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Examining Factors Using Standard Subspaces and Antiunitary Representations

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Abstract

In an effort to provide an axiomization of quantum mechanics, John von Neumann and Francis Joseph Murray developed many tools in the theory of operator algebras. One of the many objects developed during the course of their work was the von Neumann algebra, originally called a ring of operators. The purpose of this thesis is to give an overview of the classification of elementary objects, called factors, and explore connections with other mathematical objects, namely standard subspaces in Hilbert spaces and antiunitary representations. The main results presented here illustrate instances of these interconnections that are relevant in Algebraic Quantum Field Theory and point to potential further connections, currently unexplored.

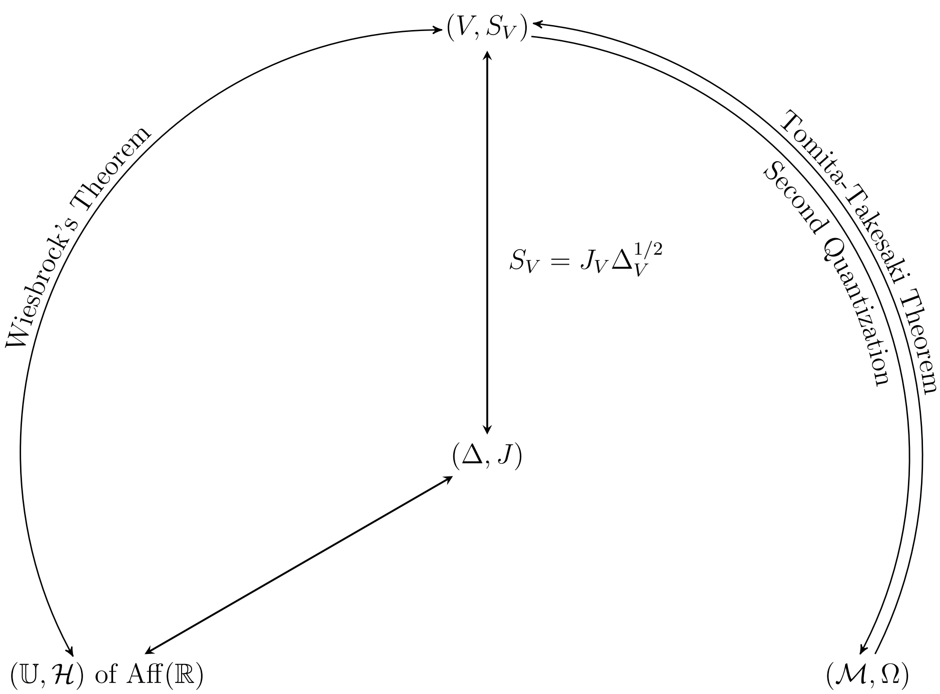

Figure 1: Interconnections between antiunitary representations $(\mathbb{U}, \mathcal{H})$ of $\mathrm{Aff}(\mathbb{R})$, modular objects (Δ, J), standard subspaces V, and von Neumann algebras and cyclic and separating vectors $(\mathcal{M}, \Omega)$

Contents

1. Introduction

Von Neumann algebras are the primary objects of interest for this thesis. These came as a development of John von Neumann and Francis Joseph Murray during their work to axiomatize quantum mechanics. Von Neumann algebras themselves were defined in 1936 in light of von Neumann's Bicommutant theorem (Thm. 3.1.11), which demonstrated the equivalence of an algebraic and topological property. The most elementary type of these algebras is the factor (Def. 3.1.14), characterized by having a trivial center.

Much as with many elementary objects, a classification theory developed for factors. Von Neumann and Murray completed the division of factors into Type I, II, and III factors in the course of four papers between 1936 and 1943 ([MV36; MN37; Neu40; MN43]), though their knowledge of Type III factors was limited. However, after the development of Tomita-Takesaki theory, the Type III factors were given a classification by Alain Connes in 1973 ([Con73]).

In addition to giving a classification of Type III factors, Tomita-Takesaki theory allowed for results on von Neumann algebras to be converted into results on the second object of interest, real standard subspaces, by considering the closed orbit of self-adjoint operators. Real standard subspaces, also called standard subspaces, form generalizations of the real and imaginary line in the complex plane, and are characterized by being closed real subspaces V of a Hilbert space $\mathcal{H}$ such that $V + iV$ is dense in $\mathcal{H}$ and $V \cap iV = \{0\}$. Similar to how standard subspaces generalize the real and imaginary line, the action of complex conjugation has a generalization in the form of the operator $S : h + ik \to h - ik$ for $h, k \in V$. By taking the polar decomposition of $S = J\Delta^{1/2}$, one obtains the third object of interest, modular objects (Δ, J). These modular objects provide a method for converting standard subspace results into representation results, since standard subspaces are in correspondence with antiunitary representations of the multiplicative group of the non-zero real numbers $\mathbb{R}^\times$ (Lem. 4.3.1). Here, powers of the form Δ^t for $t \in \mathbb{R}$ serve as a generalization of the positive real numbers while J acts as an analogue of -1. By combining the connection between standard subspaces and antiunitary representations of $\mathbb{R}^\times$ with Tomita-Takesaki theory, one can express results on von Neumann algebras using either standard subspaces or representation theory.

More connections can be drawn if one not only considers standard subspaces, but inclusions of standard subspaces. This is motivated from algebraic quantum field theory, where subset inclusions $\mathcal{O}_1 \subset \mathcal{O}_2$ of Minkowski spacetime $\mathbb{R}^{1,3}$ generates inclusions of von Neumann algebras $\mathcal{M}(\mathcal{O}_1) \subset \mathcal{M}(\mathcal{O}_2)$ associated to each region. Using Tomita-Takesaki theory, this results in inclusions of standard subspaces $V_1 \subset V_2$.

These inclusions can be extended from two standard subspaces to continuous families $(V_t)_{t\in\mathbb{R}}$, with $V_s \subset V_t$ if $s \leq t$. Because $\mathbb{R}^\times$ only provides a correspondence to a single standard subspace, providing representation theoretic results on $(V_t)_{t\in\mathbb{R}}$ requires not only considering $\mathbb{R}^\times$, but also a new parameter that ranges over the reals. This motivates the consideration of the affine group of the real numbers $\mathbb{R} \rtimes \mathbb{R}^\times$, as introducing $\mathbb{R}$ provides the needed continuous parameter.

For this paper, a particular type of inclusion, the half-sided modular inclusion, is of special interest. It was initially a notion defined on von Neumann algebras (Def. 5.4.1), but was later given a standard subspace analogue (Def. 4.4.10). Results on half-sided modular inclusions of von Neumann algebras were also extended to results on half-sided modular inclusions of standard subspaces. This includes Wiesbrock's theorem (Thm. 5.4.3, Thm. 4.5.5). However, while the von Neumann version of this theorem provides a sufficient condition for one von Neumann algebra in the theorem to be a Type III factor, the standard subspace version provides no analogue of this condition. This paper hopes to clarify some of the details of this discrepancy.

One reason for this interest in this discrepancy follows from one class of von Neumann algebras, the second quantization algebras. The physical motivation for these algebras comes from efforts to model the evolution of a system where the number of particles varies over time, which a standard treatment of quantum mechanics is incapable of doing. As for their significance to this paper, they possess a significant number of duality results with standard subspaces (Thm. 5.3.6). In particular, these duality results allow for the formulation of a condition for a second quantization algebra to be a Type III factor, expressed in terms of their associated modular objects (Thm. 5.3.10).

The thesis is outlined as follows. Chapter two provides the needed background material, including the theory of Hilbert spaces and operators, and a quick guide to standard subspaces. Chapter three gives an introduction to von Neumann algebras, including definitions and their classification, along with an introduction to Tomita-Takesaki theory, which provides a method for translating von Neumann algebras into standard subspaces. Chapter four gives a presentation on the theory of antiunitary representations of graded groups and connects it back to standard subspaces via modular objects. The chapter concludes with the presentation of half-sided modular inclusions, culminating with the presentation of Wiesbrock's theorem. Finally, chapter five is dedicated to providing constructions of von Neumann algebras and the von Neumann equivalent of the half-sided modular inclusion. The latter of these culminates with the von Neumann version of Wiesbrock's theorem, which, as mentioned previously, provides information on factors that the standard subspace version does not.

2. Background

Before beginning, concepts from the theory of Hilbert spaces and bounded operators are given, followed by information on specific types of groups and subspaces of Hilbert spaces. Concerning conventions, only the fields of real and complex numbers, denoted $\mathbb{R}$ and $\mathbb{C}$ respectively, will be considered. All algebras will be assumed to be unital.

2.1 Hilbert Spaces

To begin, information on the algebraic and topological structure of Hilbert spaces, as well as methods of generating larger Hilbert spaces is provided.

Definition 2.1.1. An *inner product space* is a vector space $\mathcal{V}$ equipped with a map, called an *inner product* $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$, which satisfies, for $h, k_1, k_2 \in \mathcal{V}$ and $z, w \in \mathbb{C}$:

(i) $\langle h, h \rangle > 0$ if and only if $h \neq 0$ (definiteness)

(ii) $\langle h, zk_1 + wk_2 \rangle = z\langle h, k_1 \rangle + w\langle h, k_2 \rangle$ (linearity in the second component)

(iii) $\langle h, k \rangle = \overline{\langle k, h \rangle}$ (conjugate symmetry)

Note that the conditions for the inner product imply conjugate linearity in the first component; that is, $\langle zh_1 + wh_2, k \rangle = \overline{z}\langle h_1, k \rangle + \overline{w}\langle h_2, k \rangle$.

Definition 2.1.2. A *norm* is a function $|| \cdot || : \mathcal{V} \to [0, \infty)$ satisfying the following properties:

(i) $||h|| = 0$ if and only if $h = 0 \in \mathcal{V}$ (definiteness)

(ii) $||zh|| = |z|||h|| \ \forall z \in \mathbb{C}$ (homogeneity)

(iii) $||h + k|| \leq ||h|| + ||k||$ (triangle inequality)

A vector space $\mathcal{V}$ equipped with a norm is called a *normed space*:

Be aware that an inner product space $\mathcal{V}$ is a normed space, with $||h|| = \sqrt{\langle h, h \rangle}$ for $h \in \mathcal{V}$. Next, topological aspects of these spaces need to be considered.

Definition 2.1.3. A *Banach space* $\mathcal{V}$ is a normed space that is complete in its metric; that is, all Cauchy sequences in $\mathcal{V}$ converge to elements in $\mathcal{V}$.

Definition 2.1.4. A *Hilbert space* $\mathcal{H}$ is an inner product space that is complete with respect to its metric.

There is a particular class of Hilbert spaces that will be of interest to us, and they are those satisfy a condition regarding the size.

Definition 2.1.5. Given a Hilbert space $\mathcal{H}$, a *Hilbert basis* consists of a family of vectors $\{h_i\}_{i\in I}$ such that:

$$\langle h_i, h_j\rangle = \begin{cases} 1 & i = j \quad \text{i.e. } h_i \text{ is normalized} \\ 0 & i \neq j \quad \text{i.e. } h_i \text{ and } h_j \text{ are orthogonal} \end{cases} \tag{2.1}$$

and $\overline{\mathrm{Span}(\{h_i\}_{i\in I})} = \mathcal{H}$.

Definition 2.1.6. A Hilbert space $\mathcal{H}$ is said to be *separable* if it has a countable Hilbert basis.

In essence, a Hilbert space being separable places a bound on the possible size of the space, similar to how dimension bounds the size of finite-dimensional vector spaces.

Forming larger spaces from smaller ones will occur regularly and, in particular, forming larger Hilbert spaces from smaller Hilbert spaces will be a consistent tool. As such, two methods for combining Hilbert spaces must be presented, those being the infinite direct sum of Hilbert spaces and the tensor product.

For the infinite direct direct sum, there's the following proposition:

Proposition 2.1.7 ([Con12], Prop. 5.6.2)**.** Given a sequence of separable Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$, consider the space:

$$\mathcal{H} := \left\{ (h_n)_{n=1}^{\infty} \,:\, h_n \in \mathcal{H}_n \text{ for all } n \geq 1 \text{ and } \sum_{n=1}^{\infty} ||h_n||^2 < \infty \right\} \tag{2.2}$$

When addition and scalar multiplication of sequences are defined componentwise, and when the following inner product is imposed:

$$\langle (h_n)_{n=1}^{\infty}, (k_n)_{n=1}^{\infty}\rangle := \sum_{n=1}^{\infty} \langle h_n, k_n\rangle$$

$$\begin{tikzcd} \mathcal{V}_1 \times \mathcal{V}_2 \arrow[r, "\odot"] \arrow[dr, "\xi"'] & \mathcal{V}_1 \odot \mathcal{V}_2 \arrow[d, "\zeta"] \\ & \mathcal{X} \end{tikzcd}$$

Figure 2.1: Universal property of algebraic tensor product

the space $\mathcal{H}$ forms a separable Hilbert space, called the direct sum of $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$, denoted:

$$\mathcal{H} := \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$$

Next, to take the tensor product of two Hilbert spaces, one starts from the following definition:

Definition 2.1.8. The *algebraic tensor product* of two vector spaces $\mathcal{V}_1, \mathcal{V}_2$ is the space $\mathcal{V}_1 \odot \mathcal{V}_2$, with a bilinear map $\odot : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathcal{V}_1 \odot \mathcal{V}_2$ satisfying the following universal property: Given an arbitrary vector space $\mathcal{X}$ and any bilinear map $\xi : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathcal{X}$, there exists a unique linear map ζ such that the diagram in Figure 2.1 commutes:

Sections 1.6 and 1.7 of [Gre78] have the details regarding the existence and uniqueness of $\odot$. Now, one performs the algebraic tensor product of the two Hilbert spaces, giving a new vector space $\mathcal{H}_1 \odot \mathcal{H}_2$. To recover the structure of a Hilbert space, the following inner product is defined on $\mathcal{H}_1 \odot \mathcal{H}_2$:

Proposition 2.1.9. Given the algebraic tensor product of two Hilbert spaces $\mathcal{H}_1 \odot \mathcal{H}_2$, the pairing given by

$$\langle h_1 \odot k_1, h_2 \odot k_2 \rangle := \langle h_1, h_2 \rangle_{\mathcal{H}_1} \langle k_1, k_2 \rangle_{\mathcal{H}_2} \tag{2.3}$$

with $h_1, h_2 \in \mathcal{H}_1$ and $k_1, k_2 \in \mathcal{H}_2$, when extended linearly, is an inner product on $\mathcal{H}_1 \odot \mathcal{H}_2$.

Proof. To prove definiteness, consider:

$$\langle h_1 \odot k_1, h_1 \odot k_1 \rangle = \langle h_1, h_1 \rangle_{\mathcal{H}_1} \langle k_1, k_1 \rangle_{\mathcal{H}_2}$$

This is 0 if and only if either $h_1 = 0_{\mathcal{H}_1}$ or $k_1 = 0_{\mathcal{H}_2}$, which would imply $h_1 \odot k_1 = 0$.

Linearity in the second component is imposed by how the function was linearly extended.

