Collective effects in stellar dynamics and plasma physics

Frank Hohl

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in
Physics

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Frank Hohl
1967
This thesis is submitted in partial fulfillment of
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in
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Approved, May 1967

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ABSTRACT

The major part of the thesis is concerned with the stability and evolution of stellar systems. One- and two-dimensional models are used to perform computer experiments tracing the evolution of stellar systems. The stationary solution of the Vlasov equation for a one-dimensional system of stars as computed for an interesting class of initial conditions is found to correspond to a minimum energy configuration. The results of the numerical experiments are compared with theory. For initial energies far from the minimum equilibrium energy the system becomes unstable and breaks up into smaller clusters. A variational principle has been applied to the one-dimensional stellar system to show that stationary distribution functions which decrease monotonically in going outward from the center of the system are stable. Other stationary distributions may be unstable as is illustrated by means of computer experiments. The one-dimensional model is of interest as an approximation to the distribution of velocity and mass normal to the galactic plane of a highly flattened galactic system. Observational results for the gravitational force normal to the galactic plane of our Galaxy agree with the results obtained from the one-dimensional model.

Numerical experiments with a simple two-dimensional rod model show that the spiral structure and other filamentary structure of galaxies may result from purely gravitational effects.

The minimum energy property investigated for a one-dimensional stellar system is found to apply also to a plasma. Numerical experiments are performed to confirm the theoretical results.

In investigating the quasi-oscillations of electrons near a sharp plasma boundary it is found that crossings of electron sheets occur which destroy the initially cold plasma oscillations. The rate at which the disturbance penetrates into the plasma and causes thermalization of the initially cold plasma oscillations is investigated by means of computer experiments with a one-dimensional sheet model. The effect of a transverse magnetic field on the thermalization is studied.
COLLECTIVE EFFECTS IN STELLAR DYNAMICS AND PLASMA PHYSICS
CHAPTER I

INTRODUCTION

In plasma physics as well as in stellar dynamics there are many problems which involve phenomena occurring in inhomogeneous systems where the interaction between the particles is fully described by a self-consistent field operating in phase space. Because the particles interact by means of the long range Coulomb force, each particle is under the simultaneous influence of a large number of other particles. Therefore, plasmas and stellar systems will respond to any perturbation in a collective manner and in studying such systems one is faced essentially with the N-body problem.

The Vlasov System

The collective phenomena do not depend on two-body collisions such as occur in ordinary gases and therefore, the collective effects will be present in collisionless systems. Since the number of particles in the system is large, a distribution function, \( f(x, \mathbf{v}, t) \) can be used to describe the density of particles in phase space. The distribution function must then satisfy the so-called collisionless Boltzmann or Vlasov equation. In using the Vlasov equation to describe, for example, a stellar system one assumes that the number of masses which make up the system becomes infinite while the total mass remains constant. While such an approach allows one to describe the system by means of a distribution function which must satisfy the Vlasov equation, solutions to the time dependent nonlinear Vlasov equation are in general very difficult to obtain. An attempt is
therefore made to condense the $10^{11}$ stars which a galaxy may contain into about $10^3$ superstars or superparticles. Numerical or computer models are then used to perform "computer experiments" simulating Vlasov phenomena by following the simultaneous motion of a number of superparticles of the order of $10^3$. One- and two-dimensional models are used to effectively solve the time-dependent nonlinear Vlasov equation for various systems.

Computer models used to study plasmas and stellar systems are essentially of two types. The first type is the Lagrangian model where one follows the self-consistent motion of large numbers of particles. In such a model the number of particles which can be treated is limited by computer storage and time. The second type of computer model is the Eulerian model where the system behaves like a fluid in phase space. The macroscopic quantities describing the system are then obtained by solving the appropriate partial differential equations on the computer. In the present study the main interest is in systems which can be described by the Vlasov equation. However, it was found advantageous to use a Lagrangian model to simulate the time development of the systems investigated. Thus, our main interest is not in solving a given differential equation for the macroscopic quantities describing a system but rather in performing numerical experiments by solving the N-body problem.

Previous Numerical Work in Plasma Physics

Direct solutions of the N-body problem have become possible only with the advent of high-speed electronic computers. After the pioneering work of Pasta and Ulam (ref. 1), many workers have used
computer models in their study of the N-body problem. Alder and Wainwright (ref. 2) have used computer models for many-body systems to investigate phase change and condensation phenomena. The same authors (ref. 3) also made nonequilibrium calculations for particles interacting through short range forces and found that only about three collisions per particle were required to establish a Maxwellian velocity distribution.

The usefulness of the one-dimensional sheet model has been well established in plasma physics. Some of the pioneering work in this area was done by Buneman (refs. 4 and 5). Buneman finds that collective interactions or collisions of charge in bulk can restore a grossly non-Maxwellian velocity distribution within a few plasma periods to a near Maxwellian distribution by means of the build-up of electrostatic instabilities. Along similar lines, the one-dimensional plasma model has been applied to specific problems by Dunn and Ho (ref. 6), Birdsall and Bridges (refs. 7 and 8), Smith and Dawson (ref. 9), Burger, Dunn, and Halsted (ref. 10), Derfler (ref. 11), and Hasegawa and Birdsall (ref. 12). The one-dimensional sheet model was used by Burger (refs. 13 and 14) to analyze the stability of DC solutions for the plane diode. A discrete one-dimensional computer model for a collisionless plasma in a magnetic field was used by Auer, Hurwitz, and Kilb (refs. 15 and 16) to examine the structure of magnetic compression waves. This work was later extended by Rossow (ref. 17).

Beginning with the work of Dawson (refs. 18 and 19) and Eldridge and Feix (refs. 20 and 21) a serious study with the one-dimensional
model was made to check the theoretical predictions of plasma behavior. Dawson (ref. 18) investigated the thermal equilibrium properties of plasmas such as the drag on a test particle, Debye shielding, diffusion in velocity space, Landau damping of Fourier modes and other parameters. In a subsequent paper (ref. 19) Dawson used similar methods to investigate the thermal relaxation of one-species plasmas and found that there is no relaxation to a Maxwellian to first order as was shown analytically by Eldridge and Feix (ref. 22). Eldridge and Feix (refs. 20 and 21) performed numerical experiments to study the properties of the one-dimensional plasma near equilibrium and related some of the one-dimensional plasma properties to the three-dimensional case. The agreement with theoretical prediction of the results obtained from the computer experiments gives considerable confidence in the simulation of Vlasov systems by only a few thousand "particles".

Most of the numerical simulations of plasma were made with the one-dimensional sheet model. However, recently several authors have used two-dimensional rod models. Hockney (ref. 23) used a two-dimensional computer model to study the diffusion of charged rods across a constant magnetic field. Wadhwa, Buneman, and Branch (ref. 24) used the two-dimensional rod model to investigate the electron-ion mixing in an ion engine. The main difficulty in the two-dimensional computer experiments is that of finding a sufficiently fast method for solving the Poisson equation to obtain the potential distribution.

Previous Numerical Simulation of Stellar Systems

With the exception of the work done by Lecar (ref. 25) with small numbers of sheets, the author is not aware of any previous numerical
experiments to simulate stellar systems with a one-dimensional sheet model. Hénon (ref. 26) used a computer model of concentric spherical mass shells to study the dynamical mixing of spherical star clusters. Lindblad (ref. 27) calculated the two-dimensional motion of a number (up to 192) of mutually attracting mass points in a given central field of force. By placing the mass points initially in a system of concentric rings with circular velocities, Lindblad investigated the mutual disturbances in such a system to simulate the spiral structure of galaxies. The remaining numerical calculations in stellar dynamics appear to be three-dimensional calculations for systems with low numbers of stars. Thus, von Hoerner (refs. 28 and 29) made extensive calculations with up to 25 stars. In such a system the effects of the individual encounters are large and cannot be smoothed by averaging over many calculations. Arseth (ref. 30) performed similar three-dimensional calculations with up to 100 stars and applied the results to clusters of galaxies. The irreversibility in stellar systems with up to 32 stars was investigated by Miller (ref. 31).
CHAPTER II

THEORY OF SELF-GRAVITATING STELLAR SYSTEMS

The first part of this study is concerned primarily with the time behavior of stellar systems. However, the analytical portion of the study is confined to an investigation of steady states for stellar systems and their stability. A computer model is then used to simulate the time behavior of the system and to study its approach to an equilibrium state. The simplest star system one can consider is one where the stars are stratified into plane parallel layers. The velocity distribution and density will then vary only in one direction. For such a model the stars can be represented by a large number of mass sheets. The motion of such sheets is sufficiently simple and the trajectories of several thousand "stars" can be followed on an electronic computer. We can thus perform numerical experiments to investigate the evolution of a stellar system and compare the results with the equilibrium conditions obtained from the self-consistent collisionless Boltzmann equation.

Nature of the Analysis

As indicated by Michie (ref. 32) there is a need to investigate the importance of orbital (or phase) mixing in the initial evolution of a system of stars. As the system of stars evolves, the gravitational field will change with time and the stars will follow complicated trajectories along which the individual stellar energies are not conserved. Hénon (ref. 26) has recently performed numerical experiments with a system of concentric spherical shells to study the relaxation
of the mean gravitational field and the resulting approach to equilibrium for a spherically symmetric star cluster. The analogy between models where the material is stratified in parallel planes and where the material is distributed in concentric spherical shells has been examined by Wooley (ref. 33).

According to Oort, (ref. 30) the velocities of stars normal to the galactic plane are decoupled from the other velocity components. Since the force on a star is approximately normal to the galactic plane the stars will oscillate perpendicular to the galactic plane with a period independent of the revolution of the galactic system. A one-dimensional model representing a system stratified in infinite parallel planes can therefore be used as an approximation to the distribution of velocity and density of stars normal to the galactic plane of a greatly flattened galactic system. Camm (ref. 35) has considered steady solutions of the collisionless Boltzmann and the Poisson equations for such a model and his results compared favorably with the observed densities. More recently Lindblad (ref. 36), Wooley (ref. 37), and Oort (ref. 38) have used the one-dimensional model to study highly flattened galactic systems. Prendergast (ref. 39) has investigated the general solutions of the steady-state one-dimensional self-gravitating system.

The present work is concerned primarily with investigating the conditions under which a system will approach a metaequilibrium state and the conditions for which a system will become unstable and break up into smaller systems. The relaxation or characteristic time for a system of stars is given by
\[ \tau_c = (4\pi G \rho)^{-1/2} \]  

where \( G \) is the gravitational constant and \( \rho \) is the mass density.

The characteristic length of interest is the Jeans or Debye length \( D \) which is defined by

\[ D = V_T \tau_c \]  

where \( V_T \) is the velocity dispersion of the system of stars. The dimension of a system of stars near equilibrium is of the order of a Debye length. Thus, for a three-dimensional system the number of stars within the Debye sphere is very large, that is, \( nD^3 \gg 1 \)

where \( n \) is the average density of stars. For example, a globular cluster contains from \( 10^5 \) to \( 10^7 \) stars so that \( nD^3 \approx 10^5 \) to \( 10^7 \).

Similarly, for the one-dimensional model the number of "stars" in a Debye length is large, or \( nD \gg 1 \). The effects of collisions between individual stars can then be neglected. That is, the distance travelled by a star before collisions significantly modifies the star's trajectory is much larger than the dimensions of the system, that is, the Debye length of the system. Thus, an actual stellar system as well as the one-dimensional model is well within the Vlasov regime. The numerical experiments using the one-dimensional model are very nearly exact for \( nD \gg 1 \) and actually give the time-dependent solution of the nonlinear Vlasov equation.

One should mention that computer calculations such as those performed by von Hoerner (refs. 28 and 29) and by Aarseth (ref. 30) involving the three-dimensional motion of up to 100 stars are not
suited to simulate purely collective or Vlasov behavior because of the large effects of collisions or graininess.

In the one-dimensional model the forces between the sheets are long range and the sheet model includes individual as well as collective behavior of the stars since the computer simply solves the equations of motion of the stars in the system. The exact behavior of a system depends on the graininess and will be affected by going to the "fluid limit" as implied by the Vlasov equation. As shown in a later section, equivalent models with different graininess give identical results for any time scales of present interest so that the graininess (or collisions) has a negligible effect on the system. Also, since the collective behavior is not affected in going to the fluid limit one would expect to get good agreement between the "experimental" results and the theoretical results obtained from the Vlasov equation for the steady state of the system.

The phrase "metaequilibrium" refers to the equilibrium state which is established by the interaction of the stars with the smoothed potential of the system in a time of the order of $2\pi \sigma_c$. The metaequilibrium can be considered a steady state only as long as collisions or graininess can be neglected. On a long time scale, binary encounters eventually cause the system to approach a state of statistical or thermal equilibrium. An actual three-dimensional stellar system can never completely attain a state of statistical equilibrium since this would involve a Gaussian velocity distribution. The absence of a potential barrier outside an actual stellar system permits the escape from the system of all stars with positive energies.
Thus the leakage of stars will prevent the establishment of a Gaussian distribution necessary for a state of statistical equilibrium. Nevertheless, the time constant with which the three-dimensional system will continually tend toward a state of statistical equilibrium (even though it can never be completely attained) is of the order $nD^3 \tau_c$, where $n$ is the density of stars. This time is of the same order of magnitude as the time constant involved in Chandrasekhar's (ref. 40) calculation of the dynamical friction. For the one-dimensional model the behavior is different because of the potential barrier outside the system. Fluctuations may now cause the establishment of a state of statistical equilibrium in a time which is at least of the order of $(nD)^2 \tau_c$.

Lecar (ref. 41) has performed numerical integrations of a one-dimensional system of stars and has shown that no thermalization exists to order $nD\tau_c$. For the case of a plasma without an external electric field this has been shown analytically by Eldridge and Feix (ref. 22). Presently only times of order $\tau_c$ are of interest during which collisional effects will be negligible.

Comparing the gravitational fields acting on stars in a one-, two-, and three-dimensional system, one finds that they are very similar with the exception of the field acting on stars near the boundary of the system. Figure 1 shows the gravitational field for a one-, two-, and three-dimensional system with constant density. Because of the $\frac{1}{r^2}$ dependence of the field outside the system in the three-dimensional case, stars can escape which is not possible for the
one-dimensional system. However, inside the system the force on a star in all three cases is proportional to the distance from the star to the center of the system.

Equilibrium Solutions

In problems concerning the structure and evolution of stellar systems Camm (ref. 43) has shown that in the Vlasov limit the difference in mass and structure of individual stars can be neglected. We, therefore, set all the star masses equal to \( m \) so that a distribution function \( f(x,v,t) \) will completely define the state of the system. When the effects of binary collisions between stars are negligible the distribution function satisfies the usual equation of stellar dynamics, that is, the Vlasov equation

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + E \frac{\partial f}{\partial v} = 0
\]  

(3)

where \( E = \frac{d^2x}{dt^2} = -\frac{\partial p}{\partial x} \) is given by the Poisson equation

\[
\frac{\partial^2 \Phi}{\partial x^2} = 4\pi Gm \int f \, dv.
\]  

(4)

The limitations of the Vlasov equation in describing stellar system are discussed by Kurth (ref. 42).

To study the time development of a system of stars one could use the distribution function at \( t = 0 \), that is, \( f(x,v,t = 0) \), as the initial condition and then solve the time dependent nonlinear Vlasov equation. Since in general this problem cannot be solved
analytically a high-speed computer is used to determine the evolution of the system and equations (3) and (4) are used only to obtain the steady-state solution for the system.

The Virial Theorem

The virial theorem can be applied to the one-dimensional system to obtain a relation between the potential and kinetic energy of the system in equilibrium. Consider the identity

$$\frac{1}{2} \frac{d^2 x^2}{dt^2} + \frac{1}{2} \frac{d^2 x}{dt^2} = m \left( \frac{dx}{dt} \right)^2 + mx \frac{d^2 x}{dt^2}.$$  \hspace{1cm} (5)

If this expression is multiplied by the distribution function and integrated over phase space the following result is obtained

$$\frac{1}{2} \frac{d^2 x^2}{dt^2} \int \int x^2 f(x,v)dx dv = \int \int m \left( \frac{dx}{dt} \right)^2 f(x,v)dx dv + \int \int mx \frac{d^2 x}{dt^2} f(x,v)dx dv.$$ \hspace{1cm} (6)

Since the system is in equilibrium the left side of equation (6) is equal to zero. Also

$$\int \int m \left( \frac{dx}{dt} \right)^2 f(x,v)dx dv = 2T$$ \hspace{1cm} (7)

where $T$ is the total kinetic energy of the system. Using

$$\rho(x) = \int m f dv \quad \text{and} \quad \frac{d^2 x}{dt^2} = -\frac{d\rho}{dx}$$

in equation (6) one obtains the expression
\[ 0 = 2T - \int x \rho(x) \frac{d\phi}{dx} \, dx. \quad (8) \]

The Poisson equation can be written as

\[ \rho(x) = \frac{1}{L\pi G} \frac{d^2 \phi}{dx^2}. \quad (9) \]

Therefore, using an integration by parts one obtains the result

\[
\int_{-X}^{X} x \rho(x) \frac{d\phi}{dx} \, dx = \frac{1}{L\pi G} \int_{-X}^{X} x \frac{d^2 \phi}{dx^2} \frac{d\phi}{dx} \, dx
\]

\[ = \frac{X}{6\pi G} \left( \frac{d\phi}{dx} \right)^2 \left| \right. _{x=-X}^{X} - \frac{1}{6\pi G} \int_{-X}^{X} \left( \frac{d\phi}{dx} \right)^2 \, dx. \quad (10) \]

The last expression is simply the potential energy \( P \) of the system. Equation (6) then becomes

\[ 2T - P = 0, \quad (11) \]

irrespective of the distribution function. In most of our numerical studies we found that the system very quickly approached conditions such that equation (11) was satisfied.

Waterbag Distribution

In the steady state \( \frac{\partial f}{\partial t} = 0 \) and the Vlasov equation takes the form
Using the method of characteristics to solve the partial differential equation given by equation (12), one obtains the subsidiary equation

\[ \frac{dx}{v} = -\frac{dv}{\frac{\partial f}{\partial \phi}} \]  

which gives the result

\[ \frac{1}{2} mv^2 + m\phi = U = \text{constant} \]  

where \( U \) is the total energy of a star. Therefore, any solution of equation (12) has the form

\[ f(x,v) = F(U). \]  

Thus, if a time-independent equilibrium state exists, then

\[ f(x,v,t \rightarrow \infty) = F(U) \]  

where \( F(U) \) may be any function of \( U \). Of course, we are interested only in function \( F(U) \) that are stable. In general, the form of \( F(U) \) depends on the initial distribution and must be obtained by following the time development of the nonlinear Vlasov equation. However, there is one type of initial distribution where \( F(U) \) is known without actually solving the Vlasov equation. For this distribution the initial \( f \) is taken to be constant over a certain region of phase space and is zero outside this region. Figure 2 illustrates the
distribution. According to the Liouville theorem $f$ remains constant along the different trajectories so that the region can only change its shape with time while keeping its area constant. For this reason the distribution function just described has been called the waterbag model by DePackh (ref. 44). When DePackh considered the waterbag model he was interested primarily in a solution for the linearized oscillations about the equilibrium state. However, a more interesting application of the waterbag model seems to be in the present study of the nonlinear problem. The waterbag model is of interest to us primarily because it allows us to calculate exactly the equilibrium configuration of the one-dimensional star gas for comparison with the computer results.

Two initial distributions considered in the present study are the waterbag model and a distribution which has a constant density over a region of the x-axis and has a Maxwellian velocity distribution. For the waterbag model we know $F(U)$; it is constant for $0 \leq U \leq m_\text{e}$ and is zero for $U$ outside of this region. $m_\text{e}$ is some maximum energy to be determined later. For the second distribution we assume that $F(U)$ is Maxwellian.

The equilibrium solution for the waterbag model will now be obtained. The initial shape of the waterbag for most of the calculations is taken to be a rectangle defined by the area in phase space between $\pm x_0$ and $\pm v_0$. $f(x,v,t \to \infty) = F(U)$ is now used in the Poisson equation (4) and the integration over $dv$ is changed to an integration over $dU$. Since we are interested in the density at a
given \( x \) we obtain \( dU = mvdv \) or \( dv = dU/ \sqrt{2m(U - m\Phi)} \). The Poisson equation, then becomes

\[
\frac{d^2 \Phi}{dx^2} = \frac{8\pi G mA}{\sqrt{2m}} \int_{m\Phi}^{m\epsilon} \frac{dU}{\sqrt{U - m\Phi}}
\]

\[
= 8\pi \sqrt{2} G mA \sqrt{\epsilon - \Phi}
\]

(17)

where \( m\epsilon \) is the energy such that \( F(U) = 0 \) for \( U > m\epsilon \) and \( F(U) = A \) for \( 0 < U < m\epsilon \). Since the area of the system in phase space remains constant the value of \( A \) for the initially rectangular waterbag is obtained from the relation

\[
\int \int A \, dxdv = N
\]

(18)

or

\[
A = \frac{N}{4x_0 v_0}.
\]

(19)

A first integration of equation (17) gives the result

\[
\left( \frac{d\Phi}{dx} \right)^2 = - \frac{8 \sqrt{2} \pi G mN}{3x_0 v_0} \left\{ (\epsilon - \Phi)^{3/2} - \epsilon^{3/2} \right\}
\]

(20)

where we have chosen \( \Phi = \frac{d\Phi}{dx} = 0 \) at \( x = 0 \), to determine the constant of integration. The condition \( \left( \frac{d\Phi}{dx} \right)_{x=0} = 0 \) is simply the statement
that the force on a star vanishes in a plane dividing the system into two equal masses. A second integration gives the final result

\[ \pm x = \left( \frac{3x o V_o}{8 \sqrt{2} \pi GmN} \right)^{1/2} \int_{0}^{\varphi} \left[ (\epsilon^{3/2} - (\epsilon - \xi)^{3/2}) \right]^{-1/2} d\zeta. \] (21)

Let \( x_s \) be the coordinate defining the boundary of the one-dimensional system, then \( \varphi(x_s) = \epsilon \) and \( \left( \frac{d\varphi}{dx} \right)_{x=x_s} = 2\pi Gm\epsilon \). If these values for \( \varphi \) and \( \frac{d\varphi}{dx} \) are used in equation (20) one obtains the result

\[ \epsilon = \left( \frac{3\pi Gnm x V_o}{2 \sqrt{2}} \right)^{2/3}. \] (22)

The value of \( x_s \) is obtained from equation (21) by taking \( \epsilon \) as the upper limit of the integral, thus

\[ x_s = \frac{1}{4} \left( \frac{9 x o^2 V_o^2}{\pi GmN} \right)^{1/3}. \] (23)

The value of the maximum velocity is given by the expression

\[ v_s = \sqrt{2} \epsilon \]

\[ = \left( \frac{3\pi Gnm x V_o}{2} \right)^{1/3}. \] (24)

Equation (17) shows that the number density is given by
\[ n(x) = \frac{N}{\sqrt{2} x_v v_0} (\varepsilon - \varphi)^{1/2} \]  \hspace{1cm} (25)

where we have made use of equation (19).

If the normalized variables

\[ x = (2\pi GmN/\varepsilon)x \]  \hspace{1cm} (26)

and

\[ \Omega = \varphi/\varepsilon \]  \hspace{1cm} (27)

are used in equation (21) one obtains the simplified equation

\[ \pm x = \int_{\Omega}^{\phi} \left( 1 - (1 - \xi)^{3/2} \right)^{-1/2} d\xi. \]  \hspace{1cm} (28)

The resulting \( \phi \) and \( E \) as functions of \( x \) are shown in figure 3.

