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Luke Mrini
William & Mary

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Black Hole Entropy in AdS/CFT and the Schwinger-Keldysh Formalism

A thesis submitted in partial fulfillment of the requirement for the degree of Bachelor of Science with Honors in Physics from the College of William and Mary in Virginia,

by

Luke Mrini

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Advisor: Prof. Joshua Erlich

Prof. Clare Pierre

Prof. Keith Griffioen

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Black Hole Entropy in AdS/CFT and the Schwinger-Keldysh Formalism

Luke Mrini\textsuperscript{1,*} and Mentor: Joshua Erlich\textsuperscript{1,†}

\textsuperscript{1}High Energy Theory Group, Department of Physics, William & Mary, Williamsburg, VA 23187-8795, USA

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Abstract

The Schwinger-Keldysh formalism for non-equilibrium field theory provides valuable tools for studying the black hole information loss paradox. In particular, there exists a Noether-like procedure to obtain the entropy density of a system by a discrete Kubo-Martin-Schwinger (KMS) variation of the action. Here, this Noether-like procedure is applied to the boundary action of an asymptotically anti-de Sitter (aAdS) black hole spacetime in maximally extended Kruskal coordinates. The result is the Kubo formula for shear viscosity, which is known in theories with an Einstein gravity dual to have a universal, constant ratio with the entropy density and is proportional to the event horizon area. In this way, the Bekenstein-Hawking entropy of a static aAdS black hole is reproduced up to a constant by a self-consistent procedure using symmetry-based, field theoretic arguments.

*lamrini@wm.edu
†jxerli@wm.edu
I. INTRODUCTION

The black hole information paradox is a longstanding, unsolved problem in physics with implications for unitarity in our universe as well as quantum gravity. Stephen Hawking famously applied the rules of quantum mechanics to black holes in the 1970’s [1] to predict that black holes evaporate and emit Hawking radiation—thermal radiation that only depends on the mass, electric charge, and angular momentum of the black hole. According to Hawking’s calculation, information about the initial physical state which led to the black hole’s formation is entirely absent in the final state of radiation. This violates unitarity, a fundamental postulate of quantum mechanics, which requires the conservation of information at all times. Either unitarity must be abandoned in a theory of quantum gravity or Hawking’s calculation was incorrect. An improved understanding of how principles from quantum mechanics and general relativity come together in the case of black hole evaporation would be a significant step towards obtaining a theory of quantum gravity.

Recent progress has been made hinting at a process by which information may be conserved during black hole evaporation. A key tool used in these developments is the AdS/CFT correspondence—an exact duality between a gravitational theory that is asymptotically Anti-de Sitter space (AdS) and a conformal field theory (CFT) that lives on the asymptotic boundary of the bulk gravitational theory. AdS/CFT is an instance of the holographic principle. First described by Gerard t’Hooft in the 1990’s [2], the holographic principle states that the observable degrees of freedom in a $d+2$-dimensional theory of quantum gravity has an equivalent description in $d+1$ dimensions [3]. The AdS/CFT correspondence has since been used to reduce the problem of the evaporating black hole to studying its holographic dual in one fewer dimension.

Since the boundary description in AdS/CFT is a unitary quantum field theory, information is guaranteed to be conserved throughout the evaporation process for a black hole with a holographic dual. The result is that emitted Hawking radiation encodes information that monotonically increases with time until it encodes the entire information content of the initial physical state. As the black hole evaporates, the emitted Hawking radiation becomes entangled with matter behind the black hole event horizon. A useful measure of the entanglement between two subsystems is entanglement entropy [4]. The entanglement entropy is zero when the state of the two subsystems is separable and increases when the subsystems
become entangled. In black hole evaporation as described by AdS/CFT, the entanglement
entropy of Hawking radiation increases monotonically until the Page time, after which it
decreases monotonically to zero when the black hole vanishes. This rise and fall of entangle-
ment entropy is described by the Page curve [5], illustrated in Fig. 1. The maximum value
of Hawking radiation entropy $S_{\text{Hawking}}$ in the Page curve corresponds to the full information
of the initial physical state.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{The entanglement entropy $S(t)$ of emitted Hawking radiation increases monotonically
until the Page time then decreases to zero, according to the Page curve. $S(t)$ is bounded from
above by the Bekenstein-Hawking entropy $S_{\text{BH}}$ of the black hole and the Hawking radiation entropy
$S_{\text{Hawking}}$ [6].}
\end{figure}

Progress has been made in the past two decades developing a formalism to calculate the
Page curve quantitatively and to provide physical insight into the evolution of entanglement
entropy in black hole evaporation. Ryu and Takayanagi [7] published their conjectured
formula for the entanglement entropy associated with a region on the asymptotic boundary in
2006. Their conjecture states that the entanglement entropy is proportional to the area of the
minimal area surface that extends into the bulk and is homologous to the boundary region.
The Ryu-Takayanagi conjecture has been shown to hold in a number of static situations
[8] but does not generalize to the time-dependent case. The Quantum Extremal Surface
prescription allows for the calculation of entanglement entropy in more general situations by
incorporating a quantum correction to the Ryu-Takayanagi formula [9]. A phase transition is identified at the Page time in the evolution of quantum extremal surfaces with evaporating black holes, consistent with the Page curve [10].

