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Pressure anisotropy effects on the stability of the guiding center model of the bumpy theta pinch

Michael J. Schmidt
College of William & Mary - Arts & Sciences

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PRESSURE ANISOTROPY EFFECTS ON THE STABILITY OF THE
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A Dissertation
Presented to
The Faculty of the Department of Physics
The College of William and Mary in Virginia

In Partial Fulfillment
Of the Requirements for the Degree of
Doctor of Philosophy

by
Michael J. Schmidt
June 1976
APPROVAL SHEET

This dissertation is submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

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ABSTRACT

The guiding center plasma model is used for a normal mode analysis of the bumpy theta pinch. Pressure anisotropies of any order in $\varepsilon$ (measures the period of the bumpiness of the magnetic field lines compared to the radius of the plasma) and $\delta$ (measures the amplitude of the bumpiness) are considered. It is shown that only $O(\varepsilon^0, \delta^0)$ pressure anisotropies will alter the growth rates previously predicted by the ideal magnetohydrodynamic model for the most unstable transverse modes. In general, for pressure anisotropy profiles that are monotonic functions of the plasma radius, the growth rates of the most unstable transverse modes decrease if the perpendicular pressure (pressure transverse to the direction of the magnetic field) is smaller than the parallel pressure. This can be qualitatively understood from the effects of the pressure anisotropy on the equilibrium fields.
PRESSURE ANISOTROPY EFFECTS ON THE STABILITY OF THE
GUIDING CENTER MODEL OF THE BUMPY THETA PINCH
I. INTRODUCTION

During the past 20 years there has been an intensive effort around the world to achieve a controlled thermonuclear reaction for the production of electrical power. This effort has taken various forms, one of which is the containment of a high energy deuterium tritium plasma in a suitably shaped "magnetic bottle".

Theoretically, the problem of magnetic containment of a plasma is quite formidable. First, in the plasma state there are a large number of time and length scales (e.g. Larmor radius and Debye length, plasma frequency and gyrofrequency, Alfvén speed and acoustic speed compared to macroscopic time and length scales, etc.) which can be the same order of magnitude and can complicate approximation procedures used in modeling he behavior of the plasma. Second, for the problem of magnetic containment, one is forced into using bounded magnetic field configurations.

Presented with this complexity, we cannot hope to construct mathematical models that embrace a substantial amount of the physics and at the same time incorporate non-trivial geometries. Therefore, one approach is to develop mathematically simpler models that give useful information in limited parameter ranges. One such model is the perfectly conducting or ideal magnetohydrodynamic (MHD) theory, which has been extensively used in plasma theory.
The theoretical justification for the validity of the MHD model requires the presence of many collisions between the particles of the plasma. In this case the particles' velocity distribution is driven towards an isotropic Maxwellian, and the particles' spatial trajectories are localized. The former is needed so that the content of all the information contained in the velocity distribution is effectively summarized by the first few velocity moments of the distribution function itself (e.g., density, fluid velocity, and scalar pressure). The latter is needed so that the properties of a group of particles composing a fluid element depends only on other fluid elements immediately surrounding it.

The ideal MHD model has demonstrated its utility by predicting phenomena in agreement with many important experimental results of the plasma fusion program, even though many of these experiments are not strictly in the MHD regime.

As plasma containment experiments are scaled up in magnetic field, in kinetic energy, and in physical size, the plasma parameters will move even further away from the collision-dominated MHD regime. The idea of many collisions localizing and isotropically thermalizing the plasma becomes less valid because the collision times are longer than or of the same order as the containment time scale of the plasma, and because of the highly anisotropic nature of the magnetic field.

However, the strong magnetic field permits the replacement of the microscopic time and length scales of collision time and mean free path with the new parameters of gyro period and Larmor radius. Using
these new microscopic time and length scales a different theoretical model, known as the Guiding Center Plasma (GCP) model, becomes appropriate.

The magnetic field localizes the particle motion in the plane perpendicular to the field (i.e. the Larmor radius is much smaller than the macroscopic length scale of interest). This localization leads to a fluid description much like the MHD description, where now the strong magnetic field assumes the role previously played by collisions. As there is no localization along the field lines, it is necessary to use a microscopic or kinetic description in this direction. In effect, this leaves the plasma only two-thirds a fluid and the disparity between the motion along and across the field lines introduces an anisotropic tensor pressure in place of the scalar pressure of the MHD model.

Even with the simplifications achieved with either the MHD or the GCP model, they still yield a set of nonlinear partial differential equations (the GCP model also has integral equations) and very little can be accomplished analytically with the full nonlinear problem. Almost all the work in both theories has been to apply linear stability analysis to specific problems. Most of these analyses have used a variational technique. More recently, normal mode analysis of the MHD equations of motion has been performed to further understand the spectral properties of the linear system of equations.

So far, the GCP model has not been studied as extensively as the MHD model and it is of current interest to see how the GCP approximation (which is in a more appropriate parameter range for larger plasma
containment devices) changes the linear stability results already predicted by MHD. To this end, we have applied a normal mode analysis to a diffuse profile bumpy theta pinch plasma using the GCP theory.

The bumpy theta pinch geometry considered assumes that the plasma is nearly cylindrical and two small parameters, \( \epsilon \) and \( \delta \), are introduced as a measure of the deviation from this cylindrical shape. \( \epsilon \) is proportional to the reciprocal of length of the bumpiness and \( \delta \) is proportional to the amplitude of the bumpiness.

There are basically two reasons for this choice. First, the bumpy theta pinch magnetic field configuration, though nontrivial, is not an exceptionally difficult geometry to analyze, so more attention can be paid to the differences in the physics between the MHD model and the GCP model. Second, there is a rather complete spectral theory on the MHD stability of the bumpy theta pinch, using the MHD model,\(^{5,6,7}\) with which this work can be compared.

To begin this problem, in Section II we discuss the GCP model in general. The simplification of this model, relative to a full kinetic theory, derives from the process of averaging over the gyro motion of the particles, replacing their actual motion with that of particles constrained to move along a field line (i.e. with the particles' guiding center motion). The kinetic equation for motion along the field lines and the macroscopic equations for motion across the field lines are presented, as well as the self consistent connection between them. Further details of the approximations that go into the GCP model are contained in Appendix A.
When we introduce the electromagnetic fields in the text, we use an unconventional system of units that dispenses with all the constants that are normally associated with Maxwell's equations. To transform the magnetic field $\mathbf{B}$, the electric field $\mathbf{E}$, and the current $\mathbf{J}$ back into Gaussian CGS units, one replaces these fields with $\frac{1}{\sqrt{4\pi}} \mathbf{B}$, $\frac{c}{\sqrt{4\pi}} \mathbf{E}$, and $\frac{\sqrt{4\pi}}{c} \mathbf{J}$ respectively.

In Section III we linearize the GCP equations introduced in Section II, and use the time independent equations to derive a bumpy theta pinch equilibrium. In the similar MHD equilibrium, the density and scalar pressure radial profiles need to be specified to solve for the magnetic field (see Appendix B). In this section it is shown that for the GCP model the density and two pressure profiles (the pressures perpendicular and parallel to the magnetic field) are needed to determine the rest of the equilibrium quantities. When the pressure anisotropy is small (in orders of $\epsilon$ and $\delta$), it is shown that the GCP equilibrium reverts back to the MHD equilibrium.

Specification of the pressure anisotropy is complicated by two local stability conditions which are needed to make the GCP model well posed. Violation of these local stability conditions are interpreted as the firehose and mirror instability because of their equivalence to threshold criteria for instabilities in an infinitely homogeneous magnetized plasma. In the context of the GCP model the firehose and mirror instability are called "exposive instabilities" since their growth rates cannot be calculated. The GCP model only predicts the onset of these instabilities by rendering the GCP equations not well posed. These conditions are further discussed in Appendix C.
In Section IV we derive ordinary differential equations from the linearized set of GCP equations, whose solution permits a determination of the spectral properties of the system. We consider only two classes of discrete modes, those for which the frequency $\tilde{\omega}$ is $0(\varepsilon^0, \delta^0)$ and $0(\varepsilon \delta)$. Transverse modes for which $\tilde{\omega}$ is $0(\varepsilon^0, \delta^0)$ are shown to be stable due to the local stability condition placed on the pressure anisotropy. Transverse modes for which $\tilde{\omega}$ is $0(\varepsilon \delta)$ are the leading order transverse unstable modes of the system. They are characterized by incompressibility of the fluid and the analogous MHD modes, play the dominant role in the stability analysis of experimental systems.\(^3\) It is noted that the same criterion for small pressure anisotropy which makes the GCP equilibrium revert back to the MHD equilibrium also makes the equations for the $\tilde{\omega} = (\varepsilon \delta)$ modes the same as the MHD modes.

In Section V, boundary conditions are developed for the set of ordinary differential equations derived in Section IV for the $\tilde{\omega} = 0(\varepsilon \delta)$ modes. These boundary conditions are for a plasma that extends to a perfectly conducting wall and are analogous to those for a similar MHD analysis. We then pick four pressure anisotropy profiles with different general characteristics to see how the MHD growth rates are modified by the GCP theory.

Section VI presents the results of these numerical calculations for the most unstable mode with one azimuthal node. Increased perpendicular pressure makes this mode more unstable, which can qualitatively be understood by considering how the equilibrium quantities are affected. Finally, there is a summary of the major results of this work.
II. THE GUIDING CENTER MODEL

The guiding center plasma model uses a one dimensional collisionless kinetic equation to describe the motion of particles that are constrained to move on magnetic field lines. The field lines themselves move according to a set of fluid equations incorporating an anisotropic pressure tensor that is consistent with the kinetic theory.

In Appendix A it is shown that for such a kinetic theory, the distribution functions \( f_s(x, v, \mu, t) \) (where \( s \) labels the particle species, \( v \) the particle velocity along the field line, and \( \mu \) the magnetic moment per unit mass) obey the one dimensional kinetic equations:

\[
\frac{\partial f_s}{\partial t} + (u + v^\parallel) \cdot \nabla f_s + \left[(u \cdot \xi) v + (\hat{\beta} \cdot \nabla)\left(\frac{1}{2} u^2 - \mu B\right) + \frac{e_s}{m_s} E_{\parallel}\right] \frac{\partial f_s}{\partial v} = 0
\]

In this equation \( u \) is the velocity of the field line perpendicular to itself (the \( E \times B \) drift), \( \hat{\beta} \) is the unit tangent to the field line, \( \xi = \hat{\beta} \cdot \nabla \hat{\beta} \) is the curvature vector of the field line, and \( \frac{e_s}{m_s} E_{\parallel} \) represents the component of the electric force per unit mass along the field line.

The normalization of \( f_s \) is such that the various physical quantities of interest are defined as:

\[
\left[p, \rho w, q, j, \rho_{\parallel}, P_{\parallel}\right] = \sum_s B \left[\int dv d\mu \left[l, v, \frac{e_s}{m_s}, \frac{\phi_s}{m_s} v, (v \cdot w)^2, \mu B\right] f_s\right]
\]

where \( p \) is the mass density, \( w \) the fluid velocity, \( q \) the charge density,
and \( j \) the current density, all parameterized by their position along a given field line. \( P_\perp \) and \( P_{\parallel} \) are the pressures perpendicular and parallel to the field lines.

