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Enumerating Switching Isomorphism Classes of Signed Graphs

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Abstract

Let Γ be a simple connected graph, and let $\{+,-\}^{E(\Gamma)}$ be the set of signatures of Γ. For σ a signature of Γ, we call the pair $\Sigma = (\Gamma, \sigma)$ a signed graph of Γ. We may define switching functions $\zeta_X \in \{+,-\}^{V(\Gamma)}$ that negate the sign of every edge $\{u, v\}$ incident with exactly one vertex in the fiber $X = \zeta_X^{-1}(-)$. The group $\mathrm{Sw}(\Gamma)$ of switching functions acts on the set of signed graphs of Γ and induces an equivalence relation of *switching classes* in its orbits; there are $2^{|E(\Gamma)|-|V(\Gamma)|+1}$ such classes. More interestingly, we may define a group $\mathrm{SwAut}(\Gamma) = \mathrm{Sw}(\Gamma) \rtimes \mathrm{Aut}(\Gamma)$ whose action on signed graphs combines both switching functions and graph automorphisms. We may also define *switching automorphism groups* $\mathrm{SwAut}(\Sigma)$ as subgroups of $\mathrm{SwAut}(\Gamma)$ that preserve individual signed graphs. The orbits of $\mathrm{SwAut}(\Gamma)$ on signed graphs represent the equivalence classes of signed graphs that are equivalent under some combination of switching and permuting vertices. We call these classes *switching isomorphism classes*, and their enumeration for an arbitrary graph is nontrivial. Following observations of Zaslavsky [19],[20], we offer algorithms for the enumeration of switching isomorphism classes, and thus provide a means for counting such classes for arbitrary graphs. We also calculate a formula for the number of switching isomorphism classes of certain species of Generalized Petersen graphs, and provide data for the number of these classes for other graphs for which no formula is yet known. Finally, we include the abstract switching automorphism groups of all switching isomorphism classes for select graphs, as determined by our program.

Acknowledgment

I should like to thank my advisor Dr. Eric Swartz for the guidance he provided me throughout my research and without whom this work would not be possible. He introduced to me an entire subject of mathematics which has captured my interest and which I hope to continue to explore.

Contents

Chapter 1

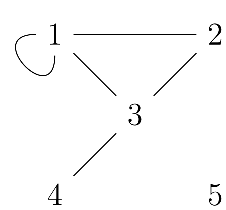
Preliminary Theory of Graphs and Groups

1.1 Basics of Graph Theory

It is important that we prepare a set of preliminary definitions.

Definition 1.1. A *graph* is an algebraic structure that can be considered in several equivalent ways. Most abstractly, a graph can be defined as some set S and a symmetric relation R defined on S. For our purposes, we will consider a geometric representation: we can define an *undirected* graph Γ as consisting of two sets, a *vertex set* $V(\Gamma)$ and an *edge set* of unordered pairs $E(\Gamma) \subseteq V(\Gamma)^2$.

Thus, a graph is a set of vertices and a set of edges connecting some (potentially none, or all) of these vertices. The sets $V(\Gamma)$ and $E(\Gamma)$ correspond to S and R respectively. Below is an example of a graph with labeled vertices.



Definition 1.2. For a graph Γ and vertices $u, v \in V(\Gamma)$, a *path* from u to v is a sequence $(r_1, r_2, ..., r_n)$ of vertices

$$\{r_i \mid 1 \leqslant i \leqslant n\} \subseteq V(\Gamma)$$

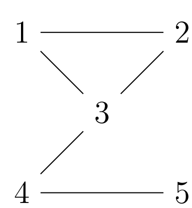
such that $r_1 = u, r_n = v$, each r_i is distinct, and $\{r_i, r_{i+1}\} \in E(\Gamma)$ for $1 \leqslant i \leqslant n-1$.

Definition 1.3. A graph is called *connected* if we can always find some path between any two vertices.

The graph above is not connected because there is no path between vertex 5 and vertex 4.

Definition 1.4. A *simple* graph is one without any loops; that is, without any edges from some vertex v to itself. This is equivalent to the condition that $\{v, v\} \notin E(\Gamma)$ for any $v \in V(\Gamma)$.

The graph above is not simple because $\{1, 1\}$ is an edge (loop). A *finite, simple connected* graph is naturally any graph Γ that is both connected and simple such that $|V(\Gamma)| \in \mathbb{N}$. It is these graphs that will concern us, so from now on general statements regarding graphs will consider only finite, simple connected graphs. Below is an example.



Definition 1.5. A *cycle* is a set of vertices

$$\{r_i \mid 1 \leqslant i \leqslant n\} \subseteq V(\Gamma)$$

such that each $(r_1, r_2, ..., r_n)$ is a path and $\{r_1, r_n\} \in E(\Gamma)$.

1.2 Graph Automorphisms

In general, a homomorphism is a function between two algebraic structures that preserves some fundamental element of each structure. The essential structure of a graph is its edge set, and so it is natural to define a graph homomorphism as some function that preserves edges.

Definition 1.6. Formally, for two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, a *graph homomorphism* ϕ is a function $\phi : V_1 \to V_2$ such that

$$\{u, v\} \in E_1 \implies \{u^\phi, v^\phi\} \in E_2$$

Intuitively, edges are mapped to edges. A *graph isomorphism* is a graph homomorphism that is also a bijection. If a graph isomorphism exists between two graphs, we call them *isomorphic* graphs. Isomorphic graphs have the same size vertex sets and the same relation structure upon said vertex sets and so can be considered, for most purposes, equivalent graphs. We call them equivalent *up to isomorphism*, and graph isomorphism is an equivalence relation.

Definition 1.7. A *graph automorphism* is a graph isomorphism from a graph to itself. That is, a graph automorphism of Γ is a function $\psi : V(\Gamma) \to V(\Gamma)$ such that

$$\{u, v\} \in E(\Gamma) \iff \{u^\psi, v^\psi\} \in E(\Gamma).$$

Lemma 1.8. *The set of all automorphisms of* Γ *forms a group under function composition.*

Proof. Let Γ be a graph. First, note that the identity function id, which maps every vertex to itself, is an automorphism of Γ, since

$$\{u, v\} \in E(\Gamma) \iff \{\mathrm{id}(u), \mathrm{id}(v)\} \in E(\Gamma).$$

Thus $\mathrm{Aut}(\Gamma)$ is nonempty. Next, suppose that $f, g \in \mathrm{Aut}(\Gamma)$. The function g is a bijection, and so it has an inverse g^{-1}. Suppose that $\{u, v\} \in E(\Gamma)$. Then

$$g^{-1}\left(f\left(\{u, v\}\right)\right) = g^{-1}\left(\{f(u), f(v)\}\right).$$

We know that $\{f(u), f(v)\} \in E(\Gamma)$, and since the inverse of an isomorphism is an isomorphism, and since the composition of isomorphisms is also an isomorphism,

$$g^{-1}\left(f\left(\{u, v\}\right)\right) = \left\{g^{-1}\left(f\left(u\right)\right), g^{-1}\left(f\left(v\right)\right)\right\} \in E(\Gamma),$$

and so $\mathrm{Aut}(\Gamma)$ is a group of automorphisms. *QED*

1.3 Group Actions

A group action is a powerful way to apply the structure of a group to a set.

Definition 1.9. Let G be a group and Ω be a set. A *group action* A is a function $A : \Omega \times G \to \Omega$ such that for all $\omega \in \Omega$ and $g, h \in G$,

$$A(\omega, 1) = \omega,$$
$$A(A(\omega, g), h) = A(\omega, gh).$$

With a specific group action A in mind, we can use the notation $\omega^g := A(\omega, g)$. Then the above parameters are equivalent to

$$\omega^1 = \omega,$$
$$(\omega^g)^h = \omega^{gh}.$$

We shall use these notations interchangeably.

Lemma 1.10. *A group action $A : \Omega \times G \to \Omega$ is equivalent to a group homomorphism $\phi : G \to \mathrm{Sym}(\Omega)$ given by $g^\phi : \omega \mapsto \omega^g$. Also, conversely, given a homomorphism $\psi : G \to \mathrm{Sym}(\Omega)$, there exists a group action $B : \Omega \times G \to \Omega$ defined by $B(\omega, g) = \omega^{(g^\psi)}$.*

Proof. Let $g, h \in G$. Then

$$(gh)^\phi : \omega \mapsto \omega^{gh}$$
$$(gh)^\phi : \omega \mapsto (\omega^g)^h .$$

Now, $g^\phi : \omega \mapsto \omega^g$ and $h^\phi : \omega^g \mapsto (\omega^g)^h$, and so $g^\phi h^\phi : \omega \mapsto (\omega^g)^h$. Thus $(gh)^\phi = g^\phi h^\phi$ and ϕ is a homomorphism.

Suppose instead we have some homomorphism $\psi : G \to \mathrm{Sym}(\Omega)$. Then for $\omega \in \Omega$,

$$\omega^{1_G} = \omega^{(1_G^\psi)} = \omega^{1_{\mathrm{Sym}(\Omega)}} = \omega,$$

since homomorphisms map identities. Also, for $g \in G$,

$$\begin{aligned}(\omega^g)^h &= (\omega^{(g^\psi)})^h = (\omega^{(g^\psi)})^{(h^\psi)} \\ &= \omega^{g^\psi h^\psi} = \omega^{(gh)^\psi} = \omega^{(gh)},\end{aligned}$$

which satisfies the axioms of a group action. *QED*

We note that the set of automorphisms $\mathrm{Aut}(\Gamma)$ naturally acts on $V(\Gamma)$ and $E(\Gamma)$ by permuting vertices.

1.3.1 Orbits and Stabilizers

Definition 1.11. Suppose that a group G is acting on some set Ω, and that $\omega \in \Omega$. Then the *orbit* of ω under the group action is defined to be

$$\omega^G = \{\omega^g \mid g \in G\}.$$

Intuitively, an element's orbit is the set of all elements that it can be mapped to by the group. There is a very powerful result regarding the number of orbits of a group action, called the *Cauchy-Frobenius Theorem*, that we will use to great effect.

Theorem 1.12. *Let G be a finite group acting on the set X. If $|X/G|$ denotes the number of orbits of this action and $\mathrm{fix}_g(X) \subseteq X$ is the set of elements in X fixed by g, then*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |\mathrm{fix}_g(X)|.$$

Proof. See Lemma 2.2.4 in Godsil, Royle [8], *QED*

An analogous concept to orbits is the *stabilizer* of ω, defined by:

$$G_\omega = \{g \in G \mid \omega^g = \omega\}.$$

The stabilizer, as the name suggests, of ω is the set of all elements in the group that do not move ω.

1.3.2 Graph Coloring

With the geometric interpretation of graphs in mind, it is intuitive to imagine painting each edge a different color.

Definition 1.13. Formally, we can define some set C as a set of colors, say, $C := \{\text{red}, \text{blue}, \text{green}\}$. An *edge coloring function* for the graph Γ is a function $f : E(\Gamma) \to C$.

The image of each edge with regards to f is the color of the edge under this coloring. Note that a similar definition can be constructed for *vertex coloring functions*. Since C has no structure — it is just a set — there is not much more for us to say about graph coloring in general without adding some constraints, such as forbidding adjacent edges to be colored similarly, or considering only isomorphically-distinct colorings.

Lemma 1.14. *Suppose that G acts on the set Ω, and let C^Ω be a set of coloring functions for Ω. Then there is a natural induced group action from G on C^Ω defined by*

$$f^g(\omega) = f(\omega^{g^{-1}})$$

for $\omega \in \Omega$, $g \in G$, and $f \in C^\Omega$.

Proof. It suffices to demonstrate that both group action axioms hold. First, note that

$$f^1(\omega) = f(\omega^1) = f(\omega).$$

Next, for $g, h \in G$,

$$\begin{aligned}(f^g)^h(\omega) &= f^g(\omega^{h^{-1}}) \\ &= f(\omega^{h^{-1}g^{-1}}) \\ &= f(\omega^{(gh)^{-1}}) \\ &= f^{(gh)}(\omega).\end{aligned}$$

QED

Specifically, there is a group action of $\mathrm{Aut}(\Gamma)$ on the set of (edge or vertex) colorings of a graph Γ. To that end we present here *Polya's Enumeration Theorem*, an application of Theorem 1.12 that we can use to determine the number of orbits of vertex-colored graphs under the graph's automorphism group.

Theorem 1.15. *Let* Γ *be a graph and* $C^{V(\Gamma)}$ *be the set of vertex coloring functions of* Γ. *Next, let* $c(\pi)$ *be the number of cycles, including fixed points, of* π *in* $\mathrm{Aut}(\Gamma)$. *Note that* $c(\pi)$ *is equivalent to the number of orbits of* $\langle \pi \rangle$ *in* $\mathrm{Sym}(|V(\Gamma)|)$. *Then if* $|C^{V(\Gamma)}/\mathrm{Aut}(\Gamma)|$ *is the number of orbits of* $\mathrm{Aut}(\Gamma)$ *on* $C^{V(\Gamma)}$*:*

$$|C^{V(\Gamma)}/\mathrm{Aut}(\Gamma)| = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\pi \in \mathrm{Aut}(\Gamma)} |C|^{c(\pi)}.$$

Proof. See Tucker [18]. *QED*

Theorem 1.15 will prove useful in Chapter 4. More immediately, we shall see that if we add some group structure to C, we can make some additional interesting observations.

Chapter 2

Signed Graphs and Switching Automorphisms

It should be noted that the general theory of signed graphs and switching contained herein, as well as much of its notation, is derived from Zaslavsky [21].

2.1 Signed Graphs

Definition 2.1. A *signed graph* is a graph wherein each edge is assigned a *sign*—either positive $+$ or negative $-$. This initial definition is equivalent to a coloring, as for a signed graph Γ we define a function

$$\sigma : E(\Gamma) \to \{+, -\}$$

that signs every edge. A graph Γ paired with some such signature σ is a signed graph denoted $\Sigma = (\Gamma, \sigma)$. We will use the notation Σ^- and Σ^+ to denote the negative and positive edges of a signed graph Σ respectively.

There are $2^{|E(\Gamma)|}$ signed graphs of Γ, since each can be identified with a subset of $E(\Gamma)$, namely the set of all the negative edges. However, if we add certain group actions to a signed graph, there are ways in which certain signed graphs are equivalent. First, let us define a group action A of $\mathrm{Aut}(\Gamma)$ upon signatures σ. Let

$$A : \{+,-\}^{E(\Gamma)} \times \mathrm{Aut}(\Gamma) \to \{+,-\}^{E(\Gamma)},$$

such that for $\phi \in \mathrm{Aut}(\Gamma)$ and $\sigma \in \{+,-\}^{E(\Gamma)}$,

$$\{u,v\}^{\sigma^{\phi}} = \{u^{\phi^{-1}}, v^{\phi^{-1}}\}^{\sigma},$$

and equivalently,

$$\{u^{\phi}, v^{\phi}\}^{\sigma^{\phi}} = \{u,v\}^{\sigma}.$$

We will ensure that this is indeed a group action. First,

$$\{u,v\}^{\sigma^{1}} = \{u,v\}^{\sigma}.$$

Next, letting $\phi, \psi \in \mathrm{Aut}(\Gamma)$, we have

$$\begin{aligned}
\{u,v\}^{(\sigma^{\phi})^{\psi}} &= \{u^{\psi^{-1}}, v^{\psi^{-1}}\}^{\sigma^{\phi}} \\
&= \{u^{\psi^{-1}\phi^{-1}}, v^{\psi^{-1}\phi^{-1}}\}^{\sigma} \\
&= \{u^{(\phi\psi)^{-1}}, v^{(\phi\psi)^{-1}}\}^{\sigma} = \{u,v\}^{\sigma^{\phi\psi}}.
\end{aligned}$$

Thus A is indeed a group action. The orbit of a signature under this action is the set of all signed graphs of Γ that can be reached by permuting Σ by $\mathrm{Aut}(\Gamma)$, ensuring that negative edges are mapped to negative edges and positive edges are mapped to positive edges. If there exists some $\phi \in \mathrm{Aut}(\Gamma)$ such that for $\Sigma_1 = (\Gamma, \sigma_1)$, $\Sigma_2 = (\Gamma, \sigma_2)$, ${\sigma_1}^{\phi} = \sigma_2$, then we say that Σ_1 and Σ_2 are *isomorphic as signed graphs*. An example of two isomorphic signed graphs is below; here and elsewhere, negative edges are dotted. Note that the second graph can be achieved by reflecting the first across the lower-left to upper-right diagonal, and that the signed graph is mapped along, via the action defined above.

Two isomorphic signed graphs

2.2 Switching

There is another interesting group action that we can define on a set of signed graphs of Γ.

Definition 2.2. Let $\{+,-\}^{V(\Gamma)}$ be the set of all functions from the vertices of Γ to $\{+,-\}$. We call these *switching functions* of Σ.

The reason for this name will become clear later. Much like the signatures we discussed earlier, these map parts of a graph to signs, but in this case we are mapping vertices, not edges. We typically denote elements of $\{+,-\}^{V(\Gamma)}$ as ζ_X, where $X \subseteq V(\Gamma)$ is the set of vertices that ζ_X takes to $-$. It turns out that we can define a group operation on these functions in $\{+,-\}^{V(\Gamma)}$. Let $\zeta_X, \zeta_Y \in \{+,-\}^{V(\Gamma)}$ for $X, Y \subseteq V(\Gamma)$. Then define $*$ by

$$\zeta_X * \zeta_Y = \zeta_{X \triangle Y},$$

where $\triangle$ refers to symmetric difference. The inverse of ζ_X is ζ_X itself. Then

$$\zeta_X * (\zeta_Y)^{-1} = \zeta_{X \triangle Y}.$$

Since subsets are closed under symmetric difference, $*$ indeed induces a group structure. Moreover, every element in this group is its own inverse, and so $(\{+,-\}^{V(\Gamma)}, *)$ is an elementary abelian 2-group. From now on, we shall denote it by $\{+,-\}^{V(\Gamma)}$.

This group has a very interesting action on signed graphs, or more precisely, on signatures of signed graphs. First we will define it formally, by

$$B : \{+,-\}^{E(\Gamma)} \times \{+,-\}^{V(\Gamma)} \to \{+,-\}^{E(\Gamma)},$$

such that for $\sigma \in \{+,-\}^{E(\Gamma)}$ and $\zeta_X \in \{+,-\}^{V(\Gamma)}$, under the induced homomorphism B_h,

$$\{u,v\}^{\sigma^{\zeta_X}} = u^{\zeta_X}\{u,v\}^{\sigma}v^{\zeta_X}.$$

Again we shall check that this is a true group action. It will be useful to realize that $\{+,-\}$, the codomain of our signatures σ and ζ_X, itself is a group isomorphic to $\mathbb{Z}_2$, with $+$ identified with 0 and $-$ with 1. Note that the identity of $\{+,-\}^{V(\Gamma)}$ is $\zeta_+ := \zeta_\varnothing$ and

$$\{u,v\}^{\sigma^{\zeta_+}} = u^{\zeta_+}\{u,v\}^{\sigma}v^{\zeta_+} = (+)\{u,v\}^{\sigma}(+) = \{u,v\}^{\sigma}.$$

Also, for $X, Y \subseteq V(\Gamma)$,

$$\begin{aligned}
\{u,v\}^{(\sigma^{\zeta_X})^{\zeta_Y}} &= u^{\zeta_Y}u^{\zeta_X}\{u,v\}^{\sigma}v^{\zeta_X}v^{\zeta_Y} \\
&= u^{\zeta_{X \triangle Y}}\{u,v\}^{\sigma}v^{\zeta_{X \triangle Y}} \\
&= \{u,v\}^{\sigma^{\zeta_{X \triangle Y}}} \\
&= \{u,v\}^{\sigma^{\zeta_X * \zeta_Y}},
\end{aligned}$$

so B is a group action. For $\Sigma = (\Gamma, \sigma)$, we shall sometimes use the notation Σ^{ζ_X} to refer to $(\Gamma, \sigma^{\zeta_X})$. There is an intuitive, geometric interpretation of this action: switching a vertex on a signed graph turns all of the negative edges incident with the vertex positive, and turns all of the positive edges incident with the vertex negative (hence, "switching"). If two adjacent vertices are switched, the shared edge retains its sign because it is first switched and then switched back. Generalizing, a switching function ζ_X switches all edges that are incident with vertices in X, except for those edges that are shared by elements of X. One can imagine grasping the vertices in X and pulling them away from the remaining vertices in $\overline{X} = V(\Gamma) - X$—the edges that remain as strands between the two sets of vertices are switched. All others are unchanged. In the example below, the vertex 2 is switched between the two graphs, with dashed lines negative:

Definition 2.3. A *switching equivalence class* containing a signed graph Σ is the set of signed graphs Π such that there exists some $\overline{\zeta_X} \in \mathrm{Sw}(\Gamma)$ and $\Pi^{\overline{\zeta_X}} = \Sigma$. Membership in these classes induces an equivalence relation among signed graphs of Γ.