Islands—quantum extremal surfaces that are disconnected and spacelike separated from the region of Hawking radiation on the boundary—are the dominant contributions of entropy after the Page time. The study of these islands is central to the vast majority of quantitative calculations of the Page curve to date. At present, calculations of the Page curve with the island prescription have been limited to modified theories of gravity or simplified models of evaporating black holes, such as generalizations of two-dimensional dilaton gravity [9, 11–14], Einstein-Gauss–Bonnet (EGB) gravity [15, 16], the black string model [17, 18], Karch-Randall models [19–21], the step-function Vaidya model [22], and bottom-up models of interface CFTs [23, 24]. Many of these calculations also incorporate an external heat bath coupled to the gravitational system by hand [25]. There was recently a derivation of a modified Page curve from the perspective of canonical quantum gravity and the Euclidean path integral [26].

The objective of this thesis is to apply the Schwinger-Keldysh (SK) formalism, which comes with an elegant procedure to directly calculate entropy densities, to evaporating black holes with holographic duals. In the past, the SK formalism has been used to study non-equilibrium steady states and non-equilibrium phase transitions in hydrodynamics, thermal fluctuations in chaotic systems, quantum many-body chaos, and even quantum gravity fluctuations in holographic systems [27]. The SK formalism describes non-equilibrium quantum systems by considering correlators of operators inserted on a doubled time contour. When the system is in local thermal equilibrium, the theory is invariant under the discrete Kubo-Martin-Schwinger (KMS) transformation. The Lagrangian varies by a total derivative following a KMS transformation. Subject to conditions related to unitarity and convergence of the functional integral, the Noether-like current resulting from this procedure is the entropy current density. The entropy current density derived in this way has a non-negative divergence and obeys the second law of thermodynamics [28].

Herzog and Son [29] study a doubled asymptotically Anti-de Sitter (aAdS) black hole geometry in order to calculate Green’s functions in the SK formalism. They do this by calculating the on-shell boundary action which determines the generating functional for SK
Green’s functions. In doing so, they set boundary conditions corresponding to one branch of the SK contour in one copy of the aAdS black hole geometry and boundary conditions corresponding to the other branch in the other copy of the geometry. In this thesis, I calculate the black hole entropy density by applying the Noether-like procedure from the SK formalism to Herzog and Son’s on-shell boundary action. The variation of the Lagrangian in this way results in an entropy density that takes the form of the Kubo formula for shear viscosity. This is consistent with the fact that, in any gauge theory with an Einstein gravity dual, the shear viscosity to entropy density ratio takes on the universal value of $1/4\pi$ [30]. Both the entropy density and shear viscosity are in turn proportional to the black hole horizon area $A_H$ [31]. Therefore, the SK Noether-like procedure reproduces (up to a constant factor) the well-known Bekenstein-Hawking entropy

$$S_{BH} = \frac{A_H}{4G},$$

(1.1)

where $G$ is the gravitational constant.

A brief introduction to the SK formalism is presented in Section II along with an overview of the Noether-like procedure for entropy current densities. This is followed by a discussion of Herzog and Son’s doubled geometry in Section III. Here, I discuss the appropriate boundary conditions for reproducing the SK Green’s functions in the doubled geometry and quote the on-shell boundary action. Section IV contains the main results of this thesis. In this section, I carry out the KMS variation of the boundary action and demonstrate that it gives the shear viscosity as the Noether-like density, which is proportional to the Bekenstein-Hawking entropy. In Section V, I summarize this result and discuss possible directions for future research.

II. THE SCHWINGER-KELDYSH FORMALISM

The Schwinger-Keldysh (SK) formalism provides a field theoretic description of systems in or out of thermodynamic equilibrium. We will generally be interested in expectation values of operators at time $t$ in a state determined by density matrix $\rho(t)$,

$$\langle \mathcal{O}(t) \rangle_\rho = \text{tr}(\mathcal{O}(t)\rho(t)).$$

(2.1)
In non-equilibrium situations, it might be that the system was in equilibrium in the distant past \( \rho(-\infty) = \rho_0 \), but time-dependent interactions or sources have since forced the system out of equilibrium [32]. In this case, we can use the unitary time evolution operator \( \mathcal{U}_{t_2,t_1} \) to write the expectation value of \( \mathcal{O}(t) \) as

\[
\langle \mathcal{O}(t) \rangle_\rho = \text{tr}(\mathcal{O}(t)\mathcal{U}_{t,-\infty}\rho\mathcal{U}_{t,-\infty}^\dagger) \\
= \text{tr}(\mathcal{U}_{t,-\infty}^\dagger\mathcal{O}(t)\mathcal{U}_{t,-\infty}\rho_0) \\
= \text{tr}(\mathcal{U}_{t,-\infty}^\dagger\mathcal{U}_{+\infty,t}\mathcal{O}(t)\mathcal{U}_{t,-\infty}\rho_0) \\
= \text{tr}(\mathcal{U}_{-\infty,+\infty}\mathcal{O}(t)\mathcal{U}_{t,-\infty}\rho_0).
\] (2.2)