The fluid equations that govern the motion of the field lines are:

\[
\left( 1 - \hat{e} \hat{e} \right) \cdot \left\{ \frac{\partial}{\partial t} + (u + w\hat{e}) \nabla \right\} \left( u + w\hat{e} + \nabla \cdot \Pi \right) = (\nabla \times \mathbf{B}) \times \mathbf{B} \tag{3}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (u \times \mathbf{B}) \tag{4}
\]

\[
\nabla \cdot \mathbf{B} = 0 \tag{5}
\]

where the pressure tensor \( \Pi \) is defined as:

\[
\Pi = \frac{1}{\beta} P_\perp + \beta \mathbf{B} \left( 1 - \sigma \right) ; \quad \sigma = 1 + \frac{P_{\parallel}}{\beta^2}
\]

For consistency and to complete the system of equations (1) through (5), one demands current and charge neutrality along the field line:

\[
q = 0 \quad \text{(6a)}
\]

\[
j = 0 \quad \text{(6b)}
\]

If one takes the velocity moments of the kinetic equation in order to obtain macroscopic equations for charge and current continuity, the constraints, Eq. (6), yield the result that the parallel electric force per unit mass, \( \frac{e}{m_s} E_{\parallel} \), is balanced by a combination of the gradients of the partial pressures of the charged particle species, i.e.
The velocity moments of the kinetic equation that describe mass and momentum conservation can be combined to give:

\[ \frac{e_S}{m_s} e_{II} = \alpha_s \hat{\beta} \left\{ \sum_t \left[ sgn(e_t) \frac{\nabla \cdot \Pi_t}{\rho_t} \right] \right\} \]

where \( \alpha_s = \frac{e_S}{m_s} \left[ \sum_t \frac{|e_t|}{m_t} \right]^{-1} \)

which can be added to Eq. (3) to form the more usual momentum conservation equation with anisotropic pressure. This explicitly shows that the component of the momentum equation that is parallel to \( \hat{\beta} \) is actually contained in the kinetic equation itself.
III. BUMPY THETA PINCH EQUILIBRIUM

We now specialize the guiding center plasma equations to describe a plasma whose charged particles are constrained to move on the field lines of a bumpy theta pinch. The equilibrium equations obtained from the previous section by setting $\frac{\partial}{\partial t}$, $u$, and $w$ equal to zero are:

$$\nabla \hat{\beta}^{(0)} \cdot \nabla f_s^{(0)} + \left[ \alpha_s \tilde{T}^{(0)} - (\hat{\beta}^{(0)} \cdot \nabla) \mu B^{(0)} \right] \frac{df_s^{(0)}}{dV} = 0 \quad (9)$$

$$\left[ \nabla - \hat{\beta}^{(0)} (\beta^{(0)} \cdot \nabla) \right] \left[ P_L^{(0)} + \frac{1}{2} (B^{(0)})^2 \right] = (B^{(0)})^2 \tau^{(0)} \chi^{(0)} \quad (10)$$

$$\nabla \cdot B^{(0)} = 0 \quad (11)$$

where the superscript $(0)$ stands for equilibrium quantities and will be dropped unless needed to avoid confusion.

From Eq. (11) and axial symmetry we can introduce a flux function $\psi$ that has the properties in cylindrical coordinates:

$$\frac{1}{r} \frac{d\psi}{dr} = B_z$$

$$-\frac{1}{r} \frac{d\psi}{dz} = B_r$$

We choose a flux function of a particular form that will yield a magnetic field having the geometry of a bumpy theta pinch. Such a flux function is:
This introduces the small expansion parameters $\varepsilon$ and $\delta$ where we assume $\varepsilon \ll \delta$. $\varepsilon$ measures the period of the bumpiness compared to the radius of the plasma and $\delta$ is related to the amplitude of the bumpiness. It is easily demonstrated (see Appendix B) that the B field lines lie on surfaces of constant $\psi$. Figure 1 depicts the bumpy theta pinch magnetic field and the surfaces of constant $\psi$.

From Eq. (12) the form for the magnetic field is then:

\[ B_r = \varepsilon \delta \, b(r) \sin \varepsilon z \]  \hspace{1cm} (13a)

\[ B_z = a(r) + \delta c(r) \cos \varepsilon z \]  \hspace{1cm} (13b)
where we have the relation:

\[(rc) = (rb)'
\]  \hspace{1cm} (14)

The prime denotes differentiation with respect to \( r \). Two more quantities of interest in Eq. (10) whose form is easily calculated from Eq. (13) are:

\[ \hat{\beta}^{(\psi)} = \left( \varepsilon \delta \frac{b}{a} \sin \varepsilon z, 0, 1 \right) + O(\varepsilon^2) \]  \hspace{1cm} (15)

\[ \kappa^{(\psi)} = \varepsilon \delta \left\{ \frac{b}{a} \left[ \cos \varepsilon z + \frac{3}{2} \left( \frac{b}{a} \right)' \right], 0, 0 \right\} + O(\varepsilon^2 \delta^2) \]  \hspace{1cm} (16)

Introducing the same form for \( P_\perp (r, z) \) as we have for the flux function, Eq. (12), namely:

\[ P_\perp (r, z) = P_\perp^{00}(r) + \delta P_\perp^{01}(r) \cos \varepsilon z \]  \hspace{1cm} (17)

we can substitute Eqs. (13), (15), (16), and (17) into Eq. (10) and group the resulting expression by orders of \( \varepsilon \) and \( \delta \). To the leading two orders in \( \delta \) we have:

\[ \frac{d}{dr} \left[ \left( P_\perp^{00} + \frac{1}{2} a^2 \right) + \delta \left( P_\perp^{01} + ac \right) \cos \varepsilon z \right] = O(\varepsilon^2) \]  \hspace{1cm} (18)

Integrating with respect to \( r \) the general form of the result is:

\[ \left[ \left( P_\perp^{00} + \frac{1}{2} a^2 \right) + \delta \left( P_\perp^{01} + ac \right) \cos \varepsilon z \right] = \text{const.} + F(\varepsilon z) + \varepsilon^2 G(r, \varepsilon z) \]  \hspace{1cm} (19)
In order to compare this to the MHD equilibrium we choose
\[ \text{const} + F(\varepsilon z) = \frac{1}{2} + \delta \cos \varepsilon z \]  
which represents the normalized value of \( \frac{1}{2} \varepsilon^2 \) (see Appendix B) at the surface of the plasma. Equation (19) then gives us two more equilibrium relations:

\[ O(\delta^0): \quad P_1^{\varepsilon\varepsilon} = \frac{1}{2} \]  
\[ O(\delta): \quad P_1^{\varepsilon\varepsilon} + \alpha c = 1 \]

Now we investigate the information contained in the equilibrium kinetic equation, Eq. (19). The first result is to note that a general functional form of the solution of the distribution function in Eq. (9) is:

\[ f_s(x, \mu, \nu) = f_s(E_s, \mu, \nu) \]  
where:

\[ E_s = \frac{1}{2} \nu^2 + \mu B + \alpha_s \phi \]

and \( \phi \) is the electric potential defined by:

\[ \alpha_s T = -\alpha_s (\hat{\beta} \cdot \nu) \phi \]

That Eq. (23) is a solution can be seen by direct substitution into Eq. (9). In the following calculation we will consider only equilibrium distribution functions that are monotone decreasing in the variable \( E_s \). This insures that the GCP equations are formally self-adjoint.  

On the application of the operator \( \sum \alpha_s B \int d\mu d\nu \frac{1}{\nu} \) to Eq. (9) we find that:

\[ \alpha_s T = 0 \]
because of the charge neutrality constraint, Eq. (6). Thus at most the electric potential $\phi$ is a constant.

The parameter $\mu$ in the functional form of the distribution function, Eq. (23), is responsible for the pressure anisotropy. This can readily be seen by considering a distribution function that has weak dependence on $\mu$, i.e.:

$$f_s = f_s(E_s, \psi) + O(\epsilon^n) \quad n \geq 1$$

Then it is easy to show by a change of integration variables in Eq. (2) that:

$$P_\perp^{(0)} = P_\parallel^{(0)} + O(\epsilon^n) = \sum_s \left\{ dE_s (E_s - \alpha_s \phi)^{3/2} f_s(E_s, \psi) + O(\epsilon^n) \right\}$$

In Eq. (24) it is instructive to note that for such weak pressure anisotropies $P_\perp^{(0)}$ is a function of $\psi$ only. Thus $P_\perp^{00}$ and $P_\perp^{01}$ in Eq. (17) are not independent functions. In fact since we have:

$$\left(\hat{\beta} \cdot \nabla\right) P(\psi) = \frac{\partial P}{\partial \psi} \hat{\beta} \cdot \nabla \psi = 0$$

because the gradient of $\psi$ is perpendicular to $\hat{\beta}$, we can write Eq. (25) in terms of equilibrium quantities to obtain:

$$\left(\hat{\beta} \cdot \nabla\right) P(\rho, \zeta) = \left[ \frac{b}{\alpha}(P_\perp^{00})' - P_\perp^{01} \right] \delta \rho \delta \zeta = 0$$

Using Eq. (21) to calculate $(P_\perp^{00})'$ one can show from Eq. (26) that:

$$P_\perp^{01} = -\alpha'b$$
Equations (14), (21), (22), (24), and (27) complete the derivation of the equilibrium quantities for weak pressure anisotropies up to \(0(\varepsilon)\). Given \(P_{\perp}^{00}(r)\), one can determine \(a(r)\), \(b(r)\), \(c(r)\), and \(P_{\perp}^{01}(r)\) in exactly the same way the MHD equilibrium is determined in Appendix B. From Eq. (24) \(P_{\parallel}(0)\) equals \(P_{\perp}(0)\) up to \(0(\varepsilon^n)\) which essentially reverts the equilibrium pressure tensor back to a scalar pressure. Thus the net effect of \(0(\varepsilon^n)\), \(n \geq 1\), pressure anisotropy is to regain an MHD equilibrium.

To obtain an equilibrium different from MHD one needs finite pressure anisotropies. A general class of distribution functions that have this property, and which we will use, are of the form:

\[
f_s = f_s(E_s + \gamma \mu, \psi)
\]

where \(\gamma\) is an arbitrary function that has to be determined. This class of distribution functions contains, for example, the two temperature Maxwellian. On substituting Eq. (28) into the equilibrium kinetic equation, Eq. (9), we conclude:

\[
\gamma = \gamma(\psi) = \gamma_0(r) + \delta \gamma_1(r) \cos \varepsilon z
\]

and it follows from \((\hat{\beta} \cdot \nabla) \psi = 0\) that:

\[
a \gamma_1 = b \gamma_0
\]

Now, \((\hat{\beta} \cdot \nabla)P_{\perp}(0) \neq 0\) since \(P_{\perp}(0)\) is no longer a function of \(\psi\) only, due to \(\gamma\) in Eq. (28). Nevertheless, \((\hat{\beta} \cdot \nabla)P_{\perp}(0)\) can still be calculated by taking the proper moment of the kinetic equation and \(P_{\perp}^{01}\) can be determined from \(P_{\perp}^{00}\) in much the same way as in Eq. (26).
Introducing the notation:

\[ P_l^{(o)} - P_\parallel^{(o)} = R(r) + \delta Q(r) \cos \varepsilon \]

the equilibrium relations can be written in the following form:

\[ P_l^{(o)} + \frac{1}{2} a^2 = \frac{1}{2} \]  (29a)

\[ \left[ \frac{(1-a^2) R}{a^2 P_{l0} - R} - a'b \right] + ac = 1 \]  (29b)

\[ rc = (rb)' \]  (29c)

\[ P_l^{(o)}(r^\parallel) = \left[ 1 + \frac{\gamma(r^\parallel)}{B(r^\parallel)} \right] P_l(r^\parallel) \]  (29d)

\[ \left[ \Delta + b \frac{\partial}{\partial r} \right] R = a Q - 2\Delta \left( \frac{\gamma_0}{\gamma_0 + a} \right) P_l^{(o)} \]  (29e)

where:

\[ \Delta = \frac{a'b - ac}{a} \]

Equations (29a), (29b), and (29c) are identical to the MHD result with the substitution of a new \( P_{l0} \). Equation (29d) comes from changing variables of integration in the moment definitions of \( P_l \) and \( P_\parallel \) as was done in Eq. (24). Equation (29e) is the information contained in \((\hat{b} \cdot \nabla) P_\parallel\) which can also be calculated from the proper moment of the kinetic equation, Eq. (9). In essence these two added equations are needed to calculate \( P_{l0} \) and \( \gamma \). The set of equations, Eqs. (29), determine the equilibrium if \( P_{l0} \) and \( P_{\parallel0} \) are given.

It turns out to be more convenient to specify \( P_{l0} \) and \( R = P_{l0} - P_{\parallel0} \) for the given information. \( R \), the zero order pressure
anisotropy has to be chosen so as not to violate the two inequalities:

\[ \alpha_0 > 0 \quad \text{and} \quad B^2 + 2P_1 + B^2 \int d \mu dv \frac{\alpha^2}{V} \frac{dP}{dV} > 0 \]

Violation of the former is interpreted as the firehose instability and the later as the mirror instability. Both are needed to make the GCP equations well posed (see Appendix C). These inequalities in terms of the equilibrium parameters become:

\[ -\alpha^2 < R < \alpha^2 P_1^{\infty} \]

Finally, we discuss the special case, \( R = 0 \). From Eq. (29d) we then deduce \( \chi_0 = 0 \). If \( \chi_0 = 0 \) then in Eq. (29e) we see that \( Q = 0 \). Equation (29b) reduces to the MHD result and we are back to the set of equations that describe the MHD equilibrium. Thus only finite pressure anisotropy (\( R \neq 0 \)) will make the GCP equilibrium different from the MHD equilibrium.
IV. STABILITY OF THE TRANSVERSE MODES

A. TRANSVERSE MODES WITH $\bar{\omega} = 0(1)$

We now proceed with a linear stability analysis of the GCP normal modes with time dependence $\exp(i\bar{\omega}t)$. To first order in the perturbation the equations can be written as

\[
(i\bar{\omega} + \nabla \cdot \nabla f_{\bar{\omega}}(u, v) \cdot \nabla f_{\bar{\omega}} - \mu(\nabla \cdot B)^2 \frac{\partial f_{\bar{\omega}}}{\partial V} + \left\{\alpha_0 T(\nabla \cdot B)^2 \right\} \frac{\partial f_{\bar{\omega}}}{\partial V} = 0
\]

\[
i\bar{\omega} \rho_0 u + \left[ \nabla \cdot (\nabla \cdot B) \right] \left( \nabla \cdot (\nabla \cdot B) \right) - \left[ \nabla \cdot (\nabla \cdot B) \right] \left( \nabla \cdot (\nabla \cdot B) \right) = \left[ \nabla \cdot (\nabla \cdot B) \right] = (32)
\]

\[
i\bar{\omega} B_{\bar{\omega}}(u) = \nabla \times (u \times B)
\]

where $u$ and quantities with superscript $(1)$ are first order quantities. The superscript $(0)$ has been dropped from all equilibrium quantities. The macroscopic equation along the field line, Eq. (8), has been omitted because the scaling of the equilibrium quantities are such that this equation will not contribute to the transverse modes.

