problem of understanding what multiplicities are possible for eigenvalues among real symmetric matrices with a given graph, constructing matrices with conjectured multiplicities is generally more difficult than finding constraining conditions. Here, the implicit function theorem method for constructing matrices with a given graph and given multiplicity list is refined and extended. In particular, the breadth of known circumstances in which the Jacobian is nonsingular is increased. This allows characterization of all multiplicity lists for binary, diametric, depth one trees. In addition the degree conjecture and a conjecture about the minimum number of multiplicities equal to 1 is proven for diametric trees. Finally, an intriguing conjecture about the eigenvalues of a matrix whose graph is a path and its submatrices is given, along with a discussion of some ides that would support a proof of the degree conjecture and the minimum number of 1's conjecture, in general.

1. Introduction

Let G be a simple, undirected graph on n vertices, and let $G(A)$ denote the graph of a real symmetric matrix A. We denote the set of real symmetric matrices A whose graph is G by

$$\mathcal{S}(G) = \{A = A^T \in M_n(\mathbb{R}) : \textit{the graph of } A \textit{ is } G\}.$$

☆ This work was supported in part by NSF Grant DMS-0751964.
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http://dx.doi.org/10.1016/j.laa.2012.11.010

The graph places no restrictions on the diagonal entries of A, other than reality. We denote the multiplicity of $\lambda \in \sigma(A)$ by $m_A(\lambda)$. We are interested in the possible lists, $L(G)$, of multiplicities of the distinct eigenvalues of matrices in $\mathcal{S}(G)$. We denote such a list as

$$m_1 \geqslant m_2 \geqslant \cdots \geqslant m_k$$

with $\sum m_i = n$.

There are many known restrictions on the multiplicity lists in $L(G)$ [3–6], as well as the interlacing inequalities. However, verifying that a matrix with a given multiplicity list actually occurs in $\mathcal{S}(G)$ is relatively difficult. Use of the implicit function theorem (IFT) to construct matrices with desired lists was pioneered in [2]. Our purpose here is to extend and refine the use of the IFT for this purpose when G is a tree. This allows us to verify certain conjectures and contribute insights into others. In addition, interesting question are raised.

Given a set $S \subset N = \{1, 2, \ldots, n\}$, we denote the principal submatrix of an n-by-n matrix A, lying in rows and columns S (resulting from the deletion of rows and columns S), by $A[S]$ $(A(S))$. In case $S = \{i\}$, we abbreviate $A(\{i\})$ by $A(i)$. When clear from the context, we will, for convenience, talk about the eigenvalues of a matrix or a submatrix as though they were eigenvalues of a corresponding graph or subgraph.

2. Background

For a tree T, $P(T)$, the *path cover number* is the minimum number of vertex disjoint paths of T that cover all the vertices of T. $M(T)$ is the maximum multiplicity of an eigenvalue in any list in $L(T)$. Then, it is known [3] that $M(T) = P(T)$; this common value is also the same as the maximum of the difference of the number of paths remaining and the number of vertices taken from T so as to induce only paths. There are always at least two eigenvalues of multiplicity 1 (the largest and smallest eigenvalue) in any matrix in $\mathcal{S}(T)$. Moreover, there are at least as many distinct eigenvalues for any matrix as the diameter (longest induced path) of T measured in vertices [4]. For some trees more than two multiplicity 1 eigenvalues are required. No characterization of this minimum number of 1's is known.

A key fact that governs multiplicities in $L(T)$ goes back to Parter [7] and is most fully discussed in [6]. If $m_A(\lambda)m_{A(i)}(\lambda) > 0$ for some i, then there is a j such that $m_{A(j)}(\lambda) = m_A(\lambda) + 1$. Since $m_{A(i)}(\lambda) \geqslant m_A(\lambda) - 1$, this implies that for any eigenvalue of multiplicity at least 2 in a real symmetric matrix whose graph is a tree, there is at least one vertex whose removal *increases* the multiplicity of λ. Moreover, in this event, there is such a vertex of degree at least three, such that the eigenvalue occurs in at least three branches. Such a "Parter" vertex is characterized by having a neighbor such that the multiplicity of the eigenvalue decreases in that branch when the neighbor is removed from its branch.

3. The implicit function theorem technique

The version of the IFT used here is the same as was presented in [2].

The general idea behind the implicit construction of multiplicities for a tree, T, given a set of eigenvalue constraints (or conditions on the determinants of submatrices), is to find a subgraph (in terms of edge containment) of T that satisfies the eigenvalue constraints for easily constructed numerical values and that serves as our initial point in the application of the IFT. Then, using the IFT, we perturb the entries of the matrix corresponding to the removed edges from zero to non-zero values. We call the entries that we manipulate to non-zero values *manual entries*, and the entries altered via the application of the IFT *implicit entries*. The difficult part of applying the IFT is making certain that the Jacobian with respect to the implicit entries is nonsingular.

To illustrate consider the following example:

Let $T = \{\{1, 2, 3, 4, 5, 6, 7\}; \{(1, 2), (2, 3), (2, 4), (4, 5), (5, 6), (5, 7)\}\}$

Our objective is to find $B \in \mathcal{S}(T)$ such that B has multiplicities 2, 2, 1, 1, 1. The following determinant conditions imply $m_B(\lambda) = m_B(\mu) = 2$.

$$b_{11} - \lambda = 0 \tag{1}$$
$$b_{33} - \lambda = 0 \tag{2}$$
$$\det(B[4, 5, 6, 7] - \lambda I_4) = 0 \tag{3}$$
$$\det(B[1, 2, 3, 4] - \mu I_4) = 0 \tag{4}$$
$$b_{66} - \mu = 0 \tag{5}$$
$$b_{77} - \mu = 0 \tag{6}$$

Note that the above conditions specify certain entries, for example $b_{11} = \lambda$. We can then think of B as a matrix-valued function of the variables $x_1, x_2, b_{12}, b_{23}, b_{24}, b_{45}, b_{56}, b_{57}$. Letting a be neither λ nor μ:

$$B = \begin{pmatrix} \lambda & b_{12} & 0 & 0 & 0 & 0 & 0 \\ b_{12} & x_1 & b_{23} & b_{24} & 0 & 0 & 0 \\ 0 & b_{23} & \lambda & 0 & 0 & 0 & 0 \\ 0 & b_{24} & 0 & x_2 & b_{45} & 0 & 0 \\ 0 & 0 & 0 & b_{45} & a & b_{56} & b_{57} \\ 0 & 0 & 0 & 0 & b_{56} & \mu & 0 \\ 0 & 0 & 0 & 0 & b_{57} & 0 & \mu \end{pmatrix}$$

If all $b_{ij} \neq 0$, then we have $B \in \mathcal{S}(T)$. Since conditions (1), (2), (5), and (6) hold for all choices of b_{ij}, let

$$F = (\det(B[4, 5, 6, 7] - \lambda I_4), \det(B[1, 2, 3, 4] - \mu I_4))$$

The Jacobian of F is

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & \det(B[5, 6, 7] - \lambda I_3) \\ \det(B[1, 3, 4] - \mu I_3) & 0 \end{pmatrix}$$

We now want to find a matrix $B^{(0)}$ whose graph is a subgraph of T, and is such that $F(B^{(0)}) = 0$ and $\det(J(B^{(0)})) \neq 0$. We see that $B^{(0)} = diag(\lambda, \mu, \lambda, \lambda, a, \mu, \mu)$ works, since

$$\begin{vmatrix} 0 & (a-\lambda)(\mu-\lambda)^2 \\ (\lambda-\mu)^3 & 0 \end{vmatrix} = (a-\lambda)(\mu-\lambda)^5 \neq 0$$

because a, λ, μ are distinct.