**Multiple Waterbag Distributions**

Another waterbag which was investigated numerically is one with a hole in the center. The only possible \( F(U) \) is then one which has a constant value \( A \) for \( m\varepsilon_o \leq U \leq m\varepsilon \) where \( m\varepsilon \) is the maximum and \( m\varepsilon_o \) is the minimum star energy in the system and \( F(U) \) is zero outside this range. The Poisson equation then takes the form

\[ \frac{d^2\varphi}{dx^2} = 8\pi Gm \sqrt{2} A \left[ \sqrt{\varepsilon - \varphi} - \sqrt{\varepsilon_o - \varphi} \right] \text{ for } x < x', \]  \hspace{1cm} (29)

and
\[ \frac{d^2 \phi}{dx^2} = 8\pi G m \sqrt{2} A \sqrt{\epsilon - \phi} \text{ for } x > x' \quad (30) \]

where \( x' \) is given by \( \phi(x') = \epsilon_o \). A first integration gives the result

\[ \left( \frac{d\phi}{dx} \right)^2 = -\frac{32}{3} \sqrt{2} \pi G m A \left[ (\epsilon - \phi)^{3/2} - (\epsilon_o - \phi)^{3/2} + C_1 \right] \text{ for } x < x' \quad (31) \]

and

\[ \left( \frac{d\phi}{dx} \right)^2 = -\frac{32}{3} \sqrt{2} \pi G m A \left[ (\epsilon - \phi)^{3/2} + C_2 \right] \text{ for } x > x' \quad (32) \]

Using the boundary conditions \( \phi(0) = 0, \left( \frac{d\phi}{dx} \right)_{x=0} = 0, \phi(x_s) = \epsilon \) and \( \left( \frac{d\phi}{dx} \right)_{x=x_s} = -2\pi G m N \) one obtains

\[ C_1 = -\left( \epsilon^{3/2} - \epsilon_o^{3/2} \right) \quad (33) \]

and

\[ C_2 = -\left( \epsilon^{3/2} - \epsilon_o^{3/2} \right) \quad (34) \]

A second integration gives the final result
\[ \pm x = \left( \frac{32 \sqrt{2} \pi m k T}{3} \right)^{1/2} \int_{0}^{\frac{x}{3/2}} \left[ (\xi - \frac{3}{2})^{3/2} - (\xi - \frac{1}{2})^{3/2} \right] \cdot (\xi^{3/2}) - \frac{1}{2} \right] d\xi \quad \text{for} \ x < x' \] (35)

and

\[ \pm x = x' + \left( \frac{32 \sqrt{2} \pi m k T}{3} \right)^{1/2} \int_{0}^{\frac{x}{3/2}} \left[ (\xi - \frac{3}{2})^{3/2} - (\xi - \frac{1}{2})^{3/2} \right] \cdot (\xi^{3/2}) - \frac{1}{2} \right] d\xi \quad \text{for} \ x > x' \] (36)

Using the normalization

\[ n = \left( \frac{32 \sqrt{2} \pi m k T}{3} \right)^{1/2} e^{-\frac{3}{2} x} \] (37)

and

\[ \Phi = \psi / \epsilon. \] (38)

Equations (35) and (36) simplify to

\[ \pm x = \int_{0}^{\frac{x}{3/2}} \left[ (1 - \xi) - (\xi - \frac{3}{2})^{3/2} - (\xi - \frac{1}{2})^{3/2} \right] \cdot (\xi^{3/2}) - \frac{1}{2} \right] d\xi \quad \text{for} \ x < x' \] (39)

and

\[ \pm x = x' + \int_{0}^{\frac{x}{3/2}} \left[ (1 - \xi) - (\xi - \frac{3}{2})^{3/2} - (\xi - \frac{1}{2})^{3/2} \right] \cdot (\xi^{3/2}) - \frac{1}{2} \right] d\xi \quad \text{for} \ x > x' \] (40)

where \( \epsilon = \sqrt{v} / \epsilon. \) Figure 4 shows the normalized equilibrium contours for four values of \( v. \) The contours are obtained from the equations

\[ \chi_{n}^{(1)}(x) = \sqrt{2(1 - \Phi(x))} \] (41)

and

\[ \chi_{n}^{(2)}(x) = \sqrt{2(1 - \Phi(x))}. \] (42)

Maxwellian Distribution

For a system at thermal equilibrium the distribution function \( \Phi(U) \) has the form \( A \exp(-U). \) Following the same method used in obtaining the solution for the waterbag model one obtains the equilibrium density

\[ n = 2 \int_{0}^{\infty} \Phi d\Phi \]

\[ = 2 \int_{0}^{\infty} A \exp(-U) \cdot \frac{dU}{\sqrt{2\pi(U - \epsilon)}} \]

\[ = 2 A \sqrt{\frac{x}{2m k T}} \exp(-\epsilon x) \cdot \] (43)

Equation (4) then becomes

\[ \frac{\partial^{2} \Phi}{\partial x^{2}} - \frac{\epsilon^{2}}{2m k T} \sqrt{\frac{\pi}{2m k T}} \exp(-\epsilon x) = 0 \] (44)

or
\[
\frac{d^2 \phi}{dx^2} - \frac{a}{2} \exp(-\phi) = 0
\]  \tag{45}

where \( \phi = \kappa \varphi \) and \( a = 16\pi Gm \kappa^2 A \sqrt{\frac{\pi}{2m}} \). A first integration of equation (44) gives the result

\[
\left( \frac{d\phi}{dx} \right)^2 - a(1 - \exp(-\phi)) = 0
\]  \tag{46}

where the constant of integration was determined by the boundary condition \( \frac{d\phi}{dx} = 0 \) at \( x = 0 \). A second integration gives the result

\[
\pm x = \frac{1}{\sqrt{a}} \ln \left[ \frac{1 - \sqrt{1 - \exp(-\phi)}}{1 + \sqrt{1 - \exp(-\phi)}} \right].
\]  \tag{47}

Solving equation (47) for \( \phi \) one obtains the equation

\[
\phi(x) = -\ln \left[ 1 - \tanh^2 \left( \frac{\alpha}{\sqrt{A}} x \right) \right].
\]  \tag{48}

One of the boundary conditions to be satisfied is

\[
\lim_{x \to \infty} \frac{d\phi}{dx} = 2\pi Gm^2 N \kappa
\]  \tag{49}

or

\[
a = \left( 2\pi Gm^2 N \right)^2.
\]  \tag{50}

The solution is then given by the equations

\[
\varphi(x) = -\frac{1}{\kappa m} \ln \left[ 1 - \tanh^2 \left( \pi Gm^2 N \kappa x \right) \right].
\]  \tag{51}
and

$$E(x) = -2\pi GmN \tanh\left(\pi Gm^2 N \kappa x\right)$$  \hspace{1cm} (52)

and

$$n(x) = \frac{1}{2} \frac{\pi Gm^2 N^2 \kappa}{\cosh^2\left(\pi Gm^2 N \kappa x\right)}$$  \hspace{1cm} (53)

where \(1/\kappa\) is the average kinetic energy per "star" at any given position. This particular solution has also been considered by Camm (ref. 35).

Figure 5 shows the variation of \(\Phi\) and \(E\) for the Maxwellian distribution where the values \(\hbar c = 1, m = 1, \kappa = 1,\) and \(N = 4\) were arbitrarily chosen.

Energy Considerations

The steady-state solution for the waterbag distribution was obtained by using conservation of area in the two-dimensional phase space. It was also assumed that irrespective of the initial conditions the system would approach an equilibrium state described by

$$f(x,v,t \to \infty) = F(U).$$

Only for the waterbag model was it possible to obtain \(F(U)\) corresponding to a given initial condition. This was accomplished by invoking the Liouville theorem. We must therefore determine whether energy considerations allow the relaxation from a function, \(f(x,v,t)\), of two independent variables to a function \(F(U)\) of the energy only. Thus, if an equilibrium state is to be reached then the initial and final energy of the system must be equal. For the initial rectangular waterbag the kinetic energy \(T\) is given by
\[ T(t = 0) = \int \int \frac{1}{2} m v^2 f(x,v,t = 0) \, dx \, dv \]

\[ = \frac{N m}{8x_0 v_0} \int_{x_0}^{x_0} \int_{v_0}^{v_0} v^2 \, dv \, dx \]

\[ = \frac{N m}{4} v_0^2. \tag{54} \]

As before \( \pm v_0 \) and \( \pm x_0 \) define the rectangular area of the initial distribution. Using equation (10) the potential energy of the initial waterbag is given by

\[ P(t = 0) = \frac{x_0}{4\pi G} (2\pi G m N)^2 - \frac{1}{4\pi G} \int_{x_0}^{x_0} \left( \frac{2\pi G m N x}{x_0} \right)^2 \, dx \]

\[ = \frac{2}{3} \pi G m^2 N^2 x_0. \tag{55} \]

The total initial energy \( W \) is then given by

\[ W(t = 0) = T(t = 0) + P(t = 0) \]

\[ = \frac{N m}{2} \left[ \frac{1}{2} v_0^2 + 2\pi G m N x_0 \right]. \tag{56} \]

For a given constant value of \( A = \frac{N}{4x_0 v_0} \), the minimum value equation (56) can attain is easily found to be given by

\[ W_{\text{min}}(t = 0) = \frac{N m}{2} \left( 2\pi G m N x_0 v_0 \right)^{2/3}. \tag{57} \]
The total energy of the equilibrium state defined by equation (21) will now be computed. The kinetic energy is given by

\[ T_{eq} = \int \int \frac{1}{2} mv^2 f \, dx \, dv \]

\[ = \frac{N}{x_0 v_0 \sqrt{2m}} \int \int_{\epsilon}^{\infty} \int_{m\epsilon}^{m\epsilon+U} \sqrt{U - m\epsilon} \, dU \, dx \]

\[ = \frac{\sqrt{2} mN}{3x_0 v_0} \int_{\epsilon}^{\infty} (\epsilon - \phi)^{3/2} \, dx \]

\[ = \frac{3}{8} Nm \left(3\pi GmN \frac{v}{\epsilon}\right)^{2/3} \int_{\epsilon}^{\infty} (1 - \phi)^{3/2} \, d\chi \quad (58) \]

where \( dv = dU / \sqrt{2m(U - m\epsilon)} \) was used and as before \( \epsilon = \phi / \epsilon \) and \( \chi = (2\pi GmN / \epsilon)x \). Using equations (10) and (20), the potential energy for the equilibrium waterbag becomes

\[ P_{eq} = \frac{x_0}{4\pi G} \left(2\pi GmN\right) \left[ e^{3/2} - (\epsilon - \phi)^{3/2} \right] \int_{\epsilon}^{\infty} \, d\chi \]

\[ = \frac{Nm}{4} \left(3\pi GmN \frac{v}{\epsilon}\right)^{2/3} \int_{\epsilon}^{\infty} (1 - \phi)^{3/2} \, d\chi \quad (59) \]

The equilibrium state then has a total energy given by the expression
\[ W = T_{eq} + P_{eq} \]

\[ W_{eq} = \frac{3}{8} \frac{\hbar m (3\pi^2 N \omega)}{\sqrt{\pi}} \int_0^\infty (1 - \phi)^{3/2} d\chi \]  \hspace{1cm} (60)

where \( \int_0^\infty (1 - \phi)^{3/2} d\chi \approx 0.983. \)

For arbitrary \( x_0 \) and \( v_0 \) it is clear that equations (56) and (60) cannot be equal. The equilibrium energy \( W_{eq} \) depends only on the product \( x_0 v_0 \) while \( W(t = 0) \) is the sum of two independent quantities that are functions of the two independent variables \( v_0 \) and \( x_0 \). The total energy as well as the area in phase space must be conserved. Thus, we have found that the only state which is compatible with the conservation of area in phase space and with the dynamics of the Vlasov equation (Liouville theorem) does not have the same energy as the initial state. Therefore, the assumption that a steady state exists for an initially rectangular waterbag distribution is incorrect. However, the equilibrium state has the interesting properties that it is a minimum energy configuration and as will be shown in a subsequent chapter, the system will do its best within the limitations of energy conservation to reach the equilibrium state.

The Minimum Energy Principle

The minimum energy property of the equilibrium waterbag distribution will now be demonstrated. First consider the initial rectangular
waterbag distribution. Equation (57) gives the minimum initial energy for such a distribution. If the ratio of equation (57) to the equilibrium energy is taken, one finds that

$$\frac{W_{\text{min}}(t = 0)}{W_{\text{eq}}} = 1.03$$

and the rectangular waterbag with the minimum energy has still more energy than the equilibrium waterbag. The minimum energy property will now be generalized to any shape of the waterbag distribution. We do this by showing that the equilibrium state has the least possible energy.

Consider the waterbag distribution shown in figure 2. Let points along the contour (C) be described by $v_+(x)$ and $v_-(x)$ where the plus sign indicates the upper contour and the minus sign indicates the lower contour. The equations of motion for $v_+(x)$ are

$$\frac{dv_+}{dt} = \frac{dv_+}{dt} + v_+ \frac{dv_+}{dx} = E.$$  \hspace{1cm} (61)

For the equilibrium state $\frac{dv_+}{dt} = 0$ and equation (61) becomes

$$v_+ \frac{dv_+}{dx} = E = 0.$$  \hspace{1cm} (62)

In order to demonstrate the minimum energy property we use a variational principle. A general function $g(x, v_+(x), \theta(x))$ with
\[ \theta = \int_{-x_s}^{x_s} v_+(x) \, dx \] is introduced such that the total energy of the system is given by

\[ W = \int_{-x_s}^{x_s} g(x, v_+, \theta) \, dx. \quad (63) \]

For the sake of simplicity the contour of the waterbag has been taken to be symmetric such that \( v_+ = -v_- \). Following Courant and Hilbert, (ref. 45) the extremum of equation (63) is found by the usual methods of variational calculus. The end points \( \pm x_s \) of the contour are held fixed and the function \( v_+(x) \) receives a variation \( \alpha \eta(x) \) where \( \alpha \) is an arbitrary constant and \( \eta(x) \) is an arbitrary function which vanishes at the end points. One then finds that

\[
\frac{\partial W}{\partial x} = \int_{-x_s}^{x_s} \left[ \frac{\partial g}{\partial v_+} \eta + \frac{\partial g}{\partial \theta} \int_{-x_s}^{x_s} \eta \, dx \right] \\
= \int_{-x_s}^{x_s} \left[ -\frac{d}{dx} \frac{\partial g}{\partial v_+} + \frac{\partial g}{\partial \theta} \int_{-x_s}^{x_s} \eta \, dx \right] \, dx. \quad (64)
\]

Since \( \eta \) is an arbitrary function the condition for obtaining an extremum is

\[
\frac{d}{dx} \left( \frac{\partial g}{\partial v_+} \right) - \frac{\partial g}{\partial \theta} = 0. \quad (65)
\]
Using the first integral in equation (10), the potential energy per unit length is given by

\[
\frac{dP}{dx} = -2mA_v+ \left[ 4\pi Gm \left( \frac{N}{2} - 2A\theta \right) \right]
\]  

(66)

where \( 4\pi Gm \left[ \frac{N}{2} - 2A\theta \right] = E \). The kinetic energy per unit length is simply \( \frac{A}{2} mv_+^3 \). Thus, for the present problem \( g \) has the form

\[
g = \frac{A}{2} mv_+^3 - 2mA_v+ \left[ 4\pi Gm \left( \frac{N}{2} - 2A\theta \right) \right]
\]  

(67)

where \( A \) is the magnitude of \( f \). If the expression for \( g \) given by equation (67) is used in equation (65), one obtains the result

\[
\frac{dv_+}{dx} + v_+ - E = 0
\]  

(68)

which is identical to equation (62). Application of the Legendre test of the second variation of \( g \) can tell us something about the extremum just calculated. One finds that \( \frac{\partial^2 g}{\partial v_+^2} = 2mA_v+ \), and since for small perturbation about the equilibrium state \( v_+ \) is positive (or zero) the Legendre test tells us that the extremum cannot be a maximum. Thus, the minimization of the total energy of a waterbag distribution leads to the equilibrium equations. The consequence of the minimum energy property is that starting from any nonequilibrium state, energy conservation will prevent the system from reaching the steady state described by equation (21) and (25). This result was to be expected from the analysis of DePackh, (ref. 44) which shows that for a plasma small perturbations of the equilibrium state are not damped.
Nevertheless, the interesting point is that our numerical results show that the system does its best within the limitation of energy conservation to approach the equilibrium state. In general, we can hope to approach the equilibrium state very closely whenever the initial energy is not too different from the energy of the equilibrium state given by equation (60).

**Multiple Waterbag Distribution**

It will now be shown that the minimum energy principle can be extended to arbitrary distribution functions. The waterbag model illustrated in figure 6 is used in the analysis. The contours \( v_+^{(k)}(x,t) \) and \( v_-^{(k)}(x,t) \) describe surfaces of constant \( f = f_k \).

According to the Liouville theorem the phase space bounded by the contours is incompressible. In the limit of a very large number of contours the waterbag model can be used to construct arbitrary distribution functions.

The distribution function

\[
f = \sum_k A_k \left[ \delta_{-1}(v - v_-^{(k)}) - \delta_{-1}(v - v_+^{(k)}) \right]
\]

(69)

with \( \delta_{-1}(z) = \int_z^\infty \delta(\zeta)d\zeta \) where \( \delta(\zeta) \) is the Dirac delta function satisfies the Vlasov equation.
\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + E \frac{\partial f}{\partial v} = \sum_k A_k \left[ \delta \left( v - v^+(k) \right) \left\{ \frac{\partial v^+(k)}{\partial t} + v \frac{\partial v^+(k)}{\partial x} - E \right\} \right. \]

\[ \left. - \delta \left( v - v^-(k) \right) \left\{ \frac{\partial v^-(k)}{\partial t} + v \frac{\partial v^-(k)}{\partial x} - E \right\} \right] = 0. \]

(70)

The force per unit mass \( E \) is obtained from the equation

\[ \frac{\partial E}{\partial x} = -4\pi G m \int f dv = -4\pi G m \sum_k A_k \left( v^+(k) - v^-(k) \right) \]

(71)

where it is assumed that there is a total of \( N \) "stars," each of mass \( m \), in the system. If equation (70) is integrated over velocity one obtains the result

\[ \sum_k A_k \left[ \left( \frac{\partial v^+(k)}{\partial t} + v^+(k) \frac{\partial v^+(k)}{\partial x} - E \right) - \left( \frac{\partial v^-(k)}{\partial t} + v^-(k) \frac{\partial v^-(k)}{\partial x} - E \right) \right] = 0. \]

(72)

To simplify the equations symmetric contours \( v^+(k) = -v^-(k) = v(k) \) are assumed in the remainder of the analysis. The results obtained are not affected by such an assumption. Equation (72) can then be simplified as

\[ \sum_k A_k \left[ \frac{\partial v(k)}{\partial t} + v(k) \frac{\partial v(k)}{\partial x} - E \right] = 0. \]

(73)

Equation (73) must hold term-by-term. The equilibrium contours are obtained from the equations.
\[ v(k) \frac{dv(k)}{dx} - E = 0. \] (74)

A new variable is now defined as

\[ \theta(k) = \int_{x_s(k)}^{x(k)} v(k)(\xi) d\xi \] (75)

where \( x_s(k) \) is the end point of the contour \( k \). In terms of \( \theta(k) \) the gravitational field is given by the equation

\[ E(x) = 4\pi Gm \left( \frac{N}{2} - 2 \sum_k A_k \theta(k) \right). \] (76)

Let \( g \) be a function such that the total energy \( W \) of the system is given by

\[ W = \sum_i \int_{x_s}^{x(k)} g(x, \theta(k), v(k)) dx. \] (77)

Extremizing the integral for \( W \) subject to the condition that

\[ N = \sum_k 2A_k \theta(k)(x_s(k)) \] (78)

is a constant requires that the contours \( v(k) \) satisfy the Euler-Lagrange equations.
\[ \frac{\partial g^*}{\partial \theta(k)} - \frac{d}{dx} \frac{\partial g^*}{\partial v(k)} = 0 \]  

(79)

where \( g^* = g + \lambda \sum_k \theta_{k}\left(x_{s}^{(k)}\right) \) and \( \lambda \) is, as yet, an undetermined Lagrange multiplier. If the end points \( x_{s}^{(k)} \) are not held fixed in the variational process, additional equations will appear which will not change the main results given below.

For a multiple waterbag the kinetic energy per unit length is

\[ \frac{dT}{dx} = \sum_k \frac{M}{2} A_k v_{k}^{3}. \]  

(80)

From equation (10) the potential energy per unit length is

\[ \frac{dP}{dx} = -2m \sum_k x A_k \left[ 4\pi G m \left( \frac{N}{2} - 2 \sum_k A_k \theta_{k}^{(k)} \right) \right]. \]  

(81)

Therefore, the expression for \( g \) is

\[ g = \frac{1}{2} m A_k v_{k}^{3} - 2m x A_k \left[ 4\pi G m \left( \frac{N}{2} - 2 \sum_k A_k \theta_{k}^{(k)} \right) \right]. \]  

(82)

The Euler-Lagrange equations then become

\[ -2m A_k \left[ v_{k}^{(k)} \frac{dv_{k}^{(k)}}{dx} - 4\pi G m \left( \frac{N}{2} - 2 \sum_k A_k \theta_{k}^{(k)} \right) \right] = 0 \]  

(83)

or
\[ v^{(k)} \frac{dv^{(k)}}{dx} - E = 0 \]  \hspace{1cm} (84)

which are the equations for the equilibrium contours.

If equations (84) are to represent a minimum energy configuration then Legendre's criterion of the second variation of \( g \) must be satisfied. Following Courant and Hilbert (ref. 45), the Legendre condition for the case of several unknown functions \( \theta^{(k)} \) is that the quadratic form whose coefficient matrix has the elements

\[ a_{ij} = \frac{\partial^2 g}{\partial v(i) \partial v(j)} \]  \hspace{1cm} (85)

must not be negative. For the present problem only the diagonal elements of equation (85) are nonzero and are given by

\[ a_{kk} = 2mA_k v^{(k)}. \]  \hspace{1cm} (86)

Since \( 2mv^{(k)} \) is never negative, the Legendre condition requires that

\[ A_k > 0. \]  \hspace{1cm} (87)

From the definition of \( f \) given by equation (69) it can be seen that equation (87) is equivalent to stating that the distribution function must always decrease in going outward from the center of the system where \( f = f_1 \) must be the largest. If equation (87) is satisfied the system is a minimum energy configuration and is always stable. However, if \( A_k > 0 \) is not satisfied for all \( k \) the system is not a minimum
energy configuration. If the Legendre condition is not satisfied the system may be unstable since two or more contours can now be deformed while keeping the total energy of the system constant. In a later chapter the results of a numerical experiment are presented which show how two-contour systems with $A_1 = -A_2$ become unstable.

Stability of a Stellar System

In investigating the stability of a system one generally studies small perturbations about some equilibrium state. The problem of gravitational instability for an infinite homogeneous system was first studied by Jeans (ref. 46) and is of fundamental importance in astronomy. The condensations of mass that are caused by the gravitational instability are considered to be the mechanism by which stellar systems such as galaxies and globular clusters and also stars are formed. Jeans' criterion for stability follows from simple principles. If the perturbations of the gravitational field caused by the density fluctuations are neglected, then the sound velocity in gaseous media is given by

$$v_0 = \sqrt{\frac{\gamma p}{\rho}}$$  \hspace{1cm} (88)

where $\gamma$ is the ratio of specific heats, $p$ is the pressure, and $\rho$ is the mass density. However, if the perturbation of the gravitational potential $\delta \phi$, given by the Poisson equation

$$\frac{d^2 \delta \phi}{dx^2} = 4\pi G \delta \rho$$  \hspace{1cm} (89)
is taken into account, the velocity of sound propagation becomes

\[ v_s = \sqrt{\frac{\gamma p}{\rho} - 4\pi G \rho \left( \frac{\lambda}{2\pi} \right)^2} = \sqrt{v_0^2 - 4\pi G \rho \left( \frac{\lambda}{2\pi} \right)^2}. \]  \hspace{1cm} (90)

Thus, the inclusion of the gravitational term decreases the velocity of sound propagation and makes it a function of the wavelength \( \lambda \). For small \( \lambda \) the decrease in velocity is inappreciable. However, if \( \lambda \) increases sufficiently \( v_s \) will become imaginary which indicates that the amplitude of the sound wave increases exponentially with time. The critical \( \lambda \), called the Jeans' length, at which \( v_s \) becomes imaginary, is given by

\[ \left( \frac{\lambda}{2\pi} \right)^2 = \frac{v_0^2}{4\pi G \rho} = D^2 \]  \hspace{1cm} (91)

where \( D \) is given by equation (1). But, stellar systems in equilibrium have dimensions near \( D \) so that no wavelength larger than \( D \) will be present. This shows that system larger than \( D \) will relax to systems with dimensions \( D \). The Jeans' length \( \frac{\lambda}{2\pi} \) is the analog of the Debye distance (ref. 17) encountered in plasma physics. The analysis of Jeans has been extended by Chandrasekhar (ref. 48) to include isotropic turbulence. Chandrasekhar (refs. 49 and 50) and Bel and Schatzman (ref. 51) have shown that rotation of a system only modified Jeans' criterion slightly.
If a stellar system can be treated as collisionless then the Vlasov equation can be used to investigate the stability of the system. We then write the distribution function as

\[ f = f_0 + f_1 \]  

(92)

where \( f_0 \) is the equilibrium distribution and \( f_1 \) is a small perturbation of the equilibrium state. Similarly for the gravitational potential we have

\[ \varphi = \varphi_0 + \varphi_1 \]  

(93)

where \( \varphi_0 \) is the equilibrium potential and \( \varphi_1 \) is the perturbation.