For any graph Γ, there are certain switching functions ζ_X that do not switch any edges, and it happens that these functions are inherent to the graph. That is, they are not particular to any signed graph of Γ. By definition, these signed graphs form the kernel of the induced homomorphism $B_h : \{+,-\}^{V(\Gamma)} \to \mathrm{Sym}\left(\{+,-\}^{E(\Gamma)}\right)$ of the group action B, since they leave every signature unchanged. Thus by the first isomorphism theorem, these switching functions form a normal subgroup of $\{+,-\}^{V(\Gamma)}$, which we will call κ_Γ.

Lemma 2.4. *For Γ a simple, connected graph, $\kappa_\Gamma = \{\zeta_+, \zeta_-\}$, where $\zeta_+ = \zeta_\varnothing$ and $\zeta_- = \zeta_{V(\Gamma)}$.*

Proof. First we demonstrate that $\zeta_+, \zeta_- \in \kappa_\Gamma$. Letting σ be some signature for Γ,

$$\{u,v\}^{\sigma^{\zeta_+}} = u^{\zeta_+}\{u,v\}^\sigma v^{\zeta_+} = (+)\{u,v\}^\sigma(+) = \{u,v\}^\sigma,$$
$$\{u,v\}^{\sigma^{\zeta_-}} = u^{\zeta_-}\{u,v\}^\sigma v^{\zeta_-} = (-)\{u,v\}^\sigma(-) = \{u,v\}^\sigma.$$

Next, let $\zeta_X \in \{+,-\}^{E(\Gamma)}$ for a nonempty $X \subset V(\Gamma)$. Then there exists some $x \in X$ and $y \in \overline{X}$, and since Γ is connected, there is a path from x to y. Thus, there exists some edge $\{w,z\}$ somewhere along this path such that $w \in X$ and $z \in \overline{X}$, and

$$\{w,z\}^{\sigma^{\zeta_X}} = w^{\zeta_X}\{w,z\}^\sigma z^{\zeta_X} = (-)\{w,z\}^\sigma(+) = -\{w,z\}^\sigma,$$

and so $\zeta_X \notin \kappa_\Gamma$. *QED*

Finally, there is a natural group action from $\mathrm{Aut}(\Gamma)$ on $\{+,-\}^{V(\Gamma)}$ which we shall here call $C : \{+,-\}^{V(\Gamma)} \times \mathrm{Aut}(\Gamma) \to \{+,-\}^{V(\Gamma)}$, defined by its induced homomorphism

$$v^{\zeta_X{}^{\phi}} = (v^{\phi^{-1}})^{\zeta_X}.$$

Equivalently, $(v^{\phi})^{\zeta_X^{\phi}} = v^{\zeta_X}$. Note that $v^{\zeta_X} = -$ if and only if $v \in X$, and so equivalently,

$$v^{\zeta_X{}^{\phi}} = v^{\zeta_{X^{\phi}}}.$$

Again we verify this action follows the axioms of group actions.

$$v^{\zeta_X^1} = v^{\zeta_{X^1}} = v^{\zeta_X}$$

For $\phi, \psi \in \mathrm{Aut}(\Gamma)$,

$$v^{(\zeta_X{}^{\phi})^{\psi}} = v^{\zeta_{X^{\phi}}^{\psi}} = v^{\zeta_{X^{\phi\psi}}} = v^{\zeta_X{}^{\phi\psi}}.$$

In summary, we have two main ways in which we can manipulate signed graphs: we can permute the vertices of Γ under $\mathrm{Aut}(\Gamma)$, which preserves signatures and so is analogous to rearranging a graph colored with two colors "positive" and "negative". We also can switch vertices, turning incident negative edges positive and vice-versa. These actions combined give us ways in which two signed graphs can be equivalent.

Definition 2.5. Let $\Sigma_1 = (\Gamma, \sigma_1)$ and $\Sigma_2 = (\Gamma, \sigma_2)$ be signed graphs of Γ. We say that Σ_1 and Σ_2 are *switching isomorphic* if there exists some $\phi \in \mathrm{Aut}(\Gamma)$ and $\zeta = \zeta_X \in \{+,-\}^{V(\Gamma)}$ such that

$$\{u,v\}^{\sigma_2} = \{u,v\}^{\sigma_1{}^{\zeta\phi}},$$

for all $\{u, v\} \in E(\Gamma)$. We then say

$$\Sigma_1 \cong \Sigma_2.$$

Note that $\sigma^{\zeta\phi}$ denotes applying the group actions B and A in turn, and is generally *not* equivalent to σ^{ζ^ϕ}. Instead, in the group of all actions on $\{+, -\}^{V(\Gamma)}$ (which we shall soon formally define), ζ^ϕ is equivalent to a conjugation of ζ by ϕ:

$$\{u, v\}^{\sigma^{\zeta^\phi}} = u^{\zeta^\phi}\{u, v\}^{\sigma} v^{\zeta^\phi} = \left(u^{\zeta}\{u, v\}^{\sigma^{\phi^{-1}}} v^{\zeta}\right)^{\phi} = \{u, v\}^{\sigma^{\phi^{-1}\zeta\phi}}.$$

The set of signed graphs that are switching isomorphic to Σ form an equivalence class. In a way, these graphs can be identified with one another, because they are indistinguishable up to our group actions. Finding all of the *switching isomorphism classes* of a graph Γ is not trivial, as testing every possible signed graph under every possible automorphism and switching combination becomes infeasible with large graphs. See Zaslavsky [21], Sivaraman [17], and Bagheri, Moghaddemfar and Ramezani [1] for previous work regarding the enumeration of the switching isomorphism classes of specific graphs.

2.3 Switching Automorphism Groups

For every unsigned graph Γ, we can define its *switching automorphism group* that contains all of its symmetries under switching and vertex permutation. We are not yet concerned about elements of this group preserving any particular signed graph of Γ in any way—they preserve the structure of Γ itself as it relates to the actions we have described. We denote this group $\mathrm{SwAut}(\Gamma)$ and can construct it as follows.

We would like $\mathrm{SwAut}(\Gamma)$ to contain within it both switchings and automorphisms. However, we have noted that for a graph Γ, not all switchings change any signed graph. We want to remove redundancies from the group $\{+, -\}^{V(\Gamma)}$ and consider only "classes" of switchings that are truly distinct, and so we quotient out κ_Γ.

Definition 2.6. The *switching group* $\mathrm{Sw}(\Gamma)$ of a graph Γ is defined by

$$\mathrm{Sw}(\Gamma) = \{+,-\}^{V(\Gamma)}/\kappa_\Gamma.$$

This is the group of switching functions of Γ, and it should be noted that (via the homomorphism described earlier) this group is isomorphic to some subgroup of the set of automorphisms of $\{+,-\}^{E(\Gamma)}$. Its members $\zeta_X\kappa_\Gamma$ we shall typically write as $\overline{\zeta_X}$. We now have two groups $\mathrm{Sw}(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ that act upon signed graphs, and there is interplay between them. Recall that we have defined a group action C of $\mathrm{Aut}(\Gamma)$ on switching functions. We will define an action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{Sw}(\Gamma)$ similarly. For $\phi \in \mathrm{Aut}(\Gamma)$,

$$\overline{\zeta_X}^{\phi} = \overline{\zeta_{X^\phi}}.$$

We want to construct a *switching automorphism group* that encodes in its structure the actions of $\mathrm{Sw}(\Gamma)$ and $\mathrm{Aut}(\Gamma)$. However, switching functions and graph automorphisms do not necessarily commute in their action on signed graphs, and so we cannot construct a direct product. Instead we can use the group action we just defined to define a semidirect product.

Definition 2.7. The *switching automorphism group* for Γ is the group

$$\mathrm{SwAut}(\Gamma) = \mathrm{Sw}(\Gamma) \rtimes_\lambda \mathrm{Aut}(\Gamma),$$

where $\lambda : \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\mathrm{Sw}(\Gamma))$ is the homomorphism induced from the group action

$$\overline{\zeta_X}^{\phi} = \overline{\zeta_{X^\phi}}.$$

The group operation $*$ is defined by

$$(\overline{\zeta_X},\phi)*(\overline{\zeta_Y},\psi)=(\overline{\zeta_X}\,\overline{\zeta_Y}^{\phi^{-1}},\phi\psi)=(\overline{\zeta_X}\,\overline{\zeta_{Y^{\phi^{-1}}}},\phi\psi)=(\overline{\zeta_{X\triangle Y^{\phi^{-1}}}},\phi\psi),$$

and we note

$$(\overline{\zeta_X},\phi)^{-1}=(\overline{\zeta_{X^\phi}},\phi^{-1}).$$

We call its elements *switching automorphisms* and often write $(\overline{\zeta_X},\phi)$ as $\overline{\zeta_X}\phi$.

This operation preserves the conjugation of $\mathrm{Aut}(\Gamma)$ on $\mathrm{Sw}(\Gamma)$ that we observed earlier, if we identify these groups with the isomorphic subgroups $\{(\zeta_+,\phi)\mid\phi\in\mathrm{Aut}(\Gamma)\}$ and $\{(\overline{\zeta_X},1)\mid\overline{\zeta_X}\in\mathrm{Sw}(\Gamma)\}$ respectively. Note that $\mathrm{SwAut}(\Gamma)$ acts on signed graphs of Γ in the natural way, i.e., combinations of switching functions and graph automorphisms.

Definition 2.8. The orbits of the action of $\mathrm{SwAut}(\Gamma)$ on the set of signed graphs of Γ form equivalence classes called *switching isomorphism classes.* Two signed graphs Σ_1 and Σ_2 are switching isomorphic (Definition 2.4) if they are in the same switching isomorphism class. Equivalently, there exists $(\overline{\zeta_X},\phi)\in\mathrm{SwAut}(\Gamma)$ such that

$$\Sigma_1^{\overline{\zeta_X}\phi}=\Sigma_2.$$

We shall thus denote the switching isomorphism classes of Γ by $\Gamma/\mathrm{SwAut}(\Gamma)$, and for Σ a signed graph of Γ, $[\Sigma]$ shall be taken to mean the class in $\Gamma/\mathrm{SwAut}(\Gamma)$ containing Σ.

Switching automorphism groups are named as such because they contain bijections between signed graphs of Γ that preserve certain of said signed graphs' fundamental properties. However, a switching automorphism need not (and generally, does not) stabilize an arbitrary signed graph. The switching automorphisms that do preserve a specific signed graph form their own group.

Definition 2.9. Let $\Sigma=(\Gamma,\sigma)$ be a signed graph of Γ. The *signed graph switching automorphism group* $\mathrm{SwAut}(\Sigma)\leqslant\mathrm{SwAut}(\Gamma)$ is the set of all switching automorphisms

$(\overline{\zeta_X}, \phi) \in \mathrm{SwAut}(\Gamma)$ such that $\sigma^{\overline{\zeta_X}\phi} = \sigma$.

Sometimes, we shall simply call these "switching automorphism groups" where the context allows it. Intuitively, $\mathrm{SwAut}(\Sigma)$ is the set of all pairs of switchings and automorphisms in $\mathrm{SwAut}(\Gamma)$ such that performing the switching on Σ and then the automorphism is equivalent, in terms of the signature, to doing nothing. In this way, it is a group of symmetries of Σ. It is not generally the case that subgroups of semidirect products are semidirect products of subgroups, and so constructing $\mathrm{SwAut}(\Sigma)$ is difficult. It is also worth noting that since elements of $\mathrm{SwAut}(\Sigma)$ stabilize the sets of negative and positive edges alike, $\mathrm{SwAut}(\Sigma) = \mathrm{SwAut}(-\Sigma)$, where $-\Sigma$ has negative edges where Σ has positive ones and vice-versa. Since it is not generally the case that $\Sigma \cong -\Sigma$, this means that two signed graphs can be switching inequivalent yet have the same group of automorphisms. The first of our two algorithms can help us find all of the signed graph switching automorphism groups for small graphs. Selected results can be found in Chapter 5.

2.4 Previous Work

Zaslavsky has written extensively about the theory of signed graphs and their automorphisms [19],[20] and has explicitly determined the switching automorphism groups for the switching isomorphism classes of the Petersen graph [21]. Bagheri, Moghaddemfar and Ramezani have enumerated the switching isomorphism classes of the Generalized Petersen graph $GP(7, 2)$ and complete graph K_5 [1], and Sivaraman [17] has enumerated those of the Heawood graph. A formula is known for the number of switching isomorphism classes for the family of complete graphs K_n as they are in direct correspondence with the sets of two-graphs and Eulerian graphs on n nodes.

More generally, Zaslavsky has noted [20] the potential use of double covering graphs to enumerate switching isomorphism classes, and Hofmeister [10] has proven a formula for the number of such covers of arbitrary graphs. Finally, Cameron, has written about the cohomology of both signed [5] and unsigned [3] graph switching, as has Seidel [16].

Chapter 3

Enumeration Algorithms

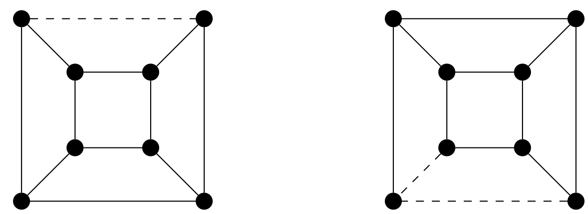
The algorithms we shall discuss were written using GAP [7] and take advantage of its GRAPE [9] and Digraphs [2] packages.

3.1 First Attempt

Definition 3.1. Let Γ be a simple connected graph. For each switching isomorphism class $[\Sigma]$ in $\Gamma/\mathrm{SwAut}(\Gamma)$, there exists at least one *minimal signed graph*, which we define as any signed graph $\Psi \in [\Sigma]$ such that for all $\Pi \in [\Sigma]$, $|\Psi^-| \leqslant |\Pi^-|$. That is, a minimal signed graph of a switching isomorphism class has the fewest negative edges of any signed graph in its class.

Note that minimal signed graphs need not be unique. An example is given below of switching-isomorphic signed graphs of the cube graph; the signed graph on the left is minimal, because switching any combination of vertices would increase or keep stable the number of negative edges. It is not unique because a rotation by 90 degrees would produce a different signed graph with exactly one negative edge. The signed graph on the right is a non-minimal representative of the same switching isomorphism class; switching the lower left vertex and rotating the graph clockwise would produce the signed graph on the left.

Two switching-isomorphic signed graphs



Our first algorithm took inspiration from Brendan McKay's Orderly Algorithm [12] and intended to produce all possible graph isomorphism-free minimal signed graphs of a given graph before sorting them for switching isomorphism. The manner in which we determined switching isomorphs relied upon a known equivalence between switching-isomorphic signed graphs and graph-isomorphic double covers.

Definition 3.2. A *signed graph double cover* is an unsigned graph $\mathcal{C}_\sigma(\Sigma)$ that we may construct from a signed graph $\Sigma = (\Gamma, \sigma)$ that encodes the negative and positive edges of Σ without using signed edges, which is easier for a computer program to examine. We construct $\mathcal{C}_\sigma(\Sigma)$ as follows. Let $\Sigma = (\Gamma, \sigma)$ be a signed graph of Γ with negative edges the fiber $\sigma^{-1}(-) \subseteq E(\Gamma)$. Then:

$$V(\mathcal{C}_\sigma(\Sigma)) = V(\Gamma) \times \{+, -\},$$

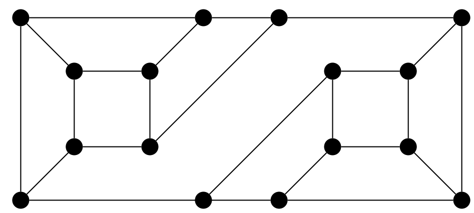
and $\{(v_1, s_1), (v_2, s_2)\} \in E(\mathcal{C}_\sigma(\Sigma))$ if and only if

$$\{v_1, v_2\} \in E(\Gamma) \text{ and } \{v_1, v_2\}^\sigma = s_1 s_2.$$

Intuitively, the vertices of $\mathcal{C}_\sigma(\Sigma)$ are two "copies" of the vertices of Γ, one "positive" and one "negative". For each positive edge $\{a, b\}$ in Σ, we add an edge between the corresponding positive pair $\{(a, +), (b, +)\}$ in the positive copy and another edge between the corresponding negative pair $\{(a, -), (b, -)\}$ in the negative copy. For all negative edges $\{a, b\}$, we add to the double cover an edge between $(a, -)$ and $(b, +)$, and likewise between $(a, +)$ and $(b, -)$. For example, the all positive signed graph corresponds to a double cover that is simply two disconnected copies of Γ:

$$\mathcal{C}_\sigma(\Sigma_+) \cong \Gamma + \Gamma.$$

Otherwise, the set of negative edges in Σ correspond to pairs of edges that span the two copies of the original graph. Below is an example for the (unlabeled) signed double cover of the cube graph signed graph on the right on the previous page.



T. Zaslavsky has proven [20] a remarkable fact on which the first algorithm rests.

Theorem 3.3. *There is a one-to-one correspondence between switching isomorphism classes of* Γ *and signed double covers of* Γ *up to isomorphism. Furthermore, two signed graphs* Σ_1 *and* Σ_2 *of* Γ *are switching isomorphic if and only if* $\mathcal{C}_\sigma(\Sigma_1) \cong \mathcal{C}_\sigma(\Sigma_2)$ *as unsigned graphs.*

Since there are computer programs (such as nauty [13]) available to determine whether two graphs are isomorphic, we now have the means to readily filter a set of signed graphs for switching isomorphism.

3.1.1 The Algorithm

Algorithm 1 takes an edge set $E(\Gamma)$ and generates an exhaustive list of candidate signed graphs for Γ before converting them to signed double covers and filtering them for isomorphism. The isomorph-free representatives are then converted back into signed graphs of Γ (precisely, the corresponding negative edge sets of signed graphs) and output by the program.

The signed graph candidates are generated by iteration on the number of edges, wherein each signed graph with n negative edges is used as a seed to generate those with $n + 1$ negative edges by considering what new edges could be added. We choose this method because it enables us to ignore redundant signed graphs, as the process for finding signed graphs of large graphs (of which there are $2^{|E(\Gamma)|}$ total) and sorting their double covers for isomorphism is time- and memory-intensive. Let Σ be some signed graph of Γ, identified with its set $X \subseteq E(\Gamma)$ of negative edges, and consider the set $Y = E(\Gamma) - X$ of possible edges that could be added to X to form a new signed graph. The stabilizer of X in $\mathrm{Aut}(\Gamma)$ acts on the set of such Y and thus induces a set of orbits. If edges $Y_1, Y_2 \in Y$ are within the same orbit, then there exists some permutation $\pi \in \mathrm{Aut}(\Gamma)_X$ such that $Y_1^\pi = Y_2$ and thus that $(X + Y_1)^\pi = X + Y_2$. We may conclude that $X + Y_1$ and $X + Y_2$ are equivalent as potential signed graphs, since they are equivlent under $\mathrm{Aut}(\Gamma)$. This fact enables us to cut down on the number of signed graphs to process by first calculating the orbits of $\mathrm{Aut}(\Gamma)_X$ on Y and then selecting one candidate edge from each.

The other method we use to profile potential signed graphs for redundancy takes advantage of the existence of a minimal signed graph for each switching isomorphism class. First, a *cut* of a graph Γ is a partition of its vertex set $V(\Gamma)$ into two sets X and $\overline{X}$. Given a cut $(X, \overline{X})$, the *cutset* is the set of edges with one vertex in X and the other in $\overline{X}$. Note that since $\overline{X} = V(\Gamma) - X$, we may define the edge cut ∇X by

$$\nabla X = \{\{u, v\} \in E(\Gamma) \mid u \in X, v \notin X\}.$$

With this concept understood we may consider the following, which says that if there exists set of vertices in Σ whose edge cut contains more negative edges than positive edges, Σ is not a minimal signed graph.