The first class of modes we consider have frequencies $\bar{\omega} = 0(\epsilon^0 \delta^0)$. A consistent scaling to $\bar{\omega} = 0(\epsilon^0 \delta^0)$ is to consider first order quantities not to be scaled in $\epsilon \delta$. That is we let first order quantities vary as $f(r) \cdot \exp(i\bar{\omega}t + i\epsilon \theta + ikz)$ and we consider
only the $O(\varepsilon^0)$ part of Eqs. (31), (32), and (33). Due to the scaling of the zero order quantities these are:

\[(\iota + i\chi \nu)T^{(i)} + (\nu + \nu T^{(i)}) \cdot \nabla f_s^{(i)} + \left\{ \alpha_s T^{(i)} \cdot \mu \left[ \left( \beta \cdot \nabla \right) B \right]^{(i)} \right\} \frac{\partial f_s}{\partial \nu} = 0 \quad (34)\]

\[i\omega \rho u + (\nabla - ik) (T^{(i)} + B^{(i)} \cdot B) = B_s^2 \sigma \kappa^{(i)} \quad (35)\]

\[i\omega B^{(i)} = - \left[ B_s (\nabla \cdot u) + (u \cdot \nabla) B_s \right] \hat{z} + i k B_s u \quad (36)\]

Recalling the definition of $\beta$ and $\kappa$ we have from Eq. (36):

\[\beta^{(i)} = \frac{k}{\omega} u \]

\[\kappa^{(i)} = \frac{i k^2}{\omega} u \]

\[\left[ \left( \beta \cdot \nabla \right) B \right]^{(i)} = - \frac{k}{\omega} B_s (\nabla \cdot u) \]

Using these identities, Eq. (34) can be rewritten:

\[i k (\nu + \frac{\omega}{K}) f_s^{(i)} + (\nu + \frac{\omega}{K}) (\nu \cdot \nabla) f_s^{(i)} + \left\{ \alpha_s T^{(i)} \cdot \mu B_s (\nu \cdot \nabla) \right\} \frac{\partial f_s}{\partial \nu} = 0 \quad (37)\]

First we operate on Eq. (37) with the operator $\sum_s \alpha_s \int \mu d\nu \frac{1}{(\nu + \frac{\omega}{K})}$ and use the constraint of charge neutrality, Eq. (6a), to obtain:

\[T^{(i)} = - \frac{k}{\omega} B_s (\nabla \cdot u) \frac{\sum_s \alpha_s \Gamma_{01}^s (\frac{\omega}{K})}{\sum_s \alpha_s^2 \Gamma_{00}^s (\frac{\omega}{K})} \]

where:

\[\Gamma_{nm}^s \left( \frac{\omega}{K} \right) \equiv \int \mu d\nu \frac{1}{(\nu + \frac{\omega}{K})} \nu^n \mu^m \frac{\partial f_s}{\partial \nu} \]

We now have an expression for the perturbed distribution function in
terms of the equilibrium quantities, that is we can rewrite Eq. (37) to read:

\[ i \omega f_s^{(1)} = - (\mathbf{u} \cdot \nabla) f_s + B_z (\mathbf{v} \cdot \mathbf{u}) \left[ \alpha_s \sum \frac{\alpha_t \Gamma_0^{t} (\omega)}{\alpha_t \Gamma_\infty^{t} (\omega)} - \mu \right] \frac{\partial f_s}{\partial \mathbf{v}} \]  

(38)

Now we proceed to Eq. (35). We use the perturbed distribution function Eq. (38) to calculate the perturbed perpendicular pressure, i.e.

\[ P_{\perp}^{(1)} = 2 B^{(1)} B \sum \left( d \mu d \mathbf{v} \mu f_s + B_z^2 \sum \left( d \mu d \mathbf{v} \mu f_s^{(1)} \right) \right) \]

\[ = - (B^{(1)} B) + \frac{1}{i \omega} (\mathbf{v} \cdot \mathbf{u}) \left[ B_z^2 + 2 P + B_z^3 M(\omega) \right] \]

(39)

where:

\[ M(\omega) = \sum \Gamma_\infty^{t} (\omega) + \left[ \frac{\alpha_s \alpha_t \Gamma_\infty^{t} (\omega)}{\alpha_t \Gamma_\infty^{t} (\omega)} \right]^2 \]

Multiplying Eq. (35) by \((-i \omega)\) and substituting in Eq. (39) we obtain an ordinary differential equation for \( \mathbf{u} \), namely:

\[ (\rho \omega^2 - \sigma k^2 B_z) \mathbf{u} - (\nabla - i k) \left\{ (\mathbf{v} \cdot \mathbf{u}) \left[ B_z^2 + 2 P + B_z^3 M(\omega) \right] \right\} = 0 \]

This equation can easily be transformed into a differential equation for \( (\nabla \cdot \mathbf{u}) \) by dividing by \((\rho \omega^2 - \sigma k^2 B_z^2)\) and taking the divergence which gives:

\[ \nabla \left\{ \frac{(\mathbf{v} \cdot \mathbf{u}) \left[ (\mathbf{v} \cdot \mathbf{u}) \left( B_z^2 + 2 P + B_z^3 M(\omega) \right) \right]}{(\rho \omega^2 - \sigma k^2 B_z^2)} \right\} + (\nabla \cdot \mathbf{u}) = 0 \]

(40)

Since the GCP equations are formally self-adjoint, we know that instabilities occur only when \( \text{Re} \{ \omega \} = 0 \). It is then reasonable
to expect that there are no unstable solutions to Eq. (40) when:

$$\left[ B_z^2 + 2P_L + B_z^3 M(\frac{\omega}{K})\right] > 0$$  \hspace{1cm} (41)

since this would exclude oscillatory solutions in a local mode analysis of Eq. (40).

Inequality (41) is in fact the local stability condition to insure the mirror instability will not exist (see Appendix C) and we conclude that the $\bar{\omega} = 0(\epsilon^0 \delta^0)$ modes are stable.

B. TRANSVERSE MODES WITH $\bar{\omega} = 0(\epsilon \delta)$

We again do a linear stability analysis, but this time we let the perturbed quantities vary as $\exp(i\bar{\omega}t)$ with $\bar{\omega} = 0(\epsilon \delta)$. By scaling considerations of the zero order quantities in the first order equations, Eq. (31), (32), and (33); non-trivial modes exist for perturbations that are functions of $\epsilon z$. This leads to perturbations that have the form:

$$u_r = i \epsilon \delta e^{iA} \left[ u_r^0(r) + \delta u_r^1(r) \cos \epsilon z + \ldots \right]$$  \hspace{1cm} (42a)

$$u_\theta = \epsilon \delta e^{iA} \left[ u_\theta^0(r) + \delta u_\theta^1(r) \cos \epsilon z + \ldots \right]$$  \hspace{1cm} (42b)

$$u_z = \mathcal{O}(\epsilon^2 \delta^2) \hspace{1cm} \text{since} \hspace{1cm} u \cdot B = 0$$  \hspace{1cm} (42c)

$$f_5^{(m)} = e^{iA} \left[ g_5^0 + g_5' \delta \cos \epsilon z + \ldots \right]$$  \hspace{1cm} (43)

$$T^{(m)} = \epsilon \delta e^{iA} \left[ iT_o(r) + T_r(r) \sin \epsilon z \right]$$  \hspace{1cm} (44)
where: \( iA = i\epsilon \delta \omega + i m \theta + i \epsilon \delta l \), The scaling for \( \omega \) has now been made explicit. From Eq. (33) we can then obtain the information:

\[
\beta_r^{(n)} = \frac{e^{iA}}{\omega} e^{i\delta} \left[ lu_r^0 + \frac{m}{\alpha} \left( \frac{y_2}{\alpha} \right) \sin \epsilon z \right] + O(e^{\delta^2})
\]

\[
\beta_\theta^{(n)} = \frac{e^{iA}}{\omega} e^{i\delta} \left[ lu_\theta^0 + \left( \frac{y_2}{\alpha} \right) \sin \epsilon z \right] + O(e^{\delta^2})
\]

\[
\beta_z^{(n)} = O(e^3)
\]

\[
\kappa_r^{(n)} = \frac{e^{iA}}{\omega} \left\{ e^{i\delta} \left[ \frac{m}{\alpha} \left( \frac{y_2}{\alpha} \right) \cos \epsilon z \right] + e^{i\delta^2} \left[ lu_r^0 + \frac{1}{2} \left( \frac{m b}{\alpha^2} y_2 \right) \right] + O(e^{\delta^2} \cos 2\epsilon z) \right\}
\]

\[
\kappa_\theta^{(n)} = \frac{e^{iA}}{\omega} \left\{ e^{i\delta} \left[ \frac{m}{\alpha} \left( \frac{y_2}{\alpha} \right) \cos \epsilon z \right] + e^{i\delta^2} \left[ lu_\theta^0 + \frac{1}{2} \left( \frac{b + b d}{\alpha^2} \right) \left( \frac{y_2}{\alpha} \right) \right] + O(e^{\delta^2} \cos 2\epsilon z) \right\}
\]

\[
\kappa_z^{(n)} = O(e^3)
\]

\[
[(\beta \cdot \nabla) B]^{(n)} = \frac{e^{iA}}{\omega} e^{i\delta} \left\{ \left[ \frac{l}{m} (\Delta u_\theta^0 - au_\theta^0) \right] - bu_\theta^0 \right\}
\]

where:

\[
y_1 = \frac{r}{m} (\Delta u_\theta^0 - au_\theta^0) - bu_\theta^0
\]

\[
y_2 = cu_\theta^0 - (b u_\theta^0)' + au_\theta^0
\]

\[
D_{01} = \frac{r}{F} \frac{d}{dr} ru_r^0 - \frac{im}{r} u_r^0
\]

\( D_0 \) and \( D_\perp \) are the \( O(\delta^0) \) and \( O(\delta) \) velocity divergence perpendicular
to the B field and are of importance for this mode. $y_1$ and $y_2$ are defined in such a way as to be convenient later in the calculation.

The relations, Eqs. (45), (46), and (47), are now inserted into the first order kinetic equation, Eq. (31). The kinetic equation can then be grouped into a part multiplied by $(\epsilon \delta)$ and another part multiplied by $(\epsilon \delta \sin \epsilon z)$. These two parts can be used to solve for $g^0_s$ and $g^1_s$, the perturbed distribution function. The results are:

$$g^0_s = -\frac{u^0}{\omega} \frac{df_s}{dr} - D_0 \frac{a}{\omega} \sum \frac{\alpha^t_I(\omega)}{\sum \alpha^t_I(\omega)} \frac{1}{(\nu + \frac{\omega}{\kappa})} \frac{df_s}{d\nu}$$

$$g^1_s = \frac{u^0}{\omega} \left[ \frac{b}{\nu} + \Delta - \frac{b}{\alpha} \frac{1}{dr} \right] \frac{df_s}{dr} - u^1 \frac{1}{\omega} \frac{df_s}{d\nu} - \frac{aD_1}{\omega} \frac{dP_1^1}{d\nu} + u^0 \frac{dP_1^1}{d\nu} + n_\perp D_0$$

where: $n_\perp$ = some non-zero linear operator on $D_0$

In these equations the constraints of charge and current neutrality along the B field, Eq. (6), were used to calculate $T_0(r)$ and $T_1(r)$ just as in the $\omega = 0(1)$ case (see paragraph following Eq. (37)). As a result of these constraints and the form of the distribution functions, Eq. (28), $T_1(r) = 0$.