For simplicity we will only treat the one-dimensional case and consider only longitudinal waves. If equations (92) and (93) are used in the Vlasov equation (3) and in the Poisson equation (4) we obtain the following first order equations

\[ \frac{\partial f_1}{\partial t} + v \frac{\partial f_1}{\partial x} = \frac{\partial \varphi_0}{\partial x} + \frac{\partial \varphi_1}{\partial x} \frac{\partial f_0}{\partial v} \frac{\partial f_1}{\partial v} = 0 \]  

(94)

since

\[ \frac{\partial f_0}{\partial t} + v \frac{\partial f_0}{\partial x} = \frac{\partial \varphi_0}{\partial x} \frac{\partial f_0}{\partial v} = 0 \]  

(95)

and the term \( \frac{\partial \varphi_1}{\partial x} \frac{\partial f_1}{\partial v} \) has been omitted since it is of second order in the perturbation.
If the gravitational perturbations in a stellar system are compared with the electrostatic electron oscillations in a plasma one finds two major differences. The first difference is that the gravitational interaction is of opposite sign to the electrostatic interaction which causes a change of sign in the dispersion relation for the waves. The second difference is that the self-gravity of a stellar system is not neutralized as is the case for an electron gas where a neutralizing ion background is present. Thus, stellar systems are inhomogeneous.

However, several authors (refs. 52, 53, and 54) have recently studied microscopic instabilities which occur at wavelengths small compared to the dimensions of the system. Thus in a small region of the inhomogeneous system all space derivatives of the zero order quantities are neglected and equation (94) becomes

\[ \frac{\partial f_1}{\partial t} + v \frac{\partial f_1}{\partial x} - \frac{\partial \varphi_1}{\partial x} \frac{\partial f_0}{\partial v} = 0. \]  

(95)

\( \varphi_1 \) is given by the Poisson equation

\[ \frac{\partial^2 \varphi_1}{\partial x^2} = 4\pi G \int f_1 \, dv. \]  

(96)

To investigate the oscillations caused by local fluctuations in the system one generally tries to find a solution for \( f_1 \) and \( \varphi_1 \) of the form

\[ f_1(x,v,t) = e^{\exp[i(\omega t - kx)]} \]  

(97)

and
\[ \phi_1(x,t) = c \exp\left[i(\omega t - kx)\right] \]  \hspace{1cm} (98)

where \( k \) is the wave vector and \( \omega \) is the complex frequency. If equations (97) and (98) are used in equations (95) and (96), one obtains the dispersion relation

\[ k^2 = -\ln(m) \int \frac{\frac{\partial f_o}{\partial v}}{v - \frac{\omega}{k}} \, dv. \]  \hspace{1cm} (99)

For example, if we choose an \( f_o \), which has a constant value of 1 between \(-v_T\) and \(v_T\) (e.g., the waterbag distribution), then the dispersion equation (99) gives the result

\[ \omega^2 = \left(v_T k \right)^2 - \left(2Av_T\right) \ln(m). \]  \hspace{1cm} (100)

Thus, \( \omega^2 \) becomes negative if

\[ \lambda = \frac{1}{k} > \frac{v_T^2}{\left(2v_TA\right) \ln(m)}. \]  \hspace{1cm} (101)

Similarly, for two cold streams of equal density \( A \) that are traveling in opposite directions with velocities \( \pm v_T \) one finds

\[ k^2 = \ln(m)A \left[ \frac{1}{\left(v_T - \frac{\omega}{k}\right)^2} + \frac{1}{\left(v_T + \frac{\omega}{k}\right)^2} \right]. \]  \hspace{1cm} (102)
A dispersion relation similar to equation (102) has been solved graphically by Kahn (ref. 55) and the result, obviously, is that all four roots for \( \omega \) are complex and must occur in pairs that are complex conjugates so that the system will always be "unstable". Similar calculations can be carried out for other distribution functions \( f_0 \), but the author believes that the neglect of the spacial variation of the zero order quantities makes the usefulness of the results very questionable in the least.

For a special cylindrical system Lynden-Bell (ref. 56) solved equations (94) and (96) exactly by using a treatment similar to that of Landau (ref. 57) and Bernstein (ref. 58). However, in his more general treatment Lynden-Bell assumes that the gravitational effects of the perturbation can be neglected so that the last term \( \Delta \varphi_1 \frac{\partial f}{\partial x} \) in equation (94) is omitted. The results of the numerical experiments with the one-dimensional model indicate that the instabilities which were numerically investigated can not be studied by neglecting either one of the last two terms in equation (94). The author is hopeful that the method of DePackh (ref. 44) can be extended to study the onset of instabilities at least for the waterbag model.
CHAPTER III

COMPUTER SIMULATION OF STELLAR SYSTEMS

The simplest system of stars we can consider is the one-dimensional case where the stars are stratified into plane parallel layers and all parameters vary only in one direction. The nonrelativistic motion of stars in one-dimension is simple enough such that the motion of large numbers of stars can be followed on an electronic computer. Since the simulation of the one-dimensional motion of a system requires relatively little computer time most of the calculations presented in this study were performed with the one-dimensional model. However, some results obtained with a two-dimensional model are also presented.

Initial distributions of the waterbag type were used extensively in the numerical experiments since they allow a comparison with the results obtained in the previous chapter. Two types of initial waterbag distributions have to be separately considered. The first type is a "fat" waterbag where the dimension of the system is near D and the total energy is near the energy of the equilibrium distribution. For this case the system quickly approaches the equilibrium state with the exception of the arms in phase space which develop to accomodate the excess energy. The second type of initial distribution is a "thin" waterbag which has a total energy much larger than the energy of the equilibrium distribution. For this second case the system cannot even come near its equilibrium state and an instability develops which causes the system to break up into smaller clusters.
Description of the One-Dimensional Model

The model consists of a system of \( N \) stars which are represented by mass sheets. These sheets are of infinite extent along the \( y \)- and \( z \)-axes and the sheets are constrained to move along the \( x \)-axis. The equations of motion of all the \( N \) stars are solved simultaneously by computing the gravitational field at the position of each star and then integrating the equations of motion for each star over a small time interval \( \Delta t \) and repeating the process. When two sheets cross they are allowed to pass freely through each other.

The evolution of the system of stars is studied by using various nonequilibrium initial distributions as input to the computer and by then following the time development of the system on the computer.

The equation of motion for the \( i \)\(^{th} \) sheet with a mass \( m_i \) per unit area is given by the expression

\[
\ddot{x_i}(t) = E(x_i, t) \tag{103}
\]

where \( x_i \) is the position of the \( i \)\(^{th} \) sheet and \( E(x, t) \) is obtained from

\[
- \frac{\partial E(x, t)}{\partial x} = \frac{h\pi G \rho(x, t)}{\Delta x}
\]

and

\[
N = \frac{h\pi G}{\Delta x} \sum_{j=1}^{N} m_j \delta^3(x - x_j(t)). \tag{104}
\]

The expression for the gravitational field is then
\[ E(x,t) = 2\pi G \sum_{j=1}^{N} m_j H(x - x_j(t)) \]  \hspace{1cm} (105)

where

\[ H(z) = \begin{cases} -1 & \text{for } z > 0 \\ 0 & \text{for } z = 0 \\ 1 & \text{for } z < 0 \end{cases} \]

Figure 7 illustrates the variation of the gravitational field for an eight-star system. In the actual numerical calculations the field was computed by first ordering the mass sheets according to their coordinate \( x_j^{(k)} \) such that

\[ x^{(k)} \leq x^{(k+1)}. \]  \hspace{1cm} (106)

The gravitational field is then given by

\[ E = 4\pi G m \left[ \frac{N}{2} - k + \frac{1}{2} \right] \]  \hspace{1cm} (107)

and the summation indicated in equation (105) does not have to be performed at every time step and for each star. The sorting routine used in ordering the stars is very fast and takes advantage of the fact that in advancing the motion of the stars for a small time interval \( \Delta t \) will not get the stars too far out of order.

The motion of the mass sheets can also be computed by a more exact method. That is, the times at which neighboring pairs of sheets cross is computed and using the shortest of these crossing times the new positions and velocities for the sheets affected is recomputed.
The next pair of sheets which crosses is then found and the process is repeated. The accelerations are constant until a pair of sheets cross. This very accurate program has been used by Lecar (ref. 41) to study certain "invariants" of the system. The exact program is also used in the present study to investigate thermalization for systems with small numbers of stars. However, the exact program takes a large amount of machine time and is not suitable for investigating the motion of large numbers of sheets.

The IBM 7094 data processing system was used to calculate the self-consistent motion of systems containing several thousand "stars."

The position and velocity of each sheet are computed at successive times $t_1, t_2, t_3, \ldots t_r$. For each time step $\Delta t = t_{n+1} - t_n$ the new positions and velocities of the sheets are computed from the equations

$$x_i(t_{n+1}) = x_i(t_n) + \frac{dx_i(t_n)}{dt} \Delta t + \frac{1}{2} \frac{d^2x_i(t_n)}{dt^2} (\Delta t)^2$$  \hspace{1cm} (108)

and

$$\frac{dx_i(t_{n+1})}{dt} = \frac{dx_i(t_n)}{dt} + \frac{d^2x_i(t_n)}{dt^2} \Delta t + \frac{1}{2} \frac{d^3x_i(t_n)}{dt^3} (\Delta t)^2$$  \hspace{1cm} (109)

where

$$\frac{d^2x_i(t_n)}{dt^2} = 2\pi G \sum_{j=1}^{N} m_i H(x_i(t_n) - x_j(t_n))$$  \hspace{1cm} (110)

and
\[
\frac{d^3 x_i(t_n)}{dt^3} = \frac{1}{\Delta t} \left[ \frac{d^2 x_i(t_n)}{dt^2} - \frac{d^2 x_i(t_{n-1})}{dt^2} \right].
\]

Introduction of the term \( \frac{d^3 x_i}{dt^3} \) allows us to take a larger time interval \( \Delta t \) in the computations while keeping the error very small.

The accuracy of the computer program was checked by comparing the results for various \( \Delta t \)'s and also by checking the reversibility of the motion. For example, the fluctuation in the computed total energy of the star system was less than 0.005 percent for the systems investigated. If \( n \) is the average density of stars and \( D \) the Debye length or dimension of the system then the numerical experiments using the model described above are very nearly exact for \( nD >> 1 \) and actually give the time-dependent solution of the nonlinear Vlasov equation.

The potential energy density for a system of \( N \) "stars," each of mass \( m \) per unit area, is given by the equation

\[
P = \frac{1}{3nG} \left\{ (2\pi Gnm)^2 (x_N - x_1) - \sum_{j=2}^{N} E_j^2(x) (x_j - x_{j-1}) \right\}
\]

where \( E_j(x) \) is defined by equation (105) such that for each \( j \), \( x_{j-1} < x < x_j \). The above definition of the potential energy is such that if all "stars" are at the same point then the potential energy of the system is zero. The kinetic energy of the system is given by the usual expression
The approximate computer program used for the calculations is included in the appendix.

The total energy $U_j$ of a star is of interest and is given by

$$U_j = \frac{1}{2} m_j \left( \frac{\text{d}x_i}{\text{d}t} \right)^2 + m_j \Phi_j.$$  \hfill (114)

The gravitational potential $\Phi_j$ is obtained after first ordering the mass sheets according to their increasing $x$ coordinates as indicated by equation (106). Thus, $\Phi_j^{(1)}$ indicates the gravitational potential of the sheet labeled $j$ which is the first sheet. Dropping the subscript $j$ we find that

$$\Phi^{\left(\frac{N}{2} + 1\right)} = \Phi^{\left(\frac{N}{2}\right)} = \pi G m \left( x^{\left(\frac{N}{2} + 1\right)} - x^{\left(\frac{N}{2}\right)} \right)$$  \hfill (115)

and

$$\Phi^{(k)} = \Phi^{\left(\frac{N}{2}\right)} + 4 \pi G m \sum_{i=\frac{N}{2}}^{\frac{N}{2} - 1} \left( x^{(i+1)} - x^{(i)} \right) \text{ for } k < \frac{N}{2}$$  \hfill (116)

and

$$\Phi^{(k)} = \Phi^{\left(\frac{N}{2}\right)} + 4 \pi G m \sum_{i=\frac{N}{2}}^{k} \left( x^{(i)} - x^{(i-1)} \right) \text{ for } k > \frac{N}{2}.$$  \hfill (117)
where \( N \) is an even integer. Also, \( \varphi \) was chosen equal to zero at the midpoint of the system which divides it into two equal masses.

The virial theorem can also be applied to a discrete system in equilibrium. Equation (6) then takes the form

\[
0 = \sum_{k=1}^{N} m\left(\frac{dx(k)}{dt}\right)^2 + \sum_{k=1}^{N} mx(k) \frac{d^2x(k)}{dt^2}
\]

\[
= 2T + \sum_{k=1}^{N} mx(k) \frac{d^2x(k)}{dt^2}
\]

(118)

where we have taken \( \frac{d^2}{dt^2} \left[ \frac{1}{2} m \sum_{k} (x(k))^2 \right] = 0 \) and \( m^k = m \) for all \( k \). Using the expression

\[
m \frac{d^2x(k)}{dt^2} = 4\pi Gm^2 \left[ \frac{N}{2} - k + \frac{1}{2} \right]
\]

(119)

we can write

\[
\sum_{k=1}^{N} mx(k) \frac{d^2x(k)}{dt^2} = 2\pi Gm^2 \left[ (N - 1)x^{(1)} + (N - 3)x^{(2)} + (N - 5)x^{(3)} \right.
\]

\[
+ \ldots + (N - 2n + 1)x^{(n)} + \ldots - (N - 3)x^{(N-1)}
\]

\[
- (N - 1)x^{(N)} \right].
\]

(120)
Assuming symmetry of the system such that 

\[ x\left(\frac{N}{2} - 2\right) = x\left(\frac{N}{2} + 2 + 1\right) \]

we can write equation (120) in the form

\[
\sum_{k=1}^{N} m x(k) \frac{d^2x(k)}{dt^2} = -\frac{1}{2} \pi G m^2 \left[ x\left(\frac{N}{2} + 1\right) + 3x\left(\frac{N}{2} + 2\right) + 5x\left(\frac{N}{2} + 3\right) + \ldots \right] \\
+ (N - 3)x^{(N - 1)} + (N - 1)x^{(N)}.
\] (121)

The potential energy of the system is given by

\[
P = \frac{1}{2} \pi G m^2 \left(\frac{N}{2}\right)^2 x^{(N)} - \sum_{k=\frac{N}{2} + 1}^{N} \left[ k - \frac{N}{2} - 1 \right] \varepsilon \left[ x^{(k)} - x^{(k-1)} \right]
\]

\[
= \frac{1}{2} \pi G m^2 \left[ x\left(\frac{N}{2} + 1\right) + 3x\left(\frac{N}{2} + 2\right) + 5x\left(\frac{N}{2} + 3\right) + \ldots \right] \\
+ (N - 3)x^{(N - 1)} + (N - 1)x^{(N)}.
\] (122)

Comparing equation (122) with equation (121) we see that

\[
\sum_{k=1}^{N} m x(k) \frac{d^2x(k)}{dt^2} = -P
\] (123)

and therefore,

\[ 2T = P \] (124)

as was shown before for a continuous system.
Initial Distributions Near Equilibrium

The one-dimensional sheet model includes individual or star-star interactions as well as the interaction of the stars with the smoothed potential of the system since the computer simply solves the equations of motion of the stars in the system. The graininess or collisional effects will be affected by going to the "fluid limit" as implied by the Vlasov equation.

The effects of graininess on the time development of a system can be determined by varying the number of sheets while keeping the total mass of the system constant. The effects of graininess must be checked to determine whether the Vlasov equation adequately describes the system under investigation. Figure 8 shows the time development of two equivalent systems with the same initial distribution but with different graininess. The curves indicate the variation in time of the kinetic energy for an initially rectangular waterbag distribution. The initial conditions are obtained by using a random number generator to give a nearly uniform distribution over a rectangular region in phase space. For all the numerical calculations the gravitational constant was normalized such that \( \bar{h} \pi G = 1 \). Also all time scales are normalized to \( \tau_c = \tau_c(t = 0) \). The initial ratio of kinetic to potential energy for both curves in figure 8 is 2, whereas the equilibrium value is 0.5. The upper curve is for a system of 1000 "stars" with \( nD \approx 1000 \) and \( m = 2 \). The lower curve is for a system of 2000 "stars" with \( nD \approx 2000 \) and \( m = 1 \). The oscillations of the kinetic energy are identical for the two systems indicating that the graininess effects do not affect the development of the system. Since
collisional effects for 1000 particle systems are negligible for time scales of present interest they will certainly be negligible for stellar systems which we want to simulate and which contain about $10^{11}$ stars. We also note from figure 8, that after only a few periods nonlinear damping or phase mixing has almost eliminated the oscillations in the kinetic energy and the kinetic energy remains near the equilibrium value indicated by the dashed line. We can thus be assured that the Liouville or Vlasov equation correctly describes the system.

To determine when the effects of graininess become important, the evolution of several equivalent systems with varying numbers of "stars" is compared. Figure 9 shows the variation of the kinetic energy for equivalent systems which contain from 20 to 500 "stars." For the system with 500 "stars" the behavior is still very similar to that shown in figure 8. The time behavior of the energy distribution function and the density for equivalent systems which contain more than 500 "stars" is also very similar. As can be seen from figure 9 when the number of stars of the system becomes 200 or less the time behavior of the kinetic energy differs markedly from that of figure 8. Also, the time development of the energy distribution function and of the density is different for the systems shown in figure 9. In general, it was found that oscillations in the various parameters of the system persisted much longer when the number of stars of the system was less than 500.

Figure 10 shows the time development of the density of the waterbag distribution for a system of 2000 "stars" with a mass $m$ per unit area equal to one. The ratio of initial to equilibrium energy (as given by
equation (60)) for this system is 1.33 and the initial ratio of kinetic to potential energy is 2. The corresponding variation of the kinetic energy is shown by the lower curve of figure 8. The dashed curve in figure 10 is the theoretical equilibrium density as obtained from equation (25). After only a few periods $2\pi\tau_c$, the "experimental" density is close to the calculated value. The time development of the energy distribution function $F(U)$ for the same system is shown in figure 11. $F(U)$ represents the relative number of sheets in an energy interval $\Delta U$. The dashed curve shown in figure 11 represents the theoretical distribution. Again, after only about six periods, $2\pi\tau_c$, the "experimental" distribution closely approximates the theoretical distribution. Thus, even for values of the ratio of initial to equilibrium energy of 1.33 the system approaches its nonstationary equilibrium state closely, in a time of the order of $2\pi\tau_c$. The best way to see the approach to equilibrium is to take a sequence of pictures showing the time development of the system in phase space. Figure 12 shows such a sequence for the system of 2000 "stars." Each "star" is represented by a small circle. While approaching the nonstationary equilibrium state indicated by the area inside the oval at time $t = 33\, \tau_c$ in figure 12, the system rotates in phase space with a frequency near $1/\tau_c$. An interesting feature of the time development shown in figure 12 is the development of "spiral arms" in phase space. Such arms were found to develop in all systems where the initial energy was larger than the equilibrium energy. The relatively few particles in the long arms shown in figure 12 are required to accommodate the excess energy and they correspond to the high energy peak shown in
figure 11 at time $t = 35 \tau_{co}$. The arms develop because the period of oscillation of the sheets increases with energy. Figure 13 shows the period as a function of energy for the equilibrium state.

The calculations just described for the 2000 sheet system with an initial waterbag distribution were also performed for an equivalent 1000 sheet system. The variation of the kinetic energy for this system is given by the upper curve in figure 8. The variation of the density with time is shown in figure 14 and is found to be very similar to that of the 2000 sheet system. The corresponding time development of the energy distribution function is shown in figure 15 which is also very similar to that of the 2000 sheet system.

The results of Hénon (ref. 26) show that the oscillations for a system of spherically concentric mass shells are damped more rapidly than for the one-dimensional system shown in figure 8. The reason for this is that the period of oscillation of a spherical shell increases much more rapidly with increasing energy than is the case for a mass sheet in the one-dimensional system. Therefore, the effectiveness of phase-mixing which depends on the change in period with changing energy is different for the two models.

In the equilibrium state the gravitational potential is time independent and the trajectories of individual "stars" form fixed closed contours in phase space as shown in figure 16. Figure 17 shows the actual trajectories for five mass sheets for the system shown in figure 12. The phase space trajectories shown in figure 17 are typical for systems which are initially near the equilibrium state. Initially, while the system is approaching the equilibrium configuration the orbits
are very much perturbed. This causes the redistribution of the orbits required to approach equilibrium. After only about two orbits the trajectories are nearly the same as the theoretical orbits.

The time behavior described so far is characteristic of the waterbag model whenever the initial energy given by equation (56) is near the equilibrium energy given by equation (60). Since the system approaches the equilibrium configuration within the limits of energy conservation it appears that the steady state given by equation (21) does have a physical meaning. The stability of the steady state has been checked using an initial distribution which satisfies equation (21). Even after many periods $2\pi\tau_c$ the system is found to remain in the equilibrium state. The effect of a perturbation of the equilibrium state has also been checked. Figure 13 shows the time development of a sinusoidal perturbation of the equilibrium state. The perturbation results in nonlinear effects and because the outermost stars are now outside the main system they have a longer period and cause the development of arms. After this small perturbation the nonstationary equilibrium state can never be completely reached.

A system with an initially constant density between $\pm x_0$ and with a Maxwellian velocity distribution has a time development similar to that for the waterbag distribution. For example, if the initial ratio of kinetic to potential energy is of the order of the equilibrium value then the system shows a time behavior very similar to that shown in figure 12. However, the arms in phase space develop more rapidly because some of the high velocity mass sheets are now able to stay outside the main system longer and therefore have a longer period. For
example, figure 19 shows the variation of the kinetic energy of such a system with an initially Maxwellian velocity distribution. The initial ratio of kinetic to potential energy is 2. We see from figure 19 that the kinetic energy of the system quickly approaches its equilibrium value. The corresponding time development of the density and energy distribution function are shown in figure 20. The dashed curves are those obtained from equations (53) and from the relation \( F(U) = A \exp(-kU) \). The "experimental" density approaches its theoretical value closely whereas the energy distribution again develops a high energy bump. Figure 21 shows the time development of the system in phase space. With the exception of the more rapid development of the "spiral arms" the time development in phase space is close to that shown in figure 12 for the waterbag model.

**Initial Distributions Far From Equilibrium**

If the energy of the initial rectangular waterbag distribution is much larger than the equilibrium energy given by equation (60) the system is unstable and will break up into smaller clusters. Figure 22 shows the time development of the kinetic energy for a waterbag distribution. The system contains 1000 "stars" and the ratio of initial to equilibrium energy is 6 whereas the ratio of initial kinetic to potential energy is 32. Figure 22 shows that the oscillations in the kinetic energy are damped only very slowly with time. The time development of the density for the same system is shown in figure 23. From the density plots we can see that the system quickly breaks up into three clusters which later combine into two clusters. Similar information is obtained from the time development of the velocity
distribution function shown in figure 24. There are large fluctuations in time of the energy distribution function of the system as shown in figure 25. Again, the time development of the system can best be observed by viewing it in phase space. Figure 26 shows the time development for the system of 1000 "stars" in phase space. The break up into three clusters is clearly demonstrated. However, after a long time the system condenses into two clusters with a halo of high energy "stars." In figure 27 the position of every 20th "star" for this system is plotted as a function of time. After only one complete oscillation the bunching of the "stars" into four distinct regions is clearly visible.