In the second line we used the cyclicity of the trace and in the third line we inserted the identity in the form \( \mathcal{U}_{+\infty,t}^\dagger\mathcal{U}_{+\infty,t} \). We can interpret Eqn. (2.2) as stating that the expectation value of \( \mathcal{O}(t) \) is obtained by its insertion on a doubled time contour (the SK contour) that runs from \( t = -\infty \) to \( +\infty \) and back, as in Fig. 2.

\[ \begin{array}{c}
-\infty \quad \mathcal{U}_{t,-\infty} \quad \mathcal{O}(t) \quad \mathcal{U}_{+\infty,t} \\
\rho_0 \end{array} \]

The classical fields \( \phi_1(x) \) and \( \phi_2(x) \) are sources for the copies \( \mathcal{O}_1(x) \) and \( \mathcal{O}_2(x) \), respectively, and \( \mathcal{P} \) denotes path ordering along the SK contour. The relative minus sign in the exponential is due to the opposite orientations of time integration on each branch of the SK contour.
Following Herzog and Son [29], we define the SK Green’s functions by variations of the partition function

\[ iG_{ab}(x - y) = -\frac{\delta^2 Z[\phi_1, \phi_2]}{\delta \phi_a(x) \delta \phi_b(y)} = i \begin{pmatrix} G_{11} & -G_{12} \\ -G_{21} & G_{22} \end{pmatrix} \]  

(2.4)

In the SK Green’s functions, operators are ordered along the SK contour. That is,

\[ iG_{11}(t, x) = \langle TO_1(t, x)O_1(0) \rangle \]
\[ iG_{12}(t, x) = \langle O_2(0)O_1(t, x) \rangle \]
\[ iG_{21}(t, x) = \langle O_2(t, x)O_1(0) \rangle \]
\[ iG_{22}(t, x) = \langle \bar{T}O_2(t, x)O_2(0) \rangle, \]  

(2.5)

where \( T \) denotes time ordering and \( \bar{T} \) denotes anti-time ordering. We also define the retarded and advanced Green’s function by

\[ iG_R(x - y) = \theta(x^0 - y^0) \langle [O(x), O(y)] \rangle, \]

\[ iG_A(x - y) = \theta(y^0 - x^0) \langle [O(y), O(x)] \rangle. \]  

(2.6)

The corresponding momentum space Green’s functions are

\[ G_R(k) = \int d^4x e^{-ik \cdot x} G_R(x), \]
\[ G_A(k) = \int d^4x e^{-ik \cdot x} G_A(x). \]  

(2.7)

Note that the advanced and retarded Green’s functions are related by \( G_A(k) = G_R(-k) \).

Consider the case of a thermal initial density matrix \( \rho_0 \), that is,

\[ \rho_0 = \frac{1}{Z_0} e^{-\beta H}, \quad Z_0 = \text{tr} (e^{-\beta H}), \]  

(2.8)

where \( \beta \) is the inverse temperature and \( H \) is the Hamiltonian. Assume also that the theory described by the Hamiltonian \( H \) is invariant under a time reversal transformation \( \Theta \), some combination of \( CPT \) that includes \( \mathcal{T} \). Then we can show that

\[ Z[\phi_1, \phi_2] = \frac{1}{Z_0} \text{tr} \left[ e^{-\beta H} \left( \bar{T} e^{-i \int dx \, O_2 \phi_2} \right) \left( T e^{i \int dx \, O_1 \phi_1} \right) \right] \]
\[ = \frac{1}{Z_0} \text{tr} \left[ e^{-(\beta-\theta)H} \left( \bar{T} e^{-i \int dx \, O_2 \phi_2} \right) e^{(\beta-\theta)H} e^{-\beta H} e^{\theta H} \left( T e^{i \int dx \, O_1 \phi_1} \right) e^{-\theta H} \right] \]
\[ = \frac{1}{Z_0} \text{tr} \left[ e^{-\beta H} \left( T e^{i \int dx \, O_1 \phi_1(t+i\theta)} \right) \left( \bar{T} e^{-i \int dx \, O_2 \phi_2(t-i(\beta-\theta))} \right) \right] \]
\[ = Z[\Theta \phi_1(t - i\theta), \Theta \phi_2(t + i(\beta - \theta))] \]  

