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C. S. Fang

S. N. Wang

W. Harrison

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2. Two-Dimensional Finite Element Analysis

C. S. FANG AND S. N. WANG

Virginia Institute of Marine Science, Gloucester Point, Virginia 23062

W. HARRISON

Erindale College, University of Toronto, Clarkson, Ontario, Canada

Abstract. Two-dimensional finite element techniques are described that model closely the complicated fluctuations observed in the water table of an ocean beach. The use of triangular elements permits the specification of more realistic boundary conditions than the use of line elements in a one-dimensional model. Also, results obtained from the two-dimensional model for the region close to the ocean compare more favorably with field data than results obtained from the one-dimensional finite element model.

The object of this study was to improve on the one-dimensional groundwater flow model of *Harrison et al.* [1971] and to examine the efficacy of a two-dimensional finite element model with triangular elements. The use of finite element methods to attack boundary value field problems was anticipated by *Zienkiewiez and Cheung* [1965]. Later, *Zienkiewiez and Cheung* [1967] gave detailed analyses of the theory as well as examples of the application of the finite element method. The application of general variational principles to the groundwater flow equation did not occur until *Neuman and Witherspoon*'s [1970*b*, 1971] studies. The application of this method has been limited to steady flow [*Neuman and Witherspoon*, 1970*a*] until now. Modeling the movement of the beach groundwater when a free surface is involved requires complete solution of the unsteady equation. Studies by *Javandel and Witherspoon* [1969] and *France et al.* [1971] were helpful in this aspect of the application of the finite element method.

As mentioned in the one-dimensional model for groundwater flow in a sandy tidal beach [*Harrison et al.*, 1971], the hydrostatic assumption is critical over the region near the ocean boundary where the effects of tidal forces and seaward directed head gradient are important. A two-dimensional finite element model was necessary for modeling the effects of tidal fluctuations in this region.

EQUATIONS OF GROUNDWATER FLOW WITH A FREE SURFACE

The governing partial differential equation of an isotropic, homogeneous porous medium in two dimensions can be represented by

$$K\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = S_s \frac{\partial h}{\partial t} \qquad (1)$$

where S_s is the specific storage and K is the hydraulic conductivity.

The initial condition for the beach groundwater flow problem can be specified as

$$h(x, y, o) = h_o(x, y) \qquad (2)$$

$$\gamma(x, o) = \gamma_o(x) \qquad (3)$$

The boundary condition for the beach groundwater flow problem for a prescribed head on the left boundary A_1 is

$$h = H(t) \qquad (4)$$

and for a prescribed flux at the bottom boundary A_2 is

$$K \frac{\partial h}{\partial y} = -V(x, t) \qquad (5)$$

where V is defined as positive downward and

$\gamma(x, t)$ represents the equation of the free surface (Figure 1).

Two conditions must be satisfied on the free surface:

$$\gamma = h \tag{6}$$

$$K\left(\frac{\partial h}{\partial x} n_x + \frac{\partial h}{\partial y} n_y\right) = \left(I - S_y \frac{\partial \gamma}{\partial t}\right) n_y \tag{7}$$

where S_y and I are the specific yield and downward infiltration through the porous medium and n_x and n_y are the x and y directional cosines of the unit outward normal along the free surface.

Before finite element analysis is employed, the variation principles must be applied to find the functional of a particular minimizing function h; thus [*Neuman and Witherspoon*, 1970*b*, 1971]

$$\omega = \iint_R \left[\tfrac{1}{2}K\left(\frac{\partial h}{\partial x}\right)^2 + \tfrac{1}{2}K\left(\frac{\partial h}{\partial y}\right)^2 + S_s h \frac{\partial h}{\partial t}\right] dx\, dy + \int_{A_2} Vh\, dA - \int_{FS} h\left(I - S_y \frac{\partial h}{\partial t}\right) n_y\, dS \tag{8}$$