The initial development of the instability depends on the initial fluctuation of the "star" density in phase space. To illustrate the effects of initial fluctuation of the phase-space density the calculations for the unstable system of 1000 stars have been repeated by using a different random number sequence to obtain the initial distribution. The results are shown in figures 28 through 31. We see that the system now breaks up into four clusters. However, after a long time the system in phase space appears very similar to the previous case in that only two clusters remain with a halo of high energy "stars."

Changing the graininess of the system does not affect the above results. To show this the calculations were again repeated for an equivalent system with 3000 "stars." The results are shown in figures 32 through 34. The system appears to break up into three clusters but it quickly condenses into two clusters by effectively winding itself
into two spirals. The Vlasov character of the system is clearly demonstrated by the continuity of the spiral arms shown in figure 34.

Initial distributions with large initial ratios of kinetic to potential energy which have a constant density between \( \pm x_0 \) and a Maxwellian velocity are also unstable and will break up into smaller clusters. However, the break up now does not occur as readily as for the waterbag distribution.

Initially "Cold" Systems

We next consider an initial distribution where the particles initially are placed a unit distance apart and have zero velocity. Each "star" is then attracted towards the center of the system with a constant acceleration. In the absence of any perturbation the crossings will not take place until the "stars" reach the center of the system. All the stars will reach the center simultaneously and cross at the same instant. After this time the force on a particular star changes sign and the system begins to expand until it reaches its original size with zero kinetic energy and the process will repeat. The results of the calculations for a 1000 "star" system are shown in figures 35 and 36. Figure 35 shows the variation of the kinetic energy with time. The curve is very similar to that shown in figure 28 except that now there is no damping of the oscillation. If the calculation were extended for a longer time the error in the numerical solution would eventually cause "stars" to get out of order and the oscillations would be damped. The variation of the density with time is shown in figure 36. As all particles approach the center of the system simultaneously the density at that point increases without limit. The
system has the interesting property that during the expanding phase the velocity of recession of the stars as viewed from any particular star is constant in either direction. This is similar to the observed expansion of the Universe.

Another initially "cold" system investigated is the two-stream case. The particles are initially uniformly spaced along \( x \) with half the particles at velocity \( V_0 \) and the other half at velocity \(-V_0\). The variation of the kinetic energy for such a system with 1000 stars is shown in figure 37. After a slight initial rise the oscillations of the kinetic energy show a strong damping. The time development in phase space for this system is shown in figure 38. The two streams wind around each other and in this manner attempt to reach some sort of equilibrium state.

In almost all cases considered so far we found that the system tried to approach a steady state and the dimensions of the system are then of the order of a Debye or Jeans length. This result is also found if we consider the dimensions of a globular cluster or even a galaxy. Thus, if we extend this simple Newtonian picture to a larger scale we find that the dimensions of the Universe are

\[
R \sim D = V_{Tc} \sim \frac{V_T}{\sqrt{G \rho}} .
\]  

(125)

Assuming a spherical geometry and uniform mass density we can write \( R \) as

\[
R \sim \frac{V_T}{\sqrt{G M/R^3}}
\]  

(126)
or
\[
\frac{GM}{RV_T^2} \sim 1
\]  
(127)

where \( M \) is the total mass in the Universe. For \( V_T \sim C \), the velocity of light, equation (127) becomes the well known relation
\[
\frac{GM}{RC^2} \sim 1
\]  
(128)

often used (refs. 59 and 60) to define the radius of the Universe.

However, the difference is that we have used a dynamic picture to obtain equation (128) instead of the static one generally used. We should also note that as soon as we know the average density of a self-gravitating system we can compute its characteristic frequency. For a typical globular cluster \( \tau_c \sim 3 \times 10^6 \) years, for a galaxy \( \tau_c \sim 10^3 \) years.

Assuming that the average density in the neighborhood of our Galaxy is characteristic of the Universe, we find that for the Universe \( \tau_c \sim 8 \times 10^{10} \) years. The characteristic period for the Universe is close to the period of oscillation generally quoted in the Big Bang and oscillating Universe theories.

**Unstable Stationary States**

In the previous chapter it was shown that if a stationary distribution function always decreases in going outward from the center of the system then the stationary state is a minimum energy state and is stable. However, if this condition is not satisfied, deformations of the waterbag contours are possible and there is a possibility that the system is unstable. The interchange instability which destroys the
stationary state has been numerically investigated for two-contour systems with \( a_1 = -a_2 \) (refer to figure 4). The results are shown in figures 39 and 40. Figure 39 illustrates the interchange instability for a system of 2000 stars with \( v \) the ration of the minimum to maximum "star" energy of the stationary state equal to 0.4. At the initial time the contours satisfy equations (41) and (42). The "equilibrium" contours are quickly distorted while the heavier outer waterbag tries to displace the inner bag. Figure 40 shows the time development of unstable system for \( v = 0.25 \). The growth of the instability is now much slower because the central waterbag or hole is now smaller.

**Thermalization Effects In a One-Dimensional Stellar System**

In a one-dimensional system crossings will always take place so that one can expect the system to approach a Maxwellian distribution. An exact double precision computer program was used to investigate thermalization effects for low numbers of "stars." By an exact program we refer to a program where the shortest crossing time for a mass sheet is determined and the system is then advanced by that time and the process is repeated. This, of course, differs from the model used for the results present so far where the system was always advance for a constant time interval \( \Delta t \), irrespective of the number of crossings during this time interval. The "exact" program is very accurate. For example, after time reversing the numerical integration of a typical system investigated it was found that the initial conditions were reproduced accurately to 12 digits.
Lecar (ref. 25) has investigated the exact one-dimensional motion of low numbers of "stars." He finds that to order $nD\tau_c$ there exists no thermalization. In the present study of thermalization effects a low number of mass sheets was chosen so that the system can be followed for a long period of time. In order to obtain meaningful results time averages of the systems are investigated. The constant time interval used in the averaging process is taken to be smaller than the average crossing time for sheets in the system. The time averages are taken over times of the order $n^2D^2\tau_c$. It should be noted that the investigation of thermalization effects is still in progress and only some of the initial numerical results are presented here.

For all of the systems investigated it was found that the time averaged potential and kinetic energies satisfied the virial theorem. That is, $<T>/<P> = 0.5$ with an accuracy of at least three digits. The $<>$ signify time-averaged quantities. As discussed by Ford (ref. 61) the Poincare recurrence time and the general behavior of the system is dependent on the initial conditions. For example, we can chose special initial conditions such that the time behavior of the system is very nearly periodic. However, for the results presented here the initial conditions were arbitrarily chosen.

Figure 41 shows the time averaged velocity distribution and density for a three-particle system. The solid line for the velocity distribution corresponds to a Maxwellian distribution and is obtained by integrating $A \exp{-\kappa U}$ over $x$. That is,

$$f(v) = \frac{N}{2} \sqrt{\frac{2m}{\pi}} \exp\left[-\frac{1}{2} m\kappa v^2\right].$$  (129)
The solid line for the density is given by equation (53). The value of $\kappa$ is obtained from

$$
\kappa = \frac{3N}{2<T> + 2<P>} = \frac{N}{2<T>}.
$$

(130)

The circles in figure 41 represent the time-averaged numerical results. The "experimental" velocity distribution is near the Maxwellian distribution. However, the variation of the density indicates that there is some near-periodic behavior of the system. Such near-periodic behavior is likely to occur for very low numbers of "stars." Figure 42 shows the results for a four-sheet system. Again, the variation of the time-averaged velocity distribution and density indicates that there is near-oscillatory behavior of the system. This oscillatory behavior has also been observed by plotting the kinetic energy of the system as a function of time. The fluctuations of the kinetic energy of the systems investigated are very violent and show no decrease in time. The fact that these fluctuations show no decrease in time indicates that a Maxwellian distribution can be reached while the fluctuations in the kinetic energy remain extremely violent. For example, figure 43 shows the time variation of the kinetic energy for a six-particle system. The corresponding time-averaged velocity distribution and density are shown in figure 44. We see that the "experimental" points are near the Maxwellian distribution for both curves. Figure 45 shows that for a ten-particle system the "experimental" velocity distribution and density points reproduce the theoretical Maxwellian curves. These results indicate that there exists thermalization on a long-time scale of the
order of $n^2D^2\tau_c$. The calculations were also performed for different initial conditions of the system but with results nearly identical to those presented.

Two-Dimensional Computer Experiments for Stellar Systems

The motion of mass rods that are of infinite extent in the $z$-direction has been computed for 400 and 500 rod systems. The force acting on a particular mass rod is obtained by summing directly over the $1/r$ force from each mass rod. This is a time consuming process and the application of fast methods of solving the Poisson equation would speed up the calculations. The relatively large grid size which would be required by the method of solving the Poisson equation will smooth the force due to the near neighbors of the particles and will affect the evolution of the system. If the force acting on a mass rod is obtained by summing directly the $1/r$ force from each particle in the system, then the effect of the near neighbors is of course included. It would be more desirable to study the evolution of a system of mass points moving in a plane. This can also be done by simply summing the $1/r^2$ force from each mass point. However, the more rapid divergence of the $1/r^2$ force for near neighbors now requires that a much smaller time step be used in the computations resulting in a considerable increase in computer time.

The system is advanced in time in the following manner. First, the force acting on all particles is computed by summing the $1/r$ force for all particles. Second, the system is advanced for a small time step $\Delta t$ and the process is repeated. The results of the calculations are displayed in $x$-$y$ coordinate space. During the calculations the total
energy and angular momentum is computed to check on the accuracy of the computations. The normalizations $4\pi G = 1$ and $m = 1$ have been used for all the calculations.

Figure 46 shows the time development of a system of 400 mass rods which has an initially rectangular distribution of uniform density in x-y space. The system has an initial thermal energy equal to $1/5$ of the initial potential energy plus an initial solid body rotation equal to nearly twice that required to oppose the gravitational force towards the center of the system. It can be seen from figure 46 that the system quickly develops into a barred spiral. However, at a later time the spiral structure has almost completely disappeared and the system approaches a configuration similar to an elliptical galaxy. The time has been normalized to $\omega_r^{-1}$, the inverse of the frequency of the initial rotation.

The remaining two-dimensional calculations were performed for 500 particle systems which have an initially uniform circular distribution in x-y space and zero thermal velocity. The evolution of such systems is then studied for various values of initial solid body rotation. The initial positions are obtained by using a random number generator which gives a nearly uniform distribution over a circular region of the x-y plane.

We now present the results for the case where the frequency of rotation, $\omega_r$, equals $\omega_g$, the frequency required such that the centrifugal force balances the gravitational force. Thus,

$$\omega_r = \omega_g = \sqrt{2\pi G \rho}$$

(131)
where $\rho$ is the mass density of the rods per unit length. The resulting evolution of the system is shown in figure 47. The time has been normalized to $\omega^{-1}_g$. Figure 47 shows that the system is relatively stable. At $\omega = 6.32 \omega^{-1}_g$ there appear four irregular spiral arms. However, at a later time the spiral arms almost completely disappear and the system takes an appearance reminiscent of an elliptical galaxy.

The results for the case of zero initial rotation are presented next. Figure 48 shows that after an initial implosion the system expands and presents some highly irregular filamentary structure. After a second implosion the temperature of the system increases due to the randomness of the initial positions. The pressure due to the temperature then tends to reduce the oscillations and the system again takes a form similar to an elliptical galaxy.

For $\omega = \frac{1}{2} \omega_0 g$ the system again contracts initially and then expands again. The results are shown in figure 49. An irregular structure appears initially which tends to disappear at a later time. Also at time $t = 1.2 \omega^{-1}_g$ the system is clustered into two aggregates which combine again at a later time.

In figure 50 the results for the case $\omega = 1.3 \omega_0 g$ are shown. The system pulsates and shows some irregular structure. The general behavior is very similar to that of the previous case for $\omega = \frac{1}{2} \omega_0 g$. The results for the two-dimensional stellar system are of a preliminary nature and additional work is required to investigate the evolution for systems with larger numbers of "stars."
CHAPTER IV

THEORY OF A BOUNDED PLASMA

In the second part of this study a plasma with a fixed neutralizing ion background is investigated. We again investigate analytically the stationary states of the system and later make use of numerical experiments to study the time development of the systems. The computer model used is semi-two-dimensional in that a magnetic field can be included in the numerical solution to investigate its effects on the oscillations in a plasma slab. The problem to be investigated is different from most of those previously investigated in that the system is strongly inhomogeneous. The fixed ion background is constant between \( \pm x_i \) and is zero outside this region. The electrons will then form a sheath at each side of the plasma slab. For the case of initially cold electrons the problem is that of quasi-oscillations in a plasma slab considered by Leavens and Leavens (ref. 62).

Equilibrium Solutions

The characteristic time for collective effects in a plasma is given by

\[
\omega_p^{-1} = \sqrt{\frac{me}{2ne^2}}
\]

and the characteristic length is the Debye length given by

\[
\lambda_D = \frac{V_T}{\omega_p} = V_T \sqrt{\frac{me}{2ne^2}}.
\]
In equations (132) and (133) \( m \) is the electron mass, \( n \) the electron density, \( e \) the magnitude of the electronic charge, \( \varepsilon_0 \) the permittivity of free space, and \( V_T \) is the thermal velocity of the electrons. The same analysis as that used for stellar systems in Chapter II will be used for the plasma. The Vlasov equation (3) now takes the form

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{eE}{m} \frac{\partial f}{\partial v} = 0
\]

(134)

where \( E = -\frac{\partial \varphi}{\partial x} \) and \( f \) is the electron distribution function.

\( \varphi \) is given by the Poisson equation

\[
\frac{\partial^2 \varphi}{\partial x^2} = -\frac{e}{\varepsilon_0} \left( n_i - \int f \, dv \right)
\]

(135)

where \( n_i \) is the fixed ion density.

**Virial Theorem**

Following the method used for a stellar system in equilibrium we now investigate identity

\[
\frac{1}{2} m \frac{d^2 x}{dt^2} x^2 = m \left( \frac{dx}{dt} \right)^2 + mx \frac{d^2 x}{dt^2}.
\]

(136)

Multiplying equation (136) by \( f \) and integrating over phase space we obtain the result

\[
\frac{1}{2} m \frac{d^2}{dt^2} \int \int x^2 f \, dx \, dv = m \int \int \left( \frac{dx}{dt} \right)^2 \, dx \, dv + m \int \int x f \frac{d^2 x}{dt^2} \, dx \, dv.
\]

(137)

Since the system is in equilibrium the lefthand side of equation (137) is zero. Also the first term on the righthand side of equation (137)
is $2T$ where $T$ is the kinetic energy of the electron system. We must now investigate the last term of equation (137). Using Newton's Law, we can replace $\frac{d^2x}{dt^2}$ by $\frac{e}{m} \frac{d\phi}{dx}$. Also, the electron density is given by

$$n_e = \int \sigma \, dv.$$  \hspace{1cm} (138)

Using the Poisson equation

$$\frac{d^2\phi}{dx^2} = -\frac{e}{\varepsilon_0} \left[ n_1 - n_e \right]$$  \hspace{1cm} (139)

we can write the electron density as

$$n_e = \frac{\varepsilon_0}{e} \frac{d^2\phi}{dx^2} + n_1. \hspace{1cm} (140)$$

The last integral of equation (137) can then be written in the form

$$m \int \int x f \frac{d^2x}{dt^2} \, dx \, dv = \varepsilon_0 \int x \left( \frac{d\phi}{dx} \right)^2 + \frac{e}{\varepsilon_0} n_1 \right) \, dx$$

$$= \varepsilon_0 \int x \frac{d\phi}{dx} \frac{d^2\phi}{dx^2} + e \int n_1 \frac{d\phi}{dx} \, dx. \hspace{1cm} (141)$$

Using an integration by parts we obtain the result

$$\varepsilon_0 \int x \frac{d\phi}{dx} \frac{d^2\phi}{dx^2} \, dx = \varepsilon_0 \left( \frac{d\phi}{dx} \right)^2 \bigg|_{x_s}^{x_e} - \varepsilon_0 \int (\frac{d\phi}{dx})^2 \, dx - \varepsilon_0 \int x \frac{d\phi}{dx} \frac{d^2\phi}{dx^2} \, dx \hspace{1cm} (142)$$

or
where \( P_e \) is the potential energy of the electron system. We must now consider the last integral in equation (141). An integration by parts gives the result

\[
\int_{x_1}^{x} e_n x \frac{d\varphi}{dx} \, dx = \left[ x_1 e_n \frac{d\varphi}{dx} \right]_{x_1}^{x} - \int_{x_1}^{x} e_n \varphi \, dx
\]

\[
= - P_1
\]  

where \( x_1 \) defines the boundary of the fixed neutralizing ion background and \( P_1 \) is related to the potential energy of the ion background. That is, \( n_i = n_0 \) for \( -x_1 < x < x_1 \) and \( n_i = 0 \) otherwise. In general the potential energy of a one-dimensional system of charges is given by

\[
P = \frac{1}{2} \sum \phi_i q_i \rightarrow \frac{1}{2} \int \rho \varphi \, dx.
\]  

Since we can always choose \( \varphi = 0 \) at \( x = x_1 \) equation (144) can be put in the same form as equation (145). However, for the cases considered we always chose \( \varphi = 0 \) at \( x = 0 \). Since we only treat systems with zero net charge \( x \left( \frac{d\varphi}{dx} \right)^2 \bigg|_{x_1}^{x} = 0 \), and equation (137) takes the form
\[ 2T = P_e + P_i. \]  \hspace{1cm} (146)

In the limit as \( x_i \to 0 \) we have two electron sheaths held together by a thin but dense positive charge. The ions then simply take the place of a positively charged grid through which the electrons can pass freely and \( P_i = 0 \). Equation (146) then takes the usual form

\[ 2T = P_e. \]  \hspace{1cm} (147)

The total energy of the system is always given by \( T + P_e \) and is a constant of the motion.

**The Waterbag Distribution**

The stationary solution for a water-bag distribution will now be obtained. Again, if a stationary state exists then it will be described by a distribution function which is a function only of \( U \), where

\[ U = \frac{1}{2}mv^2 - e\varphi \]

is the energy of a charge sheet with an electric charge per unit area of magnitude \( e \). Thus, \( F(U) = A = \text{constant for } \varphi \leq U \leq \varepsilon \) and \( F(U) = 0 \) for \( U > \varepsilon \). Following the method used in Chapter II, the Poisson equation (135) can be written as

\[
\frac{d^2 \varphi}{dx^2} = -\frac{e}{\varepsilon_0} \left[ n_i - \frac{2A}{\sqrt{2m}} \int_{-\varepsilon}^{\varepsilon} \frac{dU}{\sqrt{U + e\varphi}} \right]
\]

\[
= -\frac{e}{\varepsilon_0} \left[ n_i - \frac{4A}{\sqrt{2m}} \sqrt{\varepsilon + e\varphi} \right] \hspace{1cm} (148)
\]
where
\[ n_1 = n_0 \text{ for } |x| < |x_1| \]
and
\[ n_1 = 0 \text{ for } |x| > |x_1|. \]

We now normalize \( \varphi \) and \( x \) such that
\[ \phi = -\frac{e^\varphi}{\varepsilon} \quad (149) \]
and
\[ \zeta^2 = x^2 \left( \frac{n_0^2}{\varepsilon} \right). \quad (150) \]

Equation (148) then takes the form
\[ \frac{d^2\phi}{d\zeta^2} + 1 + \alpha \sqrt{1 - \phi} = 0 \text{ for } \zeta < \zeta_1 \quad (151) \]
and
\[ \frac{d^2\phi}{d\zeta^2} + \alpha \sqrt{1 - \phi} = 0 \text{ for } \zeta > \zeta_1 \quad (152) \]
where
\[ \alpha = \frac{2A}{n_0} \sqrt{\frac{2\varepsilon}{m}}. \quad (153) \]

A first integration of equations (151) and (152) yields
\[ \left( \frac{d\phi}{d\zeta} \right)^2 - 2 \phi + \frac{2\alpha}{3} \left( (1 - \phi)^{3/2} - 1 \right) = 0 \text{ for } \zeta < \zeta_1 \quad (154) \]
and
\[ \left( \frac{d\phi}{d\zeta} \right)^2 - \frac{4\alpha}{3} (1 - \phi)^{3/2} = 0 \text{ for } \zeta > \zeta_1. \quad (155) \]
The constants of integration in equations (154) and (155) have been evaluated by using the boundary conditions $\frac{d\Phi}{d\zeta} = 0$ at $\zeta = 0$ and $\frac{d\Phi}{d\zeta} = 0$ for $\zeta > \zeta_s$ where $\zeta_s$ defines the boundary of the system. Using the condition that $\frac{d\Phi}{d\zeta}$ and $\Phi$ are continuous at the boundary of the neutralizing ion background, $\zeta_1$, we find that

$$\Phi(\zeta_1) = \Phi_1 = \frac{2\alpha}{3}. \quad (156)$$

A second integration gives the final result

$$\pm \zeta = \int_0^\Phi \left( 2\eta + \frac{4\alpha}{3} \left[ (1 - \eta)^{3/2} - 1 \right] \right)^{-1/2} d\eta \quad \text{for} \quad \zeta < \zeta_1 \quad (157)$$

and

$$\pm \zeta = \zeta_1 + \sqrt{\frac{12}{\alpha}} \left( (1 - \frac{2\alpha}{3})^{1/4} - (1 - \Phi)^{1/4} \right) \quad \text{for} \quad \zeta > \zeta_1 \quad (158)$$

where $\zeta_1$ is given by equation (157) with the upper limit of the integral equal to $\Phi_1$. The solution of equations (157) and (158) are shown in figure 51 which gives $\Phi$ as a function of $\ln \zeta$ for various values of $\alpha$. Equations (149) and (150) can then be used to obtain $\Phi$ as a function of $x$ as follows. The relation

$$n_o = \frac{N}{2x_1} \quad (159)$$

where $N$ is the total number of "electrons" or "ions" in the system and equation (153) are used to eliminate $n_o$ and $\epsilon$ from equation (150). We then obtain

$$x_1 = \left[ \frac{\alpha \zeta_1}{L \lambda_e} \cdot \frac{N \epsilon_0}{m e} \right]^{1/3} \quad (160)$$
and the numerical values for $n_e$ and $\varepsilon$ are easily obtained. From equation (160) we find the constant of proportionality in equation (150) from the relation
$$\sqrt{\frac{\varepsilon \varepsilon_0}{n_0 e^2}} = \frac{x_1}{\xi_1}. \quad (161)$$

Figure 52 shows the variation of the electron density for three values of $\alpha$ for a system with $N = 1000$, $A = 0.24$ and $\varepsilon_0 = m = e = 1$. The corresponding variation of the electric field is shown in figure 53. The equilibrium contour in phase space $v_\pm(x)$ of the water bag is given by the equation
$$v_+(x) = -v_-(x) = \sqrt{(\varepsilon + e\varphi) \frac{2}{m}}, \quad (162)$$
and figure 54 shows the equilibrium contours for four values of $\alpha$.

The parameters used in the calculation are the same as those for the previous figure. Equations (157) and (158) cannot be used if the positive ion background reduces to a thin but dense charge and $x_1 \to 0$. We then have a positively charged permeable grid through which the electrons can pass freely. The problem then becomes that of two electron sheaths that are held together by a dense, thin positive charge. The Poisson equation (135) can then be written as
$$\frac{d^2 \varphi}{dx^2} = \frac{4eA}{\varepsilon_0 \sqrt{2m}} \sqrt{\varepsilon + e\varphi}. \quad (163)$$

The boundary conditions are that $\frac{d\varphi}{dx} = 0$ and $\varphi = -\varepsilon/e$ for $x > x_s$ and that $\lim_{x \to 0^+} \frac{d\varphi}{dx} = -\frac{Ne}{2e_0}$ and $\varphi = 0$ at $x = 0$. Using these boundary conditions, the solution for equation (163) takes the form
\[
\frac{d\psi}{dx} = -\frac{Ne}{2\epsilon_0} \left(1 + \frac{\epsilon \psi}{\epsilon}\right)^{3/4}
\]

and finally,

\[
\pm x = \left(\frac{8\epsilon \epsilon}{Ne^2}\right) \left[1 - \left(1 + \frac{\epsilon \psi}{\epsilon}\right)^{1/4}\right].
\]

The value of \( \epsilon \) is found from the relation

\[
N = 2 \int_0^x n_e(x) \, dx
\]

\[
= \frac{8A}{\sqrt{2m}} \int_0^x \sqrt{\epsilon + \epsilon \phi} \, dx
\]

and is given by

\[
\epsilon = \left[\frac{Ne}{8} \sqrt{\frac{A^3}{\epsilon^4 \alpha^2}} \sqrt{\epsilon m}\right]^{4/3}
\]

Figure 55 shows the variation of \( E = -\frac{d\psi}{dx} \) given by equation (164) as a function of \( x \).