(2.9)
where \(0 \leq \theta \leq \beta\) [28]. In the second line, the identity is inserted, making use of the cyclicity of the trace. In the third line, time evolution operators induce imaginary time translations and the remaining factor \(\exp(-\beta H)\) is cycled through the trace to the leftmost position. The remaining expression resembles the original partition function, but with the reverse time-ordering and \(\phi_1\) and \(\phi_2\) translated by imaginary times. The transformation

\[
\phi_1(t, x) \mapsto \Theta \phi_1(t - i\theta, x), \quad \phi_2(t, x) \mapsto \Theta \phi_2(t + i(\beta - \theta), x)
\]  

(2.10)

is a \(\mathbb{Z}_2\) symmetry of the theory called Kubo-Martin-Schwinger (KMS) symmetry. To quadratic order in the fields \(\phi_1\) and \(\phi_2\), KMS symmetry is equivalent to the fluctuation-dissipation theorem [28].

Define the effective action as a functional of \(\phi_1\) and \(\phi_2\),

\[
S[\phi_1, \phi_2] = -i \ln \mathcal{Z}[\phi_1, \phi_2] = \int dx \mathcal{L}(x).
\]  

(2.11)

Here we have also defined the effective Lagrangian density \(\mathcal{L}(x)\) in order to clarify the following Noether-like procedure. Unitarity and convergence of the functional integral require the following constraints [28]:

\[
\begin{align*}
(1) & \quad S[\phi_1 = \phi_2] = 0, \\
(2) & \quad S^*[\phi_1, \phi_2] = S[\phi_2, \phi_1], \\
(3) & \quad \text{Re } S[\phi_1, \phi_2] \leq 0.
\end{align*}
\]  

(2.12)

Under a KMS transformation, the effective action is invariant and the Lagrangian \(\mathcal{L}(x)\) varies by a total derivative,

\[
\mathcal{L}(x) \mapsto \mathcal{L}(x) + \partial_\mu s^\mu(x).
\]  

(2.13)

The Noether-like current \(s^\mu(x)\) is the entropy current density [28]. We say “Noether-like” to emphasize that the KMS transformation is discrete and therefore the associated density is not required to be conserved. The entropy current density derived in this way is guaranteed to obey the second law of thermodynamics,

\[
\partial_\mu s^\mu \geq 0.
\]  

(2.14)

This is a consequence of constraint (3) in Eqn. (2.12), see Ref. [28] for further details.
III. DOUBLED GEOMETRY FOR AN ASYMPTOTICALLY ADS BLACK HOLE

It has been recognized since as early as 1976 by W. Israel [33] that there is a relationship between the SK formalism and the Kruskal extension of a black hole spacetime. Herzog and Son [29] make use of the Kruskal extension of an aAdS black hole geometry in order to calculate real-time SK Green’s functions using holography. In this section, an outline of this construction is provided with the necessary details in order to calculate the entropy density by a KMS variation in Section IV. We begin with the metric of a stack of non-extremal D3-branes,

$$\text{ds}^2 = H(r)^{-1/2}[-f(r)dt^2 + dx^2] + H(r)^{1/2}[f(r)^{-1}dr^2 + r^2d\Omega_5^2],$$

(3.1)

where $$H(r) = 1 + R^4/r^4$$, $$f(r) = 1 - r_0^4/r^4$$, and $$d\Omega_5^2$$ is the metric on the five-sphere. In the near-horizon limit $$r \ll R$$, the metric reduces to that of an aAdS Schwarzschild black hole (cross $$S^5$$)

$$\text{ds}^2 = \left(\frac{\pi TR}{u}\right)^2 (-f(u)dt^2 + dx^2) + \frac{R^2}{4u^2f(u)}du^2 + R^2d\Omega_5^2$$

(3.2)

where the Hawking Temperature is $$T = r_0/\pi R^2$$ and we have introduced the coordinate $$u = r_0^2/r^2$$. This is the geometry we will study in the remainder of the paper. The holographic dual to this spacetime is $$\mathcal{N} = 4$$ SU($$N$$) supersymmetric Yang-Mills theory at finite temperature $$T$$ in the large $$N$$, strong-coupling regime. There is a black hole horizon at $$r = r_0$$ and the AdS boundary is at $$r \to \infty$$. Near to the horizon at $$r = r_0$$, the metric reduces to that of a Schwarzschild black hole in Kruskal coordinates

$$\text{ds}^2 = (\pi TR)^2 \left[-\left(1 - \frac{2M}{\rho}\right)dt^2 + \left(1 - \frac{2M}{\rho}\right)^{-1}d\rho^2 + \left(\frac{r_0\rho}{2M}\right)^2d\Omega_3^2\right] + R^2d\Omega_5^2$$

