One loop corrections to a hadronic model with vector mesons

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ONE LOOP CORRECTIONS TO A HADRONIC MODEL WITH VECTOR MESONS

A Dissertation
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The Faculty of the Department of Physics
The College of William and Mary in Virginia

In Partial Fulfillment
Of the Requirements for the Degree of
Doctor of Philosophy

by
Gary Prézeau
1999
APPROVAL SHEET

This dissertation is submitted in partial fulfillment of

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I dedicate this thesis with deep affection to my parents, and to the memory of my great uncle, Maurice Lizaire.
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Abstract

The linear $\sigma$ model and its extension, Quantum Hadrodynamics 3 (QHD-III) are discussed. QHD-III is a gauge-invariant model of the strong interaction based on the linear $\sigma$ model, which is made locally invariant under $SU(2)_L \times SU(2)_R$. Parity conservation is imposed. The gauge bosons $\rho$ and $a_1$ are made massive via a Higgs mechanism as in the standard model. The low-energy symmetries of QCD, the ability to evaluate corrections coming from meson loops and the derivation of unambiguous conserved currents is used to motivate QHD-III. A renormalized $\pi\pi$ scattering amplitude to 1-loop using Feynman diagrams in the linear $\sigma$-model is derived. The renormalized corrections due to vector boson exchange are also calculated in QHD-III and explicitly shown to be negligible when their masses become large. The pion decay constant to 1-loop is also calculated and discussed. The gauge invariance of the theory is analyzed and exploited to identify the physical pion and to considerably simplify the QHD-III lagrangian. The 1-loop effective action of the linear $\sigma$ model is derived and its predictions are shown to be identical to the amplitudes calculated using Feynman diagrams. The 1-loop effective action of QHD-III is discussed and shown to provide an explicit proof of the decoupling theorem.
Chapter 1

Introduction

The precise description of hadronic physics at low energies in terms of quarks and gluons and the interactions of QCD is a problem that remains to be solved [1]. This thesis discusses a model of the strong interactions where the effective degrees of freedom are mesons and baryons [2]. The motivation for working with the bound states of QCD instead of the fundamental degrees of freedom that are the building blocks of those bound states, stems from the non-perturbative nature of QCD at low energies and from the many phenomenological successes of effective theories of mesons and baryons. As examples of the latter, consider briefly nucleon-nucleon interactions (NN), the single-particle shell model, vector meson dominance and electromagnetic exchange currents.

Yukawa was the first physicist to point out that the NN interaction can be understood qualitatively, and quantitatively in terms of mesons as the carriers of the strong force [3]. The long range part of the NN interaction stems from the exchange of the lightest meson, the pion ($\pi^\pm, \pi^0$) with quantum numbers ($J^\pi, T = (0^-,1)$).
The intermediate-range attraction stems, to a good extent, from the exchange of a correlated pair of pions that can be parameterized as a single scalar field called the $\sigma$ ($0^+, 0$) [4]. As for the short-range repulsion, much of it can be described by the exchange of the $\omega$ ($1^-, 0$). These qualitative features of the $NN$ interactions are displayed in Fig. 1.1. Furthermore, the saturation of nuclear matter can be understood qualitatively by considering a simple relativistic quantum field theory called the $\sigma - \omega$ model that involves only three fields: the nucleon field $\psi$, the $\omega$ meson field $\omega_\mu$, and the $\sigma$ field $\phi$ which parameterizes correlated two-pion exchange [1]. The lagrangian that describes the interactions of these mesons in the $\sigma - \omega$ model is:

$$
\mathcal{L} = \bar{\psi} \left[ i\gamma^\mu (\partial_\mu - ig_\omega \omega_\mu) - (M - g_\sigma \phi) \right] \psi + \frac{1}{2} \left[ (\partial_\mu \phi)^2 - m_\sigma^2 \phi^2 \right] - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega_\mu^2
$$

(1.1)
A mean field treatment of this lagrangian (where $g_s$ and $g_\omega$ are treated as free parameters [6]) yields a saturation curve (Fig. 1.2) for nuclear matter which is robust against collapse. If one were to take the heavy nucleon limit of Eq. (1.1), then the exchange of $\sigma$'s and $\omega$'s would give rise to a nucleon-nucleon potential of the form:

$$V(r) = \frac{g_\omega^2}{4\pi} \frac{e^{-m_\omega r}}{r} - \frac{g_s^2}{4\pi} \frac{e^{-m_\sigma r}}{r}$$  \hspace{1cm} (1.2)$$

Such a static, non-relativistic, potential can be shown to be unstable against collapse by a variational calculation [1]. It is thus seen that in this relativistic mean field theory, nuclear saturation is a purely relativistic effect. By adding more
mesonic intermediate states, the $NN$ potential can replicate the measured data to a high degree of precision [7].

The single-particle shell model (SPSM) [1, 8] can also be simply reproduced by considering a mesonic relativistic mean field theory [9, 10]. In the SPSM, nuclei are composed of nucleons that occupy single particle eigenstates of a model potential. Upon filling the levels of a particular nucleus, the spin and parity of that nucleus can be predicted by ascribing all its properties to the last nucleon as it moves through the model potential. The total field seen by the last nucleon can be modeled by a square well, a harmonic potential or a combination of the two. The Schrödinger equation for that potential is then solved. The nuclear eigenstates with their corresponding spin and parity then coincide with the wave-functions of the last nucleon upon filling the lower levels. This picture is incomplete as evidenced by the fact that it yields the wrong spectrum of nuclear excited states. One must also account for the fact that as the nucleon moves through the nucleus, a spin-orbit splitting will occur. Mayer and Jensen [8] were the first to point that out and add to the Hamiltonian a term of the form:

$$H' = -\alpha(r) \hat{l} \cdot \hat{s}$$  \hspace{1cm} (1.3)

With the inclusion of this spin-orbit term, the SPSM now predicts both the magic numbers of nuclear stability, and the spins and parities of the nuclear levels with remarkable success (Fig. 1.3). This success is also reproduced by a relativistic quantum field theory of baryons and mesons such as the one given in Eq. (1.1). In the mean field treatment of this hadronic lagrangian [9, 10], when the mesonic fields are replaced by their average values, the solutions to the Dirac equation of the nucleon that
evolves through this mean field yields the correct shell structure: in this framework, the spin-orbit splitting and all the other complexities of the shell model Hamiltonian are automatically taken care of simply by fitting a few bulk properties of nuclear matter. The success of a hadronic description of the nucleus also sheds light on the applicability of hadronic field theories. Consider the fact that in the shell model, the nucleons inside a nucleus can be described as independent particles moving in a mean field created by the other nucleons. But the average distance between nucleons in a nucleus is $d \approx 1.8 \text{ fm}$ while the quarks and gluons are confined to a radius $\approx 1 \text{ fm}$ [11]; thus one would expect significant overlap between the wave-functions of the constituents of neighboring nucleons which would modify and destroy the success of the shell model. The fact that the shell model describes the data successfully suggests that a hadronic description of strong interactions works beyond the length scales one would expect such a picture to break down$^1$.

Electromagnetic interactions are also well described within an hadronic framework. For example, the vector meson dominance model (VMD) [12, 13] allows one to parameterize the electromagnetic form factors of hadrons in terms of vector meson poles. VMD is a framework which incorporates the experimental fact that electromagnetic interactions of hadrons are dominated by vector mesons when they contribute to the amplitude. The process by which vector mesons can dominate electromagnetic interactions can be visualized in terms of quarks by noting that an incoming photon can turn into a quark/anti-quark pair $q\bar{q}$ before interacting with the target hadron. This $q\bar{q}$ pair has the quantum numbers of a neutral vector meson

---

$^1$It should be noted that in the $\sigma - \omega$ model, the wavefunctions of the fields do overlap.
Figure 1.3: Level orderings in the SPSM in relativistic mean field theory. Here, \([n]\) is the total number of eigenstates available to nucleons up to a given level; \((\pi)\) is the degeneracy of a particular energy level; the notation used is \(n_l\) where \(n\) is the number of nodes in the radial wave function, \(l\) is the orbital angular momentum and \(J = l - \frac{1}{2}, l + \frac{1}{2}\). The parity of the energy level is given by \((-1)^J\). The SPSM including the spin-orbit term reproduces the magic number 2,8,20,28... of nuclear stability. Taken from [1].
which in turn interacts with the target hadron (Fig. 1.4). Thus, all the gluonic contributions and all other QCD complexities associated with the intermediate virtual $q\bar{q}$ pair can be modeled successfully by substituting in physical vector mesons.

The coupling between the photon and the vector meson is expressed by the field current identity [14] for flavor $SU(3)$:

$$j^\text{em}_\mu = -\frac{m^2_\rho}{g_\rho} \rho^{(3)}_\mu - \frac{m^2_\omega}{g_\omega} \omega_\mu - \frac{m^2_\phi}{g_\phi} \phi_\mu$$  \hspace{1cm} (1.4)$$

where $g_\rho$ is the $\rho$-nucleon coupling, $g_\omega$ is the $\omega$-nucleon coupling and $g_\phi$ is the $\phi$-nucleon coupling. These terms in the electromagnetic current allow the photon to turn into a neutral vector meson at tree level. The application of VMD to the
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determination of the electromagnetic form factors of hadrons works quite well. For example, at tree level the pion form factor in this model is given by:

$$F_\pi(q^2) = \frac{m_\rho^2}{m_\rho^2 - q^2 - i m_\rho \Gamma_\rho(q^2)}$$  \hspace{1cm} (1.5)

The last term in the denominator takes into account the finite width of the $\rho$. Comparison of $|F_\pi(q^2)|^2$ with the data in Fig. 1.5 [15] shows a very good fit in both the space-like and time-like regions; thus, VMD is applicable for both $\pi^+\pi^-$ production and $e^-\pi$ scattering. In this framework, the $\rho$ meson provides the entire structure of the pion. The derivative of the pion form factor at $q^2 = 0$ yields the pion charge radius:

$$\langle r_\pi^2 \rangle^{\frac{1}{2}} = \left( \frac{6}{d F_\pi}{dq^2}_{q^2=0} \right)^{\frac{1}{2}} = 0.62 \text{ fm}$$  \hspace{1cm} (1.6)

This also agrees well with the measured value of $\langle r_\pi^2 \rangle^{\frac{1}{2}} = (0.66 \pm 0.01) \text{ fm}$ [16, 17].

Electromagnetic exchange currents also demonstrate the usefulness of a hadronic description of nuclei [1]. Electromagnetic exchange currents are two-body contributions to the current that arise from the interactions of an incoming photon with two interacting nucleons moving through the nucleus; the photon can interact with the mesons exchanged between the nucleons or via pair creation as shown in the Feynman diagrams of Fig. 1.6 [18]. These two-body electromagnetic interactions modify the one-body current to:

$$J_\mu(x) = \sum_{i=1}^{A} J_\mu^{(1)}(x_i; x) + \sum_{i<j=1}^{A} J_\mu^{(2)}(x_i, x_j; x)$$  \hspace{1cm} (1.7)
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Figure 1.5: The pion form factor for space-like and time-like $q^2$. Taken from [15]

Figure 1.6: Diagrams contributing to the two-body exchange current sum.
The last sum can be extracted from the diagrams of Fig. 1.6. The inclusion of the exchange currents in the calculation of the magnetic moment of $^{3}\text{He}$ yields a value of $\mu = -2.078$ nm which is closer to the experimental value of $\mu = -2.127$ nm than the Schmidt value of $\mu = -1.913$ nm [1]. Consider also the magnetic form factor of $^{3}\text{He}$. Fig. 1.7 [19, 20, 21] compares the experimentally measured magnetic form factor of $^{3}\text{He}$ with calculations done with and without the inclusion of exchange currents; the dashed curve is the theoretical prediction where the exchange currents are neglected while the solid curves includes them. The data indicate that the exchange currents are necessary and even dominant at high $q^2$ where hadrons more massive than the pion can contribute; at low $q^2$, the pion exchange contributions are supported by low-energy theorems. These results demonstrate how successful a hadronic description of nuclei can be.

An update on recent progress in quantum hadrodynamics can be found in [22]. After this brief discussion of the role of hadronic theories in nuclear physics, we now turn to building the hadronic model which is the subject of this thesis. To do so, a discussion of QCD is first required.

1.1 QCD

All fundamental interactions (with the exception of gravity) are believed to emerge from the requirement that the universe be locally invariant under sets of transfor-

\footnote{Throughout this thesis, the metric \((+,-,-,-)\) is used. In this metric, space-like momenta satisfy \(q^2 < 0\) while time-like momenta satisfy \(q^2 > 0\).}
Figure 1.7: Elastic magnetic form factor for $^3$He from [20]. Two exchange current theories are shown from a) [21] and b) [19].

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mations (groups) given in table 1.1 [23, 24]. The requirements that each point in space-time be invariant under a symmetry transformation in some internal space leads to the appearance of gauge bosons. For instance, if the universe is required to be invariant when each point in space-time is assigned an arbitrary phase, electromagnetic interactions emerge: the carriers of the electromagnetic force, photons denoted $A_\mu$ in field theoretical notation, emerge out of the requirement that the universe be invariant under arbitrary $U(1)$ transformations.\(^3\) If one requires as well that the universe be invariant when each point in space-time is assigned an additional arbitrary phase in a 2-dimensional internal space (called weak isospin or $SU(2)_W$), weak interactions emerge. The combined $SU(2)_W \times U(1)_Y$ symmetry of the universe leads electromagnetic and weak interactions to form what is known as the electroweak interaction of the standard model [25, 26, 27, 28]; the carriers of the weak force are the weak bosons denoted $W^\pm_\mu$ and $Z_\mu$. The fundamental theory of strong interactions is also believed to be a gauge theory called Quantum ChromoDynamics (QCD). If the universe is required to be invariant when each point in space-time is assigned an arbitrary phase in a 3-dimensional internal space, the carriers of the strong force, the gluons denoted $G_\mu^a$, appear. The QCD lagrangian is given by:

\[\psi'(x) = e^{i\theta(x)}\psi(x)\]

\[\psi'(x) = e^{iT \cdot \theta(x)}\psi(x)\]

\[\psi'(x) = e^{iA^a_{\lambda a}(x)}\psi(x)\]

---

\(^3\)With the constraint that the transformations be analytical and differentiable.
\[ \mathcal{L} = \bar{\psi}_q [i\gamma^\mu (\partial_\mu - \frac{i}{2} g_s \lambda^a \cdot G_\mu^a) - m_q] \psi_q - \frac{1}{4} G_{\mu\nu} \cdot G^{\mu\nu}, \]  

(1.8)

where \( \psi_q \) represents the quark fields and the \( G_\mu^a \) are the gluon fields. The expression for the spinor above is:

\[
\begin{pmatrix}
  u \\
  d \\
  c \\
  s \\
  t \\
  b
\end{pmatrix}
\]

with

\[
\begin{pmatrix}
  u_r \\
  u_g \\
  u_b
\end{pmatrix}
\]

with

\[
\begin{pmatrix}
  u_{r1} \\
  u_{r2} \\
  u_{r3} \\
  u_{r4}
\end{pmatrix}
\]

(1.9)

The relations (1.9) describe six flavors of quarks (up, down, charm, strange, top, bottom) and three different kinds of charges called color (red, green, blue). There are therefore a total of 18 different quarks: 3 charges \( \times \) 6 different flavors of quarks. Each of these quarks also possesses an anti-particle which brings the total number of fermions to 36. Quarks of the same flavor but of a different color have the same mass. The mass matrix therefore has the form:

\[
m_q = \begin{pmatrix}
  m_u & 0 & 0 & 0 & 0 & 0 \\
  0 & m_d & 0 & 0 & 0 & 0 \\
  0 & 0 & m_c & 0 & 0 & 0 \\
  0 & 0 & 0 & m_s & 0 & 0 \\
  0 & 0 & 0 & 0 & m_t & 0 \\
  0 & 0 & 0 & 0 & 0 & m_b
\end{pmatrix}
\]

(1.10)

where

\[
\begin{align*}
  m_u &= 1.5 \text{ to } 5 \text{ MeV} \\
  m_d &= 3 \text{ to } 9 \text{ MeV} \\
  m_c &= 1.1 \text{ to } 1.4 \text{ GeV} \\
  m_s &= 60 \text{ to } 170 \text{ MeV} \\
  m_t &= 173.8 \pm 5.2 \text{ GeV} \\
  m_b &= 4.1 \text{ to } 4.4 \text{ GeV}
\end{align*}
\]
The mass matrix above and the corresponding values for the masses [29] is diagonal in the color subspaces. The gluon field strength is given by:

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu,$$

(1.11)

where $f^{abc}$ are the structure constants of $SU(3)$. The number of gluons is equal to the number of generators of $SU(3)$, thus $a$ runs from 1 to 8 [23].

QCD has two distinctive properties: confinement and asymptotic freedom. Both of these properties stem from the non-abelian nature of the group $SU(3)$, and the fact that the gluons carry charge and interact non-linearly as can be seen from Eq. (1.11).

Confinement is the observation that strongly interacting particles appear only in combinations that form color singlets. Hence, quarks and gluons, which carry color, can not be observed as free particles but instead must be confined to the interior of hadrons. At low energies, or at large distances, only mesons and baryons can be observed.\footnote{Exotics may also be observed.} If an individual quark is knocked out of a hadron, a $q\bar{q}$ pair can emerge from the vacuum for example. Figuratively, the $q$ will remain in the hadron while the $\bar{q}$ will pair up with the outgoing quark to form a color singlet bound-state: a meson.

Asymptotic freedom is the property that at high energies, or at short distances, the coupling constant asymptotically goes to zero [30, 31, 32, 33]; hence, strongly interacting particles can be approximately described as free particles at high energies. Therefore, at energies $> 1$ GeV, one can use perturbation theory and expand S-matrix elements in powers of the coupling constant. At low energies however, the strength of the coupling constant is larger; at energies of around 1 GeV, the coupling constant
is of $\mathcal{O}(1)$ and the perturbative expansion in powers of the coupling constant fails. To solve the strongly interacting problem at low energies, non-perturbative methods become necessary. Such a method is lattice gauge theory (LGT) \cite{34,35}. In LGT, QCD is projected unto a grid of labeled points with a spacing $a$ that represent a discretized form of space-time. The physical limit is the continuum limit when $a \to 0$ and the labeled points merge into the points of space-time; discretizing space-time allows numerical solutions to QCD to be calculated, regardless of the size of the coupling constant. This numerical approach to QCD requires huge processing power and as of now, LGT is still far from being able to describe the interacting nuclear problem and can at this time only describe some of the static properties of hadrons. Thus, in order to describe strongly interacting dynamical systems in the absence of a full solution of QCD at low energies, one is left with model building. The construction of these models must somehow incorporate both phenomenology and the known symmetries of QCD at low energies.

If the quarks were massless, in addition to the $SU(3)$ gauge symmetry, QCD would also have a global $SU(6)_L \times SU(6)_R$ symmetry that acting in flavor space. At low energies, where one can neglect the heaviest quarks and keep the two lightest, the up and down quarks, this global symmetry reduces to global $SU(2)_I \times SU(2)_I$; in other words, QCD is approximately invariant under flavor isospin rotations and chiral symmetry at low energies. In the chiral limit ($m_q = 0$), the QCD lagrangian reduces to:

\begin{equation}
\mathscr{L} = \bar{\psi}_q [i \gamma^\mu (\partial_\mu - \frac{i}{2} g_s \lambda^a \cdot G^a_\mu)] \psi_q - \frac{1}{4} G_{\mu\nu} \cdot G^{\mu\nu}, \tag{1.12}
\end{equation}
which is seen to be invariant under the transformations:

\[
\begin{align*}
\psi_q' &= e^{-\frac{i}{2} \tau \cdot \theta} \psi_q & \text{Isospin Transformation} \\
\psi_q' &= e^{-\frac{i}{2} \gamma_5 \tau \cdot \theta} \psi_q & \text{Chiral Transformation} \quad (1.13)
\end{align*}
\]

with the gluon fields invariant. Here, the transformation matrices act in a truncated two-dimensional flavor space with basis vectors composed of the top two components of Eq. (1.9):

\[
\psi_q = \begin{pmatrix} u \\ d \end{pmatrix} \quad (1.14)
\]

It is noted that the QCD lagrangian would still be invariant under isospin transformations if all the quarks had the same mass, but that mass terms always violate chiral symmetry. The violations to chiral symmetry are therefore of the order of the pion mass. The fact that Eq. (1.12) is invariant under \( SU(2)_L \times SU(2)_R \) can be seen by making a change of variables:

\[
\psi_{qR} \equiv \frac{1}{2} (1 + \gamma_5) \psi_q, \quad \psi_{qL} \equiv \frac{1}{2} (1 - \gamma_5) \psi_q, \quad \psi_q = \psi_{qR} + \psi_{qL} \quad (1.15)
\]

In terms of these new variables, Eq. (1.12) becomes:

\[
\mathcal{L} = \bar{\psi}_{qR} i \gamma^\mu (\partial_\mu - \frac{i}{2} g_s \lambda^a \cdot G^a_\mu) \psi_{qR} + \bar{\psi}_{qL} i \gamma^\mu (\partial_\mu - \frac{i}{2} g_s \lambda^a \cdot G^a_\mu) \psi_{qL} - \frac{1}{4} G_{\mu\nu} \cdot G^{\mu\nu}, \quad (1.16)
\]

Eq. (1.16) is now clearly invariant under the following right and left transformation:

\[
\begin{align*}
\psi'_{qR} &= e^{-\frac{i}{2} \tau \cdot \theta} \psi_{qR} \quad & \psi'_{qL} &= \psi_{qL} & SU(2)_R \text{ Transformation} \\
\psi'_{qR} &= \psi_{qR} \quad & \psi'_{qL} &= e^{-\frac{i}{2} \tau \cdot \theta} \psi_{qL} & SU(2)_L \text{ Transformation} \quad (1.17)
\end{align*}
\]
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Low-energy QCD also appears to undergo a dynamical breakdown of the $SU(2)_L \times SU(2)_R$ symmetry to $SU(2)_V$ [36]. This dynamical symmetry breaking leaves the ground state vacuum invariant under isospin transformations only: the generators of chiral transformations do not leave the ground state vacuum invariant. The generators are said to be broken. Goldstone's theorem [37] requires the appearance of massless fields called Goldstone bosons for each broken generator; the three pions are the Goldstone bosons corresponding to the three broken generators of chiral symmetry. The fact that pions are the Goldstone bosons of a dynamically broken approximate symmetry described by the lagrangian of the non-linear $\sigma$-model (introduced below) is reflected in their low mass, and the agreement between the predictions of the chiral symmetry breaking scenario and low-energy phenomenology5.

To model the strongly interacting problem at low energies, one must be guided by the low-energy symmetries of QCD and by phenomenology. In particular, a chiral symmetry with broken generators must be incorporated into the model. Furthermore, boson exchange models and confinement suggest that mesons mediate the strong force at low energies; it should be possible to take into account corrections due to meson loops. Phenomenologically, the existence of conserved vector and (partially) conserved axial-vector currents is observed in weak processes such as $\beta$-decay and inferred in the success of the $V-A$ theory of weak interactions [38]; these currents should be derivable from the model. Finally, to have predictive power, the number of parameters in the model should be minimized. In particular, in a hadronic model based on mesons and baryons as effective degrees of freedom, the model should be constructed in such a way as to constrain the number of mesons in a systematic way,

5 See the section on effective lagrangians below.
because of the existence of the large number of mesons that could potentially be incorporated into the model.

1.2 The $\sigma$-model and effective lagrangians

A lagrangian which would possess the features described in the last paragraph would most likely be a good model of low-energy strongly interacting physics. Such a lagrangian would constitute a low-energy approximation to the QCD lagrangian. The low-energy limit of the lagrangian of a fundamental theory such as QCD is called an effective lagrangian: effective lagrangians incorporate the high energy dynamics of the underlying theory in a few adjustable parameters that are fit to data and are generally easier to use.\(^6\)

An example of an effective lagrangian that possesses the low-energy symmetries of QCD and which gives rise to conserved vector and axial vector currents is the linear $\sigma$-model \([39, 40]\) whose lagrangian is given by:

$$\mathcal{L} = \bar{\psi} [i\gamma^\mu \partial_\mu - g_\pi (s + i\gamma_5 \tau \cdot \pi)] \psi + \frac{1}{2} (\partial_\mu s \partial^\mu s + \partial_\mu \pi \cdot \partial^\mu \pi) - \frac{1}{4} \lambda (s^2 + \pi^2 - v^2)^2 + \epsilon s + \mathcal{L}_{ct},$$

where $\psi$ is the nucleon field, $\epsilon s$ is the chiral symmetry violating term and $\mathcal{L}_{ct}$ is the counterterm lagrangian. The term $\epsilon s$ gives rise to a pion mass which violates chiral symmetry; in the chiral limit: $\epsilon = m_\pi = 0$. The fields $\pi$ and $s$ are pseudoscalar isovector and scalar isoscalar fields respectively. The lagrangian of the $\sigma$-model is

\(^6\)The current thinking is that the Standard model lagrangian is itself an effective lagrangian of a more fundamental theory whose characteristic scale is the Planck scale.
invariant under the following set of isospin infinitesimal transformations:
\[
\psi' = (1 + \frac{1}{2} \tau \cdot \alpha) \psi \\
\pi' = \pi - \alpha \times \pi
\]  
(1.19)

with $s$ invariant and $\alpha$ the transformation parameter. Eq. (1.18) is also invariant under the following chiral infinitesimal transformations in the chiral limit:
\[
\psi' = (1 + \frac{i}{2} \tau \cdot \beta \gamma_5) \psi \\
\pi' = \pi - s \beta \\
s' = s + \beta \cdot \pi
\]  
(1.20)

Noether's theorem [41, 42] states that to every symmetry of a lagrangian there corresponds a conserved current. The conserved currents can be easily derived from the transformation properties of the fields above and lead to the following conserved vector and axial-vector currents:
\[
T_{\mu} = \frac{1}{2} \bar{\psi} \gamma_{\mu} \tau \psi + \pi \times \partial_{\mu} \pi \quad \text{Vector current}
\]
\[
A_{\mu} = \frac{1}{2} \bar{\psi} \gamma_{\mu} \gamma_5 \tau \psi - \pi \partial_{\mu} s + s \partial_{\mu} \pi \quad \text{Axial vector current}
\]  
(1.21)

These currents are conserved in the chiral limit and satisfy: $\partial^\mu T_{\mu} = \partial^\mu A_{\mu} = 0$. In the physical limit when $\epsilon \neq 0$, the partially conserved axial current (PCAC) relation is obtained:
\[
\partial^\mu A_{\mu} = -\epsilon \pi
\]  
(1.22)

The symmetries of the linear $\sigma$-model can be made manifest by expressing the nucleon field in terms of left/right fields similar to those in Eq.s (1.15):
\[
\psi_R \equiv \frac{1}{2}(1 + \gamma_5)\psi, \quad \psi_L \equiv \frac{1}{2}(1 - \gamma_5)\psi, \quad \psi = \psi_R + \psi_L
\]  
(1.23)
and by combining the scalar and pseudoscalar fields to form a matrix:

\[ \chi \equiv \frac{1}{\sqrt{2}}(s - i\tau \cdot \pi) \]  

(1.24)

In terms of these variables, the linear σ-model lagrangian becomes (putting for the moment \( \epsilon = 0 \)):

\[ \mathcal{L} = i(\bar{\psi}_R\gamma_\mu \partial^\mu \psi_R + \bar{\psi}_L\gamma_\mu \partial^\mu \psi_L) - \sqrt{2}g_\sigma(\bar{\psi}_L\chi^\dagger \psi_R + \bar{\psi}_R\chi\psi_L) \]

\[ + \frac{1}{2}\text{tr}(\partial_\mu\chi^\dagger \partial^\mu \chi) + \frac{v^2\lambda}{2}\text{tr}(\chi^\dagger \chi) - \frac{\lambda}{4}\text{tr}(\chi^\dagger \chi)^2 \]  

(1.25)

It is now straightforward to verify that the linear σ-model lagrangian is invariant under the following \( SU(2)_L \times SU(2)_R \) transformations:

\[ \psi'_R = e^{-\frac{i}{2}\tau \cdot \theta_R} \psi_R; \quad \psi'_L = \psi_L; \quad \chi' = e^{-\frac{i}{2}\tau \cdot \theta_R} \chi \quad SU(2)_R \]

\[ \psi'_R = \psi_R; \quad \psi'_L = e^{-\frac{i}{2}\tau \cdot \theta_L} \psi_L; \quad \chi' = \chi e^{+\frac{i}{2}\tau \cdot \theta_L} \quad SU(2)_L \]  

(1.26)

Thus, the linear σ-model already has two desirable qualities for a candidate effective theory of low-energy strong interactions: it possesses conserved vector and axial-vector currents and it possesses the same symmetries as low-energy QCD. The linear σ-model is also an example of a theory with spontaneous symmetry breaking [23]. The potential in Eq. (1.18) has the form of a Mexican hat (Fig. 1.8) when \( v^2 > 0 \) and \( \epsilon = 0 \):

\[ V = \frac{1}{4}\lambda(s^2 + \pi^2 - v^2)^2 \]  

(1.27)

From Fig. 1.8, it is clear that the potential will be minimized only when the fields are non-zero. The vacuum is therefore filled with a condensate of the mesonic fields whose vacuum expectation value (VEV) is non-zero. The strongly interacting vacuum
Figure 1.8: Mexican hat potential. Taken from [23]
is invariant under spatial reflections and should have a positive parity, thus the VEV of the pseudoscalar fields must vanish and only the scalar field develops a non-zero condensate: \( \langle \pi \rangle = 0, \langle s \rangle \neq 0 \). Furthermore, since the vacuum now distinguishes between the scalar and pseudoscalar fields, the vacuum can only be invariant under transformations that do not mix them. From the transformation properties given in Eqs. (1.19) and (1.20), it is seen that chiral symmetry is broken, and all that remains is invariance under isospin transformations. The original \( SU(2)_L \times SU(2)_R \) symmetry has been spontaneously broken to \( SU(2)_V \).