To measure the effectiveness of phase mixing during the approach of the system to an equilibrium state it is of interest to determine the variation of the period of oscillation as a function of the total energy of a charge sheet. We will consider the case where \( x_1 = 0 \). The equation of motion for a charge sheet is

\[
m \frac{d^2x}{dt^2} = e \frac{d\psi}{dx} = -\epsilon \frac{d\psi}{dx}.
\]

Using the normalization given by equation (149) we can write equation (168) as

\[
\Phi = 1 - (1 - z)^4
\]
where \( z = x \left( \frac{\delta e}{\epsilon} \right) - 1 \). Therefore,

\[
\frac{d\phi}{dz} = 4(1 - z)^3
\]  

(170)

and equation (166) takes the form

\[
\frac{d^2 z}{dt^2} + \frac{N^2 e^4}{16m^2 \epsilon^2} (1 - z)^3 = 0.
\]  

(171)

A first integration of equation (171) gives the result

\[
\left( \frac{dz}{dt} \right)^2 = \frac{N^2 e^4}{\delta m^2 \epsilon^2} \left[ (1 - z)^4 - (1 - \delta) \right]
\]  

(172)

where \( \delta = U/\epsilon \) and \( U \) is the total energy of the sheet under consideration. A second integration of equation (171) gives the period of oscillation as

\[
\tau = \frac{\delta e}{Ne} \sqrt{2m\epsilon} \int_0^1 \left( 1 - (1 - \delta)^{1/4} \right. \left. \left( (1 - z)^4 - (1 - \delta) \right)^{-1/2} dz
\]

(173)

where \( s = 1 - z \) and the limits of integration were found from equation (172) by setting \( \frac{dx}{dt} = 0 \). The variation of period as a function of particle energy is shown in figure 56. The time \( \tau_0 \) has been arbitrarily chosen. Because of the strongly nonlinear electric field the period varies from zero to infinity. For the case where \( x_1 \) is not zero the period will still grow without bound as \( U \) approaches \( \epsilon \). The variation of period with energy as shown in
figure 13 for a stellar system shows only a slight increase of the period with increasing and is in sharp contrast with the results for the plasma sheath.

Maxwellian Distribution

The stationary state for a system which is described by a distribution function

\[ F(U) = A \exp(-\kappa U) \]  

will now be calculated. The electron density is given by the equation

\[ n_e = 2 \int_0^\infty f \, dv \]

\[ = \frac{2A}{\sqrt{2m}} \int_{-\varphi}^\infty \frac{\exp(-\kappa U)}{\sqrt{U + \varphi}} \, dU \]

\[ = 2A \sqrt{\frac{\pi}{2m\kappa}} \exp(\kappa \varphi) . \]  

(175)

Using equation (175) and the normalizations

\[ \xi^2 = \frac{e^2 n_0 \kappa}{\varepsilon_0} x^2 \]  

(176)

and

\[ \Phi = -e\varphi \]  

(177)

the Poisson equation (135) becomes

\[ \frac{d^2\Phi}{d\xi^2} = 1 - \alpha \exp(-\Phi) \text{ for } \xi < \xi_1 \]  

(178)
and
\[ \frac{d^2 \phi}{d \zeta^2} = - \alpha \exp(-\zeta) \text{ for } \zeta > \zeta_1 \]  \hspace{1cm} (179)

where
\[ \alpha = \frac{2A}{n_0} \sqrt{\frac{\pi}{2\nu k}} \]  \hspace{1cm} (180)

A first integration of the Poisson equation yields

\[ \left( \frac{d \phi}{d \zeta} \right)^2 = 2 \left( \phi + \alpha \exp(-\phi) \right) - 2\alpha \text{ for } \zeta < \zeta_1 \]  \hspace{1cm} (181)

and

\[ \left( \frac{d \phi}{d \zeta} \right)^2 = 2\alpha \exp(-\phi) + 2 \left( \phi_1 - \alpha \right) \text{ for } \zeta > \zeta_1. \]  \hspace{1cm} (182)

The constants of integration were evaluated by using the boundary conditions that \( \phi = \frac{d \phi}{d \zeta} = 0 \) at \( \zeta = 0 \) and that \( \phi \) and \( \frac{d \phi}{d \zeta} \) are continuous at \( \zeta = \zeta_1 \). To satisfy the condition \( \lim_{\zeta \to \infty} \frac{d \phi}{d \zeta} = 0 \) in equation (182) we must have \( \phi_1 = \alpha \) and equation (182) simplifies to

\[ \left( \frac{d \phi}{d \zeta} \right)^2 = 2\alpha \exp(-\phi) \text{ for } \zeta > \zeta_1. \]  \hspace{1cm} (183)

The final solution is given by

\[ \zeta = \int_0^\phi \left\{ 2(\eta + \alpha \exp(-\eta)) - 2\alpha \right\}^{-1/2} \, d\eta \text{ for } \zeta < \zeta_1 \]  \hspace{1cm} (184)

and

\[ \zeta = \sqrt{\frac{2}{\alpha}} \left( \exp(\phi/2) - \exp(\alpha/2) \right) + \zeta_1 \text{ for } \zeta > \zeta_1. \]  \hspace{1cm} (185)

For the case where \( x_1 = 0 \) the Poisson equation is
\[ \frac{d^2 \phi}{dx^2} = -\beta \exp(-\phi) \]  

(186)

where

\[ \beta = \frac{e^2 A}{\epsilon_0} \sqrt{\frac{2\pi k}{m}}. \]  

(187)

After integrating equation (186) one obtains the result

\[ x = \left( \frac{4\epsilon_0}{N\kappa e^2} \right) \left( e^{\phi/2} - 1 \right) \]  

(188)

or

\[ \phi = 2 \ln \left[ 1 + \left( \frac{4\epsilon_0}{N\kappa e^2} \right) x \right]. \]  

(189)

The constants of integration have been evaluated by using the boundary condition \( \lim_{x \to \infty} \frac{d\phi}{dx} = 0 \), \( \lim_{x \to 0^+} \frac{d\phi}{dx} = \frac{Ne^2 \kappa}{2\epsilon_0} \), and \( \phi = 0 \) at \( x = 0 \). The electron density corresponding to equation (189) is

\[ n_e(x) = \frac{Ne^2 \kappa}{8\epsilon_0} \left[ 1 + \frac{Ne^2 \kappa}{4\epsilon_0} x \right]^{-2}. \]  

(190)

Figure 57 shows \( \phi \) and \( \frac{d\phi}{dx} \) for the case where \( x_1 = 0 \) and \( \epsilon_0 = 1 \), \( \kappa = 1 \), \( e = 1 \), and \( N = 4 \).

Minimum Energy Principle

For the plasma sheath as well as for the previously considered stellar system the stationary solution for the initial waterbag distribution was obtained by conserving only area in phase space. We therefore should again compute the energy of the initially rectangular waterbag for comparison with the energy of the equilibrium state.

The kinetic energy of the initial distribution is the same as that for a stellar system,
\[ T(t = 0) = \frac{Nm}{6} v_o^2. \] (191)

The initial potential energy is given by

\[ P_e(t = 0) = \frac{e^2}{2} \int E^2(x) \, dx \]

\[ = \frac{e^2N^2}{12\varepsilon_0x_0} (x_o - x_i)^2 \] (192)

where

\[ E(x) = \frac{Ne}{2\varepsilon_0} \left( \frac{1}{x_1} - \frac{1}{x_0} \right) x \quad \text{for} \quad x < x_1 \] (193)

and

\[ E(x) = \frac{Ne}{2\varepsilon_0} \left( 1 - \frac{x}{x_0} \right) \quad \text{for} \quad x_1 < x < x_o. \] (194)

Again, \( \pm x_o \) and \( \pm v_o \) define the rectangular area of the initial distribution. The total energy of the initial distribution is given by

\[ W(t = 0) = T(t = 0) + P_e(t = 0) \]

\[ = N \left[ \frac{mv_o^2}{6} + \frac{Ne^2}{2x_0\varepsilon_0} (x_o - x_i)^2 \right]. \] (195)

The minimum value equation (195) can attain for a given value of \( A = \frac{N}{4x_0v_o} \) is given by

\[ W_{\text{min}}(t = 0) = \frac{Nm}{2} \left( \frac{Ne^2x_0v_o}{4\varepsilon_0 m} \right)^{2/3} \] (196)

where we have chosen \( x_i = 0 \).

Using equation (194) we find that the potential energy of the equilibrium state for \( x_i = 0 \) is given by
The kinetic energy is given by

\[ P_{eq} = \frac{\epsilon_o}{2} \int \left( \frac{d\rho}{dx} \right)^2 \, dx \]

\[ = \frac{16\epsilon^2}{7e} \sqrt{\frac{A\epsilon}{3\sqrt{2m\epsilon}}} . \]  

(197)

The kinetic energy is given by

\[ T_{eq} = \frac{4}{3} \int_0^{x_0} \int_0^{v_+(x)} \frac{1}{2} mv^2 A \, dv \, dx \]

\[ = \frac{8\epsilon^2}{7e} \sqrt{\frac{A\epsilon}{3\sqrt{2m\epsilon}}} \]  

(198)

where \( v_+(x) \) is given by equation (162). After using equation (167) and \( A = N/4x_0v_0 \) to eliminate \( \epsilon \) and \( A \) from equations (198) and (197) the expression for the total energy of the equilibrium state becomes

\[ W_{eq} = T_{eq} + P_{eq} \]

\[ = \frac{3}{7} \left[ \left( \frac{N}{2} \right) \left( \frac{Ne^2x_0v_0}{4\epsilon m} \right)^{2/3} \right] . \]  

(199)

The ratio of equation (196) to the total equilibrium energy is

\[ \frac{W_{\text{min}}(t = 0)}{W_{eq}} = 1.16 \]  

(200)

and again the initial rectangular distribution with the smallest possible energy has still more energy than the equilibrium distribution. The equilibrium waterbag distribution for a plasma sheath may therefore have a minimum energy property.
The demonstration of the minimum energy property for a plasma sheath is almost identical to that for the stellar system and therefore only the differences for the present case are indicated below. For a single contour waterbag the energy density for the plasma sheath is given by

\[ g(x, v^+, \theta) = \frac{A}{3} m v^+ + \frac{e_o}{2} \left[ \frac{e_n}{\epsilon_o} \left( x[u(x + x_i) - u(x - x_i)] + x_i[u(x - x_i)] + u(x + x_i) \right) \right]^2 - \frac{2 A e \theta}{e_o} \]  

(201)

where the electric field at the boundary of the system was taken to be zero and \( u(x) \) is the unit step function. Using equation (201) in the Euler-Lagrange equations (65) leads to the equilibrium equation

\[ v_+ \frac{dv^+_+}{dx} + \frac{e}{m} E = 0. \]  

(202)

The Legendre test of the second variation gives again the result

\[ \frac{\delta^2 g}{\delta x^2} = 2mAv^+ \geq 0 \]  

(203)

so that the stationary state described by equation (202) is a minimum energy state.

For a multiple waterbag system the expression for \( g \) takes the form

\[ g = \sum_k \frac{A m}{3} v_+(k)^3 + \frac{e_o}{2} \left[ \frac{e_n}{\epsilon_o} \left( x[u(x + x_i) - u(x - x_i)] + x_i[u(x - x_i)] + u(x + x_i) \right) \right]^2 - \frac{2 A e \theta}{e_o} \sum_k \frac{A_k \theta(k)}{A_k} \]  

(204)
and after using this expression in equation (65) we obtain the equations

\[ v(k) \frac{dv(k)}{dx} + \frac{e}{m} E = 0. \]  

(205)

Equations (205) are the equilibrium equations of the contours and can be directly obtained from the Valskov equation. Application of the Legendre test give the same result as obtained previously for a stellar system. That is, all \( A_k > 0 \) and therefore any stationary distribution which is always decreased in going outward from the center of the system (where \( f \) is largest) is a stable distribution.

Quasi-Oscillations of a Sharply Bounded Plasma

It has been known for some time that the oscillations near a sharply bounded plasma are nonlinear. Leavens and Leavens (ref. 62) investigated the motion of electrons near the sharp boundary in the absence of a magnetic field. By solving the equations of motion of the electrons near the boundary of an initially cold plasma they found that nonlinear phenomena appeared very quickly and that the nonoscillatory behavior of the electrons in the surface charge causes multistreaming which spreads into the rest of the plasma. The dynamics of the electrons near the boundary of a cold plasma have been previously analyzed by Tonks (refs. 63 and 64) and by Herlofson (ref. 65) who assume that all the electrons near the boundary experience the same restoring force. This assumption neglects the motion of the surface charge itself.

The Lagrangian approach introduced by Dawson (ref. 66) in his study of nonlinear plasma oscillations is used to investigate the
surface charge oscillations. The calculation of the first crossing time then follows that given by Leavens and Leavens (ref. 62) except that the additional complication of a transverse magnetic field is added.

**Equation of Motion**

In the one-dimensional model the ions are assumed to have fixed positions that are uniformly spaced between \( 0 \leq x \leq l \). The external electric field is assumed zero and a constant transverse magnetic field \( B_z \) is present. The magnetic field induced by the moving electrons is neglected. \( x(x_o,t) \) is the position of an electron and \( x_o \) is the undisturbed position of the electron. The ion density \( n_i(x) \) is given by

\[
n_i = \begin{cases} 
n_0 & 0 \leq x \leq l \\
0 & \text{otherwise}
\end{cases}
\]  

(206)

The equations of motion for the electrons is given by

\[
\frac{d^2}{dt^2} (x - x_o) = -\omega_p^2 \left[ x\gamma(x) - x_o \right] - \omega_c \frac{dy}{dt}
\]  

(207)

where

\[
\gamma(x) = \begin{cases} 
\frac{l}{x}, & x \geq l \\
1, & 0 \leq x \leq l \\
0, & x < 0
\end{cases}
\]  

(208)

and

\[
\frac{d^2y}{dt^2} = \omega_c \frac{d}{dt} (x - x_o)
\]  

(209)

and

\[
\omega_c = \frac{eB}{m}.
\]  

(210)
By means of the Laplace transform the general solution of equations (207) and (208) valid for \(0 \leq x \leq l\) is found to be

\[
x - x_0 = \left(\frac{\omega}{\omega_c}\right)^2 (x(0) - x_0) - \frac{\omega}{\omega_c^2} \dot{y}(o) + \left\{\left(\frac{\omega}{\omega_c}\right)^2 (x(0) - x_0) + \frac{\omega}{\omega_c} \dot{y}(o) \right\} \cos \omega t + \frac{\dot{x}(o)}{\omega_c} \sin \omega t
\]

(211)

and

\[
y = \left\{\ddot{y}(o) + \frac{\omega}{\omega_c} \dot{y}(o) + \frac{\omega^2}{\omega_c^2} (y(0) - \omega_c (x(0) - x_0)) \right\} \omega t - \frac{\omega}{\omega_c} \dot{y}(o) \cos \omega t
\]

\[
+ \left\{\left(\frac{\omega}{\omega_c}\right)^2 \ddot{y}(o) + \frac{\omega^2}{\omega_c^2} (y(0) - \omega_c (x(0) - x_0)) \sin \omega t \right\}
\]

(212)

where

\[
\omega^2 = \omega_c^2 + \frac{\omega^2}{p}
\]

(213)

and the dots signify differentiation with respect to time. The solution for \(x < 0\) and \(x > l\) is

\[
x - x_0 = \left\{x(0) - x_0 - \frac{\dot{y}(o)}{\omega_c} - \left(\frac{\omega}{\omega_c}\right)^2 [x\gamma(x) - x_0] \right\} + \left\{\dot{y}(o) \right\}
\]

\[
+ \left(\frac{\omega}{\omega_c}\right)^2 [x\gamma(x) - x_0] \right\} \cos \omega_c t + \frac{\dot{x}(o)}{\omega_c} \sin \omega_c t
\]

(214)

and

\[
y = \left\{\frac{\dot{x}(o)}{\omega_c} + y(o) \right\} - \left(\frac{\omega}{\omega_c}\right)^2 [x\gamma(x) - x_0] - \frac{\dot{x}(o)}{\omega_c} \cos \omega_c t + \left\{\ddot{y}(o) \right\}
\]

\[
+ \left(\frac{\omega}{\omega_c}\right)^2 [x\gamma(x) - x_0] \right\} \sin \omega_c t .
\]

(215)
Streaming Time for a Given Initial Deflection

We will consider only the motion near the boundary \( x = l \) for an initial displacement \( v > 0 \). Multistreaming or crossings will occur at the earliest time when there exists an \( x_o \) such that \( \frac{\partial x}{\partial x_o} = 0 \). Following reference 62 we call this the streaming time \( t_s \).

Using the initial conditions \( x(0) - x_o = v, x'(0) = 0, y(0) = 0, \) and \( y(0) = 0 \) we find that for \( x > l \) and \( l < x_o \):

\[
x - x_o = v - \left(\frac{\omega p}{\omega_c^2}\right)^2 (l - x_o) (1 - \cos \omega_c t)
\]

and

\[
y = -\left(\frac{\omega p}{\omega_c^2}\right)^2 (l - x_o) (\omega_c t - \sin \omega_c t).
\]

For \( 0 < x < l \) and \( x_o < l - v \) the equations are

\[
x - x_o = v - \left(\frac{\omega p}{\omega_c^2}\right)^2 (1 - \cos \omega t)
\]

and

\[
y = -\frac{\omega \omega_c^2}{\omega^2} (\omega t - \sin \omega t).
\]

There can be no crossings until the first electron returns into the ion cloud at \( x = l \) and at time

\[
t_o = \frac{1}{\omega_c} \cos^{-1} \left[ 1 + \left(\frac{\omega p}{\omega_c^2}\right)^2 \frac{l - x_o - v}{l - x_o} \right].
\]

The equations of motion for \( t > t_o \) and for \( x < l \) or \( x_o > l - v \) then become
\[ x - x_o = \left( \frac{\omega}{c} \right)^2 \nu + \left\{ l - x_o - \left( \frac{\omega}{c} \right)^2 \right\} \cos \omega (t - t_o) \]
\[- \frac{\omega^2}{\omega c} (l - x_o) \sin \omega t_o \sin \omega (t - t_o) \]

and
\[ y = \left( \frac{\omega}{c} \right)^2 (l - x_o) \left\{ \left( \frac{\omega}{c} \right)^2 \sin \omega t_o - \omega t_c \right\} - \left( \frac{\omega^2}{\omega c} \right) \omega (t - t_o) \]
\[ + \left( \frac{\omega^2}{\omega c} \right) (l - x_o) \sin \omega t_o \cos \omega (t - t_o) + \frac{\omega}{\omega} \left[ l - x_o \right] \sin \omega (t - t_o) \]
\[ - \left( \frac{\omega}{c} \right)^2 \nu \right\} \sin \omega (t - t_o) \] (222)

Applying the condition
\[ \frac{\partial (x - x_o)}{\partial x_o} \bigg|_{t=t_s} = -1 \] (223)

to equation (221) we find that
\[ -1 = \left\{ 1 + \frac{\nu}{l - x_o} \right\} \cos \omega (t_s - t_o) + \frac{\omega^2}{\omega c} \sin \omega t_o \sin \omega (t_s - t_o) \] (224)

For earliest streaming we must have
\[ \frac{\partial t_s}{\partial x_o} = 0 \] (225)
or
\[
\sin \omega (t_s - t_0) = 0 \tag{226}
\]

where it is assumed that \( \sin \omega t_0 \neq 0 \). Therefore, the smallest \( t_s \)
is given by
\[
\omega t_s = \pi + \frac{\omega}{c} \cos^{-1} \left[ 1 - \left( \frac{\omega c}{\omega_p} \right)^2 \right]. \tag{227}
\]

For small \( \omega_c \) equation (227) can be expanded in a Taylor series, and
by keeping only the first three terms we obtain the equation
\[
\sqrt{\omega_p^2 + \omega_c^2} t_s = \pi + \sqrt{\frac{\omega^2}{\omega_p^2 + \omega_c^2}} \left[ 2 + \frac{\omega_c^2}{2 \omega_p^2 - \omega_c^2} + \cdots \right]. \tag{228}
\]

In the limit as \( \omega_c \to 0 \) equation (228) reduces to the result obtained by Leavens and Leavens (ref. 62):
\[
\omega_p t_s = \pi + \sqrt{2}. \tag{229}
\]

The first crossing will occur at
\[
x = l + \nu \frac{\omega^2 - \omega_p^2}{\omega^2} \tag{230}
\]

Equation (230) shows that if \( \omega_c^2 > \omega_p^2 \) the first crossing would occur at \( x > l \) and the analysis is no longer valid. Also the analyzis near the boundary \( x = 0 \) is much more complicated and is equivalent to the case of an initial impulse of the electrons. The motion for such initial conditions will not be analyzed here since the information that can be reasonable obtained is only the first
crossing time and in the present study we are primarily interested in a computer study of the problem which will include many crossings. We can expect that crossing will always occur except in a few isolated cases such as that of a very strong magnetic field or a very small initial impulse or displacement.
CHAPTER V

NUMERICAL EXPERIMENTS WITH A ONE-DIMENSIONAL PLASMA MODEL

All the numerical computations for the sharply bounded plasma were performed with one-dimensional sheet model. In contrast to the star system we now have two species to consider, ions and electrons. If the ratio of the ion mass to the electron mass is very large then the ions can be assumed fixed and only the motion of the electrons has to be computed. Two types of problems were numerically investigated. The first problem is that of a fixed ion background between \(-x_i\) and \(x_i\), and we investigate how the electron distribution approaches an equilibrium state. The second problem is that of an initially cold plasma where again we have a fixed discrete ion background. We then study the effect of either displacing all electrons a distance \(v\) from their equilibrium position or of giving all the electrons an initial velocity \(v_c\). The resulting motion is studied with and without a transverse applied magnetic field.

The One-Dimensional Plasma Model

The plasma model consists of a system of \(N\) ions and \(N\) electrons that are represented by charge sheets. The charge sheets are of infinite extent in the \(y\) and \(z\) direction and move only in the \(x\)-direction. However, motion of the sheets in the \(y\)-direction is allowed if a magnetic field is present. When two sheets cross they are allowed to pass freely through each other. The equations of
motion of all the 2N sheets are solved simultaneously by computing
the position of each sheet and by then integrating the equations of
motion of each sheet over a small time interval $\Delta t$. The equations of
motion of a sheet with charge $\sigma_i$ and mass $m_i$ per unit area is
given by

$$\frac{d^2 x_i(t)}{dt^2} = \frac{\sigma_i}{m_i} \left[ E(x,t) + \frac{dy_i(t)}{dt} B \right] \quad (231)$$

and

$$\frac{d^2 y_i(t)}{dt^2} = \frac{\sigma_i}{m_i} \frac{dx_i(t)}{dt} B, \quad (232)$$

where $x_i$ is the position of the sheet and $B$ is the constant
transverse magnetic field. The electric field is given by

$$\frac{\partial E(x,t)}{\partial x} = \frac{1}{\varepsilon_0} \rho(x,t)$$

$$= \sum_{i=1}^{2N} \frac{\sigma_i}{\varepsilon_0} \delta(x - x_i(t)). \quad (233)$$

The magnetic field induced by the charge sheets is neglected so that
we are dealing with a "low beta" plasma. The electric field is then
given by

$$E(x,t) = \frac{1}{2\varepsilon_0} \sum \sigma_i h(x - x_i(t)) \quad (234)$$

where

$$h(z) = \begin{cases} 1 & \text{for } z > 0 \\ 0 & \text{for } z = 0 \\ -1 & \text{for } z < 0 \end{cases}. \quad (235)$$
In obtaining equation (234) it was assumed that the electric field is zero outside the system. For the computer calculations the electric field acting on a particular sheet is obtained by the net charge to the left of the sheet $x_j$, thus

$$E_j = \frac{1}{\varepsilon_0} \left[ \sum_{k=1}^{j-1} \sigma^{(k)} - \frac{1}{2} \sigma_j \right]$$

where the meaning of the superscript $k$ is given by equation (106).