By using Eq. (50) and (51) in the definition for the perturbed perpendicular pressure, this can be shown to equal:

$$P_{\perp}^{(\perp)} = -b^{(\perp)} \cdot B + \frac{e^0}{\omega} \left[ D_0 \left[ a^2 + 2P_{\perp}^{(0)} + a^3 M(\omega / \kappa) \right] + \left[ n_\perp D_0 + D_1 \left( a^2 + 2P_{\perp}^{(0)} + a^3 M(0) \right) \delta \cos \epsilon z \right] + O(\delta^2)$$

where: $n_\perp$ = some non-zero linear operator on $D_0$
We will now proceed to show, using Eq. (32), that $D_0 = 0$ so we will not explicitly need to calculate $n_1$ and $n_2$.

Using Eq. (19) and the ordering of $\hat{\beta}^{(1)}$, Eq. (45), the third term in Eq. (32) is readily calculated:

$$[\hat{\beta}^{(1)}(\hat{\beta} \cdot \nabla) + \hat{\beta}^{(1)}(\hat{\beta}^{(1)} \cdot \nabla)](P_1 + \frac{1}{2}B^2) = \hat{\beta}^{(1)} \epsilon \delta \cos ez + O(\epsilon^3)$$

Also from the definitions of $u$, Eq. (38), and $K^{(0)}$ and $K^{(1)}$, Eqs. (16) and (46), we see that $i \epsilon \delta \rho \omega u$, $K^{(0)}$, and $K^{(1)}$ all are at least $O(\epsilon^2)$. Thus, the only remaining term in Eq. (32) also has to be $O(\epsilon^2)$, i.e.

$$[\nabla - \hat{\beta}^{(1)}(\hat{\beta} \cdot \nabla)](P_1^{(1)} + B \cdot B^{(1)}) = O(\epsilon^2)$$

We recall that the operator:

$$[\nabla - \hat{\beta}^{(1)}(\hat{\beta} \cdot \nabla)] = \hat{\nu}(1 + O(\epsilon)) \frac{\partial}{\partial r} + \hat{\nu} \frac{\partial}{\partial \theta} + \hat{\nu} \epsilon \frac{\partial}{\partial z}$$

which implies in Eq. (53) that

$$[P_1^{(1)} + B \cdot B^{(1)}] = \text{const.} + f(\epsilon z) + \epsilon^2 q(r, \theta, \epsilon z)$$

If $m \neq 0$, then we know the right hand side of Eq. (54) is multiplied by a function of $\theta$, namely exp $(i m \theta)$. For this to be so we must have:

$$\text{const.} + f(\epsilon z) = 0 \quad \text{if } m \neq 0$$

The mode with $m = 0$ is a special case and is not treated here. If we substitute Eq. (52) into Eq. (54) and use Eq. (55) we have the result:
\[ O(\delta^0) : \quad D_0 \left[ a^2 + 2P_1^{00} + a^3 M \left( \frac{\omega}{q} \right) \right] = 0 \] (56)

\[ O(\delta^1) : \quad D_1 \left[ a^2 + 2P_1^{00} + a^3 M \left( 0 \right) \right] = -i\eta_i D_0 \] (57)

We have seen for the \( \tilde{\omega} = 0(1) \) calculation that \( a^2 + 2P_1^{00} + a^3 M \left( \frac{\omega}{q} \right) > 0 \) which is the local stability condition for nonexistence of the mirror instability (see Eq. (41) and Appendix C). This implies from Eq. (56) that:

\[ D_0 = 0 \]

In Eq. (57) we also have:

\[ a^2 + 2P_1^{00} + a^3 M \left( 0 \right) > 0 \]

so we are led to a similar conclusion that:

\[ D_1 = 0 \]

Thus to \( O(\delta) \) we have the result that the plasma is incompressible (i.e. \( \nabla \cdot u = 0 \)) and we can introduce a flux function \( \chi \) for the velocity \( u \). We define \( \chi \) in such a way that:

\[ \chi = e \delta e^{iA} \left[ \chi_o(r) - \delta \chi_i(r) \cos \epsilon \right] \]

\[ \frac{1}{r} \frac{\partial \chi}{\partial \theta} = u_r \] (58a)

\[ \frac{\partial \chi}{\partial r} = u_\theta \] (58b)

This incompressibility condition also simplifies the perturbed distribution function, Eq. (50) and (51), and now an easier calculation...
of the perturbed pressure anisotropy yields:

\[ p^{(1)}_x - p^{(1)}_y = \frac{\varepsilon_i}{\omega} \left\{ -u_r^0 R'_r - [u_r^0 Q'_r + u_r^1 R'_r] \delta \cos \varepsilon z \right\} + O(\delta^3) \quad (59) \]

We now have all the terms needed to derive a system of differential equations for \( \chi \) that in principle will determine values of \( \omega^2 \).

Taking the curl of the first order momentum equation, Eq. (32), in order to eliminate the gradient term, we have to \( O(\varepsilon^2 \delta^2) \):

\[ \varepsilon \delta \omega (\nabla \times \mathbf{p}_x) = \nabla \times \left( \left[ (B^2 + P_\parallel - P_\perp) \mathbf{k} \right]^{(1)} + \nabla \times (\hat{\beta}^{(1)} \delta \sin \varepsilon z) + O(\varepsilon^2 \delta^3) \right) \quad (60) \]

Before we expand this equation, we remark if \((P_\perp - P_\parallel)^{(0,1)} = 0\) in Eq. (60), this equation is exactly the equation one would obtain using the ideal MHD model for this mode.\(^6\) If the term \([(P_\perp - P_\parallel) \mathbf{k}]^{(0,1)}\) contributes to Eq. (60) then the MHD results will be modified. For this term to contribute it has to be at least \( O(\varepsilon^2 \delta^2) \). Since both \( \mathbf{k}^{(0)} \) and \( \mathbf{k}^{(1)} \) are at least \( O(\varepsilon^2 \delta) \) this means that we would need \((P_\perp - P_\parallel)^{(0,1)}\) to be either \( O(1) \) or \( O(\delta) \). By the arguments given at the end of Sec. III (i.e., if \( R = 0 \) then \( Q = 0 \)) and the form of Eq. (59), if \((P_\perp - P_\parallel)^{(0,1)}\) has an \( O(\delta) \) component then it has to have an \( O(1) \) component. Hence, we are led to the conclusion that for the long wavelength bumpy theta pinch only pressure anisotropies of \( O(1) \) will yield results different from ideal magnetohydrodynamics.

Using the flux function \( \chi \) to replace \( \mathbf{u} \) in Eq. (60) and expanding the various terms in the \( z \) component of this equation leads to the result:

...
\( O(\varepsilon^2 \delta) \): 
\[
(\sigma_0 y_2)' = -\sigma_0 (\frac{1}{\alpha} + \frac{a'}{\alpha}^2) y_2 + \frac{2m^2}{\rho_0} b [\frac{\sigma_0 a'}{\alpha} - \frac{\sigma_0}{\varepsilon^2}] \chi_0 + \sigma_0 m^2 \frac{\xi_0}{m^2} y_1. 
\](61a)

\( O(\varepsilon^2 \delta^2) \): 
\[
-\frac{3}{r} \left\{ r [\rho_0 \omega^2 - \sigma_0 a^2 \ell^2] \chi_o' \right\} + \frac{m^2}{\rho_0} b [\sigma_0 a^2 \ell^2] \chi_0 = \sigma_0 \frac{1}{\alpha} \left[ \frac{\alpha^2}{\alpha} - \Delta' \right] y_2 
+ \frac{m^2}{\rho_0} b \left[ \frac{\sigma_0 a'}{\alpha} - \frac{\sigma_0}{\varepsilon^2} \right] y_1 
+ \frac{m^2}{\rho_0} b \left[ \frac{\Delta - b}{(\alpha - b)} \frac{\sigma_0 a'}{\alpha} (\frac{\sigma_0 a'}{\alpha} - \frac{\sigma_0}{\varepsilon^2}) + \frac{\sigma_0}{\varepsilon^2} (\frac{\sigma_0}{\varepsilon^2} - \frac{\sigma_0 a'}{\alpha}) \right] \chi_0.
\](61b)

where \( \sigma_0 = (1 - R/a^2) \). If we write Eq. (48) in terms of \( \chi \) we get:
\[
y_1 = \Delta \chi_0 + a \chi_1 + b \chi_1',
\]
\[
y_2 = -c \chi_0 + (b \chi_1')' + a \chi_1.
\]

Then a simple manipulation readily yields the relation:
\[
y_1' = y_2 + \frac{a'}{\alpha} y_1 + (\frac{a'}{\alpha} \Delta - \Delta') \chi_0. 
\](61c)

As in the ideal MHD case we infer a continuous spectrum by inspection of the coefficient of the highest derivative of Eq. (61b), namely those values of \( \omega \) such that
\[
\omega^2 = \frac{\sigma_0 a^2 \ell^2}{\rho_0}
\]

Since \( \sigma_0 > 0 \) due to Eq. (30) this generalized Alfvén continuum always stays on the stable side of the spectrum. If \( \sigma_0 = 1 \) (i.e. \( R = 0 \)) then the system of equations, Eq. (61), reduces to the MHD result as was deduced using Eq. (60).
V. CALCULATION OF GROWTH RATES

In this section we numerically solve the system of equations, Eq. (61), for unstable point eigenvalues ($\omega^2$). To do this we develop, as was done for the MHD case, boundary conditions for a plasma that extends to a perfectly conducting rigid wall. This is a simpler problem than the one having a vacuum between the plasma and the wall, and in the MHD case the growth rates are not substantially affected by the presence of the vacuum region. Since Eq. (61) resembles the MHD equations and we wish to investigate how the MHD growth rates are modified, the plasma extending to the wall will be the only case considered.

A perfectly conducting wall implies a flux surface, so at the wall ($r = r_w$) we pick up the boundary condition:

$$\left[ u \cdot \nabla \psi \right]_{r=r_w} = 0 \quad (62)$$

That is $u$ is both perpendicular to the wall and the $B$ field at the wall.

Expanding Eq. (62) out in terms of $\chi$, $a(r)$, $b(r)$, and $c(r)$ we have

$$i m e^A \left[ \chi_o(r_w) - \chi_l(r_w) \delta \cos \varepsilon z \right] \left[ a(r_w) + c(r_w) \delta \cos \varepsilon z \right] = 0$$

The equation for a flux surface is easily obtained (see Appendix B) and $r_w$ turns out to be:

$$r_w(r, z) = r_o - \frac{b(r_o)}{a(r_o)} \delta \cos \varepsilon z$$
where \( r_0 \) is the mean radius of the wall. Expanding \( X(r_w) \) and \( a(r_w) \) in a Taylor's series about \( r_0 \) we obtain to \( O(\delta) \):

\[
\left\{ aX_o + \left[ \left(-\frac{gb}{a} + c \right)X_o - a\left(\frac{bX'_o}{a} + X_1 \right) \right] \cos \theta \right\}_{r = r_0} = 0
\]

We identify the second term as \(-y_1\) (see equation before Eq. (61c)) so the boundary condition, Eq. (62), becomes:

\[
X_o (r_0) = 0 \quad (63)
\]

\[
y_1 (r_0) = 0 \quad (64)
\]

The other two boundary conditions are regularity at \( r = 0 \). To obtain these we consider asymptotic solutions of \( X_0, y_1, \) and \( y_2 \) in Eq. (61) as \( r \) goes to zero. Expanding the equilibrium quantities around the origin we have:

\[
a(r) = a_o + O + a_2 \frac{r^2}{2} + \ldots \quad (65a)
\]

\[
b(r) = O + b_1 r + b_2 \frac{r^2}{2} + \ldots \quad (65b)
\]

\[
\Delta(r) = \Delta_0 + \Delta_1 r + \Delta_2 \frac{r^2}{2} + \ldots \quad (65c)
\]

\[
\sigma(r) = \sigma_{oo} + \sigma_{o1} r + \sigma_{o2} \frac{r^2}{2} + \ldots \quad (65d)
\]

where the subscript indicates the order of the derivative that is evaluated at \( r = 0 \), i.e.

\[
a_2 = \left. \frac{\partial^2 a(r)}{\partial r^2} \right|_{r=0} \quad \text{etc.}
\]
Notice that \( a_1 \) and \( b_0 \) are both zero. The former is zero because \( (P_{1}^{00})' \) will be zero at the origin. The later is zero because \( \nabla \cdot \mathbf{B} = 0 \), hence no magnetic charge.