When \( \epsilon \neq 0 \), the Mexican hat “tips” and the minimizing conditions are:

\[
\frac{\partial V}{\partial \pi^i} = \frac{\partial}{\partial \pi^i} \left[ \frac{1}{4} \lambda (s^2 + \pi^2 - \nu^2)^2 - \epsilon s \right] = 0
\]

\[
\frac{\partial V}{\partial s} = \frac{\partial}{\partial s} \left[ \frac{1}{4} \lambda (s^2 + \pi^2 - \nu^2)^2 - \epsilon s \right] = 0
\]

(1.28)

The “tipping” of the Mexican hat is at the origin of the pion mass. In solving the above equations, the VEV can be chosen in such a way that the nucleon field in the lagrangian (1.18) acquires the correct nucleon mass \( M \). With this condition in mind, after the SSB, the scalar field becomes:

\[ s = \langle s \rangle - \sigma, \quad \text{with} \quad \langle s \rangle \equiv \sigma_0 = \frac{M}{g_\pi}. \]

(1.29)

The quantum fluctuations of the \( s \) field are represented by the \( \sigma \) field which contains all the remaining dynamics of the original scalar field. Solving Eqs. (1.28) leads to the SSB potential:

\[ V = -\frac{1}{2} (m_\pi^2 \pi^2 + m_\sigma^2 \sigma^2) + \frac{m_\sigma^2 - m_\pi^2}{8 \sigma_0^2} (\sigma^2 + \pi^2) - \frac{m_\sigma^2 - m_\pi^2}{2 \sigma_0^2} \left( \sigma^2 + \pi^2 \right)^2 \]

(1.30)

where:

\[ m_\pi^2 = \frac{\epsilon}{\sigma_0}, \quad \lambda = \frac{m_\sigma^2 - m_\pi^2}{2 \sigma_0^2}. \]

(1.31)
It is seen that the explicit violation of chiral symmetry is entirely contained in the pion mass. Thus, the linear $\sigma$-model is a spontaneously-broken hadronic field theory that is globally invariant under $SU(2)_L \times SU(2)_R$ transformations and from which the corresponding conserved vector and axial-vector currents can be derived. The linear $\sigma$-model is also renormalizable and requires the measurement of only two parameters: $g_\pi$ and $m_\pi$. All these symmetry features are characteristics that were sought in a candidate effective theory of strongly interacting physics. Indeed, the linear $\sigma$-model gives a good qualitative description of low-energy strongly interacting physics at tree level; including corrections coming from loops does not improve agreement, however [43]. The reason why including loops does not improve the agreement between the predictions of the linear $\sigma$-model and the data is rooted in the fact that strongly interacting physics at all scales is predicted once just two parameters have been measured. The higher energy physics is much more complicated than that since contributions from heavier mesons and their interactions not included in the linear $\sigma$-model lagrangian could become important. Although the linear $\sigma$-model offers a systematic way of calculating corrections to tree level results, it eventually leads to wrong predictions.

Chiral perturbation theory ($\chi$PT) [44, 45, 46, 47, 48] is a method that allows the systematic calculation of loop corrections to tree level amplitudes without any assumption about the short-scale physics. In $\chi$PT, the loop expansion coincides with a momentum expansion which is a good expansion parameter at low energies. The chiral lagrangian of $\chi$PT is constructed by writing all possible terms that are consistent with the low-energy symmetries and which are of a certain order in momentum [43, 45]; here, the pion mass and the derivative of the pion field are considered to
be of the same order within the context of the momentum expansion. Thus, the \( \chi \)PT lagrangian can be thought of as a sum of lagrangians of increasing order in momentum. For example, the lowest-order chiral lagrangian is the lagrangian that is composed of terms that are quadratic in the momentum, that are invariant under \( SU(2)_L \times SU(2)_R \) and also consistent with Lorentz invariance, parity, and G-parity\(^7\). One can prove that the most general lagrangian that can be written with those requirements is the non-linear \( \sigma \)-model lagrangian [23]:

\[
\mathcal{L}_2 = F^2 \left( \frac{1}{2} \nabla_\mu U^A \nabla^\mu U^A + \chi_{ab} U^A \right)
\]  

(1.32)

\( \chi_{ab} \) is the chiral symmetry breaking term that generates a mass for the pion. \( F \) has the dimensions of mass and sets the scale of the problem; it is related to the pion decay constant [24]. \( U^A \) is a dimensionless dynamical field with the following relation to the pion field:

\[
U^A = \frac{1}{F}(\sigma, \pi), \quad U^A U^A = 1.
\]  

(1.33)

Here, repeated indices are summed and \( A = 0, \ldots, 3 \). Three observations can be made from the above constraint:

1) The scalar field in the non-linear \( \sigma \)-model also develops a condensate:

\[
\sigma = F - \frac{1}{2} \frac{\pi^2}{F} + \ldots
\]  

(1.34)

2) The pionic self-interactions come from the expansion of the scalar field.

3) From the relation \( \pi^2/F^2 < 1 \), it is seen that the actual expansion parameter is \( p/F \).

\(^7\)G-parity is an operation which combines charge conjugation with a rotation in isospace around the \( y \)-axis.
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There are many different representations of the non-linear $\sigma$-model; the above representation is called the square-root representation. It is also worth noting that the non-linear $\sigma$-model is the limit of the linear $\sigma$-model when $m_\sigma \to \infty$.\(^8\)

The covariant derivative in Eq. (1.32) is defined:

$$\nabla_\mu U^0 = \partial_\mu U^0 + g_\rho a^i_\mu U^i$$

$$\nabla_\mu U^i = \partial_\mu U^i - g_\rho \varepsilon^{ijk} \rho^j_\mu U^k - g_\rho a^i_\mu U^0 \quad i = 1, 2, 3$$ \hspace{1cm} (1.35)

The origin of this definition for the covariant derivative will become transparent when the QHD-III lagrangian is discussed below. The two isovector fields $a^i_\mu$, $\rho^i_\mu$ are pseudo-vector and vector fields respectively. They are classical, external fields that act as sources for the pion fields: functional derivatives with respects to these sources yield S-matrix elements. They can be decoupled from the problem by putting the coupling constant $g_\rho$ to 0.

The next order lagrangian is composed of terms quartic in the momentum that are invariant under $SU(2)_L \times SU(2)_R$ but that are also consistent with Lorentz invariance, parity and G-parity:

$$L_4 = l_1 (\nabla_\mu U \cdot \nabla^\mu U)^2 + l_2 (\nabla_\mu U \cdot \nabla_\nu U) (\nabla^\mu U \cdot \nabla^\nu U) + l_3 (X_{ab} \cdot U)^2$$

$$+ l_4 (\nabla_\mu X_{ab} \cdot \nabla^\mu U) + l_5 (U F^\mu_\nu F^\nu_\mu U) + l_6 (\nabla_\mu U F^\mu_\nu \nabla_\nu U)$$

$$+ l_7 (\vec{\chi} \cdot U)^2 + h_1 X_{ab} \cdot X_{ab} + h_2 \text{tr} F^\mu_\nu F^\nu_\mu + h_3 \vec{\chi} \cdot \vec{\chi} \quad (1.36)$$

As seen above, ten new parameters are needed to describe the $O(p^4)$ lagrangian.\(^9\)

These terms are the next-to-leading-order corrections due to virtual processes occurring at short length scales. The generality of $L_4$ assumes nothing about the structure

---

\(^8\)In this thesis, the non-linear $\sigma$-model is defined by Eq. (1.32).

\(^9\)The introduction of the field $\vec{\chi}$ is noted. It is proportional to the quark mass difference $(m_u - m_d)$ and therefore represents an isospin breaking term. Its contribution to most Green’s functions is negligible [44].
of the vacuum at these short length scales except for general symmetry principles and the application of quantum mechanics. The chiral lagrangian to one loop is given by:

\[ \mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 \]

(1.37)

There should also be \( \mathcal{O}(p^4) \) corrections due to loops (Fig. 1.9) stemming from the interactions of \( \mathcal{L}_2 \). Because of the generality of \( \mathcal{L}_4 \), these loop corrections must be of the form of the terms in Eq. (1.36).\(^{10}\) Thus, as first pointed out by Weinberg, any divergence coming from the loop integrals can be absorbed in the parameters of \( \mathcal{L}_4 \), thereby renormalizing them. Despite the large number of parameters that must be measured and fit to data, the \( \chi \)PT approach still possesses predictive power and works quite well [49, 50].

\(^{10}\)Power counting methods can be used to prove that loop corrections can not contribute to terms of lower order in the momentum [44, 45].


1.3 QHD-III

In the preceding sections, it was seen how low-energy strongly interacting physics has to be modeled because of the inapplicability of perturbation theory in powers of the coupling constant which is large in QCD. Because of confinement, the observable degrees of freedom at low energies are baryons and meson, not quarks and gluons; a discussion of the \( \sigma - \omega \) model, the shell model, vector meson dominance and electromagnetic exchange currents showed how successful a hadronic formulation of strongly interacting physics can be. The required criteria for a candidate effective theory of low-energy strongly interacting physics was discussed, and it was argued that any such model should be globally \( SU(2)_L \times SU(2)_R \) invariant, generate conserved vector and axial-vector currents, allow for spontaneous symmetry breaking and permit the calculation of corrections to tree level amplitudes in a systematic fashion. The linear \( \sigma \)-model was shown to possess these features but assumed too much about the structure of the short scale physics to describe the data accurately; a discussion of the chiral perturbation theory approach followed and was seen to be very successful phenomenologically due to its generality reflected in the large number of parameters that must be fit to data. A good effective model would be one that retained much of the simplicity and all of the desired features of the linear \( \sigma \)-model, but with an increased number of parameters that assumes less about the short scale physics and that incorporate important large-scale phenomenology due to the presence of heavier mesons.

Quantum Hadrodynamics 3 (QHD-III) [2] is such a model which retains all the features of the linear \( \sigma \)-model but includes additional mesonic fields that are in-
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introduced in a gauge-invariant way. The starting point for building the QHD-III lagrangian is the linear $\sigma$-model lagrangian given in Eq. (1.25). The first thing to note is that this lagrangian has an additional symmetry: it is invariant under the following strong $U(1)$ transformations:

$$
\psi'_L = e^{i\theta}\psi_L \quad \psi'_R = e^{i\theta}\psi_R \quad U(1) \text{ transformation} \quad (1.38)
$$

with all other fields invariant.\(^{11}\) One can make the linear $\sigma$-model lagrangian locally invariant under $U(1)$ transformations\(^{12}\) by introducing covariant derivatives:

$$
\partial_{\mu}\psi_{LR} \rightarrow (\partial_{\mu} + ig_{\omega}\partial_{\mu})\psi_{LR} \equiv D_{\mu}\psi_{LR} \quad (1.39)
$$

The new field $\omega_{\mu}$ has the following transformation property under $U(1)$: $\omega_{\mu} \rightarrow \omega_{\mu} + 1/g_{\omega}\partial_{\mu}\theta$. $\omega_{\mu}$ has the same parity and charge conjugation quantum numbers as the photon, and must therefore be massive since no massless meson with these quantum numbers exists. Since it is not phenomenologically necessary to retain this strong $U(1)$ gauge invariance\(^{13}\), one can add a mass term by hand which breaks the gauge invariance\(^{14}\). Substituting in the covariant derivatives (1.39) into (1.25) and adding kinetic energy and mass terms for the $\omega_{\mu}$ field leads to the lagrangian:

$$
\mathcal{L} = \frac{i(\bar{\psi}_R\gamma_{\mu}D^\mu\psi_R + \bar{\psi}_L\gamma_{\mu}D^\mu\psi_L)}{2} - \sqrt{2}g_{\omega}(\bar{\psi}_L\chi^\dagger\psi_R + \bar{\psi}_R\chi\psi_L) + \frac{1}{2}\text{tr}(\partial_{\mu}\chi^\dagger\partial^{\mu}\chi) + \frac{v^2\lambda}{2}\text{tr}(\chi^\dagger\chi) - \frac{\lambda}{4}\text{tr}(\chi^\dagger\chi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_{\omega}\omega_{\mu}\omega^\mu \quad (1.40)
$$

\(^{11}\)The pion field is not invariant under electromagnetic $U(1)$ transformations.
\(^{12}\)The transformation is said to be local when the parameter $\theta$ depends on $x$.
\(^{13}\)The QCD lagrangian (1.16) is also invariant under the transformation (1.38) with the nucleon field replaced by the quark fields; in QCD, it represents quark baryon number conservation and corresponds to a good quantum number. The QHD-III $U(1)$ local invariance is unrelated to baryon number.
\(^{14}\)The model is still renormalizable because the $\omega_{\mu}$ couples only to the conserved baryon current.
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where

\[ F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \]  

(1.41)

is the field strength tensor. This \( \omega_\mu \) can be identified with the physical meson of the same name which is phenomenologically important in strongly interacting physics. It was first introduced in theoretical nuclear physics by Nambu [51]. Since the \( \omega_\mu \) field is invariant under \( SU(2)_L \times SU(2)_R \) transformations, the lagrangian (1.40) retains this global symmetry of the linear \( \sigma \)-model.

This strong \( SU(2)_L \times SU(2)_R \) symmetry can now also be made locally invariant by letting the transformation parameters \( \theta_{LR} \) of Eq. (1.26) depend on \( x \). This local invariance allows the introduction of left and right gauge bosons in a minimal fashion through the covariant derivatives of the fields:

\[
D_\mu \psi_L \equiv (\partial_\mu + ig_\omega \omega_\mu + \frac{i}{2} G \tau \cdot l_\mu) \psi_L \\
D_\mu \psi_R \equiv (\partial_\mu + ig_\omega \omega_\mu + \frac{i}{2} G \tau \cdot r_\mu) \psi_R \\
D_\mu \chi \equiv \partial_\mu \chi + \frac{i}{2} G (\tau \cdot r_\mu) \chi - \frac{i}{2} G \chi (\tau \cdot l_\mu) 
\]  

(1.42)

The requirement that the lagrangian be invariant under parity\(^{15} \) forces the couplings to \( l_\mu \) and \( r_\mu \) to be identical.

It is now necessary to give mass to the \( l_\mu \) and \( r_\mu \) fields which will eventually represent the \( \rho \) and \( a_1 \) mesons. The masses could be inserted by hand, but that would violate \( SU(2)_L \times SU(2)_R \) gauge invariance and current conservation. Instead, in this model, the masses are generated in a gauge-invariant way via a Higgs mechanism [23] similar to what is done in the standard model.

\(^{15}\)Strong interactions are invariant under parity.
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The Higgs sector in QHD-III is composed of eight fields that come from two complex doublets: one doublet that transforms under $SU(2)_L$ while being invariant under $SU(2)_R$ transformations, and vice versa:

$$\phi_R \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_R, \quad \phi_L \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_L$$

(1.43)

Covariant derivatives are introduced that couple to the $l_\mu$ and $r_\mu$ gauge fields:

$$D_\mu \phi_L \equiv (\partial_\mu + \frac{i}{2} G^a \cdot l_\mu) \phi_L$$
$$D_\mu \phi_R \equiv (\partial_\mu + \frac{i}{2} G^a \cdot r_\mu) \phi_R$$

(1.44)

The Higgs lagrangian is given by:

$$\mathcal{L}_H = [\mathcal{D}_\mu \phi_R] (\mathcal{D}^\mu \phi_R) + [\mathcal{D}_\mu \phi_L] (\mathcal{D}^\mu \phi_L) + \frac{\mu_H^2}{2} (\phi_R^\dagger \phi_R + \phi_L^\dagger \phi_L)$$

$$-\frac{\lambda_H}{4} ([\phi_R^\dagger \phi_R]^2 + [\phi_L^\dagger \phi_L]^2)$$

(1.45)

When $\mu_H^2 > 0$, this lagrangian allows for SSB. The couplings between the gauge bosons and the Higgs fields generate mass terms for the left and right fields. Since the purpose of the Higgs sector is to generate mass terms for the gauge bosons, terms that are invariant under $SU(2)_L \times SU(2)_R$ and that directly couple the Higgs fields to the $(\sigma, \pi)$ sector are not included in the QHD-III lagrangian for the sake of simplicity.

After substituting in the covariant derivatives (1.42) into the lagrangian (1.40) and including the Higgs sector after SSB, the following QHD-III lagrangian is generated

$$\mathcal{L}_{III} = \mathcal{L}_N + \mathcal{L}_{\sigma-\omega} + \mathcal{L}_K + \mathcal{L}_H$$

(1.46)
In Eq. (1.46) $L_N$ is the nucleon lagrangian:

$$L_N = ar{\psi} \{ i \gamma^\mu [\partial_\mu + i g_\rho V_\mu + \frac{i}{2} g_\rho \tau \cdot (\rho_\mu + \gamma_5 a_\mu)] - (M - g_\rho \sigma) - i g_\pi \gamma_5 \tau \cdot \pi' \} \psi,$$  

(1.47)

where $g_\rho$ is the vector meson-nucleon coupling constant introduced earlier in Eq. (1.35) and the $a_\pi$ represents the $a_1$ meson.\(^{17}\) The meson lagrangian is given by:

$$L_{\sigma-\omega} = \frac{1}{2} \left[ (\partial_\mu \pi' + g_\rho \sigma a_\mu + g_\rho \pi' \times \rho_\mu)^2 - m_\sigma^2 \pi' \cdot \pi' \right]
+ \frac{1}{2} \left[ (\partial_\mu \sigma - g_\rho \pi' \cdot a_\mu)^2 - m_\sigma^2 \sigma^2 \right]
- g_\rho \sigma a_\mu \cdot (\partial_\mu \pi' + g_\rho \sigma a_\mu + g_\rho \pi' \times \rho_\mu)
+ \frac{m_\sigma^2 - m_\pi'^2}{2a_\omega} \sigma (\sigma^2 + \pi'^2) - \frac{m_\sigma^2 - m_\pi'^2}{8a_\omega^2} (\sigma^2 + \pi'^2)^2. \quad (1.48)$$

We note the presence of a bilinear term which couples the $a_1$ field directly to the $\pi'$ field. It is also noted that the kinetic energies of the $\pi'$ and the $\sigma$ fields involve the covariant derivatives:

$$\left( \partial_\mu \pi' + g_\rho \sigma a_\mu + g_\rho \pi' \times \rho_\mu \right)^2 \quad \text{and} \quad \left( \partial_\mu \sigma - g_\rho \pi' \cdot a_\mu \right)^2 \quad (1.49)$$

These are just the covariant derivatives introduced in Eqs. (1.35) with the substitutions $\pi'^0 \to U^i$ and $\sigma \to -U^0$.

The gauge boson kinetic energy lagrangian is given by:

$$L_K = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} m_\omega^2 V_\mu V^\mu - \frac{1}{4} R_{\mu \nu} \cdot R^{\mu \nu} + \frac{1}{2} m_\rho^2 \rho_\mu \cdot \rho^\mu
- \frac{1}{4} A_{\mu \nu} \cdot A^{\mu \nu} + \frac{1}{2} (m_\rho^2 + g_\rho^2 a_\mu^2) a_\mu \cdot a^\mu. \quad (1.50)$$

\(^{17}\)This lagrangian is expressed in terms of the physical fields and coupling constants where the following relations hold: $G = \sqrt{2} g_\rho$, $a_\mu \equiv \frac{1}{\sqrt{2}} (r_\mu - l_\mu)$ and $\rho_\mu \equiv \frac{1}{\sqrt{2}} (r_\mu + l_\mu)$. The quantum numbers of the $a_\mu$ and $\rho_\mu$ are $(1^+,1)$ and $(1^-,1)$ respectively.
The vector meson field strength are given by:

\[ A_{\mu\nu} \equiv \partial_{\mu}a_{\nu} - \partial_{\nu}a_{\mu} - g_{\rho}(\rho_{\mu} \times a_{\nu} + a_{\mu} \times \rho_{\nu}) \]
\[ R_{\mu\nu} \equiv \partial_{\mu}\rho_{\nu} - \partial_{\nu}\rho_{\mu} - g_{\rho}(\rho_{\mu} \times \rho_{\nu} + a_{\mu} \times a_{\nu}) \]  

(1.51)

We see from the above that the mass of the \( a_1 \) comes out naturally bigger than the mass of the \( \rho \). This is so because the spontaneous symmetry breaking in the \( \sigma \) sector gives the square of the \( a_1 \) mass term a contribution \( g_\sigma^2 \sigma^2 \) on top of the mass contribution coming from the Higgs sector. The final Higgs lagrangian is:

\[
L_H = \frac{1}{2}(\partial_\mu\eta\partial^\mu\eta - m_H^2\eta^2) + \frac{1}{2}(\partial_\mu\zeta\partial^\mu\zeta - m_H^2\zeta^2) + \left[ g_\rho m_\rho \eta + \frac{1}{4} g_\rho^2 (\eta^2 + \zeta^2) \right] (\rho_\mu \cdot \rho^\mu + a_\mu \cdot a^\mu) \\
+ (g_\rho m_\rho \zeta + \frac{1}{2} g_\rho^2 \eta \zeta) \rho_\mu \cdot a^\mu \\
- \left( \frac{3 m_H^2 g_\rho^2}{4 m_\rho} \eta + \frac{3 m_H^2 g_\rho^2}{16 m_\rho^2} \eta^2 \right) \zeta^2 - \frac{m_H^2 g_\rho^2}{4 m_\rho} \eta^3 - \frac{m_H^2 g_\rho^2}{32 m_\rho^2} (\eta^4 + \zeta^4) 
\]  

(1.52)

where:

\[
\mu_H^2 = \frac{1}{2} m_H^2, \quad u^2 = \frac{8 \mu_H^2}{\lambda_H} = \frac{4 m_\rho^2}{g_\rho^2}, \quad \lambda_H = \frac{m_H^2 g_\rho^2}{m_\rho^2}. 
\]  

(1.53)

From the eight original Higgs fields, only two have survived: the scalar \( \eta \) and the pseudoscalar \( \xi \). The other six fields were "eaten" by the six gauge bosons to give them their longitudinal polarization states.

To remove the bilinear term in Eq. (1.48), the following change of variables is needed:

\[
a_\mu \rightarrow a_\mu + \frac{g_\rho \sigma_5}{m_\rho^2} \partial_\mu \pi', \quad \pi' = \frac{m_\pi}{m_\rho} \pi, \quad \pi'' = \frac{m_\pi}{m_\rho} m_\pi. 
\]  

(1.54)

The resulting lagrangian after the change of variables and its corresponding Feynman
rules are given in appendix A.\textsuperscript{18} One might wonder about the need to introduce the \( a_1 \) field in a model of low-energy strong interactions since its role in nuclear physics is minimal. The role of the \( a_1 \) in this model is to make the model invariant under chiral symmetry.

The gauge invariance of the QHD-III lagrangian limits the number of mesons in the model while incorporating the most phenomenologically important vector mesons. The model allows the derivation of conserved vector and partially conserved axial-vector currents. The model is globally (as well as locally) invariant under a spontaneously-broken \( SU(2)_L \times SU(2)_R \) symmetry. Finally, the model allows the calculation of corrections stemming from meson loops and will be shown to be renormalizable to the order worked to in this thesis. Thus, the construction of a model that is consistent with phenomenology and the low-energy symmetries of QCD has been achieved.\textsuperscript{19}

1.4 Outline and results

In this thesis, only the meson sector of QHD-III is considered for the sake of simplicity. In this way, dealing with axial anomalies is also avoided. To investigate the model, it is of interest to look at three things: 1) the simplest physical processes calculated to one-loop, 2) the exploitation of the gauge invariance of QHD-III which distinguishes it from other massive Yang-Mills models and 3) its one-loop effective action. We highlight the principal new results with bullets.