To prove conjugate symmetry, the following holds:

$$\begin{aligned}\overline{\langle h_1 \odot k_1, h_2 \odot k_2\rangle} &= \overline{\langle h_1, h_2\rangle_{\mathcal{H}_1}\langle k_1, k_2\rangle_{\mathcal{H}_2}} \\ &= \overline{\langle h_1, h_2\rangle}_{\mathcal{H}_1}\ \overline{\langle k_1, k_2\rangle}_{\mathcal{H}_2} \\ &= \langle h_2, h_1\rangle_{\mathcal{H}_1}\langle k_2, k_1\rangle_{\mathcal{H}_2} \\ &= \langle h_2 \odot k_2, h_1 \odot k_1\rangle\end{aligned}$$

The function is initially defined only on elements of the form $h_1 \odot k_1$. It is then defined to be linear by:

$$\left\langle h' \odot k', \sum_j z_j(h_j \odot k_j)\right\rangle := \sum_j \langle h' \odot k', z_j(h_j \odot k_j)\rangle \tag{2.4}$$

□

With this, $\mathcal{H}_1 \odot \mathcal{H}_2$ forms an inner product space. Finally, the space needs to complete. The technicalities are not covered, but the completion is done in the same way any metric space is completed, with details being given in Theorem 43.7 of [Mun00]. At last, the tensor product of two Hilbert spaces is given by:

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \overline{\mathcal{H}_1 \odot \mathcal{H}_2}$$

This procedure can be repeated for a finite number of Hilbert spaces, giving a tensor product $\otimes_{i=1}^n \mathcal{H}_i$. When every Hilbert space in the product is the same, we shall instead denote it $\mathcal{H}^{\otimes n}$.

The next space of interest builds directly on both the direct sum and the tensor product.

Definition 2.1.10. Given a Hilbert space $\mathcal{H}$, the *Fock space* $\exp(\mathcal{H})$ is a Hilbert space defined by:

$$\exp(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n} = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus \cdots \tag{2.5}$$

This space contains more elements than will be needed. To restrict the space, the symmetry properties of the tensor product will be considered.

Given $\mathcal{H} \otimes \mathcal{H}$ for some Hilbert space $\mathcal{H}$, as well as $h, k \in \mathcal{H}$, the tensor product is not generally commutative; that is:

$$h \otimes k \neq k \otimes h$$

in general. However, having such a symmetry in the space would provide some simplifications to calculations. To remedy this issue, a new product is introduced, the symmetric tensor product $\vee$, with:

$$h \vee k := \frac{1}{\sqrt{2}}(h \otimes k + k \otimes h)$$

This can be generalized to tensor product of n vectors:

Definition 2.1.11. Given a Hilbert space $\mathcal{H}$, as well as $h_1, \dots, h_n \in \mathcal{H}$, the *symmetric tensor product* of $h_1, \dots, h_n$ is given by:

$$h_1 \vee \cdots \vee h_n := \frac{1}{\sqrt{n!}} \sum_{\Upsilon \in S_n} h_{\Upsilon(1)} \otimes \cdots \otimes h_{\Upsilon(n)} \tag{2.6}$$

where S_n is the permutation group of n elements.

Elements of this form are precisely those of interest, so the tensor product of a Hilbert space $\mathcal{H}$ is restricted as follows.

Definition 2.1.12. Consider the tensor product $\mathcal{H} \otimes \mathcal{H}$ of a Hilbert space $\mathcal{H}$. Then the *symmetric tensor product* $\mathcal{H} \vee \mathcal{H}$ is defined to be:

$$\mathcal{H} \vee \mathcal{H} := \overline{\left\{ \sum_{m=1}^{n} h_m \vee k_m \ : \ h_m, k_m \in \mathcal{H} \right\}} \subset \mathcal{H} \otimes \mathcal{H} \tag{2.7}$$

This definition generalizes readily to n symmetric tensor products, and the notation $\mathcal{H}^{\vee n}$ is adopted, similar to the notation $\mathcal{H}^{\otimes n}$. At last, the symmetric elements of the Fock space form the following subspace.

Definition 2.1.13. The *bosonic Fock space* $\exp_s(\mathcal{H})$ is:

$$\exp_s(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\vee n} = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \vee \mathcal{H}) \oplus \cdots \subset \exp(\mathcal{H}) \tag{2.8}$$

This subspace will form the Hilbert space at the cornerstone of many future duality results. This wraps up much of the algebraic material that will be needed. Next, operators and their associated topologies will be considered.

2.2 Operators: Algebra and Topology

To start, continuity conditions of operators between normed spaces are given.

Proposition 2.2.1 ([Con90], Prop 2.1; [Con12], Prop. 5.1.3)**.** Given two normed spaces $\mathcal{V}, \mathcal{X}$ and a linear transformation $T : \mathcal{V} \to \mathcal{X}$, the following are equivalent:

(i) T is bounded on a closed set

(ii) T is continuous

(iii) T is continuous at $0 \in \mathcal{V}$

(iv) T is Lipschitz; that is, for all $h \in \mathcal{V}$, we have $||Tv||_{\mathcal{X}} \leq t||h||_{\mathcal{V}}$ for some $t \in \mathbb{R}^{>0}$.

Remark 2.2.2. Continuous operators will be called bounded operators, but be aware that they do not map their domain to a bounded set. The only operator that does is the 0 operator, which maps every element of the domain to the 0 vector.

A new norm is now introduced, one that acts on operators instead of vectors.

Definition 2.2.3. The *operator norm* $|||\cdot|||$ of an operator $T : \mathcal{V} \to \mathcal{X}$, between normed spaces, is defined by the vector norms of $\mathcal{V}, \mathcal{X}$ in the following equivalent manners:

(i) $|||T||| = \sup\{||Th||_{\mathcal{X}} : h \in \mathcal{V} \text{ and } ||h||_{\mathcal{V}} \leq 1\}$

(ii) $|||T||| = \sup\left\{\frac{||Th||_{\mathcal{X}}}{||h||_{\mathcal{V}}} : h \in \mathcal{V}\backslash\{0\}\right\}$

(iii) $|||T||| = \sup\{||Th||_{\mathcal{X}} : h \in \mathcal{V} \text{ and } ||h||_{\mathcal{V}} = 1\}$

It should be noted that the equivalence of the definitions of the operator norm arises from the equivalent definitions of continuity for operators. It can be shown that $|||\cdot|||$ is a norm, called the *operator norm.* This norm needs to be applied to a space of operators.

Definition 2.2.4. The space of bounded linear operators over a Hilbert space $\mathcal{H}$ is denoted $\mathcal{B}(\mathcal{H})$. When equipped with the operator norm, $\mathcal{B}(\mathcal{H})$ forms a normed space. The metric topology induced by the operator norm is called the *uniform topology*.

This topology is by far the strongest/finest topology used. However, it will prove to be excessively strong at times. As such, new topologies are needed. In fact, the kind of topologies needed are optimized in a sense, in that they are the weakest topologies such that certain continuity conditions are satisfied. The precise construction can be found in Chapter 3 of [Bre11].

Definition 2.2.5. Given $\mathcal{B}(\mathcal{H})$, the *strong topology* on $\mathcal{B}(\mathcal{H})$ is the weakest topology such that maps of the form

$$T \mapsto Th \tag{2.9}$$

for $T \in \mathcal{B}(\mathcal{H})$ and every $h \in \mathcal{H}$ are continuous.

The *weak topology* is the weakest topology such that functionals of the form

$$T \mapsto \langle h, Tk \rangle \tag{2.10}$$

for $T \in \mathcal{B}(\mathcal{H})$ and all $h, k \in \mathcal{H}$ are continuous.

One may think of these topologies is in terms of the convergence properties of sequences of operators. For the uniform topology, a sequence of operators (T_n) converges to an operator T if

$$\lim_{n\to\infty} |||T_n - T||| = 0 \tag{2.11}$$

As such, the uniform topology can be thought of as the topology of uniform convergence.

For the strong topology, $T_n \to T$ if

$$\lim_{n\to\infty} ||T_n - Th|| = 0 \tag{2.12}$$

for all $h \in \mathcal{H}$. Thus, the strong topology can be thought of as the topology of pointwise convergence.

Finally, in the weak topology, $T_n \to T$ if, for all $h, k \in \mathcal{H}$, then

$$\lim_{n\to\infty} \langle h, T_n k \rangle = \langle h, Tk \rangle \tag{2.13}$$

The weak topology is weaker than the strong topology, which is itself weaker than the uniform topology. In particular, given a subset $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$, with weak closure $\overline{\mathcal{S}}^w$, and strong closure $\overline{\mathcal{S}}^s$, the following inclusion holds:

$$\overline{\mathcal{S}}^s \subset \overline{\mathcal{S}}^w$$

Although the next result is not fundamental to proceeding further, it provides a further characterization of the closure of strong and weak topologies.

Proposition 2.2.6. Given a convex subset $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$, and denoting the strong and weak closures of $\mathcal{S}$ as $\overline{\mathcal{S}}^w$ and $\overline{\mathcal{S}}^s$ respectively:

$$\overline{\mathcal{S}}^w = \overline{\mathcal{S}}^s$$

Because only vector spaces, which are convex sets, are used in this paper, only the closure in the uniform topology, denoted $\overline{\mathcal{S}}^u$ and the weak topology, again denoted $\overline{\mathcal{S}}^w$, will be considered.

There is one more technical matter that needs to be addressed regarding operators. Often times, it is too much to ask that an operator between normed spaces $T : \mathcal{V} \to \mathcal{X}$ be defined on the whole space $\mathcal{V}$. The best one can ask for is typically that T be defined on a dense subset of $\mathcal{V}$. This brings us to the notion of densely-defined operators.

Definition 2.2.7. An operator T between normed spaces is said to be *densely-defined* if domain, denoted $\mathcal{D}(T)$, is a dense subset of $\mathcal{V}$.

Example 2.2.8. One of the prototypical examples of a densely defined operators is the derivative operator. More precisely, consider the space $C([0,1])$ of continuous functions over the interval $[0,1]$, equipped with the supremum norm:

$$||f||_\infty := \sup_{s\in[0,1]} |f(s)|$$

and the subspace $C^1([0,1])$ of continuously differentiable functions. Because of the Stone-Weierstrass theorem, $C^1([0,1])$ is dense in $C([0,1])$. Then, consider the operator:

$$\begin{aligned} \mathrm{d} : C^1([0,1]) &\longrightarrow C([0,1]) \\ f &\longmapsto \frac{\mathrm{d}f}{\mathrm{d}x} \end{aligned}$$

This is a densely defined operator.

Another densely defined operator is given in example 2.3.4.

This section is concluded by defining some algebraic properties of operators that will constantly appear. The first is a simple twist on the notion of linearity.

Definition 2.2.9. Given a vector space $\mathcal{V}$ with $h, k \in \mathcal{V}$ and $z \in \mathbb{C}$, a map $T : \mathcal{V} \to \mathcal{V}$ is called *antilinear* if $T(h + zk) = T(h) + \overline{z}T(k)$.

The next property is ubiquitous, but its definition is provided for completeness.

Definition 2.2.10. A map U on an inner product space $\mathcal{V}$ is called an *isometry* if $\langle Uh, Uk\rangle = \langle h, k\rangle$ for all $h, k \in \mathcal{V}$.

2.3 One parameter groups

Many of the groups of interest in this paper will take a particular form. This section is dedicated to that form. To start, the complex exponential of an operator $H \in \mathcal{B}(\mathcal{H})$, denoted e^{itH}, is defined in terms of the power series:

$$e^{itH} := \mathbb{1} + itH - \frac{t^2H^2}{2!} - i\frac{t^3H^3}{3!} + \cdots \tag{2.14}$$

where $\mathbb{1}$ is the identity operator in $\mathcal{B}(\mathcal{H})$. Regarding covergence, for bounded operators, this series converges absolutely. Because, on Banach spaces, absolute convergence implies convergence, that means the complex exponential of bounded operators exists. With this definition, a group can be constructed as follows:

Definition 2.3.1. Given a self-adjoint operator $H \in \mathcal{B}(\mathcal{H})$, the family of operators:

$$\mathcal{U} := \left\{e^{itH} \,:\, t \in \mathbb{R}\right\} \tag{2.15}$$

is called the *unitary, one-parameter group*, with H called the *infinitesimal generator*.

It turns out that there is a correspondence between one-parameter, unitary groups, and the associated infinitesimal generators. This correspondence is the content of Stone's theorem on one-parameter unitary groups

Theorem 2.3.2 (Stone's theorem on one-parameter, unitary groups, [Sto32], Thm. B)**.** Given a unitary, one-parameter group $\mathcal{U}$, there exists an infinitesimal generator H for $\mathcal{U}$ such that any element of $\mathcal{U}$ can be represented as e^{itH} for some $t \in \mathbb{R}$.

Because of the relevance of this type of group, some examples are provided.

Example 2.3.3. Given a finite-dimensional vector space $\mathcal{V}$ and the set of linear transformations $\mathcal{B}(\mathcal{V})$ from $\mathcal{V}$ to itself, the special orthogonal group is defined as follows:

$$\mathrm{SO}(\mathcal{V}) = \{U \in \ \mathcal{B}(\mathcal{V}) \, : \, \det(U) = 1 \text{ and } UU^* = U^*U = \mathbb{1}\}$$

In particular, when $\mathcal{V} = \mathbb{R}^2$:

$$\mathrm{SO}\left(\mathbb{R}^2\right) = \left\{ \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}, \ \theta \in \mathbb{R} \right\} \tag{2.16}$$

This is a unitary, one-parameter group, with the infinitesimal generator being:

$$H = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

This can be seen by expanding out the Taylor series of $e^{i\theta H}$:

$$\begin{aligned} e^{i\theta H} &= \mathbb{1} + i\theta H - \frac{\theta^2 H^2}{2!} - i\frac{\theta^3 H^3}{3!} + \cdots \\ &= \begin{bmatrix} 1 - \frac{\theta^2}{2} + \dots & \theta - \frac{\theta^3}{3!} + \cdots \\ -\theta + \frac{\theta^3}{3!} - \cdots & 1 - \frac{\theta^2}{2} + \cdots \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \end{aligned}$$

Example 2.3.4. For a more complex example, we consider the translation operator $T(s_0)$ defined by

$$T(s_0)f(s) = f(s - s_0) \tag{2.17}$$

with $f(s) \in C_c^\infty(\mathbb{R})$, the space of compactly supported, smooth functions over $\mathbb{R}$, with the inner product:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(s)g(s)\mathrm{d}s \qquad \forall f, g \in C_c^\infty(\mathbb{R}) \tag{2.18}$$

The family of operators T parameterized by $s_0 \in \mathbb{R}$ forms a unitary, one-parameter group, with the operator $-i\frac{d}{ds}$ acting as the infinitesimal generator, which can be seen by Taylor expanding $f(s - s_0)$:

$$\begin{aligned} T(s_0)f(s) &= f(s - s_0) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(-s_0)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} f(s) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[is \left(-i\frac{\mathrm{d}}{\mathrm{d}s} \right) \right]^n f(s) \\ &= e^{is\left(-i\frac{\mathrm{d}}{\mathrm{d}s}\right)} \end{aligned}$$

Incidentally, the translation operator serves as another example of a densely-defined operator, as $C_c^\infty(\mathbb{R})$ forms a dense subspace of $L^2(\mathbb{R})$, the square integrable functions over the real line.