In addition to equations (108) and (109) used for the stellar system, the plasma requires the equation

$$\frac{d}{dt} \frac{dy_i(t)}{dt} + \frac{d^2 y_i(t)}{dt^2} \Delta t + \frac{1}{2} \frac{d^3 y_i(t)}{dt^3} (\Delta t)^2$$

where the third-time derivative is defined by an equation similar to equation (11). Equation (237) for $dy_i/dt$ is required only if a transverse magnetic field is present. In a more explicit form the equations of motion integrated on the computer are

$$x_j(t_{n+1}) = x_j(t_n) + \frac{dx_j(t_n)}{dt} \Delta t + \sigma_j \left[ E_j(t_n) + \frac{dy_j(t_n)}{dt} \right] (\Delta t)^2,$$

$$\frac{dx_j(t_{n+1})}{dt} = \frac{dx_j(t_n)}{dt} \left[ 1 - \frac{1}{2} \left( \frac{\sigma_j B \Delta t}{m_j} \right)^2 \right] + \frac{\sigma_j}{2m_j} \left[ 3 E_j(t_n) - E_j(t_{n-1}) \right]$$

$$+ 2B \frac{dy_j(t_n)}{dt} \Delta t$$

and
The kinetic energy of the system is given by

\[ T = \sum_{j=1}^{2N} \frac{m_j}{2} \left[ \left( \frac{dx_j}{dt} \right)^2 + \left( \frac{dy_j}{dt} \right)^2 \right], \tag{241} \]

and the potential energy is

\[ P = \frac{e^2}{2} \sum_{j=1}^{2N-1} \left( x_{j+1} - x_j \right) E_j^2. \tag{242} \]

If the ions are assumed fixed then the equations are simplified as is indicated in the computer program which is given in the appendix.

The terms \( \frac{d^3x_j}{dt^3} \) and \( \frac{d^3y_j}{dt^3} \) have been introduced in the equations to reduce the error in the computation. The effects of these terms can be illustrated by considering a system where \( \epsilon_0 \to \infty \). The electric field is then zero and the charged particles interact only with the constant magnetic field. Their motion is therefore known and is described by circles in x-y space. For simplicity we consider only electrons with cyclotron frequency \( \omega_c = o B/m \). With the third time derivative the equations of motion are

\[
\begin{bmatrix}
\dot{x}(t_{n+1}) \\
\ddot{x}(t_{n+1}) \\
\dddot{y}(t_{n+1})
\end{bmatrix} =
\begin{bmatrix}
1 & \Delta t & \frac{\omega_c}{2} (\Delta t)^2 \\
0 & 1 - \frac{1}{2} (\omega_c \Delta t)^2 & \omega_c \Delta t \\
0 & - \omega_c \Delta t & 1 - \frac{1}{2} (\omega_c \Delta t)^2
\end{bmatrix}
\begin{bmatrix}
x(t_n) \\
\dot{x}(t_n) \\
\ddot{y}(t_n)
\end{bmatrix}.
\tag{243}
\]
Without the third time derivative the equations become

\[
\begin{bmatrix}
x(t_{n+1}) \\
\dot{x}(t_{n+1}) \\
\ddot{y}(t_{n+1})
\end{bmatrix}
= \begin{bmatrix}
1 & \Delta t & \frac{\omega_c}{c} (\Delta t)^2 \\
0 & 1 & \omega_c \Delta t \\
0 & -\omega_c \Delta t & 1
\end{bmatrix}
\begin{bmatrix}
x(t_n) \\
\dot{x}(t_n) \\
\ddot{y}(t_n)
\end{bmatrix}.
\tag{244}
\]

The eigenvalues for equations (243) and (244) are

\[
\lambda = 1, 1 \pm \frac{1}{\frac{1}{4} (\omega_c \Delta t)^2}
\tag{245}
\]

and

\[
\lambda = 1, 1 \pm i\omega_c \Delta t
\tag{246}
\]

respectively. The eigenvalues for the exact equations are

\[
\lambda = \cos(\omega_c \Delta t) \pm i \sin(\omega_c \Delta t)
\]

\[
= \exp[\pm i\omega_c \Delta t].
\tag{247}
\]

with \(|\lambda| = 1\). The magnitude of two of the eigenvalues of equation (245) is

\[
|\lambda| = \sqrt{1 + \frac{1}{4} (\omega_c \Delta t)^4}
\tag{248}
\]

and the magnitude of the corresponding eigenvalues of equation (246) is

\[
|\lambda| = \sqrt{1 + (\omega_c \Delta t)^2}.
\tag{249}
\]
Thus, equations (243) and (244) are numerically unstable. After n
time steps the error in equations (243) is of the order

\[ \text{error} \sim \frac{n}{8} (\omega_c \Delta t)^4 \]  \hspace{1cm} (250)

whereas for equation (244) the error is of the order

\[ \text{error} \sim \frac{n}{2} (\omega_c \Delta t)^2 \]  \hspace{1cm} (251)

where \( \omega_c \Delta t << 1 \). Thus, the error is greatly reduced by including
the third time derivative. Computer experiments with and without
the \( \frac{d^3}{dt^3} \) terms were performed and the results confirmed that for a
given time step \( \Delta t \) the inclusion of the \( \frac{d^3}{dt^3} \) terms greatly
improves the accuracy of the computer program.

Approach to Equilibrium

In Chapter III we illustrated by means of numerical experiments
that the equilibrium state for a stellar system with an initial
waterbag distribution had a physical meaning in that the system tried
to approach the equilibrium state. This same interesting property
holds for the plasma in that the system does its best within the
limitations of energy conservation to approach the steady state. We
first present the results for a case where \( x_1 \) approaches zero. This
case corresponds to that of a permeable grid which is positively
charged and through which the electrons can pass freely. It is the
problem of two electron sheaths held together by a thin but dense
positive charge. For all the calculations we have chosen
\( e = e_0 = m = 1 \). Figure 58 shows the variation of the kinetic energy
for a system containing 1000 "electrons." The ratio of initial to
equilibrium energy for this system is 1.3. We see that the kinetic energy approaches its equilibrium value much more quickly as in the case of the stellar system shown in figure 8. This was to be expected since for the plasma sheath the variation of the particle period with energy is much stronger which causes more effective phase mixing.

One of the more important indicators of the approach to an equilibrium state is the time development of the energy distribution function. Figure 59 shows that after about six plasma periods the energy distribution function has approached the equilibrium distribution indicated by the dashed line. As was the case for the stellar system, the high-energy tail is required to conserve energy. The corresponding time development of the density is shown in figure 60. The time development of the density indicates that the density of the system quickly tends to approach its equilibrium value indicated by the continuous curve. The electric field at $t = 30 \omega_0^{-1}$ is compared with the equilibrium value in figure 61. The dashed curve represents the field of the system and the solid curve is the theoretical equilibrium field. Because of the high-energy electrons the electric field of the system does not go to zero as quickly as the theoretical value. Again it is of interest to view the time development of the system in phase space as shown in figure 62. The system now quickly develops arms which wind around the main body of the system as it rotates. Because of the very long period of rotation for particles at the outer boundary of the system the arms stretch much more quickly then for the stellar system. The calculations just presented were
repeated for a 2000 sheet system. The results were nearly identical and only the time development of the system in phase space is shown in figure 63.

Calculation for values of $x_i$ different from zero were also performed. However, it was found that even for moderate values of $x_i$ collisional effects became very important and the system approached a Maxwellian distribution. Figure 64 shows the variation of the kinetic energy for a system of 1000 sheets with $x_i = 477$. The time development of the energy distribution function is shown in figure 64 and illustrates that the system quickly approaches the Maxwellian distribution indicated by the dashed line. The corresponding results for the time development of the system in phase space are shown in figure 66.

Numerical Results for Quasi-Oscillations of Sharply Bounded Plasmas

The numerical results presented in the following section illustrate the nonoscillatory behavior of electrons near the boundary of a sharply bounded plasma. The initial oscillations are produced by giving all electrons as initial velocity $v_0$. We first investigate the behavior for systems with 500 sheets. Five hundred is a relatively large number of sheets and the evolution of the system is followed only for a short time. Results will also be given for systems containing 100 charged sheets that are integrated over a much longer time.

Figure 67 shows the electron positions as a function of time for the first 100 and the last 100 electrons of a 500 sheet system in the absence of a magnetic field. The ion positions are fixed and
the ions only supply the neutralizing positive charge. The initial velocity given to the electrons in figure 67 is \( v_{\omega} = \frac{4}{p} \). The results show that most of the electrons oscillate harmonically in the main body of the plasma and only slowly does the effect of the boundaries penetrate into the plasma. Similar results for a larger initial velocity \( v_{\omega} = -8 \), are shown in figure 68. The disturbance penetrates now much faster into the main body of the plasma. For an even larger initial velocity \( v_{\omega} = 16 \) we obtain the results shown in figure 69. In general it was found that the rate at which the disturbance travelled into the plasma and destroyed the cold plasma oscillations is of the order of \( \frac{v_{\omega} t}{p} \), where \( t \) is normalized to \( \frac{1}{p} \), the plasma period of the unperturbed system.

Systems containing 100 sheets were integrated over a long time to investigate thermalization of the initially cold plasma oscillations. Figures 70 through 73 show the electron positions as functions of time for different values of initial velocity \( v_{\omega} \) of the electrons. The figures show that the rate at which the disturbance moves into the plasma increases with increasing values of initial velocity. Also the mixing of the electron trajectories is more effective for larger values of initial impulse. The rate at which the initial energy of oscillation of the electrons is thermalized can be obtained by observing the variation of the kinetic energy as a function of time. In figures 74 through 77 we have plotted the kinetic energy versus time for the systems shown in figures 70 through 73. It can be seen that with increasing initial velocity the oscillations of the kinetic energy of the system decrease more rapidly. Thus the more violent
behavior for large initial velocity causes more effective phase mixing and thermalization. Similar information is obtained by plotting the net velocity of the electrons of the system, $\sum \dot{x}_i$, as a function of time. The results are shown in figures 78 through 81. In figure 82 we have shown the time development in phase space for the 100 sheet system with an initial velocity of $\nu \omega_p = 4$. The phase space plots show that the ordered motion is quickly destroyed.

The effect of a transverse magnetic field on the time development has also been investigated for 100 sheet systems with an initial velocity of $\nu \omega_p = 8$. Figures 83 and 84 show the results for $\omega_c = B = 0.5$ and 1, respectively. The trajectories plotted in the two figures show that the magnetic field reduces the rate at which the ordered oscillations are destroyed. The results can be seen more clearly in figures 85 and 86 where the net velocity of the electrons in the system is plotted as a function of time. Figure 85 shows that for $\omega_c = 0.5$ the system shows a somewhat smaller damping than that shown in figure 80 for the case $\omega_c = 0$. For the case $\omega_c = 1$ shown in figure 86, the oscillations in the net electron velocity of the system show only very little damping. Thus for large $\omega_c \sim 1 \sim \omega_p$, thermalization will require a much longer time than is the case in the absence of a magnetic field.

The time of first crossing and the position of first crossing for charge sheets for the case of initial displacement in the presence of a magnetic field have also been numerically obtained. The results are shown in figure 87. Systems containing only 30 sheets were used and the initial displacement was $v = 8$. The solid curves in figure 87
are obtained from equations (227) and (230) with $\omega_p = 1$. The computer results are presented by the circles and are seen to be identical to the analytical results.

After the calculations presented in the present section had been performed it was found the similar calculations had been performed by Schneider (refs. 67 and 68) using 41 electrons. However, Schneider only calculated the motion in the absence of a magnetic field and for initial displacements only.
CHAPTER VI

CONCLUSIONS

For the one-dimensional stellar system it was shown that for an initial distribution of the waterbag type with a given initial area in phase space the equilibrium state is a minimum energy configuration. The results of numerical experiments with large numbers of particles illustrate these results. Thus, starting from any nonequilibrium state the stationary state described by the time independent Vlasov equation can never be completely reached. This is because the excess initial energy causes spiral arms to develop in phase space. These arms are required to accommodate the excess energy and they wind around the main system and become longer and thinner as time progresses. The beginning of this process is illustrated in figure 12. However, after a sufficiently long time it will become increasingly more difficult to detect changes in \( f(x,v,t) \) since successive turns of the arms approach each other. It may then be possible to obtain an averaged distribution function by averaging over \( \varphi \). Such an analysis has been performed by Lynden-Bell (ref. 69). Lynden-Bell suggests that the encounterless relaxation of an unsteady system will lead to an equilibrium related to Fermi-Dirac statistics. These results have been confirmed to some extent by Hénon (ref. 70) who used a model with concentric spherical mass shells to simulate stellar systems. However, in our investigation of encounterless relaxation with a one-dimensional model we have not been able to confirm Lyndon-Bell's theory. This should not be too surprising since relaxing conditions which lead to the
distribution given by Lyndon-Bell are expected to persist only over regions where the system is dynamically unstable. Our computer calculations show that for initial distributions other than the waterbag type the results are nevertheless very similar to those for the waterbag distribution. An extension of the minimum energy property showed that all one-dimensional stationary distributions that are always decreasing in going outward from the center of the system are stable. Numerical experiments were performed for two systems which did not satisfy this criterion and these two systems were found to be unstable.

The waterbag model is found to have the interesting property that for initial energies near the equilibrium energy the system approaches its equilibrium configuration very closely. For initial energies far from the equilibrium energy the system is unstable and breaks up into smaller clusters.

The one-dimensional model is of interest as an approximation to the distribution of velocity and mass normal to the galactic plane of a highly flattened galactic system. For the two distributions investigated in this study the theoretical force \( E(x) \), normal to the galactic plane is compared in figure 88 with the results deduced by Oort from observations. The values of \( E(x) \) obtained by Oort have been tabulated by Lindblad (ref. 36). The variation of \( E(x) \) for the Gaussian distribution is in good agreement with the experimentally obtained variation. The difference for small \( x \) is probably due to the central core of the galaxy. As expected, for the waterbag
distribution the force \( E(x) \) differs markedly from the experimentally obtained variation for large \( x \).

In the study of thermalization effects in a one-dimensional model for a stellar system it was found that thermalization does take place to order \( n^2 D^2 \tau_c \). These results were obtained by taking a time average for systems containing small numbers of stars.

The simple two-dimensional model for a stellar system showed that spiral and other filamentary structure can result from purely gravitational effects. Since the spiral structure and other filamentary structure tend to disappear in time, one would conclude that the spiral galaxies are young galaxies which later develop into elliptical galaxies. However, these results should be confirmed with a computer model using point masses that are confined to move in a plane.

The results of computer experiments with the one-dimensional plasma model show that for the waterbag distribution with initial energy near the equilibrium energy the system tends to approach its equilibrium state. These results could only be confirmed for systems where the neutralizing ion background was confined to a small region. For other cases collisional effects dominated and the system quickly approached a Maxwellian distribution.

In investigating the quasi-oscillations of electrons near a sharp plasma boundary with a one-dimensional sheet model, crossings of electron trajectories were found to occur which causes a thermalization of the initial energy of oscillation of the electrons. The rate of thermalization increases with increasing initial impulse of the electrons. A transverse magnetic field decreases the rate of
thermalization. The time and position of the first crossing obtained from the computer experiments are in agreement with the theoretical results.

Several of the problems investigated in the present study require further work. For example, the waterbag model investigated by means of following the individual particles can be numerically investigated by following the equations of motion of the contours $v_+(k)$ and $v_-(k)$. Such an approach appears to be appropriate for analyzing the stability of one-dimensional system. Also, the problem of the self-consistent motion of several thousand point masses moving in a plane needs to be investigated.

Another area of future research is concerned with the analytical calculation of thermalization effects for one-dimensional models in the presence of strong gravitational (or external electric) fields.
APPENDIX

COMPUTER PROGRAM FOR THE APPROXIMATE
ONE-DIMENSIONAL MODEL

The complete computer program is reproduced in this appendix. The program is written in FORTRAN IV language and a large number of comments have been added throughout the program so that the program can be followed more easily.
$JOB
BLW 11950 D1218 I 2 004 19999
$SETUP
14 0303  SAVE 1  BIN  RO
$*
$*
$EXECUTE  IBJOB
$IBJOB  LOGIC,MAP
$GROUP  BUFC=7,OPNCT=3,-UNIT09-, -UNIT11-, -TDD80-, -FDD80-,
$ETC  -UNIT12-
$POOL  BLK=5, BUFC=12,-UNIT13-, -UNIT14-
$POOL  BLK=28, BUFC=1,-UNIT05-
$IBFTC 12180 LIST,DECK

A(1) IS 04 ON $SETUP CARD DDIONN * PLOT WITHOUT GRID
A(2) IS 05 ON $SETUP CARD OUTPUT UNIT11
A(3) IS 06 ON $SETUP CARD SCRATCH UNIT13 TRACES
B(1) IS 14 ON $SETUP CARD INPUT UNIT09
B(3) IS 16 ON $SETUP CARD SCRATCH UNIT12 SAVE X FOR E PLOTS
B(4) IS 17 ON $SETUP CARD - TRACES

INPUT
TMM(1) = MIN X FOR TRACES GROUP 1
TMM(2) = MAX X FOR TRACES GROUP1
TMM(3) = MIN X FOR TRACES GROUP2
TMM(4) = MAX X FOR TRACES GROUP2
IXDP = INTEGER, PLOT X VS XDOT EVERY IXDP ITERATION
IRHO = COMPUTE RHO EVERY IRHO CYCLE
ENK(1) = 0 USE STAR GAS RHO
NOT = 0 USE PLASMA RHO
ITRACE =0 FOR NO TRACES (TRAJECTORIES)
IALPT(1) = 1ST ALPHA FOR WHICH TRACE DESIRED GROUP 1
IALPT(2) = LAST ALPHA FOR WHICH TRACE DESIRED GROUP 1
IALPT(3) = INCREMENT GROUP 1
IALPT(4) = 1ST ALPHA FOR WHICH TRACE DESIRED GROUP 2
IALPT(5) = LAST ALPHA FOR WHICH TRACE DESIRED GROUP 2
IALPT(6) = INCREMENT GROUP 2
NOP=NUMBER OF PARTICLES, INTEGER
NP=PRINT FREQUENCY, INTEGER
NUC=NUMBER OF CYCLES, INTEGER
NUC=COMPUTE U SUB J EVERY NUC CYCLES, INTEGER
ITAPE=0 TO USE INITIAL CONDITIONS FROM TAPE, INTEGER
ITAPE NOT =0 TO USE PLOTDE OPTION
PLOTDE -1 = X=X+DX WHERE DX=RNMX/FNOP AND GAUSIAN XDOT AND YDOT
  0 = LINEAR X, GAUSIAN XDOT, YDOT FROM RAND
  +1 = LINEAR X, XDOT AND YDOT FROM RAND
  +2 = X=X+DX, LINEAR XDOT AND YDOT FROM RAND
IALPO = - FOR ALL NEGATIVE ALPHAS, INTEGER
  = 0 FOR NEGATIVE ODD ALPHAS, POSITIVE EVEN ALPHAS
  = + FOR ALL POSITIVE ALPHAS

SHEATH
PLASMA
STAR GAS
RNMXE = RANDOM NUMBER MULTIPLIER X EVEN
RNMXO = RANDOM NUMBER MULTIPLIER X ODD
IBUG = 0 FOR DEBUG PRINT OUTS, INTEGER
DELT
B
SM=M FOR ODD IALP
CM=M FOR EVEN IALP
EZ
EPS
VT= THERMAL VELOCITY
KOE= COMPUTE CONSERVATION OF ENERGY EVERY KOE ITERATION, INTEGER
S = 0 TO OMIT RHO(K) AND RHO(K)**2
S NOT = 0 TO COMPUTE RHO(K) AND RHO(K)**2

IN=2 BCD WORDS FOR DISPLAY ID FOR DDIPT
IPLT=0, NO PLOTS, INTEGER
   NOT=0, PLOTS
IXP=MAKE DENSITY PLOTS EVERY IXP ITERATION, INTEGER
IEP=PLOT CURVE OF E US X EVERY IEP ITERATION, INTEGER
FNXI=FLOATING NUMBER OF X INCREMENTS FOR DENSITY VS X PLOTS
NXI=FIXED NUMBER OF X INCREMENTS 500 OR LESS
FNXDI=FLOATING NUMBER OF X DOT INCREMENTS
NXDI=FIXED NUMBER OF X DOT INCREMENTS 500 OR LESS
FNYDI=FLOATING NUMBER OF Y DOT INCREMENTS
NYDI=FIXED NUMBER OF Y DOT INCREMENTS 500 OR LESS
XL(1)= X MIN FOR X VS XDOT, VELOCITY PLOT
XL(2)= X MAX FOR X VS XDOT
XL(3)= XDOT MIN FOR X VS XDOT, VELOCITY PLOT
XL(4)= XDOT MAX FOR X VS XDOT, VELOCITY PLOT
LODE = INTEGER IN CALL OF RAND - RANDOM NUMBER STARTER
ITPO = INTEGER SAVE TRACES EVERY ITPO CYCLE
VZE = CONSTANT INITIAL VELOCITY ADDED TO GAUSSIAN X DOT FOR EVEN AL
VZO = CONSTANT INITIAL VELOCITY ADDED TO GAUSSIAN X DOT FOR ODD ALP
GRAV = 0, FOR NO GRAVITATIONAL FIELD IN YDOT EQ FOR PLASMA
NOT = 0, FOR GRAVITATIONAL FIELD
ISYM = INTEGER USED TO PLOT E VS X
13 = DOT
14 = VECTOR
XIC = INITIAL DISPLACEMENT OF ELECTRONS

COMMON NOP, IPF, NOC, ITAPE, N(3), IBUG, IN(2)
COMMON IPLT, IXP, IEP, IALPO, NXI, NXDI, NYDI
COMMON DELT, B, CM, SM, EZ, EPS, DELX, DELXD, DELYD, RNMX, VT
COMMON EMIN, EMAX, PLOTDE, XMIN, XMAX, FNXI, FNXDI, FNYDI
COMMON KOE, NUC, IXDP, IXDPC, XL(4), LODE, ITPO, S
COMMON KKOE, NOPOT, DE, PE, FAC

COMMON X(6000), YDOT(6000), IALP(3000)
COMMON IXPC, IXP, NOCK
COMMON 1SYM
COMMON /A/TRACE, IALPT(10), ENK(21), VZE, VZ0, RNMXE, RNMXO, TMM(10)

COMMON /MAI/ IRHOK, FRHOK, NT1, NT2, NT3, FNOP, DX, X0, DELT2, ITOPC,
1 IPFC, NTPP, NUCK, K1, K2, K3, K4, RNMXDE, RNMXDO, RNMYDE, RNMYDO, IRHO, GRAV,
2 XIC, RHOKS(21), SIG, MEAN, 12, 11
DIMENSION TIME(1000), XTIME(3)

DIMENSION XDOT(3000), E(3000)
EQUIVALENCE (X(3001), XDOT(1)), (YDOT(3001), E(1))

REAL M, K1, K2, K3, K4
INTEGER PLOTDE

EQUIVALENCE (IALPT1, IALPT1), (IALPT2, IALPT2), (IALPT3, IALPT3),
(IALPT4, IALPT4), (IALPT5, IALPT5), (IALPT6, IALPT6)
DATA XTIME[1], YX, XDOT, 1HTIME, IALP=
DATA TIME/0.84, 92.12, 169, 84, 240, 24, 268, 52, 379, 80, 540, 38,
1760, 36, 943, 08, 1094, 08, 1229, 28, 1353, 04, 1467, 44, 1579, 76, 1686, 88,
21716, 00, 1747, 20, 1768, 00, 1778, 40, 1783, 60, 1788, 80, 1791, 92,
1 0, 0, 0, 0, 0, 0, 0, 0,
3 1442, 21, 1440, 4, 1438, 6, 1435, 0, 1427, 6, 1424, 0, 1405, 8, 1368, 2,
41290, 0, 1266, 8, 1172, 10, 912, 2, 790, 0, 645, 0, 456, 0, 389, 6, 299, 2,
5213, 8, 158, 0, 120, 6, 64, 4,
60, 0, 0, 0, 0, 0, 0, 0 /
DATA ITRACE/0/, IRHOK/O/, FRHOK/O/, NT1/O/, NT2/O/, NT3/O/, ITPOC/O/,

CALL INITAL

ORDER ACCORDING TO X
CALL SHIFT
XIC= INITIAL DISPLACEMENT OF ELECTRONS

IF (ITAPE.EQ.0) GO TO 500
DO 431 I=1, NOP
431 X(I)=X(I)-RNMXO/2*+RNMXE/2*+XIC
CONTINUE

KKOE=1
NOPOT=0
IEPC=0
IXDPC=0
IXPC=0
NOCK=0
FAC=EPS/EZ
FAK=FAC*CM
FAC1=EZ/2
FAC2=5*EPS*FNOP /EZ
FAC3=5*EZ*FAC2**2
FAC4=(EPS*DELT**2*5)/CM
FAC5=(EPS**5*DELT)/CM
FAC6 = FAC4 * CM  
FAC7 = FAC5 * CM  
FAC9 = EPS / SM  
CMSQ = CM ** 2  
PE = 0.  
NOP02 = NOP / 2  
NOP1 = NOP02 + 1  
NUCK = 1  

***** COUNTS FOR TRACES *****  
NT1 = 0  
NT2 = 0  
NT3 = 0  
NTPP = 0  
ITPOC = 0  
XMIN1 = 0.  
XMIN2 = 0.  
XMAX1 = 0.  
XMAX2 = 0.  