Ever since it was proved that QCD reduces to a theory of non-linearly, self-

\textsuperscript{18}A simpler representation of the QHD-III lagrangian is given in Chapter 3 of this thesis.

\textsuperscript{19}This model also contains anomalies in the baryon sector. These axial anomalies can be removed; they are discussed in detail in [2] and are only mentioned here for completeness.
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interacting mesons in the large $N_c$ limit in the 1970's [52, 53, 54], interest in hadronic theories was revived. For QHD-III in particular, this development shows that a loop expansion in the mesonic sector can be interpreted as a $1/N_c$ expansion, and is non-perturbative. Thus the first part of this thesis is concerned with the evaluation of the one-loop corrections to the tree level results of the two simplest processes: the amplitudes for $\pi\pi$ scattering and pion decay are calculated to one-loop. The first part of this thesis is essentially composed of the paper [55] that appeared in Physical Review C in September 1998.

First, the invariant $\pi\pi$ scattering amplitude $M_{\pi\pi\to\pi\pi}$ is expanded in a Feynman diagram series to 1-loop in the linear $\sigma$-model. Twenty diagrams are evaluated to $O(g^4)$ in the chiral limit, $m_\pi = 0$, with $m_\sigma^2 \gg s$.

- To renormalize this scattering amplitude, the divergent parts of the counterterms $\delta_\mu$, $\delta_\lambda$ and $\delta_z$ are extracted from the $\sigma$, $\sigma^2$ and $\sigma^3$ vertex functions, and it is shown that these counterterms cancel all the divergences in $M_{\pi\pi\to\pi\pi}$ to $O(g^4)$, as well as the divergences in the $\sigma$ 4-point function as expected in a renormalizable theory.

The result obtained in the isospin t-channel with $M_{ac,bd} = \delta_{ac}\delta_{bd} M$ is:

$$M = \left[ \beta t + \alpha_1 t^2 + \alpha_2 (s^2 + u^2) \right]$$

$$+ \frac{1}{F^4} \frac{1}{16\pi^2} \left[ - \frac{t^2}{2} \ln \frac{-t}{m_\sigma^2} - \frac{1}{12} (3s^2 + u^2 - t^2) \ln \frac{-s}{m_\sigma^2} - \frac{1}{12} (3u^2 + s^2 - t^2) \ln \frac{-u}{m_\sigma^2} \right],$$

(1.55)

\[^{20}\ln M, s, t, u \text{ are the usual Mandelstam variables.}\]
CHAPTER I. INTRODUCTION

where:

\[
\beta \equiv \frac{1}{F^2} \left\{ 1 - \delta_z - \frac{3}{16\pi^2} \frac{m_\sigma^2}{F^2} \left[ \Gamma\left(\frac{\epsilon}{2}\right) + \ln 4\pi - \ln \frac{m_\sigma^2}{\mu^2} + \frac{7}{6} \right] + 2(\delta_\mu + \delta_\lambda) \right\}, \tag{1.56}
\]

\[
\alpha_1 \equiv -\frac{1}{F^4} \frac{1}{16\pi^2} \frac{49}{18} + \frac{2}{m_\sigma^2 F^2} [2\delta_\mu + \delta_\lambda], \tag{1.57}
\]

\[
\alpha_2 \equiv -\frac{1}{F^4} \frac{1}{16\pi^2} \frac{2}{9},
\]

\[
\delta_\mu \equiv -\frac{3}{2} \frac{g_\rho^2}{M^2} \frac{m_\sigma^2}{16\pi^2} \Gamma\left(\frac{\epsilon}{2}\right); \quad \delta_\lambda \equiv -2\delta_\mu; \quad \delta_z \text{ finite.} \tag{1.58}
\]

The tree level pion decay constant is given by:

\[
F' \equiv \sigma_0 \equiv \frac{M}{g_\pi} \tag{1.59}
\]

It is noted that:

- the calculation of \( \mathcal{M} \) was repeated started from the chirally transformed \( \sigma \)-model and its very different Feynman rules and interactions, and identical results were obtained.

The \( \mathcal{O}(g_\rho^2, g_\sigma^2) \) corrections to \( \mathcal{M}_{\pi \pi \rightarrow \pi \pi} \) due to the exchange of vector mesons are next calculated within the context of QHD-III in the chiral limit with \( m_\sigma^2 \gg m_\rho^2 \gg s \).

- All divergences are cancelled and the model is renormalized with the counterterms modified by the \( g_\rho^2 \) corrections to the 1-loop diagrammatic expansions of the \( \sigma, \sigma^2 \) and \( \sigma^3 \) vertex functions.

\(^{21}\)The heavy \( \sigma \) field of the linear \( \sigma \)-model and QHD-III is different from the light phenomenological \( \sigma \) used in \( NN \) meson exchange models discussed at the beginning of this chapter.
The calculated corrections are:

\[
\delta \beta = \frac{g_\rho^2}{m_\rho^2} \left( 1 + \frac{1}{16\pi^2} \frac{m_\rho^2}{F^2} \left\{ -12 + 6 \frac{m_\rho^2}{m_\sigma^2} \ln \frac{m_\sigma^2}{m_\rho^2} \\
+ 9 \frac{m_\rho^2}{m_\sigma^2} \left[ \Gamma \left( \frac{\epsilon}{2} \right) + \ln 4\pi - \ln \frac{m_\sigma^2}{\mu^2} + \frac{131}{162} \right] \right\} \\
+ 2 \frac{g_\rho^2}{m_\rho^2} \delta_g + \left( 8 \frac{g_\rho^2}{m_\rho^2} \frac{1}{\sigma_0^2} \right) \delta_\tau + \frac{2}{\sigma_0^2} (\delta_\mu + \delta_\lambda), \right. (1.60)
\]

\[
\delta \alpha_1 = - \frac{1}{F^2} \frac{g_\rho^2}{m_\rho^2} \frac{6}{16\pi^2} \left[ \Gamma \left( \frac{\epsilon}{2} \right) + \ln 4\pi - \ln \frac{m_\sigma^2}{\mu^2} - \frac{85}{108} \right] \\
- 8 \frac{g_\rho^2}{m_\rho^2} \delta_g + 2 \frac{g_\rho^2}{m_\rho^2} \frac{\nu_\mu}{m_\rho^2} - 4 \frac{g_\rho^2}{m_\rho^2} \frac{\nu_\lambda}{m_\rho^2} + \frac{2}{m_\rho^2} (2\delta_\mu + \delta_\lambda), \quad (1.61)
\]

\[
\delta \alpha_2 = \frac{1}{F^2} \frac{g_\rho^2}{m_\rho^2} \frac{1}{16\pi^2} \left[ \frac{26}{9} + \ln \frac{m_\sigma^2}{m_\rho^2} \right] + 2 \frac{g_\rho^2}{m_\rho^2} \delta_g + \frac{3}{2} \left( g_\rho^2 - \frac{m_\sigma^2}{\sigma_0^2} \right) \frac{\Gamma \left( \frac{\epsilon}{2} \right)}{16\pi^2}; \quad \delta_\mu \doteq -2\delta_\lambda; \quad \delta_\tau \doteq \frac{8}{2} \frac{g_\rho^2 \Gamma \left( \frac{\epsilon}{2} \right)}{16\pi^2}; \quad \delta_g finite. \quad (1.62)
\]

- To 1-loop, it is evident that the \( g_\rho^2 \) corrections become negligible with respect to the result (1.55) in the limit \( m_\rho \to \infty \), and that the gauge bosons decouple.

Note that QHD-III reduces to the linear \( \sigma \) model when \( g_\rho = 0 \). This presentation of results for the 1-loop expansion of \( M_{\pi\pi} \rightarrow \pi \pi \) is followed by an analysis of pion decay.

The axial current matrix element \( \langle 0 | A^i_\mu | \pi^i \rangle \) in the \( \sigma \)-model is calculated to 1-loop; from it, the pion decay constant \( F_\pi \) is identified. The tree level result is simply the VEV of the scalar field, \( \sigma_0 \). The 1-loop corrections to \( F_\pi \) in the linear \( \sigma \)-model are:

\[
F_\pi = \sigma_0 \left\{ 1 + \frac{\delta_\tau}{2} + \frac{3}{32\pi^2} \frac{m_\rho^2}{\sigma_0^2} \left[ \Gamma \left( \frac{\epsilon}{2} \right) + \ln 4\pi - \ln \frac{m_\sigma^2}{\mu^2} + \frac{7}{6} \right] - (\delta_\mu + \delta_\lambda) \right\}. \quad (1.63)
\]

- The renormalization of the pion decay constant is accomplished by using the same counterterms evaluated to \( O(g_\rho^4) \) for \( \pi \pi \) scattering, and it is seen that the
coefficient of the tree-level term in $\mathcal{M}_{\pi\pi\rightarrow\pi\pi}$ is the inverse square of the pion decay constant to this order:

$$\beta \equiv \frac{1}{F_\pi^2}$$  \hspace{1cm} (1.64)

The matrix element $\langle 0|A^\mu_\pi|\pi^I \rangle$ is next considered in QHD-III so as to identify $F_\pi$ in that model.

- At tree-level, the pion decay constant of the $\sigma$ model is replaced by $1/F_\pi^2 = 1/\sigma_0^2 + g_\rho^2/m_\rho^2$.

- To next order, it is found that the 1-loop corrections violate local current conservation in this model; this matrix element to 1-loop is therefore neither gauge-invariant nor renormalizable: to that order, it is not an S-matrix element of the theory.

Thus the model can at best provide a phenomenological description of the strongly interacting nuclear sector. In the process of performing these calculations, the QHD-III counterterm lagrangian with coefficients $\{\delta_2, \delta_\mu, \delta_\lambda, \delta_\rho, \ldots\}$ is derived as well as all the Feynman rules.

In Chapter 3 of this thesis, an in-depth analysis of the gauge invariance of QHD-III is performed. The content is essentially the same as what appeared in Physical Review C in April 99 [56]. Chapter 3 begins by discussing the $SU(2)_L \times SU(2)_R$ locally invariant lagrangian (1.40). The gauge bosons, the $a_1$ and the $\rho$, are originally massless. This toy model is different from QHD-III since no mass terms are introduced nor generated via a Higgs sector. It is shown, however, that the SSB
that occurs in the linear $\sigma$-model still generates a mass for the $a_1$. Working in an arbitrary $\xi$ gauge, it is then shown that the field originally identified with the pion acquires a $\xi$-dependent mass, which identifies it as a fictitious particle; the field that was originally identified with the pion field is labeled as $\pi'$ to distinguish it from the physical pion. Looking at nucleon-nucleon scattering indeed demonstrates that the $\pi'$ exchange diagram is always canceled by the $\xi$ dependent part of the $a_1$ exchange diagram.

- The disappearance of the real pion in the gauged $\sigma$ model is forced by gauge invariance and demonstrates the need for massive gauge bosons with mass provided from outside this sector of the theory. Chapter 3 contains an independent illustration of that fact.

In QHD-III, pions appear as the physical Goldstone bosons. As described in the previous section, the vector meson masses are generated from a Higgs sector, and the model is gauge-invariant.

- The gauge invariance is used to diagonalize the lagrangian in an arbitrary $\xi$ gauge so as to avoid the original, momentum-dependent, diagonalization procedure used in [2].

This new diagonalization scheme produces a considerably simpler representation of QHD-III than the representation given in [2]. It is shown how, to $O(g_s^2)$, the new diagonalization procedure is equivalent to simply rescaling the pion field by the ratio $m_\rho/m_\sigma$ instead of the momentum-dependent change of variables (1.54).

In Chapter 4 of this thesis, the 1-loop effective action of the linear $\sigma$-model and
QHD-III is discussed. In this section, the background field method \cite{45, 57, 58, 59} is used to calculate all pionic matrix elements of the lagrangians to 1-loop. In this method, quantum fluctuations (1-loop corrections) about a classical field configuration are extracted:

\begin{equation}
\phi^i(x) = \bar{\phi}^i(x) + \delta \phi^i(x)
\end{equation}

where $\phi^i$ is a generic field standing in for all the fields in the model (isoscalars, isovectors, vector bosons, etc), $\bar{\phi}^i$ are the corresponding background, classical fields and $\delta \phi^i$ are the quantum fluctuations. The background fields satisfy the equations of motion; using these equations, all the heavy fields can be expressed in terms of the pion field and its derivatives. Once all the fields have been expanded about their classical solutions as in Eq. (1.65), the path integral over all the fluctuations is performed. The path integral over the fluctuations leads to an effective action of the form:

\begin{equation}
S = \int d^4 x [\bar{\mathcal{L}} + \Delta \mathcal{L}]
\end{equation}

$\bar{\mathcal{L}}$ is the original lagrangian expressed in terms of the classical fields while $\Delta \mathcal{L}$ are the new terms that arise after the path integral is performed; $\Delta \mathcal{L}$ is then expanded in powers of the momentum to $\mathcal{O}(p^4)$.

- Because of the complete generality of the $\mathcal{O}(p^4)$ chiral lagrangian of Eq. (1.36), the momentum expansion of $\Delta \mathcal{L}$ should only yield terms of the form that appear in Eq. (1.36). Indeed, this is seen to be the case in the 1-loop effective action evaluation of both the linear $\sigma$-model and QHD-III lagrangians.

Chapter 4 begins with a description of the background field method.
CHAPTER I. INTRODUCTION

- The 1-loop effective action of the linear $\sigma$-model is calculated in an arbitrary renormalization scheme.

To do this, the effective action of the linear $\sigma$-model is shown to be separable into the effective action of the non-linear $\sigma$-model and the corrections coming from virtual $\sigma$ exchange. The non-linear $\sigma$-model effective action to 1-loop is re-derived and verifies the result obtained in [43]. The corrections coming from $\sigma$ exchange in an arbitrary renormalization scheme is then calculated with the following results for the parameters appearing in Eqs. (1.32) and (1.36):

\[
\frac{F_\pi^2}{2} = \frac{\sigma_\pi}{2} + \frac{3m_\pi^2}{2(4\pi)^2}(\bar{\lambda} + \frac{7}{6})
\]

\[
F_{\pi \chi_{ab}}^2 = F_\pi^2 \chi + \frac{3m_\pi^2}{2(4\pi)^2}(\bar{\lambda} + 1)\chi
\]

\[
l_1 = \frac{1}{4g} + \frac{1}{(4\pi)^2} \left( \frac{17}{6} \bar{\lambda} + \frac{67}{36} \right)
\]

\[
l_2 = -\frac{1}{(4\pi)^2} \left( \frac{\bar{\lambda}}{3} + \frac{17}{18} \right)
\]

\[
l_3 = -\frac{1}{2g} - \frac{1}{(4\pi)^2} \left( \frac{11}{4} \bar{\lambda} + \frac{19}{12} \right)
\]

\[
l_4 = \frac{1}{2g} + \frac{1}{(4\pi)^2} \left( 5\bar{\lambda} + \frac{3}{2} \right)
\]

\[
l_5 = \frac{1}{(4\pi)^2} \left( \frac{\bar{\lambda}}{12} + \frac{5}{72} \right)
\]

\[
l_6 = \frac{1}{(4\pi)^2} \left( \frac{\bar{\lambda}}{6} + \frac{17}{36} \right)
\]

\[
l_7 = 0
\]

\[
h_1 = l_4 + \frac{1}{(4\pi)^2} \frac{1}{12}
\]

\[
h_2 = 0
\]
\[ h_3 = 0 \]  

(1.67)

where \( \lambda \) contains the divergences:

\[ \lambda \equiv \Gamma\left(\frac{\epsilon}{2}\right) + \ln 4\pi - \ln \frac{m^2}{\mu^2} \]  

(1.68)

There are also unitary log terms that give rise to the logs in Eq. (1.55). This result is shown to be equivalent to the result previously obtained by Gasser and Leutwyler with their renormalization condition [43].

- From this 1-loop effective action, the pion decay constant to 1-loop is extracted from the first equation of (1.67) and is shown to be identical to Eq. (1.63) obtained using Feynman diagrams. The \( \pi\pi \) scattering amplitude to \( \mathcal{O}(g_s^4) \) was also calculated and shown to be identical to the result previously obtained using Feynman diagrams [55];

The 1-loop effective action of the QHD-III lagrangian is next set up. The operators appearing in \( \Delta\mathcal{L} \) are calculated and the heavy fields are expressed in terms of the pion field using their equations of motion.

- The tree level result for the pion decay constant is calculated and shown to be identical to the result stemming from a Feynman diagram calculation.

- The 1-loop result for the \( \mathcal{O}(p^2) \) term of the effective action is shown to be renormalized using the counterterms calculated in the Feynman diagram expansion.

The other 1-loop terms fail to be renormalized however. The calculation is a long and difficult one and the origin of the failure is still under investigation.

\[ \text{\[22\]} \text{The term } l_4 \text{ differs by } 1/(2(4\pi)^2) \text{ from the result quoted in [43]. I believe my result to be correct.} \]
In summary, this model has been shown to possess the following features required of a good model of low-energy strong interactions: it is invariant under the symmetries of low-energy QCD, including chiral symmetry; the model allows the derivation of conserved vector currents and partially conserved axial-vector currents; this model has a minimal set of parameters from which corrections stemming from meson loops can be calculated as seen in Chapter 2 of this thesis. In particular, this gauge-invariant model is simpler than other massive Yang-Mills theories which require a momentum-dependent change of variables which is avoided in QHD-III as seen in Chapter 3 of this thesis. Lastly, this model also allows the prediction of parameters appearing in chiral phenomenological lagrangians, including corrections coming from vector meson exchanges as discussed in Chapter 4.
Chapter 2

Pion-Pion Scattering and Pion Decay in QHD

[What follows is a reproduction of the published paper [55].] QCD is very successful at describing hadronic interactions at high $Q^2$ where perturbation theory is applicable. At low $Q^2$ however, non-perturbative methods must be used and most quantitative predictions can not be extracted directly from QCD. Models that are designed to describe low $Q^2$ hadronic interactions must be guided by the symmetries of QCD and phenomenology. It is a fact that QCD possesses global $SU(2)_L \times SU(2)_R$ symmetry and that large scale processes must involve meson loops because of confinement. Phenomenology also reveals the existence of conserved vector and (partially) conserved axial-vector currents. QHD-III [2] is a relativistic quantum field theory that incorporates these features: in this model, hadrons are the effective degrees of freedom and the vector mesons that couple to these currents, the $\rho$ and the $a_1$, are introduced as the gauge bosons of a local $SU(2)_L \times SU(2)_R$ $\sigma - \omega$ model with pions (see below). The vector mesons are made massive via the Higgs mechanism (although the Higgs scalars do not contribute to the order considered in this paper)
and the model is renormalizable.

A further motivation for QHD-III is the fact that simpler versions of QHD based on \( \{N; \sigma, \omega\} \) and \( \{N; \sigma, \omega, \pi\} \) have had significant phenomenological success [1]; hence, it is of interest to see how far this description can be extended. A similar model with vector mesons was developed in [60]. For models with vector mesons in nonlinear chiral lagrangians, see [61, 5, 62, 63, 64]. For early work on the subject, see [69]. The consequences of the present model have not yet been explored. In the final analysis, the currents of QHD-III allow the theoretical exploration of strongly interacting systems and processes, while incorporating meson loop corrections.

This paper calculates to 1-loop the two simplest amplitudes in the meson sector of this model (the baryon sector is not included in this initial investigation): \( \pi \pi \) scattering and pion decay. First, the invariant \( \pi \pi \) scattering amplitude \( M_{\pi \pi \to \pi \pi} \) is calculated to 1-loop to \( O(g^4) \) and to \( O(g^2 g^2_\rho) \) in the limit \( m_\sigma^2 \gg m_\rho^2 \gg s \) with \( m_\pi = 0 \). To renormalize this scattering amplitude, the divergent parts of the counterterms are extracted from the \( \sigma, \sigma^2 \) and \( \sigma^3 \) vertex functions, and it is shown that these counterterms cancel all the divergences in \( M_{\pi \pi \to \pi \pi} \) to \( O(g^4) \) and to \( O(g^2 g^2_\rho) \), as well as the divergences in the \( \sigma \) 4-point function as expected in a renormalizable theory. To 1-loop, it is also argued that the gauge bosons decouple in the limit \( m_\rho \to \infty \).

Note that QHD-III reduces to the linear \( \sigma \) model when \( g_\rho = 0 \). Second, pion decay is analysed by looking at the axial current matrix element \( \langle 0 | A_{\mu}^i | \pi^j \rangle \) in the \( \sigma \) model to 1-loop; from it, the pion decay constant is identified. To renormalize the pion decay constant, the same counterterms evaluated to \( O(g^4) \) are used, and it is verified that the coefficient of the tree-level term in \( M_{\pi \pi \to \pi \pi} \) is the inverse square of the pion decay constant to this order. The matrix element \( \langle 0 | A_{\mu}^i | \pi^j \rangle \) is next considered in QHD-III.
so as to identify the pion decay constant in that model. At tree-level, the pion decay constant of the \( \sigma \) model is replaced by \( 1/F_\pi^2 = 1/\sigma_s^2 + g_\rho^2/m_\rho^2 \). To next order, it is found that the 1-loop corrections violate local current conservation in this model; this matrix element to 1-loop is therefore neither gauge-invariant nor renormalizable: to that order, it is not an \( S \)-matrix element of the theory. Thus the model can at best provide a phenomenological description of the strongly interacting nuclear sector. In the process of performing these calculations, the QHD-III counterterm lagrangian with coefficients \( \{ \delta_2, \delta_\mu, \delta_\lambda, \delta_\sigma, \ldots \} \) is derived.

2.1 QHD-III

The model is constructed as follows: start with the \( \sigma - \omega \) model with pions [1] (we use the conventions in [2]):

\[
\mathcal{L} = \bar{\psi}[i\gamma^\mu(\partial_\mu + ig_VV_\mu) - g_\pi(s + i\gamma_5\tau\cdot\pi)]\psi + \frac{1}{2}(\partial_\mu s\partial^\mu s + \partial_\mu \pi\partial^\mu \pi)
- \frac{1}{4}\lambda(s^2 + \pi^2 - v^2)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_\rho^2V_\mu V^\mu + \epsilon_\pi + \mathcal{L}_{ct},
\]

where \( \epsilon_\pi \) is the chiral symmetry violating term and \( \mathcal{L}_{ct} \) is the counterterm lagrangian. The QHD-III lagrangian is constructed as follows (details are given in [2]): this lagrangian is first made locally invariant under \( SU(2)_L \times SU(2)_R \); this results in the appearance of the \( l_\mu \) and \( r_\mu \) gauge bosons coupled to conserved currents. These bosons are given mass through the Higgs mechanism. The mass matrix is then diagonalized and the \( l_\mu \) and \( r_\mu \) fields are replaced by the new generalized coordinates, the \( \rho_\mu \) and \( a_\mu \). The \( O(4) \) symmetry is spontaneously broken by giving the scalar field a vacuum expectation value \( s = \sigma_0 - \sigma \) with \( \sigma_0 \equiv M/g_\sigma \). This in turn yields a bilinear term in the lagrangian that must now be diagonalized by redefining the
pion and $a_1$ field. The end result for the meson sector is:

$$\mathcal{L}_\sigma = \frac{1}{2} \left[ \left(1 - \frac{m_a^2}{m_p^2}\right) \partial_\mu \pi \partial^\mu \pi + \left(\frac{m_a}{m_p} \partial_\mu \pi + g_\rho \sigma a_\mu + g_\rho \pi \times \rho_\mu\right)^2 \right] + \frac{1}{2} \left[ (\partial_\mu \sigma - g_\rho \frac{m_a}{m_p} \pi \cdot a_\mu)^2 - m_\sigma^2 \sigma^2 \right] - g_\sigma^2 \sigma \cdot a_\mu \cdot (\sigma a_\mu + \frac{m_a}{m_p} \pi \times \rho_\mu) + g_\sigma \frac{m_\pi^2}{2M} \left( \sigma^2 + \frac{m_\pi^2}{m_p^2} \right) - g_\sigma^2 \frac{m_\pi^2}{8M^2} \left( \sigma^2 + \frac{m_\pi^2}{m_p^2} \right)^2 + \mathcal{L}_\sigma'. \quad (2.2)$$

In the above, the $a_1$ mass is given by $m_a^2 = m_p^2 + g_\rho^2 \sigma_0^2$ with $m_p$ the $\rho$ mass and $g_\rho$ the $\rho$-nucleon coupling constant. Eq. (2.2) is referred to as the "diagonalized lagrangian" and reduces to the linear $\sigma$ model when $g_\rho = 0$. $\mathcal{L}_\sigma'$ contains the "new" interactions that appear when the pion and $a_1$ fields are redefined; to $O(g_\rho^2)$ it is given by:

$$\mathcal{L}_\sigma' = \frac{g_\sigma^2 \sigma}{m_p^2} [\sigma \partial_\mu \pi \cdot \partial^\mu \pi - \pi \cdot \partial_\mu \pi \partial^\mu \sigma]. \quad (2.3)$$

### 2.2 Results

The scattering amplitude is considered first to $O(g_\rho^4)$ by putting $g_\rho = 0$ and then to $O(g_\rho^2 g_\sigma^2)$. The loop integrals are done using dimensional regularization in the metric $(+,-,-,-)$, and it is assumed that $m_\pi = \epsilon = 0$ as well as $m_\rho^2 \gg m_\pi^2 \gg s, t, u$ where $s, t$ and $u$ are the Mandelstam variables (for discussions regarding the $m_\sigma \to \infty$ limit of the linear $\sigma$ model, see [1, 43, 65, 66]). To $O(g_\rho^4)$ in the $t$-channel, the amplitude is

$$M_{ac, bd} = M \delta_{ac} \delta_{bd}$$

with:
\[ \mathcal{M} = \left[ \beta t + \alpha_1 t^2 + \alpha_2 (s^2 + u^2) \right] + \frac{1}{F^4} \left[ \frac{1}{16\pi^2} \left[ \frac{\ln \frac{-t}{m^2}}{m^2} - \frac{1}{12} (3s^2 + u^2 - t^2) \ln \frac{-s}{m^2} \right] - \frac{1}{12} (3u^2 + s^2 - t^2) \ln \frac{-u}{m^2} \right] \]

\[ \beta \equiv \frac{1}{F^2} \left\{ 1 - \delta_z - \frac{3}{16\pi^2} \frac{m^2}{F^2} \left[ \Gamma \left( \frac{\xi}{2} \right) + \ln 4\pi - \ln \frac{m^2}{\mu^2} + \frac{7}{6} \right] + 2(\delta_\mu + \delta_\lambda) \right\} \]

\[ \alpha_1 \equiv -\frac{1}{F^4} \frac{1}{16\pi^2} \frac{49}{18} + \frac{2}{m^2 F^2} \left[ 2\delta_\mu + \delta_\lambda \right], \quad \alpha_2 \equiv -\frac{1}{F^4} \frac{1}{16\pi^2} \frac{2}{9}. \]

Here, \( 1/F^2 \equiv 1/\sigma_0^2 \equiv g_\sigma^2/M^2 \), \{\delta_z, \delta_\mu, \delta_\lambda\} are of \( \mathcal{O}(g_\sigma^2) \), and \( \mu \) parametrizes the renormalization conditions. The amplitude \( \mathcal{M} \) does not depend on \( \mu \); the counterterms and the running coupling constant conspire to ensure that. The subscripts are defined in Fig. 2.1 and twenty diagrams contributed to the above result.