Since the determinant is a polynomial, and hence continuously differentiable, we can apply the IFT. We choose $y = (b_{12}, b_{23}, b_{24}, b_{45}, b_{56}, b_{57})$ to be sufficiently small, such that F is satisfied for some pair (x_1, x_2).

Thus the matrix $B((x_1, x_2), y) \in \mathcal{S}(T)$, and has eigenvalues λ and μ, each with multiplicity 2.

Although in the example checking the nonsingularity of the Jacobian was simple, in general it is a difficult task. To facilitate this check, we have the following lemmas from [2]:

Lemma 1. *Let T be a tree and $F = (f_k)$ be a vector of r determinant conditions with r implicit entries identified. Suppose that a symmetric matrix $A^{(0)}$, whose graph is a subgraph of T, is the direct sum of irreducible matrices $A_1^{(0)}, A_2^{(0)}, \ldots, A_p^{(0)}$. Let $J(A^{(0)})$ be the Jacobian matrix of F with respect to the implicit entries, evaluated at $A^{(0)}$, and suppose*

(i) *every off-diagonal implicit entry in* $A^{(0)}$ *has a non-zero value,*
(ii) *for every* $k = 1, \ldots, r$, $f_k(A_\ell^{(0)}) = 0$ *for precisely one* $\ell \in \{1, \ldots, p\}$,
(iii) *for every* $\ell = 1, \ldots, p$, *the columns of* $J(A^{(0)})$ *associated with the implicit entries of* $A_\ell^{(0)}$ *are linearly independent,*

then $J(A^{(0)})$ *is nonsingular.*

Lemma 2. *Let* $F = (f_k)$ *be a vector of r determinant conditions, and let* $A^{(0)}$ *be a diagonal matrix. Suppose that for every* $k = 1, \ldots, r$, $f_k(A^{(0)}[\ell]) = 0$ *for precisely one* $\ell \in \{1, \ldots, n\}$. *Take* $a_{\ell\ell}$ *to be an implicit entry if and only if* $f_k(A^{(0)}[\ell]) = 0$ *for some k. If there are then r implicit entries, then the Jacobian of F with respect to the implicit entries evaluated at* $A^{(0)}$ *is nonsingular.*

Also, in our example, we were lucky enough to find an initial matrix that was completely edgeless. In general, we cannot be sure that this will be the case. Indeed, in some cases the only way our initial matrix can satisfy the eigenvalue constraints is if it contains a path. We will concern ourselves with the case in which the graph of the initial matrix has non-adjacent edges. Note that these correspond to 2-by-2 direct summands in the initial matrix. In the case in which a 2-by-2 direct summand is required, we say that our initial matrix is of degree 2.

As before, the difficulty in applying the IFT lies in determining whether the Jacobian is nonsingular. The following lemma is useful in this regard.

Lemma 3. *Let* $F = (f_k)$ *be a vector of r determinant conditions, and let* $A^{(0)}$ *be the direct sum of 1-by-1 and 2-by-2 symmetric irreducible matrices* $A_1^{(0)}, \ldots, A_p^{(0)}$. *Suppose that for every* $k = 1, \ldots, r$, $f_k(A_\ell^{(0)}) = 0$ *for precisely one* $\ell \in \{1, \ldots, p\}$. *If*

(i) *both diagonal entries,* a_{m_1} *and* a_{m_2}, *and the off-diagonal entry,* b_m, *of each 2-by-2 direct summand of* $A^{(0)}$ *are implicit,*
(ii) *for any 2-by-2 direct summand* $A_m^{(0)} = A^{(0)}[m_1, m_2]$, *there is at least one determinant condition* $f_i = det(A[S_i] - \lambda_i I)$ *such that* $m_1 \in S_i$ *and* $m_2 \notin S_i$, *and there are at least two determinant conditions* $f_j = det(A[S_j] - \lambda_j I)$ *and* $f_k = det(A[S_k] - \lambda_k I)$ *such that* $\lambda_j \neq \lambda_k$ *and* $m_1, m_2 \in S_j$ *and* $m_1, m_2 \in S_k$,
(iii) $\lambda_i, \lambda_j, \lambda_k$ *are not eigenvalues of* $A^{(0)}[S_i \backslash \{m_i\}], A^{(0)}[S_j \backslash \{m_1, m_2\}], A^{(0)}[S_k \backslash \{m_1, m_2\}]$ *respectively,*
(iv) *there are r implicit entries total,*

then the Jacobian of F with respect to the implicit entries evaluated at $A^{(0)}$ *is nonsingular.*

Proof. We apply Lemmas 1 and 2. From Lemma 2 we have that if $A_\ell^{(0)}$ is a 1-by-1 direct summand, then the columns of $J(A^{(0)})$ associated with the implicit entries in $A^{(0)}$ (if any) are linearly independent. So, we only need to check that the columns of $J(A^{(0)})$ associated with the implicit entries in any 2-by-2 direct summand, $A_m^{(0)}$, are linearly independent. To do so, let f_i, f_j, f_k satisfy condition (ii). We then consider the following submatrix of the Jacobian with respect to the diagonal entries a_{m_1}, a_{m_2} and the off-diagonal entry b_m:

$$\begin{pmatrix} \frac{\partial f_i}{\partial a_{m_1}} & \frac{\partial f_i}{\partial a_{m_2}} & \frac{\partial f_i}{\partial b_m} \\ \frac{\partial f_j}{\partial a_{m_1}} & \frac{\partial f_j}{\partial a_{m_2}} & \frac{\partial f_j}{\partial b_m} \\ \frac{\partial f_k}{\partial a_{m_1}} & \frac{\partial f_k}{\partial a_{m_2}} & \frac{\partial f_k}{\partial b_m} \end{pmatrix}$$