FINITE ELEMENT ANALYSIS

Assume that the flow region is divided into many triangular elements (Figure 1), where i, j, and m represent the first, second, and third nodes of an element e. The o' is the centroid of element e, and x' and y' are the element coordinates through the centroid, if no transformation angle is assumed. The total head h within each triangular element can be uniquely defined linearly by

$$h = \alpha_1 + \alpha_2 x + \alpha_3 y \tag{9}$$

When the coordinates and the total heads for the three nodes of each element are substituted into equation 9, h can be represented in terms of the coordinates and the total heads at three nodes in matrix form as

$$h = [N][h]^e \tag{10}$$

where

$$[h]^e = \begin{bmatrix} h_i \\ h_j \\ h_m \end{bmatrix} \tag{11}$$

and

$$[N] = [N_i N_j N_m] \tag{12}$$

where $N_I = (1/2\,\Delta)(a_I + b_I\, x + c_I\, y)$, $I = i$, j, and m, and Δ is the area of the triangular element. Letters a, b, and c with subscripts i, j, and m are short notations for

$$\begin{aligned} a_i &= x_j y_m - x_m y_j \\ b_i &= y_j - y_m \\ c_i &= x_m - x_j \end{aligned} \tag{13}$$

and the corresponding coefficients for each element are obtained by a cyclic permutation of the subscripts in the order i, j, and m.

Similarly, the time derivatives within each element can be represented by

$$\frac{\partial h}{\partial t} = [N]\left[\frac{\partial h}{\partial t}\right]^e \tag{14}$$

where

$$\left[\frac{\partial h}{\partial t}\right]^e = \begin{bmatrix} \left(\frac{\partial h}{\partial t}\right)_i \\ \left(\frac{\partial h}{\partial t}\right)_j \\ \left(\frac{\partial h}{\partial t}\right)_m \end{bmatrix} \tag{15}$$

When the whole region is divided into many small elements, the functional of the region can be expressed by the summation of the functional of all the individual elements as

$$\omega = \sum_{e=1}^{K_1} \omega_{FS}^e + \sum_{e=K_1+1}^{K_2} \omega^e + \sum_{e=K_2+1}^{M} \omega_{A_2}^e \tag{16}$$

where ω_{FS}^e, ω^e, and $\omega_{A_2}^e$ are the element functionals of the elements along the free surface, in the inside region, and along the boundary of the prescribed flux, respectively; K_1 is the total number of elements along the free surface, $K_2 - K_1$ is the total number of inner elements; $M - K_2$ is the total number of elements along the prescribed flux boundary; and M is the total number of elements in the whole region.

In (8) the integral along the prescribed flux boundary $\int_{A_2} Vh\, dA$ existed only in the element functionals $\omega_{A_2}^e$, and the integral along the free surface vanished except in ω_{FS}^e. Therefore the element functional can be rewritten as