Lemma 3.4. *Let Σ be a signed graph of the graph Γ. If there exists some edge cut ∇X such that $|\nabla X \cap \Sigma^-| > |\nabla X \cap \Sigma^+|$, then Σ is not a minimal signed graph of $[\Sigma]$.*

Proof. Let $\Sigma = (\Gamma, \sigma)$ be a signed graph such that for some $X \subseteq V(\Gamma)$, $|\nabla X \cap \Sigma^-| > |\nabla X \cap \Sigma^+|$. Let $E(X)$ denote the edges $\{x, y\}$ such that $x, y \in X$ and $E(\overline{X})$ denote the edges $\{x, y\}$ such that $x, y, \notin X$. Then we see that $E(X), E(\overline{X})$, and ∇X partition $E(\Gamma)$. We examine the effects of the switching function ζ_X on the signs of edges $\{u, v\}$ in Σ.

$$\{u,v\}^{\sigma\zeta_X} = u^{\zeta_X}\{u,v\}^{\sigma}v^{\zeta_X} = \begin{cases} \{u,v\}^{\sigma}, & u^{\zeta_X} = v^{\zeta_X} \\ -\{u,v\}^{\sigma}, & u^{\zeta_X} = -v^{\zeta_X}. \end{cases}$$

Thus the sign of an edge in Σ is changed if and only if the edge is in ∇X. We know that there are more negative edges in ∇X than positive edges, and so after the switch there are more positive edges than negative edges.

$$|\nabla X \cap (\Sigma^{\zeta_X})^-| = |\nabla X \cap \Sigma^+| < |\nabla X \cap \Sigma^-| = |\nabla X \cap (\Sigma^{\zeta_X})^+|.$$

However, as edges outside of ∇X are not switched, Σ and Σ^{ζ_X} have the same number of negative edges in $E(X) \cup E(\overline{X})$. Thus $|(\Sigma^{\zeta_X})^-| < |\Sigma^-|$, and since $\Sigma^{\zeta_X} \in [\Sigma]$, Σ is not minimal by definition. *QED*

This fact gives us a method by which we can quickly discard certain potential signed graphs of length n (that is, with n negative edges) as not minimal, and thus necessarily redundant, if we have collected all possible signed graphs of length $m < n$.

Determining all possible cuts of a given graph is not trivial, and it would not be practical to test the minimality of a every new signed graph. However, we can test whether or not adding the edge e to a signed graph X will make a signed graph that could be switched to a smaller signed graph by switching less than or equal to k vertices, for some $k \geqslant 1$. The more selective we are for new edges e (the higher the k), the longer it will take to compute whether e is eligible, and so there is a trade-off in increasing k and generating fewer potential signed graphs. The final code screens new edges with $k = 4$; that is, for a signed graph X, it discards all edges e if $X + e$ could be switched to a smaller signed graph by switching fewer than 5 vertices. This is performed by first ensuring that adding e would not make any vertex in X incident with more negative (in X) than positive edges and then calculating the negative-positive "balance" of vertices adjacent to each end of e.

For example, let v be a vertex of e that is incident with $n-1$ negative edges (not including e) and $n+1$ positive edges. Suppose we add e to X, making v incident with n positive and

n negative edges. If u is adjacent to v along a positive edge, and u too is incident with n positive and n negative edges, then switching u and v would lessen the number of negative edges, but switching either individually would not. The algorithm prevents such $X + e$ from being considered as signed graphs, and more strictly, it examines any 3-path of positive that begins at a candidate edge e, finds the negative-positive balance of each vertex along this path, and uses this information to determine whether switching some subset of those 4 vertices would make $|X + e| < |X| + 1$, discarding e as a candidate if it would.

On one specific note, consider a graph Γ wherein each vertex is incident with 3 (or fewer) edges. Then if Σ is a signed graph of Γ such that some vertex v in Σ is incident with two negative edges, switching v would make those edges negative and the positive edge incident with v (if one even exists) negative, necessarily reducing Σ to a smaller signed graph. This demonstrates that in so-called (sub-)cubic graphs, every minimal signed graph must contain no incident negative edges (subgraphs with non-incident edges are called *matchings*). Therefore, when using this algorithm to find signed graphs of (sub-)cubic graphs, we can automatically discard as a candidate new-edge for a signed graph X any edge e that shares a vertex with a negative edge in X. We have included as an input to our algorithm the potential for Γ to be (sub-)cubic; if we indicate that it is, our algorithm will save time by only considering matchings as potential signed graphs.

Regardless, during the algorithm, the construction of signed graphs continues until no more edges could be added to any existing signed graphs without violating one of the constraints we have discussed. At this point, each signed graph is converted into a signed double cover. We then take advantage of the computer algebra program GAP [7], its GRAPE package [9], and the

`GraphIsomorphismClassRepresentatives`

function, which uses nauty to produce one representative for each isomorphism class of a list of graphs. Inputting our list of signed double covers produces such representatives. We can then either convert these back into signed graphs or simply take the length of the list to count how many classes exist. By Theorem 3.3, this number is equal to the number of switching isomorphism classes.

This program is considerably less efficient than the next in calculating the number of switching isomorphism classes of an arbitrary graph. However, its advantage is that it involves explicit construction and thus produces an exhaustive set of switching-isomorphic-free signed graphs. In addition to allowing us to see signed graphs for each switching isomorphism class, this enables us to readily calculate the switching automorphism groups for each class.

Lemma 3.5. *Let Σ_1, Σ_2 be signed graphs of Γ such that $\Sigma_1 \cong \Sigma_2$; that is, they belong to the same switching isomorphism class of Γ. Then* $\mathrm{SwAut}(\Sigma_1) \cong \mathrm{SwAut}(\Sigma_2)$.

Proof. Since $\Sigma_1 \cong \Sigma_2$, there exists some $\overline{\zeta_X}\phi \in \mathrm{SwAut}(\Gamma)$ such that

$$\Sigma_1^{\overline{\zeta_X}\phi} = \Sigma_2.$$

We define $m : \mathrm{SwAut}(\Sigma_1) \to \mathrm{SwAut}(\Sigma_2)$ by

$$(\overline{\zeta_A}\alpha)^m = (\overline{\zeta_A}\alpha)^{(\overline{\zeta_X}\phi)} = (\overline{\zeta_{X^\phi}}\phi^{-1})(\overline{\zeta_A}\alpha)(\overline{\zeta_X}\phi),$$

where $\overline{\zeta_A} \in \mathrm{SwAut}(\Sigma_1)$, $(\overline{\zeta_X}\phi)^{-1} = \overline{\zeta_{X^\phi}}\phi^{-1}$, and $(\overline{\zeta_A}\alpha)^{(\overline{\zeta_X}\phi)}$ denotes the action of conjugation in $\mathrm{SwAut}(\Gamma)$. Now,

$$\Sigma_2^{(\overline{\zeta_{X^\phi}}\phi^{-1})(\overline{\zeta_A}\alpha)(\overline{\zeta_X}\phi)} = \Sigma_1^{(\overline{\zeta_A}\alpha)(\overline{\zeta_X}\phi)} = \Sigma_1^{(\overline{\zeta_X}\phi)} = \Sigma_2,$$

and so m is well-defined. Next, suppose $\overline{\zeta_B}\beta \in \mathrm{SwAut}(\Sigma_2)$. Then

$$\Sigma_1^{(\overline{\zeta_X}\phi)(\overline{\zeta_B}\beta)(\overline{\zeta_{X^\phi}}\phi^{-1})} = \Sigma_2^{(\overline{\zeta_B}\beta)(\overline{\zeta_{X^\phi}}\phi^{-1})} = \Sigma_2^{(\overline{\zeta_{X^\phi}}\phi^{-1})} = \Sigma_1,$$

so $(\overline{\zeta_X}\phi)(\overline{\zeta_B}\beta)(\overline{\zeta_{X^\phi}}\phi^{-1}) \in \mathrm{SwAut}(\Sigma_1)$ such that $((\overline{\zeta_X}\phi)(\overline{\zeta_B}\beta)(\overline{\zeta_{X^\phi}}\phi^{-1}))^m = \overline{\zeta_B}\beta$ and m is surjective. Finally, if $\overline{\zeta_C}\gamma \in \mathrm{SwAut}(\Sigma_1)$ such that $\overline{\zeta_C}\gamma \in \ker(m)$, then

$$\begin{aligned}(\overline{\zeta_C}\gamma)^m &= (\overline{\zeta_{X^\phi}}\phi^{-1})(\overline{\zeta_C}\gamma)(\overline{\zeta_X}\phi) = 1,\\ (\overline{\zeta_X}\phi)(\overline{\zeta_{X^\phi}}\phi^{-1})(\overline{\zeta_C}\gamma)(\overline{\zeta_X}\phi)(\overline{\zeta_{X^\phi}}\phi^{-1}) &= (\overline{\zeta_X}\phi)(\overline{\zeta_{X^\phi}}\phi^{-1}) = 1,\\ \overline{\zeta_C}\gamma &= 1.\end{aligned}$$

Therefore, m is a bijection, and so $\mathrm{SwAut}(\Sigma_1), \mathrm{SwAut}(\Sigma_2)$ are conjugate subgroups and hence isomorphic subgroups of $\mathrm{SwAut}(\Gamma)$. *QED*

Thus we need only calculate the switching automorphism group for one representative of each switching isomorphism class.

Theorem 3.6. *Let Γ be a simple, connected graph. Let Σ be a signed graph of Γ and let $\mathcal{C}_\sigma(\Sigma)$ be its signed double cover. Let*

$$P = \{\{(v,+),(v,-)\} \mid v \in V(\Gamma)\}$$

be a partition of the vertices of $\mathcal{C}_\sigma(\Gamma)$. Finally, let $\kappa \in \mathrm{Aut}(\mathcal{C}_\sigma(\Sigma))$ be defined by $\{(v,s)\}^\kappa = \{(v,-s)\}$. Then:

$$\mathrm{SwAut}(\Sigma) \cong \mathrm{Aut}(\mathcal{C}_\sigma(\Sigma))_P/\langle\kappa\rangle,$$

where $\mathrm{Aut}(\mathcal{C}_\sigma(\Sigma))_P$ denotes the setwise stabilizer of P in $\mathrm{Aut}(\mathcal{C}_\sigma(\Sigma))$.

Proof. We define a projection $\delta : V(\mathcal{C}_\sigma(\Gamma)) \to V(\Gamma)$ by $(v,s)^\delta = v$. Let $G = \mathrm{Aut}(\mathcal{C}_\sigma(\Gamma))$, and G_P be the setwise stabilizer of P in G. Let $f : G_P \to \mathrm{SwAut}(\Sigma)$ be defined by

$$\pi^f = \overline{\zeta_{X_\pi}}\phi_\pi,$$

where X_π is defined by

$$X_\pi = \{v \in V(\Gamma) \mid \exists u \in V(\Gamma), (v,+)^\pi = (u,-)\},$$

and ϕ_π is defined by

$$v^{\phi_\pi} = ((v,s)^\pi)^\delta.$$

It is clear that $X_\pi \subseteq V(\Gamma)$, and so $\overline{\zeta_{X_\pi}} \in \mathrm{Sw}(\Gamma)$. Next, note that

$$((v,+)^\pi)^\delta = ((v,-)^\pi)^\delta$$

by the definition of G_P. Now,

$$\{(a,s),(b,r)\} \in E(\mathcal{C}_\sigma(\Gamma)) \implies \{a,b\} \in E(\Gamma),$$

and also, since $\pi \in G_P$,

$$\begin{aligned}
\{(a,s),(b,r)\} \in E(\mathcal{C}_\sigma(\Gamma)) &\iff \{(a,s)^\pi,(b,r)^\pi\} \in E(\mathcal{C}_\sigma(\Gamma)) \\
&\implies \{((a,s)^\pi)^\delta,((b,r)^\pi)^\delta\} \in E(\Gamma) \\
&\iff \{a^{\phi_\pi}, b^{\phi_\pi}\} \in E(\Gamma).
\end{aligned}$$

Thus $\phi_\pi \in \mathrm{Aut}(\Gamma)$. Finally, let $(u,v) \in E(\Gamma)$ with sign $s = \{u,v\}^\sigma$. Then

$$\begin{aligned}
\{(u,+),(v,s)\} &\in E(\mathcal{C}_\sigma(\Gamma)) \\
\{(u,+)^\pi,(v,s)^\pi\} &\in E(\mathcal{C}_\sigma(\Gamma)) \\
\{(u^{\phi_\pi}, u^{\zeta_{X_\pi}})(v^{\phi_\pi}, v^{\zeta_{X_\pi}} s)\} &\in E(\mathcal{C}_\sigma(\Gamma)).
\end{aligned}$$

Thus $\{u^{\phi_\pi}, v^{\phi_\pi}\} \in E(\Gamma)$ and $\{u^{\phi_\pi}, v^{\phi_\pi}\}^\sigma = u^{\zeta_{X_\pi}} v^{\zeta_{X_\pi}} s$, and so

$$\begin{aligned}
u^{\zeta_{X_\pi}}\{u^{\phi_\pi}, v^{\phi_\pi}\}^\sigma v^{\zeta_{X_\pi}} &= \{u, v\}^\sigma \\
(u^{\phi_\pi^{-1}})^{\zeta_{X_\pi}}\{u, v\}^\sigma (v^{\phi_\pi^{-1}})^{\zeta_{X_\pi}} &= \{u^{\phi_\pi^{-1}}, v^{\phi_\pi^{-1}}\}^\sigma \\
\{u, v\}^\sigma &= (u^{\phi_\pi^{-1}})^{\zeta_{X_\pi}}\{u^{\phi_\pi^{-1}}, v^{\phi_\pi^{-1}}\}^\sigma (v^{\phi_\pi^{-1}})^{\zeta_{X_\pi}} \\
\{u, v\}^\sigma &= u^{\zeta_{(X_\pi)^{\phi_\pi}}}\{u^{\phi_\pi^{-1}}, v^{\phi_\pi^{-1}}\}^\sigma v^{\zeta_{(X_\pi)^{\phi_\pi}}} \\
\{u, v\}^\sigma &= \{u, v\}^{\sigma^{\overline{\zeta_{X_\pi}}\phi_\pi}}.
\end{aligned}$$

Therefore, $\overline{\zeta_{X_\pi}}\phi_\pi \in \mathrm{SwAut}(\Sigma)$ and f is well defined.

We next want to show that f is a surjective homomorphism. We will first prove f is a homomorphism. First, let $g, h \in G_P$. Then

$$(gh)^f = \overline{\zeta_{X_{gh}}}\phi_{gh}.$$

Suppose that $x \in X_{gh}$. Then $(x, +)^{gh} = (y, -)$ for some y, and since G_P stabilizes P setwise by definition, $(x, -)^{gh} = (y, +)$. Now, if $x \in X_g$ and $x^{\phi_g} \in X_h$, then

$$(x, +)^{gh} = (x^{\phi_g}, -)^h = ((x^{\phi_g})^{\phi_h}, +),$$

and so $x \notin X_{gh}$. Furthermore, if $x \notin X_g$ and $x^{\phi_g} \notin X_h$, then

$$(x, +)^{gh} = (x^{\phi_g}, +)^h = ((x^{\phi_g})^{\phi_h}, +),$$

so as before, $x \notin X_{gh}$. However, let $x \in X_g$ and $x^{\phi_g} \notin X_h$, so

$$(x, +)^{gh} = (x^{\phi_g}, -)^h = ((x^{\phi_g})^{\phi_h}, -).$$

Likewise, if $x \notin X_g$ and $x^{\phi_g} \in X_h$, then

$$(x,+)^{gh} = (x^{\phi_g},+)^h = ((x^{\phi_g})^{\phi_h},-).$$

Therefore, exactly one of $x \in X_g$ and $x^{\phi_g} \in X_h$ holds if $x \in X_{gh}$. If $x^{\phi_g} \in X_h$, then $x \in (X_h)^{\phi_g^{-1}}$, so we may conclude $X_{gh} = X_g \triangle X_h^{\phi_g^{-1}}$. Next, note that as

$$(v,s)^{gh} = ((v,s)^g)^h,$$

we can conclude that

$$((v,s)^{gh})^\delta = (((v,s)^g)^h)^\delta,$$

and thus that

$$\phi_{gh} = \phi_g \phi_h.$$

Therefore, by the definition of the binary operation of $\mathrm{SwAut}(\Sigma)$,

$$(g^f)(h^f) = (\overline{\zeta_{X_g}}\phi_g)(\overline{\zeta_{X_h}}\phi_h) = \overline{\zeta_{X_g \triangle (X_h)^{\phi_g^{-1}}}}\phi_g\phi_h = \overline{\zeta_{X\,gh}}\phi_{gh} = (gh)^f,$$

and so f is a homomorphism.

We next will prove f is surjective. Let $\overline{\zeta_X}\phi \in \mathrm{SwAut}(\Sigma)$. Then for all $\{u,v\} \in E(\Gamma)$,

$$\{u,v\}^\sigma = \{u,v\}^{\sigma^{\overline{\zeta_X}\phi}} = u^{\zeta_X\phi}\{u^{\phi^{-1}}, v^{\phi^{-1}}\}^\sigma v^{\zeta_X\phi}$$
$$\{u^\phi, v^\phi\}^\sigma = \{u^\phi, v^\phi\}^{\sigma^{\overline{\zeta_X}\phi}} = (u^\phi)^{\zeta_X\phi}\{u,v\}^\sigma (v^\phi)^{\zeta_x\phi} = u^{\zeta_X}\{u,v\}^\sigma v^{\zeta_X},$$

and equivalently,

$$\{u,v\}^{\sigma} = u^{\zeta_X}\{u^{\phi}, v^{\phi}\}^{\sigma} v^{\zeta_X}.$$

Next, define $\xi : V(\mathcal{C}_\sigma(\Gamma)) \to V(\mathcal{C}_\sigma(\Gamma))$ by $(v,s)^{\xi} = (v^{\phi}, v^{\zeta_X} s)$.

$$\begin{aligned}
\{(a,r),(b,s)\} \in E(\mathcal{C}_\sigma(\Gamma)) &\iff \{a,b\} \in E(\Gamma) \text{ and } rs = \{a,b\}^{\sigma} \\
&\iff \{a^{\phi}, b^{\phi}\} \in E(\Gamma) \text{ and } rs = a^{\zeta_X}\{a^{\phi}, b^{\phi}\}^{\sigma} b^{\zeta_X} \\
&\iff \{a^{\phi}, b^{\phi}\} \in E(\Gamma) \text{ and } (a^{\zeta_X} r)(b^{\zeta_X} s) = \{a^{\phi}, b^{\phi}\}^{\sigma} \\
&\iff \{(a,r),(b,s)\}^{\xi} \in E(\mathcal{C}_\sigma(\Gamma)).
\end{aligned}$$

Thus $\xi \in G_P$. Note that by the definition of ξ, $(v,s)^{\xi} = (v,-s)$ if and only if $v^{\zeta_X} = -$; that is, if and only if $v \in X$. Also, $(v,s)^{\xi} = (v^{\phi}, v^{\zeta_X} s)$, so $((v,s)^{\xi})^{\delta} = v^{\phi}$, and $\xi^{f} = \overline{\zeta_X}\phi$ by the definition of f. Consequently, f is a surjection.

Finally, let $\kappa \in \ker(f)$, so

$$\kappa^{f} = \overline{\zeta_{X_\kappa}}\phi_\kappa = \overline{\zeta_+}\,\mathrm{id}.$$

Then $\zeta_{X_\kappa} \in \overline{\zeta_+}$ and so $X_\kappa = \varnothing$ or $X_\kappa = V(\Gamma)$. Also, $v^{\phi_\kappa} = ((v,s)^{\kappa})^{\delta} = v$, and so $(v,s)^{\kappa} = (v,r)$ for all v. Thus for all v, we may define κ^+ and κ^- by

$$\begin{aligned}
(v,s)^{\kappa^+} &= (v,s) \\
(v,s)^{\kappa^-} &= (v,-s),
\end{aligned}$$

corresponding to $X_\kappa = \varnothing$ and $X_\kappa = V(\Gamma)$, respectively. Accordingly, $\ker(f) = \{\kappa^+, \kappa^-\}$. By the First Isomorphism Theorem,

$$G_P/\{\kappa^+, \kappa^-\} \cong \mathrm{SwAut}(\Sigma).$$

QED

Our first algorithm can use this result to great effect. Once signed double cover representatives are found for each switching isomorphism class, we may use GAP methods to determine its automorphism group, the stabilizer of P therein, and its quotient by the group generated by κ. This enables us to quickly find the abstract group isomorphic to $\mathrm{SwAut}(\Sigma)$ for Σ a representative in each switching isomorphism class. The code for Algorithm 1 can be found in Appendix A, and selected examples of its use in determining switching automorphism groups can be found in Chapter 5.

3.2 An More Efficient Approach to Enumeration

We shall introduce some new concepts that will help convert the problem of enumerating switching isomorphism classes to a problem that is easier to conceptualize and compute.

Definition 3.7. Let Γ be a simple connected graph. The *cycle space* $\mathcal{C}_\Gamma$ is the set of all subgraphs of Γ wherein every vertex is incident with an even number of edges. These subgraphs are called *Eulerian subgraphs.* $\mathcal{C}_\Gamma$ forms a vector space over $\mathbb{F}_2$ with addition defined as symmetric difference on the edges of the vector cycles.

The elements of cycle spaces can be constructed from all the linear combinations of its basis cycles. We can find basis cycles for any graph with the help of a special subgraph known as a spanning tree.