We then evaluate it at $A^{(0)}$:

$$\begin{pmatrix} \det(A^{(0)}[S_i \setminus \{m_1\}] - \lambda_i I) & 0 & 0 \\ (a_{m_2} - \lambda_j) \det(A^{(0)}[S_j^*] - \lambda_j I) & (a_{m_1} - \lambda_j) \det(A^{(0)}[S_j^*] - \lambda_j I) & -2b_m \det(A^{(0)}[S_j^*] - \lambda_j I) \\ (a_{m_2} - \lambda_k) \det(A^{(0)}[S_k^*] - \lambda_k I) & (a_{m_1} - \lambda_k) \det(A^{(0)}[S_k^*] - \lambda_j I) & -2b_m \det(A^{(0)}[S_k^*] - \lambda_k I) \end{pmatrix}$$

in which $S_p^* = S_p \setminus \{m_1, m_2\}$. Because of condition (iii), we can reduce this to:

$$\begin{pmatrix} 1 & 0 & 0 \\ a_{m_2} - \lambda_j & a_{m_1} - \lambda_j & -2b_m \\ a_{m_2} - \lambda_k & a_{m_1} - \lambda_k & -2b_m \end{pmatrix}$$

This can be further reduced to:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{m_1} - \lambda_j & -2b_m \\ 0 & a_{m_1} - \lambda_k & -2b_m \end{pmatrix}$$

To show that these columns are linearly independent, assume the opposite and set the determinant equal to zero:

$$(1)[-2b_m(a_{m_1} - \lambda_j) - (-2b_m)(a_{m_1} - \lambda_k)] = 0$$

Since $A_m^{(0)}$ is not a diagonal matrix, we have $b_m \neq 0$, which implies

$$(a_{m_1} - \lambda_j) - (a_{m_1} - \lambda_k) = 0$$

which gives us $\lambda_j = \lambda_k$, a contradiction to condition (ii). □

4. Multiplicity lists for certain classes of trees

We say a tree is *binary* if every vertex has degree at most 3. A tree is *diametric* provided there exists a longest path along which all vertices of degree $\geqslant 3$ lie. If every vertex is at distance one from a vertex on this path, the tree is called *depth one*.

In this section we provide results regarding the possible multiplicity lists of binary, diametric, depth one trees.

Recall that a *strong Parter vertex* for λ, is a vertex, v, such that v is Parter for λ, $deg(v) \geqslant 3$, and λ is an eigenvalue of at least 3 branches at v. In [6] it was shown that if $m_A(\lambda) \geqslant 2$, then there exists a strong Parter vertex for λ.

Lemma 4. *If the graph of a symmetric matrix A is a binary, diametric, depth one tree T, and $\lambda_1, \ldots, \lambda_\ell$ are the distinct eigenvalues of A, then T has at least $\sum_{i=1}^{\ell}(m_A(\lambda_i) - 1)$ degree 3 vertices.*

Proof. We induct on the multiplicity of λ in A, and show that λ has $m_A(\lambda) - 1$ strong Parter vertices. Note that since a strong Parter vertex has degree at least 3, in a binary tree it has degree precisely 3. If $m_A(\lambda) = 2$, then there exists a strong Parter vertex i for λ, with λ being an eigenvalue of the three branches of $A(i)$. Note that one of the branches corresponds to a single vertex, where λ must have multiplicity 1. Therefore, the multiplicity of λ in each of the other two branches is less than $m_A(\lambda)$, but the sum of the multiplicities in both branches is $m_A(\lambda)$. Now, we assume the result to be true whenever $2 \leqslant m_A(\lambda) < k$, and let $m_A(\lambda) = k$. We know that λ has a strong Parter vertex. Then λ is an eigenvalue in all three branches, and the sum of the multiplicities in two of them is $m_A(\lambda)$. If the multiplicity of λ in one of these branches is 1, then the other branch has multiplicity $m_A(\lambda) - 1$. By the induction hypothesis, there are $m_A(\lambda) - 2$ strong Parter vertices in that branch, giving us a total of $m_A(\lambda) - 1$ strong Parter vertices. If, on the other hand, the multiplicity in both remaining branches is greater than 1, then they have a total of $m_A(\lambda) - 2$ strong Parter vertices between them, by the induction hypothesis. Then, including the original strong Parter vertex, there are a total of $m_A(\lambda) - 1$ strong Parter vertices for λ. Since a vertex can only be a strong Parter vertex for a single eigenvalue, we have a total of $\sum_{i=1}^{\ell}(m_A(\lambda_i) - 1)$ strong Parter vertices, and so at least that many degree 3 vertices. □

Lemma 5. *If the graph of a symmetric matrix $A = (a_{ij})$ is a binary, diametric, depth one tree T and λ is an eigenvalue of A such that $m_A(\lambda) \geqslant 2$, then no two strong Parter vertices for λ can be adjacent.*

Proof. Let i be strong Parter vertex for λ. If the multiplicity of λ in any branch at i is at least 2, then within that branch there is a strong Parter vertex j for λ, which means there must be 3 branches at j, which cannot be true if j is adjacent to i. □

Let $D_k(T)$ denote number of degree k vertices in T, and let $S_k(T)$ be the maximum number of vertices in a set of non-adjacent degree k vertices.

Theorem 6. *Let T be a binary, diametric, depth one tree on n vertices and suppose that*

$$m_1, \ldots, m_\ell, 1, \ldots 1$$

is a list that partitions n with $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_\ell \geqslant 2$. If

(i) $\sum_{i=1}^{\ell}(m_i - 1) \leqslant D_3(T)$
(ii) *for $i = 1, \ldots, \ell$ we have $m_i - 1 \leqslant S_3(T)$*

then there exists a symmetric matrix $A \in \mathcal{S}(T)$ with the given multiplicities.

Proof. Choose any distinct numerical values $\lambda_1, \ldots, \lambda_\ell$.

Identify a diameter of T, placing one end on the "left" and the other on the "right." We will identify $m_k - 1$ separated degree 3 vertices which will be Parter for λ_k in our matrix. For convenience, we will refer to these as Parter vertices, even though we have not yet constructed a matrix. For each λ_i, we will distribute the vertices in V_i amongst the degree 3 vertices as evenly as possible. We let the left-most degree 3 vertex be Parter for λ_1, unless it is not adjacent to a degree 3 vertex and the right-most degree 3 vertex is, in which case we let the right-most degree 3 vertex be Parter for λ_1. For simplicity, we may assume that we have labeled the left-most vertex, otherwise, we "flip" our graph so that the right-most vertex is now the left-most vertex. We then label the next degree 3 vertex as Parter for λ_2, the next as Parter for λ_3, continuing this process until we have one Parter vertex for each λ_i. We then begin this process again starting at the unlabeled degree 3 vertex the furthest left (next to the Parter vertex for λ_ℓ). However, this time, if $m_i - 1 = 1$, we will not assign another Parter vertex for λ_i. We continue this process of cycling through our list of eigenvalues to label the degree 3 vertices until we have reached a point where λ_i has $m_i - 1$ Parter vertices, for each $i = 1, \ldots, \ell$. Note that the right hand section of the graph could have a sequence of adjacent Parter vertices all of which are Parter for λ_1

Our vector of determinant conditions, F, has $\sum_{i=1}^{\ell}(2m_i - 1)$ entries, since deleting $m_i - 1$ Parter vertices for λ_i from T will increase the multiplicity of λ_i by $m_i - 1$. Each entry of F is of the from $\det(A[S] - \lambda_i I)$, where S identifies one of the branches obtained from the deletion of the Parter vertices for λ_i.