Note the first term in \( \beta \) is just the tree-level amplitude. Note also that the unitary corrections in the second line of Eq. (2.4) has the log structure in Mandelstam variables first derived by Lehmann [67]. The divergent parts of the counterterms are obtained by considering the \( \sigma \) 1-point, 2-point and 3-point functions. The result to
1-loop is:

\[
\delta_\mu = - \frac{3}{2} \frac{g_\rho^2}{M^2} \frac{m_\rho^2}{16\pi^2} \Gamma\left(\frac{\xi}{2}\right); \quad \delta_\lambda = -2\delta_\mu,
\]

(2.7)

with \( \delta_z \) finite. One can check by direct substitution that the above counterterms determined from the scalar sector cancel the divergences occurring in the \( \pi \pi \) scattering amplitude. This verifies the counterterms calculated in [23].

For the scattering amplitude to \( \mathcal{O}(g_\rho^2g_\rho^2) \), the gauge bosons from QHD-III contribute another fifty diagrams. The corrections to the parameters \( \beta, \alpha_1 \) and \( \alpha_2 \) are:

\[
\delta \beta = \frac{g_\rho^2}{m_\rho^2} \left( 1 + \frac{1}{16\pi^2} \frac{m_\rho^2}{F^2} \left\{ -12 + 6 \frac{m_\rho^2}{m_\sigma^2} \ln \frac{m_\rho^2}{m_\sigma^2} + 9 \frac{m_\rho^2}{m_\sigma^2} \left( \Gamma\left(\frac{\xi}{2}\right) + \ln 4\pi - \ln \frac{m_\rho^2}{\mu^2} + \frac{131}{162} \right) \right\} \right)
\]

\[
+ 2 \frac{g_\rho^2}{m_\rho^2} \delta_\rho + \left( 8 \frac{g_\rho^2}{m_\rho^2} - \frac{1}{\sigma_\rho^2} \right) \delta_z + \frac{2}{\sigma_\rho^2} (\delta_\mu + \delta_\lambda),
\]

(2.8)

\[
\delta \alpha_1 = - \frac{1}{F^2} \frac{g_\rho^2}{m_\rho^2} \frac{6}{16\pi^2} \left[ \Gamma\left(\frac{\xi}{2}\right) + \ln 4\pi - \ln \frac{m_\rho^2}{\mu^2} - \frac{85}{108} \right]
\]

\[
- 8 \frac{g_\rho^2}{m_\rho^2} \frac{\delta_\rho}{m_\sigma^2} + 2 \frac{g_\rho^2}{m_\rho^2} \delta_z + 4 \frac{g_\rho^2}{m_\rho^2} \frac{\delta_\mu}{m_\sigma^2} + \frac{2}{m_\sigma^2 \sigma_\rho^2} (2\delta_\mu + \delta_\lambda),
\]

(2.9)

\[
\delta \alpha_2 = \frac{1}{F^2} \frac{g_\rho^2}{m_\rho^2} \frac{1}{16\pi^2} \left[ \frac{26}{9} + \ln \frac{m_\rho^2}{m_\sigma^2} \right] + 2 \frac{g_\rho^2}{m_\rho^2} \frac{\delta_\rho}{m_\sigma^2}.
\]

(2.10)

Now, in Eqs. (2.8)-(2.10) and in the coefficient of the log terms in Eq. (2.4),

\[
1/F^2 \equiv 1/\sigma_\rho^2 + g_\rho^2/m_\rho^2 \text{ to } \mathcal{O}(g_\rho^2g_\rho^2) \text{ as required by unitarity. The counterterms become:}
\]

\[
\delta_\mu = \frac{3}{2} \left( g_\rho^2 - \frac{m_\rho^2}{\sigma_\rho^2} \right) \frac{\Gamma\left(\frac{\xi}{2}\right)}{16\pi^2}; \quad \delta_\lambda = -2\delta_\mu; \quad \delta_z = 6g_\rho^2 \frac{\Gamma\left(\frac{\xi}{2}\right)}{16\pi^2}.
\]

(2.11)

The calculation is carried out in the unitary gauge. It is shown that the divergent contributions to amplitudes in the scalar sector are gauge-invariant; this is proved.
using the non-diagonalized lagrangian which maintains explicit current conservation at each step. Here, $\delta_{g_{\rho}}$ can be determined from $\rho$ decay, and it is finite. Note that $\delta_{z}$ has acquired a divergence. Upon substitution of Eqs. (2.11) into Eqs. (2.8) and (2.9), it is found that the amplitude is now finite to $\mathcal{O}(g_{4}^{2})$; note that the $\sigma$ 4-point function is also made finite with these counterterms. Hence, the $\pi\pi$ scattering amplitude is now rid of all infinities (as is the entire scalar sector).

Consider pion decay in the $\sigma$ model. From the axial current, $F_{\pi}$ is calculated to 1-loop to be:

$$F_{\pi} = \sigma_{o} \left\{ 1 + \frac{\delta_{z}}{2} + \frac{3}{32\pi^{2}} \frac{m_{\rho}^{2}}{\sigma_{o}^{2}} \left[ \Gamma\left(\frac{1}{2}\right) + \ln 4\pi - \ln \frac{m_{\rho}^{2}}{\mu^{2}} + \frac{7}{6} \right] - (\delta_{\mu} + \delta_{\lambda}) \right\}. \quad (2.12)$$

First, note that the counterterms given in Eqs. (2.7) cancel the divergences in $F_{\pi}$. Second, notice that to $O(g_{4}^{2})$, $F_{\pi}^{-2}$ is identical to $\beta$ given in Eq. (2.5). This verifies a well known property of the $\sigma$ model in the chiral limit [24].

From the axial current of QHD-III, the tree level pion decay constant is found to be:

$$F_{\pi} = \frac{m_{\rho}}{m_{a}} \sigma_{o} \left( 1 + \frac{g_{\rho}^{2}}{m_{\rho}^{2} \sigma_{o}^{2}} \right)^{-\frac{1}{2}} \sigma_{o}, \quad (2.13)$$

or $1/F_{\pi}^{2} = 1/\sigma_{o}^{2} + g_{\rho}^{2}/m_{\rho}^{2}$. This result was first obtained by Gasiorowicz and Geffen [60] (see also [4b]). From the first term in Eq. (2.8), it is seen that the relationship between the pion decay constant and the $\pi\pi$ scattering amplitude in the chiral limit is also verified in QHD-III at tree-level.
2.3 Discussion and conclusions

Consider the $O(g^2\rho^2)$ corrections to the $\pi\pi$ scattering amplitude, Eqs. (2.8)-(2.10). As $m_\rho \to \infty$, $\delta\alpha_1, \delta\alpha_2 \to 0$. This can be seen directly from Eqs. (2.9) and (2.10) by noticing that to this order, i) $\delta g_\rho$ is finite in this limit and ii) the $O(1/m^2_\rho)$ terms are to be neglected so that the only surviving contributions from the counterterms are those of the $\sigma$ model in Eqs. (2.7). Note also that the tree-level correction of $\beta$ [the first term of Eq. (2.8)] goes to zero in that limit. However, in this amplitude, the limit $m_\rho \to \infty$ must be taken without violating the constraint $m^2_\rho/m^2_\sigma \geq 1$ but finite. This constraint implies the appearance of a new (quadratic) divergence in $\beta$ proportional to $m^2_\sigma$. The $\ln m^2_\rho/\mu^2$ divergences in $\beta$ and $\delta\beta$ are already absorbed in the renormalization of the parameters of the lagrangian and need not be considered further. The new quadratic divergence in $\beta$ renormalizes the pion decay constant to 1-loop in exactly the same fashion; in [6,7], it is shown quite generally that the linear $\sigma$ model reduces to the non-linear $\sigma$ model in the limit $m_\sigma \to \infty$ and that quadratic divergences have no observable effect to 1-loop. In $\delta\beta$, note that the 1-loop corrections are finite constants in the heavy mass limit subject to the above constraint. They are thus negligible with respect to the quadratic contributions in $\beta$. Hence, the gauge boson contribution to the scattering amplitude becomes negligible in the heavy mass limit: the QHD-III scattering amplitude reduces to the $\sigma$ model amplitude in the limit $m_\rho \to \infty$ with $m^2_\rho/m^2_\sigma \gg 1$ but finite. ¹

¹Since the 1-loop corrections in $\beta$ and $\delta\beta$ become larger then the tree-level terms in the limit $m_\rho, m_\sigma \to \infty$, taking these limits in our scattering amplitude is questionable since we used perturbation theory to obtain our result. However, 4-point functions can be used to construct effective lagrangians in the heavy mass limit [65], and our scattering amplitude would have the same structure as that effective lagrangian.

The decoupling of the $a_1$ and the $\rho$ can be understood more generally as follows:
from the lagrangian given in Eq. (2.2) and the definition of \( m_\sigma \), it is seen that the gauge bosons decouple from the \( \pi \) and the \( \sigma \) when the gauge fields are rescaled according to \( \rho_\mu = \rho_\mu/m_\rho \) and \( a_\mu = a_\mu/m_\rho \) and the limit \( m_\rho \to \infty \) is taken; this procedure results in the lagrangian of the linear \( \sigma \) model (for a similar discussion in the linear \( \sigma \) model see [1]). This suppression in inverse powers of the mass is essentially the decoupling theorem [68].

Now consider pion decay in QHD-III. When the 1-loop correction of the matrix element \( \langle 0 | A_\mu^i | \pi^2 \rangle \) is evaluated using the axial current derived from the QHD-III lagrangian, it is found that the counterterms given in Eqs. (2.11) fail to cancel all of the divergences; it is also explicitly found that this matrix element is not gauge invariant to 1-loop in QHD-III. This violation of current conservation occurs because the pion acquires a strong isospin charge when the vector mesons are introduced as gauge bosons as is the case in QHD-III; hence, in a process which destroys strong isospin charge such as pion decay, vector and axial-vector isospin current conservation is violated,\(^2\) and that process is not an S-matrix element of QHD-III.

In summary, the renormalized \( \pi \pi \) scattering amplitude has been calculated to 1-loop in the \( \sigma \) model and in QHD-III in the chiral limit for small external momenta with respect to the masses. The pion decay constant has been calculated to 1-loop in the \( \sigma \) model and it is shown explicitly that \( F_\pi^{-2} = \beta \). It is also shown that this relation holds at tree-level in QHD-III. It is argued that the gauge bosons decouple from the pion and the \( \sigma \) in the heavy mass limit. The 1-loop correction to \( F_\pi \) in QHD-III is seen to violate strong isospin current conservation; thus, the model can\(^2\)

---

\(^2\)Electromagnetic charge also disappears in pion decay. Of course, it is carried off in the lepton sector. The author knows of no simple way to fix up strong vector and axial-vector isospin current conservation in this model for pion decay.
at best describe the strongly interacting nuclear sector.
Chapter 3

Gauge Invariance in QHD

[What follows is a reproduction of the published paper [56].] Hadronic models have had significant phenomenological success in describing the many-body strongly interacting system at low energies [1]. These models take hadrons as their effective degrees of freedom. The lagrangians of hadronic models are constructed in such a way that they reflect the symmetries of QCD, while incorporating low-energy phenomenology. In particular, these models must give rise to conserved vector and partially conserved axial-vector currents, and they should include the exchange of mesons which are known to carry the strong force at low energies. The mesons should be introduced such that corrections coming from meson loops are consistently calculable within the model. Since there is a large number of mesons, any model based on meson exchange must find a consistent way to choose the relevant mesons; this can be achieved by introducing them as gauge bosons.

Briefly, QHD-III [2] is a gauge-invariant hadronic quantum field theory based on the gauged $\sigma - \omega$ model with pions. The $\sigma - \omega$ model is built from the linear $\sigma$ model with the $\omega$ introduced as a massive $U(1)$ gauge boson (see below). When the global
CHAPTER 3. GAUGE INVARIANCE IN QHD

$SU(2)_L \times SU(2)_R$ symmetry of the $\sigma - \omega$ model is made local and parity conservation is imposed, the $\rho$ and its chiral partner, the $a_1$, appear as the gauge bosons. The $\rho$ and the $a_1$ are made massive by the inclusion of a Higgs sector composed of two complex doublets: one transforming under $SU(2)_L$ and the other under $SU(2)_R$; the doublets couple to the gauge bosons through their covariant derivatives. This procedure for giving mass to the $\rho$ and the $a_1$ is very similar to the one used to give mass to the weak bosons in the standard model, and preserves the gauge invariance of the model. Keeping the gauge invariance allows the unambiguous derivation of the strong conserved currents. A small symmetry-breaking term is included to yield massive pions. The resulting lagrangian has a minimal number of massive mesons which couple to conserved vector and axial-vector currents; in this model, the $a_1$ naturally comes out heavier than the $\rho$. The lagrangian is also renormalizable [2].

As shown below, the physical pion disappears in a hadronic model based on the gauged $\sigma$ model.\footnote{This point has also been previously independently noted in [5]} To retain the pion, the gauge bosons must be given mass. In contrast to QHD-III, the usual procedure [69, 60, 61, 70] is to put in the same mass, $m_\rho$, by hand for both the $\rho$ and the $a_1$. The spontaneous symmetry-breaking (SSB) that occurs in the $\sigma$ sector provides an extra contribution to the mass of the $a_1$ making the $a_1$ mass, $m_{a_1}$, larger than the $\rho$ mass, $m_\rho$. However, introducing mass terms for the gauge bosons by hand violates current conservation, and loops can no longer be calculated unambiguously. Furthermore, because of the SSB in the $\sigma$ sector, the $a_1$ and the gradient of the pion mix, and the resulting lagrangian must be diagonalized. The diagonalization of the lagrangian is carried out by making a change of variables involving the $a_1$ and the gradient of the pion. The final lagrangian
is complicated because of the introduction of momentum-dependent vertices due to the gradient of the pion.

In a gauge-invariant quantum field theory such as QHD-III [2], one can make a gauge transformation to diagonalize the lagrangian instead of making the above-mentioned change of variables. The result is a considerably simpler lagrangian where no new momentum dependent vertices appear. In particular, to $O(g^2_\rho)$, the diagonalization of the lagrangian is equivalent to the rescaling of the pion field by the ratio $m_\rho/m_\pi$. This work details the derivation of this new representation of the QHD-III lagrangian.

This paper begins by discussing the $SU(2)_L \times SU(2)_R$ locally invariant $\sigma - \omega$ model with pions. The gauge bosons, the $a_1$ and the $\rho$, are originally massless. It is shown that the SSB gives a mass to the $a_1$. Working in an arbitrary $\xi$ gauge, it is shown that the field originally identified with the pion acquires a $\xi$-dependent mass, which identifies it as a fictitious particle; what looks like a pion is labeled as $\pi'$ to distinguish it from the physical pion. By looking at nucleon-nucleon scattering, it is shown that the $\pi'$ exchange diagram is always canceled by the $\xi$ dependent part of the $a_1$ exchange diagram. The disappearance of the real pion is forced by gauge invariance and demonstrates the need for massive gauge bosons with mass provided from outside this sector of the theory.

QHD-III is then reviewed. In this locally gauge-invariant model, pions appear as the physical Goldstone bosons. Here, the vector meson masses are generated from a Higgs sector, as in the $\sigma$ model. Gauge invariance is used to diagonalize the lagrangian in an arbitrary $\xi$ gauge so as to avoid the original, momentum-dependent, diagonalization procedure used in [2]. This new diagonalization scheme produces a
considerably simpler representation of QHD-III than the representation given in [2]. It is shown how, to \( O(g^2) \), the new diagonalization procedure is equivalent to simply rescaling the pion field by the ratio \( m_\rho/m_\sigma \).

To demonstrate the need for massive vector mesons, consider the \( \sigma - \omega \) model with pions (QHD-II) [2]:

\[
\mathcal{L}_{\sigma-\omega} = \bar{\psi}[i\gamma^\mu(\partial_\mu + ig_\nu V_\mu) - g_\pi(s + ig_\nu \tau \cdot \pi)]\psi + \frac{1}{2}(\partial_\mu s\partial^\mu s + \partial_\mu \pi \cdot \partial^\mu \pi)
\]

\[
- \frac{1}{4}\lambda(s^2 + \pi^2 - v^2)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_\nu^2 V_\mu V^\mu + \epsilon s. \tag{3.1}
\]

In Eq. (3.1), \( V^\mu \) represents the \( \omega \) field, and \( \epsilon s \) is the chiral symmetry violating term that gives a mass to the pion. The global \( SU(2)_L \times SU(2)_R \) symmetry of \( \mathcal{L}_{\sigma-\omega} \) is now made local, and the scalar field is given a vacuum expectation value \( s = \sigma_0 - \sigma \) with \( \sigma_0 \equiv M/g_\pi \). This yields for the lagrangian, \( \mathcal{L}_s \), of the gauged \( \sigma - \omega \) model with pions:

\[
\mathcal{L}_s = \bar{\psi}[i\gamma^\mu(\partial_\mu + ig_\nu V_\mu + \frac{i}{2}g_\rho \tau \cdot (\rho_\mu + \gamma_5 a_\mu)] - (M - g_\sigma \sigma - ig_\pi \gamma_5 \tau \cdot \pi')\psi
\]

\[
+ \frac{1}{2}[(\partial_\mu \pi' + g_\rho \sigma a_\mu + g_\rho \pi' \times \rho_\mu)^2 - m_\pi^2 \pi' \cdot \pi']
\]

\[
+ \frac{1}{2}[(\partial_\mu \sigma - g_\rho \pi' \cdot a_\mu)^2 - m_\sigma^2 \sigma^2]
\]

\[
- g_\rho \sigma_0 a_\mu \cdot (\partial_\mu \pi' + g_\rho \sigma a_\mu + g_\rho \pi' \times \rho_\mu)
\]

\[
+ \frac{m^2 - m_\rho^2}{2\sigma_0} \sigma (\sigma^2 + \pi'^2) - \frac{m^2 - m_\rho^2}{8\sigma_0^2} (\sigma^2 + \pi'^2)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_\nu^2 V_\mu V^\mu
\]

\[
- \frac{1}{4}R_{\mu\nu} \cdot R^{\mu\nu} - \frac{1}{4}A_{\mu\nu} \cdot A^{\mu\nu} + \frac{1}{2}g_\rho^2 \sigma_0^2 a_\mu \cdot a_\mu. \tag{3.2}
\]

From Eq. (3.2), we see that the SSB in the \( \sigma \) sector has given a mass \( M \) to the nucleon and a mass \( g_\rho \sigma_0 \) to the \( a_1 \). The \( \sigma \) mass is \( m_\sigma \), and the \( \rho \) remains massless. We also note the presence of a bilinear term, \(-g_\rho \sigma_0 a_\mu \cdot \partial_\mu \pi'\), and thus the need for
diagonalization. Most importantly, the pion has disappeared and has been replaced by the auxiliary field, $\pi'$. The fact that $\pi'$ is an auxiliary field can be seen either by counting the degrees of freedom before and after the SSB, or by making the change of variables:

$$a_\mu \rightarrow a_\mu + \frac{1}{g_\rho \sigma_s} \partial_\mu \pi'.$$

(3.3)

This change of variables both diagonalizes the lagrangian and forces a cancelation of the kinetic energy term, $\partial_\mu \pi' \cdot \partial^\mu \pi'$, clearly identifying $\pi'$ as an auxiliary field.

The initial pion was “eaten” by the $a_1$. $\mathcal{L}_g$ can also be diagonalized using a gauge-fixing function and working in a $\xi$ gauge [23]. For simplicity, take the chiral limit ($m_\pi = 0$) and add to the lagrangian the gauge-fixing function, $-\frac{1}{2} G^2$, where:

$$G = \frac{1}{\sqrt{\xi}} (\partial_\mu a^\mu + \xi g_\rho \sigma_s \pi').$$

(3.4)

The gauge-fixed lagrangian, $\mathcal{L}_g^{gf}$, becomes:

$$\mathcal{L}_g^{gf} = \mathcal{L}_g - g_\rho \sigma_s \pi' \cdot \partial_\mu a^\mu - \frac{1}{2\xi} (\partial_\mu a^\mu)^2 - \frac{1}{2} \xi g_\rho^2 \sigma_s^2 \pi'^2.$$

(3.5)

It is noted that the second term in (3.5) cancels the bilinear term in (3.2) after a partial integration, and that the $\pi'$ has acquired a mass that depends on the gauge parameter, $\xi$. From (3.5), the propagators in momentum space of the $a_\mu$ and the $\pi'$ are found to be respectively:

$$\Delta^{ij}_{\mu\nu} = \frac{-i\delta^{ij}}{q^2 - m_a^2} \left[ g_{\mu\nu} - \frac{q_\mu q_\nu}{m_a^2} \right] - \frac{i\delta^{ij}}{q^2 - \xi m_a^2} \frac{q_\mu q_\nu}{m_a^2},$$

(3.6)

$$\Delta^{ij} = \frac{i\delta^{ij}}{q^2 - \xi m_a^2}.$$

(3.7)

Here, $m_a = g_\rho \sigma_s$ and we have separated out the $\xi$-dependent part of the $a_\mu$ propagator in the last term of (3.6). In the limit $\xi \rightarrow \infty$, the $\pi'$ decouples from
the problem, while the $a_\mu$ propagator goes into the unitary gauge. In an arbitrary $\xi$ gauge, the $\xi$-dependent part of the $a_\mu$ propagator always cancels the contribution coming from $\pi'$ exchange. This can be seen in nucleon-nucleon scattering at tree level in Fig. 3.1 [23]. The $\pi'$ exchange diagram is precisely canceled by the $\xi$-dependent part of the $a_1$ propagator as is easily verified. The fact that the $\pi'$ does not contribute to physical processes, and the fact that the gauge invariance is preserved, are visibly true because the $\pi'$ has the correct mass: $m_{\pi'}^2 = \xi m_\sigma^2$. \footnote{Since the non-linear $\sigma$ model is the limit of the linear $\sigma$ model as $m_\sigma \to \infty$, the pion also disappears in the gauged non-linear $\sigma$ model.}

To retain the physical pion in the model, the gauge bosons must develop a mass from outside the $\sigma$ sector. In QHD-III [2], a Higgs sector composed of left and right complex doublets is included to preserve the gauge invariance of the model:

$$\mathcal{L}_H = \partial_\mu \phi_R^\dagger \partial^\mu \phi_R + \partial_\mu \phi_L^\dagger \partial^\mu \phi_L + \mu_H^2 (\phi_R^\dagger \phi_R + \phi_L^\dagger \phi_L)$$  (3.8)
As detailed in [2], $\mathcal{L}_H$ is then made locally invariant by minimal substitution. After the SSB in the full theory, and the elimination of the Goldstone bosons from the Higgs sector by going into the unitary gauge, the Higgs lagrangian becomes:

$$\mathcal{L}_H = \frac{\lambda_H}{4}[(\phi_R^\dagger \phi_R)^2 + (\phi_L^\dagger \phi_L)^2].$$

In the equation above, $u$ is twice the vacuum expectation value of the scalar fields in the Higgs sector. The Higgs fields $\eta$ and $\zeta$ are respectively scalar and pseudoscalar fields. The QHD-III lagrangian, $\mathcal{L}_{III}$, is given by:

$$\mathcal{L}_{III} = \mathcal{L}_g + \mathcal{L}_H$$

where:

$$\mu_H^2 = \frac{1}{2}m_H^2, \quad u^2 = \frac{8\mu_H^2}{\lambda_H} = \frac{4m_R^2}{g_R^2}, \quad \lambda_H = \frac{m_H g_R^2}{m_R^2}.$$ (3.10)

In the equation above, $u$ is twice the vacuum expectation value of the scalar fields in the Higgs sector. The Higgs fields $\eta$ and $\zeta$ are respectively scalar and pseudoscalar fields. The QHD-III lagrangian, $\mathcal{L}_{III}$, is given by:

$$\mathcal{L}_{III} = \mathcal{L}_g + \mathcal{L}_H$$

It is seen that the $a_1$ obtains contributions to its mass, $m_{a_1}$, from both the $\sigma$ sector and from the Higgs sector. Thus, the $a_1$ comes out naturally more massive than the $\rho$ with:

$$m_{a_1}^2 = m_\rho^2 + g_\rho^2 \sigma_0^2 > m_\rho^2$$

The pion in QHD-III is now real and $\mathcal{L}_{III}$ must be diagonalized further to remove
the term $-g_\rho \sigma_\omega a_\mu \cdot \partial_\mu \pi'$ in $\mathcal{L}_2$. This is achieved by performing the change of variables:

$$a_\mu \rightarrow a_\mu + \frac{g_\rho \sigma_\omega}{m_a^2} \partial_\mu \pi', \quad \pi' = \frac{m_a}{m_\rho} \pi, \quad m_\pi' = \frac{m_\rho}{m_a} m_\pi. \quad (3.13)$$

$\pi$ is now the physical pion field. Since the $a_1$ appears in many places in $\mathcal{L}_{III}$, and the change of variables involves the gradient of the pion, the diagonalized lagrangian is quite complicated with momentum-dependent vertices showing up everywhere.