Definition 3.8. For a connected graph Γ, a *spanning tree* T is a connected subgraph of Γ such that $V(T) = V(\Gamma)$ and T contains no cycles. That means given any two vertices $u, v \in V(T)$, there is exactly one path of edges from u to v in $E(T)$. A *leaf* of a tree is a vertex of degree 1.

Lemma 3.9. *Any spanning tree for a connected graph* Γ *has* $|V(\Gamma)| - 1$ *edges.*

Proof. Let T be a spanning tree for Γ. Then T has $|V(\Gamma)|$ vertices, so it is sufficient to prove that any tree with n vertices has $n-1$ edges. We proceed by induction. First, suppose $n = 1$. Then clearly T_1 has no edges, since an edge requires 2 vertices. Assume that if $|V(T_n)| = n$, then $|E(T_n)| = n - 1$. We consider a tree T_{n+1} with $n + 1$ vertices. There must exist a vertex that is incident with only one edge. This is because if every vertex were adjacent to 2 or more edges, then we could create a cycle by choosing any vertex v_1, traveling to an adjacent vertex v_2, then from v_2 to v_3 and so on (which is possible because every vertex is adjacent to at least 2 others). Eventually a vertex must be repeated, lest the graph have infinite vertices. So, let $v \in V(T_{n+1})$ be incident with one edge. Then removing v would only remove the edge incident with v, so the resulting graph $T_{n+1} - v$ would still be connected. Of course, removing a vertex and edge would not create a cycle, so $T_{n+1} - v$ would be a tree on n vertices. From the induction hypothesis, $T_{n+1} - v$ has $n - 1$ edges, and so we conclude that T_{n+1} had n edges. *QED*

Definition 3.10. Let T be a spanning tree for the graph Γ, and consider the set $S = E(\Gamma) - E(T)$ of edges of Γ that are not in T. For each $e \in S$, there exists one cycle in Γ that consists of e and edges in T. These cycles are called *fundamental basis cycles* for $\mathcal{C}_\Gamma$.

Lemma 3.11. *Fundamental basis cycles are unique for a given spanning tree, and they indeed form a basis of* $\mathcal{C}_\Gamma$.

Proof. Let T be a spanning tree for Γ, and let $\{u, v\} \in S = E(\Gamma) - E(T)$. Since T spans Γ, $u, v \in V(T)$, and since T is connected, there exists a path from u to v within T, which forms a cycle with $\{u, v\}$. If there existed another path from u to v within T, then it with the first path would contain a cycle, contradicting that T was a tree, so the path (and thus cycle) must be unique.

Let $\mathcal{B}$ be the set of all such cycles. Since each contains a unique edge from $S = E(\Gamma) - E(T)$ by construction, they are linearly independent under symmetric difference (the additive operation of $\mathcal{C}_\Gamma$). It remains only to prove that $\mathcal{B}$ spans $\mathcal{C}_\Gamma$. Let $C \in \mathcal{C}_\Gamma$ be an arbitrary

Eulerian subgraph of Γ, and let $X = E(C) \cap S$ be the set of edges in C that are not in T. Then if we let C_x denote the unique cycle in $\mathcal{B}$ containing x,

$$\sum_{x \in X} C_x + C \in \mathcal{C}_\Gamma,$$

with addition the normal symmetric difference, since each element in the sum is a member of $\mathcal{C}_\Gamma$ and vector spaces are closed. Note that

$$S \cap \left(\sum_{x \in X} C_x + C \right) = \varnothing,$$

because every edge in C not contained in T is eliminated by some cycle in X, by construction. So $\sum_{x \in X} C_x + C$ is a subgraph of T. If a tree T contains no cycles (by definition), then any subgraph of T cannot contain any cycles, so $\sum_{x \in X} C_x + C$ contains no cycles. So if $\sum_{x \in X} C_x + C$ contains any edges, there must be some vertex in $\sum_{x \in X} C_x + C$ that is incident with only 1 edge, as we discussed in the proof of Lemma 3.7. However, a vertex incident with only 1 edge would contradict that $\sum_{x \in X} C_x + C \in \mathcal{C}_\Gamma$, so we can conclude that this graph had no edges. That is,

$$\sum_{x \in X} C_x + C = 0,$$

for 0 representing the empty Eulerian graph. That is, $\sum_{x \in X} C_x = C$, so C can be written as the linear combination of $\mathcal{B}$. This makes $\mathcal{B}$ a basis for $\mathcal{C}_\Gamma$. *QED*

We note that $\mathcal{C}_\Gamma$ has dimension $|\mathcal{B}| = |E(\Gamma)| - |V(\Gamma)| + 1$, by Lemma 3.7, as it is formed by the set of edges not included in a spanning tree of Γ.

It is likely unclear what cycle spaces and Eulerian subgraphs have to do with enumerating the switching isomorphism classes of signed graphs. However, there is a fundamental relationship between the two concepts. It turns out that signed graphs and switching functions can be defined using a similar cohomological framework as concepts called *Seidel switching* and *two-graphs*, whose own "switching classes" correspond directly with Eulerian graphs (cf. Cameron [3] and [5]). For our purposes, we are concerned with the following result, which is essential to our second algorithm.

Proposition 3.12. *There is a one-to-one correspondence between switching isomorphism classes of* Γ *and* $\mathrm{Aut}(\Gamma)$*-isomorphism classes of Eulerian subgraphs of* Γ*.*

This powerful theorem has been demonstrated by Zaslavsky [19], [20] and is achievable using more general methods of Cameron [3] [5], Hofmeister [10], and Seidel [16]. We shall prove some preliminary results ourselves that will give context to this theorem.

Let Γ be a simple, connected graph, and fix a spanning tree T. Let X be the set of edges of Γ not in T.

Lemma 3.13. *Every signed graph* Σ *is switching equivalent to a signed graph* $\Pi = (\Gamma, \sigma)$ *such that* $\{u, v\}^\sigma = +$ *for all* $\{u, v\} \in E(T)$*.*

Proof. Let $\Sigma = (\Gamma, \sigma)$ be a signed graph of Γ such that $|E(T)| = 1$ and $\{u, v\} \in E(T)$. If $\{u, v\}^\sigma = -$, then we may switch v such that $\{u, v\}^{\sigma^{\zeta_{\{v\}}}} = +$. Next, let Γ be such that $|E(T)| = k$, and assume that there is some switching of Σ that will make every edge in T positive. Then consider Γ such that $|E(T)| = k+1$. The spanning tree T must contain some edge incident with a leaf, say e. The tree T' with $E(T') = E(T) - \{e\}$ has $|E(T')| = k$ edges, and by the induction hypothesis, we may switch some vertices in $V(\Gamma)$ to make the edges of T' positive. Then, e is either positive or negative; if negative, switch the leaf incident with e. This leaves $\{u, v\}$ positive for all $\{u, v\} \in E(T)$, for any size spanning tree T by induction. *QED*

Lemma 3.14. *There is a one-to-one correspondence between switching equivalence classes of* Γ *and signed graphs* (Γ, σ) *such that* $\{u, v\}^\sigma = +$ *for all* $\{u, v\} \in E(T)$*.*

Proof. Let $\Sigma = (\Gamma, \sigma)$ be a signed graph of Γ. By Lemma 3.13, the switching equivalence class containing Σ contains a signed graph that signs every edge in T positive.

Suppose Σ, Π are switching-equivalent signed graphs of Γ such that T is all positive on each. By definition, there exists some $\overline{\zeta_Y} \in \mathrm{Sw}(\Gamma)$ such that $\Sigma^{\zeta_Y} = \Pi$, and $\overline{\zeta_Y}$ does not change the sign of any $t \in E(T)$. That is, for every $t \in T$, either both vertices are switched by $\overline{\zeta_Y}$, or neither is. Since T is connected, any nontrivial partition of $V(T)$ will induce an edge cut, and so it must be that either $Y = \varnothing$ or $V(T) \subseteq Y$. But T spans $V(\Gamma)$, so either $Y = \varnothing$ or $Y = V(\Gamma)$, and thus $\overline{\zeta_X} = \overline{\zeta_+}$. Thus $\Sigma = \Pi$, and signed graphs with positive signs on T are unique for each switching equivalence class. Of course, if two signed graphs $A = (\Gamma, \sigma_A)$ and $B = (\Gamma, \sigma_B)$, where $\{u,v\}^{\sigma_A} = \{u,v\}^{\sigma_B} = +$ for all $\{u,v\} \in E(T)$, are equal, they are in the same switching equivalence class, and so there is a bijection between switching equivalence classes and signed graphs with T positive. *QED*

Lemma 3.15. *We call a cycle balanced if the product of the signs of its edges is positive. For each switching equivalence class, there is a set of balanced cycles that is unique to the class, and every member of the class has exactly those balanced cycles.*

Proof. From Lemma 3.14, two signed graphs Σ, Π that are not switching equivalent can each be switched to a graph that is all positive on T, and these two new signed graphs Σ^T, Π^T are not identical. That is, there is some edge e in $E(\Gamma) - E(T)$ that is negative on, say, Σ^T, and positive on Π^T. The cycle formed by e and the unique path between its vertices in T is a cycle, one positive in Π^T but negative in Σ^T, and so Σ^T and Π^T do not have the same balanced cycles. Switching does not change the set of balanced cycles, because if a cycle contains a switched vertex v, then both edges in the cycle that are incident with v change signs, which does not change the sign of the cycle. Since every member of a switching equivalence class can be switched to the identical signed graph with all positive T, all members of the class have the same set of balanced cycles. *QED*

Lemma 3.16. *The set of balanced cycles in a signed graph Σ is determined by the signs of the edges in $X = E(\Gamma) - E(T)$ once Σ is switched to make T all-positive.*

Proof. As before, let $\mathcal{B}$ be the set of cycles formed by each edge $e \in X$ and the unique path between the vertices of e in T. Every cycle is an Eulerian subgraph, and so is formed by some sum of basis cycles in $\mathcal{B}$. If two cycles C_1, C_2 are added to make another cycle, then it must be that $E(C_1) \cap E(C_2)$ is some path; let it be p. Now, for σ denoting the sign of a subgraph calculated by multiplying the signs of each edge in the subgraph, we have that

$$C_1^\sigma = (C_1 - p)^\sigma \cdot p^\sigma,$$

$$C_2^\sigma = (C_2 - p)^\sigma \cdot p^\sigma.$$

The sign of $C_1 + C_2$ is:

$$(C_1 + C_2)^\sigma = (C_1 - p)^\sigma \cdot (C_2 - p)^\sigma = (C_1 - p)^\sigma \cdot (C_2 - p)^\sigma \cdot (p^\sigma)^2 = C_1^\sigma \cdot C_2^\sigma,$$

so the sign of any balanced cycle is determined by the product of the signs the basis cycles that form it. *QED*

Theorem 3.17. *There is a one-to-one correspondence between* $\Gamma/\mathrm{Sw}(\Gamma)$*, the set of switching equivalence classes of* Γ*, and* $\mathcal{C}_\Gamma$*, the set of Eulerian subgraphs of* Γ*.*

Proof. Let Ψ be a switching equivalence class of Γ. From Lemmas 3.14 and 3.15, we know that there exists a unique signed graph $\overline{\Psi} = (\Gamma, \sigma) \in \Psi$ such that for $\{u, v\} \in E(T)$,

$$\{u, v\}^\sigma = +.$$

Let $\overline{\Psi}^- \subseteq X$ be the set of edges $\{u, v\} \in X = E(\Gamma) - E(T)$ such that

$$\{u, v\}^\sigma = -.$$

That is, $\overline{\Psi}^-$ is the set of negative edges in the unique signed graph of Ψ wherein every edge of T is signed positive. Recall from Lemma 3.11 that each edge $x \in X$ forms a cycle C_x with

T in $\mathcal{C}_\Gamma$, the set of Eulerian subgraphs of Γ, and these C_x form a basis $\mathcal{B}$ of $\mathcal{C}_\Gamma$. Thus

$$\sum_{x \in \overline{\Psi}^-} C_x$$

is some Eulerian subgraph in $\mathcal{C}_\Gamma$. Let $\Gamma/\mathrm{Sw}(\Gamma)$ refer to the switching equivalence classes of Γ. Then we define $\mu : \Gamma/\mathrm{Sw}(\Gamma) \to \mathcal{C}_\Gamma$ by

$$\Psi^\mu = \sum_{x \in \overline{\Psi}^-} C_x,$$

which formalizes the correspondence we just discussed. We know μ is well-defined since the representative of Ψ that signs every edge in $E(T)$ positive is unique, and so are the sums of basis cycles $C_x \in \mathcal{B}$.

Next, if $\mathcal{E}$ is some Eulerian subgraph of Γ, then

$$\mathcal{E} = \sum_{x \in Y} C_x$$

for some $Y \subseteq X$, since $\mathcal{B}$ is a basis for $\mathcal{C}_\Gamma$. Clearly, σ defined by

$$\{u,v\}^\sigma = \begin{cases} -, & \{u,v\} \in Y \\ +, & \{u,v\} \in E(\Gamma) - Y, \end{cases}$$

forms a signed graph $\Sigma := (\Gamma, \sigma)$. By definition, Σ is positive on every edge in $E(T)$, so $\Sigma = \overline{\Omega}$ for Ω the switching equivalence class that contains Σ, and Σ is the unique representative of Ω that is all-positive on T. Thus, $\Omega^\mu = \mathcal{E}$ and μ is surjective.

Finally, suppose we have two switching equivalence classes A and B such that

$$A^\mu = B^\mu = E.$$

Since $E \in \mathcal{C}_\Gamma$,

$$E = \sum_{x \in Z} C_x$$

for some $Z \subseteq X$. As $A^\mu = B^\mu = E$, we know $\overline{A}^- = \overline{B}^-$, so $\overline{A} = \overline{B}$. If two equivalence classes share a member, then they must be identical, so $A = B$ and μ is a bijection. *QED*

We also note that the association of switching equivalence classes with sets of balanced circles can be observed by noting that the set of positive edges in $P := E(\overline{\Psi}) \cap X$ corresponds to a set of basis cycles $\{C_x \mid x \in P\} \subseteq \mathcal{B}$. From Lemmas 3.15, 3.16, the balanced cycles in $\mathcal{B}$ determine uniquely the balanced cycles for all $\Sigma \in \Psi$.

Theorem 3.18. *There is a one-to-one correspondence between switching isomorphism classes of* Γ *and* $\mathrm{Aut}(\Gamma)$*-isomorphism classes of Eulerian subgraphs of* Γ*.*

Proof. See Zaslavsky [19] and Cameron [5, 3]. *QED*

The pertinence of this theorem cannot be overstated, as it is now sufficient for us to enumerate the Eulerian subgraphs of Γ, up to $\mathrm{Aut}(\Gamma)$-isomorphism, to count its switching isomorphism classes.

3.2.1 The Algorithm

This program counts the number of Eulerian subgraphs of a given graph up to $\mathrm{Aut}(\Gamma)$-isomorphism (we shall often say Eulerian graphs are *isomorphic* to mean they are *isomorphic under* $\mathrm{Aut}(\Gamma)$ when Γ is understood by context). That is, it determines the number $|\mathcal{C}_\Gamma/\mathrm{Aut}(\Gamma)|$ of orbits of the set $\mathcal{C}_\Gamma$ of all such graphs under $\mathrm{Aut}(\Gamma)$. By Theorem 1.12:

$$|\mathcal{C}_\Gamma/\mathrm{Aut}(\Gamma)| = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\pi \in \mathrm{Aut}(\Gamma)} |\mathrm{fix}_\pi(\mathcal{C}_\Gamma)|.$$

Since the action on $\mathcal{C}_\Gamma$ is a representation of $\mathrm{Aut}(\Gamma)$, each permutation $\pi \in \mathrm{Aut}(\Gamma)$ is a linear transformation from $\mathcal{C}_\Gamma$ to itself, and can be associated with a matrix M_π over $\mathbb{F}_2$ defined in terms of $\mathcal{B}$. Furthermore, $\text{fix}_\pi(\mathcal{C}_\Gamma)$ can be identified with the set of vectors in $\mathbb{F}_2^{|\mathcal{B}|}$ fixed by M_π.

$$\begin{aligned}|\text{fix}_\pi(\mathcal{C}_\Gamma)| &= |\{\lambda \in \mathbb{F}_2^{|\mathcal{B}|} \mid \lambda M_\pi = \lambda\}| \\ &= |\{\lambda \in \mathbb{F}_2^{|\mathcal{B}|} \mid \lambda(M_\pi - I) = 0\}| \\ &= |\ker(M_\pi - I)|.\end{aligned}$$

Substituting, we have that

$$|\mathcal{C}_\Gamma/\text{Aut}(\Gamma)| = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{\pi \in \text{Aut}(\Gamma)} |\ker(M_\pi - I)|.$$

The computer algebra system GAP has a function

```
BasisNullspaceModN(M,n)
```

that returns a basis for the nullspace of a matrix M over the ring $\mathbb{Z}/n\mathbb{Z}$. Generally, if V is a vector space over $\mathbb{F}_p$ with finite dimension $\dim(V)$, then

$$|V| = p^{\dim(V)}.$$

Therefore, we have that if $\mathcal{N}_\pi$ is a basis for the nullspace of $M_\pi - I$, then

$$|\ker(M_\pi - I)| = 2^{\text{null}(M_\pi - I)} = 2^{|\mathcal{N}_\pi|},$$

$$|\mathcal{C}_\Gamma/\text{Aut}(\Gamma)| = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{\pi \in \text{Aut}(\Gamma)} 2^{|\mathcal{N}_\pi|}.$$

Computationally, we may calculate $|\mathcal{N}_\pi|$ as

```
Length(BasisNullspaceModN(M,2))
```

in GAP, and if we can calculate M_π for each $\pi \in \text{Aut}(\Gamma)$, the enumeration of Eulerian subgraphs will quickly follow. To accomplish this, we first find a spanning tree for Γ and define its fundamental basis cycles $\mathcal{B}$ using GAP's Digraphs package [2]. We then calculate the image of each such basis cycle under π and write it as a linear combination of $\mathcal{B}$. Note that it is easy to determine which basis vectors appear in the linear combination defining an arbitrary $C \in \mathcal{C}_\Gamma$ by noting which edges in C are not in the spanning tree used to define $\mathcal{B}$. Once we write the image of each basis cycle as a vector, we form M_π with such vectors as its columns. This process is efficient, and typically limited by the order of the automorphism group of the graph in question. It turns out that we need not create a matrix for every permutation.

Lemma 3.19. *If π and ψ are in the same conjugacy class of* $\mathrm{Aut}(\Gamma)$*, then* $|\mathcal{N}_\pi| = |\mathcal{N}_\psi|$.

Proof. By definition of conjugacy, there exists some $\gamma \in \mathrm{Aut}(\Gamma)$ such that

$$\gamma\pi\gamma^{-1} = \psi.$$

Since each permutation is analogous to a matrix in the representation $\mathcal{C}_\Gamma$ of $\mathrm{Aut}(\Gamma)$ over $\mathbb{F}_2$:

$$\begin{aligned} M_\gamma M_\pi M_{\gamma^{-1}} &= M_\gamma M_\pi M_\gamma^{-1} = M_\psi, \\ M_\gamma M_\pi M_\gamma^{-1} - I &= M_\psi - I, \\ M_\gamma M_\pi M_\gamma^{-1} - M_\gamma M_\gamma^{-1} &= M_\gamma(M_\pi - I)M_\gamma^{-1} = M_\psi - I. \end{aligned}$$

So $M_\pi - I$ and $M_\psi - I$ are similar matrices, and they must have the same nullity. By definition, $|\mathcal{N}_\pi| = |\mathcal{N}_\psi|$. *QED*

This fact helps us reduce the number of necessary computations. Let $\mathrm{Cl}(\Gamma)$ be the set of conjugacy classes of $\mathrm{Aut}(\Gamma)$, where $\mathrm{Cl}(\pi)$ refers to the conjugacy class containing π. Since we have shown $|\mathcal{N}_\pi|$ is constant across these classes:

$$|\mathcal{C}_\Gamma/\mathrm{Aut}(\Gamma)| = \frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\pi\in\mathrm{Aut}(\Gamma)} 2^{|\mathcal{N}_\pi|} = \frac{1}{|\mathrm{Aut}(\Gamma)|}\sum_{\mathrm{Cl}(\pi)\in\mathrm{Cl}(\Gamma)} |\mathrm{Cl}(\pi)| \cdot 2^{|\mathcal{N}_\pi|}.$$

This refinement reduces the number of nullity computations we must make from $|\mathrm{Aut}(\Gamma)|$ to $|\mathrm{Cl}(\Gamma)|$. This completes the algorithm, and by Theorem 3.18, it calculates the number of switching isomorphism classes for any graph. Selected results using this program can be found in Chapter 5, and the code itself is listed in Appendix B.

Chapter 4

Switching Classes of Certain Generalized Petersen Graphs

In this chapter we shall use Theorems 1.12 and 1.15 to derive an explicit formula for the number of switching isomorphism classes for certain species of a family of graphs called Generalized Petersen graphs.