Now we will construct our initial matrix. For each vertex that we have identified as a Parter vertex for λ_i, label the neighbor on the diameter immediately to the right, and also the adjacent pendant vertex with λ_i. Next we begin a process which we will call "left-labeling", label the left-most vertex on the diameter with λ_1. For $i = 2, \ldots, \ell$, label the next Parter vertex to the right with λ_i. Note that in this way we will not label a vertex which is Parter for λ_i with a λ_i, since our assignment of Parter vertices ensures that it is always the Parter vertex following the vertex labeled with λ_i which is Parter for λ_i. In this way, no vertex is labeled more than twice, and if a vertex is labeled twice, it is labeled with two distinct eigenvalues, say λ_i and λ_j, and is Parter for some other eigenvalues, say λ_k. In this case, we label the edge connecting the Parter vertex to its adjacent pendant vertex with λ_i and λ_j.

Now, construct the initial matrix $A^{(0)}$ by setting $a_{kk} = \lambda_i$ if vertex k is labeled with λ_i. If the edge connecting vertices u and v is labeled with λ_i and λ_k, then we let $A^{(0)}[u, v]$ have eigenvalues λ_i and λ_j. Note that this construction requires a particular ordering of some of the eigenvalues. Since one of the diagonal entries, a_{uu} of $A^{(0)}[u, v]$ is equal to some eigenvalue, λ_h, that is not equal to λ_i or λ_j, interlacing tells us that $\lambda_i < \lambda_h < \lambda_j$. To find the entries of $A^{(0)}[u, v]$, we use the trace condition to find that $a_{vv} = \lambda_i + \lambda_j - \lambda_h$. The off diagonal entry can be calculated using the determinant condition, i.e., $a_{uv} = \sqrt{\lambda_i\lambda_h + \lambda_j\lambda_h - \lambda_h^2 - \lambda_i\lambda_j}$.

However, this restriction on the numerical ordering of our eigenvalues necessitates a check to make sure our restrictions on the values of the eigenvalues are mutually compatible. First, note that if we have 2 consecutive Parter vertices for λ_i, the second Parter vertex will be labeled with λ_i, and its pendant vertex will also be labeled with λ_i. If the second Parter vertex is left-labeled with some other eigenvalue, we are presented with an impossible ordering to fulfill, i.e., we must have $\lambda_i < \lambda_i$. However, the only i for which this could happen is $i = 1$. Furthermore, left-labeling ends within the first ℓ Parter vertices, but we only have consecutive Parter vertices for λ_1 after the first ℓ Parter vertices, and so this issue never arises.

Next, if we have a vertex, v, that is labeled with two distinct eigenvalues, λ_x and λ_y, and is Parter for some other eigenvalue, λ_z, we must check that there is no vertex w that is either

(i) labeled with λ_x and λ_z and is Parter for λ_y
(ii) labeled with λ_y and λ_z and is Parter for λ_x

We may assume that v is to the left of w. We will show both that cases (i) and (ii) cannot happen. Also, recall that all double-labeling of vertices must happen within the first ℓ Parter vertices. Let the Parter vertex immediately to the left of v be denoted v_0, and the vertex immediately to the left of w be denoted w_0

Suppose case (i). Since v is to the left of w, and w is Parter for λ_y, we know that $z < y$ in our index (again, this is because double labeled vertices only occur in the first ℓ vertices, and if we restrict our attention to the first ℓ Parter vertices, by our construction, if a Parter vertex for λ is to the left of a Parter vertex for μ, the index of λ is less than the index of μ). This implies that v_0 must be Parter for λ_x, and that v must be left labeled with λ_y. However, this implies that w cannot be left labeled with λ_z (since $z < y$), and so w_0 must be Parter for λ_z. But now consider the left-labeling sequence we must see: we must left-label a vertex with λ_y before we left-label a vertex with λ_x, giving us $y < x$. But recall that we know that v_0 is Parter for λ_x, implying that $x < y$, but this is a contradiction.

Similarly, we can show that case (ii) cannot happen. Thus our labeling is feasible.

We designate as implicit entries those corresponding to labeled vertices, the diagonal entries of the 2-by-2 matrices corresponding to vertices on the diameter, and the off-diagonal entries of the 2-by-2 matrices. Thus there are a total of $\sum_{i=1}^{\ell}(2m_i - 1)$ implicit entries.

Because $F(A^{(0)}) = 0$, and there are as many implicit entries in $A^{(0)}$ as determinant conditions in F, and the Jacobian is nonsingular at $A^{(0)}$ (by Lemma 3), we can use the IFT to infer the existence of a matrix $A \in \mathcal{S}(T)$ such that $F(A) = 0$.

Since λ_i is an eigenvalue of each of the $2m_i - 1$ direct summands of $A(V_i)$, we have by interlacing $m_A(\lambda_i) \geqslant (2m_i - 1) - |V_i| = (2m_i - 1) - (m_i - 1) = m_i$.

To show $m_A(\lambda_i) = m_i$, we place an appropriate upper bound on $m_A(\lambda_i)$. If $m_A(\lambda_i)$ were greater than m_i, then λ_i would be a multiple eigenvalue of one of the direct summands of $A(V_i)$, where V_i is the set of Parter vertices for λ_i. However the multiplicity of λ_i in each direct summand of $A^{(0)}(V_i)$ is at most 1, so by choosing our perturbations to be small enough (since the eigenvalues are a continuous function of the entries in the matrix), we can guarantee that λ_i is not a multiple eigenvalue of any direct summand of $A(V_i)$. This gives us that $m_A(\lambda_i) = m_i$.