In contrast, the gauge invariance of QHD-III allows another, simpler diagonalization procedure to work. Consider Eq. (3.9): to obtain $\mathcal{L}_H$, the Goldstone bosons of the Higgs sector were eliminated from the theory by going into the unitary gauge, as is done in the standard model: these Goldstone bosons are fictitious since they can be removed by a gauge transformation. This can be seen by counting the degrees of freedom: initially, before the SSB in the Higgs sector, there are two complex doublets (the Higgs fields) which yield eight degrees of freedom, and two massless isovector fields (the $\rho$ and the $a_1$) which add 12 degrees of freedom; this yields a total of 20 degrees of freedom. After the SSB, there are two isoscalar Higgs fields (the $\eta$ and the $\zeta$), and two massive isovector fields for a total 20 degrees of freedom. Thus, two isovector fields “disappeared” from the Higgs sector to become the longitudinal polarization states of the $\rho$ and the $a_1$; these are the Goldstone bosons. If one does not work in the unitary gauge, the Goldstone bosons must still couple to the other fields to maintain gauge invariance. One of the isovector fields must couple directly to the $\rho$ and must therefore be a scalar field because of parity conservation, while the other isovector field which we denote as $\chi'$ must couple directly to the $a_1$ and must be a pseudoscalar field. The scalar isovector field that was eaten by the $\rho$ is not needed for this discussion, and can be decoupled from the problem independently of
the other Goldstone bosons. As for the pseudoscalar Goldstone bosons, the contributions to $\mathcal{L}_H$ stemming from working in an arbitrary gauge, and therefore keeping the $\chi'$ are:

$$\mathcal{L}'_H = \mathcal{L}_H + \partial_\mu \chi' \cdot \partial^\mu \chi' + m_p a_\mu \cdot \partial^\mu \chi' + \Delta \mathcal{L}'_H.$$  

(3.14)

$\Delta \mathcal{L}'_H$ represents terms given in appendix B for completeness. The presence of the bilinear term in Eq. (3.14) is noted and is typically removed with a gauge-fixing function similar to the one given in (3.4). The $\chi'$ is then decoupled by taking $\xi$ to infinity as discussed in the case of the $\pi'$ below equation (3.7). This is what was done in [2].

Consider instead the following gauge-fixing function:

$$G_a = \frac{1}{\sqrt{\xi}} (\partial_\mu a^\mu - \xi m_p \chi' + \xi g_\sigma \sigma \pi')$$  

(3.15)

Adding $-\frac{1}{2} G_a^2$ to $\mathcal{L}_{III} = \mathcal{L}'_H + \mathcal{L}_g$ cancels the bilinear terms: $-g_\sigma \sigma \alpha_a \cdot \partial_\mu \pi'$ and $m_p a_\mu \cdot \partial^\mu \chi'$. The propagator of the $a_1$ is exactly as in Eq. (3.6) with $m_a$ now given by (3.12). What is left from the gauge-fixing function is:

$$-\frac{1}{2} G_a^2 \doteq -\frac{1}{2} \xi m_p^2 \chi' \cdot \chi' - \frac{1}{2} \xi g_\sigma^2 \sigma \sigma \pi' \cdot \pi' + \xi g_\sigma \sigma m_p \pi' \cdot \chi'.$$  

(3.16)

Eq. (3.16) needs a further diagonalization to cancel the last term; this can be achieved by making the following change of variables:

$$\chi' = a \pi + b \chi; \quad \pi' = c \pi + d \chi.$$  

(3.17)

The parameters $\{a, b, c, d\}$ are constrained by the requirements that all bilinear terms that couple the $\pi$ and the $\chi$ be canceled, and that the kinetic energies be normalized:

$$\partial_\mu \pi' \cdot \partial^\mu \pi' + \partial_\mu \chi' \cdot \partial^\mu \chi' = \partial_\mu \pi \cdot \partial^\mu \pi + \partial_\mu \chi \cdot \partial^\mu \chi$$  

(3.18)
These equations have the solution:

\[
\begin{align*}
    a &= \frac{g_\rho \sigma_0}{m_a}, \\
    b &= \frac{m_\rho}{m_a}, \\
    c &= \frac{m_\rho}{m_a}, \\
    d &= -\frac{g_\rho \sigma_0}{m_a}
\end{align*}
\]  
(3.19)

It is found that the masses of the \(\chi, \pi\) fields are respectively:

\[
\begin{align*}
    m_\chi^2 &= \xi (m_\rho^2 + g_\rho^2 \sigma_0^2) = \xi m_a^2, \\
    m_\pi &= \frac{m_\rho}{m_a} m_\pi'.
\end{align*}
\]  
(3.20)

The mass of the \(\chi\) is exactly the mass needed to cancel the \(\xi\)-dependent part of the \(a_1\) propagator given in equation (3.6), as discussed below Eq. (3.7). The \(\chi\) is the field that provided the longitudinal polarization states of the \(a_1\) after the SSB.

\(\chi\) can be decoupled by taking \(\xi \to \infty\) and Eq. (3.17) becomes:

\[
\chi' = \frac{g_\rho \sigma_0}{m_a} \pi; \quad \pi' = \frac{m_\rho}{m_a} \pi.
\]  
(3.21)

It is thus seen that the diagonalization is achieved by a constant rescaling of the pion field. Because of the constraint (3.18), the pion kinetic energy is not rescaled. This completely avoids the introduction of new momentum-dependent vertices.

Through the first equation of (3.21), the pion couples directly to the Higgs fields.\(^3\)

The pion-Higgs vertices generally involve a high power of \(g_\rho\) as seen in appendix B. By inspection, it is seen that any amplitude that does not involve both a Higgs field and a gauge boson as external legs will not contribute to \(O(g_\rho^2)\). Hence, to order \(O(g_\rho^2)\), all S-matrix elements that do not involve a Higgs field as either an incoming or an outgoing field, can be calculated by ignoring the Higgs sector, and rescaling the pion field by the constant factor: \(m_\rho/m_a\). In the chiral limit, when \(m_\pi = 0\), this new, simpler representation of the QHD-III lagrangian is completely equivalent to the one given in [2]; when \(m_\pi = 0\), the two representations lead to exactly the same

\(^3\)This was also the case in the change of variable (3.13) as is seen by substituting (3.13) into Eq. (3.9).
physical predictions. In particular, a calculation in QHD-III of an amplitude such as $\pi\pi$ scattering to one loop in the chiral limit [55] is considerably simplified.

When $m_\pi \neq 0$, the two representations will differ slightly in their physical predictions since the pion mass term violates chiral symmetry; i.e. the contribution of the chiral symmetry breaking term is gauge dependent. Although the pion mass term is gauge-dependent, the symmetry breaking term in Eq. (3.1) is always chosen so as to result in the physical pion mass once the gauge has been picked. This is what was done in Eq. (3.20).

In summary, it is shown that a quantum field model of the strong interaction, based on an $SU(2)_L \times SU(2)_R$ locally invariant $\sigma$ model, is not realistic if the corresponding gauge bosons are massless before the SSB; in such a model, the pion disappears. We saw how putting masses by hand violates gauge invariance and forces a complicated diagonalization procedure on the model. QHD-III is reviewed, and the gauge invariance of the model is exploited to provide a new, considerably simpler representation of the model that makes it more accessible. The pion here appears as a physical degree of freedom.
Chapter 4

One-Loop Effective Action in QHD

4.1 Introduction

In this section, the 1-loop effective action of the linear $\sigma$-model is calculated in an arbitrary renormalization scheme and shown to be equivalent to the result previously obtained by Gasser and Leutwyler in their scheme [43]. From this 1-loop effective action, the pion decay constant $F_\pi$ to 1-loop is extracted from the $O(p^2)$ lagrangian $\mathcal{L}_2$ that appears in the effective action of the linear $\sigma$-model, and it is explicitly shown to be identical to the $F_\pi$ obtained using Feynman diagrams and the axial-vector current in Chapter 2 of this thesis. The $\pi\pi$ scattering amplitude extracted from the 1-loop effective action is calculated and shown to be identical to the result obtained in Chapter 2.

The 1-loop effective action of the QHD-III lagrangian is next set up. The operators appearing in $\Delta\mathcal{L}$ are calculated and the heavy fields are expressed in terms of the pion field using their equations of motion. The tree level result for the pion decay constant is calculated and shown to be identical to the result stemming from a

\footnote{There is a small difference in one of the terms which is discussed below.}
Feynman diagram calculation. It is noted that the 1-loop result for the $\mathcal{O}(p^2)$ term of the effective action is made finite using the counterterms calculated in the Feynman diagram expansion. The other 1-loop terms fail to be renormalized. The origin of the failure is so far unknown but is still under investigation.

The background field method [45, 57, 58, 59] is first used to calculate the 1-loop effective action of the linear $\sigma$-model. In this method, quantum fluctuations (1-loop corrections) about a classical field configuration are extracted:

$$
\phi^i(x) = \bar{\phi}^i(x) + \delta \phi^i(x)
$$

(4.1)

where $\phi^i$ is a generic field standing in for all the fields in the model (isoscalars, isovectors, vector bosons, etc), $\bar{\phi}^i$ are the corresponding background, classical fields and $\delta \phi^i$ are the quantum fluctuations. The superscript $i$ represents all the components of the field. The background fields are chosen to satisfy the classical equations of motion; using these equations, all the heavy fields can be expressed in terms of the pion field and its derivatives. Once all the fields have been expanded to second order in the fluctuations about their classical solutions as in Eq. (4.1), the path integral over all the fluctuations is performed. The path integral over the fluctuations leads to an effective action of the form:

$$
S = \int d^4x [\bar{\mathcal{L}} + \Delta \mathcal{L}]
$$

(4.2)

$\bar{\mathcal{L}}$ is the original lagrangian expressed in terms of the classical fields while $\Delta \mathcal{L}$ are the new terms that arise after the path integral was performed. $\Delta \mathcal{L}$ is then expanded in powers of the momentum to $\mathcal{O}(p^4)$. Because of the complete generality of the $\mathcal{O}(p^4)$ $\chi$PT lagrangian, the momentum expansion of $\Delta \mathcal{L}$ should only yield terms of a form that is invariant under the low-energy symmetries of strong interactions, i.e., under
$SU(2)_L \times SU(2)_R$, parity and G-parity [see Eq. (1.36)]. Indeed, this is seen to be the case in the 1-loop effective action evaluation of both the linear $\sigma$-model and QHD-III lagrangians.

### 4.2 One-loop expansion of the linear $\sigma$-model effective action

In the following, the nucleon sector is excluded. The linear $\sigma$-model lagrangian is rewritten here in the notation found in [43]:

\[
\mathcal{L}_\sigma = \frac{1}{2} \nabla_\mu \phi^A \nabla^\mu \phi^A + \frac{1}{2} m^2 \phi^A \phi^A - \frac{g}{4} (\phi^A \phi^A)^2 + f^A \phi^A
\]  

(4.3)

$f^A \phi^A$ is the symmetry breaking term that gives mass to the pion and $\phi^A$ is a vector with the following components:

\[
\phi^A = (\epsilon, \pi_1, \pi_2, \pi_3), \quad A = 0, 1, 2, 3
\]  

(4.4)

The previous expression given for the linear $\sigma$-model lagrangian [Eq. (1.18)] is related to equation (4.3) by the following relations:

\[
\partial_\mu \rightarrow \nabla_\mu, \quad \psi^2 \lambda \rightarrow m^2, \quad \lambda \rightarrow g
\]  

(4.5)

In particular, the chiral symmetry breaking term in equation (1.18) is obtained from $f^A \phi^A$ in the special case:

\[
f^A = (\epsilon, 0, 0, 0)
\]  

(4.6)

Here repeated capital latin letters are summed from 0 to 3; the notation $\phi^A \phi^A = \phi \cdot \phi$ is also employed. Sources\(^2\) are introduced in the form of classical, non-dynamical

\(^2\)Matrix elements of the S-matrix are calculated in the standard fashion by applying functional derivatives with respect to the source fields to the generating functional.
vector fields through covariant derivatives like the ones found in the gauged linear \( \sigma \)-model lagrangian of Eq. (3.2); in Eq. (4.3) they are defined:

\[
\nabla_\mu \phi^0 = \partial_\mu \phi^0 + g_\sigma a^i_\mu \phi^i \\
\nabla_\mu \phi^i = \partial_\mu \phi^i - g_\rho \epsilon^{ijk} \rho^j_\mu \phi^k - g_\rho a^i_\mu \phi^0 \quad i = 1, 2, 3
\]

(4.7)

Repeated latin superscripts are here summed from 1 to 3. The origin of the definitions (4.7) can be understood in terms of the covariant derivatives defined in Eq. (1.42) which lead to the covariant derivatives of the \( \sigma = \sigma_0 - s \) and pion fields appearing in Eq. (1.48).

Because the \( \rho^\mu \) and \( a^\mu \) are classical external fields that act as sources, only the \( \phi^A \) will develop quantum fluctuations in the background field method:

\[
\phi^A = \phi^A + y^A
\]

(4.8)

where the \( y^A \) are the quantum fluctuations about the classical field \( \phi^A \). Substituting Eq. (4.8) into (4.3) yields the following result quadratic in the fluctuations:

\[
\mathcal{L}_\sigma = \mathcal{L}_\sigma[\phi^A] + y^A D^{AB} y^B + \ldots
\]

(4.9)

where:

\[
D^{AB} \equiv \delta^{AB} [\nabla_\mu \nabla^\mu + (m^2 + g\phi \cdot \phi) + 2g\phi^A \phi^B]
\]

(4.10)

There are no terms linear in the fluctuations since the \( \phi^A \) satisfy their own equations of motion. When the path integral is performed over the quadratic fluctuations \( y^A \) using standard gaussian quadrature [23], the result for the action is:

\[
S_\sigma = \int d^4 x \mathcal{L}_\sigma[\phi^A] + \frac{i}{2} \ln \det D
\]

(4.11)

\footnote{Gasser and Leutwyler include a kinetic energy term for the sources, \( \pi F_{\mu\nu} F^{\mu\nu} \). This term is not included in the current treatment because it is not needed (nor is it included in appendix B of the monograph by Donoghue, Golowich and Holstein [45]).}
The last term $i/2 \ln \det D$ contains the loop corrections due to the quantum fluctuations.

It is known that the linear $\sigma$-model reduces to the non-linear $\sigma$-model in the limit that the $\sigma$ mass goes to infinity. Since an expansion in powers of the momentum of $i/2 \ln \det D$ will eventually be performed to $O(p^4)$ with $p \ll m_\sigma$, it is useful to write the classical solution in such a way that the non-linear $\sigma$ model is easily separable from the linear $\sigma$ model. The following spherical parametrization is therefore introduced for the classical field $\tilde{\phi}^A$:

$$\tilde{\phi}^A(x) = \frac{m}{\sqrt{g}} R(x) U^A(x) \quad \text{with: } U^A U^A = 1 \quad (4.12)$$

Here, the $U^A$ is the dimensionless field introduced in the introduction in the discussion of the non-linear $\sigma$-model [Eq. (1.32)]. To understand this parametrization, it is useful to express the field $R(x)$ in terms of the pion field using the equation of motion for $R$. Substituting equation (4.12) into $\mathcal{L}_\sigma[\tilde{\phi}^A]$ yields:

$$\mathcal{L}_\sigma[\tilde{\phi}^A] = \frac{m^2}{2g} (R^2 \nabla_\mu U^A \nabla^\mu U^A + \nabla_\mu R \nabla^\mu R) + \frac{m^4}{2g} R^2 (1 - \frac{R^2}{2}) + \frac{m}{\sqrt{g}} f^A U^A \quad (4.13)$$

The second term in Eq. (4.13) is of sixth order in powers of the derivative, or of $O(p^6)$, as will be seen below, and can therefore be neglected in the present analysis. Using the Euler-Lagrange equations for $R$ derived from (4.13), the equation of motion for $R$ is found to be:

$$R^2 = 1 + \frac{1}{m^2} \nabla_\mu U^A \nabla^\mu U^A + \frac{\sqrt{g}}{R m^3} f^A U^A$$

$$\approx 1 + \frac{1}{m^2} \nabla_\mu U^A \nabla^\mu U^A + \frac{\sqrt{g}}{m^3} f^A U^A$$

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From the above, it is clear that $\nabla_\mu R \nabla^\mu R$ is of $O(p^6)$ as claimed above. In the linear $\sigma$-model, the $\sigma$ mass is given by:

$$m_\sigma^2 = 2m^2$$

Thus,

$$\lim_{m_\sigma \to \infty} R = 1 \quad \text{and} \quad \lim_{m_\sigma \to \infty} \bar{\phi}^A = \frac{m}{\sqrt{g}} U^A$$

This shows that the parametrization (4.12) has a simple relationship to the nonlinear $\sigma$ model and motivates the introduction of this parametrization. The physics of this is that as the scalar field $R$ becomes heavier, the fluctuations of $\phi^A$ in the radial direction shrink to zero and $R$ becomes frozen at the value 1.

With the parametrization (4.12), the fluctuations $\delta \phi^A$ are defined by:

$$\delta \phi^A \equiv (\delta R) U^A + R (\delta U^A)$$

This implies that the $y^A = \delta \phi^A$ have the following representation:

$$y^A = \xi U^A + \sum_{i=1}^{3} \epsilon_i^A \theta^i$$

The $\xi$ represents the fluctuations of the $R$ field in the radial direction $\hat{r}$; it essentially corresponds to the fluctuations of the $\sigma$ field. The $\theta^i$'s represent the fluctuations of the pions. The $\epsilon_i$'s constitute an orthonormal set of basis vectors along which the pion fields fluctuate. They satisfy the following completeness and orthonormality relations:

$$\sum_{i=1}^{3} \epsilon_i^A \epsilon_i^B + U^A U^B = \delta^{AB}, \quad \epsilon_i^A \epsilon_j^A = \delta_{ij}, \quad \epsilon_i^A U^A = 0.$$
Fig. 4.1 illustrates this situation: the $\xi$ represent fluctuations of the field vector $\phi^A$ along the radial direction $\hat{r}$. While the $\theta^i$ are the fluctuations on the surface of the 4-sphere. From Fig. 4.1, the following representation for the basis vectors can be extracted:

$$\epsilon^0_i = -U^0, \quad \epsilon^i_i = \delta^{ij}U^0 - \epsilon^{ijk}U^k$$

(4.20)

It is easy to check that the $\epsilon_i$ satisfy the relations (4.19). This representation leads to erroneous results when vector mesons are included, however, because the basis vectors are not eigenstates of parity. The parity transformation evidently produces (recall that the pion is a pseudoscalar field):

$$P : U^0 \rightarrow U^0, \quad P : U^i \rightarrow -U^i$$

$$P : \epsilon^0_i \rightarrow -\epsilon^0_i, \quad P : \epsilon^i_i \rightarrow \epsilon^i_i.$$ 

(4.21)

Since strong interactions are parity invariant, the $\epsilon_i$ must also be parity eigenstates. A more suitable representation is the following:

$$\epsilon^0_i = -U^0, \quad \epsilon^i_i = \delta^{ij} - \frac{U^iU^j}{1 + U^0}$$

(4.22)

with the following transformation properties under parity:

$$P : \epsilon^0_i \rightarrow -\epsilon^0_i, \quad P : \epsilon^i_i \rightarrow \epsilon^i_i.$$ 

(4.23)

Having expanded the $\phi^A$ about the classical background field, the operator $D$ can be expressed in the basis of the fluctuations. To do so, we introduce the notation

---

5. This leads to an effective lagrangian that is not invariant under parity.
Figure 4.1: The field $\phi^A (A = 0, \ldots, 3)$ fluctuates about the background field $\bar{\phi}(x)$ at a particular point in space-time $x$. The fluctuations are both along the radial direction $\hat{r}$ and along the basis vectors $\varepsilon_i (i = 1, 2, 3)$ on the surface of a 4-sphere.
used in [43]:

\[(y, Dy) \equiv \int d^4x d^4y(x) D(x, z)y(z)\]

\[= \int d^4x \left[ -\nabla_\mu(y \cdot \nabla^\mu y) + (-m^2 + g\bar{\phi} \cdot \phi) y \cdot y + 2g(\bar{\phi} \cdot y)^2 \right] \tag{4.24}\]

In this notation, \(D(x, z)\) is both a matrix in fluctuation space as well as a matrix in coordinate space similar to a propagator in coordinate space.\(^7\)

Eq. (4.18) is used to expand the matrix \(D\):

\[(y, Dy) = (\xi, \hat{d}\xi) + 2(\xi, \hat{\delta}^i \theta^i) + (\theta^i, \hat{D}^{ij} \theta^j) \tag{4.25}\]

The operators \(\hat{d}, \hat{\delta}^i\) and \(\hat{D}^{ij}\) appearing in expression (4.25) are matrices in coordinate space and are defined by the following relations:

\[(\xi, \hat{d}\xi) = \int d^4x d^4y \xi(x)\hat{d}(x, y)\xi(y)\]

\[= \int d^4x \left[ \partial^\mu \xi \partial_\mu \xi + f^\mu f^\nu \xi^2 + m^2(1 - 3R^2)\xi^2 \right] = (\hat{d}\xi, \xi) \tag{4.26}\]

\[(\xi, \hat{\delta}^i \theta^i) = \int d^4x d^4y \xi(x)\hat{\delta}^i(x, y)\theta^i(y)\]

\[= \int d^4x \left[ -\xi f^k f^\mu\delta^{\mu\nu} \delta^i - \xi \partial^\mu(f^\mu\theta^i) \right] = (\hat{\delta}^i \theta^i, \xi) \tag{4.27}\]

\[(\theta^i, \hat{D}^{ij} \theta^j) = \int d^4x d^4y \theta^i(x)\hat{D}^{ij}(x, y)\theta^j(y)\]

\[= \int d^4x [D^{ik}_{\mu} \theta^i D^{\mu\nu} \theta^j + f^{\mu\nu} \theta^i \theta^j + m^2(1 - R^2)\theta^i \theta^j] \tag{4.28}\]

The following definitions have been introduced:

\[D^i_{\mu k} \theta^i \equiv \partial^i \theta^k + \Gamma^i_{\mu k} \theta^i \tag{4.29}\]

\(^6\)We assume that the fields vanish at infinity and that one can perform partial integrations.

\(^7\)To work with matrices in coordinate space, it is simpler to work in discretized space-time where space-time is divided into unit cells labeled by an index \(\alpha\) and of volume \(\epsilon^4\). An expression of the form \(\int d^4x d^4y \phi(x)K(x - y)\phi(y)\) becomes a matrix product of the form \(\sum_{\alpha\beta} \phi_\alpha K_{\alpha\beta} \phi_\beta \). The continuum limit is taken at the end of the calculation; the integral \(\int d^4x\) is obtained from the following limit: \(\lim_{\epsilon \to 0} \sum_{\alpha} \epsilon^4 \to \int d^4x\). Problem 27.8 of [1] discusses this discretization procedure in greater detail.
CHAPTER 4. ONE-LOOP EFFECTIVE ACTION IN QHD

\[ \Gamma_{\mu}^{k} \equiv \epsilon_{k} \cdot (\nabla_{\mu} \epsilon_{i}) = -\epsilon_{i} \cdot (\nabla_{\mu} \epsilon_{k}) \]  
(4.30)

\[ f_{\mu}^{i} \equiv U_{\epsilon} (\nabla_{\mu} \epsilon_{i}) = -\epsilon_{i} \cdot (\nabla_{\mu} U) \]  
(4.31)

Note that repeated Greek Lorentz indices are summed from 0 to 3. From Eq. (4.15), it is noted that \((\xi, \hat{d} \xi)\) is proportional to the square of the \(\sigma\) mass, \(m^{2}_{\sigma}\); the operator \(\hat{d}\) is proportional to the inverse propagator of the \(\sigma\) field. From the definition of \((\theta^{i}, \hat{D}^{ij} \theta^{j})\), it is also noted that the operator \(\hat{D}^{ij}\) couples only the fluctuations of the non-linear \(\sigma\)-model field, \(U^{A}\); \(\hat{D}^{ij}\) is proportional to the inverse pion propagator and is the quadratic operator of the 1-loop effective action of the non-linear \(\sigma\)-model lagrangian [43] just as \(D\) is the quadratic operator of the 1-loop effective action of the linear \(\sigma\)-model lagrangian.

To perform the gaussian path integral over the fluctuations and obtain the result (4.11), the matrix \(D\) defined by equation (4.25) must be diagonalized in the fluctuations basis \((\xi, \theta^{i})\):

\[ \int d[\xi] d[\theta^{i}] e^{(\xi, \hat{d} \xi) + 2(\xi, \theta^{j}) \cdot (\theta^{i}, \hat{D}^{ij} \theta^{j})} = \int d[\xi'] d[\theta^{i}] e^{(\xi', \hat{d} \xi') + (\theta^{i}, \hat{D}^{ij} \theta^{j})} \]  
(4.32)

where \(\hat{d}'\) and \(\hat{D}'^{ij}\) are the diagonalized operators in the new fluctuations basis \(\xi', \theta^{i}\). In this form, the path integral over the fluctuations can be performed in the standard fashion.

This diagonalization can be accomplished by introducing the new fluctuation variable:

\[ \xi' = \xi + \hat{d}^{-1} \delta^{i} \theta^{i} \]  
(4.33)
Upon substitution,\(^8\) the fluctuation matrix (4.25) is diagonalized and becomes:

\[
(y, D y) = (\xi', \hat{d} \xi') + (\theta^i, \hat{D}^i \theta^i) - (\theta^i, [\hat{\delta}^T \hat{d}^{-1} \hat{\delta}^i] \theta^i)
\]  

(4.34)

Here the inverse operator \(\hat{d}^{-1}\) is defined:

\[
(\xi, \hat{d}^{-1} \hat{d} \xi) = (\xi, \xi) = \int d^4 x d^4 y \xi(x) \delta^4(x - y) \xi(y) = \int d^4 x \xi^2
\]  

(4.35)

In addition, the operator \((\hat{\delta}^T)^i\) is defined:

\[
(\theta^i, (\hat{\delta}^T)^i \xi) = \int d^4 x d^4 y \theta^i(x) \hat{\delta}^T_i(x, y) \xi(y) = \int d^4 x \left[ \theta^i (f^{i\mu}_\mu \partial^\mu \xi + \Gamma^{ij\mu} f^{j\mu}_\mu \xi) + \theta^i \partial^\mu (f^{i\mu}_\mu \xi) \right]
\]  

(4.36)

To derive this result, consider the identity:

\[
(\xi', \hat{d} \xi') = (\xi + \hat{d}^{-1} \hat{\delta}^i \theta^i, \hat{d} \xi + \hat{d}^{-1} \hat{\delta}^i \theta^i)
\]  

(4.37)

The first three terms resemble the first two terms of equation (4.25). The third term is in fact equal to the second term:

\[
(\hat{d}^{-1} \hat{\delta}^i \theta^i, \hat{d} \xi) = (\hat{\delta}^i \theta^i, \xi) \quad \text{using eqns (4.26) and (4.35)}
\]  

(4.38)

The last term can also be rewritten using partial integration. In what follows, we define \(\zeta \equiv \hat{d}^{-1} \hat{\delta}^i \theta^i\) to simplify the notation:

\[
(\hat{d}^{-1} \hat{\delta}^i \theta^i, \hat{d} \xi') = (\hat{d}^{-1} \hat{\delta}^i \theta^i, \hat{d} \hat{d}^{-1} \hat{\delta}^i \theta^i)
\]  

\(^8\)This is a straightforward shift in the path integral variable which does not involve fermions. It is clear that the Jacobian is 1.
\[ = (\hat{\delta}^i \theta^i, \hat{\delta}^{-1} \hat{\delta}^j \theta^j) \]
\[ = (\hat{\delta}^i \theta^i, \zeta) \]
\[ = \int d^4 x [ - \zeta f^i_\mu D^{ij} \partial^\mu \theta^j - \zeta \partial^\mu (f^i_\mu \theta^j)] \]
\[ = \int d^4 x [ - \zeta f^i_\mu (\partial^\mu \theta^i + \Gamma^{ij} \theta^j) - \zeta \partial^\mu (f^i_\mu \theta^j)] \]
\[ = \int d^4 x [\theta^i (f^i_\mu \partial^\mu \zeta + \Gamma^{ij} f^i_\mu \zeta) + \theta^i \partial^\mu (f^i_\mu \zeta)] \]
\[ \equiv (\theta^i, \hat{\delta}^i \zeta) \]
\[ (\hat{\delta}^{-1} \hat{\delta}^i \theta^i, \hat{\delta}^i \theta^i) = (\theta^i, [\hat{\delta}^i \hat{\delta}^{-1} \hat{\delta}^j] \theta^j) \] (4.40)

From the above calculation, it is clear that \( \hat{\delta}^T \) is just \( \hat{\delta} \) partially integrated. Using Eqs. (4.38) and (4.40) Eq. (4.37) becomes:

\[ (\xi', \hat{d} \xi') = (\xi, \hat{d} \xi) + 2(\xi, \hat{\delta}^i \theta^i) + (\theta^i, [\hat{\delta}^T i \hat{\delta}^{-1} \hat{\delta}^j] \theta^j) \] (4.41)

Eq. (4.41) is then used in Eq. (4.25) to obtain Eq. (4.34).