Asymptotically \( \chi_0 \), \( y_1 \) and \( y_2 \) will have solutions that go like a power of \( r \), that is they will have functional form:

\[
\begin{align*}
\chi_0 &= C_1 r^q \\
y_1 &= C_2 r^s \\
y_2 &= C_3 r^t
\end{align*}
\] (66a) (66b) (66c)

Substituting these and Eq. (65) into Eq. (61) and keeping the lowest order powers of \( r \) one can determine \( (q, s, t) \) and \( (C_1, C_2, C_3) \) up to an arbitrary constant. The two regular solutions obtained in this manner are:

\[
\begin{align*}
\chi_0 &= r^m \\
y_1 &= \frac{m^2 \sigma_{oo} b_1 + \sigma_{oo} (m+1) \Delta_1}{\sigma_{oo} (2m+1)} \ r^{m+1} \\
y_2 &= \frac{m^2 [\Delta_1 \sigma_{oo} + \sigma_{oo} b_1 (m+1)]}{\sigma_{oo} (2m+1)} \ r^{m-1}
\end{align*}
\] (67a) (67b) (67c)

and:

\[
\chi_0 = \frac{1}{2} \frac{m^2 \sigma_{oo} b_1 + m \sigma_{oo} \Delta_1}{(p_0 \omega^2 + \sigma_{oo} a^2 k^2) (2m+1)} \ r^{m+1}
\] (67d)
\[ y_1 = r^m \]  
\[ y_2 = mr^{m-1} \]

These are the asymptotic solutions to Eq. (61) if \( \frac{dR}{dr} \bigg|_{r=0} \neq 0 \) (recall that \( R = P_{ij}^{00} - P_{ii}^{00} \), the zero order pressure anisotropy). If, on the other hand, \( \frac{dR}{dr} \bigg|_{r=0} = 0 \) then by Eq. (29b) and the definition of \( \sigma_0 \) we have:

\[ \Delta_1 = 0 \quad \text{and} \quad \sigma_{01} = 0 \quad \text{if} \quad \frac{dR}{dr} \bigg|_{r=0} = 0 \]

This modifies Eqs. (65b) and (65c) and leads to a new set of asymptotic solutions which are:

\[ \chi_o = r^m \]  
\[ y_1 = \frac{2m^2 L - (m+2)\sigma_{oo} \mathcal{J}}{-4\sigma_{oo} (m+1)} r^{m+2} \]  
\[ y_2 = \frac{2m^2 (2+m) L - m^2 \sigma_{oo} \mathcal{J}}{-4\sigma_{oo} (m+1)} r^{m+1} \]

and

\[ \chi_o = \frac{m^2 L + \frac{1}{2} m \sigma_{oo} \mathcal{J}}{-4(m+1)(\rho \omega^2 - \sigma_{oo} \alpha^2 \beta^3)} r^{m+2} \]  
\[ y_1 = r^m \]  
\[ y_2 = mr^{m-1} \]
where: \[ J = \frac{a_2 \Delta \phi}{a_0} - \Delta_2 \quad \text{and} \quad L = b \left( \frac{a_2 \phi}{a_0} + \frac{a_2^2}{2} \right) \]

Since \( \frac{dR}{dr} \) is specified information, either the first set, Eq. (67), or second set, Eq. (68), of asymptotic solutions has to be chosen appropriately for the given problem.

Next, we choose functions for the equilibrium pressures and the density profile \((P_{100}, P_{n00}, \rho_0)\) or equivalently for \((P_{100}, R, \rho_0)\) to define all the other equilibrium quantities.

We make our choice to be comparable to published results for the MHD case. Thus we choose:

\[ \rho_0 = \beta e^{-x^2} \]  
\[ P_{100} = \frac{1}{2} \beta e^{-x^2} \]

where \( x^2 = r^2/r^2 \) and \( r^2 \) is a length characteristic of the plasma radius.

The quantity \( \beta \) in Eq. (70) is the normalized pressure defined as:

\[ \beta = \frac{P_{100}(0)}{P_{100} + \frac{1}{2} \alpha^2} \]

\( \beta \) measures the penetration of the magnetic field into the plasma and is one of the chief parameters characterizing a plasma in the magnetized state.

Also to compare to the MHD case, we let the value of \( x \) at the wall be \( x_w = r_0/R = 4.2 \). Using Eqs. (21) and (70) the limiting values of \( R \), Eq. (30), become:

\[ \frac{1}{2} \beta e^{-x^2}(1 - \beta e^{-x^2}) > R > (\beta e^{-x^2} - 1) \]
which excludes R from the shaded regions in Fig. 2. Also in Fig. 2 are the four R-profiles that were used for computation. These are:

A. \[ R = \frac{K}{2}(1-\beta e^{-x^2})\beta e^{x^2} \] (71a)

B. \[ R = K(\beta e^{-x^2} - 1) \] (71b)

c. \[ R = K \frac{(\beta - 1)}{4.2} x \] (71c)

d. \[ R = \frac{K}{2} e^{-(x-2.1)^2} \] (71d)

The reasons for these choices are explained in the next section. K is a multiplicative constant to turn R on from zero. It should be kept in mind that when \( R = 0 \) the numerical results should duplicate the previous MHD results.

This completes the specification of all the information needed to numerically solve the set of ordinary differential equations, Eq. (61). Equations (69), (70), and (71) are used to determine the equilibrium quantities as shown in Sec. III. These are used to calculate the coefficients in Eq. (61). Equation (67) is used to initialize the numerical solution depending on the value of \( \frac{dR}{dr} \) at the origin. The value of \( \omega^2 \) is then adjusted to satisfy the boundary conditions at the wall, Eq. (63) and (64). This determines \( \omega^2 \) for one particular mode.

The numerical solution is normalized the following way. Equation (15) has already normalized the pressures and magnetic field
Fig. 2: Pressure anisotropy profiles (R-profiles) used in the calculation of the growth rates for the $\omega = 0(\epsilon \delta)$ transverse modes. The labels A, B, C, and D correspond to Eqs. (71a), (71b), (71c), and (71d) respectively.
to \( (P + \frac{1}{2} B^2) \) or equivalently to the value of \( \frac{1}{2} B^2 \) on the surface of the plasma. We normalize the length scale by setting \( \tilde{r} = 1 \) in Eq. (69) and (70). The density is normalized by setting \( \tilde{\rho} = 1 \) in Eq. (69). These three normalizations imply that the time scale is measured in units of the Alfvén transit time defined as

\[
t_A = \left[ \frac{2}{\tilde{r}_*^2} \cdot \frac{P_{\text{ref}} + \frac{1}{2} \alpha^2}{\tilde{\rho}} \right]^{-\frac{1}{2}}.
\]

The numerical value of \( \omega^2 \) converted back to real time would then be:

\[
\omega^2 [\text{in sec}^{-2}] = \frac{\omega^2}{\epsilon \Delta \omega^2 t_A^2}
\]

To calculate \( \epsilon \) and \( \Delta \) from experimental dimensions see Appendix B.
VI. RESULTS AND CONCLUSIONS

Figures 3, 4, 5, and 6 show the growth rate squared as a function of $\beta$ for R-profiles, Eqs. (71a), (71b), (71c), and (71d) respectively for the most unstable $\ell = 0, m = 1$ mode. In every case, it turned out, this was the mode for which $\chi_0$ had no nodes. The figures show the effect of increasing $K$ from zero.

The first R-profile, Eq. (71a), is an example of increasing the perpendicular pressure compared to the parallel pressure. As seen in Fig. 3, this has the effect of rapidly increasing the growth rate. This mode seems sensitive to the mirror instability.

The second R-profile, Eq. (71b), represents a plasma that has an enhanced parallel pressure. This profile tends to stabilize the plasma as seen in Fig. 4.

The third R-profile, Eq. (71c), was considered because it has non-zero slope at the origin and different asymptotics are needed as indicated in the last section. Even though different asymptotics were used, as $K \to 0$ the growth rate went over to the MHD result (see Fig. 5). This profile also reduced the growth rate of the plasma, supporting the conjecture from Fig. 4 that increased parallel pressure has a stabilizing effect.

The final R-profile, Eq. (71d) indicates a plasma that has an outer ring of enhanced parallel pressure. As Fig. 6 shows, this makes the plasma more unstable for low $\beta$ and less unstable for high $\beta$. 

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Fig. 3: Growth rate squared \((-\omega^2)\) versus for a plasma with enhanced perpendicular pressure, Eq. (71a). K is a multiplicative constant that adjusts the degree of pressure anisotropy.

Fig. 4: Growth rate squared \((-\omega^2)\) versus for a plasma with enhanced parallel pressure, Eq. (71b). K is a multiplicative constant that adjusts the degree of pressure anisotropy.
Fig. 5: Growth rate squared ($-\omega^2$) versus for a plasma with enhanced parallel pressure where $R'(0) \neq 0$, Eq. (71c). $K$ is a multiplicative constant that adjusts the degree of pressure anisotropy.

Fig. 6: Growth rate squared ($-\omega^2$) versus for a plasma with a ring of enhanced parallel pressure, Eq. (71d). $K$ is a multiplicative constant that adjusts the degree of pressure anisotropy.
As an indication of why the first R-profile is more unstable than the second we can refer to Fig. 7 which plots the equilibrium quantities c(r) and b(r) for these two profiles, for the parameter values \( \beta = 0.5 \) and \( K = 0.8 \). The radial magnetic field, b(r), is reduced for the second R-profile, Eq. (71b), indicating a reduction in the equilibrium curvature (see Eq. (16)). On the other hand, the first R-profile increases b(r) thus increases the curvature. In general, the greater the outward curvature the less stable the plasma.

The \( \partial (\delta B) \) field, c(r), is also shown to be larger at the origin for the first R-profile, and reduced for the second. An increase in c(r) leads to an increase in the magnetic mirror force (i.e. \(-\mu(\hat{\beta} \cdot \nabla)B\)) along the field lines inside the plasma. In fact from the equilibrium relations, Eq. (29b), this force can be written:

\[
-\mu(\hat{\beta} \cdot \nabla)B = -\mu \Delta e\delta \sin \varepsilon = -\frac{\mu aP_{ii}^{oo}}{\alpha^2 R^{oo} - R} e\delta \sin \varepsilon \quad (72)
\]

As a pressure anisotropy profile like the first R-profile tends toward the mirror instability region, this force will tend to infinity due to the denominator in Eq. (72). This supports the conjecture that the first R-profile is sensitive to the mirror instability.

In general then, this work has studied the effects of finite pressure anisotropy in a GCP stability analysis of a bumpy theta pinch. The results of this work were then compared to similar MHD calculations. The main conclusion is that in order to modify the MHD growth rates of the long wavelength bumpy theta pinch plasma, pressure anisotropies of
Fig. 7: Effects of pressure anisotropy on the equilibrium magnetic field. The A profile corresponds to enhanced perpendicular pressure, Eq. (71a). The B profile corresponds to enhanced parallel pressure, Eq. (71b). Both are for the case $\beta = 0.5$ and $K = 0.8$. 
Fig. 7: Effects of pressure anisotropy on the equilibrium magnetic field. The A profile corresponds to enhanced perpendicular pressure, Eq. (71a). The B profile corresponds to enhanced parallel pressure, Eq. (71b). Both are for the case $\beta = 0.5$ and $K = 0.8$. 

A profile
B profile
(R=0) profile
finite order are needed. This supports Weitzner's result that reported no difference from MHD growth rates for a helical plasma with small pressure anisotropy.\textsuperscript{12}

When the pressure anisotropy is of finite order, it has to be specified in the equilibrium along with the perpendicular pressure and density (only the scalar pressure and density have to be specified in the MHD case). For the Gaussian profiles chosen in Sec. V, the numerical solutions show that if the perpendicular pressure increases over the parallel pressure, the curvature of the bumpiness increases and the mirror force \((- \mu (\hat{\beta} \cdot \nabla) B)\) inside the plasma increases. Both these effects tend to destabilize the plasma.
REFERENCES


APPENDIX A

The GCP Kinetic Equation

I. EQUATION OF MOTION AND INVARIANCE OF THE MAGNETIC MOMENT

We start the derivation of the kinetic equation by considering the motion of a charged particle in an electric and magnetic field. The equation of motion for such a charged particle is:

\[ \frac{dx}{dt} = U \]  \hspace{1cm} (A1)

\[ \frac{dU}{dt} = \frac{e}{m}E + \frac{e}{m}(U \times B) \]  \hspace{1cm} (A2)

where \( x \) is the position of the particle, \( U \) the velocity of the particle, \( e \) and \( m \) the charge and mass of the particle, and \( E \) and \( B \) the electric and magnetic fields.

It is well known that the basic motion of such a particle is a helical or gyro motion about the magnetic field lines. If the radius of this gyro motion and the period for one revolution are small compared to the length and time scale of the fields, one can divide the particle trajectory into a guiding center motion and a circular motion about the guiding center. Mathematically it can be shown that this division of the trajectory represents the leading terms of an asymptotic expansion of the exact solution to Eq. (A1) and (A2) as the formal expansion parameter \( \frac{1}{N} = \frac{m}{eB} \) tends to zero. The validity of this solution depends on certain
continuity and boundedness properties of the quantities $E$, $B$, $B$, and $\Omega(\dot{\beta} \cdot E)$.