Next, consider the remaining eigenvalues, that are intended to have multiplicity 1. To see that these eigenvalues must have multiplicity 1, it suffices to show that no eigenvalues other than $\lambda_1, \ldots, \lambda_\ell$ has a strong Parter vertex. For binary trees, no two eigenvalues may share a Parter vertex, so consider a degree 3 vertex, v, that is not a Parter vertex for any λ_i. The vertex v is adjacent to a pendant vertex u, whose corresponding entry is neither implicit nor manual, i.e., $a_{uu} = a_{uu}^{(0)}$. By choosing the perturbation to be sufficiently small, A can be guaranteed not to have a_{uu} as an eigenvalues of any other direct summand of $A(v)$. This guarantees that v is not a Parter vertex for any eigenvalue. □

To illustrate the construction of the tree described in the proof of the Theorem, consider the following tree T= ({1, 2,..., 16} ; {(1, 2), (2, 3), (2, 4), (4, 5), (4, 6), (6, 7), (7, 8), (7, 9), (9, 10), (9, 11), (11, 12), (11, 13), (13, 14), (14, 15), (14, 16) })

According to Theorem 6, there exists $A \in \mathcal{S}(T)$ with multiplicities $4, 3, 2, 1, \ldots, 1$. Let the three multiple eigenvalues be denoted λ, μ and ν, where $m(\lambda) = 4$, $m(\mu) = 3$ and $m(\nu) = 2$. To begin, we will assign our Parter vertices. Vertices 2, 9 and 14 will be Parter for λ, vertices 4 and 11 will be Parter for μ and vertex 7 will be Parter for ν

Then, for each vertex that is Parter for λ, we label the vertex directly above, directly to the right with λ. We do the same for μ and ν. We then perform our "left-labeling" process.

Vertex 4 is labeled twice with λ and ν, so we remove both of those values from the vertex and instead label the edge connecting vertices 4 and 5 with λ and ν. We then remove the unlabeled edges.

This labeling allows us to construct a second order initial matrix $A^{(0)}$, whose graph is a subgraph of T, and which has the desired multiplicities. We can then use $A^{(0)}$ and the implicit function theorem to show that there exists $A \in \mathcal{S}(T)$ with these multiplicities.

Now, we will show that the multiplicity lists that can occur among symmetric matrices whose graph is a binary, diametric, depth one tree T may be succinctly described by characteristics of T.

Lemma 7. *Let T be a binary, diametric, depth one tree. Then* $p(T) = S_3(T) + 1$.

Proof. We use the fact that $P(T) = \max\{p - q\}$, where the maximum is taken over all ways in which q vertices can be deleted from T to form p paths. We locate a maximal set of non-adjacent degree 3 vertices in T, which has $S_3(T)$ vertices. Note that any degree 3 vertex not in the set must be adjacent to at least one vertex in the set, or the set would not be maximal. Thus, when we remove our maximal set, the only vertices remaining will have degree at most 2, and so by deleting this set of vertices, we leave only paths. The number of these paths is $2S_3(T) + 1$, since there is a path to the left and above each deleted vertex, and one path to the right of the right-most deleted vertex. Thus for this set of vertices $p - q = S_3(T) + 1$.

Since not deleting any of these vertices would leave branches that are not paths, it only remains to show that deleting any other vertices will not increase this number. Deleting any degree 1 vertex will not increase this number, since it can only make an existing path shorter. Deleting any degree 2 vertex will also not increase this number, since it can only make an existing path shorter or divide an existing path into two paths. Since the deletion of our maximal set leaves only paths, deleting any other vertex will not increase $p - q$. Therefore, $P(T) = S_3(T) + 1$. □

Given a sequence $a = a_1 \geqslant \cdots \geqslant a_n$, we say that *a majorizes* $b = b_1 \geqslant \cdots \geqslant b_n$ provided

$$\sum_{i=1}^{k} a_i \geqslant \sum_{i=1}^{k} b_i, \quad \text{for } k = 1, \ldots, n-1$$

and

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$$

Theorem 8. *The possible multiplicities for a binary, diametric, depth one tree T on n vertices are the sequences of positive integers that are majorized by* $p(T), d(T) - p(T) - D_2(T), 1, \ldots, 1$, *a partition of n.*

Proof. First, we show that this list satisfies the conditions of Theorem 6. For condition (i) we have

$$\sum_{i=1}^{\ell}(m_i - 1) = (P(T) - 1) + (d(T) - P(T) - D_2(T) - 1) = d(T) - D_2(T) - 2$$

Since $d(T) = D_3(T) + D_2(T) + 2$, we have

$$(D_3(T) + D_2(T) + 2) - D_2(T) - 2 = D_3(T)$$

Thus condition (i) is satisfied. For condition (ii), we use Lemma 7. Since $P(T) = S_3(T) + 1$, for m_1 we have

$$m_1 - 1 = P(T) - 1 = S_3(T)$$

For m_2 we have.

$$\begin{aligned} m_2 - 1 &= d(T) - P(T) - D_2(T) - 1 \\ &= (D_3 + D_2(T) + 2) - (S_3(T) + 1) - D_2(T) - 1 \\ &= D_3(T) - S_3(T) \end{aligned}$$

Since $D_3(T) \leqslant 2S_3(T)$, we see that condition (ii) is satisfied.

Any other list, b, majorized by $a = P(T), d(T) - P(T) - D_2(T), 1, \ldots, 1$ will also satisfy the conditions of Theorem 6, by definition of majorization. □

5. The degree conjecture

We say a vertex, v, is of *high degree* provided $deg(v) \geqslant 3$. Given a tree T, the *high degree sequence* of that tree is the list of degrees of all the high degree vertices arranged in non-increasing order.

Conjecture 9. *Given a tree, T, with high degree sequence $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_k > 2$, there exists a matrix $A \in \mathcal{S}(T)$ with the unordered multiplicity list $d_1 - 1, d_2 - 1, \ldots, d_k - 1, 1, \ldots, 1$.*

We call this conjecture the *degree conjecture*. In this section we provide a verification of the degree conjecture for diametric trees.

Theorem 10. *Let T be a diametric tree with high degree sequence $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_k > 2$. Then there exists a symmetric matrix $A \in \mathcal{S}(T)$ with the multiplicity list $d_1 - 1, d_2 - 1, \ldots, d_k - 1, 1, \ldots, 1$.*

Proof. Here we construct an initial matrix and use the implicit function theorem, but we also account for all of the single eigenvalues to show that we can always get exactly this multiplicity list. To do so, we specify all but two eigenvalues, the largest and smallest, which must have multiplicity 1.

Choose any distinct numerical values $\lambda_1, \ldots, \lambda_k$ to be the multiple eigenvalues. Identify a diameter of T, placing one end on the "left" and the other on the "right." Each λ_i will have exactly one Parter vertex, which can be easily identified. If λ_i has multiplicity m_i, then its Parter vertex will be the left-most vertex with degree $d_i = m_i + 1$, which we denote v_i.

The vector of determinant conditions has $\sum_{i=1}^{k} d_i$ entries corresponding to the multiple eigenvalues. These entries will be of the from $\det(A[S] - \lambda I)$ in which λ is a multiple eigenvalue of A, and S identifies one of the branches obtained from the deletion of the Parter vertex for λ. We also have $n - \sum_{i=1}^{k}(d_i - 1) - 2 = n + k - 2 - \sum_{i=1}^{k} d_i$ determinant conditions corresponding to all but the largest and smallest single eigenvalues. These entries will be of the from $\det(A - \lambda I)$, in which λ is a desired single eigenvalue. Thus there are a total of $\left[n + k - 2 - \sum_{i=1}^{k} d_i\right] + \sum_{i=1}^{k} d_i = n + k - 2$ determinant conditions.