After the path integral, one obtains the term \( i/2 \ln \det D \) as explained before Eq. (4.11). This term can be rewritten:

\[ \frac{i}{2} \ln \det D = \frac{i}{2} \text{Tr} \ln D = \frac{i}{2} [\text{tr} \ln \hat{d} + \text{tr} \ln (\hat{\delta}^T \hat{\delta}^{-1} \hat{\delta})] \] (4.42)

where the trace Tr involves both a sum over the diagonal elements of the fluctuations matrix (4.25) and a trace tr in coordinate space. To obtain (4.42), Eq. (4.34) was used to take the sum over the diagonal elements of (4.25). For a generic operator \( \hat{O}(x, y) \), tr is defined:

\[ \text{tr} \hat{O} \equiv \int d^4 x \hat{O}(x, x) \] (4.43)

This trace in coordinate space will be made more clear below with a sample calcu-
lation. At the present, consider from Eq. (4.42):

\[
\frac{i}{2} \ln \det D = \frac{i}{2} [\text{tr} \ln \hat{d} + \text{tr} \ln (\hat{D} - \hat{\delta^T \delta}^{-1} \hat{\delta})]
\]

\[
= \frac{i}{2} \text{tr} \ln \hat{d} + \frac{i}{2} \text{tr} \ln [\hat{D}(1 - \hat{\delta}^{-1} \hat{\delta^T \delta}^{-1} \hat{\delta})]
\]

\[
= \frac{i}{2} \text{tr} \ln \hat{d} + \frac{i}{2} \text{tr} \ln (1 - \hat{\delta}^{-1} \hat{\delta^T \delta}^{-1} \hat{\delta}) + \frac{i}{2} \text{tr} \ln \hat{D}
\]

(4.44)

where the indices were suppressed for simplicity of notation. Thus, the calculation of the 1-loop corrections of the effective action has been broken down to the calculation of the three traces of logs above. Since what follows is very formal and abstract, here is a summary of what has been done so far, and of what remains to be done. What has been done so far is:

- The action was expanded in powers of the quantum fluctuations about the classical fields; the expansion was carried out to second order in powers of the fluctuations.

- The quadratic operator that contains the loop corrections was diagonalized in an orthonormal set of basis vectors along which the fields fluctuate.

In the calculation of the traces, the low-energy expansions will be carried out with respect to the two mass scales: \( F^2 \) and \( m_\sigma^2 \). The power counting in the calculation of the 1-loop effective action of the linear \( \sigma \)-model is as follows:

- \( M^2 \), the mass of the pion, is counted as of \( \mathcal{O}(p^2) \).

- \( m_\sigma^2 \) is much larger than any other masses in the problem and all terms of \( \mathcal{O}(m_\sigma^{-1}) \) are neglected. Contributions from \( \sigma \) exchange are non-zero, however, since terms of \( \mathcal{O}(m_\sigma^2) \) and \( \mathcal{O}(m_\sigma^0) \) contribute. As \( m_\sigma \to \infty \), the terms of \( \mathcal{O}(m_\sigma^2) \) become
new divergences that are not canceled by the counterterms of the theory, and the renormalizable linear $\sigma$-model becomes the non-renormalizable non-linear $\sigma$-model.

- Since $U \cdot U = 1$, then $|U^A| < 1$. Thus, an expansion in powers of $U^A = \phi^A / F$ is possible. By dimensional analysis, any operator of $\mathcal{O}[(U^A)^n]$ should be of $\mathcal{O}(p^4)$; an expansion in powers of $U^A$ corresponds to a momentum expansion. In what follows, all operators of $\mathcal{O}[(U^A)^n]$ with $n > 4$ shall be neglected: only the $\mathcal{O}(p^4)$ effective action is retained.

- In all the calculations, the vector fields $a_\mu$ and $\rho_\mu$ are counted as of $\mathcal{O}(U^i)$ and $\mathcal{O}[(U^i)^2]$ respectively. This assignment will be demonstrated by using the equations of motion to re-express the vector fields in terms of the pion fields when the 1-loop effective action of QHD-III is considered.

What remains is therefore:

- The calculation of each of the three traces in Eq. (4.44).

- The calculation of $\mathcal{L}(\phi)$ using the equation of motion for $R$.

- The renormalization of the effective action.

These manipulations should eventually yield a lagrangian of the same form as the chiral perturbation theory lagrangian given in the introduction and rewritten here:

$$
\mathcal{L}_4 = l_1(\nabla_\mu U \cdot \nabla^\mu U)^2 + l_2(\nabla_\mu U \cdot \nabla_\nu U)(\nabla^\mu U \cdot \nabla^\nu U) + l_3(\chi_{ab} \cdot U)^2
+ l_4(\nabla_\mu \chi_{ab} \cdot \nabla^\mu U) + l_5(U F_{\mu\nu} F^{\mu\nu} U) + l_6(\nabla_\mu U F^{\mu\nu} \nabla_\nu U)
+ l_7(\bar{\chi} \cdot U)^2 + l_8 \text{tr} F_{\mu\nu} F^{\mu\nu} + h_3 \bar{\chi} \cdot \bar{\chi}
$$

(4.45)
This is so because Eq. (4.45) is the most general covariant lagrangian that can be written that is invariant under $SU(2)_L \times SU(2)_R$, parity and G-parity.

The author has calculated separately each of the three traces in Eq. (4.44). The calculations are well defined at this point, but very lengthy (120 pages of the author's notes). The sum of the three traces is given in [43] which provides an overall check on the calculations. Here we simply state the results of the calculation of the three individual traces in this thesis.

Consider the first trace in Eq. (4.44). Note that it involves the operator $\hat{d}$ which is related to the scalar field propagator. The log can be expanded in inverse powers of $m_\sigma$. The differential operator $\hat{d}$ can be extracted from Eq. (4.26), and the first trace of (4.44) can be rewritten:

$$\text{tr} \ln \hat{d} = \text{tr} \ln \left[ -\partial_\mu \partial_\mu + f_\mu f^{\mu} + m^2 (1 - 3R^2) \right]$$

$$= \text{tr} \ln \left[ -\partial_\mu \partial_\mu - m_\sigma^2 + f_\mu f^{\mu} + 3m^2 (1 - R^2) \right]$$

$$= \text{tr} \ln \left\{ \left( -\partial_\mu \partial_\mu - m_\sigma^2 \right) [1 + (-\partial_\mu \partial_\mu - m_\sigma^2)^{-1}] \sigma \right\}$$

$$= \text{tr} \ln d_0 + \text{tr} \ln (1 + d_0^{-1} \sigma) \quad (4.46)$$

with the following definitions:

$$d_0 = [-\partial_\mu \partial_\mu - m_\sigma^2] \quad (4.47)$$

$$\sigma = f_\mu f^{\mu} + 3m^2 (1 - R^2). \quad (4.48)$$

Here, $d_0^{-1}$ is the inverse of the operator $[-\partial_\mu \partial_\mu - m_\sigma^2]$ in coordinate space and satisfies:

$$[-\partial_\mu \partial_\mu - m_\sigma^2]d_0^{-1}(x) = \delta^4(x) \quad (4.49)$$

$\delta^4(x)$ is the unit matrix in coordinate space and $d_0^{-1}(x)$ is the scalar propagator when the boundary condition are included, i.e., when a small, negative imaginary term is

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added to the mass. It is given by:

$$d_0^{-1}(x - y) = \int \frac{d^4l}{(2\pi)^4} \frac{e^{-i l \cdot (x - y)}}{l^2 - m^2 + i\epsilon} \tag{4.50}$$

The first term of Eq. (4.46) is a constant which does not involve the pion fields; it can therefore be ignored. The second term involves the log of an operator. The \(\ln(1 + d_0^{-1}\sigma_1)\) is an example of the operator \(\hat{O}(x, y)\) in equation (4.43) and is expanded in inverse powers of \(m_\sigma\):\(^9\)

$$\hat{O}(x, y) = \ln(1 + d_0^{-1}\sigma_1)$$

$$= d_0^{-1}(x - y)\sigma_1(y)$$

$$= -\frac{1}{2} \int d^4z d_0^{-1}(x - z)\sigma_1(z)d_0^{-1}(z - y)\sigma_1(y) + \mathcal{O}(m^2) \tag{4.51}$$

These are the only terms that contribute in the expansion of the log; all other terms are suppressed by inverse powers of \(m_\sigma\). Note that the second term in (4.51) is simply matrix multiplication in coordinate space.

From the definition of tr in Eq. (4.43), we have:

$$\text{tr} \ln \hat{d} = \text{tr} \ln(1 + d_0^{-1}\sigma_1)$$

$$= \int d^4x \{d_0^{-1}(0)\sigma_1(x)$$

$$- \frac{1}{2} \int d^4z d_0^{-1}(x - z)\sigma_1(z)d_0^{-1}(z - x)\sigma_1(x)\} \tag{4.52}$$

As a sample calculation, consider the first term of Eq. (4.52) which represents tadpole diagrams.\(^10\) It is given by:

$$\text{tr}(d_0^{-1}\sigma_1) = \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2 - m^2 + i\epsilon} \int d^4x \{f_{\mu}(x)f^{\mu}(x) + 3m^2[1 - R^2(x)]\}$$

\(^9\)Recall from Eqs. (4.14) and (4.15) that \(R = 1 + \mathcal{O}(m^2)\).

\(^10\)The fact that this trace represents tadpole diagrams can be seen by noting the presence of only one scalar propagator coupled to pion fields.
where $\lambda$ is defined:

$$\lambda \equiv \Gamma \left( \frac{\epsilon}{2} \right) + \ln 4\pi - \ln \frac{m^2}{\mu^2}$$

The momentum integral was performed using dimensional regularization with the dimensionality of the integral given by: $d = 4 - \epsilon$. The above can be rewritten by making use of Eq. (4.14) and the following identity (everything is evaluated at the point $x$ which is suppressed for ease of reading):

$$f^i_{\mu}f^{\mu i} = (\varepsilon^A_i \nabla_{\mu} U^A)(\varepsilon^B_i \nabla_{\mu} U^B)$$

$$= \nabla_\mu U \cdot \nabla^\mu U$$

The completeness relation (4.19) is used to prove the above result. Thus, Eq. (4.53) becomes:

$$\text{tr}(d_0^{-1}\sigma_1) = \frac{i}{(4\pi)^2} m^2 \left( \lambda + 1 \right) \int d^4x \left\{ \nabla_\mu U \cdot \nabla^\mu U - 3m^2 \left[ \frac{1}{m^2} \nabla_\mu U^A \nabla^\mu U^A + \frac{\sqrt{g}}{m^3} f^A U^A \right] \right\}$$

$$= \frac{i}{(4\pi)^2} m^2 \left( \lambda + 1 \right) \int d^4x \left( -2 \nabla_\mu U \cdot \nabla^\mu U - 3 \frac{\sqrt{g}}{m} f^A U^A \right)$$

$$= \frac{i}{(4\pi)^2} \left( \lambda + 1 \right) \int d^4x \left( 2 \nabla_\mu U \cdot \nabla^\mu U + 3 \frac{\sqrt{g}}{m} f^A U^A \right)$$

\(^{11}\text{In Eq. (4.14), there are terms of } \mathcal{O}(p^4) \text{ that contribute. These terms cancel in the renormalization and are ignored at this stage for simplicity.}\)
CHAPTER 4. ONE-LOOP EFFECTIVE ACTION IN QHD

As mentioned earlier, the 1-loop effective action should have the structure of the terms that appear in Eq. (4.45). It is noted that the terms of Eq. (4.56) are of that form. Making the identification:

\[ \chi^A \equiv \frac{\sqrt{\theta}}{m} f^A \]  

Eq. (4.56) can be rewritten using (4.57):

\[ \text{tr}(d_0^{-1}\sigma_1) = \frac{i}{(4\pi)^2}(\overline{\lambda} + 1) \int d^4x - m_0^2(2\nabla_\mu U \cdot \nabla^\mu U + 3\chi \cdot U) \]  

The second term in equation (4.52) is evaluated in a similar fashion. In the process of calculating it, a term of the form \( \chi \cdot U(\nabla_\mu U \cdot \nabla^\mu U) \) will be obtained. At a first glance, this term is not of the form given in Eq. (4.45). This term can in fact be rewritten using the equations of motion of the non-linear \( \sigma \) model whose lagrangian is rewritten here:

\[ \mathcal{L}_2 = F^2 \left( \frac{1}{2} \nabla_\mu U^A \nabla^\mu U^A + \chi^A U^A \right) \]  

The derivation of the equation of motion for \( U \) is straightforward\(^{12}\) and yields:

\[ \nabla_\mu \nabla^\mu U^A - U^A[U \cdot (\nabla_\mu \nabla^\mu U)] = \chi^A - U^A(U \cdot \chi) \]  

(4.60)

Multiplying Eq. (4.60) by \( \chi^A \) yields:

\[ \chi \cdot U[U \cdot (\nabla_\mu \nabla^\mu U)] = \chi \cdot \nabla_\mu \nabla^\mu U - \chi \cdot \chi + (U \cdot \chi)^2 \]  

(4.61)

Eq. (4.61) combined with partial integrations in the action can be used to re-express \( \chi \cdot U(\nabla_\mu U \cdot \nabla^\mu U) \) in the form of the terms of Eq. (4.45).

\(^{12}\)The constraint \( U \cdot U = 1 \) must be taken into account.
The calculation of the first trace of Eq. (4.44) can now be completed straightforwardly and the end result is:

\[
\text{tr} \ln \hat{d} = \int d^4x [\alpha_1 \nabla_\mu U \cdot \nabla^\mu U + \beta_1 \chi \cdot U + c_1 (\nabla_\mu U \cdot \nabla^\mu U)^2 + d_1 \nabla_\mu \chi \cdot \nabla^\mu U + e_1 \chi \cdot \chi + f_1 (\chi \cdot U)^2] \tag{4.62}
\]

with

\[
\begin{align*}
\alpha_1 &= -\frac{i}{(4\pi)^2} \left( \tilde{\lambda} + 1 \right) 2m_\sigma^2 \\
\beta_1 &= -\frac{i}{(4\pi)^2} \left( \tilde{\lambda} + 1 \right) 3m_\sigma^2 \\
c_1 &= -2 \frac{i}{(4\pi)^2} \tilde{\lambda} \\
d_1 &= -6 \frac{i}{(4\pi)^2} \tilde{\lambda} \\
e_1 &= -6 \frac{i}{(4\pi)^2} \tilde{\lambda} \\
f_1 &= \frac{3}{2} \frac{i}{(4\pi)^2} \tilde{\lambda}
\end{align*}
\]

Consider now the second trace in Eq. (4.44) expanded in inverse powers of \(m_\sigma\):

\[
\text{tr} \ln (1 - \hat{D}^{-1} \hat{d}^{-1} \hat{d}^{-1}) = -\text{tr} (\hat{D}^{-1} \hat{d}^{-1} \hat{d}^{-1}) - \frac{1}{2} \text{tr} [(\hat{D}^{-1} \hat{d}^{-1} \hat{d}^{-1})^2] + \mathcal{O}(m_\sigma^{-2}) \tag{4.64}
\]

The evaluation of this trace is significantly more complex than the previous calculation. In particular, the calculation of the first term in Eq. (4.64) requires two additional types of expansions. First the operators \(\hat{D}^{-1}\) and \(\hat{d}^{-1}\) in Eqs. (4.26) and (4.28) must be expanded. This can be more easily done by first defining:

\[
\begin{align*}
D_M^{ij} &= -(\partial_\mu \partial^\mu - M^2 + i\epsilon) \delta^{ij} \tag{4.65} \\
\delta^{ij} &= -2 \Gamma^{ij} \partial^\mu - \partial^\mu \Gamma^{ij} - \Gamma^{ik} \Gamma^{kj} + f_\mu^i f_\nu^j + m^2 (1 - R^2) \delta^{ij} + M^2 \delta^{ij} \tag{4.66}
\end{align*}
\]
Then:

\[
(\hat{D}^{-1})^{ij}(x,y) = [-D^{ik}_{\mu}D^{kj}_\mu - (M^2 - i\epsilon)\delta^{ij} + f^i_\mu m^j_\mu + m^2(1 - R^2\delta^{ij}) + M^2\delta^{ij}]^{-1} \\
= [D^{ij}_M + \delta^{ij}]^{-1} \\
= \int d^4 z(x)(D^{-1}_M)^{ik}(x-z)[\delta^{kl}(z-y)
- \delta^{kl}(z)(D^{-1}_M)^{lk}(z-y) + \cdots] 
\]

(4.67)

where the propagator of the pion field is given by:

\[
(D^{-1}_M)^{ij}(x-y) = \int \frac{d^4 l}{(2\pi)^4} \frac{e^{-il(x-y)}\delta^{ij}}{l^2 - M^2 + i\epsilon} 
\]

(4.68)

Note that the Eq. (4.67) is an explicit example of the p/F expansion: each non-constant term is of \(O(p^2/F^2)\). The expansion of \(d^{-1}\) is given by:

\[
d^{-1}(x,y) = [d_0 + \sigma_1]^{-1} \\
= \int d^4 z d^{-1}_0(x-z)[\delta^4(z-y) - \sigma_1(z)d^{-1}_0(z-y) + \cdots] 
\]

(4.69)

To evaluate the term \(\text{tr}(\hat{D}^{-1}\hat{T}d^{-1}\hat{T})\) one must substitute the expansions (4.67) and (4.69). For example, considering only the zeroth order terms in (4.67) and (4.69) and using Eqs. (4.27) and (4.36) [noting that \(\hat{T}^{ij}(x,y)\) and \(\hat{\delta}^i(x,y)\) are proportional to \(\delta^4(x-y)\)] yields:

\[
\text{tr}(\hat{D}^{-1}\hat{T}d^{-1}\hat{T}) \approx \int d^4 x d^4 y d^4 z d^4 \omega \\

= \int \frac{d^4 l}{(2\pi)^4} \frac{d^4 p}{(2\pi)^4} (\frac{e^{-il(x-y)}}{(l^2 - M^2 + i\epsilon)(p^2 - m_y^2 + i\epsilon)} \\
\times [-2if_{\mu}^i(x)p^\mu + \partial^\mu f_{\mu}^i(x) + \Gamma^{i\mu\nu}(x)f_{\mu}^k(x)] \\
\times [2if_{\mu}^i \Gamma^{\nu\rho}(y) - \partial^\nu f_{\nu}^i(y) - 2f_{\nu}^i(y)\Gamma^{i\nu}(y)] \}
\]

(4.70)
In this low-energy effective theory, functions such as $f^i_\mu$ evaluated at the point $x$ can only differ from their value at the point $y$ by powers of the derivative. To evaluate equation (4.70), one must go to Jacobi coordinates:

$$v = \frac{1}{2}(x + y), \quad u = x - y, \quad \Rightarrow \quad x = v + \frac{u}{2}, \quad y = v - \frac{u}{2} \quad (4.71)$$

and expand all the functions that appear in (4.70) about the point $v$. This Taylor series is a momentum expansion and is the second expansion that must be performed on $\text{tr}(\hat{D}^{-1}\delta\hat{T}\hat{d}^{-1}\delta)$. These two expansions complicate the calculation of the trace substantially and we simply quote the end result for the second trace of Eq. (4.44):

$$\text{tr} \ln(1 - \hat{D}^{-1}\delta\hat{T}\hat{d}^{-1}\delta) = \int d^4x [\alpha_2 \nabla_\mu U \cdot \nabla^\mu U + \beta_2 \chi \cdot U$$

$$+ c_2 (\nabla_\mu U \cdot \nabla^\mu U)^2 + g_2 (\nabla_\mu U \cdot \nabla^\mu U)^2$$

$$+ d_2 \nabla_\mu \chi \cdot \nabla^\mu U + e_2 \chi \cdot \chi + f_2 (\chi \cdot U)^2$$

$$+ h_2 U F_{\mu\nu} F^{\mu\nu} U + i_2 \nabla_\mu U F^{\mu\nu} \nabla_\nu U] \quad (4.72)$$

with

$$\alpha_2 = -\frac{i}{(4\pi)^2}(\tilde{\lambda} + \frac{3}{2})m_\sigma^2$$

$$\beta_2 = 0$$

$$c_2 = -\frac{i}{(4\pi)^2}\left(\frac{11}{3}\tilde{\lambda} + \frac{67}{18}\right)$$

$$d_2 = -\frac{i}{(4\pi)^2}(4\tilde{\lambda} + 3)$$

$$e_2 = -\frac{i}{(4\pi)^2}(4\tilde{\lambda} + \frac{19}{6})$$

$$f_2 = \frac{i}{(4\pi)^2}(4\tilde{\lambda} + \frac{19}{6})$$

$$g_2 = \frac{i}{(4\pi)^2}\left(\frac{2}{3}\tilde{\lambda} + \frac{17}{9}\right)$$

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The last trace that must be evaluated in Eq. (4.44) is the trace of the non-linear \( \sigma \) model 1-loop operator.

\[
\begin{align*}
\text{tr} \ln \hat{D} &= \text{tr} \ln[ - D^{ik}_{\mu} D^{kj}_{\mu} + f^{\mu}_{ij} f^{ij} + m^2 (1 - R^2) ] \\
&= \text{tr} \ln[ - \partial_{\mu} \partial^{\mu} - M^2 ] \delta^{ij} \\
&\quad + m - 2 \Gamma^{ij}_{\mu} \partial^\mu - \partial^\mu \Gamma^{ij}_{\mu} - \Gamma^{ik}_{\mu} \Gamma^{kj}_{\mu} + f^{\mu}_{ij} f^{\mu ij} + m^2 (1 - R^2) \delta^{ij} + M^2 \delta^{ij} \\
&= \text{tr} \ln[D_M + \hat{\sigma}] 
\end{align*}
\]  

(4.74)

With this result, \( \text{tr} \ln \hat{D} \) can now be expanded in powers of \( p/F \):

\[
\text{tr} \ln \hat{D} = \text{tr} \ln[D_M (1 + D^{-1}_M \hat{\sigma})] \\
= \text{tr}(D^{-1}_M \hat{\sigma}) - \frac{1}{2} \text{tr}(D^{-1}_M \hat{\sigma} D^{-1}_M \hat{\sigma}) + \mathcal{O}(\hat{\sigma}^6) 
\]  

(4.75)

Eq. (4.75) can now be evaluated using the expression for the pion propagator, \( \hat{\sigma} \) and the definition for tr. We quote the result:

\[
\begin{align*}
\text{tr} \ln \hat{D} &= \int d^4x \left[ \alpha_3 \nabla_\mu U \cdot \nabla^\mu U + \beta_3 \chi \cdot U \right. \\
&\quad + c_3 (\nabla_\mu U \cdot \nabla^\mu U)^2 + g_3 (\nabla_\mu U \cdot \nabla_\nu U)^2 + d_3 \nabla_\mu \chi \cdot \nabla^\mu U \\
&\quad + e_3 \chi \cdot \chi + f_3 (\chi \cdot U)^2 + h_3 U F_{\mu\nu} F^{\mu\nu} U + i_3 \nabla_\mu U F^{\mu\nu} \nabla_\nu U \\
&\quad \left. + Z_u \right] \\
\end{align*}
\]  

(4.76)

with

\[
\alpha_3 = -2 \frac{i}{(4\pi)^2} M^2
\]
\[ \beta_3 = -3 \frac{i}{(4\pi)^2} M^2 \]
\[ c_3 = -\frac{i}{(4\pi)^2} \left( \frac{1}{3} \tilde{\lambda}_0 - \frac{1}{9} \right) \]
\[ d_3 = -2 \frac{i}{(4\pi)^2} \tilde{\lambda}_0 \]
\[ e_3 = -2 \frac{i}{(4\pi)^2} \tilde{\lambda}_0 \]
\[ f_3 = \frac{i}{(4\pi)^2} \frac{1}{2} \tilde{\lambda}_0 \]
\[ g_3 = -\frac{i}{(4\pi)^2} \left( \frac{2}{3} \tilde{\lambda}_0 + \frac{1}{9} \right) \]
\[ h_3 = \frac{i}{(4\pi)^2} \left( \frac{1}{6} \tilde{\lambda}_0 + \frac{1}{9} \right) \]
\[ i_3 = \frac{i}{(4\pi)^2} \left( \frac{1}{3} \tilde{\lambda}_0 + \frac{2}{9} \right) \]

\[ \tilde{\lambda}_0 \equiv \Gamma \left( \frac{\xi}{2} \right) + \ln 4\pi \] (4.78)

In Eq. (4.76), \( Z_u \) contains the unitary logs and is given in appendix C.

The last thing that must be done to evaluate the effective action (4.11) is to calculate \( \mathcal{L}(\tilde{\phi}) \) using Eqs. (4.13) and (4.14). The algebra is straightforward and yields:

\[ \mathcal{L}(\tilde{\phi}) = \frac{1}{2} \frac{m^2}{g} \nabla_\mu U \cdot \nabla^\mu U + \frac{m^2}{g} \chi \cdot U \]
\[ + \frac{1}{4g} \left( \nabla_\mu U \cdot \nabla^\mu U \right)^2 + \frac{1}{2g} \left( \nabla_\mu \chi \cdot \nabla^\mu U + \chi \cdot U \right) - \frac{1}{4g} \left( \chi \cdot U \right)^2 \] (4.79)

where \( \chi \) is given in equation (4.57).