There are many excellent derivations of the guiding center solution to Eq. (A1) and (A2) in the literature. We will present here only those points that help in the understanding of the physical meaning of the approximations.

In the guiding center approximation we are not interested in the actual gyro motion of the particle, but only consider its averaged contribution to the guiding center motion. To do this averaging, we first break $U$ in Eq. (A2) into parts parallel and perpendicular to the magnetic field as follows:

$$\frac{d}{dt} [u_x + \nu\beta] = \frac{e}{m} E + \frac{e}{m} \left\{ u_x \times \left[ B_0 + (r(t) \cdot \nabla_0) B + \ldots \right] \right\} \quad \text{(A3)}$$

Here we have anticipated the basic gyro motion of small radius around a $B$ line and have expanded the $B$ field in a Taylor's series using a coordinate system whose origin moves with the guiding center motion as shown in Fig. 8:

![Fig. 8: Gyro motion of a charged particle in a magnetic field.](image-url)
\( B_0 \) is a constant field which is equal to the field at the guiding center.

\( (r \cdot \nabla_0)B \) is the gradient of the B field at the guiding center dotted into the particle position vector in the local guiding center coordinate system.

We define \( u_{\text{gyro}} \) as that circular motion that satisfies the equation:

\[
\frac{d}{dt} u_{\text{gyro}} = \frac{e}{m} (u_{\text{gyro}} \times B_0)
\]  

(A4)

Breaking \( u_\perp \) in Eq. (A3) into:

\[
u_\perp = u_{\text{gyro}} + u_{\text{drift}}
\]  

(A5)

we then have the equation:

\[
\frac{d}{dt} (v_\perp^2 + u_{\text{drift}}) = \frac{e}{m} E + \frac{e}{m} [u_{\text{drift}} \times B_0] + \frac{e}{m} [(u_{\text{gyro}} + u_{\text{drift}}) \times (r \cdot \nabla_0)B]
\]  

(A6)

The produce of \( u_{\text{gyro}} \times (r \cdot \nabla_0)B \) in the last term in Eq. (A6) contains a constant term in the direction of \( B \), and a constant term plus a harmonic term perpendicular to \( B \). It is these constant terms that feed back into Eq. (A6) as a new force due to gradients in the B field. The effect of the harmonic term tends to cancel itself out after one gyro orbit. To calculate this new force then, we average over one gyro period. The result is surprisingly simple:

\[
\left\langle \frac{e}{m} u_{\text{gyro}} \times (r \cdot \nabla_0)B \right\rangle \approx \frac{\Omega}{2\pi} \int_0^{2\pi} \left\langle \frac{e}{m} \left[ u_{\text{gyro}} \times (r \cdot \nabla_0)B \right] \right\rangle dt = -\mu \nabla B + O(\frac{1}{n})
\]

where \( \mu = \frac{1}{2} \left\langle \frac{1}{2} |u_{\text{gyro}}|^2 \right\rangle \) is the magnetic moment per unit mass of the particle. To the same order we can average Eq. (A6) to obtain:
\[
\frac{d}{dt}(v\hat{\beta} + u_{\text{drift}}) = \frac{e}{m}E - \mu \nabla B + \frac{e}{m}(u_{\text{drift}} \times B)
\]  

(A7)

where now:

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + (v\hat{\beta} + u_{\text{drift}}) \cdot \nabla
\]

and the part of \(u_{\text{drift}}\) left in Eq. (A7) is the guiding center motion perpendicular to the field line. It is a property of this equation that \(u_{\text{drift}}\) is simply related to the terms in Eq. (A7) that are perpendicular to the B field. That is we let:

\[
u_{\text{drift}} = \frac{E \times B}{B^2} + \frac{1}{\eta} \left[-\mu \nabla B + \frac{d}{dt}(v\hat{\beta} + u_{\text{drift}})\right] \times \hat{\beta}
\]  

(A8)

Using this expression for \(u_{\text{drift}}\) in Eq. (A7) we obtain the result:

\[
\hat{\beta} \cdot \left[ \frac{d}{dt}(v\hat{\beta} + u_{\text{drift}}) \right] = \hat{\beta} \left[ \frac{e}{m}E - \mu \nabla B \right]
\]  

(A9)

This shows that the solution to \(u_{\text{drift}}\), Eq. (A8), satisfies the part of Eq. (A7) that is perpendicular to the B field.

Rearranging Eq. (A9) in the form of an equation of motion, we have

\[
\frac{dv}{dt} = -u_{\text{drift}} \cdot \frac{d\hat{\beta}}{dt} + \frac{e}{m}(\hat{\beta} \cdot E) - \mu(\hat{\beta} \cdot \nabla)B
\]  

(A10)

Equations (A8) and (A10) give the motion of the guiding center and contain only the averaged effects of the gyro motion through the magnetic moment \(\mu\).

It is also well known\(^{11}\) that the magnetic moment is an adiabatic constant of the motion. It turns out that to the approximations
used in deriving Eqs. (A8) and (A10) \( \mu \) is an exact constant of the
motion. This can be shown by considering the averaged time rate of
change of the gyro motion energy \( \frac{1}{2} \langle |u_{\text{gyro}}|^2 \rangle \).

From Eq. (A2) we have:

\[
\frac{1}{2} \frac{d}{dt} |U|^2 = \frac{e}{m} \bar{V} \cdot \vec{E}
\]

The average of this equation over a gyro period can be shown to be:

\[
\frac{d}{dt} \frac{1}{2} \langle |u_{\text{gyro}}|^2 \rangle + \frac{1}{2} \frac{d}{dt} |U_{\text{ac}}|^2 = \frac{e}{m} \langle u_{\text{gyro}} \cdot \bar{E} \rangle + \frac{e}{m} (U_{\text{ac}} \cdot \bar{E}) \tag{A11}
\]

where:

\( U_{\text{ac}} = U_{\text{drift}} + \sqrt{\beta} \)

The quantity \( \langle u_{\text{gyro}} \cdot \bar{E} \rangle \) essentially measures the change of the \( \bar{E} \) field
along the circular path of the gyro motion. Using Faraday's law of in-
duction this is equal to the time rate of change of \( \bar{B} \) through this loop.

Specifically we have:

\[
\frac{e}{m} \langle u_{\text{gyro}} \cdot \bar{E} \rangle = \frac{\Omega}{2\pi} \left( \frac{e}{m} \bar{E} \cdot (u_{\text{gyro}} dt) \right) = \frac{e \Omega}{2\pi m} \left( \frac{dB}{dt} \cdot dA \right) = \mu \frac{dB}{dt} \tag{A12}
\]

The remaining two terms in (A12) can also be manipulated in
an equally simple form using Eqs. (A8) and (A10). After a good deal of
work it can be shown:

\[
-\frac{1}{2} \frac{d}{dt} |U_{\text{ac}}|^2 + \frac{e}{m} U_{\text{ac}} \cdot \bar{E} = \mu \left[ U_{\text{ac}} \cdot \nabla \bar{B} \right] \tag{A13}
\]

Combining Eqs. (A12) and (A13) into Eq. (A11) we have:

\[
\frac{d}{dt} \frac{1}{2} \langle |u_{\text{gyro}}|^2 \rangle = \mu \left[ \frac{\partial}{\partial t} \bar{B} + \bar{V} \cdot (\nabla \times \bar{B}) \right] = \mu \frac{dB}{dt} \tag{A14}
\]
But we also have from the definition of $\mu$:

$$\frac{d}{dt} \frac{1}{2} \langle |\mathbf{u}_y|^2 \rangle = \frac{d}{dt} (\mu B) = B \frac{d\mu}{dt} + \mu \frac{dB}{dt} \quad (A15)$$

Hence we conclude that the time rate of change of $\mu$ is zero and therefore $\mu$ is a constant of the motion.

This interpretation of $\mu$ immediately puts a lower bound on the quantity $B$ as a condition for the validity of the guiding center approximation, since if $B$ went to zero the magnetic moment would go to infinity.

The other troublesome quantity in this approximation scheme is the second term on the right hand side of Eq. (A10):

$$\frac{e}{m} (\hat{\beta} \cdot \mathbf{E})$$

This term represents the parallel acceleration of the particle along the $B$ field by the electric field. The quantity $(e/m)$ goes like the inverse of the formal small expansion parameter if we define $B$ as an $O(1)$ quantity. If the $E$ field parallel to $B$ is finite this term will invalidate the expansion. This rather annoying problem is treated in the next section.

II. KINETIC AND FLUID EQUATIONS

Once the equation of motion is known there is a standard procedure to write down a (collisionless) kinetic equation that describes how a collection of a large number of such particles will behave. Here we are interested in using Eq. (A10) as the equation of motion of a fictitious particle that moves with a velocity $v$ along a $B$ line, and
that possesses an internal energy through the quantity \( \mu \). The \( B \) line itself moves with the lowest order drift velocity perpendicular to itself, namely:

\[
\mathbf{u} = \frac{\mathbf{E} \times \mathbf{B}}{\mathbf{B}^2}
\]

Thus the kinetic equation we will construct is a description of a one dimensional gas constrained to move on magnetic lines of force whose motion is given.

 Appropriately, we construct a phase space of quantities \( \{x, v, \mu\} \) where \( \mu \) takes the place of the perpendicular velocity of the particle that is related to its kinetic energy. In this phase space we construct a probability density function \( F_s(x, v, \mu, t) \) for each charge species \( s \) (labels the species) that satisfies the collisionless Vlasov type kinetic equation:

\[
\frac{dF_s}{dt} + \sum_{i=1}^{3} \left( \frac{\partial}{\partial x_i} \frac{dU}{dt} + \frac{\partial}{\partial v_i} \frac{dU}{dt} \right) F_s = 0 \quad (A16)
\]

where \( x_i \) and \( v_i \) stand for the position and velocity of the particle predicted by the equation of motion. Using the phase space of these particles, the fact that \( \mu \) is a constant of the motion, and Eq. (A10), Eq. (A16) takes the form:

\[
\frac{dF_s}{dt} + \frac{\partial}{\partial x} (u \cdot \mathbf{v}) F_s + \frac{\partial}{\partial v} \left[ -u \cdot \frac{\partial}{\partial t} \frac{\mathbf{B}}{m_s} - \frac{e}{m_s} \left( \mathbf{\beta} \cdot \mathbf{E} \right) - \mu \left( \mathbf{\beta} \cdot \mathbf{E} \right) \right] F_s = 0 \quad (A17)
\]

It is important to remember that \( u \) and \( B \) are to be treated as "known" quantities in Eq. (A17). This necessitates the need for another set of equations to determine these quantities. We tentatively introduce a set that is closely related to the more familiar MHD equations:
\[
\rho \left[ \frac{d}{dt} + (u + w \hat{\beta}) \cdot \nabla \right] (u + w \hat{\beta}) + \nabla \cdot \Pi = \mathbf{J} \times \mathbf{B} \quad (A18)
\]

\[
\frac{d \mathbf{B}}{dt} = -(\nabla \times \mathbf{E}) \quad (A19)
\]

\[
\nabla \cdot \mathbf{B} = 0 \quad (A20)
\]

\[
\mathbf{E}_\perp + \mathbf{u} \times \mathbf{B} = 0 \quad (A21)
\]

\[
\mathbf{J} = \nabla \times \mathbf{B} \quad (A22)
\]

The only difference between these equations and the corresponding MHD subset is the introduction of a tensor pressure. There can only be a difference in the pressure parallel and perpendicular to the magnetic field so the pressure tensor has the general form:

\[
\Pi = P_\perp \mathbb{1} - \frac{\mathbb{B} \cdot (P_{\parallel} - P_{\perp})}{\mathbb{B} \cdot \mathbb{B}}
\]

The quantities \( \rho \), \( v \), \( P_\perp \), and \( P_{\parallel} \) can be obtained from the distribution function in the obvious way:

\[
[\rho, \rho v, P_\parallel, P_\perp] = \sum \int d\mu d\nu \left[ 1, v, (v-w)^2, \mu B \right] f_\Sigma
\]

This equation also defines the normalization we use for \( f_\Sigma \).

These moment definition encourage us to take the velocity moments of the kinetic equation. The first two moments readily yield the result:

\[
\hat{\beta} \cdot \left\{ \rho \left[ \frac{d}{dt} + (u + w \hat{\beta}) \cdot \nabla \right] (u + w \hat{\beta}) + \nabla \cdot \Pi \right\} = 0
\]
which is just the parallel part of Eq. (A18) and can be subtracted from this equation since the kinetic equation already contains this information.