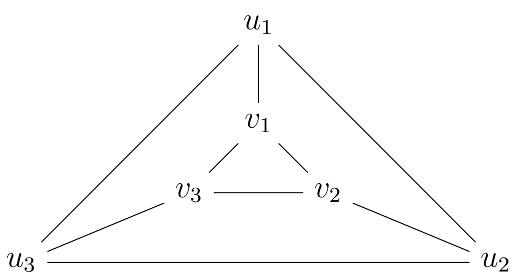
Definition 4.1. Let $n \geqslant 3$ and $k < \frac{n}{2}$. Then the *Generalized Petersen graph* $GP(n,k)$ has vertices

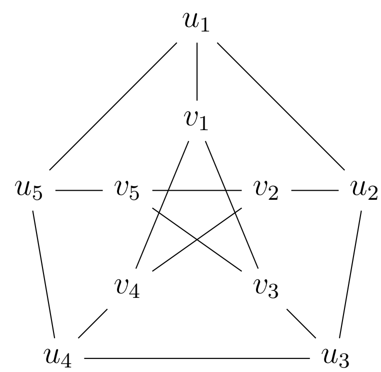
$$V(GP(n,k)) = \{u_i \mid 1 \leqslant i \leqslant n\} \cup \{v_i \mid 1 \leqslant i \leqslant n\}.$$

with edge set defined by

$$\begin{aligned}
\{u_i, u_j\} \in E(GP(n,k)) &\iff j - i \equiv 1 \pmod{n}, \\
\{v_i, v_j\} \in E(GP(n,k)) &\iff j - i \equiv k \pmod{n}, \\
\{u_i, v_j\} \in E(GP(n,k)) &\iff j = i.
\end{aligned}$$

Geometrically, $GP(n,k)$ consists of an outer ring of n vertices u_i that are each connected to an inner set of n vertices v_i by a series of spoke edges $\{u_i, v_i\}$. Given an inner vertex v_i, we travel k spaces around the inner ring and add the edge $\{v_i, v_{i+_n k}\}$, etc., where $+_n$ denotes addition modulo n. Below are examples for $GP(3,1)$ and $GP(5,2)$; the latter is the original Petersen graph that lends its name to the family.





Theorem 4.2 (Frucht [6]). *Let $GP(n,k)$ be a Generalized Petersen graph. Then*

$$\mathrm{Aut}(GP(n,k)) \cong \begin{cases} D_n \rtimes \mathbb{Z}_2 & k^2 \equiv 1 \pmod n \\ \mathbb{Z}_n \rtimes \mathbb{Z}_4 & k^2 \equiv -1 \pmod n \\ D_n & k^2 \not\equiv \pm 1 \pmod n, \end{cases}$$

where D_n denotes (here and elsewhere) the dihedral group of an n-gon. There are seven exceptions: $GP(4,1), GP(5,2), GP(8,3), GP(10,2), GP(10,3), GP(12,5)$, and $GP(24,5)$.

Corollary 4.3 (Frucht [6]). *For Generalized Petersen graphs $GP(n,k)$, excluding the seven exceptions, there exists no permutation $\pi \in \mathrm{Aut}(GP(n,k))$ such that $\{u_i, v_i\}^\pi = \{u_a, u_b\}$ or $\{u_i, v_i\}^\pi = \{v_a, v_b\}$. That is, the spoke edges form a single orbit under $\mathrm{Aut}(GP(n,k))$.*

Definition 4.4. The 2-*colored bracelet number* on n points, denoted $B_{n,2}$, is the number of distinct bracelets with n black or white beads, where two such bracelets are considered indistinct or *equivalent* if one can be obtained by the other via rotation or reflection (turning-over). Formally, an $n-$bracelet is a cyclic graph C_n and a coloring function $f : V(C_n) \to \{\text{black, white}\}$.

We can imagine a ring of n beads, each colored white or black. The set of actions we can perform to this bracelet, viz. rotating it by some number or flipping it over, form a group of symmetries, called the dihedral group D_n. Note that the following two bracelets would not be considered distinct by our definition, as rotating the left bracelet clockwise by three beads and then flipping vertically (a valid action) produces the bracelet on the right.

We can use Theorem 1.15 to construct an explicit formula for $B_{n,2}$.

Lemma 4.5. *Let ϕ denote Euler's totient function. Then*

$$B_{n,2} = \begin{cases} \frac{1}{n}\left(\sum_{d|n}\phi(d)2^{\frac{n}{d}-1}\right) + 3\cdot 2^{\frac{n-4}{2}} & n \text{ even} \\ \frac{1}{n}\left(\sum_{d|n}\phi(d)2^{\frac{n}{d}-1}\right) + 2^{\frac{n-1}{2}} & n \text{ odd.} \end{cases}$$

Proof. An n-bracelet can be thought of as the graph C_n with automorphism group D_n, consisting of rotations ρ and reflections γ. We shall apply Theorem 1.15 to the action of D_n on 2^{C_n}, the set of 2-coloring functions on the vertices of C_n:

$$B_{n,2} = |2^{C_n}/D_n| = \frac{1}{|D_n|}\sum_{\pi\in D_n} 2^{c(\pi)}.$$

We note that $D_n = R_1 \cup R_2$, where R_1 is the set of rotations, including 1, and R_2 is the set of reflections.

$$B_{n,2} = \frac{1}{2n}\left(\sum_{\pi\in R_1} 2^{c(\pi)} + \sum_{\pi\in R_2} 2^{c(\pi)}\right).$$

Let $\rho_i \in R_1$ be a rotation such that $\{u_k, v_k\}^{\rho_1} = \{u_{k+_n i}, v_{k+_n i}\}$. Then ρ_i has order $\frac{n}{\gcd(i,n)}$, and has a cycle representation consisting of $\gcd(i,n)$ cycles of length $\frac{n}{\gcd(i,n)}$. Let $d|n$. Then there are $\phi(\frac{n}{d})$ numbers $i \leqslant n$ such that $\gcd(i,n) = d$. This is because we can multiply d by each number coprime to $\frac{n}{d}$, which produces numbers divisible by d but with no additional prime factors of n. Equivalently, for $d|n$, there are $\phi(d)$ numbers $i \leqslant n$ such that $\gcd(i, n) = \frac{n}{d}$. This means that for each $d|n$, there are $\phi(d)$ distinct i such that ρ_i contains $\frac{n}{d}$ cycles. So,

$$\sum_{\pi \in R_1} 2^{c(\pi)} = \sum_{d|n} \phi(d) 2^{\frac{n}{d}}.$$

If n is even, there are two types of reflections: those whose axes pass through two points of C_n, and those whose axes pass through none. There are $\frac{n}{2}$ of each, corresponding to the number of possible axes. The first species has two fixed points (the points through which the axis travels) and $\frac{n-2}{2}$ transpositions, for a total of $2 + \frac{n-2}{2} = \frac{n+2}{2}$ cycles. The second species fixes no points, and so has $\frac{n}{2}$ transpositions. Thus for n even:

$$\sum_{\pi \in R_2} 2^{c(\pi)} = \frac{n}{2}\left(2^{\frac{n+2}{2}} + 2^{\frac{n}{2}}\right) = n\left(2^{\frac{n}{2}} + 2^{\frac{n-2}{2}}\right) = 3n \cdot 2^{\frac{n-2}{2}},$$

$$B_{n,2} = \frac{1}{2n}\left(\sum_{d|n} \phi(d) 2^{\frac{n}{d}} + 3n 2^{\frac{n-2}{2}}\right) = \frac{1}{n}\sum_{d|n} \phi(d) 2^{\frac{n}{d}-1} + 3 \cdot 2^{\frac{n-4}{2}}.$$

If n is odd, then there is one type of reflection; that which travels through one point. There are n of these, and each fixes one point and reflects the remaining $\frac{n-1}{2}$ for a total of $\frac{n+1}{2}$ cycles. Thus for n odd:

$$\sum_{\pi \in R_2} 2^{c(\pi)} = n 2^{\frac{n+1}{2}},$$

$$B_{n,2} = \frac{1}{2n}\left(\sum_{d|n} \phi(d) 2^{\frac{n}{d}} + n 2^{\frac{n+1}{2}}\right) = \frac{1}{n}\sum_{d|n} \phi(d) 2^{\frac{n}{d}-1} + 2^{\frac{n-1}{2}}.$$

QED

Definition 4.6. Let $B_{n,2}$ be a bracelet, and let $B'_{n,2}$ be the bracelet achieved by turning every white bead in $B_{n,2}$ black and every black bead white. This is the *complement bracelet* of $B_{n,2}$. A *self-complementary bracelet* is one that is equivalent (under rotations and turning-over) to its complement.

Note that self-complementary bracelets must have the same number of black and white beads and so must be even in length. The example bracelet on page 46 is self-complementary.

Lemma 4.7. *Let $S_{n,2}$ denote the number of self-complementary bracelets on n beads.*

$$S_{n,2} = \begin{cases} \frac{1}{n}\sum_{d|\frac{n}{2}} \phi(2d)2^{\frac{n}{2d}-1} + 2^{\frac{n-4}{2}} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Proof. If n is odd, then there cannot be as many black beads as white beads, so no odd bracelet can be self-complimentary. For the proof of $S_{2n,2}$, which applies Theorem 1.15, see Palmer and Robinson [14]. *QED*

With this terminology we can begin to construct concise formulas for the number of switching isomorphism classes of $GP(n,k)$. By Theorem 3.18, it is sufficient to count the number of isomorphism-free Eulerian subgraphs. We know that the dimension of the cycle space $\mathcal{C}$ of any Generalized Petersen graph $GP(n,k)$ is

$$|E(GP(n,k))| - |V(GP(n,k))| + 1 = 3n - 2n + 1 = n + 1.$$

Thus any $n+1$ linearly independent Eulerian subgraphs form a basis for $\mathcal{C}(GP(n,k))$. With this understanding, we shall prove formulas for the number of switching isomorphism classes for three species of Generalized Petersen graphs.

4.0.1 Prism Graphs

A Generalized Petersen graph of the form $GP(n,1)$ consists of two concentric regular n-gons connected at each vertex. Because of their similarity to the three-dimensional figures, they are often called *prism graphs*.

Lemma 4.8. *Let $\mathcal{Q}$ consist of the n cycles $\{\{u_i, u_{i+_n1}\}, \{u_i, v_i\}, \{v_i, v_{i+_n1}\}, \{u_{i+_n1}, v_{i+_n1}\}\}$, which are the 4-cycles formed by each pair of adjacent spoke edges. Let O be the outer n-cycle*

$\{\{u_1, u_2\}, \{u_2, u_3\}, ...\{u_n, u_1\}\}$. Then $\mathcal{B} = \mathcal{Q} \cup \{O\}$ forms a basis for $\mathcal{C}$, the cycle space of $GP(n, 1)$.

Proof. Since $|B| = n + 1$, it is sufficient to prove that the cycles in $\mathcal{B}$ are linearly independent. Each cycle $q_i = \{\{u_i, u_{i+_n1}\}, \{u_i, v_i\}, \{v_i, v_{i+_n1}\}, \{u_{i+_n1}, v_{i+_n1}\}\} \in \mathcal{Q}$ contains an edge $\{u_i, u_{i+_n1}\}$ that is not contained within any other q_j by construction. Since addition in $\mathcal{C}$ is symmetric difference of edges, the cycles in $\mathcal{Q}$ are linearly independent. O contains every such edge $\{u_i, u_{i+_n1}\}$, and so if

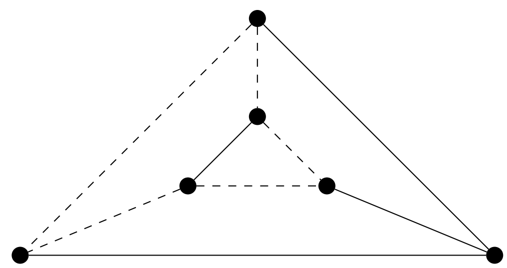
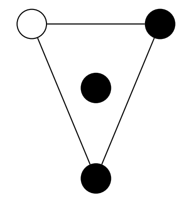
$$f_1 q_1 + f_2 q_2 + \cdots + f_n q_n = O,$$

for $f_i \in \mathbb{F}_2$, then it must be that $f_1, ..., f_n = 1$. However, the inner edges $\{v_i, v_{i+_n1}\}$ are also contained in exactly one $q_i \in Q$, and so

$$\{\{v_1, v_2\}, \{v_2, v_3\}, ...\{v_n, v_1\}\} \subset q_1 + q_2 + \cdots + q_n.$$

Since $\{\{v_1, v_2\}, \{v_2, v_3\}, ...\{v_n, v_1\}\} \notin O$, we know $q_1 + q_2 + \cdots + q_n \neq O$. So $\mathcal{B}$ is linearly independent. *QED*

Thus each Eulerian subgraph $C \in \mathcal{C}$ can be written as some linear combination of $\mathcal{B}$. We can represent each such combination as a black and white bracelet with n beads, and a decision to include O or not. Each bead is black if the cycle to which it corresponds has coefficient 1 in the linear combination of $\mathcal{B}$ and white if the coefficient is 0. Below is an example of an Eulerian subgraph (dashed) of $GP(3, 1)$ and the corresponding bracelet representation, where we add a center "bead" colored black to denote that we are including O.

Recall that $\mathrm{Aut}(GP(n,1)) \cong D_n \rtimes \mathbb{Z}_2$. Every permutation $\pi \in D_n \rtimes \mathbb{Z}_2$ can be written as $\delta\xi^k$ for $\delta \in D_n$ and $\langle \xi \rangle \cong \mathbb{Z}_2$. For our purposes, this means every permutation of $GP(n,1)$ is some combination of a normal bracelet permutation (that is, rotations and reflections) and then a decision to turn the graph inside-out, or not. Specifically, identifying D_n with its isomorphic subgroup of $\mathrm{Aut}(GP(n,1))$:

$$\mathrm{Aut}(GP(n,1)) \cong D_n \rtimes \langle (u_1\, v_1)(u_2\, v_2) \cdots (u_n\, v_n) \rangle.$$

Lemma 4.9. *There exists no isomorphism class of $\mathcal{C}$ that contains an Eulerian subgraph with O as a basis vector and an Eulerian subgraph without O as a basis vector.*

Proof. Since (Corollary 4.3) the spokes of $GP(n,1)$ form an orbit, for $q \in \mathcal{Q}$, $q^\pi \in \mathcal{Q}$ for all $\pi \in \mathrm{Aut}(GP(n,1))$. However, the image of O under any permutation in D_n is O, and the image of O under $\xi = (u_1v_1)(u_2v_2)\cdots(u_nv_n)$ is the inner cycle $I := \{\{v_1,v_2\},\{v_2,v_3\},...,\{v_n,v_1\}\}$. It is easily verified that

$$I = q_1 + q_2 + \cdots + q_n + O.$$

Letting $f_i, g_i \in \mathbb{F}_2$, if $\sum_i f_i q_i + O \sim \sum_i g_i q_i$, then

$$\left(\sum_i f_i q_i + O\right)^\pi = \sum_i g_i q_i.$$

So, if $\pi \in D_n$, then

$$\begin{aligned}\sum_i g_i q_i &= \left(\sum_i f_i q_i\right)^\pi + O^\pi \\ &= \sum_i h_i q_i + O,\end{aligned}$$

for some $h_i \in \mathbb{F}_2$. If instead $\pi = \delta\xi$, then

$$\begin{aligned}\sum_i g_i q_i &= \left(\sum_i f_i q_i\right)^\pi + O^\pi \\ &= \sum_i h_i q_i + I \\ &= \sum_i (h_i + 1) q_i + O,\end{aligned}$$

for some $h_i \in \mathbb{F}_2$. Writing the same cycle as two different linear combinations contradicts the linear independence of $\mathcal{B}$, so we may conclude that $\sum_i f_i q_i \not\sim \sum_i g_i q_i + O$ *QED*

We may thus count the equivalence classes in $\mathcal{C}$ by considering Eulerian subgraphs with and without O in turn. Let $C \in \mathcal{C}$ such that O has coefficient 0 in the construction of C as a combination of $\mathcal{B}$. This means that C is a linear combination of cycles in $\mathcal{Q}$. Note that under the action of ξ, $q_i^\xi = q_i$, and so $C^{\delta\xi} = C^\delta$, it is sufficient to determine the number of isomorph-free Eulerian subgraphs of combinations of $\mathcal{C}$ under D_n. As we mentioned, each such combination is a bracelet, and D_n is the automorphism group of bracelets. There are thus $B_{n,2}$ distinct Eulerian subgraphs in $\mathcal{C}$ that do not contain O. Next, let $C \in \mathcal{C}$ such that O is included in the construction of C as a sum of cycles in $\mathcal{B}$.

Lemma 4.10. *Let $\sum_i f_i q_i + O$ be a cycle in $\mathcal{C}$. Then $\sum_i f_i q_i + O \sim \sum_i (f_i + 1) q_i + O$.*

Proof. The result follows from the proof of Lemma 4.9. *QED*

Proposition 4.11. *Let $|GP(n,1)/\mathrm{SwAut}(GP(n,1))|$ denote the number of switching iso-*

morphism classes of the graph $GP(n,1)$ for $n \geqslant 3, n \neq 4$. Then

$$|GP(n,1)/\mathrm{SwAut}(GP(n,1))| = \frac{3B_{n,2} + S_{n,2}}{2}.$$

Equivalently,

$$|GP(n,1)/\mathrm{SwAut}(GP(n,1))| = \begin{cases} \frac{3}{n}\sum_{d|n}\phi(d)2^{\frac{n}{d}-2} + \frac{1}{n}\sum_{d|\frac{n}{2}}\phi(2d)2^{\frac{n}{2d}-2} + 5\cdot 2^{\frac{n-4}{2}} & n \text{ even} \\ \frac{3}{n}\sum_{d|n}\phi(d)2^{\frac{n}{d}-2} + 3\cdot 2^{\frac{n-3}{2}} & n \text{ odd.} \end{cases}$$

Proof. If $\sum_i f_i q_i$ is a bracelet, then the beads are colored black when $f_i = 1$ and white when $f_i = 0$. Thus the colors change when $f_i \mapsto f_i + 1$, and so the bracelet for $\sum_i (f_i+1)q_i$ is the complement of that for $\sum_i f_i q_i$. Now, we know that there are $B_{n,2}$ distinct beads under D_n, and thus there are $B_{n,2}$ distinct Eulerian subgraphs of the form $\sum_i f_i q_i + O$ if we do not consider the permutation ξ. We consider it now.

Let $C = \sum_i f_i q_i + O \in \mathcal{C}$ such that the bracelet for $\sum_i f_i q_i$ is self-complementary. Then by Lemma 4.10,

$$C^\xi = \sum_i (f_i+1)q_i + O = \sum_i f_i q_i + O = C,$$

and so $C^\xi = C^\delta$ for $\delta = 1 \in D_n$. That is, the image of C under ξ is not a "new" Eulerian subgraph. Alternatively, if the bracelet representation of $C' \in \mathcal{C}$ is not self-complementary, then $C'^\xi \neq C'^\delta$ for any $\delta \in D_n$. Therefore, C'^ξ and C' are equivalent under $\mathrm{Aut}(GP(n,1))$ but not under D_n. Since $|\xi| = 2$, there are half as many isomorphism classes for $C' = \sum f_i q_i + O$ where C' is not self-complementary, because ξ brings two classes that were disjoint under D_n together. Naturally, there are $B_{n,2} - S_{n,2}$ cycles $C' \in \mathcal{C}$ with non-self-complementary bracelets, so they form $\frac{B_{n,2} - S_{n,2}}{2}$ isomorphism classes. To this we add the $S_{n,2}$ isomorphism classes formed by the self-complementary bracelets, so there are

$$\frac{B_{n,2} + S_{n,2}}{2}$$

orbits of $\mathcal{C}$ under $\text{Aut}(GP(n,1))$ whose representatives contain O. Recall that there were $B_{n,2}$ orbits of Eulerian subgraphs that did *not* contain O, and since we've proven these cases disjoint,

$$|GP(n,1)/\text{SwAut}(GP(n,1))| = \frac{B_{n,2}+S_{n,2}}{2} + B_{n,2} = \frac{3B_{n,2}+S_{n,2}}{2}.$$

Applying Lemmas 4.5 and 4.7, we can write this more explicitly as

$$|GP(n,1)/\text{SwAut}(GP(n,1))| = \begin{cases} \frac{3}{n}\sum_{d|n}\phi(d)2^{\frac{n}{d}-2} + \frac{1}{n}\sum_{d|\frac{n}{2}}\phi(2d)2^{\frac{n}{2d}-2} + 5\cdot 2^{\frac{n-4}{2}} & n \text{ even} \\ \frac{3}{n}\sum_{d|n}\phi(d)2^{\frac{n}{d}-2} + 3\cdot 2^{\frac{n-3}{2}} & n \text{ odd.} \end{cases}$$

QED

4.0.2 Generalized Petersen graphs for $k^2 \not\equiv \pm 1 \pmod{n}$

From Theorem 4.2, Generalized Petersen graphs of this species have the automorphism group D_n. That is, we need not be concerned about any permutations that switch the interior and exterior edges, but rather only rotations and reflections.