It should be noted that the definition given for one-parameter groups is not the most general possible, with a broader definition found in Definition 2.27 of [NÓ17]. However, that definition is unnecessarily complicated for our purposes, so we restrict ourselves to this simpler version.

One special kind of one-parameter group will be particularly relevant.

Definition 2.3.5. We shall call a one-parameter group $\mathcal{U}$ a *positive-energy group* if the generator H is positive semidefinite, denoted $H \geq 0$. If H is positive definite, denoted $H > 0$, then $\mathcal{U}$ is called a *strictly-positive energy group*.

There is one more set of objects that need to be introduced.

2.4 Introduction to Standard Subspaces and Modular Objects

Because of the fundamental nature of standard subspaces and modular objects to this paper, they are introduced here.

Definition 2.4.1. A *real subspace* of a complex vector space $\mathcal{V}$ is a subspace that is closed under scalar multiplication by elements of $\mathbb{R} \subset \mathbb{C}$.

In essence, what is desired with real subspaces is that one stop thinking of the vector space as a vector space over $\mathbb{C}$ but instead as a vector space over $\mathbb{R}$, much as how $\mathbb{C}$ can be thought of as a two-dimensional vector space over $\mathbb{R}$ via $\mathbb{C} \cong \mathbb{R}^2$, instead of as a one-dimensional vector space over $\mathbb{C}$. With this, we already have enough to define a standard subspace.

Definition 2.4.2. A *real standard subspace*, or just *standard subspace*, is a closed real subspace $V \subset \mathcal{H}$ such that both of the following conditions are satisfied:

- $V + iV$ is dense in $\mathcal{H}$ (cyclic)
- $V \cap iV = \{0\}$ (separating)

Here,

$$V + iV = \{h + ik \,:\, h, k \in V\}$$

The following is the prototypical example of a standard subspace.

Proposition 2.4.3. The line of the form

$$V = \{re^{i\theta} \in \mathbb{C} \,:\, r \in \mathbb{R} \text{ and fixed } \theta \in \mathbb{R}\}$$

is a standard subspace in $\mathbb{C}$. In particular, the real line $\mathbb{R}$ is standard in $\mathbb{C}$.

Proof. For $\mathbb{R}$, because $\mathbb{R} + i\mathbb{R} = \{s + ti \,|\, s, t \in \mathbb{R}\} = \mathbb{C}$, we automatically have that $\mathbb{R}$ is standard in $\mathbb{C}$. The case for $re^{i\theta}$ is essentially the same with more trigonometric functions. $\square$

To introduce modular objects, we next need to consider the following antilinear involution:

$$\begin{aligned} S_V \,:\, \mathcal{D}(S_V) := V + iV &\longrightarrow \mathcal{H} \\ x + iy &\longmapsto x - iy \end{aligned}$$

Notice that S acts as a generalization of complex conjugation $\mathbb{C}$. This operator is clearly densely defined over $\mathcal{H}$, since it is defined over $V + iV$, with V a standard subspace. Moreover, S_V is closed. By then taking the polar decomposition of S_V, one gets $S_V = J_V \Delta_V^{1/2}$, with the operators (Δ_V, J_V) satisfying the modular relation:

$$J_V \Delta_V J_V = \Delta_V^{-1} \tag{2.19}$$

Here, J_V is an antilinear isometry satisfying $J_V^2 = 1$, while $\Delta_V > 0$ is a self-adjoint operator. These properties motivate the following definition.

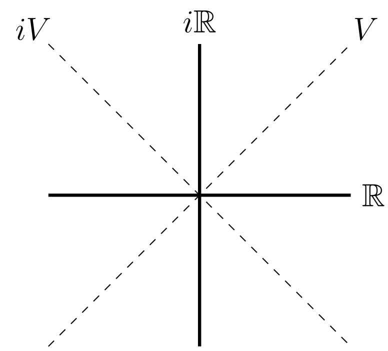

Figure 2.2: Example graph for arbitrary standard subspace in $\mathbb{C}$ given in Proposition 2.4.3.

Definition 2.4.4. Given a Hilbert space $\mathcal{H}$, a pair of objects (Δ, J) are called *modular objects* if J is an antilinear isometry satisfying $J^2 = 1$, while $\Delta > 0$ is a self-adjoint operator, with both satisfying the modular relation:

$$J\Delta J = \Delta^{-1} \tag{2.20}$$

With all this, we finally have enough background to begin discussing the theory of von Neumann algebras.

3. Von Neumann Algebras: Definitions and Classification

With the background material complete, we can discuss von Neumann algebras. Basic definitions and an overview of the classification of factors is given, including Tomita-Takesaki theory, which will prove to be fundamental to mapping between von Neumann algebras and standard subspaces.

3.1 Basic Definitions

Definition 3.1.1. A complex algebra $\mathcal{A}$ is called a *C^*-algebra* if it is a Banach space such that the norm satisfies the following conditions for all $a, b \in \mathcal{A}$:

- $||ab|| \leq ||a||||b||$ (submultiplicativity)
- $||a^*a|| = ||a||^2$ (C^* identity)

This definition is rather abstract, so one might wish to instead work over more familiar spaces. For that, consider, for a Hilbert space $\mathcal{H}$, the bounded operators $\mathcal{B}(\mathcal{H})$, equipped with the involution:

$$\begin{aligned} {}^* \,:\, \mathcal{B}(\mathcal{H}) &\longrightarrow \mathcal{B}(\mathcal{H}) \\ T &\longmapsto T^* \end{aligned}$$

where T^* is the adjoint of the operator T. We have the following definition.

Definition 3.1.2. A *representation* of a C^*-algebra $\mathcal{A}$ is a *-homomorphism

$$\varrho \,:\, \mathcal{A} \to \mathcal{B}(\mathcal{H}) \tag{3.1}$$

Note that, by *-homomorphism, we mean an algebraic homomorphism such that:

$$\varrho(a^*) = \varrho(a)^* \;\forall a \in \mathcal{A} \tag{3.2}$$

A representation of a C^*-algebra that is injective is called *faithful*.

In the event these representations are not surjective, their range will be a subalgebra of $\mathcal{B}(\mathcal{H})$. However, because these representations are $*$-homomorphism, the subalgebras will inherit additional structure, as given in the following definition.

Definition 3.1.3. A subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is called a **-subalgebra* if it is closed under taking adjoints; that is, if $a \in \mathcal{A}$, then $a^* \in \mathcal{A}$.

So far, these representations have only been carrying algebraic information, but they can also carry topological information about $\mathcal{H}$.

Definition 3.1.4. A representation $\varrho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is called *cyclic* if there exists a vector $\Omega \in \mathcal{H}$ such that:

$$\varrho(\mathcal{A})\Omega = \{\varrho(a)\Omega \,:\, a \in \mathcal{A}\}$$

is dense in $\mathcal{H}$. In this case, Ω is called a *cyclic vector*.

One might ask if it is always possible to find a representation of any C^*-algebra. A construction by Gelfand, Naimark, and Segal, the GNS construction, answers that question in the affirmative, as seen in Theorem 3.1.8. Thus, every C^*-algebra can be realized as a subalgebra of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. To introduce this construction, a couple more preliminaries are needed, the first being the spectrum of an element $a \in \mathcal{A}$.

Definition 3.1.5. Given a C^*-algebra $\mathcal{A}$ and an element $a \in \mathcal{A}$, the *spectrum* of a, denoted $\mathrm{Sp}(a)$, is defined to be

$$\mathrm{Sp}(a) := \{\lambda \in \mathbb{C} \,:\, a - \lambda\mathbb{1} \text{ is not invertible}\}$$

Next, a specific set of elements and a new kind of map need to be considered.

Definition 3.1.6 ([Dix77], Prop. 1.6.1, Def. 1.6.5)**.** Given a C^*-algebra $\mathcal{A}$, we call an element $b \in \mathcal{A}$ *positive* if it satisfies any of the following equivalent conditions:

(i) $\mathrm{Sp}(b) \subset \mathbb{R}_{\geq 0}$

(ii) $b = aa^*$ for some $a \in \mathcal{A}$

(iii) $b = a^2$ for some self-adjoint element $a \in \mathcal{A}$

The set of positive elements of $\mathcal{A}$ is denoted $\mathcal{A}^+$.

Definition 3.1.7. A *state* on an algebra $\mathcal{A}$ is a linear map:

$$\omega \,:\, \mathcal{A}^+ \to [0, \infty] \tag{3.3}$$

such that $\omega(\mathbb{1}) = 1$. A state ω is called *faithful* if $\omega(a^*a) = 0$ implies $a = 0$.

We now present the GNS construction.

Theorem 3.1.8 (GNS Construction, [Dix77], Thm. 2.6.1)**.** Given a C^*-algebra $\mathcal{A}$, a *-representation ϱ of $\mathcal{A}$ over a Hilbert space $\mathcal{H}$, and a unit, cyclic vector $\Omega \in \mathcal{H}$, then the map:

$$a \mapsto \langle \Omega, \varrho(a)\Omega \rangle \tag{3.4}$$

is a state on $\mathcal{A}$.

Conversely, given a state ω on $\mathcal{A}$, there exists a cyclic representation ϱ_ω of $\mathcal{A}$ on $\mathcal{H}$ and a cyclic vector $\Omega_\omega \in \mathcal{H}$ for ϱ_ω such that:

$$\omega(a) = \langle \Omega_\omega, \varrho_\omega(a)\Omega_\omega \rangle \tag{3.5}$$

Thus, to every state ω and C^*-algebra $\mathcal{A}$, a triplet $(\varrho_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is associated. Thus, any C^*-algebra can be represented as a subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. This subalgebra is closed in the uniform topology and under taking adjoints.

Example 3.1.9. For an example of a state, in $M_n(\mathbb{C})$, the algebra of n-by-n matrices, every state φ can be represented as:

$$\varphi(\cdot) = \mathrm{Tr}(\rho\ \cdot)$$

where ρ is a positive definite matrix with $\mathrm{Tr}(\rho) = 1$. The state being a trace functional follows from the Riesz Representation Theorem when applied to $M_n(\mathbb{C})$ equipped with the Frobenius inner product:

$$\langle T_2, T_1 \rangle := \mathrm{Tr}(T_2^* T_1) \quad \text{for} \quad T_1, T_2 \in M_n(\mathbb{C}) \tag{3.6}$$

Next, a particular type of subset of $\mathcal{B}(\mathcal{H})$ needs to be considered, in anticipation of a mjaor result.

Definition 3.1.10. Given a subset of an algebra $\mathcal{S} \subset \mathcal{A}$, the *commutant*of $\mathcal{S}$ is:

$$\mathcal{S}' = \{a \in \mathcal{A} \,:\, ab = ba\ \forall b \in \mathcal{S}\} \tag{3.7}$$

Similarly, the commutant of the commutant, called the *bicommutant* and denoted $\mathcal{S}'' = (\mathcal{S}')'$, is defined as:

$$\mathcal{S}'' = \{a \in \mathcal{A} \,:\, ab = ba\ \forall b \in \mathcal{S}'\} \tag{3.8}$$

With that, the Bicommutant theorem can be given. It is a remarkable result proved by John von Neumann in Theorem V of [Neu30], with an English translation provided in Chapter 3 of [Neu95]. What makes this result striking is that it links a purely algebraic property of $\mathcal{B}(\mathcal{H})$ to a purely topological property.

Theorem 3.1.11 (Bicommutant Theorem)**.** Given a subalgebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ which has an identity operator and is closed under taking adjoints, called a unital $*$-subalgebra, the following conditions are equivalent:

- $\mathcal{M} = \overline{\mathcal{M}}^w = \overline{\mathcal{M}}^s$
- $\mathcal{M} = \mathcal{M}''$

This theorem is at the heart of the definition of von Neumann algebras, which is given now.

Definition 3.1.12. A *von Neumann algebra* is a unital *-subalgebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ satisfying either of the equivalent conditions given in Theorem 3.1.11; that is, a von Neumann algebra $\mathcal{M}$ satisfies:

$$\mathcal{M} = \mathcal{M}'' = \overline{\mathcal{M}}^w = \overline{\mathcal{M}}^s \tag{3.9}$$

With von Neumann algebras being fundamental to this paper, it would be best to provide examples of these objects.

Example 3.1.13. Given any Hilbert space $\mathcal{H}$, the space $\mathcal{B}(\mathcal{H})$ is itself a von Neumann algebra, as the whole space is closed in any topology, including the weak topology. As a particular example, when $\mathcal{H} = \mathbb{C}^n$, where $n \in \mathbb{Z}_{>0}$, then $\mathcal{B}(\mathcal{H}) = M_n(\mathbb{C})$.

Constructing more complicated examples will require significantly more effort, and is dealt with in Chapter 5. Next, one of the most fundamental objects in the theory of von Neumann algebras, the factor, is introduced.

Definition 3.1.14. A von Neumann algebra $\mathcal{M}$ is called a *factor* if

$$\mathcal{M} \cap \mathcal{M}' = \mathbb{C}\mathbb{1}_{\mathcal{B}(\mathcal{H})} \tag{3.10}$$

Much as with many fundamental structures, there is a reduction theory for factors. Given a separable Hilbert space $\mathcal{H}$, consider the set of all factors in $\mathcal{B}(\mathcal{H})$, denoted $\mathcal{F}$. Then one can define a probability measure on $\mathcal{F}$ and, therefore, an integration procedure over the set, called the direct integral. One does this by taking a measure space $(\mathcal{Y}, \sigma, \mu)$ with μ a probability measure and taking a map:

$$\begin{aligned} \mathcal{Y} &\longrightarrow \mathcal{F} \\ t &\longmapsto \mathcal{M}(t) \end{aligned}$$

Details of the full procedure can be found in [Neu49] or Chapter 5.1 of [Con94]. This construction can be summarized with the notation:

$$\mathcal{M} = \int^{\oplus} \mathcal{M}(t) \mathrm{d}\mu(t) \tag{3.11}$$

Any von Neumann algebra over a separable Hilbert space can be decomposed into factors, themselves over separable Hilbert spaces, using the direct integral. Thus, the classification theory of von Neumann algebras can be reduced to the classification of factors. This classification is the focus of this chapter.

The completion of this classification is on its own a grand journey, requiring the efforts of von Neumann, Murray, Powers, Araki, Wood, Krieger, and Connes to complete. There were many steps in the classification, but there are two steps of interest here: the division of factors into Type I, II, and III, and the classification of Type III_λ factors. The classification of of Type I, II, and III factors can be done immediately. However, the classification of Type III_λ factors requires a detour into Tomita-Takesaki theory to complete.

3.2 Classification of Factors: Type I, II, and III

The original classification of factors into Types was completed by Francis Joseph Murray and John von Neumann in a series of papers [MV36; MN37; Neu40; MN43], with the primary tool being the theory of projections. Thus, a few definitions regarding projections are in order.

Definition 3.2.1. An operator $E \in \mathcal{B}(\mathcal{H})$ is called a *projection* if $E^2 = E$.

This definition is applicable to spaces without an inner product, such as normed spaces. However, the existence of an inner product provides a notion of orthogonality, and so more can be said about the geometry of the operators.

Definition 3.2.2. A projection operator $E \in \mathcal{B}(\mathcal{H})$ is called an *orthogonal projection* if $E^* = E$.