With Eqs. (4.63), (4.73), (4.77) and (4.79), the parameters of the chiral lagrangian
in equation (4.45) have now been calculated in the linear \( \sigma \)-model. The result is:

\[
\alpha = \frac{m^2}{2g} + \frac{1}{(4\pi)^2} \left( \frac{3}{2} m^2 \lambda + \frac{7}{4} m^2 - M^2 \right) \\
\beta = \frac{m^2}{g} + \frac{1}{(4\pi)^2} \left( \frac{3}{2} m^2 \lambda + \frac{3}{2} m^2 + \frac{3}{2} M^2 \right) \\
l_1 = \frac{1}{4g} + \frac{1}{(4\pi)^2} \left( \frac{17}{6} \lambda + \frac{\lambda_0}{6} + \frac{65}{36} \right) \\
l_2 = \frac{1}{(4\pi)^2} \left[ -\frac{1}{3} (\lambda - \lambda_0) - \frac{8}{9} \right] \\
l_3 = -l_4 + \frac{1}{4g} + \frac{1}{(4\pi)^2} \left( \frac{9}{4} \lambda + \frac{3}{4} \lambda_0 - \frac{1}{12} \right) \\
l_4 = \frac{1}{2g} + \frac{1}{(4\pi)^2} \left( 5\lambda + \lambda_0 + \frac{3}{2} \right) \\
l_5 = \frac{1}{(4\pi)^2} \left[ \frac{1}{12} (\lambda - \lambda_0) + \frac{1}{72} \right] \\
l_6 = \frac{1}{(4\pi)^2} \left[ \frac{1}{6} (\lambda - \lambda_0) + \frac{13}{36} \right] \\
l_7 = 0 \\
h_1 = l_4 + \frac{1}{(4\pi)^2} \frac{1}{12} \\
h_2 = 0 \\
h_3 = 0
\]

where \( \lambda \) and \( \lambda_0 \) were defined in Eqs. (4.54) and (4.78). The 1-loop effective action is finite since the linear \( \sigma \)-model is renormalizable. To demonstrate this, the counterterms calculated in Chapter 2 that renormalize the \( \pi \pi \) scattering amplitude in the linear \( \sigma \)-model are rewritten here using the relation \( g_{\pi}/M^2 \equiv g/m^2 \):\(^{13}\)

\[
\delta_{\mu} = -6g \frac{1}{16\pi^2} \Gamma\left(\frac{\xi}{2}\right); \quad \delta_{\lambda} = -2\delta_{\mu}; \quad \delta_{\pi} \text{ finite.} \quad (4.92)
\]

\(^{13}\) Note the extra factor of 2 with respect to the original counterterms; it arises because \( m_\pi^2 = 2m^2 \).
We write:

\[ m_r^2 = m^2(1 - \delta_\mu), \quad g_r = g(1 + \delta_\lambda), \quad (4.93) \]

where the subscript \( r \) means renormalized. The following relations are easily derived:

\[ \frac{1}{g} = \frac{1}{g_r} - 12\frac{1}{(4\pi)^2} \Gamma\left(\frac{\epsilon}{2}\right) \quad (4.94) \]
\[ \frac{m^2}{g} = \frac{m_r^2}{g_r} - 6\frac{1}{(4\pi)^2} \Gamma\left(\frac{\epsilon}{2}\right) \quad (4.95) \]

From Eq. (4.80), it is clear that \( m^2/g_r \) is the tree-level pion decay constant and must be finite. Since \( m_r^2 \) is proportional to \( m^2_r \), so must \( g_r \). Therefore, terms such as \( 1/g_r \) must be neglected. Furthermore, in renormalizing the effective action, special care must be taken with the term with coefficient \( \beta \):

\[ \frac{m^2}{g} \chi \cdot U = \frac{m}{\sqrt{g}} f \cdot U \quad (4.96) \]

It is the ratio \( m/\sqrt{g} \) which must be renormalized.

In the terms that are multiplied by the coefficients \( \{l_3, l_4, h_1\} \), the chiral symmetry breaking field \( \chi \) is already renormalized; that was the purpose of ignoring the \( \mathcal{O}(p^4) \) in (4.56). Thus, only Eq. (4.94) is required to renormalize those terms. It is now straightforward to verify that the effective action is finite.

It is also possible to give an alternate renormalization scheme to the minimal subtraction scheme (MS) given above where only the divergences are canceled. For instance, if one were to add a finite piece to the mass counterterm:

\[ \delta_\mu = -6g\frac{1}{16\pi^2}[\Gamma\left(\frac{\epsilon}{2}\right) + \gamma] \quad (4.97) \]

then one could pick \( \gamma \) to satisfy particular renormalization conditions. In [43], Gasser
and Leutwyler picked the following condition:

$$F_\pi = \frac{m_r^2}{g_r}$$

(4.98)

For this renormalization condition, it was found that:

$$\gamma = \frac{1}{(4\pi)^2} \frac{1}{12}$$

(4.99)

With this renormalization condition, the contributions to the parameters of the chiral lagrangian due to $\sigma$ exchange are:

$$\alpha = \frac{m_r^2}{2g_r}$$

$$\beta = \frac{m_r^2}{2g_r} \left(1 - \frac{g_r}{32\pi^2}\right)$$

$$l_1 = \frac{1}{4g} + \frac{1}{(4\pi)^2} \left(\frac{17}{6} \lambda + \frac{67}{36}\right)$$

$$l_2 = -\frac{1}{(4\pi)^2} \left(\frac{\lambda}{3} + \frac{17}{18}\right)$$

$$l_3 = -\frac{1}{2g} - \frac{1}{(4\pi)^2} \left(\frac{11}{4} \lambda + \frac{19}{12}\right)$$

$$l_4 = \frac{1}{2g} + \frac{1}{(4\pi)^2} \left(5\lambda + \frac{3}{2}\right)$$

$$l_5 = \frac{1}{(4\pi)^2} \left(\frac{\lambda}{12} + \frac{5}{72}\right)$$

$$l_6 = \frac{1}{(4\pi)^2} \left(\frac{\lambda}{6} + \frac{17}{36}\right)$$

$$l_7 = 0$$

$$h_1 = l_4 + \frac{1}{(4\pi)^2} \frac{1}{12}$$

$$h_2 = 0$$

$$h_3 = 0$$

(4.100)

\textsuperscript{14}The contributions from $\sigma$ exchange are quoted separately as was done in [43] for ease of comparison.
which agrees with the result given in [43]. All these terms agree with the result quoted in [43] except for $l_4$; in [43] they have a 2 instead of a $3/2$ within the parenthesis of the equation for $l_4$. The author believes his result to be correct.

As an application of the 1-loop effective action, note that the pion decay constant to 1-loop can be read off from $\alpha$ in the chiral limit $M^2 = 0$:

$$F^2_\pi = 2\alpha$$

$$= \frac{m^2}{g} + \frac{1}{(4\pi)^2} \left( 3m_\sigma^2 \lambda + \frac{7}{2}m_\rho^2 \right)$$

$$= \sigma_0^2 \left[ 1 + \frac{3}{(4\pi)^2} \frac{m_\sigma^2}{\sigma_0^2} \left( \lambda + \frac{7}{6} \right) \right]$$

(4.101)

where $\sigma_0^2 = m^2/g = M^2/g^2$. This result is identical to the result obtained using the Feynman diagram expansion in Chapter 2.

The $\pi\pi$ scattering amplitude was also extracted from the effective action; this was done by taking four functional derivatives of the generating functional with respect to the pseudoscalar source $\chi^i$. One can also extract the scattering amplitude by expressing the $\chi^i$ field in terms of the pion field using the equation of motion (4.60), and calculating the Feynman rules for the pionic self-interactions. The result obtained for the $\pi\pi$ scattering amplitude is also identical to the result obtained in Chapter 2.

In the predicted parameters from (4.80) to (4.91), it is noted that the parameters $\{l_3, l_4, h_1\}$ are all related. These relations are induced by the equation of motion (4.61). These relations are not reflected by the data [43] and appear because the linear $\sigma$-model assumes that physics at all scales can be described by the measurement of only two parameters, $m_\sigma$ and $g$. QHD-III contains extra parameters, $m_\rho$ and $g_\rho$,
which may improve upon the predictions of the linear $\sigma$-model.

Thus, after having derived the renormalized 1-loop effective action of the linear $\sigma$-model, and after having shown in an example how to extract physical predictions, consider now the 1-loop effective action of QHD-III.

4.3 One-loop effective action of QHD-III

In this section, the 1-loop effective action of QHD-III to $O(g_2^2)$ is formulated. To this order, the Higgs fields do not contribute and the model is equivalent to a massive Yang-Mills theory where the gauge boson masses are introduced by hand. In QHD-III, the fields $a_\mu$ and $\rho_\mu$ are no longer mere external fields, they are now dynamical fields that participate in processes. To $O(g_2^2)$, the QHD-III lagrangian can be written before the spontaneous symmetry breaking:

$$\mathcal{L}_{III} = \frac{1}{2} \nabla_\mu \phi^A \nabla^\mu \phi^A + \frac{1}{2} m^2 \phi^A \phi^A - \frac{g}{4} (\phi^A \phi^A)^2 + f^A \phi^A - \frac{1}{4} A_{\mu \nu} \cdot A^{\mu \nu} - \frac{1}{4} R_{\mu \nu} \cdot R^{\mu \nu} + \frac{1}{2} m_2 a_\mu \cdot a^\mu + \frac{1}{2} m_\rho \rho_\mu \cdot \rho^\mu$$

(4.102)

where $A_{\mu \nu}$, $R_{\mu \nu}$ are given by:

$$A_{\mu \nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu - g_\rho (\rho_\mu \times a_\nu + a_\mu \times \rho_\nu)$$

$$R_{\mu \nu} \equiv \partial_\mu \rho_\nu - \partial_\nu \rho_\mu - g_\rho (\rho_\mu \times \rho_\nu + a_\mu \times a_\nu)$$

(4.103)

In applying the background field method, not only does the $\phi$ field fluctuate, but so do the gauge fields:

$$a^i_\mu = \bar{a}^i_\mu + \alpha^i_\mu$$

(4.104)

$$\rho^i_\mu = \bar{\rho}^i_\mu + \tau^i_\mu$$

(4.105)
where $a^i_\mu$ and $r^i_\mu$ are the quantum fluctuations of $a^i_\mu$ and $\rho^i_\mu$ respectively. The above expansion of the gauge fields about their classical fields will be substituted into the QHD-III lagrangian as was done in the linear $\sigma$-model. QHD-III involves the new mass scale $m_\rho$ as well as the coupling constant $g_\rho$. The power counting described after Eq. (4.44) must now be amended to take these facts into account. In addition to the previous power counting rules, the following rules now also apply:

- $m_\rho^2 \gg m_\rho^2 \gg p^2$ and terms of $O(m_\rho^{-4})$ are retained.
- Terms of $O(g_\rho^n)$ with $n > 2$ are neglected.

With these rules in mind, $\bar{a}^i_\mu$ and $\tilde{\rho}^i_\mu$ can be expressed in terms of the pion fields by using the equations of motion (in what follows, the definition $F^2 = m^2/g = M^2/g_\sigma^2$ is used):

$$
\partial_\nu A^{\mu i} = m_\rho^2 \bar{a}^{\mu i} - g_\rho F^2 R^2 (U^0 \partial^\mu U^i - U^i \partial^\mu U^0) + O(g_\rho^3) \quad (4.106)
$$

$$
\partial_\nu R^{\mu i} = m_\rho^2 \tilde{\rho}^{\mu i} + g_\rho F^2 R^2 \epsilon^{ijk} \partial^\mu U^j U^k + O(g_\rho^3) \quad (4.107)
$$

Thus:

$$
\bar{a}^{\mu i} = \frac{g_\rho}{m_\rho^2} F^2 R^3 (U^0 \partial^\mu U^i - U^i \partial^\mu U^0) + O(g_\rho^3, p^3) \quad (4.108)
$$

$$
\tilde{\rho}^{\mu i} = - \frac{g_\rho}{m_\rho^2} F^2 R^3 \epsilon^{ijk} \partial^\mu U^j U^k + O(g_\rho^3, p^3) \quad (4.109)
$$

The above result demonstrates three things:

- The gauge bosons are of $O(g_\rho)$
- $a^i_\mu$ and $\rho^i_\mu$ are of $O(U^i)$ and $O[(U^i)^2]$ respectively. This verifies the power counting rules that follow Eq. (4.44).
• The gauge bosons are inversely proportional to $m^2$. This proves that in the limit $m^2 \to \infty$, the gauge bosons decouple.

With an expression for the classical gauge fields, equation (4.104) can be substituted into the QHD-III lagrangian; after integration over the fluctuations, one obtains:

$$S = \int dx \mathcal{L}_{III}[\delta^A; \bar{a}^i, \bar{\rho}^i] + \frac{i}{2} \ln \det D_3$$

(4.110)

$D_3$ is the QHD-III operator that contains the 1-loop corrections. One can use the expressions for $a^i$ and $\rho^i$ in terms of the pion fields in the QHD-III lagrangian:

$$\mathcal{L}_{III}[\delta^A; \bar{a}^i, \bar{\rho}^i] = \frac{F^2}{2} \left(1 - \frac{g^2}{m^2} F^2\right) \partial_\mu U \cdot \partial^\mu U + \cdots$$

(4.111)

From the term above, the tree-level pion decay constant can be extracted:

$$\frac{1}{F^2} = \frac{1}{F^2 \left(1 - \frac{g^2}{m^2} F^2\right)} \approx \frac{g_s^2}{M^2} + \frac{g^2}{m^2}$$

(4.112)

• This is precisely the tree-level pion decay constant obtained in QHD-III using Feynman diagrams in Chapter 2.

The QHD-III operator is given by:

$$\text{tr} \ln D_3 = \text{tr} \ln \tilde{D} + \text{tr} \ln \tilde{d} - \text{tr} (\tilde{D}^{-1} \tilde{d}) - \frac{1}{2} \text{tr} [(\tilde{D}^{-1} \tilde{d})^2]$$

$$+ \text{tr} \ln D^{\mu\nu}_{ij} + \text{tr} \ln \Delta^{\mu\nu}_{ij} - \text{tr}(\xi_{i} D^{ij} F_{\mu\nu}) - \text{tr}(\xi_{i} D^{ij} F_{\mu\nu})$$

$$- \text{tr} \{\tilde{D}^{m-1} \{(\xi_{i} D^{ij} F_{\mu\nu}) + \xi_{i} D^{ij} F_{\mu\nu}\} d^{-1} \tilde{d} \cdot d^{-1} \tilde{d}\}$$

$$- 2 \{\xi_{i} D^{ij} F_{\mu\nu} + \xi_{i} D^{ij} F_{\mu\nu}\} d^{-1} \tilde{d} \cdot d^{-1} \tilde{d}\}$$

$$+ \tilde{D} d^{-1} (\xi_{i} D^{ij} F_{\mu\nu} + \xi_{i} D^{ij} F_{\mu\nu} d^{-1} \tilde{d} \cdot d^{-1} \tilde{d}\}$$

$$+ \tilde{D} d^{-1} (\xi_{i} D^{ij} F_{\mu\nu} + \xi_{i} D^{ij} F_{\mu\nu} d^{-1} \tilde{d} \cdot d^{-1} \tilde{d}\}$$
The operators $\hat{D}$, $\hat{d}$, $\hat{\delta}$ and $\hat{\delta}^T$ are the operators calculated for the effective action of the linear $\sigma$-model; thus, the first line of Eq. (4.113) has already been calculated. The new operators are those associated with the fluctuations of the gauge bosons. The notation used is that a subscript $r$ implies an operator associated with the $\rho_\mu$ field while a subscript $\alpha$ implies an operator associated with the $a_\mu$ field. It is noted that operators with subscript $r$ and $\alpha$ come in pairs: the sum of these operators yields a gauge-invariant result. These operators are defined by the following relations:

\[
\begin{align*}
(\alpha^i_\mu, \Delta^{ij}_{\mu\nu} \alpha^j_\nu) &= \int d^4x d^4ya^i_\mu(x) \Delta^{ij}_{\mu\nu}(x, y) \alpha^j_\nu(y) \\
&= \int d^4x a^i_\mu \{ \delta^{ij} [g^{\mu\nu}(\Box + m_\rho^2) - \partial^\mu \partial^\nu] \\
&\quad + g_\rho^2 F^2 R^2 g^{\mu\nu} [U^i U^j + (U^0)^2 \delta^{ij}] \} \alpha^j_\nu \\
(r^i_\mu, D^{ij}_{\mu\nu} r^j_\nu) &= \int d^4x d^4y r^i_\mu(x) \Delta^{ij}_{\mu\nu}(x, y) r^j_\nu(y) \\
&= \int d^4x r^i_\mu \{ \delta^{ij} [g^{\mu\nu}(\Box + m_\rho^2) - \partial^\mu \partial^\nu] \\
&\quad + g_\rho^2 F^2 R^2 g^{\mu\nu} (U^k U^k \delta^{ij} - U^i U^j) \} r^j_\nu
\end{align*}
\]

These are the inverse propagators of the $a_\mu$ and $\rho_\mu$ fields respectively. The other operators are defined:

\[
\begin{align*}
(\alpha^i_\mu, \xi^{\mu i}_\alpha) &= \int d^4x d^4ya^i_\mu(x) \xi^{\mu i}_\alpha(x, y) \xi_\alpha(y) \\
&= 2g_\rho F \int d^4x R a^i_\mu(U^i \partial^\mu U^0 - U^0 \partial^\mu U^i) \xi = (\xi^{\mu i}_\alpha \alpha^i_\mu, \xi) \\
(r^i_\mu, \xi^{\mu i}_r) &= \int d^4x d^4y r^i_\mu(x) \xi^{\mu i}_r(x, y) \xi(y) \\
&= -2g_\rho F \int d^4x r^i_\mu R e^{ijk} U^j \partial^\mu U^k \xi = (\xi^{\mu i}_r r^i_\mu, \xi)
\end{align*}
\]
CHAPTER 4. ONE-LOOP EFFECTIVE ACTION IN QHD

\[ (\alpha^i_{\mu}, \theta^i_{\alpha}) = \int d^4 x d^4 y \alpha^i_{\mu}(x) [\theta^i_{\alpha}(x, y)] \theta^i(y) \]
\[ = g_s F \int d^4 x \alpha^i_{\mu} R [\varepsilon^i_{\mu} \partial^\mu U^0 + U^0 (\partial^\mu \varepsilon^i_{\mu} + \varepsilon^i_{\mu} \partial^\mu)] \theta^i \]
\[ = g_s F \int d^4 x \alpha^i_{\mu} R [\varepsilon^i_{\mu} \partial^\mu U^0 + U^0 (\partial^\mu \varepsilon^i_{\mu} + \varepsilon^i_{\mu} \partial^\mu)] \theta^i \quad (4.118) \]

\[ (\gamma^j_{\mu}, \theta^j_{\gamma}) = \int d^4 x d^4 y \gamma^j_{\mu}(x) [\theta^j_{\gamma}(x, y)] \theta^j(y) \]
\[ = g_s F \int d^4 x \gamma^j_{\mu} R [\varepsilon^j_{\mu} \partial^\mu U^0 + U^0 (\partial^\mu \varepsilon^j_{\mu} + \varepsilon^j_{\mu} \partial^\mu)] \theta^j \quad (4.119) \]

All the transposed operators can be derived from the above by partial integration. The above equations represent the starting point for the evaluation of the 1-loop effective action of QHD-III.

The effective QHD-III action (4.113) was calculated. The result is inconclusive so far because the counterterms that renormalize the \(\pi\pi\) scattering amplitude in QHD-III calculated in Chapter 2 fail to renormalize that effective action; only the \(O(p^2)\) term is renormalized. The origin of the failure is under investigation (see also appendix C).

In summary of this chapter, the 1-loop effective action of the linear \(\sigma\)-model is calculated. The pion decay constant to 1-loop is extracted from the effective action and shown to be identical to the result previously obtained using Feynman diagrams; the \(\pi\pi\) scattering amplitude extracted from the effective action is also identical to the previously obtained result. Using the counterterms derived in the Feynman diagrammatic expansion of Chapter 2, the parameters of the effective action are shown to be finite and the effective action is renormalized. The QHD-III effective action is considered next. An expression for the gauge fields in terms of the pion field is given. It is shown how the gauge fields decouple in the limit that their masses become large. The tree-level pion decay constant calculated in Chapter 2 in QHD-III

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is reproduced using the effective action. Lastly, the starting point for the calculation of the 1-loop effective action of QHD-III is given.
Chapter 5

Conclusions

In this thesis, two hadronic quantum field theories are investigated: the $\sigma$-model (both the linear and non-linear representations) and its extension, a model of the strong interaction called QHD-III. QHD-III is a relativistic hadronic quantum field theory based on the linear $\sigma$-model. It is a massive Yang-Mills theory that undergoes spontaneous symmetry breaking and possesses the low-energy symmetries of QCD.

One of the important features of QHD-III derived in this thesis is:

- QHD-III is shown to be simpler than other massive Yang-Mills models of nuclear physics where momentum-dependent interactions induced by $\pi a_1$ mixing are important while they are avoided through the use of gauge invariance in QHD-III.

From this result, it follows that the tree level amplitudes of QHD-III have a simpler expression and can be manipulated more easily than in other massive Yang-Mills hadronic models of the strong interaction. Therefore, any formal approach to solving the many-body strongly interacting problem that relies on the tree level amplitudes of a model will be simplified in this new representation of the QHD-III lagrangian, as
CHAPTER 5. CONCLUSIONS

compared to other massive Yang-Mills models. In particular, the QHD-III lagrangian constitutes a good starting point for the calculation of the bulk properties of nuclear matter in mean field theory in the presence of vector mesons as well as for evaluating the contributions of vector mesons to exchange currents processes discussed in the introduction.

The 1-loop structure of QHD-III was also explored in Chapter 2 and Chapter 4 of this thesis using perturbation theory:

- A 1-loop expansion to $O(g_4^4)$ was performed to calculate the $\pi\pi$ scattering amplitude and the pion decay constant to one loop in the linear $\sigma$-model.
- A 1-loop expansion to $O(g_\pi^2g_\rho^2)$ of the corrections to the $\pi\pi$ scattering amplitude stemming from the exchange of vector mesons were also calculated in QHD-III.
- The corrections stemming from vector meson exchange were shown to be negligible when their masses go to infinity.
- QHD-III was shown to be renormalizable to this order.

Perturbation theory may not be applicable if the coupling constants are large. The results presented in Chapter 2 and Chapter 4 remain valid in certain limits:

- To know the 1-loop formal structure of the theory where nothing is assumed a priori about the coupling constants is useful. The results obtained in Chapter 2 and Chapter 4 exclude the baryon sector: the results are those that emerge from the mesonic sector of QHD-III. QCD on the other hand, reduces to a theory of weakly interacting mesons in the large $N_c$ limit. The 1-loop structure of the
mesonic sector of QHD-III can therefore, in principle, be compared and linked to the large $N_c$ limit of QCD.\footnote{Since the amplitudes presented in this thesis were calculated in the chiral limit, only the large $N_c$ limit of massless QCD could be directly compared.}

- At large distances, the strong interaction resemble a Yukawa potential and diminish in strength exponentially. Therefore, at large distances, the large coupling constants are suppressed by factors of $\exp\{-m_r\}$ that emerge. Thus, the perturbation theory results for the scattering amplitude are correct at large distances at tree level and they incorporate the long range part of virtual processes that involve vector mesons in loops.

In Chapter 4, the effective actions of the linear $\sigma$-model and QHD-III are analyzed and the following results are obtained:

- The 1-loop effective action of the linear $\sigma$-model is derived in an arbitrary renormalization scheme and shown to be finite when the counterterms calculated using Feynman diagrams are inserted. The overall result from [43] is also reproduced in the renormalization scheme used in that work.

- The pion decay constant and the $\pi\pi$ scattering amplitude to 1-loop are extracted from the linear $\sigma$-model effective action and shown to be identical to the result obtained from the Feynman diagrammatic expansion.

- The QHD-III effective action at tree level is shown to reproduce the tree-level pion decay constant obtained from Feynman diagrams.

- Using their equations of motion, the vector mesons were explicitly shown to decouple from the problem in the limit that their masses go to infinity.
Although the QHD-III 1-loop effective action was also calculated, it was not made finite using the counterterms obtained from Feynman diagrams, except for the coefficient of the kinetic energy of the pion in the chiral lagrangian. The origin of this failure is under investigation and is briefly discussed in Appendix C.

There remains a number of areas of theoretical nuclear physics where QHD-III could be applied:

- Electromagnetic interactions with hadrons can be investigated by making the QHD-III lagrangian gauge-invariant under $U(1)$ transformations. In particular, electromagnetic exchange currents can be extracted from the tree level interactions of a $U(1)$ gauge-invariant QHD-III lagrangian and used to calculate the contributions coming from vector meson exchange. Also, once the 1-loop effective action of the QHD-III lagrangian is obtained, the result can be used to calculate electromagnetic processes involving pions, as well as the electromagnetic form factor of the pion itself.

- The strong interaction has a profound influence on the structure and the behavior of matter for systems which in size span more than twenty four orders of magnitude. From baryons and mesons whose characteristic length scale is $10^{-15}$m to the equation of state (EOS) of stellar objects such as the sun with a diameter $\equiv 10^8$m, strong interactions are central to our understanding of these objects. Compact stellar objects such as neutron stars are basically big neutron nuclei and are the closest thing to nuclear matter that exists in the universe. QHD-III can be used to calculate the EOS of nuclear matter explicitly including the important mesonic degrees of freedom in nuclear physics. This EOS can
be inserted into the Oppenheimer-Volkoff Eq. [71] to compute the mass of a neutron star as a function of the central density of the nuclear matter. This would provide an estimate of the effects of vector mesons in such a system.

- An $SU(3) \times SU(3)$ extension of QHD-III could be used to investigate the properties of strange nuclear matter. Witten [72] was the first to point out that adding strange quarks lowers the Fermi energy of nuclear matter. In light of this, it is reasonable to consider the possibility of the existence of bound strange nuclear matter [73, 74]. Hadronic models are suitable for the investigation of strange nuclear matter since a QCD description of the interacting many-body nuclear problem remains far into the future.