Lemma 4.12. *Let $\mathcal{Q}$ consist of the n cycles*

$$\{\{u_i,v_i\},\{v_i,v_{i+_nk}\},\{u_{i+_nk},v_{i+_nk}\},\{u_i,u_{i+_n1}\},\ldots,\{u_{i+_nk-1},u_{i+_nk}\}\}$$

Let O be the outer n cycle $\{\{u_1,u_2\},\{u_2,u_3\},\ldots,\{u_n,u_1\}\}$. Then $\mathcal{B}=\mathcal{Q}\cup\{O\}$ forms a basis for $\mathcal{C}$.

Proof. It is sufficient to prove the $n+1$ cycles are linearly independent. Since each $q_i \in \mathcal{Q}$ contains the unique edge $\{v_i, v_{i+_nk}\}$, $\mathcal{Q}$ is a linearly independent set. Suppose that

$$f_1q_1 + f_2q_2 + \cdots + f_nq_n = O$$

for some $f_i \in \mathbb{F}_2$. Then $\{u_1,u_2\} \in \sum f_iq_i$, so $f_j = 1$ for some j. But $\{v_j, v_{j+_nk}\} \in q_j$, and since this edge is unique to q_j, it cannot be removed by the addition of any other q_i, and so $\{v_j, v_{j+_n k}\} \in \sum f_i q_i$. This contradicts that $\sum f_i q_i = O$. So $\mathcal{B}$ is linearly independent. *QED*

Proposition 4.13. *Let $k^2 \not\equiv \pm 1 \pmod{n}$. Then $|GP(n,k)/\text{SwAut}(GP(n,k))| = 2B_{n,2}$, excluding the special cases. Equivalently,*

$$|GP(n,k)/\text{SwAut}(GP(n,k))| = \begin{cases} \frac{1}{n}\sum_{d|n}\phi(d)2^{\frac{n}{d}} + 3\cdot 2^{\frac{n-2}{2}} & n \textit{ even} \\ \frac{1}{n}\sum_{d|n}\phi(d)2^{\frac{n}{d}} + 2^{\frac{n+1}{2}} & n \textit{ odd.} \end{cases}$$

Proof. As in the previous case, we can represent each $C \in \mathcal{C}$ as a bracelet denoting the coefficients of $\mathcal{Q}$ paired with a decision to include O or not. Since $\text{Aut}(GP(n,k)) \cong D_n$, the only permutations on $\mathcal{B}$ are rotations or reflections of $\mathcal{Q}$; O is fixed by every permutation. We know that there are $B_{n,2}$ distinct bracelets on n beads, and for each, we have the choice to include O or not. Thus, there are $2B_{n,2}$ distinct Eulerian subgraphs in $\mathcal{C}$, and

$$|GP(n,k)/\text{SwAut}(GP(n,k))| = 2B_{n,2}.$$

This is equivalent to the formula

$$|GP(n,k)/\text{SwAut}(GP(n,k))| = \begin{cases} \frac{1}{n}\sum_{d|n}\phi(d)2^{\frac{n}{d}} + 3\cdot 2^{\frac{n-2}{2}} & n \text{ even} \\ \frac{1}{n}\sum_{d|n}\phi(d)2^{\frac{n}{d}} + 2^{\frac{n+1}{2}} & n \text{ odd.} \end{cases}$$

by Lemma 4.5. *QED*

4.0.3 Generalized Petersen graphs for $k^2 \equiv -1 \pmod{n}$

Recall from Theorem 4.2 that $\text{Aut}(GP(n,k)) \cong \mathbb{Z}_n \rtimes \mathbb{Z}_4$, and let $\tau \in \text{Aut}(GP(n,k))$ permute the interior and exterior vertices (respectively, edges, since spokes are stabilized setwise) of the graph. Then $u_1^\pi = v_m$ for some $1 \leqslant m \leqslant n$. Since u_2 is adjacent to u_1, then $u_2^\tau = v_{m+_n k}$

or $u_2^\tau = v_{m-_nk}$, whence

$$u_i^\tau = v_{(i-1)k+m} \text{ and } v_i^\tau = u_{(i-1)k+m},$$

or

$$u_i^\tau = v_{(1-i)k+m} \text{ and } v_i^\tau = u_{(1-i)k+m}.$$

We will denote permutations of these forms as τ and τ' respectively, and let

$$i^\tau = (i-1)k+m \qquad i^{\tau'} = (1-i)k+m$$

stand for the action of τ, τ' on the spoke edges $\{u_i, v_i\}$.

First, we observe the following facts, which will prove useful:

$$\tau^2 : i \mapsto 2-i+(m-1)(k+1),$$

$$(\tau')^2 : i \mapsto 2-i-(m-1)(k-1).$$

Lemma 4.14. *Every spoke edge $\{u_i, v_i\}$ is contained within an orbit of size 4 or smaller under the action by $\langle\tau\rangle$ or $\langle\tau'\rangle$. Specifically, orbits have size 1, 2, or 4.*

Proof. It is sufficient to prove that $i^{\tau^4} = i^{(\tau')^4} = i$.

$$i^{\tau^4} = (i^{\tau^2})^{\tau^2} = (2-i+(m-1)(k+1))^{\tau^2} = 2-2+i-(m-1)(k+1)+(m-1)(k+1) = i$$

$$i^{(\tau')^4} = (i^{(\tau')^2})^{(\tau')^2} = (2-i-(m-1)(k+1))^{(\tau')^2} = 2-2+i+(m-1)(k+1)-(m-1)(k+1) = i$$

The second result is a consequence of the Orbit-Stabilizer Theorem. *QED*

We shall now divide $GP(n,k), k^2 \equiv -1 \pmod{n}$ into cases and perform a series of calculations to prove similar results for each.

Lemma 4.15. *Let $GP(n,k)$ be a Generalized Petersen graph such that $k^2 \equiv -1 \pmod{n}$ and n is odd. Then under the action of $\langle\tau\rangle$ or $\langle\tau'\rangle$, one spoke (u_i, v_i) is fixed, whereas all* *other spokes are contained within an orbit of size 4, and there are $\frac{n-1}{2}$ such orbits.*

Proof. Every square number $k^2 \equiv 0$ or $1 \pmod 4$. Since n is odd and $k^2 \equiv -1 \pmod n$, it must be that k^2 is even, and so we can conclude that $k^2 \equiv 0 \pmod 4$. Thus k is even.

Now, suppose that $i^{\tau^2} = i$. Then as shown,

$$i \equiv 2 - i + (m-1)(k+1) \pmod n$$

$$2(i-1) \equiv (m-1)(k+1) \pmod n.$$

Recall that we defined m as the image of 1 under τ or τ'. Suppose that m is odd. Then $m-1$ is even.

$$i - 1 \equiv \left(\frac{m-1}{2}\right)(k+1) \pmod{\frac{n}{\gcd(n,2)}}.$$

Since n is odd:

$$i \equiv \left(\frac{m-1}{2}\right)(k+1) + 1 \pmod n.$$

This is the condition for $i^{\tau^2} = i$. However, observe what happens to such an i under τ:

$$\begin{aligned}
i^\tau &\equiv \left(\left(\frac{m-1}{2}\right)(k+1)+1\right)^\tau \pmod n\\
&\equiv \left(\left(\frac{m-1}{2}\right)(k+1)+1-1\right)k+m \pmod n\\
&\equiv \left(\frac{m-1}{2}\right)(k^2+k)+m \pmod n\\
&\equiv \left(\frac{m-1}{2}\right)(k-1)+m-1+1 \pmod n\\
&\equiv \left(\frac{m-1}{2}\right)k+\left(\frac{m-1}{2}\right)+1 \pmod n\\
&\equiv \left(\frac{m-1}{2}\right)(k+1)+1 \pmod n\\
&\equiv i \pmod n.
\end{aligned}$$

Thus the i for which $i^{\tau^2} = i$ satisfies $i^\tau = i$. That is, there is one spoke $\left(\frac{m-1}{2}\right)(k+1)+1$

(mod n) that is fixed by τ, and no spoke orbits of size 2 exist. Since every orbit is of size $1, 2$, or 4, it must be that all spokes but the one that is fixed are in orbits of size 4. Orbits are disjoint, and so of the $n-1$ moved points, there are $\frac{n-1}{2}$ orbits of size 4.

Now suppose instead that m is even and again that $i^{\tau^2} = i$ for some i. Recall

$$2(i-1) \equiv (m-1)(k+1) \pmod{n}.$$

We add $n \equiv 0$ to each side:

$$2(i-1) \equiv (m-1)(k+1) + n \pmod{n}.$$

Since m is even, $m-1$ is odd, as are $k+1$ and n, so $(m-1)(k+1)+n$ is even.

$$i - 1 \equiv \frac{(m-1)(k+1)+n}{2} \quad (\text{mod } \frac{n}{\gcd(n,2)})$$

$$i \equiv \frac{(m-1)(k+1)+n}{2} + 1 \pmod{n}.$$

Again we observe the image of this i under τ:

$$\begin{aligned}
i^\tau &\equiv \left(\frac{(m-1)(k+1)+n}{2} + 1\right)^\tau \pmod{n} \\
&\equiv \left(\frac{(m-1)(k+1)+n}{2} + 1 - 1\right) k + m \pmod{n} \\
&\equiv \left(\frac{(m-1)(k+1)+n}{2}\right) k + m - 1 + 1 \pmod{n} \\
2i^\tau &\equiv (m-1)(k^2+k) + kn + 2(m-1) + 2 \pmod{n} \\
&\equiv (m-1)(k-1) + kn + 2(m-1) + 2 \pmod{n} \\
&\equiv k(m-1) - (m-1) + 2(m-1) + 2 \pmod{n} \\
&\equiv (m-1)(k+1) + 2 \pmod{n}.
\end{aligned}$$

We add $n \equiv 0 \pmod{n}$ to make the expression on the right even:

$$i^\tau \equiv \frac{(m-1)(k+1)+2+n}{2} \pmod{\frac{n}{\gcd(n,2)}}$$
$$i^\tau \equiv \frac{(m-1)(k+1)+2+n}{2} \equiv i \pmod{n}.$$

As before, if $i^{\tau^2} = i$, then $i^\tau = i$. There is one fixed point $\frac{(m-1)(k+1)+2+n}{2} \pmod{n}$ and no orbits of size 2. The remaining spokes are the $\frac{n-1}{2}$ orbits of size 4. The lemma is proven for τ.

We will now briefly repeat the proof for τ'. Recall that

$$i^{(\tau')^2} = 2 - i - (m-1)(k-1) \pmod{n}.$$

Then if $i^{(\tau')^2} = i$:

$$i \equiv 2 - i - (m-1)(k-1) \pmod{n}$$
$$2(1-i) \equiv (m-1)(k-1) \pmod{n}.$$

Let m be odd, so $m-1$ is even.

$$1 - i \equiv \left(\frac{m-1}{2}\right)(k-1) \pmod{\frac{n}{\gcd(n,2)}}$$
$$i \equiv 1 - \left(\frac{m-1}{2}\right)(k-1) \pmod{n}.$$

We calculate $i^{\tau'}$:

$$\begin{aligned}
i^{\tau'} &\equiv \left(1 - 1 + \left(\frac{m-1}{2}\right)(k-1)\right)k + m \pmod{n} \\
&\equiv \left(\frac{m-1}{2}\right)(k^2 - k) + m \pmod{n} \\
&\equiv \left(\frac{m-1}{2}\right)(-1 - k) + m - 1 + 1 \pmod{n} \\
&\equiv (-k)\left(\frac{m-1}{2}\right) + \left(\frac{m-1}{2}\right) + 1 \pmod{n} \\
&\equiv 1 - \left(\frac{m-1}{2}\right)(k-1) \pmod{n} \\
&\equiv i \pmod{n}.
\end{aligned}$$

So under τ', there is one fixed point and no orbits of size 2 for m odd. Let m be even and $i^{\tau'^2} = i$. Then:

$$\begin{aligned}
2(1-i) &\equiv (m-1)(k-1) \pmod{n} \\
&\equiv (m-1)(k-1) + n \pmod{n} \\
i &\equiv 1 - \frac{(m-1)(k-1)+n}{2} \pmod{n}.
\end{aligned}$$

And again, we see that this point is fixed:

$$\begin{aligned}
i^{\tau'} &\equiv \left(1 - \frac{(m-1)(k-1)+n}{2}\right)^{\tau'} \pmod n \\
&\equiv \left(\frac{(m-1)(k-1)+n}{2}\right) k + m \pmod n \\
2i^{\tau'} &\equiv (m-1)(-1-k) + kn + 2m \pmod n \\
&\equiv (m-1)(1-k) + 2 \pmod n \\
i^{\tau'} &\equiv \frac{(m-1)(1-k)+2+n}{2} \pmod n \\
&\equiv 1 - \frac{(m-1)(k-1)+n}{2} \pmod n \\
&\equiv i \pmod n.
\end{aligned}$$

Thus with τ' as with τ, for n odd, there is one spoke orbit of size 1 and $\frac{n-1}{4}$ of size 4. *QED*

Lemma 4.16. *Let n be even and m odd. Then under $\langle\tau\rangle, \langle\tau'\rangle$, there is one spoke orbit of size 2 and $\frac{n-2}{4}$ of size 4.*

Proof. Since n is even and $k^2 \equiv -1 \pmod n$, k^2 and k are odd. Suppose that $i^{\tau^2} = i$. Then

$$2(i-1) \equiv (m-1)(k+1) \pmod n.$$

Since $k+1$ is even, so is $(m-1)(k+1)$:

$$i - 1 \equiv \frac{(m-1)(k+1)}{2} \quad (\text{mod } \frac{n}{\gcd(n,2)}),$$

$$i \equiv \frac{(m-1)(k+1)}{2} + 1 \quad (\text{mod } \frac{n}{2}).$$

Thus there are two spokes that are fixed under τ^2:

$$i_1 = \frac{(m-1)(k+1)}{2} + 1 \quad (\text{mod } \frac{n}{2}), \quad i_2 = i_1 + \frac{n}{2}.$$

Equivalently, switching i_1 and i_2 if necessary:

$$i_1 = \frac{(m-1)(k+1)}{2} + 1 \pmod{n}, \quad i_2 = i_1 + \frac{n}{2} \pmod{n}.$$

We will consider the images of these points under τ, but first, note the following. We know that

$$(k+1)k = k-1 \pmod{n}.$$

Since n is even, we cannot conclude that

$$\frac{(k+1)k}{2} = \frac{k-1}{2} \pmod{n}.$$

However, we *may* conclude that

$$x(k+1)k = x(k-1) \pmod{n},$$

and similarly that

$$x(k-1)k = x(-1-k) \pmod{n}.$$

for any integer x. This distinction will prove important.

Since k is odd, i and $(i-1)k$ have different parity, and so i and $i^\tau = (i-1)k + m$ have different parity when m is even. But reduction modulo n preserves parity when n is even. So we may conclude that $i = i^\tau$ is only possible when m is odd. That is, if m is even, then i_1 and i_2 are not fixed points. We have already shown that they are the only two points of order 2, and so it must be that they are partners in a transposition. All other spokes must be in an orbit of size 4, since otherwise they would be fixed under τ^2, and there are $\frac{n-2}{4}$ such orbits.

We repeat the proof for τ'. If $i^{\tau'^2} = i$:

$$2(1-i) \equiv (m-1)(k-1) \pmod{n}.$$

Since $k-1$ is even, we know $(m-1)(k-1)$ is even so

$$i = 1 - \frac{(m-1)(k-1)}{2} \pmod{\frac{n}{2}}.$$

Thus there are two points that are fixed under $(\tau')^2$:

$$i_1 = 1 - \frac{(m-1)(k-1)}{2} \pmod{n},\ i_2 = i_1 + \frac{n}{2} \pmod{n}.$$

As m is even, then there are no solutions to $i = i^{\tau'} = (1-i)k + m$, since i and $1-i$ are of opposite parity $\pmod{n}$. We conclude that i_1, i_2 are not fixed but rather in a transposition, and the remaining spokes are in $\frac{n-2}{4}$ orbits of size 4. *QED*

We now examine the final condition, that when n is even and m is odd.

Lemma 4.17. *Let n be even and m odd. Then under $\langle\tau\rangle, \langle\tau'\rangle$, there are two fixed spokes and $\frac{n-2}{4}$ orbits of size 4.*

Proof. m is odd, so $m-1$ is even, and so $\frac{m-1}{2}$ is an integer. As discussed, this means that

$$\left(\frac{m-1}{2}\right)(k+1)k \equiv \left(\frac{m-1}{2}\right)(k-1) \pmod{n}.$$

Recall from the proof of Lemma 4.16 that there are two solutions to $i^{\tau^2} = i$, a fact that did not rely upon the parity of m:

$$i_1 = \frac{(m-1)(k+1)}{2} + 1 \pmod{n}, \quad i_2 = i_1 + \frac{n}{2} \pmod{n}.$$

Then

$$\begin{aligned}
i_1^{\tau} &\equiv (i_1 - 1)k + m \pmod{n} \\
&\equiv \left(\frac{(m-1)(k+1)}{2}\right) k + m \pmod{n} \\
&\equiv \left(\frac{m-1}{2}\right)(k+1)k + m \pmod{n} \\
&\equiv \left(\frac{m-1}{2}\right)(k-1) + m - 1 + 1 \pmod{n}.
\end{aligned}$$

The last equality owing to the distinction we made earlier. So

$$i_1^{\tau} \equiv \left(\frac{(m-1)(k+1)}{2}\right) + 1 \equiv i_1 \pmod{n}.$$

Thus i_1 is fixed under τ, and so must be i_2, having no possible partners. All other spokes are not fixed under τ^2 and must be in orbits of size 4. Finally, recall the two points fixed under $(\tau')^2$ for any m:

$$i_1 = 1 - \frac{(m-1)(k-1)}{2} \pmod{n},\ i_2 = i_1 + \frac{n}{2} \pmod{n}.$$

Then

$$\begin{aligned}
i_1^{\tau'} &\equiv (1 - i_1)k + m \pmod{n} \\
&\equiv \left(\frac{(m-1)(k-1)}{2}\right) k + m \pmod{n} \\
&\equiv \left(\frac{m-1}{2}\right)(k-1)k + m \pmod{n} \\
&\equiv \left(\frac{m-1}{2}\right)(-1-k) + m - 1 + 1 \pmod{n}.
\end{aligned}$$

We could say that $\left(\frac{m-1}{2}\right)(k-1)k \equiv \left(\frac{m-1}{2}\right)(-1-k) \pmod{n}$ because $\frac{m-1}{2}$ was an integer.

$$i_1^{\tau'} \equiv (-k)\left(\frac{m-1}{2}\right) + \left(\frac{m-1}{2}\right) + 1 \equiv 1 - \frac{(m-1)(k-1)}{2} \equiv i_1 \pmod{n}.$$

Thus i_1 is fixed under τ', and so must be i_2, having no possible partners. All other spokes are not fixed under τ^2 and must be in orbits of size 4, and there are $\frac{n-2}{4}$ such orbits. *QED*

We now have the machinery to prove the main result for $GP(n,k), k^2 \equiv -1 \pmod{n}$, for which we will use a more straightforward application of Theorem 1.12. Let $\mathcal{B}$ be the basis for $\mathcal{C}$ that was defined in Lemma 4.13. These are the n cycles defined by their interior edge and the one outer cycle.

Lemma 4.18. *Every permutation* $\pi \in \mathrm{Aut}(GP(n,k))$ *can be written* $\pi = \rho\tau_1^k$ *for* ρ *a rotation and* τ *as defined before, with* $\tau_1 : 1 \mapsto 1$.