C   WRITE (6, 23) (X(I), I = 1, NOP)  
23 FORMAT (7H X=(7E17.8) )  
C   WRITE (6, 24) (XDOT(I), I = 1, NOP)  
24 FORMAT (10H XDOT=(7E17.8) )  
C   WRITE (6, 25) (YDOT(I), I = 1, NOP)  
25 FORMAT (10H YDOT=(7E17.8) )  
C   WRITE (6, 26) (IALP(I), I = 1, NOP)  
26 FORMAT (10H IALP=(10I10) )  
C   IF (IPLOT .EQ. 0) GO TO 22  

C   PLOT INITIAL ARRAYS  
C   CALL PLOT  
C
22 CONTINUE  
C   COMPUTE X, XDOT, YDOT  
C
C   FIRST = 0.  
C
***** HERE TO START NEW CYCLE  

39 SUM = 0  
YD0TM1 = 0.  
C   IF (IALPO .EQ. 0) GO TO 239  
DIF = X(NOP) - X(I)  
IF (IALPO .EQ. 0) AND (B .EQ. 0) GO TO 1000  
IF (NUCK .NE. NUC ) GO TO 239  
C
C   COMPUTE PHI AND U  
C
C   PUT PHI SUB J IN YDOT ARRAY  
C
C
C
1000 CONTINUE  
C   IF (IALPO .EQ. -1) GO TO 400  
FN02 = FN0P ** 5
NO2M1=NOPO2-1
PHINO2=(FAK/4.)*(X(NO2P1)-X(NOPO2))
NO2P2=NOPO2+2
K=NO2M1+1
FK=FNPO2
FJ=0.
DO 248 J=1,NOP
FJ=FJ+1.0
IF(J.GE.NOPO2)GO TO 247
K=K-1
FK=FK-1.0
KP1=K+1
KM1=K-1
IF(K.EQ.NO2M1)GO TO 252
YDOT(K)=YDOT(KP1)+((FN02-2)*S*(X(KP1)-X(K))*FAK
GO TO 241
252 YDOT(K)=PHINO2+FAK*(S*(FK-2)*S*(X(KP1)-X(K))
241 CONTINUE
GO TO 248
247 IF(J.LE.NO2P1)GO TO 246
JM1=J-1
IF(J.NE.NO2P2)GO TO 242
YDOT(J)=PHINO2+FAK*(FJ-2)*S*(X(J)-X(JM1))
GO TO 248
242 YDOT(J)=YDOT(JM1)+FAK*(FJ-2)*S*(X(J)-X(JM1))
246 IF(J.EQ.NOPO2)YDOT(J)=PHINO2
IF(J.EQ.NO2P1)YDOT(J)=PHINO2
248 CONTINUE
DO 249 J=1,NOP
C
C PUT U SUB J IN YDOT ARRAY
C
249 YDOT(J)=S*CM*YDOT(J)**2+EPS*YDOT(J)*CM
IF(NOCK.EQ.0) CALL PLOT
GO TO 239
C
SHEATH COMPUTATION F(U) CALCULATION
C
400 J=NOPO/2
402 IF(X(J).LE.X0) GO TO 404
J=J-1
GO TO 402
404 IF(X(J+1).GE.X0) GO TO 405
J=J+1
GO TO 404
405 CONTINUE
C
PHIP=0.
XMM1=X0
N=NOPO/2
C
DO 410 M1=1,J
JP1MM=J+1-M1
410 CONTINUE
I = (X(JP1MM)/DELX)

IF (I*LE*N AND N*EQ*0) GO TO 409

408 IF (I*GE*N OR N*EQ*0) GO TO 409

R = N

RR = JP1MM

PHII = PHI + FAC*(R - RR + .5)*(XMM1 - R*DELX)

PHI = PHIII

XMM1 = R*DELX

N = N - 1

GO TO 408

409 R = N

RR = JP1MM

YDOT(JP1MM) = PHI + FAC*(R - RR - .5)*(XMM1 - X(JP1MM))

C PHI = YDOT

PHI = PHI + FAC*(R + .5)*(X(L) - XMM1)

410 XMM1 = X(L)

N = NOP/2

XMM1 = X0

PHI = 0

C

JP1 = J + 1

DO 415 L = JP1, NOP

I = (X(L)/DELX)

407 IF (I*LE*N OR N*EQ*NOP) GO TO 406

N = N + 1

R = N - L

RR = N

PHII = PHI + FAC*(R + .5)*(RR*DELX - XMM1)

XMM1 = RR*DELX

GO TO 407

406 R = N - L

YDOT(L) = PHI + FAC*(R + .5)*(X(L) - XMM1)

PHI = YDOT(L)

415 XMM1 = X(L)

C

DO 416 J = 1, NOP

416 YDOT(J) = .5*SM*XDOT(J)**2 + EPS*YDOT(J)

C IF (NOCK*EQ*0) CALL PLOT

C 239 CONTINUE

C PLASMA

FJ = 0.

IF (IALPO*NE*1) GO TO 420

IF (K*KOE*NE*KOE) GO TO 419

XMM1 = X(I)

K = 0

C POTENTIAL ENERGY SHEATH

C X(1) AND X(NOP) ARE ELECTRONS
DO 421 L=2*NOP  
1=(X(L)/DELX)
J=1-K
IF(J.LE.0) GO TO 423
425 IF(K.EQ.NOP) GO TO 423
FK=K+1
XN= FK*DELX
R=(K-L+1)
PE=PE + (XN-XNM1)*R**2
XNM1=XN
K=K+1
J=1-K
IF(J.EQ.0) GO TO 423
GO TO 425
423 R=(K-L+1)
PE= PE + (X(L)-XNM1)*R**2
421 XNM1=X(L)
PE= PE*FAC**2
C KINETIC ENERGY SHEATH
DO 427 J=1+NOP
XDAV=XDAV+XDOT(J)
427 K3= K3+XDOT(J)**2
IF(B.EQ.0.)GO TO 419
DO 426 J=1+NOP
YDAV=YDAV+YDOT(J)
426 K4= K4+YDOT(J)**2
ACCING =(DELT2/2.)*((EPS*B)/SM)**2
C COMPUTE ELECTRIC FIELD AND MOTION
419 DO 428 L=1*NOP
I=(X(L)/DELX)
IF(I.LE.0) FI=0
IF(I.GT.0.*AND.I.LE.NOP) FI=I
IF(I.GT.NOP) FI = FNO P
FN=L
ENRM1=E(L)
E(L)=FAC*(s-FN+FI)
IF(FIRST.EQ.0.) ENRM1=E(L)
C ** SAVEX FOR X VS E PLOT **
XDD = FAC9 *(E(L)+ YDOT(L)*B)
XDOTM1 = XDOT(L)
XDOT(L)= XDOTM1*(1.-ACCING)- FAC9*(s*(3.*E(L)-ENRM1)
1 + YDOT(L)*B) *DELT
X(L)= XDD+5*DELT2 + XDOTM1*DELT +X(L)
IF(B.EQ.0.) GO TO 428
YDOT(L)= FAC9*B*XDOTM1*DELT + YDOT(L)- YDOT(L)*ACCING+GRAV
428 CONTINUE
GO TO 430
C STAR GAS AND PLASMA LOOP
420 DO 40 J=1,NOP
FJ=FJ+1.*0
C
JM1=J-1
IF (IALP0.NE.0) GO TO 240
IF (IALP(J).LT.0) GO TO 41
M=CM
EXP=.5
GO TO 142
41 M=-SM
EXP=-.5
142 IF (JM1.EQ.0) GO TO 146
145 IF (IALP(JM1).LT.0) GO TO 143
EXP1=+1.
GO TO 144
143 EXP1=-1.
144 SUM=SUM+EXP1
146 EJP=FAC*SUM
EJRM1=E(J)
IF (IALP0.NE.0) GO TO 240
E FOR + AND - ALPHAS
E(J)=EJP+FAC*EXP
240 CONTINUE
IF (KK0E.NE.K0E) GO TO 49
IF (JM1.EQ.0) GO TO 244
FOR + AND - ALPHAS
IF (IALP0.EQ.0) PE=PE+(X(J)-XJM1)*EJP**2
FOR ALL + ALPHAS
IF (IALP0.EQ.0) PE=PE+(FNOP*.5-FJ+1.)**2*(X(J)-XJM1)
244 XJM1=X(J)
COMPUTE KINETIC ENERGY AND POTENTIAL ENERGY
IF (IALP(J).LT.0) GO TO 48
K1=K1+XDOT(J)**2
IF (B*NE.0.) K2=K2+YDOT(J)**2
GO TO 49
48 K3=K3+XDOT(J)**2
IF (B*NE.0.) K4=K4+YDOT(J)**2
49 CONTINUE
IF (IALP0.EQ.0) GO TO 42
EJRM1=E(J)
E(J)=FAK*(.5*FNOP-FJ+5)
IF (FIRST*EQ.0) EJRM1=E(J)

***** STAR GAS *****
***** SAVE X FOR X VS E PLOT *****

X(J) = X(J) + XDOT(J) * DELT + FAC6 * E(J)
XDOT(J) = XDOT(J) + FAC7 * (3 * E(J) - EJRM1)
GO TO 158

***** PLASMA *****

42 XDD = (EPS/M) * (E(J) + YDOT(J) * B)
IF (FIRST.EQ.0) EJRM1 = E(J)
ACCINC = DELT**2 / 2. * ((EPS*B)/M)**2
XDOTM1 = XDOT(J)
XDOT(J) = XDOTM1 * (1. - ACCINC) + (EPS/M) * (5*(3.*E(J) - EJRM1) + YDOT(J)*B)
1*DELT
X(J) = XDD * 5*DELT**2 + XDOTM1*DELT + X(J)
YDOT(J) = -(EPS/M)*B*XD0TM1*DELT + YDOT(J) - YDOT(J) * ACCINC + GRAV
243 CONTINUE

158 CONTINUE
IF (IBUG.NE.0) GO TO 40
WRITE (6,47)
47 FORMAT (7H DEBUG/6B J, JM1, M, EXP, EXP1, SUM, E(J), XD0TM1, XDOT(J),
1*YDOT(J), X(J), EJM1, EJP//)
WRITE (6,50) J, JM1, M, EXP, EXP1, SUM, E(J), XD0TM1, XDOT(J), YDOT(J),
1*X(J), EJM1, EJP
WRITE (6,50) IALP(J)
50 FORMAT (215/ (7E17.8))
40 CONTINUE

430 CONTINUE
IF (IEPC + 2.EQ.1EP) WRITE (12) (X(L), L = 1, NOP)
IF (IEPC + 1.NE.1EP) GO TO 262
WRITE (12) (X(L), L = 1, NOP)
RE_WIND 12
READ (12) (X(L), L = 1, NOP)
IEPC = 0

***** GET X(-1) VS E PLOT ****)
CALL PLOT

IEPC = -10
READ (12) (X(L), L = 1, NOP)
RE_WIND 12
262 CONTINUE
FIRST = 1.
IF (IALPO) 254, 254, 255
C EVEN OR POSITIVE ALPHAS
255 CONTINUE
PE = (FAC3*DIF - PE*FAC1)*CMSQ
GO TO 254
C NEGATIVE OR ODD ALPHAS
CALL SHIFT
IF (KKOE.EQ.KOE) GO TO 58
KKOE=KKOE+1
GO TO 59
58 KKOE=1
K1=.5*CM*K1
K2=.5*CM*K2
K3=.5*SM*K3
K4=.5*SM*K4
IF (IALPO.LE.0) PE=FAC1*PE
S4K=K1+K2+K3+K4
P4K=S4K+PE
WRITE (6,1) NOCK, K1, K2, K3, K4, PE, P4K, XDAV, YDAV
1 FORMAT(7H CYCLE 15.33H KINETIC AND POTENTIAL ENERGY/10H K1(XDE)=E15.8,10H
K2(YDE)=E15.8,10H K3(XDO)=E15.8,10H K4(YDO)=E15.8
28,4H P=E15.8/10H SUM=E15.8,10H SUM 4K=E15.8,10H XDOTAV=E
315.8,10H YDOTAV=E15.8//)
59 CONTINUE
PE=0.
K1=0.
K2=0.
K3=0.
K4=0.
XDAV=0.
YDAV=0.
CONTINUE
NOCK=NOCK+1
NUCK=NUCK+1
IF (NUCK.GT.NUC) NUCK=1
IPFC=IPFC+1
IXPC=IXPC+1
IEPC=IEPC+1
IXDPC=IXDPC+1
C
C S = 0 TO OMIT RHO(K) AND RHO(K)**2 COMPUTATION
C IRHO = COMPUTE RHO EVERY IRHO CYCLE
C
C IF (S.EQ.0.) GO TO 157
IRHOK=IRHOK+1
IF (IRHOK.NE.IRHO) GO TO 157
FRHOK=FRHOK+1.0
IRHOK=0
SRFN=SQR(FNOP)
SUMX=0.
DO 77 J=1,NOP
77 SUMX=SUMX+X(J)
XAVE=SUMX/FNOP
WRITE (6,260) NOP, S, NOCK, XAVE
260 FORMAT(3H N=15.4H S=E15.8,8H CYCLE=14.7H XAVE=E15.8/19H K*RHOK,)
{RH0\textsuperscript{3}+RHO\textsuperscript{2}/
CC1=-1.
DO 257 K=1,21
COP=0.
RHOI=0.
RHOR=0.
SIP =0.
CC1=CC1+1.
C1=CC1*S
DO 258 J = 1,NOP
IF (1ALP(J)\textgreater 0 ) AFAC=+1.0
IF (1ALP(J)\textless 0 ) AFAC=-1.0
ENK(1) = 0 USE STAR GAS RHO
NOT = 0 USE PLASMA RHO
IF(ENK(1)\textneq 0 ) GO TO 266
***** STAR GAS *****
RHOI=RHOI + AFAC*SIN(C1* (X(J)-XAVE) )
GO TO 258
***** PLASMA *****
266 CONTINUE
COP=COP+COS(C1*X(J))*AFAC
SIP=SIP+SIN(C1*X(J))*AFAC
258 CONTINUE
IF (ENK(1)\textneq 0 ) GO TO 79
RHOI=-RHOI/SRFN
RHOS=2.* RHOI**2
GO TO180
79 COSXB=COS(C1*XAVE)
SINXB=SIN(C1*XAVE)
***** RHO REAL *****
***** ENK(21) TERM IN REAL PART OF RHO(K) Eq. ZER0 UNLESS INPUT
RHOR=COSXB*COP/ SRFN+SINXB*SIP/ SRFN
***** RHO IMAGINARY *****
RHOI=SINXB*COP/ SRFN-COSXB*SIP/ SRFN
RHOS=RHOR**2+RHOI**2
***** SUM OF RHOS(K) OVER CYCLES *****
180 CONTINUE
RHOKS(K)=RHOKS(K)+RHOS
KM1=K-1
259 FORMAT(15,E19.8) WRITE(6,259)KM1,RHOR,RHOI,RHOS
257 CONTINUE
157 IF (ITRACE\textneq 0 ) GO TO 159
C ITP0 = INTEGER SAVE TRACES EVERY ITP0 CYCLE
ITPOC=ITPOC+1
IF (ITPOC\textneq ITP0) GO TO 159
ITPOC=0
IF (NTPP\textneq 1000) WRITE(6,1151)
1151 FORMAT(16H TIME DIM.*=1000)
IF(NTPP.EQ.1000) GO TO 159
NTPP=NTPP+1
CONTINUE

151

***** SAVE TIME AND X FOR TRACES *****

AT EVERY CYCLE(TIME) PICK OFF X OF ALPHA PARTICLES

FNOCK=NOCK
TIME(NTPP)=FNOCK*DELT
DO 150 I=1,NOP
DO 152 K=IALPT4,IALPT5,IALPT6
IF(IABS(ALP(I)).NE.K) GO TO 152

***** COUNT FIRST TIME THRU ONLY

IF(NT3.EQ.0) NT2=NT2+1
WRITE(13) X(I)
IF(X(I).LT.XMIN2) XMIN2=X(I)
IF(X(I).GT.XMAX2) XMAX2=X(I)
GO TO 154

152

CONTINUE

DO 155 L=IALPT1,IALPT2,IALPT3
IF(IABS(ALP(I)).NE.L) GO TO 155
IF(NT3.EQ.0) NT1=NT1+1
IF(NTPP.EQ.1000) GO TO 150
WRITE(14) X(I)
IF(X(I).LT.XMIN1) XMIN1=X(I)
IF(X(I).GT.XMAX1) XMAX1=X(I)
GO TO 150

155

CONTINUE

150

CONTINUE

NT3=1

159 IF(IPFC.NE.IPF) GO TO 54

PRINT

IPFC=0
WRITE(6,51) NOCK
51 FORMAT(10H CYCLE15)
IF(NUC.EQ.0) GO TO 72
WRITE(6,73)
73 FORMAT(92H J ALPHA X X DOT
1U SUB J E XMI//)
GO TO 156
72

CONTINUE
WRITE(6,56)
56 FORMAT(76H J ALPHA X X DOT
1Y DOT E//)
156 CONTINUE

DO 53 J=1,NOP
WRITE(6,52) J,IALP(J),X(J),XDOT(J),YDOT(J),F(J)
52 FORMAT(218,4E17.8)
53

CONTINUE

54 IF(IPLT.EQ.0) GO TO 139
IF(IXP.EQ.IXP) IXPC=0
IF(IEP.EQ.IEP) IEP=0
IF (IXDPCEQIXDP) IXDPC=0

CALL PLOT

IF (IEPCLT0) IEPC=0

139 IF (NOCKNENOC) GO TO 39

60 CONTINUE

WRITE(6,55) NOCK, NOC

55 FORMAT(9HNOCKI5,5HI5,14HTERMINATE//)

DO 153 J=1,NOP

WRITE(6,52) J, IALP(J), X(J), XDOT(J), YDOT(J), E(J)

153 CONTINUE

WRITE ARRAYS ON TAPE TO SAVE

71 WRITE(11) NOP, (X(I), I=1,NOP), (XDOT(I), I=1,NOP), (YDOT(I), I=1,NOP),

1(IALP(I), I=1,NOP)

WRITE(6,174) FRHOK

174 FORMAT(8HFRHOK=E17.8)

WRITE(6,74)

74 FORMAT(44HSUM(RHO(K))**2 SUM(RHO(K))**2/FRHOK//)

DO 75 K=1,21

BUM=RHOKS(K)/FRHOK

75 WRITE(6,76) K, RHOKS(K), BUM

76 FORMAT(15,2E19.8)

N=0

NT3=NT1+NT2

IF (ITRACEEQ0) GO TO 167

***** PLOT TRACES *****

WRITE(6,91) NTPP, NT1, NT2

91 FORMAT(20HTPS PER TRAJ PLOT=I5/I5 OF TRAJ=I5/I5//)

WRITE(6,92) XMIN1, XMAX1, XMIN2, XMAX2

92 FORMAT(8H XMIN1=E17.8, 8H XMAX1=E17.8, 8H XMIN2=E17.8, 8H XMAX2=E17.8)

REWIND 13

192 IPC=0

DO 162 K=1,NTPP

DO 162 I=1,NT2

IPC=IPC+1

READ(13) X(IPC)

YDOT(IPC)=TIME(K)

IF (KEQNTPP) AND (IEQNT2) N=1

IF ((IPC+EQ6000) OR (NEQ1)) GO TO 193

GO TO 162

193 CALL DDIONN(N, IN, IPC, YDOT(1), X(1), TIME(1), TIME(NTPP), TMM(3), TMM(4)

1.3*TIME(1), YX, 13)

IPC=0

162 CONTINUE

REWIND 13

N=0
REWIND 14
DO 163 K=1,NTPP
DO 163 I=1,NT1
IPC=IPC+1
READ(14) X(IPC)
YDOT(IPC)=TIME(K)
IF((K.EQ.NTPP).AND.(I.EQ.NT1)) N=1
IF((IPC.EQ.6000).OR.(N.EQ.1)) GO TO 194
GO TO 163
194 CALL DDIONN(N,IN,IPC,YDOT(1),TIME(1),TIME(NTPP),TMM(1),TMM(2)
1,3,XTIME,1,YX,13)
IPC=0
163 CONTINUE
REWIND 14
166 CONTINUE
167 WRITE(6,70)
70 FORMAT(5H END)
CALL EXIT
STOP
END
5*
SUBROUTINE PLOT

COMMON NOP, IPF, NOC, I TAPE, N (3), I BUG, IN (2)
COMMON I PLOT, I XP, I EP, IALP, NXI, NXDI, NYDI
COMMON DELT, B, CM, SM, EZ, EPS, DELX, DELXD, DELYD, RNM X, VT
COMMON KOE, NUC, I XDP, I XDPCL, X (4), LODE, ITPO, S
COMMON KKOE, NOPOT, DE, PE, FAC
COMMON X (3000), XDOT (3000), YDOT (3000), E (3000), I ALP (3000)
COMMON I XPC, I EPC, NOCK, ISYM

DIMENSION D (500), C (500)
DIMENSION CP (500)
DIMENSION YM (2), XME (5), XMO (5), XDME (5), XMOM (5), YDME (5), YDMO (5)
DIMENSION XLAB (3), XDLAB (3), EXM (3)
DIMENSION XI (45), Y1 (45), Y2 (45)

DATA XLAB (1) , XDLAB (1) / 13HX FOR CYCLE, 13HXDOT, VELOCITY/
DATA G / 0.0 /
DATA EXM (1) , EYM, YM (1) / 13HX FOR CYCLE, 13HDENSITY/
DATA XME (1) , XMO (1) , XDME (1) , XMOM (1) , YDME (1) , YDMO (1) /