Geometrically, in addition to mapping spaces onto subspaces, orthogonal projections have kernels $\ker(E)$ that are orthogonal to their range $\operatorname{ran}(E)$. Explicitly, for an orthogonal projection E:

$$\ker(E) = \operatorname{ran}(E)^{\perp} := \{h \in \mathcal{H} \,:\, \forall k \in \operatorname{Im}(E),\ \langle k, h\rangle = 0\}$$

Examples of both orthogonal and non-orthogonal projections in $\mathbb{R}^2$ are provided.

Example 3.2.3. For a non-orthogonal projection, consider:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \tag{3.12}$$

To observe its non-orthogonality, consider its image and kernel, illustrated in Figure 3.2.3.

$$\ker\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix} t \,:\, t \in \mathbb{R}\right\} \qquad \operatorname{ran}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} t \\ 0 \end{bmatrix} t \,:\, t \in \mathbb{R}\right\}$$

Example 3.2.4. For an orthogonal projection, consider:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.13}$$

To observe its orthogonality, consider its range and kernel, illustrated in Figure 3.2.3.

$$\ker\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} 0 \\ t \end{bmatrix} t \,:\, t \in \mathbb{R}\right\} \qquad \operatorname{ran}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} t \\ 0 \end{bmatrix} t \,:\, t \in \mathbb{R}\right\}$$

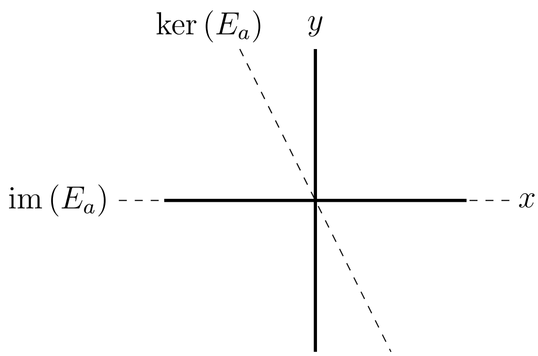

Figure 3.1: Image and kernel of Equation 3.12

The range of different projections also allows one to introduce a partial order structure on the set of projections in $\mathcal{B}(\mathcal{H})$. This partial ordering is given by, for projections $E_1, E_2 \in \mathcal{B}(\mathcal{H})$:

$$E_1 \leq E_2 \quad \Longleftrightarrow \quad \mathrm{ran}\,(E_1) \subset \mathrm{ran}\,(E_1) \tag{3.14}$$

The next definition provides a generalization of the notion of an isometry.

Definition 3.2.5. A *partial isometry* is a linear map between Hilbert spaces, $U : \mathcal{H} \to \mathcal{K}$ such that U is an isometry on the orthogonal complement of $\ker(U)$.

The reason for this generalization is that, on the kernel of a linear map, preserving the geometry is not a useful condition, as the entire kernel will be reduced to a point in the range. Thus, one can safely remove the requirement that the isometry U preserves the geometry of the kernel.

Definition 3.2.6. Given a von Neumann algebra $\mathcal{M}$ and orthogonal projections E_1 and E_2:

(i) Two projections E_1, E_2 are called *equivalent* and denoted $E_1 \sim E_2$ if there exists a partial isometry $U \in \mathcal{M}$ such that $U^*U = E_1$ and $UU^* = E_2$

(ii) If there is a partial isometry $U \in \mathcal{M}$ such that:

$$U^*U = E_1 \quad \text{and} \quad UU^* \leq E_2$$

then they are denoted $E_1 \preceq E_2$;

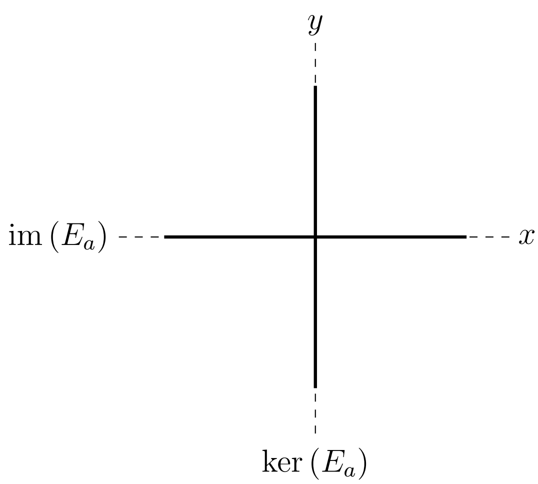

Figure 3.2: Range and kernel of Equation 3.13

(iii) If, for all projections $E_2 \in \mathcal{M}$:

$$E_2 \leq E_1 \Longrightarrow E_2 = 0 \text{ or } E_2 = E_1$$

E_1 is said to be *minimal*

(iv) A projection E_1 is called *finite* if:

$$E_1 \sim E_2 \leq E_1 \Longrightarrow E_1 = E_2$$

Definition 3.2.7. Given a von Neumann algebra $\mathcal{M}$, we say

(i) $\mathcal{M}$ is a *Type I* factor if there is a minimal projection in $\mathcal{M}$.

(ii) $\mathcal{M}$ is a *Type II* factor if there exists a finite but no minimal projection of $\mathcal{M}$.

(iii) $\mathcal{M}$ is a *Type III* factor if there is no finite projection of $\mathcal{M}$.

This completes the classification of Type I, II, and III factors. However, observe how Type III factors are defined in terms of not being Type I or Type II. This is how Murray and von Neumann thought of Type III factors, and these factors were the least understood. At the completion of their investigation, Murray and von Neumann had only found two non-*-isomorphic factors of Type III. The completion of the classification of Type III factors required more time and effort, eventually culminating with the work of Connes. However, to do this classification, an understanding of Tomita-Takesaki theory is necessary.

3.3 Tomita-Takesaki Theory

Although this theory was developed for the purpose of helping to finish the classification of factors, it also incidentally provides a way of moving from the theory of von Neumann algebras, which has been seen to be very complex, to the theory of standard subspaces, a much less difficult object to study. This section is spent eluciating this connection.

Definition 3.3.1. Given a Hilbert space $\mathcal{H}$ and a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, a unit vector $\Omega \in \mathcal{H}$ is called:

- *cyclic* if $\mathcal{M}\Omega$ is dense in $\mathcal{H}$
- *separating* if the map $\mathcal{M} \to \mathcal{H}$ with $a \mapsto a\Omega$ is injective.

We now come to one of the theorems most fundamental to this paper.

Theorem 3.3.2 (Tomita-Takesaki Theorem)**.** Consider a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and a cyclic and separating vector $\Omega \in \mathcal{H}$ for $\mathcal{M}$. Let

$$\mathcal{M}_h := \{a \in \mathcal{M} \,:\, a^* = a\} \tag{3.15}$$

be the real subspace of self-adjoint elements in $\mathcal{M}$. Then $V := \overline{\mathcal{M}_h\Omega}$ is a standard subspace. The modular objects (Δ_V, J_V) associated to V satisfy:

(i) $J_V\mathcal{M}J_V = \mathcal{M}'$ and $\Delta_V^{it}\mathcal{M}\Delta_V^{-it} = \mathcal{M}$

(ii) $J_V\Omega = \Omega, \quad \Delta_V\Omega = \Omega, \quad$ and $\Delta_V^{it}\Omega = \Omega$ for all $t \in \mathbb{R}$

(iii) For $a \in \mathcal{M} \cap \mathcal{M}'$, the following hold: $J_V a J_V = a^*$ and $\Delta_V^{it} a \Delta_V^{-it} = a$ for $t \in \mathbb{R}$

Next, some notation common in Tomita-Takesaki theory is introduced:

Returning to the connections between standard subspaces and von Neumann algebras, recall from Definition 2.4.2 that a closed real subspace had two potential properties called cyclic and separating, and satisfying both made the subspace standard. The naming choice when compared to Definition 3.3.1 is no coincidence, as the following correspondence shows:

$$\begin{aligned} \Omega \text{ cyclic } &\longleftrightarrow V \text{ cyclic} \\ \Omega \text{ separating } &\longleftrightarrow V \text{ separating} \end{aligned}$$

One might wonder if one is even guaranteed a cyclic and separating vector for a von Neumann algebra. Going back to the discussion of states, the GNS construction provided a method for passing between states and cyclic vectors. If the state is also faithful, the associated vector is separating, giving the correspondences:

$$\begin{array}{ccccc} \varphi \text{ state} & \longleftrightarrow & \Omega \text{ cyclic} & \longleftrightarrow & V \text{ cyclic} \\ \varphi \text{ faithful} & \longleftrightarrow & \Omega \text{ separating} & \longleftrightarrow & V \text{ separating} \end{array}$$

There is an additional condition, called normality, that is needed for the state. It is not strictly needed for the existence of a cyclic and separating vector, but it is necessary to ensure continuity of the state in the weak topology. However, the definition of normality is not used here, so it is not presented, though interested readers can see Proposition 2.5.6 in [BR87].

This only shifts the question of existence from cyclic and separating vectors to faithful normal states. This existence question is unnecessarily complicated for our purposes, and so we just note that von Neumann algebras over separable Hilbert spaces have cyclic and separating vectors.

Notation 3.3.3. Because of the connection between faithful normal states φ, cyclic and separating vectors Ω, and standard subspaces V, one can identify the modular objects (Δ, J) with either a faithful normal state φ, denoted $(\Delta_\varphi, J_\varphi)$, or a standard subspace V, denoted (Δ_V, J_V).

3.4 Classification of Factors: Type III_λ Factors

With Tomita-Takesaki Theory discussed, the classification of Type III_λ factors can be discussed. A full exposition of this step is far from simple, and it was the culmination of much work done by Connes in [Con73]. However, the crux of the classification comes down to the following quantity, called the Connes invariant:

$$S(\mathcal{M}) := \bigcap \{\mathrm{Sp}\,(\Delta_\omega) \,:\, \omega \text{ a faithful normal state on } \mathcal{M}\} \tag{3.16}$$

It turns out this quantity can only take on three possible values, given below, and the value it takes dictates the type III factor $\mathcal{M}$ is:

$$\begin{array}{cc} \mathrm{III}_0 & S(\mathcal{M}) = \{0,1\} \\ \mathrm{III}_\lambda \text{ with } \lambda \in (0,1) & S(\mathcal{M}) = \lambda^{\mathbb{Z}} \cup \{0\} \\ \mathrm{III}_1 & S(\mathcal{M}) = [0,\infty) \end{array}$$

This completes the classification of Type III_λ factors. For completeness, the final classification of factors is provided:

(i) Type I_n for $n \in \mathbb{Z}_{>0}$: These factors are all isomorphic to $M_n(\mathbb{C})$.

(ii) Type I_∞: These factors are isomorphic to $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$

(iii) Type II_1: An example of this factor is given in Example 5.1.2.

(iv) Type II_∞: Every factor of this type can be expressed as the tensor product of a Type I_∞ and Type II_1 factor. Details about the tensor product $\otimes$ are covered at the start of Chapter 5.

(v) Type III_0

(vi) Type III_λ for $\lambda \in (0,1)$

(vii) Type III_1

4. Antiunitary Representations and Standard Subspaces

The representation theory of graded groups, the structure of standard subspaces, modular objects, and inclusions of standard subspaces are considered next. It will be found that certain representations of graded groups are in one-to-one correspondence with standard subspaces. Moreover, when inclusions of standard spaces are considered, it will turn out that certain representations will also provide information on those inclusions, and vice versa.

4.1 Graded Groups

Starting with graded groups, the key idea is to generalize the multiplicative structure of positive and negative numbers in $\mathbb{R}^\times$

Definition 4.1.1. A *graded group* is a pair $(\mathcal{G}, \varepsilon_\mathcal{G})$ where $\mathcal{G}$ is a group and $\varepsilon_\mathcal{G}$ is a surjective homomorphism:

$$\varepsilon_\mathcal{G} : \mathcal{G} \to \{\pm 1\} \tag{4.1}$$

called the *parity mapping*. The kernel is denoted $\mathcal{G}_1 := \ker(\varepsilon_\mathcal{G})$ and its complement $\mathcal{G}_{-1} := \mathcal{G} \backslash \mathcal{G}_1$ so that the following equations hold:

$$\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_{-1} \quad \text{ and } \quad \mathcal{G}_j \mathcal{G}_k = \mathcal{G}_{jk} \tag{4.2}$$

where $j, k \in \{\pm 1\}$.

Example 4.1.2. Two crucial examples of graded groups are given.

- $\mathbb{R}^\times$ is the multiplicative group of nonzero, real numbers. Its identity component is $\mathbb{R}^\times_{>0}$, the set of positive real numbers.

  This group is the prototype for graded groups, with the nature of multiplication between positive and negative numbers being the basis for how elements of the two components of graded groups interact, as seen in the definition.

- $\mathrm{Aff}(\mathbb{R}) = \mathbb{R} \rtimes \mathbb{R}^\times$ is the affine group over the real numbers, also known as the $ax + b$ group. Its identity component is given by $\mathbb{R} \rtimes \mathbb{R}^\times_{>0}$.

  This group is of crucial significance, and so we shall revisit it again in much more detail in a later section.

One might wish to map between graded groups while preserving the graded structure, hence the following definition:

Definition 4.1.3. A *morphism of graded groups*

$$\eta : (\mathcal{J}, \varepsilon_{\mathcal{J}}) \to (\mathcal{G}, \varepsilon_{\mathcal{G}}) \tag{4.3}$$

is a homomorphism $\eta : \mathcal{J} \to \mathcal{G}$ with $\varepsilon_{\mathcal{G}} \circ \eta = \varepsilon_{\mathcal{J}}$.

Next, to be able to work with representations of graded groups, operators that behave in a similar manner to elements of said graded groups are needed. For that, recall antilinear maps from Definition 2.2.9.

Definition 4.1.4. Given a Hilbert space $\mathcal{H}$, the set of surjective isometries that are either linear or antilinear, denoted $\mathrm{AU}(\mathcal{H})$, is called the *antiunitary group* over $\mathcal{H}$. We denote the subset of linear operators with $\mathrm{U}(\mathcal{H})$ and call them *unitary operators*, while antilinear elements are called *antiunitary operators*.

Example 4.1.5. Examples of both unitary and antiunitary operators are given. Unitary operators are a familiar sight, with:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{C})$$

when acting on $\mathbb{R}^2$, being one of the simplest examples. An infinite dimensional example is the Fourier transform operator:

$$F(f) = \int_{-\infty}^{\infty} e^{-2\pi i x p} f(x) \mathrm{d}x$$

when acting on $f \in L^2(\mathbb{R})$. One can refer to Example 2.3.4 for another unitary operator, or rather a family of unitary operators.

Example 4.1.6. The best example of an antiunitary operator is the complex conjugation operator $\overline{\cdot}$, when acting on $\mathbb{C}$. By composing complex conjugation with unitary operators, one can also obtain a large class of antiunitary operators. For instance, the following twist on the Fourier transform is antiunitary:

$$\tilde{F}(f) = \int_{-\infty}^{\infty} e^{-2\pi i x p} \overline{f(x)} \mathrm{d}x$$

The reason for discussing $\mathrm{AU}(\mathcal{H})$ is that it will be at the heart of the representation theory of graded groups, in part because it is a graded group.