Proof. Recall from Theorem 4.2 that $\mathrm{Aut}(GP(n,k)) \cong \mathbb{Z}_n \rtimes \mathbb{Z}_4$. Thus there is a subgroup of order 4 in $\mathrm{Aut}(GP(n,k))$. We have already shown that $\langle\tau\rangle \leqslant \mathrm{Aut}(GP(n,k))$ has order 4 on the vertices. Also, since $k^2 \equiv -1 \pmod{n}$, we know that $4 \nmid n$. So there is no rotation that generates a subgroup of order 4, and $\langle\tau\rangle$ must be the $\mathbb{Z}_4$ for some τ. For simplicity we let it be τ_1. Next, $\mathbb{Z}_n \cap \mathbb{Z}_4 = \{1\}$, so $\mathbb{Z}_n$ cannot contain any elements of $\langle\tau_1\rangle$, and only (certain) rotations can generate $\mathbb{Z}_n \leqslant \mathrm{Aut}(GP(n,k))$, so we know $\mathbb{Z}_n$ is the subgroup of rotations. By properties of semidirect products, every permutation π can be written as the unique product $\rho\tau_1^k$ for ρ a rotation and $\tau_1^k \in \langle\tau_1\rangle$. *QED*

Finally, consider τ_1^2 and τ_1^3:

$$i^{\tau_1^2} \equiv ((i-1)k+1)^{\tau_1} \equiv (i-1)k^2+1 \equiv 2-i \pmod{n},$$
$$i^{\tau_1^3} \equiv (2-i)^{\tau_1} \equiv (1-i)k+1 \equiv i^{\tau_1'} \pmod{n}.$$

So we conclude that $\langle\tau_1\rangle = \{1, \tau_1, \gamma, \tau_1'\}$ where γ is a reflection of $GP(n,k)$ across the axis through $\{u_1, v_1\}$. With all this information we may now apply Theorem 1.12.

Let R_1 be the set of rotations in $\mathrm{Aut}(GP(n,k))$, including 1, and let R_2 be the set of reflections. Furthermore, let $R_3 = R_1\tau_1$ and $R_4 = R_1\tau_1'$. Then as a consequence of Lemma

4.18, $\mathrm{Aut}(GP(n,k)) = R_1 \cup R_2 \cup R_3 \cup R_4$ and $R_1 \cup R_2 \cong D_n$. Thus

$$M := |GP(n,k)/\mathrm{SwAut}(GP(n,k))| = \frac{1}{|GP(n,k)|}\left(\sum_{\pi\in\mathrm{Aut}(GP(n,k))} |\mathrm{fix}_\pi(\mathcal{C})|\right)$$
$$M = \frac{1}{4n}\left(\sum_{\pi\in R_1}|\mathrm{fix}_\pi(\mathcal{C})| + \sum_{\pi\in R_2}|\mathrm{fix}_\pi(\mathcal{C})| + \sum_{\pi\in R_3}|\mathrm{fix}_\pi(\mathcal{C})| + \sum_{\pi\in R_4}|\mathrm{fix}_\pi(\mathcal{C})|\right)$$
$$M = \left(\frac{1}{2}\right)\left(\frac{1}{2n}\left(\sum_{\pi\in R_1}|\mathrm{fix}_\pi(\mathcal{C})| + \sum_{\pi\in R_2}|\mathrm{fix}_\pi(\mathcal{C})|\right)\right) + \frac{1}{4n}\left(\sum_{\pi\in R_3}|\mathrm{fix}_\pi(\mathcal{C})| + \sum_{\pi\in R_4}|\mathrm{fix}_\pi(\mathcal{C})|\right)$$
$$M = \left(\frac{1}{2}\right)\left(\frac{1}{2n}\left(\sum_{\pi\in D_n}|\mathrm{fix}_\pi(\mathcal{C})|\right)\right) + \frac{1}{4n}\left(\sum_{\pi\in R_3}|\mathrm{fix}_\pi(\mathcal{C})| + \sum_{\pi\in R_4}|\mathrm{fix}_\pi(\mathcal{C})|\right).$$

Recall that there are $2B_{n,2}$ distinct combinations of $\mathcal{B}$ under D_n, because each distinct bracelet can be paired with O, or not.

$$M = \left(\frac{1}{2}\right)(2B_{n,2}) + \frac{1}{4n}\left(\sum_{\pi\in R_3}|\mathrm{fix}_\pi(\mathcal{C})| + \sum_{\pi\in R_4}|\mathrm{fix}_\pi(\mathcal{C})|\right)$$
$$M = B_{n,2} + \frac{1}{4n}\sum_{\pi\in R_3}|\mathrm{fix}_\pi(\mathcal{C})| + \frac{1}{4n}\sum_{\pi\in R_4}|\mathrm{fix}_\pi(\mathcal{C})|.$$

It now suffices to determine the number of cycles fixed by $\rho\tau_1 \in R_3$ and $\rho\tau_1' \in R_4$.

Lemma 4.19. *For all rotations ρ, $\rho\tau_1 = \tau_{m_1}$ for some m_1, and $\rho\tau_1' = \tau_{m_2}$ for some m_2.*

Proof. Let $i^\rho = i +_n r$ for some $1 \leqslant r \leqslant n$.

$$i^{\rho\tau_1} \equiv (i^\rho)^{\tau_1} \equiv (i+r-1)k + m \equiv (i-1)k + m + rk \equiv i^{\tau_{m+_n rk}} \pmod{n}.$$

Likewise:

$$i^{\rho\tau_1'} \equiv (i^\rho)^{\tau_1'} \equiv (1-i-r)k + m \equiv (1-i)k + m - rk \equiv i^{\tau'_{m-_n rk}} \pmod{n}.$$

QED

Therefore, we may apply to $\rho\tau_1$ and $\rho\tau_1'$ the structure for τ, τ' we discussed in Lemmas

4.15-4.17. Namely,

- If n is odd, then $\langle \rho\tau_1 \rangle$ and $\langle \rho\tau_1' \rangle$ induce on the set of spoke edges 1 fixed point and $\frac{n-1}{2}$ orbits of size 4.

- If n is even, then $\langle \rho\tau_1 \rangle$ and $\langle \rho\tau_1' \rangle$ induce on the set of spoke edges $\frac{n-2}{2}$ orbits of size 4, and either two fixed points (if $1^\tau, 1^{\tau'}$ is odd) or one orbit of size 2.

Suppose that C is some Eulerian subgraph of $\mathcal{C}$ that is fixed by $\rho\tau_1$, and let $S \subset C$ be the spoke edges (if any) that C contains. Then S must be a union of orbits of spokes under $\langle \rho\tau_1 \rangle$, because if $\{u_i, v_i\} \in S$, so must be $\{u_i, v_i\}^{\rho\tau_1}$. The same is true for any C fixed by $\rho\tau_1'$.

Lemma 4.20. *Let S be some set of spoke edges of $GP(n, k)$ formed as the (possibly empty) union of 4-orbits under $\langle \rho\tau_1 \rangle$ or $\langle \rho\tau_1' \rangle$. Then there are exactly two Eulerian subgraphs that contain exactly those spoke edges and are fixed by $\rho\tau_1$ or $\rho\tau_1'$.*

Proof. We are building Eulerian subgraphs E with spokes S. If S is empty, then the empty subgraph and the subgraph

$$C = q_1 + q_2 + \cdots + q_n + O$$

which consists of the inner and outer n-cycles, are both fixed. If S is nonempty, let $s \in S$ be the edge $\{u_i, v_i\}$ such that i is minimal. Let s_+ be the edge $\{u_j, v_j\}$ such that j is minimal in $S - \{s\}$, and s_- be the edge $\{u_k, v_k\}$ such that k is maximal in S. These are the spokes in S directly clockwise and counterclockwise from s, respectively. Since every vertex in an Eulerian subgraph has even degree, u_i must have degree 2 (it already has one edge), and so either $\{u_i, u_{i+_n 1}\} \in E$ or $\{u_i, u_{i-_n 1}\} \in E$. Continuing, either the path

$$u_i ... u_j = \{\{u_i, u_{i+_n 1}\}, \{u_{i+_n 1}, u_{i+_n 2}\}, ..., \{u_{j-_n 1}, u_j\}\}$$

or else the path

$$u_i ... u_k = \{\{u_i, u_{i-_n 1}\}, \{u_{i-_n 1}, u_{i-_n 2}\}, ..., \{u_{k+_n 1}, u_k\}\}$$

is in C. Assume the former is true. Then $\{u_j, u_{j+_n 1}\} \notin C$, because otherwise u_j would have degree three. Then if $\{u_l, v_l\}$ is the next spoke clockwise from $\{u_j, v_j\}$, we know $\{u_l, u_{l+_n 1}\} \in C$, because $\{u_l, u_{l-_n 1}\}$ cannot be. Continuing this process, we see that the disjoint paths along O between spoke edges alternate between being include and not included in C, and this completes evenly, leaving no vertex with odd degree, because $|S|$ is even (and there are $|S|$ such paths). The set of these paths that are included is decided entirely by our choice whether $u_i...u_j$ or $u_i...u_k$ was included. Note that if C is the graph containing $u_i...u_j$, then $C+O$ is the one including $u_i...u_k$, because adding O takes the complement of the outer path.

Once the edges along O are determined, then in order for C to be fixed, we must add the images of each path to C. That is, if $u_i...u_j \in C$, then we must add $(u_i...u_j)^{\rho\tau_1}$ (or $(u_i...u_j)^{\rho\tau_1'}$) to C, and repeat for each outer path. Since $(\rho\tau_1)^2$ is a reflection, the spokes in S are symmetric across the axis of reflection, and so are the paths between them. So we know that indeed $(u_i...u_j)^{(\rho\tau_1)^2} \in C$ given $u_i...u_j \in C$ (likewise for $\rho\tau_1'$). Thus the interior paths $v_a...v_b$ are determined by the two possible choices for outer paths, and there are only two Eulerian subgraphs fixed. *QED*

Finally, we rule out the possibility of any fixed Eulerian subgraph containing spokes in an orbit of size 1 or 2. If n is odd, then there is only one spoke in an orbit of size 1, and so any subgraph whose set of spokes includes this fixed spoke has an odd number of spokes. However, it is impossible to add a series of alternating external paths in O between an odd number of spokes, so if n is odd, any Eulerian subgraph fixed by $\langle\rho\tau_1\rangle$ or $\langle\rho\tau_1'\rangle$ has spokes created by some union of 4-orbits.

If n is even, then a fixed Eulerian subgraph C cannot contain just one of the two fixed spokes (for m odd) by a similar logic. So suppose C contains both fixed spokes, or the two spokes in the 2-orbit (for m even). Since C is fixed under $\rho\tau_1$ (or $\rho\tau_1'$), it is clearly fixed under the reflection $(\rho\tau_1)^2$ (or $(\rho\tau_1)^2$). If we let (u_i, v_i) be one of the spokes fixed under this reflection, we know $\{u_i, u_{i\pm_n 1}\} \in C$ to ensure u_i has even degree. However, $(u_i, u_{i\pm_n 1})$ is not fixed by the reflection, but rather mapped to $(u_i, u_{i\mp_n 1})$. This contradicts that C was fixed under $(\rho\tau_1)^2$ (likewise, $(\rho\tau_1')^2$). Therefore, no Eulerian subgraph containing a spoke in an orbit of size 1 or 2 can be fixed by any permutation $\rho\tau_1$ or $\rho\tau_1'$. Lemma 4.21 confirms, on the other hand, that for every union of 4-orbits of spokes, there are two Eulerian subgraphs fixed under $\rho\tau_1$ (or $\rho\tau_1'$). If n is odd, there are $\frac{n-1}{4}$ such 4-orbits for each permutation, and if n is even, there are $\frac{n-2}{4}$. Since $n \equiv 1, 2 \pmod 4$ respectively, we can say there are $\lfloor \frac{n}{4} \rfloor$ 4-orbits for an arbitrary n. That is, there are $2^{\lfloor \frac{n}{4} \rfloor}$ possible unions of these orbits and so $2 \times 2^{\lfloor \frac{n}{4} \rfloor}$ Eulerian subgraphs fixed for each permutation. With this, we can now state our final result.

Proposition 4.21. *Let $k^2 \equiv -1 \pmod n$. Then $|GP(n,k)/\mathrm{SwAut}(GP(n,k))| = B_{n,2} + 2^{\lfloor \frac{n}{4} \rfloor}$, excluding the special cases. Equivalently,*

$$|GP(n,k)/\mathrm{SwAut}(GP(n,k))| = \begin{cases} \frac{1}{n}\sum_{d|n} \phi(d) 2^{\frac{n}{d}-1} + 3 \cdot 2^{\frac{n-4}{2}} + 2^{\frac{n-2}{4}} & n \text{ even} \\ \frac{1}{n}\sum_{d|n} \phi(d) 2^{\frac{n}{d}-1} + 2^{\frac{n-1}{2}} + 2^{\frac{n-1}{4}} & n \text{ odd} \end{cases}$$

Proof. We use our recent conclusions on the number of stabilized Eulerian subgraphs to resolve our sum from earlier:

$$M = B_{n,2} + \frac{1}{4n}\sum_{\pi \in R_3} |\mathrm{fix}_\pi(\mathcal{C})| + \frac{1}{4n}\sum_{\pi \in R_4} |\mathrm{fix}_\pi(\mathcal{C})|$$

$$M = B_{n,2} + \frac{1}{4n}\sum_{\pi \in R_3} 2 \times 2^{\lfloor \frac{n}{4} \rfloor} + \frac{1}{4n}\sum_{\pi \in R_4} 2 \times 2^{\lfloor \frac{n}{4} \rfloor}.$$

There are n rotations ρ, and so $|R_3| = |\{\rho\tau\}| = n$, and likewise for R_4.

$$M = B_{n,2} + \frac{n}{4n} 2 \times 2^{\lfloor \frac{n}{4} \rfloor} + \frac{n}{4n} 2 \times 2^{\lfloor \frac{n}{4} \rfloor} = B_{n,2} + \frac{2 \times 2^{\lfloor \frac{n}{4} \rfloor}}{2}$$

$$|GP(n,k)/\mathrm{SwAut}(GP(n,k))| = B_{n,2} + 2^{\lfloor \frac{n}{4} \rfloor}.$$

We can write this explicitly with Lemma 4.5 as:

$$|GP(n,k)/\mathrm{SwAut}(GP(n,k))| = \begin{cases} \frac{1}{n}\sum_{d|n}\phi(d)2^{\frac{n}{d}-1} + 3\cdot 2^{\frac{n-4}{2}} + 2^{\frac{n-2}{4}} & n \text{ even} \\ \frac{1}{n}\sum_{d|n}\phi(d)2^{\frac{n}{d}-1} + 2^{\frac{n-1}{2}} + 2^{\frac{n-1}{4}} & n \text{ odd} \end{cases}$$

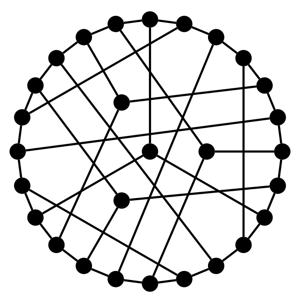
QED

The automorphism structures of Generalized Petersen graphs $GP(n,k)$ such that $k^2 \equiv 1 \pmod{n}$, $k \neq 1$, are significantly more complicated, and we have not yet been able to construct a similar formula for this final species.

Chapter 5

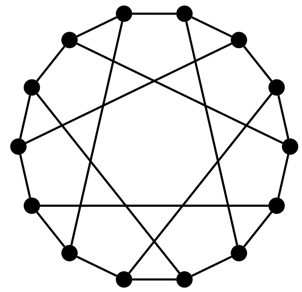
Selected Results

5.1 Switching Automorphism Groups



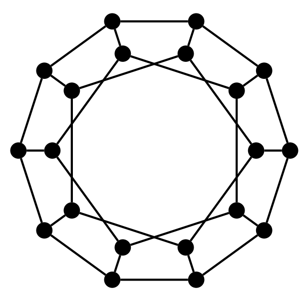
The switching automorphism groups of the 167 *switching isomorphism classes of the Coxeter graph.*

G	Number of S.I. Classes $[\Sigma]$ with $\text{SwAut}(\Sigma) \cong G$
1	57
$\mathbb{Z}_2$	65
$\mathbb{Z}_3$	3
$\mathbb{Z}_2 \times \mathbb{Z}_2$	19
S_3	4
A_4	1
D_4	6
D_6	6
D_7	2
D_8	2
$\text{PSL}(3,2) \rtimes \mathbb{Z}_2$	2



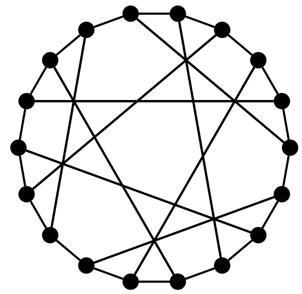
The switching automorphism groups of the 7 switching isomorphism classes of the Heawood graph. Note that Sivaraman [17] first proved there were seven switching isomorphism classes.

G	Number of S.I. Classes $[\Sigma]$ with $\text{SwAut}(\Sigma) \cong G$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	1
S_3	1
D_4	1
D_6	1
D_7	1
D_8	1
$\text{PSL}(3,2) \rtimes \mathbb{Z}_2$	1



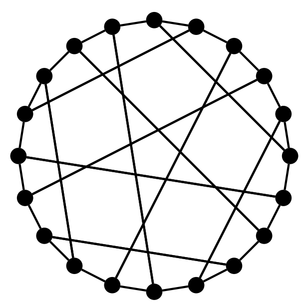
The switching automorphism groups of the 46 switching isomorphism classes of the Dodecahedron graph.

G	Number of S.I. Classes $[\Sigma]$ with $\text{SwAut}(\Sigma) \cong G$
1	4
$\mathbb{Z}_2$	18
$\mathbb{Z}_2 \times \mathbb{Z}_2$	12
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	2
S_3	2
D_5	2
D_6	2
D_{10}	2
$\mathbb{Z}_2 \times A_5$	2



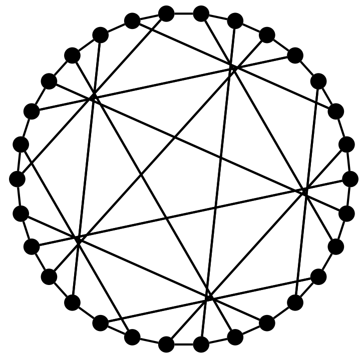
The switching automorphism groups of the 19 *switching isomorphism classes of the Pappus graph.*

G	Number of S.I. Classes $[\Sigma]$ with $\mathrm{SwAut}(\Sigma) \cong G$
$\mathbb{Z}_2$	6
$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
S_3	1
$S_3 \times S_3$	1
D_4	2
D_6	3
D_{12}	1
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$	1



The switching automorphism groups of the 31 *switching isomorphism classes of the Desargues graph.*

G	Number of S.I. Classes $[\Sigma]$ $\mathrm{SwAut}(\Sigma) \cong G$
1	2
$\mathbb{Z}_2$	6
$\mathbb{Z}_2 \times \mathbb{Z}_2$	10
S_3	1
D_4	4
D_5	1
D_{10}	1
$\mathbb{Z}_2 \times S_5$	2
$\mathbb{Z}_2 \times D_4$	2
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$	2



The switching automorphism groups of the 93 *switching isomorphism classes of the Tutte* 8*-cage.*

G	Number of S.I. Classes $[\Sigma]$ with $\mathrm{SwAut}(\Sigma) \cong G$
1	19
$\mathbb{Z}_2$	44
$\mathbb{Z}_4$	1
$\mathbb{Z}_6$	1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	10
$\mathbb{Z}_4 \times \mathbb{Z}_2$	1
S_3	1
D_4	5
D_5	3
D_6	1
D_8	2
D_{10}	1
$\mathbb{Z}_8 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$	1
$\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$	1
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	1
$(A_6 \,.\, \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	1

5.2 Switching Isomorphism Classes

Definition 5.1. A *complete bipartite graph* $K_{n,n}$ has vertices $V(K_{n,n}) = [n] \times [2]$ and $\{(a,b),(c,d)\} \in E(K_{n,n})$ if and only if $b \neq d$.

Here is an example of a complete bipartite graph, specifically $K_{5,5}$:

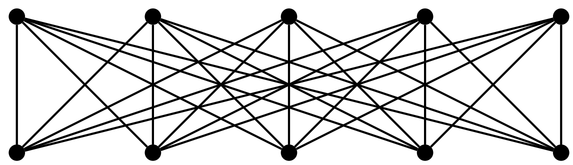


TABLE 1

The number of switching isomorphism classes for complete bipartite graphs $K_{n,n}$

n	$\lvert K_{n,n}/\mathrm{SwAut}(K_{n,n})\rvert$
1	1
2	2
3	3
4	10
5	30
6	242
7	4,386
8	332,513
9	99,976,108
10	112,351,999,472
11	446,983,927,046,926
12	6,198,676,214,116,269,010
13	299,048,546,994,431,406,208,782
14	50,353,545,654,171,448,522,501,436,878
15	29,748,767,946,569,482,487,754,208,177,072,729
16	62,040,859,710,632,592,888,548,097,650,689,577,975,126
17	459,536,316,632,167,030,937,007,189,372,281,636,670,907,163,480
18	12,161,085,637,962,934,969,207,487,980,681,818,705,221,569,753,393,078,738

TABLE 2

The number of switching isomorphism classes for Generalized Petersen graphs $GP(n,k)$. Stars $*$ mark those among the seven graphs with unusual automorphism groups. Note that the results for $GP(5,2)$ and $GP(7,2)$ were previously shown by Zaslavsky [21] *and Bagheri, Moghaddamfar and Ramezani* [1]*, respectively.*

(n,k)	1	2	3	4	5	6	7	8	9
1									
2									
3	6								
4	6*								
5	12	6*							
6	21	26							
7	27	36							
8	48	60	20*						
9	69	92	92						
10	122	46*	31*	156					
11	189	252	252						
12	346	448	448	448	124*				
13	570	760	760	760					
14	1,049	1,374	1,374	1,374	1,374				
15	1,836	2,448	2,448	1,320	2,448	2,448			
16	3,412	4,500	4,500	4,500		4,500	2,448		
17	6,168	8,224	8,224	4,128	8,224				
18	11,599	15,370	15,370	15,370	15,370	15,370	15,370		
19	21,465	28,620	28,620	28,620			28,620		
20	40,660	54,024	54,024	54,024	54,024	54,024		54,024	28,276

Definition 5.2. A *Kneser graph* $K(n,k)$ has vertices $V(K(n,k)) = [n]^k$, the k-subsets of $[n]$, with $\{A,B\} \in E(K(n,k))$ if and only if $A \cap B = \varnothing$.