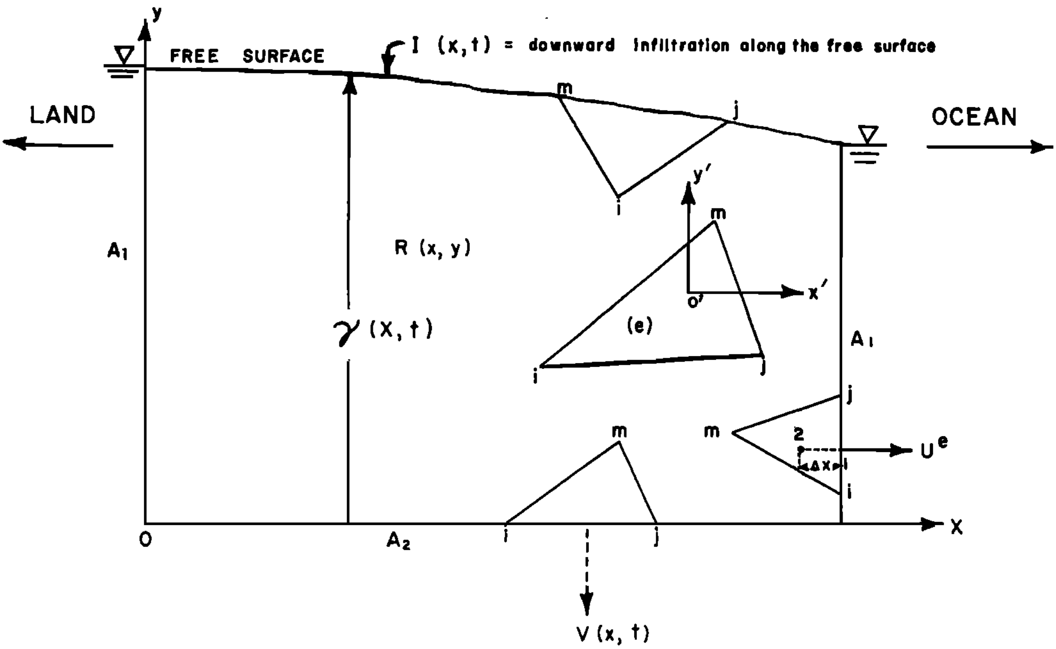

Fig. 1. Definition sketch for the triangular elements and nodes, initial conditions, and boundary conditions.

$$\omega^e = \iint_{R^e} \left[\tfrac{1}{2} K^e \left(\frac{\partial h^e}{\partial x} \right)^2 + \tfrac{1}{2} K^e \left(\frac{\partial h^e}{\partial y} \right)^2 + S_s^e h^e \frac{\partial h^e}{\partial t} \right] dx\, dy \tag{17}$$

$$\omega_{A_2}{}^e = \iint_{R^e} \left[\tfrac{1}{2} K^e \left(\frac{\partial h^e}{\partial x} \right)^2 + \tfrac{1}{2} K^e \left(\frac{\partial h^e}{\partial y} \right)^2 + S_s^e h^e \frac{\partial h^e}{\partial t} \right] dx\, dy + \int_{A_2} V^e h^e \, dA \tag{18}$$

$$\omega_{FS}{}^e = \iint_{R^e} \left[\tfrac{1}{2} K^e \left(\frac{\partial h^e}{\partial x} \right)^2 + \tfrac{1}{2} K^e \left(\frac{\partial h^e}{\partial y} \right)^2 + S_s^e h^e \frac{\partial h^e}{\partial t} \right] dx\, dy - \int_{FS} h^e \left(I^e - S_y{}^e \frac{\partial h^e}{\partial t} \right) n_y{}^e \, dS \tag{19}$$

where the superscript e indicates the parameters of an element e under consideration and R^e means to take the area integral of the element being considered.

If the function h is defined uniquely and continuously throughout the region, then the functional can be minimized with respect to all nodal values of the total head h_i; that is,

$$\frac{\partial \omega}{\partial h_i} = \sum_{e=1}^{K_1} \frac{\partial \omega_{FS}{}^e}{\partial h_i} + \sum_{e=K_1+1}^{K_2} \frac{\partial \omega^e}{\partial h_i} + \sum_{e=K_2+1}^{M} \frac{\partial \omega_{A_2}{}^e}{\partial h_i} = 0 \tag{20}$$

If the linear variation of h between two adjacent nodes of an element and the uniform flux through the boundary of an element along A_2 are assumed and there are N nodes in the whole region, then (20) becomes a linear system of N equations at any particular time step, which can be written as

$$[P][h]_t + [Q]\left[\frac{\partial h}{\partial t}\right]_t = [R] \tag{21}$$

where $[P]$ and $[Q]$, called the overall matrices, are $N \times N$ square matrices; $[h]_t$ is an $N \times 1$ row matrix formed by the total head of all the nodes at the particular time step being considered, and so is $[\partial h/\partial t]_t$; and $[R]$ is an $N \times 1$ constant row matrix obtained owing to the existence of the prescribed flux and the infiltration flux.