To construct the initial matrix $A^{(0)} = (a_{ij}^{(0)})$, which is a direct sum of 1-by-1 and 2-by-2 matrices, for $i = 1, \ldots, k$, identify the Parter vertex for λ_i. Label every adjacent vertex off the diameter with λ_i. Then label the next Parter vertex tot he left and the next Parter vertex to the right each with λ_i. In this way, no vertex is labeled more than twice, and if a vertex is labeled twice, it is labeled with 2 distinct eigenvalues, λ_i and λ_j, and is Parter for some other eigenvalue λ_k. Then, instead of labeling the vertex twice we can label the edge connecting the Parter vertex to any of its adjacent off-diameter vertices with λ_i and λ_j. We then use the remaining vertices to specify our single eigenvalues. Note that all Parter vertices except the left-most and the right-most were labeled twice. Thus there are $n - \left[\sum_{i=1}^{k} d_i - (k-2)\right] = n - \sum_{i=1}^{k} d_i + k - 2$ vertices that have not been labeled. We then choose $m = n - \sum_{i=1}^{k} d_i + k - 2$ distinct numerical values $\mu_1, \ldots, \mu_m$ for the single eigenvalues such that $\min_{1\leqslant i\leqslant k} \lambda_i < \mu_j < \max_{1\leqslant i\leqslant k} \lambda_i$ for any j, and $\mu_i \neq \lambda_j$ for any i and j. Label the remaining vertices with the μ_i. Now, construct the initial matrix $A^{(0)}$ by setting $a_{kk}^{(0)} = \lambda_i\ (\mu_i)$ if vertex k is labeled with $\lambda_i\ (\mu_i)$, and ensure $A^{(0)}[u, v]$ has eigenvalues λ_x, λ_y if the edge connecting vertices u and v is labeled with λ_x and λ_y Note that this construction requires a particular ordering of some of the eigenvalues. Since one of the diagonal entries, $a_{uu}^{(0)}$ is equal to some eigenvalue, λ_w, that is not equal to λ_x or λ_y, we know $\lambda_x < \lambda_w < \lambda_y$, by interlacing. This also requires a check that there is not a vertex which is Parter for λ_x, and has an edge leading to one of its pedant vertices labeled with λ_w and λ_y. But since there is precisely one Parter vertex for each multiple eigenvalue, this situation cannot occur. The remaining entries of $A^{(0)}[u, v]$ can be calculated using the trace and determinant conditions.

The implicit entries are those corresponding to the labeled vertices, both diagonal entries of the 2-by-2 matrices, and the off-diagonal entries of the 2-by-2 matrices. There are a total of $n + k - 2$ implicit entries.

Because $F(A^{(0)}) = 0$, there are as many implicit entries in $A^{(0)}$ as determinant conditions in F, and the Jacobian is nonsingular at $A^{(0)}$ (by Lemma 3), we known that there exists a matrix $A = (a_{ij})$ with graph T such that $F(A) = 0$. Thus for each i, λ_i is an eigenvalue of each of the d_i direct summands of $A(v_i)$. By interlacing, $m_A(\lambda_i) \geqslant d_i - 1$. Note for each j, μ_j is a single eigenvalue of A. This gives us at least $\sum_{i=1}^{k}(d_i - 1) + n - \sum_{i=1}^{k} d_i + k - 2 = n - 2$ eigenvalues already accounted for. But recall that we did not specify the largest and smallest eigenvalues, which must both be single eigenvalues. Thus all eigenvalues are accounted for, and we have each λ_i has multiplicity $d_i - 1$, and each μ_i has multiplicity 1. □

6. The minimum number of 1's in a multiplicity list

Let $U(T)$ denote the minimum number of 1's appearing in any multiplicity list for T.

It has been conjectured by Johnson, Leal-Duarte and Saiago that $U(T) \leqslant 2 + D_2(T)$ for all trees. We prove this conjecture here for diametric trees.

Theorem 11. *Let T be a diametric tree. Then $U(T) \leqslant 2 + D_2(T)$.*

Proof. We use the degree list in Theorem 9, and count the number of 1's in the list to provide our list.

Since the sum of all the degrees of all the vertices of a tree is $2n - 2$, we have that the sum of the degrees of all the high degree vertices is $2n - 2 - D_1(T) - 2D_2(T)$. Note that the number of high degree vertices is $n - D_1(T) - D_2(T)$. By Theorem 9, there exists a matrix $A \in \mathcal{S}(T)$ such that the sum of the multiplicities of the multiple eigenvalues is $2n - 2 - D_1(T) - 2D_2(T) - (n - D_1(T) - D_2(T)) = n - 2 - D_2(T)$. Thus, the number of eigenvalues with multiplicity 1 is $n - (n - 2 - D_2(T)) = 2 + D_2(T)$. This gives us $U(T) \leqslant 2 + D_2(T)$. □

Also note that if the degree conjecture holds true for all trees, then Theorem 10 holds for all trees. Since the sum of all the degrees of all the vertices of a tree is $2n-2$, we have that the sum of the degrees of all the high degree vertices is $2n - 2 - D_1(T) - 2D_2(T)$. Note that the number of high degree vertices is $n - D_1(T) - D_2(T)$. According to the degree conjecture, there exists a matrix $A \in \mathcal{S}(T)$ such that the sum of the multiplicities of the multiple eigenvalues is $2n-2-D_1(T)-2D_2(T)-(n-D_1(T)-D_2(T)) = n-2-D_2(T)$. Thus, the number of eigenvalues with multiplicity 1 is $n-(n-2-D_2(T)) = 2+D_2(T)$. This gives us $U(T) \leqslant 2 + D_2(T)$.

Note that this inequality can be strict. Consider the generalized star $G = (\{1, 2, \ldots, 7\}; \{(1, 2), (2, 3), (3, 4), (4, 5), (3, 6), (6, 7)\})$.

G has 3 degree 2 vertices, so Theorem 10 provides an upper bound on $U(G)$ of 5. However, by assigning λ, μ as the eigenvalues of each component of G minus vertex 3, we have a multiplicity list 2, 2, 1, 1, 1, so in fact $U(G) \leqslant 3 < 5$.

7. The degree conjecture in the general case

In this section we present two conjectures, and show how these conjectures imply the degree conjecture for all trees. The first is most interesting by itself, given the considerable spectral information already known about tridiagonal matrices.