Two other moments of physical interest are the charge density and parallel current:

\[
[q_x, j_z] = \sum_s \frac{e_s}{m_s} \int d\mu d\nu \left[ v, v \right] F_s.
\]

These quantities are multiplied by the large quantity \( \frac{e_s}{m_s} \) as was the quantity \( \frac{e}{m} (\hat{\beta} \cdot \mathbf{E}) \) we previously discussed and will tend to invalidate the guiding center approximation. The proper way around this difficulty is to postulate the constraints:

\[
q_x = 0 \quad (A23a)
\]

\[
j_z = 0 \quad (A23b)
\]

The first constraint is consistent with Eq. (A22) due to charge neutrality. The second constraint, while being consistent with Eqs. (A18) through (A22) (i.e., only the perpendicular current contributes to Eq. (A18)), it also makes \( \frac{e_s}{m_s} (\hat{\beta} \cdot \mathbf{E}) \) a finite quantity. This is shown by operation on the kinetic equation with the operator \( \sum_s \frac{e_s}{m_s} \int d\mu d\nu \). The result is:

\[
\frac{e_s}{m_s} (\hat{\beta} \cdot \mathbf{E}) = \alpha_s \left[ \hat{\beta} \cdot \sum \sum \left( e_t \right) \frac{\nabla \cdot \mathbf{u}}{P_t} \right] \quad (A24)
\]

where

\[
\alpha_s = \frac{e_s}{m_s} \left[ \sum \frac{|e_t|}{m_t} \right]^{-1}
\]

This makes the guiding center set of equations self-consistent with the original expansion used to obtain the equation of motion.
Finally, Eq. (A21), which is the equation of perfect conductivity, only gives information about the $E$ field perpendicular to the magnetic field. This is consistent with the fact we are letting:

$$u = \frac{E \times B}{B^2}$$

which can easily be seen by substitution.

Equations (A19) and (A21) can be combined to give:

$$\frac{dB}{dt} = \nabla \times (u \times B) \quad (A25)$$

which when manipulated yields the result:

$$u \cdot \frac{d\hat{\beta}}{dt} = \frac{1}{2} (\hat{\beta} \cdot \nabla) u^2 + v(\hat{u} \cdot \kappa) \quad (A26)$$

where $\kappa = (\hat{\beta} \cdot \nabla) \hat{\beta}$ is the curvature vector of the $B$ line. This result is customarily substituted back into the kinetic equation, Eq. (A17).

Another standard procedure which is used in the main text is to renormalize the distribution function $F_s$ to be:

$$f_s = \frac{1}{B} F_s$$

Thus through a calculation similar to the one in Eq. (A25) and (A26), the new distribution function $f_s$ satisfies the following equation:

$$\frac{df_s}{dt} + (u \cdot v \hat{\beta}) \cdot \nabla f_s + \left[ \frac{e_s}{m_s} (\hat{\beta} \cdot \nabla) + (\hat{\beta} \cdot \nabla) (\frac{1}{2} u^2 - \mu B) \right] \frac{df_s}{dy} = 0 \quad (A28)$$
APPENDIX B

MHD Equilibrium

The bumpy theta pinch geometry is one where the equilibrium B field has the form in cylindrical coordinates:

\[ \mathbf{B} = \begin{bmatrix} B_r(r,z), 0, B_z(r,z) \end{bmatrix} \quad \text{where} \quad |B_r| \ll |B_z| \quad (B1) \]

Because there is no \( \theta \) dependence, the field is axially symmetric.

Given a scalar pressure profile \( P \), the problem is to construct a \( \mathbf{B} \) of the form of Eq. (B1) consistent with the ideal MHD equilibrium equations which are:

\[ \nabla \cdot \mathbf{B} = 0 \quad (B2) \]

\[ \nabla P = (\nabla \times \mathbf{B}) \times \mathbf{B} \quad (B3) \]

Equation (B2) and axial symmetry tell us that there exists a flux function \( \psi(r,z) \) such that:

\[ \frac{1}{r} \frac{\partial \psi}{\partial z} = -B_r \quad (B4) \]

\[ \frac{1}{r} \frac{\partial \psi}{\partial r} = B_z \quad (B5) \]

\( \psi(r,z) \) has the property that the surfaces of constant \( \psi \) are everywhere parallel to the \( \mathbf{B} \) since:
The standard form of the flux function used in the bumpy theta pinch is:

$$\psi(r, \theta) = \psi_0(r) + \delta \psi_1(r) \cos \theta$$

which introduces the small expansion parameters $\epsilon$ and $\delta$ where $\epsilon \ll \delta$. $\epsilon$ measures the periodicity of the bumpiness in the field lines and $\delta$ is related to the amplitude of the bumpiness as shown in Fig. 1:

![Fig. 1: Flux surfaces for the Bumpy Theta Pinch](image)

Using Eq. (B6) in Eqs. (B4) and (B5) determines the form of the $B$ field, namely:

$$B_r = \epsilon \delta \frac{1}{r} \psi_1(r) \sin \theta = \epsilon \delta b(r) \sin \theta$$

(B7)
\[ B_z = \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial \psi}{\partial r} \cos \varepsilon z = a(r) + \delta c(r) \cos \varepsilon z \]  \hspace{1cm} (B8)

which immediately gives the relation

\[ r C(r) = \frac{d}{dr} \left[ r b(r) \right] \]  \hspace{1cm} (B9)

The equation for the flux surface can readily be calculated from the formula:

\[ \frac{dr}{B_r} = \frac{dz}{B_z} \]

which by using Eqs. (B7) and (B8) can be rewritten:

\[ \frac{dr}{dz} = \delta \frac{b(r_0)}{a(r_0)} \sin \varepsilon z + O(\delta^c) \]

Integrating with respect to \( z \) then gives:

\[ r(z) = r_0 - \delta \frac{b(r_0)}{a(r_0)} \cos \varepsilon z \]  \hspace{1cm} (B10)

Hence the amplitude of the bumpiness in a flux surface of mean radius \( r_0 \) is \( \delta \frac{b(r_0)}{a(r_0)} \).

We now turn our attention to Eq. (B3). The first property we note is that \( P \) is a function of \( \psi \) only since:

\[ \mathbf{B} \cdot \nabla P = \frac{1}{r} \frac{\partial (\psi P)}{\partial (r\tilde{z})} = 0 \]

We can expand \( P(\psi) \) around \( \psi_0(r) \) indicating that \( P(r,z) \) has the functional form:

\[ P(r,z) = P^{(0)}(r) + \delta P^{(1)}(r) \cos \varepsilon z \]  \hspace{1cm} (B11)
\( \mathcal{P}^{00} \) and \( \mathcal{P}^{01} \) are not independent functions since \( \mathbf{B} \cdot \nabla \mathcal{P} = 0 \) implies

\[
\frac{d}{dr}(\mathcal{P}^{00}) = a \mathcal{P}^{01} \tag{B12}
\]

Next we can use some vector identities to rewrite Eq. (B3) in the following form:

\[
[\nabla - \hat{\beta} (\hat{\beta} \cdot \nabla)](\mathcal{P} + \frac{1}{2} \mathcal{B}^2) = \mathcal{B}^2 (\hat{\beta} \cdot \nabla) \hat{\beta} \tag{B13}
\]

where \( \mathcal{B} = (\mathcal{B} \cdot \mathcal{B})^{1/2} \) and \( \hat{\beta} = \mathcal{B}/\mathcal{B} \). Using Eqs. (B7), (B8), and (B10) this equation to \( O(\epsilon^0 \delta) \) is:

\[
\frac{d}{dr} \left[ (\mathcal{P}^{00} + \frac{1}{2} \alpha^2) + \delta (\mathcal{P}^{01} + \alpha c) \cos \epsilon \right] = 0 \tag{B14}
\]

Equation (B14) can be integrated in \( r \) to give:

\[
\mathcal{P}^{00}(r) + \frac{1}{2} \alpha^2 = \frac{1}{2} \tag{B15}
\]

\[
\mathcal{P}^{01}(r) + \alpha c = 1 \tag{B16}
\]

The \( 1/2 \) and 1 are arbitrarily chosen constants. Physically they mean we have normalized the \( \mathcal{B} \) field in such a way that at the surface of the plasma (\( \mathcal{P} = 0 \)) we have:

\[
\frac{1}{2} \mathcal{B}^2 = \frac{1}{2} + \delta \cos \epsilon \tag{B17}
\]

The equilibrium equations are then Eqs. (B9), (B15), and (B16). Rewritten using Eq. (B12) they are:

\[
\mathcal{P}^{00}(r) + \alpha^2 = \frac{1}{2} \tag{B17}
\]
- \frac{\partial a}{\partial r} b + ac = 1 \quad (B18)

r_c = \frac{d}{dr}(rb) \quad (B19)

Given the pressure profile $P^{00}(r)$, the B field (a, b, and c) can now be determined. $a(r)$ can be directly calculated from $P^{00}(r)$ using Eq. (B17). Equation (B19) can be substituted into Eq. (B18) resulting in a differential equation for $b(r)$:

$$\frac{db}{dr} + \left(\frac{1}{r} - a \frac{da}{dr}\right)b - \frac{1}{a} = 0 \quad (B20)$$

The boundary condition for $b(r)$ at the origin is:

$$b(0) = 0$$

since $\nabla \cdot B = 0$ so there are no sources or sinks for the magnetic field. Equation (B20) can then be integrated numerically and the solution used in Eq. (B19) to determine $c(r)$. 


The technique for establishing necessary and sufficient conditions for the initial value problem of a set of partial differential equations to be well posed is well known.\(^{15}\) By well posed we mean that the solution to the initial value problem depends continuously on the initial conditions. Essentially, the technique is to determine that all the characteristics of the set of partial differential equations are real. In particular, if each term in the set of partial differential equations has precisely either one space or time derivative, an equivalent procedure is to test for the exponential stability of the system linearized about an infinite homogeneous background whose values correspond to local values of the general problem to be considered. In this sense then, well posedness can be viewed on as a local stability condition for the general problem. An excellent example of this procedure is the MHD set of equations.\(^{16}\)

The GCP equations are a set of integro-differential equations and the methods for showing well posedness are not as clear. Grad\(^ {17}\) has shown that the conditions for exponential stability of the infinitely homogeneous guiding center plasma (first derived by Kadish\(^ {18}\)) are necessary conditions for local stability in the general problem. Although it has not been proven, it seems likely that the violation of the conditions for local stability will result in the general problem not being
well posed. Here, we take this viewpoint, and derive the conditions for local stability using the much simpler technique of testing for exponential stability of the infinitely homogeneous problem.

As presented in Sec. II of the main text, the GCP equations are:

$$\frac{\partial \phi}{\partial t} + (u + v \beta) \cdot \nabla \phi + \left[ \alpha_s \Gamma + (\beta \cdot \nabla)(\frac{1}{2} u^2 - \mu B) + v(u \cdot \nabla) \right] \frac{\partial \phi}{\partial y} = 0 \tag{C1}$$

$$\left( \frac{1}{l} - \beta \right) \left( \frac{\partial}{\partial t} + (u + w \beta) \cdot \nabla \right) (u + w \beta) + \nabla \cdot \Pi = (\nabla \times B) \times B \tag{C2}$$

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) \tag{C3}$$

where the pressure tensor is:

$$\Pi = \frac{1}{B} - BB \frac{P - P_{\parallel}}{B^2}$$

and the effective electric field is

$$\alpha_s \Gamma = \alpha_s \beta \cdot \left[ \sum \frac{q_{\nu} \gamma_n (e_{\nu}) \nabla \cdot \Pi_{\nu}}{P_{\nu}} \right]; \quad \alpha_s = \frac{e_s}{m_s} \left[ \sum \frac{1}{m_k} \right]^{-1}$$

and the moment definitions are:

$$\begin{bmatrix} \rho \\ \rho w \\ 0 \\ 0 \\ P_{\parallel} \\ P_{\perp} \end{bmatrix} = \sum \begin{bmatrix} \rho_s \\ \rho_{sws} \\ q_s \\ j_s \\ P_{\parallel s} \\ P_{\perp s} \end{bmatrix} = \sum B \int du dv \begin{bmatrix} 1 \\ \nu \frac{e_s}{m_s} \\ \nu \frac{e_s}{m_s} v \\ (v - \omega)^2 \\ \mu B \end{bmatrix} f_s \tag{C4}$$
We linearize this set of equations by assuming

\[ y = (u_x, u_y, 0) \]
\[ B = (B_x^{(b)}, B_y^{(b)}, B_0 + B_z^{(b)}) \]
\[ P_1 = P_1^{(a)} + P_1^{(b)} \]
\[ P_\perp = P_\perp^{(a)} + P_\perp^{(b)} \]
\[ f_\perp = f_\perp^{(a)} + f_\perp^{(b)} \]

where \( B_0 \) and quantities with superscript \((0)\) are zero order quantities that are constant in space and time. \( u \) and quantities with superscript \((1)\) are first order in the perturbation amplitude and have spatial dependence \( \exp(ik \cdot x) \), where \( k = (k_x, k_y, k_z) \).