Proposition 4.1.7. Given a Hilbert space $\mathcal{H}$, the set $\mathrm{AU}(\mathcal{H})$ is a graded group, with the parity mapping $\varepsilon_{\mathrm{AU}(\mathcal{H})}$ given by:

$$\varepsilon_{\mathrm{AU}(\mathcal{H})}(U) = \begin{cases} +1 & U \in \mathrm{U}(\mathcal{H}) \\ -1 & U \in \mathrm{AU}(\mathcal{H}) \backslash \mathrm{U}(\mathcal{H}) \end{cases} \tag{4.4}$$

where $U(\mathcal{H})$ is the set of unitary operators over $\mathcal{H}$.

Proof. The identity operator is unitary and associativity is inherited from the nature of linear transformations. The key point in this proof is that composing linear and antilinear maps produces either a linear or antilinear map. Starting with two unitary operators U_1, U_2 and $h, k \in \mathcal{H}$, we get:

$$U_2U_1(h + zk) = U_2U_1h + U_2(zU_1k) = U_2U_1h + zU_2U_1k \tag{4.5}$$

Thus, multiplying two unitary operators gives another unitary operator, as expected. Now, considering a unitary and antiunitary operator denoted respectively U, A:

$$UA(h + zk) = UAh + U(\overline{z}Ak) = UAh + \overline{z}UAk \tag{4.6}$$

Thus, multiplying a unitary and antiunitary operator gives another antiunitary operator. A quick calculation shows that swapping the order of multiplication does not change the antilinearity of the output. Finally, given two antiunitary maps A_1, A_2:

$$A_2A_1(h + zk) = A_2A_1h + A_2(\overline{z}A_1k) = A_2A_1h + zA_2A_1k \tag{4.7}$$

Thus, composing two antiunitary operators outputs a unitary operator. Hence, $\mathrm{AU}(\mathcal{H})$ is closed under composition of operators, and so $\mathrm{AU}(\mathcal{H})$ is a group.

Finally, the map $\varepsilon_{\mathrm{AU}(\mathcal{H})}$ needs to be a surjective homomorphism. Surjectivity follows from the existence of unitary and antiunitary operators U and A, which map as follows:

$$\varepsilon_{\mathrm{AU}(\mathcal{H})}(U) = +1 \qquad \varepsilon_{\mathrm{AU}(\mathcal{H})}(A) = -1$$

For the homomorphism, there are the following equalities for the unitary operators U, U_1, U_2 and antiunitary operators A, A_1, A_2:

$$\varepsilon_{\mathrm{AU}(\mathcal{H})}(U_1U_2) = +1 = (+1)(+1) = \varepsilon_{\mathrm{AU}(\mathcal{H})}(U_1)\varepsilon_{\mathrm{AU}(\mathcal{H})}(U_2) \tag{4.8}$$
$$\varepsilon_{\mathrm{AU}(\mathcal{H})}(UA) = -1 = (+1)(-1) = \varepsilon_{\mathrm{AU}(\mathcal{H})}(U)\varepsilon_{\mathrm{AU}(\mathcal{H})}(A) \tag{4.9}$$
$$\varepsilon_{\mathrm{AU}(\mathcal{H})}(AU) = -1 = (-1)(+1) = \varepsilon_{\mathrm{AU}(\mathcal{H})}(A)\varepsilon_{\mathrm{AU}(\mathcal{H})}(U) \tag{4.10}$$
$$\varepsilon_{\mathrm{AU}(\mathcal{H})}(A_1A_2) = +1 = (-1)(-1) = \varepsilon_{\mathrm{AU}(\mathcal{H})}(A_1)\varepsilon_{\mathrm{AU}(\mathcal{H})}(A_2) \tag{4.11}$$

with Equation 4.8 following from Equation 4.5, Equation 4.9 and 4.10 from Equation 4.6, and Equation 4.11 from 4.7. □

With this, the representation theory of graded groups can now be given.

Definition 4.1.8. An *antiunitary representation* of a graded group $(\mathcal{G}, \varepsilon_{\mathcal{G}})$ is a pair $(\mathbb{U}, \mathcal{H})$ with

$$\mathbb{U} : \mathcal{G} \longrightarrow \mathrm{AU}(\mathcal{H}) \tag{4.12}$$

being a morphism of graded groups.

Remark 4.1.9. Antiunitary representations whose range has an identity component isomorphic to a one-parameter, unitary group will be of particular interest in Lemma 4.3.1. The infinitesimal generator of this one-parameter, unitary group, as afforded by Theorem 2.3.2, will provide needed structure.

4.2 Structure of $\mathrm{Aff}(\mathbb{R})$

This section is devoted to $\mathrm{Aff}(\mathbb{R})$ (cf. Example 4.1.2) and its representations, as they form one of the four pillars of this paper. Thus, this section covers details regarding the group and subgroup structure, and its matrix representations.

4.2.1 Definition and Representation

Definition 4.2.1. The *affine group of the real line*, denoted $\mathrm{Aff}(\mathbb{R})$, is the semidirect product:

$$\mathrm{Aff}(\mathbb{R}) := \mathbb{R} \rtimes \mathbb{R}^\times \tag{4.13}$$

with the group action of $\mathbb{R}^\times$ on $\mathbb{R}$ defined by:

$$\alpha \cdot \beta = \alpha\beta \tag{4.14}$$

for $\alpha \in \mathbb{R}^\times$ and $\beta \in \mathbb{R}$. Thus, composition in $\mathrm{Aff}(\mathbb{R})$ is defined as follows:

$$(\beta_1, \alpha_1)(\beta_2, \alpha_2) = (\alpha_2\beta_1 + \beta_2, \alpha_1\alpha_2)$$

Proposition 4.2.2. The group $\mathrm{Aff}(\mathbb{R})$ is isomorphic to the $ax + b$ group; that is, the group of transformations on the real line $\mathbb{R}$ of the form:

$$f_{a,b} \,:\, t \longmapsto at + b \tag{4.15}$$

with $a \in \mathbb{R}^\times$ and $b \in \mathbb{R}$.

Proof. To start, the composition of f_{a_1,b_1} and f_{a_2,b_2} is evaluated:

$$\begin{aligned}(f_{a_2,b_2} \circ f_{a_1,b_1})(t) &= f_{a_2,b_2}(a_1 t + b_1) \\ &= a_2(a_1 t + b_1) + b_2 \\ &= a_1 a_2 t + (a_2 b_1 + b_2) \\ &= f_{a_1a_2, a_2b_1+b_2}(t)\end{aligned}$$

Thus, $f_{a_1,b_1} f_{a_2,b_2} = f_{a_1a_2,a_2b_1+b_2}$.

The isomorphism is given by:

$$\Xi \,:\, (\beta, \alpha) \longmapsto f_{\alpha,\beta}$$

The key point is Ξ is a homomorphism, which follows from the following equalities:

$$\begin{aligned}\Xi\big((\alpha_2\beta_1 + \beta_2, \alpha_1\alpha_2)\big) &= f_{\alpha_1\alpha_2, \alpha_2\beta_1+\beta_2} \\ &= f_{\alpha_1,\beta_1} f_{\alpha_2,\beta_2} \\ &= \Xi\big((\beta_1, \alpha_1)\big)\Xi\big((\beta_2, \alpha_2)\big)\end{aligned}$$

□

Now, it is time to consider the matrix representations of $\mathrm{Aff}(\mathbb{R})$. However, to do so, the space on which $\mathrm{Aff}(\mathbb{R})$ acts needs to be extended from $\mathbb{R}$ to a higher-dimensional vector space. This can be done by extending to $\mathbb{R}^2$, and having $\mathrm{Aff}(\mathbb{R})$ represented by a subset of $\mathrm{GL}(2, \mathbb{R})$.

Proposition 4.2.3. There is a matrix representation for $\mathrm{Aff}(\mathbb{R})$ acting on $\mathbb{R}^2$ given by the following map:

$$\begin{gathered}\mathbb{A} \,:\, \mathrm{Aff}(\mathbb{R}) \to \mathrm{GL}(2, \mathbb{R}) \\ (\beta, \alpha) \mapsto \begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix}\end{gathered}$$

Moreover, $\mathbb{A}$ is a faithful representation.

Proof. To prove injectivity of $\mathbb{A}$, consider two matrices

$$\mathbb{A}\big((\beta_1, \alpha_1)\big) = \begin{bmatrix} \alpha_1 & \beta_1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbb{A}\big((\beta_1, \alpha_1)\big) = \begin{bmatrix} \alpha_2 & \beta_2 \\ 0 & 1 \end{bmatrix}$$

and assume

$$\begin{bmatrix} \alpha_1 & \beta_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha_2 & \beta_2 \\ 0 & 1 \end{bmatrix}$$

Then that must mean $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, and so $(\beta_1, \alpha_1) = (\beta_2, \alpha_2)$.

For the homomorphism:

$$\begin{aligned} \mathbb{A}\big((\beta_2, \alpha_2)(\beta_1, \alpha_1)\big) &= \mathbb{A}\big((\alpha_2\beta_1 + \beta_2, \alpha_1\alpha_2)\big) \\ &= \begin{bmatrix} \alpha_1\alpha_2 & \alpha_2\beta_1 + \beta_2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha_2 & \beta_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 & \beta_1 \\ 0 & 1 \end{bmatrix} \\ &= \mathbb{A}\big((\beta_2, \alpha_2)\big)\mathbb{A}\big((\beta_1, \alpha_1)\big) \end{aligned}$$

□

With the basic notions in place, there are a few subgroups that will prove to be of particular interest, and so those are considered next.

4.2.2 Subgroups

The first two subgroups of $\mathrm{Aff}(\mathbb{R})$ to consider are the two that $\mathrm{Aff}(\mathbb{R})$ can be built from.

Definition 4.2.4. The *translation subgroup*, denoted $(\mathbb{R}, 1)$, is defined as follows:

$$\mathbb{R} \rtimes \{1\} := \{(\beta, 1) \in \mathrm{Aff}(\mathbb{R}) \,:\, \beta \in \mathbb{R}\} \tag{4.16}$$

The *dilation subgroup*, denoted $(0, \mathbb{R}^\times)$, is defined as follows:

$$\{0\} \rtimes \mathbb{R}^\times := \{(0, \alpha) \in \mathrm{Aff}(\mathbb{R}) \,:\, \alpha \in \mathbb{R}^\times\} \tag{4.17}$$

The meaning behind these names is apparent when considering the action of $\mathrm{Aff}(\mathbb{R})$ on $\mathbb{R}$. Elements $(\beta, 1) \in (\mathbb{R}, 1)$ manifest as follows:

$$(\beta, 1)t = t + \beta$$

Thus, when an element of the translation subgroup acts on $\mathbb{R}$, it translates the elements of $\mathbb{R}$. As for the dilation subgroup, its elements act on $\mathbb{R}$ as follows:

$$(0, \alpha)t = \alpha t$$

and so elements of the dilation group rescale the real line.

As for the representations of these subgroups, elements of the translation subgroup map to matrices of the form:

$$\mathbb{A}\big((\beta, 1)\big) = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$$

while elements of the dilation subgroup map to matrices of the form:

$$\mathbb{A}\big((0, \alpha)\big) = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix representations of the translation and dilation subgroup shall be denoted as $\mathbf{N}$ and $\mathbf{A}$ respectively.

The third and last subgroup to consider is the following:

Proposition 4.2.5. The following subset of $\mathrm{Aff}(\mathbb{R})$:

$$\delta_{1,\infty} := \{(1 - \alpha, \alpha) \in \mathrm{Aff}(\mathbb{R}) \,:\, \alpha \in \mathbb{R}^\times\} \tag{4.18}$$

forms a subgroup.

Proof. The identity element is obtained by setting $\alpha = 1$.

For closure under composition:

$$\begin{aligned}(1 - \alpha_1, \alpha_1)(1 - \alpha_2, \alpha_2) &= \big(\alpha_2(1 - \alpha_1) + 1 - \alpha_2, \alpha_1\alpha_2\big) \\ &= (1 - \alpha_1\alpha_2, \alpha_1\alpha_2)\end{aligned}$$

□

4.3 Modular Objects

It is time to revisit modular objects and see how they connect to antiunitary representations of $\mathbb{R}^\times$. For the following lemma, recall Remark 4.1.9.

Lemma 4.3.1. Let $(\mathbb{U}, \mathcal{H})$ of $\mathbb{R}^\times$ be a continuous antiunitary representation and H the infinitesimal generator defined by:

$$\mathbb{U}(e^t) := e^{itH}. \tag{4.19}$$

Then,

$$\Delta := e^{2\pi H} \tag{4.20}$$

is a positive definite operator, and

$$J := \mathbb{U}(-1) \tag{4.21}$$

is an antiunitary operator such that $J^2 = 1$. Moreover, the pair (Δ, J) satisfies the modular relation:

$$J\Delta J = \Delta^{-1}. \tag{4.22}$$

Conversely, any pair of objects (Δ, J) satisfying the modular relation defines an antiunitary representation $\mathbb{U}$ of $\mathbb{R}^\times$ by

$$\mathbb{U}(e^t) := \Delta^{-it/2\pi} \quad \text{and} \quad \mathbb{U}(-1) := J \tag{4.23}$$

The (Δ, J) in this lemma are the same modular objects given in Definition 2.4.4, with all the same properties as mentioned before. Thus, there are two different formulations of modular objects, one using a polar decomposition of an antilinear involution, and one using the antiunitary representation of $\mathbb{R}^\times$. However, $\mathbb{R}^\times$ is not the only group with this connection. Because morphisms between graded groups provide a connection between $\mathbb{R}^\times$ and any other graded group, modular objects can be connected to any graded group.

Proposition 4.3.2. For any continuous morphism of graded groups:

$$\eta : (\mathbb{R}^\times, \mathrm{sign}) \to (\mathcal{G}, \varepsilon_{\mathcal{G}})$$

and any continuous antiunitary representation $(\mathbb{U}, \mathcal{H})$ of $(\mathcal{G}, \varepsilon_{\mathcal{G}})$, a pair of modular objects (Δ_η, J_η) is obtained from the antiunitary representation $\mathbb{U} \circ \eta$ of $\mathbb{R}^\times$ as follows:

$$\Delta_\eta = \mathbb{U}\big(\eta(e^t)\big) \qquad J_\eta = \mathbb{U}\big(\eta(-1)\big) \tag{4.24}$$

In particular, mapping between $\mathbb{R}^\times$ and $\mathrm{Aff}(\mathbb{R})$ will be at the forefront of a key result in the next section, which will revisit standard subspaces, their connection to modular objects, and expound on their connection to antiunitary representations of $\mathrm{Aff}(\mathbb{R})$.

4.4 Standard Subspaces Revisited

Recall from Definition 2.4.2 that standard subspaces are closed real subspaces $V \subset \mathcal{H}$ of a Hilbert space $\mathcal{H}$ that are both cyclic and separating:

- $V + iV$ is dense in $\mathcal{H}$ (cyclic)
- $V \cap iV = 0$ (separating)

One object not introduced in the background material is the following:

Definition 4.4.1. Given a standard subspace $V \subset \mathcal{H}$, the *symplectic orthogonal complement* is given by:

$$V' := \{h \in \mathcal{H} \,:\, (\forall k \in V)\ \mathrm{Im}\langle k, h\rangle = 0\} = iV^{\perp_{\mathbb{R}}} \tag{4.25}$$

where

$$V^{\perp_{\mathbb{R}}} = \{h \in \mathcal{H} \,:\, (\forall k \in V)\ \mathrm{Re}\langle k, h\rangle = 0\} \tag{4.26}$$

is the real orthogonal complement.