When the central finite difference approximation for $[\partial h/\partial t]_t$ is made as

$$\left[\frac{\partial h}{\partial t}\right]_t = -\left[\frac{\partial h}{\partial t}\right]_{t-\Delta t} + ([h]_t - [h]_{t-\Delta t})\frac{2}{\Delta t} \tag{22}$$

then (21) becomes

$$[D][h]_t = [E] \tag{23}$$

where

$$[D] = [P] + \frac{2}{\Delta t}[Q] \tag{24}$$

$$[E] = [R] + [Q]\left(\frac{2}{\Delta t}[h]_{t-\Delta t} + \left[\frac{\partial h}{\partial t}\right]_{t-\Delta t}\right)$$

and Δt is the time step increment. Then the total head of all nodes h_i, $i = 1, 2, \cdots, N$, should be found from (23) instead of (21). If there are just N' nodes ($N' < N$) with an unknown total head, and these are numbered first, then only the first N' nonredundant equations in (23) are useful. The prescribed values of all given h_i should be substituted into (23) and the constant terms moved to the right-hand side of the equation to get a new linear system of N' equations:

$$[D'][h]_t = [E'] \tag{25}$$

where

$$D_{ij}' = D_{ij} \qquad i = 1, 2, \cdots, N' \qquad j = 1, 2, \cdots, N'$$

$$E_i' = E_i - \sum_{j=N'+1}^{N} D_{ij}h_j \qquad i = 1, 2, \cdots, N' \tag{26}$$

To obtain overall matrices and the total heads of all nodes shown in the above procedures, the terms $\partial\omega_{FS}{}^e/\partial h_i$, $\partial\omega^e/\partial h_i$, and $\partial\omega_{A_2}{}^e/\partial h_i$ would be obtained first as expressed in (20). Equation 20 is the source of element matrices and the reason why element matrices must be found before overall matrices.

The detailed formulation of the element matrices can be obtained in *Zienkiewicz and Cheung* [1967]. The results are as follows.

For inner elements

$$\left[\frac{\partial \omega}{\partial h}\right]^e = \begin{bmatrix} \dfrac{\partial \omega^e}{\partial h_i} \\ \dfrac{\partial \omega^e}{\partial h_j} \\ \dfrac{\partial \omega^e}{\partial h_m} \end{bmatrix} = [BC][h]^e + [SKN]\left[\frac{\partial h}{\partial t}\right]^e \tag{27}$$

For elements along the boundary of prescribed flux

$$\left[\frac{\partial \omega}{\partial h}\right]^e_{A_2} = [BC][h]^e + [SKN]\left[\frac{\partial h}{\partial t}\right]^e + \tfrac{1}{2}V^e\,\Delta x^e \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{28}$$

For elements along the free surface

$$\left[\frac{\partial \omega}{\partial h}\right]^e_{FS} = [BC][h]^e + [SKN]\left[\frac{\partial h}{\partial t}\right]^e + \frac{Sy^e}{6}(x_j - x_m)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\left[\frac{\partial h}{\partial t}\right]^e - \frac{I^e}{2}(x_j - x_m)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{29}$$

where

$$[BC] = \frac{K^e}{4\Delta}\begin{bmatrix} b_ib_i & b_ib_j & b_ib_m \\ b_jb_i & b_jb_j & b_jb_m \\ b_mb_i & b_mb_j & b_mb_m \end{bmatrix} + \frac{K^e}{4\Delta}\begin{bmatrix} C_iC_i & C_iC_j & C_iC_m \\ C_jC_i & C_jC_j & C_jC_m \\ C_mC_i & C_mC_j & C_mC_m \end{bmatrix} \tag{30}$$

and

$$[SKN] = S_s^e \begin{bmatrix} \iint_{R^e} N_i N_i \, dx\, dy & \iint_{R^e} N_i N_j \, dx\, dy & \iint_{R^e} N_i N_m \, dx\, dy \\ \iint_{R^e} N_j N_i \, dx\, dy & \iint_{R^e} N_j N_j \, dx\, dy & \iint_{R^e} N_j N_m \, dx\, dy \\ \iint_{R^e} N_m N_i \, dx\, dy & \iint_{R^e} N_m N_j \, dx\, dy & \iint_{R^e} N_m N_m \, dx\, dy \end{bmatrix} \quad (31)$$

and Δx is the length of the side of an element along the prescribed flux boundary.