Conjecture 12. *Let S be a set of $2n-1$ distinct real numbers. Then there exists a symmetric tridiagonal matrix A such that $\sigma(A) \subseteq S$, and for each $k = 1, \ldots, n-1$, $A[\{1, \ldots, k\}]$ has an eigenvalue $\lambda_k \in S \setminus \sigma(A)$, with $\lambda_i \neq \lambda_j$ for $i \neq j$.*

Conjecture 13. *Let $F = (f_k)$ be a vector of r determinant conditions, and let $A^{(0)}$ be the direct sum of tridiagonal, symmetric, irreducible matrices $A_1^{(0)}, \ldots, A_p^{(0)}$. Suppose that for every $k = 1, \ldots, r$, $f_k(A_\ell^{(0)}) = 0$ for precisely one $\ell \in \{1, \ldots, p\}$. If*

(i) *if $A_\ell^{(0)}$ is a direct summand of a size larger than 1-by-1, then every entry is implicit,*
(ii) *for any j-by-j direct summand $A_m^{(0)} = A^{(0)}[\{m_1, \ldots, m_j\}]$, there is at least one determinant condition $f_i = det(A[S_i] - \lambda_i I)$ such that $\{m_1, \ldots, m_q\} \subseteq S_i$ and $\{m_{q+1}, \ldots, m_j\} \not\subseteq S_i$, and λ_i is not an eigenvalue of $A^{(0)}[S_i \setminus \{m_1, \ldots, m_j\}]$ for every $q = 1, \ldots, j-1$,*
(iii) *for any j-by-j direct summand $A_m^{(0)} = A^{(0)}[\{m_1, \ldots, m_j\}]$, there are at least j determinant conditions $f_{i_1}, \ldots, f_{i_j}$, each of the form $f_{i_\ell} = det(A[S_{i_\ell} - \lambda_{i_\ell} I)$, such that $\lambda_{i_s} \neq \lambda_{i_t}$ is $s \neq t$, $\{m_1, \ldots, m_j\} \in S_{i_\ell}$, and λ_{i_ℓ} is not an eigenvalue of $A^{(0)}[S_{i_\ell} \setminus \{m_1, \ldots, m_j\}]$ for all $\ell = 1, \ldots, j$,*
(iv) *there are r implicit entries total,*

then the Jacobian of F with respect to the implicit entries evaluated at $A^{(0)}$ is nonsingular.

Lemma 14. *Conjecture 13 is valid in case the largest summand is at most 3-by-3.*

Proof. We apply Lemmas 1–3. Lemmas 2 and 3 tell us that if $A_\ell^{(0)}$ is a 1-by-1 or 2-by-2 direct summand, then the columns of $J(A^{(0)})$ associated with the implicit entries of $A_\ell^{(0)}$ are linearly independent. So, we only need to show that the columns of $J(A^{(0)})$ associated with the implicit entries in any 3-by-3 direct summand $A_m^{(0)}$ are linearly independent. To do so, we consider $f_{i_1}, \ldots, f_{i_5}$, the first two of which satisfy condition (ii) and the remaining three which satisfy condition (iii). We then consider the submatrix of the Jacobian of F with respect to the implicit entries $a_{m_{11}}, a_{m_{12}}, a_{m_{22}}, a_{m_{23}}, a_{m_{33}}$, then evaluate it at $A_m^{(0)}$, and row reduce to obtain the following:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -2a_{m_{12}} & -2a_{m_{12}}(a_{m_{33}}-\lambda_{i_3}) & -2a_{m_{12}}(a_{m_{33}}-\lambda_{i_4}) & -2a_{m_{12}}(a_{m_{33}}-\lambda_{i_5}) \\ 0 & a_{m_{11}}-\lambda_{i_2} & (a_{m_{11}}-\lambda_{i_3})(a_{m_{33}}-\lambda_{i_3}) & (a_{m_{11}}-\lambda_{i_4})(a_{m_{33}}-\lambda_{i_4}) & (a_{m_{11}}-\lambda_{i_5})(a_{m_{33}}-\lambda_{i_5}) \\ 0 & 0 & -2a_{m_{23}}(a_{m_{11}}-\lambda_{i_3}) & -2a_{m_{23}}(a_{m_{11}}-\lambda_{i_4}) & -2a_{m_{23}}(a_{m_{11}}-\lambda_{i_5}) \\ 0 & 0 & -a_{m_{12}}^2+(a_{m_{11}}-\lambda_{i_3})(a_{m_{22}}-\lambda_{i_3}) & -a_{m_{12}}^2+(a_{m_{11}}-\lambda_{i_4})(a_{m_{22}}-\lambda_{i_4}) & -a_{m_{12}}^2+(a_{m_{11}}-\lambda_{i_5})(a_{m_{22}}-\lambda_{i_5}) \end{pmatrix}$$

continuing to row reduce,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & (a_{m_{33}} - \lambda_{i_3}) & (a_{m_{33}} - \lambda_{i_4}) & (a_{m_{33}} - \lambda_{i_5}) \\ 0 & 0 & (\lambda_{i_2} - \lambda_{i_3})(a_{m_{33}} - \lambda_{i_3}) & (\lambda_{i_2} - \lambda_{i_4})(a_{m_{33}} - \lambda_{i_4}) & (\lambda_{i_2} - \lambda_{i_5})(a_{m_{33}} - \lambda_{i_5}) \\ 0 & 0 & (a_{m_{11}} - \lambda_{i_3}) & (a_{m_{11}} - \lambda_{i_4}) & (a_{m_{11}} - \lambda_{i_5}) \\ 0 & 0 & (\lambda_{i_3} - \lambda_{i_2})(\lambda_{i_3} - \mu) & (\lambda_{i_4} - \lambda_{i_2})(\lambda_{i_4} - \mu) & (\lambda_{i_5} - \lambda_{i_2})(\lambda_{i_5} - \mu) \end{pmatrix}$$

where μ is the other eigenvalue of the upper left principal submatrix. We continue to row reduce to get

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & (\lambda_{i_2} - \lambda_{i_3})(a_{m_{33}} - \lambda_{i_3}) & (\lambda_{i_2} - \lambda_{i_4})(a_{m_{33}} - \lambda_{i_4}) & (\lambda_{i_2} - \lambda_{i_5})(a_{m_{33}} - \lambda_{i_5}) \\ 0 & 0 & (a_{m_{11}} - \lambda_{i_3}) & (a_{m_{11}} - \lambda_{i_4}) & (a_{m_{11}} - \lambda_{i_5}) \\ 0 & 0 & (\lambda_{i_3} - \lambda_{i_2})(\lambda_{i_3} - \mu) & (\lambda_{i_4} - \lambda_{i_2})(\lambda_{i_4} - \mu) & (\lambda_{i_5} - \lambda_{i_2})(\lambda_{i_5} - \mu) \end{pmatrix}$$

This continues to reduce to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & \lambda_{i_3} & \lambda_{i_4} & \lambda_{i_5} \\ 0 & 0 & \lambda_{i_3}^2 & \lambda_{i_4}^2 & \lambda_{i_5}^2 \end{pmatrix}$$

If these rows were linearly dependent, we would have a quadratic $ax^2 + bx + c$ with each λ_{i_k} for $k = 3, 4, 5$ as a root. However, since the λ_{i_k} are distinct, and there are three of them, this is not possible. Therefore, the rows are linearly independent. □

Given a high degree vertex v, let the i—th branch degree of periphery of v in the branch T_i at v, denoted $r_{T_i}(v)$, be the maximum number of high degree vertices in any path in T_i, including v. Let $r(v)$, the degree of periphery of v, be the second largest branch degree of periphery of v over all T_i at v.