(3.3)

where $$\rho = 2Mr^2/r_0^2$$. In the asymptotic limit $$r \to \infty$$, the metric is that of AdS$_5$ in Poincaré coordinates,

$$\text{ds}^2 = -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}dr^2 + \frac{r^2}{R^2}dx^2 + R^2d\Omega_5^2$$

(3.4)

In the AdS/CFT correspondence, the partition function for the boundary field theory is obtained from the on-shell action $$S_{\text{bulk}}$$ of the bulk gravitational theory evaluated at the boundary $$r = r_B$$. The regularized boundary coordinate $$r_B$$ will be taken to infinity in the end. For a local scalar operator $$\mathcal{O}(x)$$ in the boundary theory, there is a dual scalar
field $\phi(x,r)$ in the gravitational theory where the coordinate $r$ parameterizes the additional dimension in the bulk. The boundary partition function is given by

$$Z[\phi(x)] = e^{iS_{bulk}[\phi(x,r)]}|_{r=r_B}. \quad (3.5)$$

In order to obtain the on-shell bulk action, we need to solve the equations of motion for a scalar field subject to the appropriate boundary conditions. The equation of motion for a scalar field $\phi(x,r)$ is

$$0 = \frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\phi) - m^2 \phi. \quad (3.6)$$

In order to be able to calculate Green’s functions at all orders, one needs to use a coordinate system that covers the full Penrose diagram [29]. One choice is the transformation

$$U = -4Me^{-(t-r_*)/4M}, \quad V = 4Me^{(t+r_*)/4M}, \quad r_* = \rho + 2M \ln |(\rho/2M) - 1| \quad (3.7)$$

which, near to the horizon, is the same as Kruskal coordinates for a Schwarzschild black hole. In these coordinates, the Penrose diagram has four quadrants when we allow $U$ and $V$ to run negative, see Fig. 3. We set boundary conditions so that $\phi$ is equal to $\phi_1$ on the asymptotic boundary in the R quadrant and $\phi_2$ on the asymptotic boundary in the L quadrant. It is

![Penrose Diagram](image)

FIG. 3: The full Penrose diagram for aAdS-Schwarzschild spacetime has four quadrants: left (L) and right (R) quadrants, and future (F) and past (P) quadrants which have singularities [29].
in this way that the doubled SK structure of the boundary field theory is mapped into the doubled aAdS black hole geometry. We also impose the boundary condition that, at the horizon in the R quadrant, positive frequency modes should be purely ingoing and negative frequency modes should be purely outgoing [29].

Subject to these boundary conditions, we can write the solution for \( \phi \) in terms of a set of functions \( f_k(r) \). The \( f_k(r) \) are defined so that \( \exp(ik \cdot x)f_k(r) \) is a solution to the scalar wave equation

\[
0 = 4u^3\partial_u\left(\frac{f(u)}{u}\partial_u\phi(k, u)\right) + \frac{u}{(\pi T)^2 f(u)}(\omega^2 - f(u)|k|^2)\phi(k, u) - m^2R^2\phi(k, u),
\]

(3.8)

and to behave like \( e^{ikr} \) near the horizon. This equation cannot be solved analytically, but the asymptotic behavior of its solutions can be analyzed. This is the same scalar wave equation as Eqn. (3.6) evaluated explicitly in the metric components of Eqn. (3.2) and written in terms of the partial Fourier transform

\[
\phi(k, u) = \int d^4xe^{-ik\cdot x}\phi(x, u).
\]

(3.9)

Then the solutions for \( \phi(k, r) \) in the L and R quadrants take the form

\[
\begin{align*}
\phi(k, r)|_R &= ((n + 1)f^*_k(r_R) - nf_k(r_R))\phi_1(k) + \sqrt{n(n + 1)}(f_k(r_R) - f^*_k(r_R))\phi_2(k), \\
\phi(k, r)|_L &= \sqrt{n(n + 1)}(f^*_k(r_L) - f_k(r_L))\phi_1(k) + ((n + 1)f_k(r_L) - nf^*_k(r_L))\phi_2(k),
\end{align*}
\]

(3.10)

(3.11)

where \( n = (e^{\beta \omega} - 1)^{-1} \) and \( r_R \) and \( r_L \) correspond to the \( r \) coordinate in the R and L quadrants, respectively. Son and Starinets [34] conjecture that the retarded and advanced Green’s functions are related to the functions \( f_k(r) \) in the following manner

\[
G_R(k) = -K[\sqrt{-g}g^{rr}f_k(r)\partial_rf^*_k(r)]_{r_B}, \quad G_A(k) = -K[\sqrt{-g}g^{rr}f^*_k(r)\partial_r f_k(r)]_{r_B},
\]