In deriving the above element matrices, we numbered the nodes of elements along the free surface and the boundary of the prescribed flux as shown in Figure 1. Also, we assumed that h varied linearly from one node to another.

BOUNDARY CONDITIONS

Mathematical modeling of groundwater flow requires a knowledge of hydraulic conductivity and specific storage as well as appropriate boundary conditions. The lack of precise measurements of these two geohydrologic parameters will cause uncertainty in preparing mathematical models. *Freeze and Witherspoon* [1968] tried to estimate the parameters by a trial and error process of matching calculated and measured data at various points. *Kleinecke* [1971] attempted to employ linear programing to achieve the same purpose. Hydrologic records over large regions were needed to use even the simplest forms of boundary conditions. Therefore, it is necessary to replace the semi-infinite, unconfined aquifer with a finite region.

There are two types of boundaries, prescribed head and prescribed flux boundaries. Such boundaries are used in most studies; however, for the beach groundwater problem, if a finite region is selected, portions of the boundaries will not possess such simple forms. Both the bottom boundary and the ocean (right) boundary (Figure 1) conditions are complicated functions of space and time due to the seaward directed head gradient and tidal fluctuations. The assumption of the existence of the hydrostatic condition required for the one-dimensional model on the ocean boundary [*Harrison, et al.,* 1971] was somewhat weak; it may be replaced here by a prescribed boundary condition (see below).

The drainage velocity $V(x, t)$, calculated from the field data [*Harrison et al.,* 1971] by the finite difference method, was used to impose the effects of the tide on each element of the bottom boundary. If there is any infiltration in the system it will be lumped into this term.

For the present two-dimensional model, the landward (left) boundary was assumed to be hydrostatic. The ocean boundary was approximated by imposing a uniform horizontal flux along the boundary of each element (Figure 1).

This horizontal flux could be approximated by Darcy's law as

$$U^e = -\frac{K^e}{\Delta x}(h_1 - h_2) \quad (32)$$

where point 1 is any point on the boundary and Δx is selected as small as possible. Because point 2 has the same altitude as point 1 and (10) holds for any element, the uniform boundary flux becomes

$$U^e = -\frac{K^e}{\Delta x}[b][h]^e \quad (33)$$

The effects of this flux must be considered, as shown in (28), to be expressed implicitly in terms of the unknown nodal heads before the given nodal heads are substituted into (23). The same procedure is also followed for the left boundary, whether a prescribed head or a hydrostatic condition is imposed.

FREE SURFACE

The most difficult problem with the free surface boundary for the beach groundwater problem lies in treating the free surface as a moving boundary to preserve the accuracy of the model. At each time step the iteration method was chosen for relocating the position of the free surface and then recalculating its element matrices. Over several time steps, the position of the free surface can change considerably. It is necessary, therefore, to reset nodes and elements or shift nodes every few steps.

For simplification, shifting was restricted to vertical coordinates; thus

$$yN_i = y_i + (yN_j - y_j)$$

where yN_i and y_i represent the y coordinate of node i under consideration, before and after shifting, and yN_j and y_j represent the new and old y coordinates of the free surface node j, which is directly above the inner node i. Then, the new total head h_i' after a shift can be found by averaging the values obtained from equation 10 (Figure 2):

$$\begin{aligned} h_i' = \frac{1}{4\Delta^{(e)}} [(a_i &+ b_i x_i + C_i yN_i)h_i \\ &+ (a_j + b_j x_i + c_j yN_i)h_j \\ &+ (a_m + b_m x_i + c_m yN_i)h_m]^{(e)} \\ &+ \frac{1}{4\Delta^{(e+1)}} [(a_i + b_i x_i + c_i yN_i)h_i \\ &+ (a_k + b_k x_i + c_k yN_i)h_k \\ &+ (a_m + b_m x_i + c_m yN_i)h_m]^{(e+1)} \end{aligned} \tag{34}$$

where the superscripts (e) and $(e + 1)$ correspond to transformed elements e and $e + 1$, respectively.