Example 4.4.2. The symplectic orthogonal complement of $\mathbb{R} \subset \mathbb{C}$ will be calculated here. First note that the inner product on $\mathbb{C}$ is given by $\langle w, z\rangle := \overline{w}z$. Then, express w and z with real and imaginary parts as follows:

$$w = s_1 + t_1 i \qquad z = s_2 + t_2 i$$

at which point the inner product expands as:

$$\langle w, z\rangle = \overline{w}z = (s_1 s_2 + t_1 t_2) + (s_1 t_2 - s_2 t_1)i$$

Assuming $w \in \mathbb{R}$ means that $t_1 = 0$. To then ensure

$$\mathrm{Im}\langle w, z\rangle = s_1 t_2 - s_2 t_1 = 0$$

either s_1 or t_2 to be 0. However, $s_1 = 0$ would just imply $w = 0$, which means it would not represent $\mathbb{R}$. Thus, t_2 must be 0, which implies $z \in \mathbb{R}$. Thus, the symplectic orthogonal complement of $\mathbb{R}$ is itself.

One can show that the symplectic orthogonal complement of any line of the form $e^{it}\mathbb{R}$, with $t \in \mathbb{R}$, is itself. Because of this, the real orthogonal complement $V^{\perp_{\mathbb{R}}}$ is the natural object to consider when dealing with geometric notions of orthogonality.

Symplectic orthogonal complements will turn out to be, in many aspects, dual to standard subspaces. This is immediately demonstrated by the following proposition, which is a summary of results given on page 46 of [Lon08].

Proposition 4.4.3. Given real subspaces V, V_1, V_2:

(i) V is cyclic if and only if V' is separating

(ii) V is a standard subspace if and only if V' is a standard subspace.

(iii) $\overline{V} = V''$

(iv) $V_1 \subset V_2$ if and only if $V_2' \subset V_1'$

Returning to the connection between standard subspaces and modular objects, recall the antilinear involution S which maps as follows:

$$\begin{aligned} S \,:\, \mathcal{D}(S) := V + iV &\to \mathcal{H} \\ h + ik &\mapsto h - ik \end{aligned}$$

with $h, k \in V$. Because V is standard, S is well-defined and densely defined, with $S^2 = \mathbb{1}_{\mathcal{D}(S)}$. Moreover, it is closed, which follows from the standard subspace being a closed subspace. These involutions are in fact in correspondence with the set of standard subspaces of $\mathcal{H}$.

Proposition 4.4.4 ([Lon08], Prop 3.2)**.** There is a bijection between the set of standard subspaces of a Hilbert space $\mathcal{H}$ and the closed, densely defined, antilinear involutions on $\mathcal{H}$.

In addition to this bijection between standard subspaces V and their associated maps S, the following proposition and, specifically, the uniqueness of the polar decomposition, provide a bijection between S and its associated modular objects (Δ, J).

Proposition 4.4.5 ([Lon08], Prop. 3.3)**.** Given a standard subspace V and its associated operator S_V, take the polar decomposition:

$$S_V = J_V \Delta_V^{1/2} \tag{4.27}$$

Then the following hold:

(i) J_V is an antiunitary involution:

$$J_V = J_V^* = J_V^{-1} \tag{4.28}$$

(ii) Δ_V is a positive, nonsingular, selfadjoint operator and:

$$J_V \Delta_V J_V = \Delta_V^{-1} \tag{4.29}$$

(iii) $J_{V'} = J_V$ and $\Delta_{V'} = \Delta_V^{-1}$

Combining all these results with the correspondences between modular objects and antiunitary representations of $\mathbb{R}^\times$ gives the following bijections:

Proposition 4.4.6. There is a one-to-one correspondence between the data of:

(i) A standard subspace V

(ii) The closed, densely defined, antilinear involution S acting on V

(iii) The modular objects (Δ, J) acquired from the polar decomposition of S

(iv) Antiunitary representations of $\mathbb{R}^\times$

Proof. These correspondences follow from prior results, as outlined in the diagram below:

$$(i) \xleftrightarrow{Prop.\ 4.4.4} (ii) \xleftrightarrow{Eqn.\ 4.27} (iii) \xleftrightarrow{Lemma\ 4.3.1} (iv)$$

□

Notation 4.4.7. By combining Proposition 4.4.6(i, iii, iv), we can now refer to the antiunitary representation $\mathbb{U}_V$ of $\mathbb{R}^\times$ associated to a standard subspace V, with:

$$\mathbb{U}_V(e^t) := \Delta_V^{-it/2\pi} \quad \text{and} \quad \mathbb{U}_V(-1) := J_V \tag{4.30}$$

The next result gives the result of the action of Δ^{it} and J on V.

Theorem 4.4.8 ([Lon08], Thm. 3.4)**.** Given a standard subspace V and its associated modular objects (Δ_V, J_V), as well as any $t \in \mathbb{R}$, both $\Delta_V^{it} V = V$ and $J_V V = V'$.

With this proposition, it should start being clear why the symplectic orthogonal complement V' is the other space of interest when considering standard subspaces. They form the space that V maps into when acted on by J, similar to how $\mathcal{M}'$ forms the space $\mathcal{M}$ maps to when conjugated by J: $J\mathcal{M}J = \mathcal{M}'$.

From here, material needs to be built up to present a representation result on $\mathrm{Aff}(\mathbb{R})$. In particular, inclusions between different standard subspaces will need to be considered. There are two different methods for tackling these inclusions. The first, pioneered by Borchers in [Bor92], involves the use of antiunitary representations acting on standard subspaces. The second method, developed by Wiesbrock in [Wie93], primarily makes use of modular operators, and eventually leads to the notion of half-sided modular inclusions. The definitions of both are given now, after which results on Borchers' pairs are given.

Definition 4.4.9. Given a continuous unitary one-parameter group $(\mathcal{U}_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$ and a standard subspace $V \subset \mathcal{H}$, we call $(\mathcal{U}, V)$ a $\pm$*Borchers pair* if $\mathcal{U}_t V \subset V$ holds for $t \geq 0$ and $\mathcal{U}_t = e^{itH}$ with $\pm H \geq 0$.

Definition 4.4.10. Given standard subspaces $X \subset \mathcal{H}$ and $V \subset \mathcal{H}$, we say that $X \subset V$ is a $\pm$*half-sided modular (hsm) inclusion* if:

$$\Delta_V^{-it} X \subset X \tag{4.31}$$

for $\pm t \geq 0$. When we refer to a hsm inclusion, it will be a +hsm inclusion unless otherwise stated.

For some clarification on the definition on the hsm inclusion, because V is a larger subspace of $\mathcal{H}$ than X, Δ_V has a larger domain. In general, it is not guaranteed that Δ_V^{-it} would map X back into X, but if it so happens to for half of the real line, it is called a hsm inclusion.

This next result presents the specific connection between antiunitary representations of $\mathrm{Aff}(\mathbb{R})$ and Borchers pairs.

Theorem 4.4.11 (Borchers' Theorem, [NÓ17], Thm 3.12)**.** If $(\mathcal{U}, V)$ is a $\pm$Borchers pair, then there is an antiunitary positive energy representation $(\mathbb{U}, \mathcal{H})$ of $\mathrm{Aff}(\mathbb{R})$ defined by $\mathbb{U}_{(t,s)} := \mathcal{U}_t \mathbb{U}_V(s)$, meaning:

$$\mathbb{U}_V(s)\mathcal{U}_t\mathbb{U}_V(s)^{-1} = \mathcal{U}\left(s^{\pm 1}t\right) \qquad \text{for } s \in \mathbb{R}^\times, t \in \mathbb{R} \tag{4.32}$$

Although this theorem already provides connections between antiunitary representations, it acts as a lemma for a more comprehensive result.

Theorem 4.4.12 ([NÓ17], Thm. 3.13)**.** Let $(\mathbb{U}, \mathcal{H})$ be an antiunitary representation of $\mathrm{Aff}(\mathbb{R})$. For each $t \in \mathbb{R}$, consider the homomorphism:

$$\gamma_t : \mathbb{R}^\times \to \mathrm{Aff}(\mathbb{R}), \quad \gamma_t(s) := (t,1)(0,s)(-t,1) = ((1-s)t, s)$$

and the corresponding family of standard subspaces $(V_t)_{t\in\mathbb{R}}$ determined by $\mathbb{U}_{V_t} = \mathbb{U} \circ \gamma_t$. Then the following assertions hold:

(i) $\mathbb{U}(\beta, \alpha)V_t = V_{\alpha t+\beta}$ and $\mathbb{U}(\beta, -\alpha)V_t = V'_{-\alpha t+\beta}$ for $\beta, t \in \mathbb{R}, \alpha > 0$

(ii) The following are equivalent:

  (a) $\mathbb{U}$ is a positive energy representation

  (b) $V_s \subset V_0$ for $s \geq 0$

  (c) $V_s \subset V_t$ for $s \geq t$

  (d) $(\mathcal{U}, V_0)$, where $\mathcal{U}_t := \mathbb{U}_{(t,1)}$ is a positive Borchers pair

  (e) $V_1 \subset V_0$ is a hsm-inclusion

(iii) $V_t = V_0$ for all $t \in \mathbb{R}$ is equivalent to $\mathbb{U}_{(\beta,1)} = 1$ for every $\beta \in \mathbb{R}$

(iv) $V_\infty := \bigcap_{t\in\mathbb{R}} V_t = \{h \in V_0 : (\forall\beta \in \mathbb{R}),\ \mathbb{U}_{(\beta,1)}h = h\}$ is the fixed point space for the translations

(v) $V_0 \cap V_0' = \mathcal{H}^{\mathrm{Aff}(\mathbb{R})} = \{h \in \mathcal{H} : (\forall(\beta, \alpha) \in \mathrm{Aff}(\mathbb{R})),\ \mathbb{U}(\beta, \alpha)h = h\}$

Going through these points more thoroughly:

- Point (i) indicates that the set of standard subspaces V_t behaves much like the real line when acted on by a representation of $\mathrm{Aff}(\mathbb{R})$.

- Point (ii) connects Borchers pairs and positive energy representations to a host of inclusions relations, including hsm inclusions.

- Point (iii) provides a formulation of trivial representations of the translation subgroup $\mathbb{R} \rtimes \{1\}$ in terms of standard subspaces.

- Finally, points (iv) and (v) provide expressions for the fixed point spaces of both $\mathbb{R} \rtimes \{1\}$ and $\mathrm{Aff}(\mathbb{R})$.

Remark 4.4.13. The two spaces V_0 and V_1 are going to be of particular interest, so we note that the elements of $\mathrm{Aff}(\mathbb{R})$ that they correspond to are the dilation subgroup $\{0\} \rtimes \mathbb{R}^\times$ and $\delta_{1,\infty}$ respectively.

Remark 4.4.14. The space $V_0 \cap V_0'$ given in Theorem 4.4.12(v), or rather its generalization to arbitrary standard subspaces V:

$$V \cap V' \tag{4.33}$$

will be of great significance when covering the duality between standard subspaces and factors, as seen in Theorem 5.3.6(vi).

4.5 Half Sided Modular Inclusions

Theorem 4.4.12(ii(e)) gives a condition for the existence of a hsm inclusion $V_1 \subset V_0$ in terms of an antiunitary representation of $\mathrm{Aff}(\mathbb{R})$. The next major result to present, Wiesbrock's theorem, inverts this implication, starting with the hsm inclusion $V_1 \subset V_0$ and demonstrating the existence of an antiunitary representation of $\mathrm{Aff}(\mathbb{R})$.

To do so, and because inclusions of standard subspaces are being used, it would be best if maps preserved the structure of standard subspaces. That motivates the following definition.

Definition 4.5.1. Given Hilbert spaces $\mathcal{H}, \mathcal{K}$ and standard subspaces $V \subset \mathcal{H}$ and $X \subset \mathcal{K}$, a bounded linear map $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is called a *$X - V$-real map* if:

$$TX \subset V \tag{4.34}$$

Specific strips in the complex plane will also prove relevant:

$$\mathbb{S}_t := \{z \in \mathbb{C} \,:\, 0 < \mathrm{Im}(z) < t\} \tag{4.35}$$

The presence of these strips is not artificial. It is a natural consequence of the close connection between the modular automorphism groups (σ_t^φ) and what is known as the KMS condition in quantum statistical mechanics. Because the KMS condition does not play a role in this paper, it is not considered here and, instead, the reader is directed toward [Kub57; MS59] for the original formulation, [HHW67] for a relevant reformulation, and sections 10 and 13 of [Kas70; Tak70] respectively for the connection between modular automorphism groups and the KMS condition.

There will be two real maps of interest, given below, though much of the buildup to this material has been skipped.

Corollary 4.5.2 ([Lon08], Cor. 3.19)**.** Let $X \subset V$ be standard subspaces of $\mathcal{H}$. Then the map:

$$W(s) := \Delta_V^{-is} \Delta_X^{is} \tag{4.36}$$

with $s \in \mathbb{R}$ extends to a strongly continuous map on $\overline{\mathbb{S}_{1/2}}$, analytic in $\mathbb{S}_{1/2}$, such that:

$$W\left(s + \frac{i}{2}\right) = J_V W(s) J_X$$

and $W\left(s + \frac{i}{2}\right)$ is $X' - V'$-real.

Theorem 4.5.3. Let $V \subset \mathcal{H}$ and $X \subset \mathcal{K}$ be standard subspaces. Assume the map:

$$\begin{aligned} T \,:\, \overline{\mathbb{S}_{1/2}} \backslash \{i/2\} &\to \mathcal{B}(\mathcal{K}, \mathcal{H}) \\ z &\mapsto T(z) \end{aligned}$$

is bounded, weakly continuous on $\overline{\mathbb{S}_{1/2}} \backslash \{i/2\}$, analytic on $\mathbb{S}_{1/2}$, and satisfies:

- $T(s)$ is $X - V$-real for all $s \in \mathbb{R}$
- $J_X T\left(s + \frac{i}{2}\right) J_V$ is $V - X$-real for all $s \in \mathbb{R} \backslash \{0\}$

Then there exists a $X - V$-real operator $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that:

$$T(s) = \Delta_V^{-is} T \Delta_X^{is} \quad s \in \mathbb{R}$$

Moreover $T(\cdot)$ extends to a strongly continuous map on $\overline{\mathbb{S}_{1/2}}$ and satisfies:

$$T(z + t) = \Delta_V^{-it} T(z) \Delta_X^{it} \quad z \in \overline{\mathbb{S}_{1/2}}, t \in \mathbb{R}$$

$$T\left(s + \frac{i}{2}\right) = J_V T(s) J_X \quad s \in \mathbb{R}$$

With this, there is enough machinery to state the theorem of interest, although it comes in two parts.