(3.12)

where \( K \) is a normalization constant. Finally, the on-shell boundary action \( S[\phi_1, \phi_2] \equiv S_{bulk}[\phi(x, r)]|_{r_B} \) is written in terms of the retarded and advanced Green’s functions:

\[
S[\phi_1, \phi_2] = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4}\left[ ((1 + n)G_R(k) - nG_A(k))\phi_1(-k)\phi_1(k) - ((1 + n)G_A(k) - nG_R(k))\phi_2(-k)\phi_2(k) \\
+ \sqrt{n(1 + n)}(G_A(k) - G_R(k))(\phi_1(-k)\phi_2(k) + \phi_2(-k)\phi_1(k)) \right].
\]

(3.13)

As will be shown in the next section, this action is invariant under a KMS transformation in the fields \( \phi_1 \) and \( \phi_2 \). The quadratic Green’s functions derived from this on-shell action satisfy the fluctuation-dissipation theorem as a consequence of KMS symmetry [29].
IV. KMS VARIATION OF THE BOUNDARY ACTION

In this section, I apply the SK Noether-like procedure to the boundary action Eqn. (3.13) by performing a KMS transformation. In particular, I consider here the equilibrium case of the static, aAdS black hole. We expect the KMS variation of the Lagrangian to be the time derivative of the entropy density. There will be no contribution from a spatial entropy current due to translation invariance in equilibrium. The on-shell boundary action can be written in terms of a Lagrangian $L(x)$ or its Fourier transform $\tilde{L}(k)$,

$$S[\phi_1, \phi_2] = \int d^4x L(x) = \int d^4k \delta^4(k) \tilde{L}(k).$$  \hspace{1cm} (4.1)

The variation of $L(x)$ is related to the variation of $\tilde{L}(k)$ by

$$S[\phi_1, \phi_2] \rightarrow \int d^4x [L(x) + \partial_0 s(x)] = \int d^4k \delta^4(k) [\tilde{L}(k) - i\omega \tilde{s}(k)].$$  \hspace{1cm} (4.2)

The function $s(x)$ is the entropy density of the boundary theory and $\tilde{s}(k)$ is its Fourier transform. Since the boundary theory and gravitational theory are holographic duals, they have the same information content and $s(x)$ is also the entropy density of the black hole. When $s(x) = s$ is a constant—as in the static, equilibrium case—$\tilde{s}(k)$ is proportional to a delta function

$$\tilde{s}(k) = (2\pi)^4 s \delta^4(k).$$  \hspace{1cm} (4.3)

I demonstrate in this section that the KMS variation of the on-shell boundary action Eqn. (3.13) results in an entropy density that takes this form.

The boundary conditions chosen in Section III select $\theta = \beta/2$ in the form of the KMS transformations in Eqn. (2.10) [29]. It is possible in general to choose an arbitrary $\theta \in [0, \beta]$ by an alternate construction [27, 35]. As a Lorentz invariant, local quantum field theory with a Hermitian Hamiltonian, the theory is invariant under the time-reversal transformation $\Theta = CPT$ by the CPT theorem [36]. We will use this choice of $\Theta$ in the form of the KMS transformations. The KMS transformations of the Fourier transformed fields $\phi_1(k)$ and $\phi_2(k)$ with $\theta = \beta/2$ are then

$$\phi_1(k) \rightarrow e^{\beta \omega/2} \phi_1(-k), \quad \phi_2(k) \rightarrow e^{-\beta \omega/2} \phi_2(-k).$$  \hspace{1cm} (4.4)

The terms proportional to $\phi_1(-k)\phi_1(k)$ and $\phi_2(-k)\phi_2(k)$ in the boundary action are independently KMS-invariant. The factors involving the retarded and advanced Green’s functions...
are unaffected by the KMS transformation. The remaining terms transform as

\[ \phi_1(-k)\phi_2(k) + \phi_2(-k)\phi_1(k) \mapsto e^{-\beta\omega}\phi_1(k)\phi_2(-k) + e^{\beta\omega}\phi_2(k)\phi_1(-k) \quad (4.5) \]

Overall, the integrand in Eqn. (3.13) transforms into itself plus an additional term

\[ \delta^4(k)\tilde{L}(k) \mapsto \delta^4(k)\tilde{L}(k) - \frac{1}{2}\sqrt{n(1+n)(G_A(k) - G_R(k))}((e^{-\beta\omega} - 1)\phi_1(k)\phi_2(-k) + (e^{\beta\omega} - 1)\phi_2(k)\phi_1(-k)). \]

(4.6)

To see that this takes the form of Eqn. (4.2), we take the low frequency limit \( \beta\omega \ll 1 \), valid in equilibrium. In this limit, the leading order contribution to Eqn. (4.6) is

\[ \delta^4(k)[L(k) - i\omega\tilde{s}(k)] \]