To employ (34), all elements around any shifted node should be given; therefore the numbering system has to be read in as one of initial setup.

APPLICATION OF METHOD

To facilitate computer programing, the nodes on the free surface were numbered first (nodes 1–19, Figure 3). Then all nodes with an unknown total head were numbered (nodes 20–69), after which nodes with a given head were numbered (nodes 70–74). Similarly, the elements along the free surface were numbered first, from 1 to 40, in the element-numbering system. The elements along the free surface should be smaller than the other elements to obtain accurate results.

Once the node element configuration is decided, data cards for the numbering system should be prepared. The initial total head and the coordinates are also needed. The starting time was 0645 EDT, August 11, 1969 [*Harrison and Fausak*, 1970].

The computer program continuously seeks the element matrices, which are based on (30) and (31), before obtaining the overall matrices (21). Then it finds the time derivatives of the total head at time $t - \Delta t$ before reaching (23). The prescribed values of h, which are obtained from the field data [*Harrison and Fausak*, 1970], are then applied on the nodes of the landward boundary (nodes 71–74, Figure 3), where the hydrostatic state is assumed to exist. A prescribed value of h is also applied at the upper node (70) of the ocean boundary, where the hydrostatic condition does not exist.

The prescribed head values were substituted into (23) to obtain a linear system of simultaneous equations such as (25). The equations were solved by the elimination method, the largest pivotal divisor being used to obtain the total head for all nodes at any time step under consideration. The program then proceeded to consider the effects of the moving free surface. The necessity of resetting the node element configuration was checked every five cycles. The results of each time step were used as initial values for the next time step. This procedure was followed as long as desired.

Fourier coefficients for bottom drainage velocities were read at the initial setup; a subroutine was called to find the drainage velocities from the given coefficients for each time step. The following data were also read in: total nodes, 74; total elements, 110; porosity, 34%; hydraulic conductivity, 0.014 cm/sec; specific storage, 0.003125 l/cm (liters per centimeter); and time increment for each step, 15 minutes.

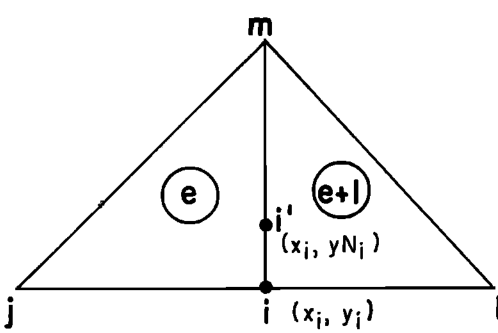

Fig. 2. Definition sketch for shifting vertical coordinates.