Lemma 15. *Let T be a tree. Then there is at most one high degree vertex, v_k, in T such that* $r(v_k) = \max_i r_{T_i}(v_k)$.

Proof. Assume that there are two vertices, v_i and v_j, such that $r(v_i) = \max_k r_{T_k}(v_i)$ and $r(v_j) = \max_k r_{T_k}(v_j)$. Identify two paths, P_{i_1} and P_{i_2}, in different branches at v_i, each having $r(v_i)$ high degree vertices. If v_j is in one of those paths, say P_{i_1}, then $r(v_j) > r(v_i)$, since there is a path starting at v_j that includes v_i and P_{i_2}. But that means that P_{i_1} has more than $r(v_i)$ high degree vertices, since it includes one of the paths of v_j with $r(v_j)$ high degree vertices. Thus, v_j cannot be in P_{i_1} or P_{i_2}. However, if v_j is in some other path, then, again, $r(v_j) > r(v_i)$, since some path starting at v_j contains v_i and both P_{i_1} and P_{i_2}. But then there is a path starting at v_i containing v_j and one of its paths containing $r(v_j)$ high degree vertices. Thus $r(v_i)$ is not maximal. □

If there exists a vertex, v, such that $r(v) = \max_i r_{T_i}(v)$, we call v the *center vertex*, denoted v_c.

Statement 16. The degree conjecture follows from Conjectures 12 and 13.

Proof. Choose any distinct numerical values $\lambda_1, \ldots, \lambda_k$ to be the multiple eigenvalues. Each λ_i will have exactly one Parter vertex, which can be easily identified: If λ_i has multiplicity m_i, then its Parter vertex will be the vertex with degree $d_i = m_i + 1$, denoted v_i.

The vector of determinant conditions has $\sum_{i=1}^{k} d_i$ entries corresponding to the multiple eigenvalues. These entries will be of the from $\det(A[S] - \lambda I)$, in which λ is a desired multiple eigenvalue of A, and S identifies one of the branches obtained from the deletion of the Parter vertex for λ. We will also have $n + k - 2 - \sum_{i=1}^{k} d_i$ determinant conditions corresponding to all but the largest and smallest single eigenvalues. These entries will be of the from $\det(A - \lambda I)$, where λ is a desired single eigenvalue of A. Thus, there are a total of $\sum_{i=1}^{k} d_i + \left[n + k - 2 - \sum_{i=1}^{k} d_i\right] = n + k - 2$ determinant conditions.

To construct the initial matrix $A^{(0)}$, for $i = 1, \ldots, k$, identify the Parter vertex for λ_i. If $v_i \neq v_c$, then in every branch that does not contain the path with more than $r(v_i)$ high degree vertices, label the closest high degree vertex, or the vertex adjacent to v_i if there is no high degree vertex, with λ_i. Then, moving clockwise, label the next high degree vertex on the same level of periphery as v_i with λ_i. If $v_i = v_c$, then in every branch, label the closest high degree vertex with λ_i, or, if $r(v_i) = 1$, label the vertex adjacent to v_i with λ_i. Finally, in any of v_c's branches, remove the labeled eigenvalue on the high degree vertex closest to v_c, whose Parter vertex is not v_c, and label v_c with it. This is to prevent a contradiction in the numerical ordering of the eigenvalues. In this way, no vertex is labeled more than twice, and if a vertex is labeled twice, it is labeled with two distinct eigenvalues, λ_i and λ_j, and is Parter for some other eigenvalue, λ_k. We then use the remaining vertices to specify our single eigenvalues. All but two Parter vertices are labeled twice. Thus, there are $n - \left[\sum_{i=1}^{k} d_i - (k-2)\right] = n + k - 2 - \sum_{i=1}^{k} d_i$ vertices that have not been labeled, which is equal to the number of single eigenvalues we need to specify. We then choose $m = n + k - 2 - \sum_{i=1}^{k} d_i$ distinct numerical values $\mu_1, \ldots, \mu_m$ for the single eigenvalues such that $\min_{1 \leqslant i \leqslant k} \lambda_i < \mu_j < \max_{1 \leqslant i \leqslant k} \lambda_i$ for any j, and $\mu_j \neq \lambda_i$ for any j and i, and label the remaining vertices with them. Now construct $A^{(0)}$ by setting $a_{kk}^{(0)} = \lambda_i$ or μ_i if vertex k is labeled with λ_i of μ_i. If there is a path on $w, \ldots, v, u$ such that every vertex, except u is labeled twice, and $r(w) < r(u)$, then we make $A^{(0)}[\{u, v \ldots, w\}]$ a tridiagonal matrix where:

(i) $A^{(0)}[\{u, v, \ldots, w\}]$ has eigenvalues $\lambda_x, \ldots, \lambda_y$, where vertices $v, \ldots, w$ are labeled with $\lambda_x, \ldots, \lambda_y$, whose Parter vertices $\notin \{v, \ldots, w\}$.

(ii) The leading principal submatrix of $A^{(0)}[\{u, \ldots, \ell, m, \ldots, w\}]$, $A^{(0)}[\{u, \ldots, \ell\}]$ has λ_i as one of its eigenvalues if vertex ℓ is labeled with λ_i, where m is Parter for λ_i.

The implicit entries are those corresponding to vertices labeled once, and every entry of the tridiagonal matrices. There are a total of $n + k - 2$ implicit entries. Because $F(A^{(0)}) = 0$, and there are as many implicit entries in $A^{(0)}$ as determinant conditions in F, and the Jacobian is nonsingular at $A^{(0)}$ by Conjecture 12, we known that there exists a matrix $A \in \mathcal{S}(T)$ such that $F(A) = 0$. Thus, for each i, λ_i is an eigenvalue of each of the d_i direct summands of $A(v_i)$. By the interlacing inequalities, $m_A(\lambda_i) \geqslant d_i - 1$. However, for each j, μ_j is a single eigenvalue of A. This gives us at least $\sum_{i=1}(d_i - 1) + \left[n + k - 2 - \sum_{i=1}^{k} d_i\right] = n - 2$ eigenvalues. Since we have not specified the largest and smallest eigenvalues, which must both be single eigenvalues, each λ_i must have multiplicity $d_i - 1$. □

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