Theorem 4.5.4 (Wiesbrock's Theorem, [Lon08], Thm. 3.21)**.** Let $X \subset V \subset \mathcal{H}$ be a hsm-inclusion. Then there exists a positive energy unitary representation $\mathbb{U}$ of $\mathrm{Aff}(\mathbb{R})$ on $\mathcal{H}$ determined by:

$$\mathbb{U}\big((0,s)\big) = \Delta_V^{-is/2\pi}, \quad \mathbb{U}\big((1-s,s)\big) = \Delta_X^{-is/2\pi} \tag{4.37}$$

The translation unitaries $\mathcal{U}_t := \mathbb{U}\big((t,1)\big)$ are defined by:

$$\mathcal{U}_{e^{2\pi t}-1} = \Delta_V^{-it}\Delta_X^{it} \tag{4.38}$$

and satisfy $\mathcal{U}_s V \subset V$, with $s \geq 0$, and $X = U(1)V$.

This is the original work by Longo. It only contained information for unitary representations, but this work was later extended to antiunitary representations in [NÓ17].

Theorem 4.5.5 (Wiesbrock's Theorem, [NÓ17], Thm. 3.15)**.** $X \subset V \subset \mathcal{H}$ is a hsm inclusion of standard subspaces if and only if there exists an antiunitary positive energy representation $(\mathbb{U}, \mathcal{H})$ of $\mathrm{Aff}(\mathbb{R})$ with $X = V_1$ and $V = V_0$, with V_t defined as in Theorem 4.4.12.

Here, we will outline the proof method used to prove the version in [Lon08].

Proof.

(i) Start with the real map $W(z)$ with domain $\overline{\mathbb{S}_{1/2}}$, defined in Corollary 4.5.2, and identify the following reality properties:

  (a) $W(z)$ is $X - X$ real for $z \in (0,\infty)$

  (b) $W(z)$ is $X' - X'$ real for $z \in (-\infty, 0)$

  (c) $W(z)$ is $X' - X'$ real for $z \in \mathbb{R} + \frac{i}{2}$

(ii) Express the real maps $T(s)$ in Theorem 4.5.3 in terms of $W(s)$, that meaning $T(z) := W(h(z))$, using the conformal function:

$$\begin{aligned} h\,:\, \overline{\mathbb{S}_{1/2}}\backslash\{i/2\} &\longrightarrow \overline{\mathbb{S}_{1/2}}\backslash\{0\} \\ z &\longmapsto \frac{1}{2\pi}\log\left(1+e^{2\pi z}\right) \end{aligned}$$

(iii) Prove $W(s)$ forms a commutative family.

(iv) Define $\mathcal{U}_{e^{2\pi s}-1} := W(s)$ and prove $\mathcal{U}$ forms a positive-energy, one-parameter group.

(v) Create the following unitary representation of $\mathrm{Aff}(\mathbb{R})$ in $\mathcal{U}$:

$$\begin{aligned} \mathbb{V} : \mathrm{Aff}(\mathbb{R}) &\longrightarrow \mathcal{U} \\ (\beta, \alpha) &\longmapsto \mathcal{U}_\beta \Delta_V^{-i\alpha/2\pi} \end{aligned}$$

(vi) Use properties of $\mathcal{U}$ and the modular objects to prove the remaining identities.

$\square$

Wiesbrock's theorem is the primary quarry of this paper. To see why, however, von Neumann algebras need to be revisited. In particular, an older formulation of hsm inclusions needs to be considered.

5. Von Neumann Algebras: Constructions and Duality

The theory of von Neumann algebras is now revisited, as their connection to standard subspaces currently only manifests through the Tomita-Takesaki theorem. This theorem passes from von Neumann algebras to standard subspaces, but not the other way around. To go backwards, a specific class of von Neumann algebras, the second quantization algebras, need to be considered. Thus, the necessity of these algebras will be used as an excuse to, in the next few sections, present different constructions for large classes of von Neumann algebras. These constructions will allow for the introduction of examples of von Neumann algebras that are not finite-dimensional matrix algebras. From here, further connections between von Neumann algebras and standard subspaces will be drawn. In particular, we provide an analog of the hsm inclusions for von Neumann algebras.

Before beginning, one construction that will repeatedly appear is the tensor product of von Neumann algebras. Taking the tensor product of von Neumann algebras is not a trivial process, and so the construction is given here.

Starting with two von Neumann algebras $\mathcal{M}_1, \mathcal{M}_2$ over Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, the space that the new algebra will act on needs to be determined first. This space is in fact the tensor product of the underlying Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$, constructed in Section 2.1. To construct the tensor product of the algebras, one needs to endow the algebraic tensor product $\mathcal{M}_1 \odot \mathcal{M}_2$ with a multiplicative structure. This is done by defining:

$$(a \odot b)(a' \odot b') := (aa' \odot bb')$$

The bicommutant, or equivalently the weak closure, is then taken to form a von Neumann algebra. Thus, one has the tensor product of two von Neumann algebras.

$$\mathcal{M}_1 \otimes \mathcal{M}_2 := (\mathcal{M}_1 \odot \mathcal{M}_2)'' = \overline{\mathcal{M}_1 \odot \mathcal{M}_2}^{w} \tag{5.1}$$

5.1 Hyperfinite von Neumann Algebras

The largest class of algebras of interest are called the hyperfinite algebras. The construction of these algebras makes use of inclusions of matrix algebras.

This is part of their appeal as a class of algebras, as this construction allows for approximation results in terms of matrix algebras.

To give the construction, begin with a sequence of finite-dimensional matrix algebras, treated as C^*-algebras, and states $(\mathcal{A}_n, \varphi_n)_{n\in\mathbb{N}}$. One then considers the tensor product:

$$\mathcal{C}_m := \bigotimes_{n=1}^{m} \mathcal{A}_n \tag{5.2}$$

From here, one embeds $\mathcal{C}_m$ into $\mathcal{C}_{m+1}$ by means of the map

$$a \longmapsto a \otimes \mathbb{1}_{\mathcal{A}_{m+1}}$$

with $a \in \mathcal{C}_m$ and $\mathbb{1}_{\mathcal{A}_{m+1}}$ being the identity operator associated with $\mathcal{A}_{m+1}$. By taking the limit of this procedure, one obtains the inductive limit:

$$\mathcal{C} := \lim_{m\to\infty} \mathcal{C}_m \tag{5.3}$$

which is a C^*-algebra. However, one needs a state associated with $\mathcal{C}$ to ensure the Hilbert space $\mathcal{C}$ acts on is not too large. One obtains this state $\varphi := \bigotimes_{n=1}^{\infty} \varphi_n$ by defining:

$$\varphi(a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes \mathbb{1}_{\mathcal{M}_{n+1}} \otimes \mathbb{1}_{\mathcal{M}_{n+2}} \otimes \cdots) = \varphi(a_1)\varphi_2(a_2)\cdots\varphi_n(a_n) \tag{5.4}$$

This state is faithful if every φ_n is faithful. Thus, there is now a pair $(\mathcal{C}, \varphi)$ of a C^*-algebra and a state. By then performing the GNS construction, one obtains the triplet $(\varrho_\varphi, \mathcal{H}_\varphi, \Omega_\varphi)$. Lastly, take the bicommutant:

$$\mathcal{M} := \varrho_\varphi(\mathcal{C})'' \subset \mathcal{B}(\mathcal{H}_\varphi) \tag{5.5}$$

where

$$\varrho_\varphi(\mathcal{C}) := \{\varrho(a) \,:\, a \in \mathcal{C}\} \tag{5.6}$$

Because the bicommutant is taken, $\mathcal{M}$ is a von Neumann algebra, and it is said that it is generated by an increasing sequence of finite-dimensional subalgebras. With the construction complete, the definition of hyperfinite algebras is now given.

Definition 5.1.1. A von Neumann algebra is called *hyperfinite* or *approximately finite-dimensional (AFD)* if it is generated by an increasing sequence of finite-dimensional subalgebras.

One might reasonably think that this method, by embedding matrix algebras and defining an infinite tensor product of states, seems rather arbitrary, and ask what the purpose of this specific procedure is. The point of using embeddings of matrix algebras is to ensure the final algebra can be reasonably approximated by finite-dimensional algebras, hence the name AFD.

As for the tensor product of states, the problem is that if one is not careful with infinite tensor products, the final Hilbert space on which a von Neumann algebra acts might be nonseparable, making it quite difficult to perform analysis on the algebra. Thus, the definition chosen for the infinite tensor product of states is meant to place restrictions on the Hilbert space generated by the GNS construction, so that it remained separable.

With these constructions explained, examples of Type II and Type III factors can now be given. The first example will run through the procedure again to illustrate how the matrix algebras and states interact. All examples after that will be presented with just the algebras and associated states used in the inductive limit, which is enough information to perform the construction.

Example 5.1.2. For the first example, one embed $M_n(\mathbb{C})$ into $M_{2n}(\mathbb{C})$ via:

$$\begin{aligned} M_n(\mathbb{C}) &\longrightarrow M_{2n}(\mathbb{C}) \\ a &\longmapsto \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \end{aligned}$$

Then, there is a series of inclusions:

$$M_2(\mathbb{C}) \hookrightarrow M_{2^2}(\mathbb{C}) \hookrightarrow \cdots \hookrightarrow M_{2^n}(\mathbb{C}) \hookrightarrow \cdots$$

and thus a C^*-algebra:

$$\mathcal{C} := \bigcup_{n\geq 1} M_{2^n}(\mathbb{C}) \tag{5.7}$$

and a faithful state:

$$\varphi(a) := \frac{1}{2^n}\mathrm{Tr}(a) \tag{5.8}$$

for $a \in M_{2^n}(\mathbb{C})$. By performing the GNS construction on $(\mathcal{C}, \varphi)$, one gets a representation $(\varrho_\varphi, \mathcal{H}_\varphi, \Omega_\varphi)$ for $\mathcal{C}$, at which point:

$$\mathcal{M} := \varrho(\mathcal{C})'' \subset \mathcal{B}(\mathcal{H})$$

is a von Neumann algebra. This von Neumann algebra is fairly special. It is a Type II_1 factor and, in fact, the only hyperfinite Type II_1 factor acting on a separable Hilbert space, up to *-isomorphism. See Chapter IV of [MN43] for the original work by Murray and von Neumann and [AP] for an updated presentation.

Example 5.1.3. Using $\mathcal{A}_n = M_3(\mathbb{C})$, with:

$$\varphi_n(\cdot) = \mathrm{Tr}(\rho\,\cdot)\ , \qquad \rho = \frac{1}{1+\lambda_1+\lambda_2}\mathrm{diag}(1,\lambda_1,\lambda_2)\ , \qquad \frac{\log(\lambda_1)}{\log(\lambda_2)} \notin \mathbb{Q} \tag{5.9}$$

one obtains a Type III_1 factor. In fact, one obtains the unique up to *-isomorphism hyperfinite Type III_1, called the Araki-Woods factor. This construction is given in [Haa16].

Example 5.1.4. By once again using $\mathcal{A}_n = M_2(\mathbb{C})$ as the matrix algebras, but with the states given by:

$$\varphi_n(a) = \mathrm{Tr}\left(\begin{bmatrix} \frac{\lambda}{\lambda+1} & 0 \\ 0 & \frac{1}{\lambda+1} \end{bmatrix} a\right)$$

for all $n \in \mathbb{N}$, and with $\lambda \in (0,1)$, one obtains a set of pairwise, non-isomorphic factors. These are in fact the Powers factors, which are Type III_λ, and originally given in Definition 4.2 of [Pow67].

Remark 5.1.5. With these examples, as well as $M_n(\mathbb{C})$ for Type I_n, and $\mathcal{B}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$, which is Type I_∞, most of the hyperfinite factors have been presented. The only ones missing are the unique Type II_∞ and the Type III_0 factors, called the Krieger factors. However, finding an inductive limit construction for either type is difficult. For Type II_∞, the difficulty arises from it being expressible as the tensor product of Tyype II_1 and Type I_∞ factors. Thus, anything that could be said about Type II_∞ can be said in terms of Type II_1 factors. For the Krieger factors, they derive from ergodic theory. However, it would be nice from a completeness standpoint to have said constructions.

5.2 Araki-Woods Factors

The next construction provided takes a different approach to generating algebras. Instead of embedding matrix algebras, this procedure starts with the tensor product of von Neumann algebras. As it turns out, when done properly, this procedure, if repeated infinitely, generates another class of von Neumann algebras.

Recalling from the beginning of this chapter the construction of the tensor product of two von Neumann algebras $\mathcal{M}_1 \otimes \mathcal{M}_2$, this process can be repeated a finite number of times without any complications. However, the situation becomes significantly more difficult when trying to generalize this procedure to infinite tensor products. The issue, much as with hyperfinite algebras, arises from the fact that doing this procedure over an infinite number of Hilbert spaces will typically result in a nonseparable Hilbert space. This is a delicate matter, but thankfully, von Neumann tackled the intricacies in [Neu39]. The methods by which he solved the technicalities of the Hilbert space are not covered here. Instead, a procedure developed by Araki and Woods in [AW68] is presented instead, one that makes significant use of von Neumann's work.

The procedure starts with defining the Hilbert space $\mathcal{H}$ that the final algebra will act on. To do that, consider the algebraic tensor product of a sequence of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$:

$$\bigotimes_{n\in\mathbb{N}} \mathcal{H}_n \tag{5.10}$$

which will form the equivalent of the space of sequences of vectors. To avoid creating a nonseparable Hilbert space, fix a special sequence of vectors $\{\Omega_n\}_{n\in\mathbb{N}}$ such that $\Omega_n \in \mathcal{H}_n$ for $n \in \mathbb{N}$,

$$0 < \prod_{n\in\mathbb{N}} ||\Omega_n|| < \infty \tag{5.11}$$

and consider the product vector

$$\bigotimes_{n\in\mathbb{N}} \Omega_n := \Omega_0 \otimes \Omega_1 \otimes \cdots \tag{5.12}$$

The condition in Equation 5.11 ensures that the product vector will have finite norm. Then define $\mathcal{H}$ as the space generated by combinations of vectors of the form:

$$\bigotimes_{n\in\mathbb{N}} h_n := h_0 \otimes h_1 \otimes \cdots \tag{5.13}$$

satisfying

$$\sum_{n\in\mathbb{N}} ||\Omega_n - h_n|| < \infty \tag{5.14}$$

or, equivalently:

$$\sum_{n\in\mathbb{N}} |1 - \langle h_n, \Omega_n\rangle| < \infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} |1 - ||h_n||\ | < \infty \tag{5.15}$$

Of course, this is not a Hilbert space without an inner product. For $\mathcal{H}$, impose the following inner product:

$$\left\langle \bigotimes_{n\in\mathbb{N}} h_n, \bigotimes_{n\in\mathbb{N}} k_n \right\rangle := \prod_{n\in\mathbb{N}} \langle h_n, k_n\rangle \tag{5.16}$$

All these conditions ensure $\mathcal{H}$ is separable. Next, the operators are introduced.

Definition 5.2.1. Consider the canonical injection:

$$\begin{aligned} \iota \,:\, \mathcal{B}(\mathcal{H}_n) &\to \mathcal{B}(\mathcal{H}) \\ a &\mapsto \bigodot_{m\neq n} \mathbb{1}_m \odot a \end{aligned}$$

where $\mathbb{1}_m$ is the identity operator for $\mathcal{B}(\mathcal{H}_m)$. Given a subset $\mathcal{A} \subset \mathcal{B}(\mathcal{H}_n)$, we define:

$$\iota(\mathcal{A}) = \{\iota(a) \,:\, a \in \mathcal{A}\} \tag{5.17}$$

With all these definitions in place, the countable tensor product of von Neumann algebras can be defined.