(4.7)

where

\[ \delta^4(k)\tilde{s}(k) = \lim_{\omega \to 0} \frac{1}{2\omega i}(G_A(k) - G_R(k))(-\phi_1(k)\phi_2(-k) + \phi_2(k)\phi_1(-k)) \]

(4.8)

In this final expression we have substituted the low frequency limit of the \( \sqrt{n(1+n)} \) factor. Next, we plug in the solutions to the equations of motion for the fields \( \phi_1 \) and \( \phi_2 \). The equilibrium solution is \( \phi_1(x) = \phi_2(x) = \phi_0 \), where \( \phi_0 \) is a constant. The Fourier transform of this solution is a delta function, but straightforwardly substituting this into Eqn. (4.8) gives zero for the entropy density. Instead, we take the near-equilibrium solution

\[ \phi_1(k) = 8\pi^{5/2}\delta(\omega + \epsilon)\delta^3(k), \quad \phi_2(k) = 8\pi^{5/2}\delta(\omega - \epsilon)\delta^3(k) \]

(4.9)

where we take the constant \( \epsilon > 0 \) to zero in the equilibrium limit. This choice is motivated by the fact that the boundary conditions described in Section III are such that outgoing modes at the horizon in the R (L) quadrant have negative (positive) frequency. Any additional factors of \( \phi_0 \) or \( 2\pi \) have been absorbed into the normalization constant \( K \) in the form of the retarded and advanced Green’s functions in Eqn. (3.12). The remaining factors are chosen to match known results at the end of the calculation. With this choice of a solution we have

\[ \delta^4(k)\tilde{s}(k) = 4\pi(2\pi)^4[\delta^3(k)]^2 \lim_{\omega \to 0} \frac{1}{2\omega i}(G_A(k) - G_R(k))(-[\delta(\omega + \epsilon)]^2 + [\delta(\omega - \epsilon)]^2). \]

(4.10)
There is a discontinuity at $\omega = 0$ resulting from our choice of a near-equilibrium solution, forcing us to choose a direction from which to take the limit $\omega \to 0$. Taking the limit from the positive frequency side and using Eqn. (4.3) we identify

$$s = 4\pi \eta,$$

where

$$\eta = \lim_{\omega \to 0^+} \frac{1}{2\omega^4} (G_A(\omega, 0) - G_R(\omega, 0))$$

is the shear viscosity as expressed by the Kubo formula. The limit from the negative frequency side simply introduces an overall minus sign. There is an extra factor of $\delta^4(k)$ in Eqn. (4.10) that is effectively cancelled from both sides in arriving at the Kubo formula. This step is justified by considering the boundary theory to live in a finite volume so that the delta function can be approximated as a well-behaved, finite function and taking the infinite volume limit at the end of the calculation. Eqn. (4.11) is in agreement with the well-known result that the shear viscosity to entropy density ratio takes a universal, constant value $\eta/s = 1/(4\pi)$ (in natural units) in gauge theories with a holographic Einstein gravity dual [30].

The shear viscosity—and in turn the black hole entropy density—are known to be proportional to the black hole horizon area $A_H$ [31]. The entropy density Eqn. (4.11) calculated by the SK Noether-like procedure is then proportional to the Bekenstein-Hawking entropy $S_{BH} = A_H/4G$. To see this, note that the Kubo relation for a kinetic coefficient is more commonly expressed in terms of correlation functions of the corresponding current,

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt dx e^{i\omega t} \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle,$$

where $T_{xy}$ is the shear stress component of the stress-energy tensor and the spatial momentum is taken to vanish $k = 0$ in a theory with translation invariance. The correlation function is taken as an average over the equilibrium thermal ensemble [31]. A related quantity is the absorption cross section of a graviton polarized parallel to one of the D3 branes,

$$\sigma(\omega) = \frac{8\pi G}{\omega} \int dt dx e^{i\omega t} \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle,$$

where $G$ is the ten-dimensional gravitational constant. Gravitons polarized parallel to the brane are coupled to the stress-energy tensor of the brane, so in the dual gauge theory the
graviton absorption rate is related to the stress tensor correlator in Eqn. (4.13). The shear viscosity is proportional to the \( \omega \to 0 \) limit of the graviton absorption cross section

\[
\eta = \frac{1}{16\pi G}\sigma(0).
\]

(4.15)

The cross section \( \sigma(\omega) \) can be obtained in the gravitational theory by solving the scalar wave equation associated with the metric Eqn. (3.2) and calculating the probability of a graviton being absorbed by the brane. Policastro, Son, and Starinets [31] carry out this procedure and demonstrate that \( \sigma(0) \) is equal to the horizon area \( A_H \).