130HDELTA XIONS CYCLE, 30HDELTA X ELECTRONS CYC
3 2 30HDELTA XDOTIONS CYC, 30HDELTA XDOT ELECTRONS CYC

*** FOLLOWING DATA CARDS USED TO PLOT ELLIPSE WITH VELOCITY
PLOTS ***

C DATA XI$/-791.92,-788.80,-783.60,-778.40,-768.00,-747.20,-716.00,
C 1-686.88,-579.76,-467.44,-353.04,-229.20,-94.08,56.92,231.64,
C 2 2459.62,620.20,731.48,759.76,930.16,879.88,915.08,1000.00,
C 3 3 1084.92,1120.12,1159.84,1240.24,1268.52,1379.80,1540.38,
C 1 1176.36,1943.08,2094.08,2229.28,2353.04,2467.44,2579.76,2686.88,
C 5 2 5216.00,2747.20,2768.00,2778.40,2783.60,2788.80,2791.029/,
C DATA Y1/0.0,64.4,120.6,158.0,213.8,299.2,389.6,456.0,645.0,790.00,
C 1 1912.2,1019.8,1117.2,1205.8,1290.0,1368.2,1405.8,1424.0,1427.6,
C 2 1435.0,1438.6,1440.4,1442.2,
C 3 3 1440.4,1438.6,1435.0,1427.6,1424.0,1405.8,1368.2,1290.00,
C 1 11206.8,1117.2,1019.8,912.2,790.0,645.0,456.0,389.6,299.2,213.8,
C 2 1158.0,120.6,644.0,2,
C DATA Y2/0.0,-64.4,-120.6,-158.0,-213.8,-299.2,-389.6,-456.0,-645.0,0,
C 1-790.0,-912.2,-1019.8,-1117.2,-1205.8,-1290.0,-1368.2,-1405.8,
C 2-1435.0,-1438.6,-1435.0,-1427.6,-1424.0,-1405.8,-1368.2,-1290.00,
C 3-1435.0,-1427.6,-1424.0,-1405.8,-1368.2,-1290.0,-1206.8,-1117.2,
C 4-1019.8,-912.2,-790.0,-645.0,-456.0,-389.6,-299.2,-213.8,-158.00,
C 5-120.6,-644.0,
C GO TO 81 THIS USED TO PLOT ELLIPSE AND VELOCITY

DELXX=(X(NOP)-X(1))/FNXI
IF (IXPC.EQ.0) GO TO 10
IF (IXDPC.NE.0) GO TO 20

HERE TO COLLECT DATA FOR X,X DOT, Y DOT DENSITY PLOTS

10 DELXT=X(1)+DELXX

COUNT TIMES ENTERING DENSITY PLOT ROUTINE

XDMIN=0*
XDMAX=0*
YDMIN=0*
YDMAX=0*
I1=0
COUNT=0
COUNTP=0*

FIND MAX AND MIN VALUES

DO 114 I=1,NOP
IF (XDOTT(I)*GT*XDMIN) GO TO 111
XDMIN=XDOT(I)
111 IF (XDOTT(I)*LT*XDMAX) GO TO 112
XDMAX=XDOT(I)
112 IF (YDOTT(I)*GT*YDMIN) GO TO 113
YDMIN=YDOT(I)
113 IF (YDOTT(I)*LT*YDMAX) GO TO 114
YDMAX=YDOT(I)
114 CONTINUE
IF (IXPC.EQ.0) GO TO 71
IF (IXDPC.EQ.0) GO TO 81

BEGIN X PLOT

71 DO 70 I=1,NOP
70 CONTINUE
IF (X(I)*GT*DELXT) GO TO 72
IF (I3ALP(I)*LT.0) GO TO 21
COUNT +ALPHAS OR IONS
COUNTP=COUNTP+1*
GO TO 77
COUNT -ALPHAS OR ELECTRONS
21 COUNT=COUNT+1*
77 IF (I*NE*NOP) GO TO 70
72 IF (I*LE*500) GO TO 73
WRITE(6,74) I
74 FORMAT(BH 11=16,18H CHANGE DIMENSION//) GO TO 79
73 I=I+1
D(I)=DELXT
IF (I*DELXT/X(NOP)*LT.1E-5) COUNT=COUNT+1*
$C(i) = \text{COUNT}$

$CP(i) = \text{COUNTP}$

75 $\text{DELX} = \text{DELX} + \text{DELXX}$

COUNT $= 0$

COUNTP $= 0$

IF $(\text{DELX} \leq X(\text{NOP}))$ GO TO 80

70 CONTINUE

79 CONTINUE

CALL CVT1(NOCK, XMOD(4), NP)

XME(5) $= XMO(5)$

IF $(\text{IALPO} \cdot GT.0)$ GO TO 93

CALL DDIPLT(IN, XI, D, XMOD, XDMIN, XDMAX, YM, 14)

IF $(\text{IALPO} \cdot LT.0)$ GO TO 1933

93 CALL DDIPLT(IN, XI, D, CP, XMOD, XDMIN, XDMAX, YM, 14)

1933 CONTINUE

WRITE(6, 1114) DELXX, FNXI, DELX, XDMIN, XDMAX, YM, 14

1114 FORMAT(1H x DELXX, FNXI, DELX, XDMIN, XDMAX, YM)

193 WRITE(6, 200) NKXI, (D(K), K = 1, NKXI), (C(K), K = 1, NKXI), (CP(K), K = 1, NKXI)

200 FORMAT(5I, (7E17.8))

BEGIN X DOT PLOT INFO

DELXD $= (XDMAX - XDMIN) / FNXI$

DELXDT $= XDMIN + DELXD$

DM1 $= XDMIN$

DO 76 J = 1, NXDI

IF $(J \cdot EQ. N X D I )$ DELXDT $= \text{DELXDT} + .00001$

COUNT $= 0$

COUNTP $= 0$

70 CONTINUE

D(J) $= \text{DELXDT}$

C(J) $= \text{COUNT}$

CP(J) $= \text{COUNTP}$

DM1 $= \text{DELXDT}$

76 DELXDT $= \text{DELXDT} + \text{DELXD}$

CALL CVT1(NOCK, XMOD(4), NP)

XME(5) $= XMOD(5)$

IF $(\text{IALPO} \cdot GT.0)$ GO TO 94

CALL DDIPLT(IN, XI, D, XMOD, XDMIN, XDMAX, YM, 14)

IF $(\text{IALPO} \cdot LT.0)$ GO TO 1944

94 CALL DDIPLT(IN, XI, D, CP, XMOD, XDMIN, XDMAX, YM, 14)

1944 CONTINUE

194 WRITE(6, 200) NKXI, (D(K), K = 1, NKXI), (C(K), K = 1, NKXI), (CP(K), K = 1, NKXI)

IF (IXDPC \cdot NE. 0) GO TO 86

C
C  *** GET VELOCITY PLOTS X VS XDOT, EVERY IXDP CYCLE *****
C  ***** PLOT X VS XDOT *****
C  XL(1)= X MIN FOR X VS XDOT, VELOCITY PLOT
C  XL(2)= X MAX FOR X VS XDOT
C  XL(3)= XDOT MIN FOR X VS XDOT, VELOCITY PLOT
C  XL(4)= XDOT MAX FOR X VS XDOT, VELOCITY PLOT
C
81 CONTINUE
CALL CVTI(NOCK,XLAB(2),NP)
CALL DDIONN(1,IN,NOP,X,XDOT, XL(1),XL(2),XL(3),XL(4),3,XLAB,3,XDLAB
1,11)
C  ***** NEXT 4 CARDS USED TO PLOT ELLIPSE AND VELOCITY ONLY *****
C  CALL DDIPLT(0,IN,NOP,X,XDOT,A,F,H,0,3,XLAB,3,XDLAB,11)
C  CALL DDIPLT(0,IN,45,X1,Y1,A,F,H,0,3,XLAB,3,XDLAB,14)
C  CALL DDIPLT(1,IN,45,X1,Y2,A,F,H,0,3,XLAB,3,XDLAB,14)
C RETURN
IF(IXPC.NE.0) GO TO 20
BEGIN Y DOTT PLOT INFO
C
86 CONTINUE
NOPOT=NOPOT+1
DELYD=(YDMAX-YDMIN)/FNYDI
DELYDT=YDMIN+DELYD
DM1=YDMIN
DO 83 J=1,NYDI
IF(J.EQ.NYDI) DELYDT=DELYDT+.00001
COUNT=0
COUNTP=0
DO 85 I=1,NOP
IF(YDOT(I).GT.DELYDT.OR.YDOT(I).LT.DM1) GO TO 85
IF(IALP(I).LT.C) GO TO 25
COUNTP=COUNTP+1.
GO TO 85
85 COUNT=COUNT+1.
83 DELYDT=DELYDT+DELYD
CALL CVTI(NOCK,YDMO(4),NP)
YDME(5)=YDMO(5)
IF(IALPO.GT.0) GO TO 95
CALL DDIPLT(1,IN,NYDI,0,C,O,0,O,G,0,5,YDMO(1),2,YM(1),14)
IF(IALPO.LT.0) GO TO 92
95 CALL DDIPLT(1,IN,NYDI,0,CP,C,O,G,0,5,YDME(1),2,YM(1),14)
92 CONTINUE
WRITE(6,200)NYDI, (D(K),K=1,1,NYDI), (C(K),K=1,1,NYDI), (CP(K),K=1,1,NYDI)
WRITE(6,183)DELYD,DELYDT,DELYD
183 FORMAT(27H DELXD,DELYD,DELYDT/4E17.8)
20 IF(IPEC .NE. 0) GO TO 15
   IF(NOCK .EQ. NOC) GO TO 16
12 CONTINUE

C PLOT E VS X AT EVERY IEP ITERATION

C CALL CVTI(NOCK, EXM(2), NP)
C ISYM = 13 = DOT
C ISYM = 14 = VECTOR
14 CALL DDIPLT(1, IN, NOP, X, E, 0, 0, 0, 3, EXM, 1, EYM, ISYM)
   GO TO 90

C 15 IF(NOCK .NE. NOC) GO TO 90

C FINAL PT AT LAST ITERATION

C PLOT E VS X FOR LAST ITERATION

16 CALL CVTI(NOCK, EXM(2), NP)
   CALL DDIPLT(1, IN, NOP, X, E, 0, 0, 0, 3, EXM, 1, EYM, 14)
   WRITE(6, 91) NOP
91 FORMAT(2I18) THERE SHOULD BE 15,14H DENSITY SETS //
90 RETURN

END
SUBROUTINE NATAL DECK

COMMON NOP, IPF, NOC, ITAPE, N(3), IBUG, IN(2)
COMMON IPlot, XP, IEP, IALPO, NXI, NDX, NYD
COMMON DELT, B, CM, SM, EZ, EPS, DELX, DELXD, DELYD, RNMX, VT
COMMON EMIN, EMAX, PLOTDE, XMIN, XMAX, FXI, FNXD, MND
COMMON KOE, NUC, IXDP, IXDPC, XL(4), LODE, ITPO, S
COMMON KKO, NOPOT, D, PE, FAC

COMMON X(6000), YDOT(6000), IALP(3000)
COMMON XP, EPC, NUCK, ISYM

COMMON /A/ TRACE, IALPT(10), ENK(21), VZE, VZO, RNMXE, RNMXO, TM(10)
COMMON /MA/ RHOK, FRHOK, NT1, NT2, NT3, FNP, DX, XO, DELT, ITOPC
1 IPFC, NT, NUCK, K1, K2, K3, K4, RNMD, RNMXOE, RNMDOE, RNMDOE, IRHO, GRAV
2 XIC, RHOKS(21), SIG, MEAN, 12, 11

DIMENSION XDOT(3000), E(3000)

EQUIVALENCE (IALPT1, IALPT(1)), (IALPT2, IALPT(2)), (IALPT3, IALPT(3)),
(IALPT4, IALPT(4)), (IALPT5, IALPT(5)), (IALPT6, IALPT(6))

DATA SIG/1.0/, MEAN/0.0/, 12/0.0, 11/1/
DO 1 K=1, 21
ENK(K)=0.0
1 RHOKS(K)=0.0
READ(5, 110) IN
110 FORMAT(2A6)

7-19-66 INPUT CHANGED TO READ

READ(5, 555) NOP, NOC, IPF, ITAPE, N(3), PLOTDE, IALPO, NUCK, IBUG,
1 KOE, LODE, IPlot, XP, IEP, ITPO, NXI,
2 NDX, NYD, TRACE, (IALPT(I), I=1, 5),
3 IALPT(6), IRHO, ISYM

555 FORMAT(8I10)
READ(5, 557) RNMXE, RNMXO, DELT, B, SM,
1 CM, EZ, EPS, VT, VZE,
2 VZO, S, (TM(1), I=1, 3),
3 TM(4), GRAV, (XL(1), I=1, 3),
4 XL(4), FNXD, MND, FXI, ENK(1),
5 XIC
557 FORMAT(5E15.8)
556 FORMAT(5E16.8)
11 WRITE(6,12)
12 FORMAT(115H
      NUMERICAL EXPERIMENT TO SIMULATE DIFFUSION OF BAR
      IUM CLOUD OR SIMULATE COLLISIONLESS ULASOV EQUATION-HOHJ-BLW//)
C
      WRITE(6,310)
310 FORMAT(80H NOP, NOC, IPF, ITOPE, PLOTDE, I
1ALPO, NUC, IBUG)
      WRITE(6,555) NOP, NOC, IPF, ITOPE, PLOTDE, IALPO, NUC, IBUG
      WRITE(6,311)
311 FORMAT(80H KOE, LODE, ILOT, IXP, IEP, ITOPO, NXI)
      WRITE(6,555) KOE, LODE, ILOT, IXP, IEP, ITOPO, NXI
      WRITE(6,312)
312 FORMAT(80H NXD, NYDI, ITRACE, IALPT(1), IALPT(2), IALP
1T(3), IALPT(4), IALPT(5))
      WRITE(6,555) NXD, NYDI, ITRACE, (IALPT(I), I=1,5)
      WRITE(6,313)
313 FORMAT(30H PLOTDE, IXP, IALPT(2), IALP
S YM)
      WRITE(6,555) IALPT(6), IRHO, ISYM
      WRITE(6,304)
304 FORMAT(75H RNMXE, RNMXO, DELT, B, SM)
      WRITE(6,556) RNMXE, RNMXO, DELT, B, SM
      WRITE(6,305)
305 FORMAT(75H CM, EZ, EPS, VT, VZE)
      WRITE(6,556) CM, EZ, EPS, VT, VZE
      WRITE(6,306)
306 FORMAT(75H VZO, S, TMM(1), TM
1M(2), TMM(3))
      WRITE(6,556) VZO, S, (TMM(I), I=1,3)
      WRITE(6,307)
307 FORMAT(75H TMM(4), GRAV, X L(1), X
1L(2), XL(3))
      WRITE(6,556) TMM(4), GRAV, (XL(I), I=1,3)
      WRITE(6,308)
308 FORMAT(75H XL(4), FNXDI, FNYDI
FINX, ENK(1))
      WRITE(6,556) XL(4), FNXDI, FNYDI, FINX, ENK(1)
      WRITE(6,309)
309 FORMAT(15H XIC)
      WRITE(6,556) XIC
      WRITE(6,110) IN
C
119 CONTINUE
C
      FNOP=NOP
      DX=RNMXO/FNOP
      DELX=RNMXE/FNOP
X0 = RNMXE/2. + DELX/2.
DELT2=DELT**2

C IF ITAPE=0, READ IC FROM TAPE, WHEN NOT=0 RANDOM GENERATE

C IF (ITAPE .NE. 0) GO TO 15
READ (9) NOPT, (X(I), I=1, NOPT), (XDOT(I), I=1, NOPT), (YDOT(I), I=1, NOPT)
1, (IALP(I), I=1, NOPT)
REWD 9
IF (NOP .EQ. NOPT) GO TO 21
WRITE (6, 17) NOP, NOPT
17 FORMAT (6H NOP=15, 7H NOPT=15, 22H THEY SHOULD BE EQUAL)
CALL EXIT
C GENERATE RANDOM NUMBERS FOR X XDOT AND YDOT
15 CONTINUE
C ****** GAUSIAN MULTIPLIERS *****
RNMXED = SQRT(SM/CM)*VT
RNMXDD = VT
RNMYDE = RNMXDE
RMYDD = VT
I2 = 0
[I = 1
C SET UP ALPHAS *****
IF (IALPO.GT.300.301,302
301 WRITE (6, 282)
282 FORMAT (35H ALPHAS ALTERNATELY 'PLUS AND MINUS/
GO TO 303
302 WRITE (6, 281)
281 FORMAT (21H ALL POSITIVE ALPHAS/
GO TO 303
300 WRITE (6, 280)
280 FORMAT (21H ALL NEGATIVE ALPHAS/
303 DO 20 1 = 1, NOPT
IM1 = I - 1
IF (IALPO.GT.80,81,82
80 IALP(I) = -1
GO TO 83
82 IALP(I) = 1
GO TO 83
81 I2 = I2 + 1
[I = I + I1
IALP(I) = I1*I2
C CONTINUE
C PLOTDE -1 = X,X+DX WHERE DX=RNMX/FNOP AND GAUSIAN XDOT AND YDOT
C 0 = LINEAR X, GAUSIAN XDOT, YDOT FROM RAND
C +1 = LINEAR X, XDOT AND YDOT FROM RAND
C +2 = X,X+DX, LINEAR XDOT AND YDOT FROM RAND
C IF (PLOTDE .EQ. 0) GO TO 89
IF( PLOTDE .EQ. 1 ) GO TO 89
IF( I .EQ. 1 ) X(I) = DX
IF( I .NE. 1 ) X(I) = X(I-1) + DX
GO TO 88
89 CALL RAND(MEAN, SIG, LODE, ANS, ANSI)
X(I) = ANS
88 CALL RAND(MEAN, SIG, LODE, ANS, ANSI)
IF( PLOTDE .LE. 0 ) XDOT(I) = ANS
IF( PLOTDE .GT. 0 ) XDOT(I) = ANSI
CALL RAND(MEAN, SIG, LODE, ANS, ANSI)
IF( PLOTDE .LE. 0 ) YDOT(I) = ANS
IF( PLOTDE .GT. 0 ) YDOT(I) = ANSI
20 CONTINUE
IF( IBUG .NE. 0 ) GO TO 28
WRITE(6, 23) (X(I), I = 1, NOP)
WRITE(6, 24) (XDOT(I), I = 1, NOP)
WRITE(6, 25) (YDOT(I), I = 1, NOP)
WRITE(6, 26) (IALP(I), I = 1, NOP)
23 FORMAT(7H X = / (7E 17.8))
24 FORMAT(1OH XDOT = / (7E 17.8))
25 FORMAT(1OH YDOT = / (7E 17.8))
26 FORMAT(1OH IALP = / (10110))
28 CONTINUE
IF( PLOTDE .EQ. -1 ) GO TO 228
DO 229 I = 1, NOP
C
***** LINEAR X *****
IF( PLOTDE .EQ. 2 ) GO TO 182
IF( IALP(I) .LT. 0 ) X(I) = X(I) * RNMXO
IF( IALP(I) .GT. 0 ) X(I) = X(I) * RNMXE
IF( PLOTDE .LE. 0 ) GO TO 229
C
***** LINEAR XDOT AND YDOT *****
182 CONTINUE
IF( IALP(I) .LT. 0 ) GO TO 181
XDOT(I) = (XDOT(I) - 5) * RNMXDE + VZE
YDOT(I) = (YDOT(I) - 5) * RNMXDE
GO TO 229
181 XDOT(I) = (XDOT(I) - 5) * RNMXD0 + VZO
YDOT(I) = (YDOT(I) - 5) * RNMXD0
229 CONTINUE
IF( PLOTDE .GT. 0 ) GO TO 21
228 CONTINUE
C
***** GAUSIAN XDOT, YDOT *****
DO 19 I = 1, NOP
128 IF( IALP(I) .LT. 0 ) GO TO 18
XDOT(I) = RNMXDF * XDOT(I) + VZE
YDOT(I) = RNMYDF * YDOT(I)
GO TO 19
18 XDOT(I) = RNMXDO * XDOT(I) + VZO
YDOT(I) = RNMYDO * YDOT(I)
19 CONTINUE
C

21 CONTINUE
RETURN
END
SUBROUTINE SHIFT

SUBROUTINE TO ORDER ACCORDING TO X

COMMON NOP, IPF, NOC, ITAPE, N(3), IBUG, IN(2)
COMMON IPLD, ID, IE, IP, ALPO, NXI, NXD, NYD
COMMON DELT, B, CM, S, EZ, EPS, DELX, DELYD, RNMX, V1
COMMON EMIN, EMAX, PILOTDE, XMIN, XMAX, FNXI, FNXDI, FNYDI
COMMON KOE, NUC, IDP, IDPC, X(4), LODE, ITPO, S
COMMON KKOE, NOPOT, DE, PE, FAC
COMMON X(3000), XDOD(3000), YDOD(3000), E(3000), IALP(3000)

30 CONTINUE

31 DO 40 J1 = 2, NOP
   J2 = J1 - 1
   IF (X(J1) .GE. X(J2)) GO TO 40
   XSAVE = X(J1)
   ESAVE = E(J1)
   XDSAVE = XDOD(J1)
   YDOD = YDOD(J1)
   IALPS = IALP(J1)
   X(J1) = X(J2)
   E(J1) = E(J2)
   XDOD(J1) = XDOD(J2)
   YDOD(J1) = YDOD(J2)
   IALP(J1) = IALP(J2)
   J3 = J2 - 1
   IF (J3 .NE. 0) GO TO 32
   X(J2) = XSAVE
   E(J2) = ESAVE
   XDOD(J2) = XDSAVE
   YDOD(J2) = YDSAVE
   IALP(J2) = IALPS
   GO TO 40
32 IF (XSAVE .LT. X(J3)) 34, 33, 33

33 J4 = J3 + 1
   X(J4) = XSAVE
   E(J4) = ESAVE
   XDOD(J4) = XDSAVE
   YDOD(J4) = YDSAVE
   IALP(J4) = IALPS
   GO TO 40
34 J4 = J3 + 1
   X(J4) = X(J3)
   E(J4) = E(J3)
   XDOD(J4) = XDOD(J3)
YDOT(J4) = YDOT(J3)
IALP(J4) = IALP(J3)
J3 = J3 - 1
IF (J3) 32, 35, 32
35 X(1) = XSAVE
   E(1) = ESAVE
   XDOT(1) = XDSAVE
   YDOT(1) = YDSAVE
   IALP(1) = IALPS
40 CONTINUE
RETURN
END
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\begin{verbatim}
FAO   \texttt{A+2}
FDP   \texttt{TEMP}
XCA   \texttt{CHA}
CMS   \texttt{FAV}
FAD   \texttt{VI}
STO   \texttt{TEMPI}
CLA   \texttt{UI}
FSB   \texttt{CONS}
TPL   \texttt{PLUS}
CLS   \texttt{TEMP}
TRA   \texttt{**+2}
PLUS  \texttt{CLA TEMPI}
      \texttt{XCA}
SIG   \texttt{FMP **}
MEAN  \texttt{FAD **}
ANS   \texttt{STO **}
RETURN RAND
FLOAT OCT  011060471625.2000000000000.0
\texttt{CONS DEC} \texttt{.5\texttt{1**-2**}}
A \texttt{DEC} \texttt{.01032848028532.515577}
B \texttt{DEC} \texttt{.00130841892691.432188}
U! \texttt{PZE}
VI \texttt{PZE}
TEMP \texttt{PZE}
DOG \texttt{PZE}
CAT \texttt{PZE}
END
\end{verbatim}
REFERENCES


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The author, Frank Hohl, was born in Mülheim, Germany, on January 28, 1934. After completing elementary school training he began a three year apprenticeship to become an industrial electrician. In 1953 the author immigrated to the United States and shortly thereafter joined the United States Air Force. Upon completion of a four year tour of duty with the Air Force he attended the University of Florida and received the degree of Bachelor of Electrical Engineering in June 1961. From 1961 to 1962 the author was employed by the National Aeronautics and Space Administration, Langley Research Center, Hampton, Virginia, as an aerospace engineer. In 1962 the author became a member of the technical staff of Bell Telephone Laboratories, Holmdel, New Jersey. In 1963 he received the degree of Master of Electrical Engineering from New York University and returned to the National Aeronautics and Space Administration, Hampton, Virginia, as an aerospace engineer. The author received the degree of Master of Science in Physics in 1965 from the College of William and Mary. Upon admission to candidacy for the degree of Doctor of Philosophy the author began work on his thesis project which was sponsored by the National Aeronautics and Space Administration.
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$\textbf{Figure 10.}$  

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- $t = 31.5\, \tau_{co}$
- $t = 46.5\, \tau_{co}$
- $t = 19.5\, \tau_{co}$
- $t = 37.5\, \tau_{co}$
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- $\alpha = 0.9$
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Kinet. energy $\frac{1}{2} \frac{m}{\mu} \frac{1}{2} \frac{1}{1}$
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$8 \times 10^{3}$
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