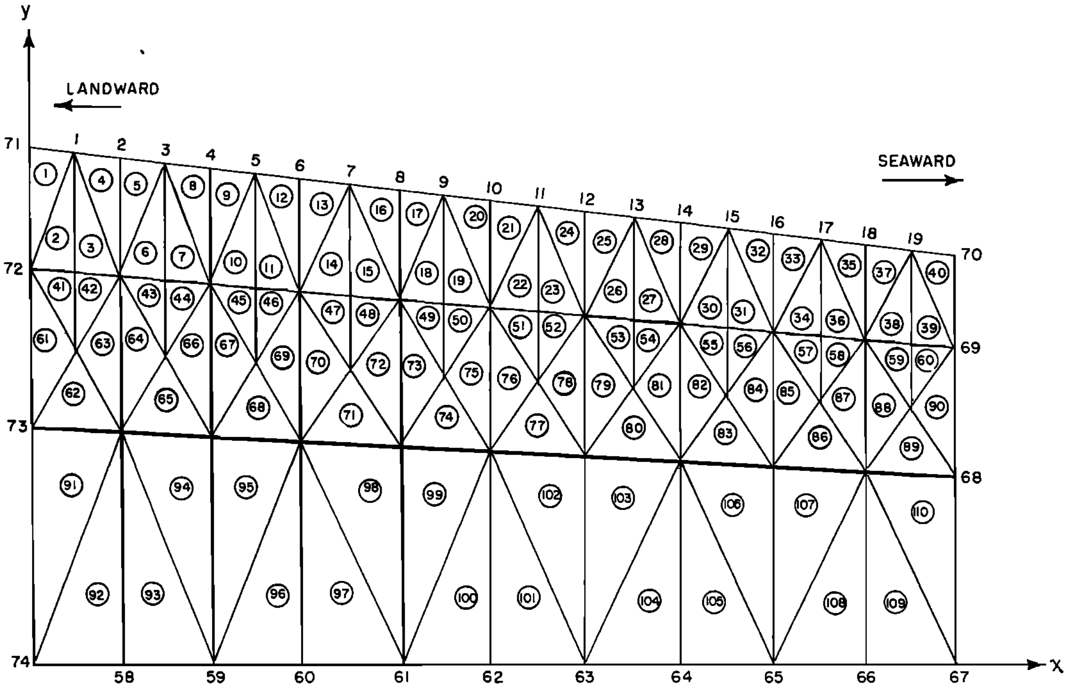

Fig. 3. Definition sketch for numbering nodes and elements.

The storage coefficient may be considered equal to the specific yield for an unconfined flow with a free surface [*Chow,* 1964]. Since only field tests of porosity and hydraulic conductivity were made, Chow's Figure 13-2 was chosen to find the specific yield. This was found to be 25% for the given porosity. The specific storage was taken as 0.003125 l/cm for an average flow region assumed to be 80 cm thick by virtue of the fact that the storage coefficient is equal to the product of the specific storage and the thickness of the aquifer when the aquifer is homogeneous and uniformly thick [*Jacob,* 1940, 1950].

RESULTS AND DISCUSSION

As shown in Figure 4, the two-dimensional finite element method has provided an accurate solution for the complicated beach water table fluctuations of groundwater flow. A compromise decision was made relative to the assumed positions of the boundaries, the value of the specific storage, and the average drainage velocity. A Fourier series was used to describe the mean regional drainage velocity characteristics and the beach water table response to the input tidal fluctuations. Comparison of the results (Figure 4) for the one-dimensional case, the two-dimensional case, and the field data indicates that the two-dimensional finite element method is more accurate for modeling the fluctuations of the beach groundwater table than the one-dimensional method.

Even though one-dimensional field data were used as the boundary conditions for the two-dimensional case, the results still exhibit less fluctuation after many time steps near the ocean boundary (Figure 4), where the effects of the tidal fluctuations are large. The small discrepancies (Figure 4) can be further reduced by using smaller elements over this region, since no matter what combinations of element sizes are used, a system of linear matrix equations will finally result from the two-dimensional finite element method. For the one-dimensional case, a system of nonlinear functions was obtained, and the equations were solved by the Newton-Raphson iteration method [*Harrison et al.,* 1971]. The Newton-Raphson method, according to our testing, seems restricted to