From the structure of the \( B \) field we have:

\[ B = B_o + B_z^{(b)} \]
\[ \beta = \frac{B}{B_o} = \left( \frac{B_x^{(b)}}{B_o}, \frac{B_y^{(b)}}{B_o}, 1 \right) \]

It is then straightforward to linearize Eqs. (C1), (C2), and (C3) to obtain:

\[ \left( \frac{d}{dt} + i k_z V \right) f_\perp^{(b)} + \left[ \alpha_s \Gamma^{(a)} - i k_x B_x^{(b)} \mu \right] \frac{\partial f_\perp^{(a)}}{\partial V} = 0 \]  

\[ B_0 \frac{d u_x}{dt} + \frac{2P_\perp^{(a)} + B_z^{2} i k_x B_z^{(b)} + ik_x \mu}{P} - \frac{P_\perp^{(a)} - P_\perp^{(a)} + B_z^{2} i k_x B_z^{(b)}}{P} = 0 \]
\[
\frac{\delta \Phi}{\delta \eta} = \left( \frac{\partial \delta \eta}{\partial \eta} - \frac{\partial \delta \rho}{\partial \rho} \right) - \frac{\delta \Phi}{\delta \rho} - \frac{\delta \Phi}{\delta \rho}
\]

\[
\frac{3z}{(2\pi)^2} = \frac{\lambda \rho}{(2\pi)^2} \left( \frac{\partial \Phi}{\partial \rho} \right) - \frac{\partial \Phi}{\partial \rho} + \frac{\partial \Phi}{\partial \rho}
\]

The equations by using these transforms are identical. We will solve the initial value problem on these transformed coordinates.

\[
\begin{align*}
\frac{3z}{(2\pi)^2} &= \langle \eta' \rangle \\
\frac{\partial \Phi}{\partial \rho} &= \langle \eta' \rangle \\
\end{align*}
\]

where we define:

\[
\begin{align*}
O &= \frac{\delta \Phi}{\delta \eta} + \frac{\delta \Phi}{\delta \rho} \\
O &= \frac{\delta \Phi}{\delta \eta} - \frac{\delta \Phi}{\delta \rho} \\
O &= \frac{\delta \Phi}{\delta \eta} - \frac{\delta \Phi}{\delta \rho} \\
O &= \frac{\delta \Phi}{\delta \eta} - \frac{\delta \Phi}{\delta \rho}
\end{align*}
\]
\[-\hat{z} \hat{B}_y + \lambda \left( \frac{2\hat{\rho}_x - \hat{B}_x^2}{\hat{p}} \right) \hat{B}_y + \lambda \langle \hat{\mu} \rangle - \left( \frac{\hat{B}_y^2 \hat{\sigma}}{\hat{q}} \right) \hat{B}_y = \frac{\hat{B}_y(0)}{i\hat{k}_z} \quad \text{(C11b)}\]

\[-\hat{z} \hat{B}_x - \hat{B}_x \hat{u}_x = \frac{B_x(0)}{i\hat{k}_z} \quad \text{(C12a)}\]

\[-\hat{z} \hat{B}_y - \hat{B}_x \hat{u}_y = \frac{B_y(0)}{i\hat{k}_z} \quad \text{(C12b)}\]

\[-\hat{z} \hat{B}_z - \hat{\lambda} \hat{B}_x \hat{u}_z = \frac{B_z(0)}{i\hat{k}_z} \quad \text{(C12c)}\]

where

\[\hat{\sigma} = 1 + \frac{\hat{p}_x - \hat{p}_y}{\hat{B}_z^2}\]

and we have divided through by \(i\hat{k}_z\), set \(\hat{k}_z = 0\) due to symmetry in the \(xy\) plane, and defined:

\[\frac{\hat{S}}{i\hat{k}_x} = -\hat{z} \quad \text{and} \quad \frac{\hat{k}_l}{\hat{k}_x} = \hat{\lambda}\]

The quantities on the right hand side of Eqs. (C10), (C11), and (C12) are the initial data.

First, we will solve for \(T^{(1)}\) in Eq. (C10) by using the constraints that the total charge \(q\) along the field lines is zero (see Eq. (C4)). Dividing Eq. (C10) by \(v - z\) and operating with the operator \[\sum S \alpha_s \int d\mu dy\] we obtain:

\[\hat{T}^{(1)} = \hat{B}_z \frac{\sum S \alpha_s \hat{J}_{01}(z)}{\sum S \alpha_s \hat{J}_{00}(z)} + \frac{1}{i\hat{k}_z} \frac{q(0)}{\sum S \alpha_s \hat{J}_{00}(z)} \quad \text{(C13)}\]
Using this expression for $T^{(1)}$ we can rewrite Eq. (C10) to obtain a formula for $f^{(1)}_s$, namely:

$$
\hat{f}^{(t)}_s = B^{(t)}_z \left[ \mu - \frac{\sum s \alpha_s I^{(t)}_m(z)}{\sum s \alpha_s I^{(t)}_m(z)} \right] + C(\mu, V, z) \tag{C14}
$$

where $C(\mu, V, z)$ contains initial conditions and the exact form of this term will not be important in the subsequent calculation.

Next, we substitute Eq. (C14) into Eq. (C8) to obtain an expression for $\langle \hat{\mu} \rangle$ in Eq. (C11b). This eliminates $\langle \hat{\mu} \rangle$ in favor of $B^{(1)}_z$. Having done this, Eqs. (C10), (C11), and (C12) can be written in the following matrix form:

$$
A(z, \lambda) \cdot S(z) = \frac{1}{i\kappa_z} J(0) \tag{C15}
$$

where:

$$
A(z, \lambda) = \begin{bmatrix}
-\lambda & 0 & 0 & 0 \\
0 & -\lambda & 0 & 0 \\
0 & \frac{2B_0^{(t)} + B_0^2}{\rho} - \lambda & 0 & 0 \\
0 & 0 & 0 & -\lambda \end{bmatrix}
$$
\[ s(z) = \begin{bmatrix}
\hat{B}_{x}^{(1)} \\
\hat{B}_{y}^{(1)} \\
\hat{B}_{y}^{(1)} \\
\hat{B}_{x}^{(1)} \\
\hat{B}_{o}^{(1)} \\
\end{bmatrix} \quad J(0) = \begin{bmatrix}
B_{x}^{(0)}(0) \\
B_{y}^{(0)}(0) \\
B_{x}(0) \\
B_{y}(0) \\
B_{o}(0) \\
\end{bmatrix} \]

\[ M(z) = \sum_{s} I_{o2}(z) - \frac{\left[ \sum_{s} \alpha_{s} I_{o1}^{s}(z) \right]^{2}}{\sum_{s} \alpha_{s} I_{o1}^{s}(z)} \]  
(C16)

The general procedure to solve Eq. (C15) is to invert the matrix \( \hat{A}(z, \lambda) \) and take the inverse Laplace transform of both sides. When this is done the terms on the right hand side contain the determinant of \( \hat{A}(z, \lambda) \) in their denominator. This determinant will be a fifth order polynomial in \( z \). The roots of this polynomial (i.e., those values of \( z \) that make the polynomial zero) correspond to the normal modes of the system. If any of these roots are complex then we have exponential instability.

This determinant is easily calculated and turns out to be:

\[ \text{Det} \left[ \hat{A}(z, \lambda) \right] = -z\left( z^{2} - \frac{\sigma B_{o}^{2}}{\rho} \right) D(z, \lambda^{2}) \]  
(C17)

where

\[ D(z, \lambda^{2}) = \left\{ z^{2} - \frac{\sigma B_{o}^{2}}{\rho} - \lambda^{2} \left[ \frac{2P^{(0)}_{*} B_{o}^{2}}{\rho} + \frac{B_{o}^{3}}{\rho} M(z) \right] \right\} \]
The first two roots in Eq. (C17) are easily recognized. For $z$ to be real we must have:

$$\sigma > 0$$  \hspace{1cm} (C18)

The second two roots come from the zeros of $D(z, \lambda^2)$. To analyze this function we need to impose various conditions on $f_s^{(0)}$ to handle the singular integrals $(I_{0n}^s)$ contained in $M(z)$.

First, we require the integrals:

$$\int d\mu f_s^{(0)} \mu^n \hspace{1cm} n = 0, 1, 2$$

to be continuously differentiable in $\nu$. This permits us to define the integrals $(I_{0n}^s)$, which are analytic in the upper half $z$ plane, continuously up to the real axis. For $z$ on the real axis these integrals are defined as:

$$\left\{ d\mu d\nu \frac{\mu^n}{\nu - x} \left. \frac{df_s^{(0)}}{d\nu} \right|_{\nu = x} + \text{P.V.} \left\{ d\mu d\nu \frac{\mu^n}{\nu - x} \frac{df_s^{(0)}}{d\nu} \right\} \right\}$$  \hspace{1cm} (C19)

where P.V. stands for Cauchy principal value.

In general we cannot continue these integrals into the lower half $z$ plane so $k_z$ must be restricted to the half space $k_z > 0$, or $k_z < 0$. The result of either case is essentially the same since if $D(z, \lambda^2)$ is zero for some $s$ when $k_z > 0$, then by inspection $D(z, \lambda^2)$ is zero for $-s$ when $k_z < 0$. Thus for exponential stability all we need to show is that $D(z, \lambda^2)$ has no zeros in the upper half $z$ plane when $k_z > 0$.  

The second condition on $f_s^{(0)}$ is that it be an even function of $v$, specifically of the form $f_s^{(0)}(\mu, v^2)$. This introduces a symmetry into $D(z, \lambda^2)$ such that:

$$D(z, \lambda^2) = D^*(z, \lambda^2); \quad * \text{ stands for complex conjugate} \quad (C20)$$

In addition to this property if we do an asymptotic expansion of $(I_{On}^s)$ we can show:

$$\lim_{|z| \to \infty} \frac{D(z, \lambda^2)}{\Im z > 0} = \text{ negative real number} \quad (C21)$$

Using the information in Eqs. (C20) and (C21) and the evenness of $f_s^{(0)}$ in $v$, we can get the general idea of how a semicircular region in the upper half $z$ plane maps onto the $D(z, \lambda^2)$ plane as shown in Fig. 9.

Fig. 9: A complex mapping of the z plane onto the $D(z, \lambda^2)$ plane
All points in the $z$ plane where $|z| \gg 1$ get mapped into a small region around the point $D$ in the $D(z, \lambda^2)$ plane due to Eq. (C21). The boundary of the region in the $D(z, \lambda^2)$ plane is symmetric about the real axis due to Eq. (C20). The point $B$ ($z = 0$) in the $z$ plane is a real number in the $D(z, \lambda^2)$ plane due to $f_s(0)$ being an even function of $v$. For $D(z, \lambda^2)$ to have a complex zero the image of the semicircular region in the $D(z, \lambda^2)$ plane must contain the origin. Without going into detail of more complex mappings than we have depicted in Fig. 9, it can be shown that a necessary condition for $D(z, \lambda^2)$ to have no complex zero is that

$$D(0, \lambda^2) < 0$$

(C22)

Referring back to the definition of $D(z, \lambda^2)$ for inequality (C22) to be true for all $\lambda^2 \geq 0$ we must have:

$$\left[ 2P_1^{(\omega)} + B_\omega^2 + B_\omega^3 M(0) \right] > 0$$

Furthermore, if $f_s(0)$ is a function of $v^2$ then:

$$\sum_s \alpha_s I_{\alpha s}^s (0) = 0$$

due to charge neutrality. This eliminates one of the two terms in $M(0)$, Eq. (C16). Writing the remaining term out we finally have the condition:

$$2P_1^{(\omega)} + B_\omega^2 + B_\omega^3 \int d\mu d\nu \mu^2 \frac{df_s(\omega)}{dv} > 0$$

(C23)

where the principal value notation is dropped (see Eq. (C19)) because:
\[ \frac{1}{\nu} \frac{df_s^{(b)}(\mu, \nu^2)}{\partial \nu} = \frac{1}{2} \frac{\partial}{\partial (\nu^2)} f_s^{(a)}(\mu, \nu^1) \]

and so $\frac{1}{\nu}$ is no longer a pole.

Equations (C18) and (C23) are the necessary conditions for local stability referred to in the text.