Definition 5.2.2. Given the algebraic tensor product

$$\mathcal{H} = \bigodot_{n\in\mathbb{N}} (\mathcal{H}_n, \Omega_n) \tag{5.18}$$

and von Neumann algebras $\mathcal{M}_n \subset \mathcal{B}(\mathcal{H}_n)$, we define the tensor product:

$$\bigotimes_{n\in\mathbb{N}} \mathcal{M}_n := \{\iota(\mathcal{M}_n)\}'' \tag{5.19}$$

This procedure generates von Neumann algebras, denoted $\mathcal{M}(\mathcal{H}_n, \mathcal{M}_n, \Omega_n)$, but the algebras of interest are factors. Thus, the von Neumann algebras $\mathcal{M}_n$ are required to be Type I_n, with $2 \leq n < \infty$. However, the class of algebras of interest are slightly more general, as seen in the following definition.

Definition 5.2.3. Let every von Neumann algebra $\mathcal{M}_n$ in Definition 5.2.2 be a Type I_n factor, with $2 \leq n < \infty$. A factor $\mathcal{N}$ is called an *Araki-Woods factor* or *Infinite Tensor Products of Factors of Type I (ITPFI) factor* if it is unitarily equivalent to a factor of the form $\mathcal{M} := \mathcal{M}(\mathcal{H}_n, \mathcal{M}_n, \Omega_n)$; that is, $\mathcal{N} = U\mathcal{M}U^*$ for some unitary operator $U \in \mathcal{B}(\mathcal{H})$.

The point of introducing this procedure is that every Araki-Woods factor is in fact a hyperfinite factor, as noted in [AW68] below Definitoin 2.6. Thus, for Araki-Woods factors, there are two equivalent methods of characterizing them, using either an embedding procedure or tensor products.

This next fact is not strictly necessary for further developments, but it provides yet more connections to Tomita-Takesaki theory.

Proposition 5.2.4 ([AW68], Lemma 3.15)**.** Given an Araki-Woods factor $\mathcal{M}(\mathcal{K}_n, \mathcal{N}_n, \Omega'_n)$, there exists a factor $\mathcal{M}(\mathcal{H}_n, \mathcal{M}_n, \Omega_n)$, where $\Omega_n \in \mathcal{H}_n$ are cyclic separating vectors, such that $\mathcal{M}(\mathcal{K}_n, \mathcal{N}_n, \Omega'_n)$ and $\mathcal{M}(\mathcal{H}_n, \mathcal{M}_n, \Omega_n)$ are *-isomorphic and $S(\mathcal{M}) = S(\mathcal{N})$.

Be aware that the original statement of this lemma was done using a different invariant r_∞, defined by Araki and Woods for the purpose of analyzing ITPFI factors. However, for ITFPI factors specifically, this invariant is equal to the Connes invariant:

$$S(\mathcal{M}) = r_\infty(\mathcal{M}) \tag{5.20}$$

5.3 Second Quantization

We now arrive at he algebras we are most interested in, the second quantization algebras. They originate from the development of algebraic quantum field theory and, specifically, from a desire to provide a representation for the canonical commutation relations. The details of this development are not given here, but those interested should check Chapter 21.3 of [BLT75].

To construct these algebras, start with a real subspace $V \subset \mathcal{H}$ of a Hilbert space $\mathcal{H}$ and the bosonic Fock space $\exp_s(\mathcal{H})$, as given in Definition 2.1.13. A particular subset of vectors from $\exp_s(\mathcal{H})$ need to be considered.

Definition 5.3.1. The *coherent vectors* in $\exp_s(\mathcal{H})$ are those of the form:

$$\exp(h) := \bigoplus_{n=0}^{\infty} \frac{h^{\otimes n}}{\sqrt{n!}} \tag{5.21}$$

These vectors form a total family in $\exp_s(\mathcal{H})$; that is, their span forms a dense subset of $\exp_s(\mathcal{H})$. One vector is of particular importance.

Definition 5.3.2. Given a bosonic Fock space $\exp_s(\mathcal{H})$, the *vacuum vector* is given by $\Omega := \exp(0)$.

Next, a specific class of operators needs to be introduced:

Definition 5.3.3. Given a vector $h \in \mathcal{H}$ and the bosonic Fock space $\exp_s(\mathcal{H})$, the *Weyl operator* $W(h)$, which is defined on the coherent vectors of $\exp_s(\mathcal{H})$, is defined by how it maps the vacuum vector Ω:

$$W(h)\Omega := e^{-\frac{1}{4}||h||^2} \exp\left(\frac{i}{\sqrt{2}}h\right) \tag{5.22}$$

Composition of Weyl operators is defined by the *canonical commutation relations*:

$$W(h)W(k) := e^{-\frac{i}{2}\mathrm{Im}\langle h,k\rangle} W(h+k) \tag{5.23}$$

for all $h, k \in \mathcal{H}$.

Imposing Equations 5.22 and 5.23 means Weyl operators are well defined. As noted, they are initially defined on the dense set spanned by the total family of coherent vectors. They are closable and, in fact, can be closed and extended to operators defined on all of $\exp_s(\mathcal{H})$. A von Neumann algebra is then obtained as follows:

Definition 5.3.4. For a real subspace $V \subset \mathcal{H}$, the *second quantization algebra* is given by:

$$\mathcal{M}(V) := \{W(h) \,:\, h \in V\}'' \tag{5.24}$$

with $W(h)$ being the Weyl operators given in Definition 5.3.3.

Although the initial definition for second quantization algebras only needed real subspaces, the following result shows that one can, without loss of generality, restrict to closed real subspaces.

Lemma 5.3.5 ([NÓ17], Lemma 6.2(iii))**.** Given a real subspace $V \subset \mathcal{H}$ of a Hilbert space and the associated second quantization algebra $\mathcal{M}(V)$,

$$\mathcal{M}(V) = \mathcal{M}(\overline{V}) \tag{5.25}$$

There is a great deal of duality between second quantization algebras and their closed real subspaces, especially if those subspaces turn out to be standard. The relevant results are summarized here.

Theorem 5.3.6. Given a closed real subspace $V \subset \mathcal{H}$, the following hold:

(ii) The vacuum vector Ω is cyclic for $\mathcal{M}(V)$ if and only if $V + iV$ is dense in $\mathcal{H}$ ([NÓ17], Lemma 6.2(iv))

(iii) Ω is separating for $\mathcal{M}(V)$ if and only if $V \cap iV = \{0\}$ ([NÓ17], Lemma 6.2(v))

(iv) Ω is cyclic and separating if and only if V is standard ([NÓ17], Lemma 6.2(vi))

(v) $\mathcal{M}(V)' = \mathcal{M}(V')$ ([NÓ17], Theorem 6.4(iv))

(vi) $\mathcal{M}(V) \cap \mathcal{M}(V') = \mathcal{M}(V \cap V')$. In particular, $\mathcal{M}(V)$ is a factor if and only if $V \cap V' = \{0\}$ ([NÓ17], Theorem 6.4(v))

Theorem 5.3.6(vi) in particular provides a biconditional for a second quantization algebra to be a factor. Because of this condition, the following terminology is adopted.

Definition 5.3.7. Closed real subspace V satisfying the equality $V \cap V' = \{0\}$ are called *factor subspaces*.

Thus, second quantization factors act as natural algebras when considering the duality between closed real subspaces and factors, with the duality involving standard subspaces manifesting if Ω turns out to be cyclic and separating. What is especially useful is that the condition for a von Neumann algebra to be a factor can be entirely formulated in terms of real subspaces instead. However, not only is the necessary and sufficient conditions for a second quantization algebra to be a factor known, it is in fact known what kind of factors can be obtained as second quantization algebras. This is via Theorem 1.3(ii and iii) in [FG94]. However, to present this result, another important class of operators acting on $\exp_s(\mathcal{H})$ must be defined: the second quantization operators.

Definition 5.3.8. Given a Hilbert space $\mathcal{H}$ and its associated bosonic Fock space $\exp_s(\mathcal{H})$, as well as a closed, densely-defined operator M on $\mathcal{H}$, the *second quantization operator* associated to M is given by:

$$\exp(M) := \bigoplus_{n=0}^{\infty} M^{\otimes n} \tag{5.26}$$

and acts on $\exp_s(\mathcal{H})$.

Modular objects associated with the second quantization factor can be derived from the modular objects associated with the original standard subspace.

Theorem 5.3.9 ([FG94], Thm 1.1)**.** Given a standard subspace V, its modular objects (Δ_V, J_V) and involution $S_V = J_V \Delta_V^{1/2}$, and its associated second quantization algebra $\mathcal{M}(V)$, the modular objects of $\mathcal{M}(V)$ are:

$$\begin{aligned} J_{\mathcal{M}(V)} &= \exp(J_V) \\ \Delta_{\mathcal{M}(V)} &= \exp(\Delta_V) \\ S_{\mathcal{M}(V)} &= \exp(S_V) = J_{\mathcal{M}(V)} \Delta_{\mathcal{M}(V)}^{1/2} \end{aligned}$$

With this theorem in tow, the result constraining the Type of second quantization factors possible can now be given.

Theorem 5.3.10 ([FG94], Thm. 1.3(ii, iii))**.** Given a standard factor subspace V and its associated bosonic Fock space $\mathcal{M}(V)$, one of the following is true.

- The second quantization factor $\mathcal{M}(V)$ is Type I if and only if $\Delta_V\big|_{[0,1]}$ is trace class.
- If $\mathcal{M}(V)$ is not Type I, then it is Type III.

Here, $\Delta_V\big|_{[0,1]}$ derives from Borel Functional Calculus, which is not explained here. See Chapter 5 of [KR97] for details.

Remark 5.3.11. As a fun aside, the previously mentioned Powers factors can be represented as second quantization factors, as shown in Section 2 of [FG94].

Although the duality present with second quantization algebras provides a method for passing between closed real subspaces and von Neumann algebras, not every von Neumann algebras is *-isomorphic to a second quantization algebra, with ITPFI and hyperfinite algebras providing a myriad of examples. This duality breaks down when considering arbitrary von Neumann algebras.

Given that second quantization algebras provide the desired duality between factors and standard subspaces, one could ask why so much time was spent on hyperfinite factors and ITPFI factors. However, the following theorem justifies the work.

Theorem 5.3.12 ([FG94], Thm. 1.3(i))**.** Second quantization factors are Araki-Woods factors.

Thus, by combining Theorem 5.3.12 with the fact that Araki-Woods factors are hyperfinite, there are the following increasingly large classes of factors:

$$\text{Second quantization} \subset \text{ITPFI} \subset \text{Hyperfinite}$$

Thus, not only can second quantization algebras be characterized in terms of Weyl operators and Fock spaces, but also in terms of infinite tensor products of the matrix algebras, or via embedding matrix algebras. This is important, as the original construction of second quantization algebras can seem quite opaque without the background motivation of the canonical commutation relations. However, now, one can work with them as ITPFI factors or hyperfinite factors.

With all these constructions complete, it is time to return to the original target, Wiesbrock's theorem.

5.4 Half Sided Modular Inclusions

Given a von Neumann algebra $\mathcal{M}$, a von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$, and a shared cyclic and separating vector Ω, the standard subspace $V_{\mathcal{N}} := \overline{\mathcal{N}_h \Omega}$ is associated to $\mathcal{N}$, similarly to how $V_{\mathcal{M}}$ is associated to $\mathcal{M}$.

Definition 5.4.1. Let Ω be a simultaneous cyclic and separating vector of von Neumann algebras $\mathcal{N} \subset \mathcal{M}$. Then the triplet $(\mathcal{M}, \mathcal{N}, \Omega)$ is called a $\pm$*half-sided modular (hsm) inclusion* of von Neumann algebras if:

$$\Delta_{\mathcal{M}}^{-it} \mathcal{N} \Delta_{\mathcal{M}}^{it} \subset \mathcal{N} \quad \pm t \geq 0 \tag{5.27}$$

This is the original definition given for hsm inclusion, with the version for standard subspaces coming later. Because Tomita-Takesaki theory allows one to pass from von Neumann algebras to standard subspaces, and second quantization provides, to a certain degree, an inversion of that procedure, there is the following correspondence.

Theorem 5.4.2 ([NÓ17], Lemma 4.9)**.** If $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ are von Neumann algebras with common cyclic and separating vector $\Omega \in \mathcal{H}$, then $(\mathcal{M}, \mathcal{N}, \Omega)$ is a $\pm$hsm inclusion if and only if the corresponding standard subspaces $V_{\mathcal{N}} \subset V_{\mathcal{M}}$ form a $\pm$hsm inclusion.

This correspondence between hsm inclusions of von Neumann algebras and standard subspaces extends to any results on hsm inclusions. Specifically, Wiesbrock's Theorem's original formulation, given now, was in terms of von Neumann algebras.

Theorem 5.4.3 (Wiesbrock's Theorem)**.** Given a hsm inclusion $(\mathcal{M}, \mathcal{N}, \Omega)$ over $\mathcal{H}$ and the associated standard subspaces and modular objects, then there exists a one-parameter, unitary group $\mathcal{U}(s)$ on $\mathcal{H}$ with $s \in \mathbb{R}$, a generator

$$\frac{1}{2\pi}\big(\log(\Delta_{\mathcal{N}}) - \log(\Delta_{\mathcal{M}})\big) \geq 0 \tag{5.28}$$

and:

(i) $\Delta_{\mathcal{M}}^{it}\mathcal{U}(s)\Delta_{\mathcal{M}}^{-it} = \Delta_{\mathcal{N}}^{it}\mathcal{U}(s)\Delta_{\mathcal{N}}^{-it} = \mathcal{U}\left(e^{-2\pi t}s\right)$ for all $s, t \in \mathbb{R}$.

(ii) $J_{\mathcal{N}}J_{\mathcal{M}} = \mathcal{U}(2)$

(iii) $\forall t \in \mathbb{R}, \Delta_{\mathcal{N}}^{it} = \mathcal{U}(1)\Delta_{\mathcal{M}}^{it}\mathcal{U}(-1)$

(iv) $\mathcal{N} = \mathcal{U}(1)\mathcal{M}\mathcal{U}(-1)$

(v) If Ω is cyclic for $\mathcal{N}' \cap \mathcal{M}$ and $\mathcal{M}$ is a factor, then $\mathcal{M}$ is of Type III_1

The original theorem was stated in [Wie93]. However, a gap in the proof was found, later filled in [AZ05]. The result we are particularly interested in is Theorem 5.4.3(v), which appears as Theorem 12 in [Wie93], which uses a special case of Theorem 4 in [Lon82]. However, this result itself makes use of Theorem 3 in [Lon79]. One could reasonably ask why so much time was spent getting to this result, and what is so special about Theorem 5.4.3(v). What is special about this result is that it has no reformulation in terms of standard subspaces or antiunitary representations. Every other result has been translated, but this one has yet to be given an equivalent.

Another reason this discrepancy is intriguing goes back to Theorem 5.3.10. This result dictates that second quantization factors can only be Type I or Type III. The fact that there is a known condition on second quantization factors to make them Type III, and especially given it is in terms of the modular operator of the associated standard factor subspace V, hints at this being a special case of the more general result in Theorem 5.4.3.

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