TABLE 3

The number of switching isomorphism classes for Kneser Graphs $K(n,2)$

n	$\lvert K(n,2)/\mathrm{SwAut}(K(n,2))\rvert$
2	1
3	1
4	1
5	6
6	3,004,784
7	7,675,719,496,506,999,510,208
8	304,066,575,568,628,741,752,193,615,294,429,683,699,846,121,840,640
9	49,377,088,672,350,181,008,761,955,038,986,869,476,221,461,681,669,243,793,976,362, 336,410,824,461,907,370,412,926,721,324,220,416
10	69,793,411,350,435,141,886,561,557,959,688,801,420,848,352,480,059,778,467,000,352, 890,102,645,541,980,614,440,345,703,829,156,254,428,537,034,046,080,007,006,546,185, 315,088,888,958,473,144,937,186,305,725,800,745,690,726,400

Chapter 6

Future Research

There are many potential routes of further research that I hope to pursue. I would like to study cohomology that I might better understand signed switching from a more abstract and foundational perspective as constructed by Cameron [3], [5]. This would also permit me to understand more completely the relationship between signed switching and Seidel switching [16], two-graphs, equiangular lines, and Eulerian graphs, which may provide inspiration for new approaches to enumeration. On that note I plan to research the methods by which the formula for the number of switching classes of complete graphs K_n was discovered, and investigate whether or not it may be applicable to other graphs, such as $K_{n,n}$. Finally, I plan to research oligomorphic permutation groups, as I believe they are related to signed graphs and there exist methods for studying their orbits [4], and matroids, as I understand that others have applied their theory to the study of signed graphs and cycle spaces [17].

Most immediately, I would like to optimize Algorithm 1, so that it may be able to compute specific minimal signed graphs, and thus switching automorphism groups, for more complicated graphs. One likely improvement would be an implementation of the theory learned in the design of Algorithm 2; namely, the fact that with a spanning tree fixed, we need only consider signatures whose negative edges are found outside of the tree. Additionally, I would like to explore whether or not existing graph cut programs could be exploited to better screen potential minimum signatures for redundancy.

Appendix A

Algorithm 1

```
LoadPackage("grape");

grapher := function(E)
    local G;
    G := Graph(Group(()), [1..VertexCount(E)], OnPoints,\\
    function(x,y) return [x,y] in E or [y,x] in E; end);
    return G;
end;

Deg := function(X, v)
    local i, x;
    x := 0;
    for i in X do
        if v in i then
            x := x+1;
        fi;
    od;
```

```
    return x;
end;

Mood := function(X,E, v)
    local x,i,j;
    x := 0;
    for i in X do
        if v in i then
            x := x-1;
        fi;
    od;
    for j in Difference(E, X) do
        if v in j then
            x := x+1;
        fi;
    od;
    return x;
end;

Friends := function(X,E,v)
    local x, i;
    x := [];
    for i in Difference(E,X) do
        if i[1] = v then
            Add(x, i[2]);
        elif i[2] = v then
            Add(x, i[1]);
```

```
        fi;
    od;
    return x;
end;

VertexCount := function(E)
    local i, AllVertices;
    AllVertices := [];
    for i in E do
        Add(AllVertices, i[1]);
        Add(AllVertices, i[2]);
    od;
    return Length(DuplicateFreeList(AllVertices));
end;

UpperObjects := function(X, E, cube)
    local Uppers, i, Cands, Stab, S, A, Orbs, o;
    Uppers := [];
    S := grapher(E);
    A := AutomorphismGroup(S);
    Stab := Stabilizer(A, X, OnSetsSets);
    Orbs := Orbits(Stab, Eligibles(X,E, cube), OnSets);
    for o in Orbs do
        Add(Uppers, Union(X, [o[1]]));
    od;
    return Uppers;
end;
```

```
scan_step := function(X, E, G, cube)
    local C, Uppers, upper, fprimeimage, y, c, orbs, reps, o, negs,\\
    u,s,v,sigs,graphcovers,iso_free,r;
    C := [];
    Uppers := UpperObjects(X, E, cube);
    if not Uppers = [] then
        for upper in Uppers do
            if not upper  = [] then
                Add(C, upper);
            fi;
        od;
        return C;
    else
        return [];
    fi;
end;


scan := function(X, E, G, cube)
    local counter, i, A, C, c, s, remain, graphcovers, iso_free,negs,u,v,r;
    remain := true;
    counter := 1;
    C:=[];
    C[1] := [X];
    while remain do
        C[counter + 1] := [];
        remain := false;
```

```
    for c in C[counter] do
        s := scan_step(c, E, G, cube);
        if not s = [] then
            remain := true;
            Add(C[counter + 1], s);
        fi;
    od;
    C[counter + 1]:=Concatenation(C[counter + 1]);
orbs := Orbits(G, AsSet(C[counter+1]), OnSetsSets);
reps := [];
for o in orbs do
    Add(reps, o[1]);
od;
graphcovers := [];
for c in reps do
    Add(graphcovers, Covers(E, VertexCount(E), c));
od;
iso_free := GraphIsomorphismClassRepresentatives(graphcovers);
C[counter+1] := [];
for l in iso_free do
    negs := [];
    for edge in UndirectedEdges(l) do
        if edge[1]>VertexCount(E) and edge[2] <= VertexCount(E) then
            Add(negs, AsSet([edge[1]-VertexCount(E), edge[2]]));
        elif edge[2]>VertexCount(E) and edge[1] <= VertexCount(E) then
            Add(negs, AsSet([edge[2]-VertexCount(E), edge[1]]));
        fi;
```

```
        od;
        Add(C[counter+1], DuplicateFreeList(negs));
    od;
        counter := counter + 1;
    od;
    A:=[];
    for i in [1..counter] do
        A:=Union(A, C[i]);
    od;
    return A;
end;


Covers := function(e, v, m)
    local Negs, i, j, k, AllEdges, PosEdges, Strands, Edges;
    Negs := [];
    for i in m do
        Add(Negs, i);
        Add(Negs, [i[1] + v, i[2] + v]);
    od;
    AllEdges := [];
    for j in e do
        Add(AllEdges, j);
        Add(AllEdges, [j[1]+v, j[2] +v]);
    od;
    PosEdges := Difference(AllEdges, Negs);
    Strands := [];
    for k in m do
```

```
        Add(Strands, [k[1], k[2] + v]);

        Add(Strands, [k[1]+v, k[2]]);

    od;

    Edges := Concatenation(PosEdges, Strands);

    return Graph(Group(()), [1..2*v], OnPoints, function(x,y)\\

    return [x,y] in Edges or [y,x] in Edges; end);

end;


SIClasses := function(X, E, cube)

    local x, i, S, A,s;

    S := grapher(E);

    A := AutomorphismGroup(S);

    x := [];

    s := scan(X, E, A, cube);

    for i in s do

        Add(x, Covers(E, VertexCount(E), i));

    od;

    return Length(GraphIsomorphismClassRepresentatives(x));

end;


SIClassReps := function(X, E, cube)

    local x, i, S, A;

    S := grapher(E);

    A := AutomorphismGroup(S);

    x := [];

    for i in scan(X, E, A, cube) do

        Add(x, Covers(E, VertexCount(E), i));
```

```
    od;
    return GraphIsomorphismClassRepresentatives(x);
end;

Eligibles := function(X,E, cube)
    local Candidates, Rejects, V;
    if cube = 1 then
        Candidates:=Difference(E,X);
        Rejects:=[];
        V := DuplicateFreeList(Union(X));
        for i in Candidates do
            if not Intersection(V, i) = [] then
                Add(Rejects, i);
            fi;
        od;
        return Difference(Candidates, Rejects);
    else
        Candidates := [];
        Rejects := [];
        for i in Difference(E,X) do
            if Deg(E,i[1]) - Deg(X, i[1]) > Deg(X, i[1]) + 1 and\\
            Deg(E, i[2]) - Deg(X, i[2]) > Deg(X, i[2]) + 1 then
                Add(Candidates, i);
            fi;
        od;
        for i in Candidates do
            if not i in Rejects then
```

```
for c in Difference(Friends(X,E,i[1]), [i[2]]) do
    for d in Difference(Friends(X,E,c), [i[1], i[2]]) do
        for x in\\
        Difference(Friends(X,E,i[1]), [i[2], c, d]) do
            y := AsSet([x, d]);
            if y in Difference(E,X) then
                if AsSet([c,x]) in Difference(E,X) then
                    if AsSet([d,i[1]]) in Difference(E,X)\\
                    then
                        if Mood(X,E,i[1])+Mood(X,E,x)+ \\
                        Mood(X,E,c)+ Mood(X,E,d) - 14 < 0\\
                        then
                            Add(Rejects, i);
                        fi;
                    elif not AsSet([d,i[1]]) in E then
                        if Mood(X,E,i[1])+Mood(X,E,x)+\\
                        Mood(X,E,c)+Mood(X,E,d) - 12 < 0\\
                        then
                            Add(Rejects, i);
                        fi;
                    fi;
                elif not AsSet([c,x]) in E then
                    if AsSet([d,i[1]]) in Difference(E,X)\\
                    then
                        if Mood(X,E,i[1])+Mood(X,E,x)+\\
                        Mood(X,E,c)+Mood(X,E,d) - 12 < 0\\
                        then
```

```
                Add(Rejects, i);
            fi;
        elif not AsSet([d,i[1]]) in E then
            if Mood(X,E,i[1])+Mood(X,E,x)+\\
            Mood(X,E,c)+Mood(X,E,d) - 10 < 0\\
            then
                Add(Rejects, i);
            fi;
        fi;
fi;
elif not y in E then
    if AsSet([c,x]) in Difference(E,X) then
        if AsSet([d,i[1]]) in Difference(E,X)\\
        then
            if Mood(X,E,i[1])+Mood(X,E,x)+\\
            Mood(X,E,c)+Mood(X,E,d) - 12 < 0\\
            then
                Add(Rejects, i);
            fi;
        elif not AsSet([d,i[1]]) in E then
            if Mood(X,E,i[1])+Mood(X,E,x)+\\
            Mood(X,E,c)+Mood(X,E,d) - 10 < 0\\
            then
                Add(Rejects, i);
            fi;
        fi;
elif not AsSet([c,x]) in E then
```

```
        if AsSet([d,i[1]]) in Difference(E,X)\\
        then
                if Mood(X,E,i[1])+Mood(X,E,x)+\\
                Mood(X,E,c)+Mood(X,E,d) - 10 < 0\\
                then
                    Add(Rejects, i);
                fi;
            elif not AsSet([d,i[1]]) in E then
                if Mood(X,E,i[1])+Mood(X,E,x)+\\
                Mood(X,E,c)+Mood(X,E,d) - 8 < 0\\
                then
                    Add(Rejects, i);
                fi;
            fi;
        fi;
    elif y in X then
        if AsSet([c,x]) in Difference(E,X) then
            if AsSet([d,i[1]]) in Difference(E,X)\\
            then
                if Mood(X,E,i[1])+Mood(X,E,x)+\\
                Mood(X,E,c)+Mood(X,E,d) - 10 < 0\\
                then
                    Add(Rejects, i);
                fi;
            elif not AsSet([d,i[1]]) in E then
                if Mood(X,E,i[1])+Mood(X,E,x)+\\
                Mood(X,E,c)+Mood(X,E,d) - 8 < 0\\
```

```
                then
                    Add(Rejects, i);
                fi;
            fi;
        elif not AsSet([c,x]) in E then
            if AsSet([d,i[1]]) in Difference(E,X)\\
            then
                if Mood(X,E,i[1])+Mood(X,E,x)+\\
                Mood(X,E,c)+Mood(X,E,d) - 8 < 0\\
                then
                    Add(Rejects, i);
                fi;
            elif not AsSet([d,i[1]]) in E then
                if Mood(X,E,i[1])+Mood(X,E,x)+\\
                Mood(X,E,c)+Mood(X,E,d) - 6 < 0\\
                then
                    Add(Rejects, i);
                fi;
            fi;
        fi;
    fi;
od;
od;
od;
fi;
if not i in Rejects then
        for c in Difference(Friends(X,E,i[2]), [i[1]]) do
```

```
for d in \\
Difference(Friends(X,E,c), [i[1], i[2]]) do
    for x in \\
    Difference(Friends(X,E,i[2]), [i[1], c, d]) do
    y := AsSet([x, d]);
    if y in Difference(E,X) then
        if AsSet([c,x]) in Difference(E,X) then
            if AsSet([d,i[2]]) in Difference(E,X)\\
            then
                if Mood(X,E,i[2])+Mood(X,E,x)+\\
                Mood(X,E,c)+Mood(X,E,d) - 14 < 0\\
                then
                    Add(Rejects, i);
                fi;
            elif not AsSet([d,i[2]]) in E then
                if Mood(X,E,i[2])+Mood(X,E,x)+\\
                Mood(X,E,c)+Mood(X,E,d) - 12 < 0\\
                then
                    Add(Rejects, i);
                fi;
            fi;
        elif not AsSet([c,x]) in E then
            if AsSet([d,i[2]]) in Difference(E,X)\\
            then
                if Mood(X,E,i[2])+Mood(X,E,x)+\\
                Mood(X,E,c)+Mood(X,E,d) - 12 < 0\\
                then
```

```
                Add(Rejects, i);
            fi;
        elif not AsSet([d,i[2]]) in E then
            if Mood(X,E,i[2])+Mood(X,E,x)+\\
            Mood(X,E,c)+Mood(X,E,d) - 10 < 0\\
            then
                Add(Rejects, i);
            fi;
        fi;
    fi;
elif not y in E then
    if AsSet([c,x]) in Difference(E,X) then
        if AsSet([d,i[2]]) in Difference(E,X)\\
        then
            if Mood(X,E,i[2])+Mood(X,E,x)+\\
            Mood(X,E,c)+Mood(X,E,d) - 12 < 0\\
            then
                Add(Rejects, i);
            fi;
        elif not AsSet([d,i[2]]) in E then
            if Mood(X,E,i[2])+Mood(X,E,x)+\\
            Mood(X,E,c)+Mood(X,E,d) - 10 < 0\\
            then
                Add(Rejects, i);
            fi;
        fi;
    elif not AsSet([c,x]) in E then
```

```
                if AsSet([d,i[2]]) in Difference(E,X)\\
                then
                    if Mood(X,E,i[2])+Mood(X,E,x)+\\
                    Mood(X,E,c)+Mood(X,E,d) - 10 < 0\\
                    then
                        Add(Rejects, i);
                    fi;
                elif not AsSet([d,i[2]]) in E then
                    if Mood(X,E,i[2])+Mood(X,E,x)+\\
                    Mood(X,E,c)+Mood(X,E,d) - 8 < 0\\
                    then
                        Add(Rejects, i);
                    fi;
                fi;
            fi;
        elif y in X then
            if AsSet([c,x]) in Difference(E,X) then
                if AsSet([d,i[2]]) in Difference(E,X)\\
                then
                    if Mood(X,E,i[2])+Mood(X,E,x)+\\
                    Mood(X,E,c)+Mood(X,E,d) - 10 < 0\\
                    then
                        Add(Rejects, i);
                    fi;
                elif not AsSet([d,i[2]]) in E then
                    if Mood(X,E,i[2])+Mood(X,E,x)+\\
                    Mood(X,E,c)+Mood(X,E,d) - 8 < 0\\
```

```
                                  then
                                      Add(Rejects, i);
                                  fi;
                              fi;
                          elif not AsSet([c,x]) in E then
                              if AsSet([d,i[2]]) in Difference(E,X)\\
                              then
                                  if Mood(X,E,i[2])+Mood(X,E,x)+\\
                                  Mood(X,E,c)+Mood(X,E,d) - 8 < 0\\
                                  then
                                      Add(Rejects, i);
                                  fi;
                              elif not AsSet([d,i[2]]) in E then
                                  if Mood(X,E,i[2])+Mood(X,E,x)+\\
                                  Mood(X,E,c)+Mood(X,E,d) - 6 < 0\\
                                  then
                                      Add(Rejects, i);
                                  fi;
                              fi;
                          fi;
                      fi;
                  od;
              od;
          od;
      fi;
od;
fi;
```

```
    return Difference(Candidates, Rejects);
end;

Groups := function(X,E, cube)
    local groups, covers, P, n, i, k, K, a, A, S, F;
    groups:=[];
    covers := SIClassReps(X,E, cube);
    P := [];
    n := VertexCount(E);
    for i in [1..n] do
        Add(P, [i, i+n]);
    od;
    k := MappingPermListList([1..n],[n+1..2*n]);
    K := Group(k);
    for a in covers do
        A := AutomorphismGroup(a);
        S := Stabilizer(A, P, OnSetsSets);
        F:=FactorGroup(S,K);
        Add(groups, F);
    od;
    return List(groups, StructureDescription);
end;

Signings := function(X,E,cube)
    local S, sigs, n, strands, edges, x, s;
    S := SIClassReps(X,E,cube);
    sigs := [];
```

```
    n := VertexCount(E);
    for s in S do
        strands:=[];
        edges := UndirectedEdges(s);
        for x in edges do
            if x[1] <= n and x[2] > n then
                Add(strands, AsSet([x[1], x[2]-n]));
            fi;
        od;
        Add(sigs, DuplicateFreeList(strands));
    od;
    return sigs;
end;
```

Appendix B

Algorithm 2

```
LoadPackage("grape");
LoadPackage("digraphs");

grapher := function(E)
    local G;
    G := Graph(Group(()), [1..VertexCount(E)], OnPoints,\\
    function(x,y) return [x,y] in E or [y,x] in E; end);
    return G;
end;

digrapher := function(E)
    local G;
    G:=Digraph(Group(()), [1..VertexCount(E)], OnPoints,\\
    function(x,y) return [x,y] in E or [y,x] in E; end);
    return G;
end;
```

```
cyclebuilder := function(edge, tree)
    local path, x,i;
    path := DigraphPath(tree, edge[1], edge[2])[1];
    x := [];
    for i in [1..Length(path)-1] do
        if path[i] < path[i+1] then
            Add(x, [path[i], path[i+1]]);
        else
            Add(x, [path[i+1], path[i]]);
        fi;
    od;
    Add(x, edge);
    Sort(x);
    return x;
end;

basisbuilder := function(E)
    local G, T, TrueT, t, seeds, cycles, s;
    G := digrapher(E);
    T := UndirectedSpanningTree(G);
    TrueT := [];
    for t in DigraphEdges(T) do
        if t[1]<t[2] then
            Add(TrueT, t);
        fi;
    od;
    seeds := Difference(E, TrueT);
```

```
    cycles := [];
    for s in seeds do
        Add(cycles, cyclebuilder(s, T));
    od;
    return [cycles, seeds];
end;

factor := function(S, B, c)
    local x, i;
    x:=[];
    for i in S do
        if i in c then
            Add(x, 1);
        else
            Add(x, 0);
        fi;
    od;
    return x;
end;

Counter := function(G, B, S)
    local lengths, g, M, c, image, f;
    z:=0;
    C:=ConjugacyClasses(G);
    for x in C do
        M := [];
        g := Representative(x);
```

```
        for c in B do
            image := OnSetsSets(c, g);
            f := factor(S, B, image);
            Add(M, f);
        od;
        TransposedMatDestructive(M);
        n := DimensionsMat(M)[1];
        K := IdentityMat(n)-M;
        z := z+Size(x)*2^Length(BasisNullspaceModN(K,2));
    od;
    answer := z/Order(G);
    return answer;
end;

SICounter := function(E)
    local A,B,S;
    A := AutomorphismGroup(grapher(E));
    B := basisbuilder(E)[1];
    S := basisbuilder(E)[2];
    return Counter(A,B,S);
end;
```

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