V. DISCUSSION

In summary, the SK Noether-like procedure applied to Herzog and Son’s doubled aAdS black hole geometry reproduces the Bekenstein-Hawking entropy, up to a constant. This result was obtained by a symmetry-based, field-theoretic procedure as follows. A system in local thermal equilibrium as described by the SK formalism has a discrete \( \mathbb{Z}_2 \) symmetry in its fields called KMS symmetry. The Noether-like current obtained from a KMS variation of the action is the entropy current density. An action for the SK doubled degrees of freedom is obtained for the static aAdS black hole by considering the doubled geometry given by its Kruskal extension. We then set boundary conditions corresponding to each branch of the SK contour in each copy of the geometry. The boundary action obtained in this way is KMS invariant. The density obtained by following the Noether-like procedure and performing a KMS variation is the Kubo formula for shear viscosity. In gauge theories with Einstein gravity duals, the shear viscosity to entropy density ratio takes a universal, constant ratio. Both the shear viscosity and the entropy density are proportional to the event horizon area so that the Bekenstein-Hawking entropy is reproduced.

The calculations of the previous section demonstrate that it is possible to determine the entropy density of a static aAdS black hole by the SK Noether-like procedure together with a particular choice of a near-equilibrium on-shell solution in Eqn. (4.9). This approach offers a new way to understand the origin of black hole entropy in the context of holography and the SK formalism. While this approach offers a self-consistent approach to calculating the black hole entropy density that agrees with the Bekenstein-Hawking entropy, it remains to show that the choice of a near-equilibrium solution in Section IV is justified physically.
Additionally, there are subtleties in the calculation involving identifying coefficients of delta functions and taking the limit towards the discontinuity at zero frequency. Applying this procedure to another black hole configuration, such as a Kerr-Newman black hole or a non-static evaporating black hole, would provide evidence that the self-consistent argument presented here generalizes.

The SK Noether-like procedure for entropy densities introduced in Section II applies whenever the system is in local thermodynamic equilibrium. This includes the case of the non-static, evaporating black hole. It should be possible, in principle, to derive the entropy of Hawking radiation as a function of time for the evaporating black hole using the SK Noether-like procedure. A derivation of the Page curve in this way would offer a direct calculation within AdS/CFT and Einstein gravity without relying on any form of modified gravity or simplified models. Such a derivation would also provide new insight into a microscopic physical interpretation of the evaporation process from the perspective of the exterior of the black hole.

To derive the Page curve within the present framework, a couple of generalizations are required. The aAdS black hole metric of Eqn. (3.2) should be abandoned in favor of a time-dependent counterpart to model the evaporating black hole. In this new metric, the construction of the doubled geometry in Section III will be altered. In particular, the retarded and advanced Green’s functions evaluated at the asymptotic boundary will take on a new, time-dependent form as opposed to Eqn. (3.12). Then the KMS variation of the boundary action will give the time derivative of the entropy density as a function of time. The resulting differential equation should have a solution that gives the quantitative form of the Page curve.

It is also of interest to study entanglement entropies in this framework in order to provide new tools to better understand the Ryu-Takayanagi conjecture and the island prescription. The entanglement entropy of a subsystem \( A \) is determined by its reduced density matrix \( \rho_A \), defined as the partial trace \( \text{tr}_{A^c}[\rho] \) of the density matrix for the whole system over the complementary degrees of freedom \( A^c \). One approach is then to obtain the effective boundary action associated with the reduced density matrix. The KMS variation of this effective action should give the entanglement entropy by the Noether-like procedure. Another possible approach is to follow the replica trick for calculating entanglement entropies [37].
The replica trick involves calculating the $\alpha$-th Rényi entropy

$$S_\alpha = \frac{1}{1 - \alpha} \log \text{tr}[(\rho_A^\alpha)],$$

(5.1)

where $\rho_A$ is the reduced density matrix for a subsystem $A$. The limit $\alpha \to 1$ gives the entanglement entropy of the subsystem $A$. Calculating the $\alpha$-th Rényi entropy involves constructing the replica geometry by gluing together $\alpha$ copies of the original geometry along cuts at $A$ in each copy [37], see Fig. 4. It is possible the SK boundary action analogous to Eqn. (3.13) could be obtained for the replica geometry and used to calculate the Rényi entropies by the SK Noether-like procedure.

![Replica geometry](image)

FIG. 4: The $\alpha$-th Rényi entropy of a region $A$ is typically calculated by a path integral over the $\alpha$-sheeted replica geometry. Cuts are made along a spacelike hypersurface $\Sigma_t$ at $A$ and $\alpha$ copies of the geometry are glued together to construct the replica geometry [37].

Another potential benefit of the framework presented in this paper would be an analysis taking place purely within the gravitational theory and not referring to holography. KMS symmetry has a non-perturbative dual in the gravitational theory. This was confirmed half a century ago by Gibbons and Perry [38], formalized in terms of constraints between Green’s functions. It would be illuminating to identify the parallel procedure to a KMS variation and the SK Noether-like procedure in the bulk gravitational theory.

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References