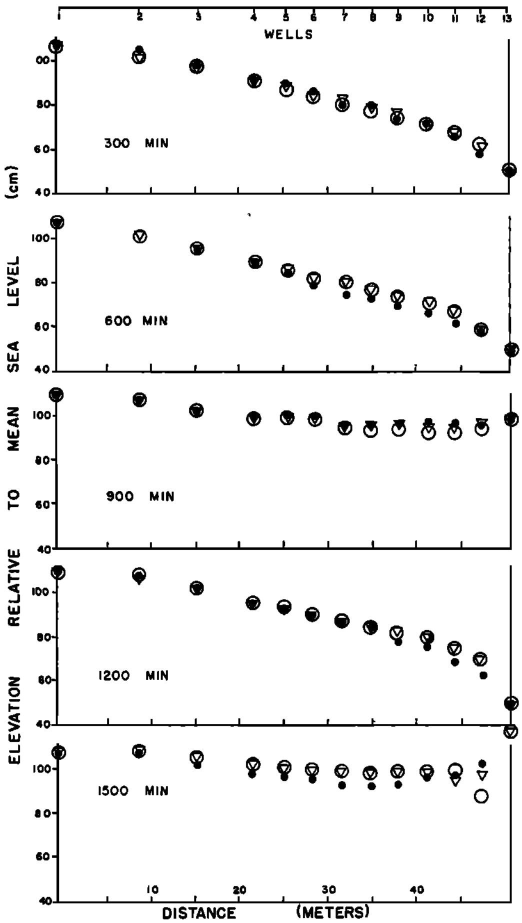

Fig. 4. Comparison between field data (solid circles), one-dimensional model results (open circles), and two-dimensional model output (triangles).

elements of even length; otherwise the iteration solution would easily become divergent.

The small differences existing between the field data and the theoretical results can be attributed to the effects of variables in (2) of *Harrison et al.* [1971, p. 1315]. The effects of capillarity and groundwater density gradients are probably also important. It also seems certain that if the specific storage can be measured precisely, the model results will more closely parallel the field data.

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REFERENCES

Chow, V. T., *Handbook of Applied Hydrology*, McGraw-Hill, New York, 1964.

France, P. W., C. J. Parekh, J. C. Peters, and C. Taylor, Numerical analysis of free surface seepage problems, *J. Irrig. Drain. Div., Amer. Soc. Civil Eng., 97*(1), 165–178, 1971.

Freeze, R. A., and P. A. Witherspoon, Theoretical analysis of regional groundwater flow, 3, Quantitative interpretations, *Water Resour. Res., 4*(3), 581–590, 1968.

Harrison, W., and L. E. Fausak, A time series from the beach environment—2, *Data Rep. 7*, 1–96, Va. Inst. of Mar. Sci., Gloucester Point, 1970.

Harrison, W., C. S. Fang, and S. N. Wang, Groundwater flow in a sandy tidal beach, 1, One-dimensional finite element analysis, *Water Resour. Res.*, 7(5), 1313–1322, 1971.

Jacob, C. E., The flow of water in an elastic artesian aquifer, *Eos Trans. AGU*, 574–586, 1940.

Jacob, C. E., *Engineering Hydraulics*, edited by H. Rouse, chap. 5, pp. 321–386, John Wiley, New York, 1950.

Javandel, I., and P. A. Witherspoon, A method of analyzing transient fluid flow in multilayered aquifers, *Water Resour. Res., 5*(4), 856–869, 1969.

Kleinecke, D., Use of linear programing for estimating geohydrologic parameters of groundwater basins, *Water Resour. Res.*, 7(2), 367–374, 1971.

Neuman, S. P., and P. A. Witherspoon, Finite element method of analyzing steady seepage with a free surface, *Water Resour. Res., 6*(3), 889–897, 1970*a*.

Neuman, S. P., and P. A. Witherspoon, Variational principles for confined and unconfined flow of groundwater, *Water Resour. Res., 6*(5), 1376–1382, 1970*b*.

Neuman, S. P., and P. A. Witherspoon, Variational principles for fluid flow in porous media, *J. Eng. Mech. Div., Amer. Soc. Civ. Eng., 97*(2), 359–374, 1971.

Zienkiewiez, O., and Y. K. Cheung, Finite elements in the solution of field problems, *Engineering*, 507–510, 1965.

Zienkiewiez, O., and Y. K. Cheung, *The Finite Element Method in Structure and Continuum Mechanics*, McGraw-Hill, New York, 1967.

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