In the final analysis, QHD-III and its possible extensions remain a laboratory for the investigation of strong interactions.
Appendix A

Diagonal QHD-III Lagrangian

The QHD-III lagrangian is given by:

\[ \mathcal{L}_{III} = \mathcal{L}_N + \mathcal{L}_{\sigma - \omega} + \mathcal{L}_K + \mathcal{L}_H. \]  \hspace{1cm} (A.1)

With the nucleon lagrangian given by:

\[ \mathcal{L}_N = \bar{\psi} \left\{ i\gamma^\mu [\partial_\mu + ig_\omega V_\mu + i\frac{g_\rho}{2}\tau \cdot (\rho_\mu + \gamma_5 a_\mu)] \right\}
- (M - g_\pi \sigma) - i g_\pi \gamma_5 \tau \cdot \pi' \psi, \]  \hspace{1cm} (A.2)

The mesonic lagrangian:

\[ \mathcal{L}_{\sigma - \omega} = \frac{1}{2} \left[ \left( \partial_\mu \pi' + g_\rho \sigma a_\mu + g_\rho \pi' \times \rho_\mu \right)^2 - m_{\pi'}^2 \pi' \cdot \pi' \right]
+ \frac{1}{2} \left[ \left( \partial_\mu \sigma - g_\rho \pi' \cdot a_\mu \right)^2 - m_\sigma^2 \sigma^2 \right]
- g_\rho \sigma \sigma \cdot (\partial_\mu \pi' + g_\rho \sigma a_\mu + g_\rho \pi' \times \rho_\mu)
+ \frac{m_\sigma^2 - m_{\pi'}^2}{2\sigma_0} \left( \sigma^2 + \pi'^2 \right)
- \frac{m_\sigma^2 - m_{\pi'}^2}{8\sigma_0^2} \left( \sigma^2 + \pi'^2 \right)^2. \]  \hspace{1cm} (A.3)

The kinetic energies of the fields are:

\[ \mathcal{L}_K = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_\sigma^2 V_\mu V^\mu
- \frac{1}{4} R_{\mu\nu} \cdot R^{\mu\nu} + \frac{1}{2} m_\rho^2 \rho_\mu \cdot \rho^\mu \]
where the field strengths $F_{\mu\nu}$, $R_{\mu\nu}$, and $A_{\mu\nu}$ are those of the $\omega_\mu$, the $\rho$ and the $a_1$ respectively and are given in Eqs. (1.41) and (4.103).

The Higgs sector is given by:

$$\mathcal{L}_H = \frac{1}{2} (\partial_\mu \eta \partial^\mu \eta - m_\eta^2 \eta^2) + \frac{1}{2} (\partial_\mu \zeta \partial^\mu \zeta - m_H^2 \zeta^2) + \frac{1}{2} [g_\rho m_\rho \eta + \frac{1}{4} g_\rho^2 (\zeta^2 + \zeta^\dagger \zeta)] (\rho_\mu \cdot \rho^\mu + \alpha_\mu \cdot \alpha^\mu)$$

$$+ (g_\rho m_\rho \zeta + \frac{1}{2} g_\rho^2 \zeta \zeta) \rho_\mu \cdot \alpha^\mu$$

$$- \left( \frac{3 m_H^2 g_\rho}{4 m_\rho} \epsilon + \frac{3 m_H^2 g_\rho^2}{16 m_\rho^2} \eta^2 \right) \zeta^2 - m_\rho^2 \eta \zeta - \frac{m_H^2 g_\rho^2}{32 m_\rho^2} (\eta^4 + \zeta^4)$$

The presence of the bilinear term coupling the $a_1$ to the $\pi$ in the mesonic lagrangian is removed using the change of variables:

$$a_\mu \rightarrow a_\mu + \frac{g_\rho \sigma_\rho}{m_\sigma^2} \partial_\mu \pi', \quad \pi' = \frac{m_\sigma}{m_\rho} \pi, \quad m_\pi' = \frac{m_\rho}{m_\sigma} m_\pi.$$

Here, $m_\sigma^2 \equiv m_\rho^2 + g_\rho^2 \sigma_\rho^2$ and as before $\sigma_\rho \equiv M/g_\rho$.

After this change of variables, the QHD-III lagrangian becomes:

$$\mathcal{L}_{III} = \mathcal{L}_N' + \mathcal{L}_{\sigma - \omega} + \mathcal{L}_K + \mathcal{L}_H + \Delta \mathcal{L}_N + \Delta \mathcal{L}_{\sigma - \omega} + \Delta \mathcal{L}_K + \Delta \mathcal{L}_H$$

Here $\mathcal{L}_N'$ and $\mathcal{L}_{\sigma - \omega}$ are given by:

$$\mathcal{L}_N' = \bar{\psi} \{ i \gamma^\mu [\partial_\mu + i g_\rho V_\mu + \frac{i}{2} g_\rho \sigma \cdot (\rho_\mu + \gamma_5 a_\mu)]
- (M - g_\rho \sigma) - ig_\rho \gamma_5 \tau \cdot \frac{m_\sigma}{m_\rho} \pi \} \psi,$$

and:

$$\mathcal{L}_{\sigma - \omega}' = \frac{1}{2} [(\partial_\mu \pi + g_\rho \sigma a_\mu + g_\rho \frac{m_\sigma}{m_\rho} \pi \times \rho_\mu)^2 - m_\pi^2 \pi \cdot \pi]$$
\begin{align}
&\frac{1}{2}[(\partial_\mu \sigma - g_\rho m_\rho \pi \cdot a_\mu)^2 - m_\rho^2 \sigma^2] - g_\rho^2 \sigma_0 a_\mu \cdot (\sigma a_\mu + \frac{m_\rho}{m_\rho} \pi \times \rho_\mu) \\
&+ \frac{m_\rho^2}{2m_\rho m_{\pi}} \sigma \left( \sigma^2 + \frac{m_\rho}{m_\rho} \pi^2 \right) - \frac{m_\rho^2}{8m_\rho m_{\pi}} \left( \sigma^2 + \frac{m_\rho}{m_\rho} \pi^2 \right)^2. \quad (A.9)
\end{align}

The $\Delta L$'s are the extra terms that appear because of the gradient of the pion in the change of variables.

The new terms involving the nucleons are:

$$
\Delta L_N = -\frac{g_\rho^2 \sigma_0}{2m_\rho m_\rho} \bar{\psi} \gamma^\nu \gamma_5 \tau \cdot \partial_\mu \pi \psi \quad (A.10)
$$

The new mesonic terms are:

$$
\Delta L_{\sigma-\omega} = -\frac{g_\rho^2 \sigma_0}{2m_\rho m_\rho} \left[ 2(\partial_\mu \sigma - g_\rho m_\rho \pi \cdot a_\mu) \pi \cdot \partial_\mu \pi - \frac{g_\rho^2 \sigma_0}{m_\rho^2} (\pi \cdot \partial_\mu \pi)^2 \right] \\
+ \frac{g_\rho^2 \sigma_0}{2m_\rho m_\rho} \left[ 2\sigma \partial_\mu \pi \left( \frac{m_\rho}{m_\rho} \partial_\mu \pi + g_\rho \sigma a_\mu + g_\rho \frac{m_\rho}{m_\rho} \pi \times \rho_\mu \right) + \frac{g_\rho^2 \sigma_0}{m_\rho m_\rho} (\sigma \partial_\mu \pi)^2 \right] \\
- \frac{g_\rho^2 \sigma_0}{m_\rho m_\rho} \partial_\mu \pi \left( 2\sigma a_\mu + \frac{m_\rho}{m_\rho} \pi \times \rho_\mu \right) - \frac{g_\rho^2 \sigma_0}{m_\rho m_\rho^2} \sigma \partial_\mu \pi \cdot \partial_\mu \pi \quad (A.11)
$$

The new terms that stem from the kinetic energies are:

$$
\Delta L_K = \frac{g_\rho^2 \sigma_0}{2m_\rho m_\rho} \left[ \left( 2a_\mu \times \partial_\nu \pi + \frac{g_\rho \sigma_0}{m_\rho m_\rho} \partial_\mu \pi \times \partial_\nu \pi \right) \cdot \mathcal{R}^{\mu\nu} \\
+ (\partial_\mu a_\nu - \partial_\nu a_\mu) \cdot \left( \rho^\mu \times \partial_\nu \pi + \partial_\mu \pi \times \rho_\nu \right) \right] \\
- \frac{g_\rho^2 \sigma_0}{4m_\rho m_\rho} \left[ \frac{g_\rho \sigma_0}{m_\rho m_\rho} (\rho^\mu \times \partial_\nu \pi + \partial_\mu \pi \times \rho_\nu)^2 \\
+ 2(\rho^\mu \times \partial_\nu \pi + \partial_\mu \pi \times \rho_\nu) \cdot (\rho_\mu \times a_\nu + a_\mu \times \rho_\nu) \right]
$$
The new terms involving the Higgs fields are:

\[ \Delta L_H = \frac{g_2^2 \sigma_0}{2 \rho_0 \rho_a} \left[ m_\rho \eta + \frac{1}{4} g_\rho (\eta^2 + \xi^2) \right] \left[ 2 a_\mu \cdot \partial^\mu \pi + \frac{g_\rho \sigma_0}{m_\rho m_a} \partial_\mu \pi \cdot \partial^\mu \pi \right] \]

\[ + \frac{g_2^2 \sigma_0}{m_\rho m_a} \left[ m_\rho \xi + \frac{1}{2} g_\rho \eta \xi \right] \rho_\mu \cdot \partial^\mu \pi \]  

This lagrangian is to be contrasted with the QHD-III representation given in Chapter 3 of this thesis.

A.1 Feynman rules

The Feynman rules of the diagonalized lagrangian\(^1\) are now given. In the notation used for the Feynman rules, the terms in boldface are the vertex contributions coming from the terms in the \( \Delta L \)'s, and the arrows in the boson fields indicate the direction of the momenta.

\(^1\)The Higgs sector is excluded since it does not contribute to the order considered in this thesis.
APPENDIX A. DIAGONAL QHD-III LAGRANGIAN

Propagators

\[ \alpha, i \longrightarrow \beta, j \]

- - -

Nucleon: \( iS_f^{ij}(p)_{\alpha\beta} = \left( \frac{-i\gamma^\nu}{p - m + i\epsilon} \right)_{\alpha\beta} \)

\[ i \longrightarrow j \]

\[ \sigma: \Delta(p) = [p^2 - m_\sigma]^{-1} \]

\[ i, \mu \longrightarrow j, \nu \]

\[ \rho: \Delta^{\mu\nu}_{ij}(p) = \frac{\delta_{ij}}{p^2 - m_\rho^2} \left( -g^{\mu\nu} + \frac{p^\mu p^\nu}{m_\rho^2} \right) \]

\[ i, \mu \longrightarrow j, \nu \]

\[ a_1: D^{\mu\nu}_{ij}(p) = \frac{\delta_{ij}}{p^2 - m_a^2} \left( -g^{\mu\nu} + \frac{p^\mu p^\nu}{m_a^2} \right) \]

\[ \mu \longrightarrow \nu \]

\[ \omega_\mu: \omega^{\mu\nu}(p) = \frac{1}{p^2 - m_\omega^2} \left( -g^{\mu\nu} + \frac{p^\mu p^\nu}{m_\omega^2} \right) \]

Nucleon Interactions

\[ -ig_\mu \gamma_\mu \]

\[ -\frac{i}{2}g_\rho \tau^a \gamma_\mu \]

\[ -\frac{i}{2}g_\rho \tau^a \gamma_\mu \gamma_5 \]

\[ ig_\pi \]

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APPENDIX A. DIAGONAL QHD-III LAGRANGIAN

\[ = g_\pi \frac{m_a}{m_p} \gamma_5 \tau^a + - \frac{g_\sigma \sigma_\omega}{2 m_p m_a} q_\mu \gamma^\mu \gamma_5 \tau^a \]

Mesonic Interactions

\[ = g_\rho (q_\mu - k_\mu) \frac{m_a}{m_p} \delta_{ab} - 2 \frac{g_2^a \sigma_2^b}{m_p m_a} \delta_{ab} q_\mu \]

\[ = g_\rho \epsilon_{abc}(q_\mu - k_\mu) \frac{m_a}{m_p} \delta_{ab} - \frac{g_2^a \sigma_2^b}{m_p^2} \epsilon_{abc}(q_\mu - k_\mu) \]

\[ - \frac{g_2^a \sigma_2^b}{m_p^2 m_a} \epsilon_{abc}(q \cdot r k_\mu - k \cdot r q_\mu) \]

\[ = i g_\sigma \frac{(m^2 - m_x)}{M} \frac{m_a}{m_p} \delta_{ab} \delta_{ab} \]

\[ - \frac{g_2^a \sigma_2^b}{m_p^2} \left[ r \cdot (q + k) - 2 q \cdot k \left( 1 - \frac{g_2^a \sigma_2^b}{m_p^2} \right) \right] \]

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\[ -ig_\rho^2 \sigma_0 \frac{m_a}{m_p m_q} g_{\mu \nu} \epsilon_{abc} \]
\[ -i \frac{g_\rho^2 \sigma_0}{m_p m_a} \epsilon_{abc} [q_{\nu} k_{\mu} - \omega_{\nu} q_{\mu} - q \cdot (k - \omega) g_{\mu \nu}] \]

\[ = ig_\rho^2 m_a^2 g_{\mu \nu} (\delta_{ab} \delta_{cd} + \delta_{bc} \delta_{ad}) \]
\[ + \frac{g_\rho^4 \sigma_0^2}{m_p^2 m_a^2} [\epsilon_{ace} \epsilon_{bde} (q_{\mu} k_{\nu} - q_{\nu} k_{\mu}) \]
\[ + \epsilon_{bae} \epsilon_{dce} (q \cdot k g_{\mu \nu} - q_{\nu} k_{\mu}) \]
\[ + \epsilon_{dae} \epsilon_{bce} (q \cdot k g_{\mu \nu} - q_{\mu} k_{\nu})] \]

\[ = ig_\rho^2 m_a^2 g_{\mu \nu} (\epsilon_{abe} \epsilon_{ced} + \epsilon_{cbe} \epsilon_{ead}) \]
\[ + \frac{g_\rho^4 \sigma_0^2}{m_p^2 m_a^2} [\epsilon_{ace} \epsilon_{bde} (q_{\mu} k_{\nu} - q_{\nu} k_{\mu}) \]
\[ + \epsilon_{abe} \epsilon_{cde} (q \cdot k g_{\mu \nu} - q_{\nu} k_{\mu}) \]
\[ + \epsilon_{ade} \epsilon_{cbe} (q \cdot k g_{\mu \nu} - q_{\mu} k_{\nu})] \]

\[ = ig_\rho^2 m_a^2 g_{\mu \nu} \epsilon_{abc} \]
\[ -i \frac{(m_2^2 - m_3^2)}{\sigma^2} \frac{m_2^2}{m_3^2} \delta_{ab} - 2ig^4_p \frac{\sigma^2}{m_3^2 m_3^2} q \cdot \kappa \delta_{ab} \]

\[ = -i \frac{(m_2^2 - m_3^2)}{\sigma^2} \frac{m_2^4}{m_3^2} (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \]

\[ -i \frac{g^4_p \sigma^2}{m_3^2 m_3^2} [(q \cdot r + k \cdot \omega)(\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \]

\[ + (q \cdot \omega + k \cdot r)(\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd}) \]

\[ + (q \cdot k + \omega \cdot r)(\delta_{ab} \delta_{cd} + \delta_{ad} \delta_{bc})] \]

\[ -i \frac{g^4_p \sigma^2}{m_3^2 m_3^2} [\epsilon_{ace} \epsilon_{bde}(q \cdot r k \cdot \omega - q \cdot \omega k \cdot r) \]

\[ + \epsilon_{bae} \epsilon_{dce}(q \cdot k r \cdot \omega - q \cdot \omega k \cdot r) \]

\[ + \epsilon_{dae} \epsilon_{bce}(q \cdot k r \cdot \omega - q \cdot r k \cdot \omega)] \]
New interactions due to change of variables

\[
-ig_0^2 [\epsilon_{abc} \epsilon_{cde} (g_{\mu \sigma} g_{\nu \sigma} - g_{\mu \sigma} g_{\nu \sigma}) + \epsilon_{ace} \epsilon_{cbd} (g_{\mu \sigma} g_{\nu \sigma} - g_{\mu \sigma} g_{\nu \sigma}) + \epsilon_{adc} \epsilon_{ceb} (g_{\mu \sigma} g_{\nu \sigma} - g_{\mu \sigma} g_{\nu \sigma})]
\]

\[
= - \frac{g_0^2}{m_\rho m_\alpha} \left[ \epsilon_{abc} \epsilon_{cde} (g_{\sigma \mu} q_\nu - g_{\sigma \nu} q_\mu) + \epsilon_{acd} \epsilon_{bce} (g_{\sigma \nu} q_\sigma - g_{\sigma \nu} q_\sigma) \right]
\]

\[
= - \frac{g_0^2}{2m_\rho m_\alpha} \left[ 2 \epsilon_{abc} \epsilon_{cde} (g_{\sigma \mu} q_\nu - g_{\sigma \nu} q_\mu) + \epsilon_{acd} \epsilon_{bce} (g_{\sigma \nu} q_\sigma - g_{\sigma \nu} q_\sigma) \right]
\]
\[ \frac{2g_0^2\delta m_a}{m_p} \delta_{ab} q_{\mu} \]

\[ = \frac{g_0^2m_a}{m_p} \left[ \delta_{ab}\delta_{cd}(k + r)_\mu + \delta_{cd}\delta_{ad}(q + r)_\mu + \delta_{db}\delta_{ac}(k + q)_\mu \right] \\
+ \frac{g_0^2m_a}{m_p} \left[ \epsilon_{ack}\epsilon_{bk}(k \cdot r q_{\mu} - k_{\mu} q \cdot r) + \epsilon_{abk}\epsilon_{cdk}(q \cdot r k_{\mu} - r_{\mu} q \cdot k) \\
+ \epsilon_{adk}\epsilon_{bc}(k \cdot r q_{\mu} - r_{\mu} q \cdot k) \right] \]
A.2 Counterterms

The counterterm lagrangian\(^2\) is obtained in the following way [23]:

- Count the total number of parameters.
- Subtract out the divergences from each vertex appearing in the lagrangian. This can be done by assigning a counterterm to each vertex which is then rendered finite.
- If the total number of counterterms exceeds the total number of parameters, there must be relationships between the counterterms which must be derived.
- Write out the full counterterm lagrangian.

One starts from the QHD-III lagrangian before SSB discussed between Eqs. (1.40) through (1.45) of the introduction. The following parameters and fields appear:

\[
\{(s, \pi^i); (l^i_\mu, r^i_\mu); (\phi_L, \phi_R); g_p; \lambda; v^2; \mu_H; \lambda_H\}
\]

(A.14)

Note that the masses \(m_\sigma, m_\rho\) and \(m_H\) are all expressed in terms of the above parameters. In the above set, it is noted that the \(s\) and \(\pi\) fields transform into each other under chiral transformations and the left and right fields transform into each other under the parity. Fields that transform into each other must have the same wave function renormalization.

- There are therefore 8 independent counterterms.

From each field, the divergent part can be extracted in the following manner:

\(^2\)The nucleon and the \(\omega^\mu\) fields are excluded from this analysis and \(m_\sigma = 0\) is assumed.
\[ Z^{1/2}(s, \pi^i) = (1 + \delta_Z)(s, \pi^i) \quad (A.15) \]
\[ Z^{1/2}_V(l^i_\mu, r^i_\mu) = (1 + \delta_V)(l^i_\mu, r^i_\mu) \quad (A.16) \]
\[ Z^{1/2}_H(\phi_L, \phi_R) = (1 + \delta_H)(\phi_L, \phi_R) \quad (A.17) \]

In Eqs. (A.15) through (A.17), the fields are all finite and all the infinities are in the counterterms: \( \{ \delta_Z, \delta_V, \delta_H \} \). For each of the vertices that appear in the QHD-III lagrangian before SSB, the same procedure is applied. For example, the \( ss\pi\pi \) vertex in the lagrangian is given by:

\[ ss\pi\pi \text{ Vertex} = -\frac{\lambda_{\text{bare}}}{2} s_{\text{bare}}^2 \pi_{\text{bare}}^2 \]
\[ = \frac{\lambda_{\text{bare}} Z^2}{2} s^2 \pi^2 \]
\[ = \frac{(\lambda + \delta_\lambda)}{2} s^2 \pi^2 \quad (A.18) \]

In the above, \( \text{bare} \) means the bare quantity that diverges. All the infinities of this vertex are now contained in \( \delta_\lambda \) of Eq. (A.18) which defines it:

\[ \delta_\lambda = \lambda_{\text{bare}} Z^2 - \lambda \quad (A.19) \]

For each vertex, the same procedure is applied. In what follows, "M" denotes the meson fields \((s, \pi^i)\), "V" denotes the vector fields \((l^i_\mu, r^i_\mu)\) \(\rightarrow (a^i_\mu, \rho^i_\mu)\) and "H" denotes...
the Higgs fields \((\phi_L, \phi_R)\). One obtains the following counterterms:

\[
\begin{align*}
\delta_Z &= Z - 1 && \text{M wavefunction renormalization} \\
\delta_V &= Z_V - 1 && \text{V wavefunction renormalization} \\
\delta_H &= Z_H - 1 && \text{H wavefunction renormalization} \\
\delta_\lambda &= \lambda_b Z^2 - \lambda && \text{M}^4 \text{ vertex renormalization} \\
-\delta_\mu &= \lambda_b \nu_b^2 Z - \nu^2 \lambda && \text{M}^2 \text{ vertex renormalization} \\
\delta_{g_p} &= Z_{1/2} V g_{p_b} - g_p && \text{VM}^2 \text{ vertex renormalization} \\
\delta_{g_v} &= Z_V g_{p_b}^2 - g_p^2 && \text{V}^2 \text{M}^2 \text{ vertex renormalization} \\
\delta_{V^3} &= Z_{V^3} g_{p_b}^2 - g_p^2 && \text{V}^3 \text{ vertex renormalization} \\
\delta_{V^4} &= Z_{V^4} g_{p_b}^2 - g_p^2 && \text{V}^4 \text{ vertex renormalization} \\
\delta_{g_H} &= Z_H Z_V g_{p_b}^2 - g_p^2 && \text{V}^2 \text{H}^2 \text{ vertex renormalization} \\
\delta_{\mu_H} &= Z_H \mu_H^2 - \mu_H^2 && \text{H}^2 \text{ vertex renormalization} \\
\delta_{\lambda_H} &= Z_H \lambda_H^2 - \lambda_H^2 && \text{H}^4 \text{ vertex renormalization}
\end{align*}
\]

Here, the subscript \(b\) means bare. Twelve counterterms were introduced and are defined by the above relations. Note that the power of \(Z^{1/2}\) are given by the powers of the corresponding field M, V or H.

These 12 counterterms can not be independent since there are only 8 fields and parameters that can absorb divergences. Indeed, relations can be found between the counterterms by taking total differentials of the above equations and inspecting the results.

The relations between the counterterms are:

\[
\begin{align*}
g_p^2 \delta_Z &= 2 g_p \delta_{g_p} - \delta_{g_p^2} \quad (A.21) \\
g_p^2 \delta_V &= 2 g_p \delta_{g_v} - \delta_{V^2} \quad (A.22)
\end{align*}
\]
The spontaneous symmetry breaking of the lagrangian can now proceed with the following result for the counterterm lagrangian after the change of variables (A.6):

$$L^\delta = L^\delta_{\sigma-\omega} + L^\delta_K + L^\delta_H$$  \hspace{1cm} (A.25)

With:

$$L^\delta_{\sigma-\omega} = \left[ \frac{\delta z}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{\delta \mu}{2} \sigma^2 - \frac{3}{2} \delta \lambda \sigma^2 \sigma^2 \right]$$

$$+ \left[ \frac{\delta z}{2} \partial_\mu \pi \cdot \partial^\mu \pi - \frac{1}{2} (\delta \nu \pi + \sigma^2 \delta \lambda) \pi^2 \right]$$

$$+ \delta \lambda \sigma \sigma \pi^2 - \frac{\delta \lambda}{4} (\pi^4 + \sigma^4 + 2 \sigma^2 \pi^2) - \delta \rho \sigma \sigma \partial^\mu \pi$$

$$+ \delta \rho \pi (\partial^\mu \pi - \partial^\mu \sigma) + \delta \rho \partial^\mu \pi \cdot \rho \pi \times \partial^\mu \pi$$

$$+ \frac{1}{2} \left( - \sigma \sigma \partial_\mu \partial^\mu + \frac{1}{2} \sigma^2 \partial_\mu \partial^\mu + (\sigma - \sigma \partial_\mu \partial^\mu \pi \times \rho \pi$$

$$+ \frac{1}{2} \left( \sigma \pi^2 + \rho \pi \pi^2 \right) \right]$$  \hspace{1cm} (A.26)

The kinetic energy counterterm lagrangian is given by:

$$L_K = -\frac{\delta \nu}{4} (\partial_\mu a_\nu - \partial_\nu a_\mu)^2 + \left( \frac{\delta g_\pi}{2g_\rho^2} m_\pi^2 + \frac{\delta g_\rho}{2g_\rho^2} \right) a_\mu a^\mu$$

$$- \frac{\delta \nu}{4} (\partial_\mu \rho_\nu - \partial_\nu \rho_\mu)^2 + \frac{\delta g_\pi}{2g_\rho^2} m_\pi^2 \rho_\mu \cdot \rho^\mu$$

$$+ \frac{1}{2} \delta g_\nu [(\partial_\mu a_\nu - \partial_\nu a_\mu) \cdot (\rho^\mu \times a^\nu + a^\mu \times \rho^\nu)$$

$$+ (\partial_\mu \rho_\nu - \partial_\nu \rho_\mu) \cdot (\rho^\mu \times \rho^\nu + a^\mu \times a^\nu)]$$

$$- \frac{1}{4} \delta \nu [(\rho^\mu \times a^\nu + a^\mu \times \rho^\nu)^2 + (\rho^\mu \times \rho^\nu + a^\mu \times a^\nu)^2]$$  \hspace{1cm} (A.27)
The Higgs field counterterm lagrangian is given by:

\[ \mathcal{L}_H = \left[ \frac{\delta_H}{2} \partial_\mu \eta \partial^\mu \eta + \left( \frac{\delta_H}{2} - \frac{3}{4} \frac{\lambda_H}{g_p^2} \right) \eta^2 \right] \\
+ \left[ \frac{\delta_H}{2} \partial_\mu \xi \partial^\mu \xi + \left( \frac{\delta_H}{2} - \frac{3}{4} \frac{\lambda_H}{g_p^2} \right) \xi^2 \right] \\
+ 2 \frac{m_p}{g_p} \left( \delta_{\mu H} - \frac{m_p^2}{2g_p^2} \delta_{\lambda H} \right) \eta - \frac{3}{16} \delta_{\lambda H} \eta^2 \xi^2 - \frac{3}{4} \frac{m_p}{g_p} \delta_{\lambda H} \eta \xi^2 - \frac{\delta_{\lambda H}}{4} \frac{m_p}{g_p} \eta^3 \\
- \frac{\delta_{\lambda H}}{32} (\eta^4 + \xi^4) + \frac{\delta_{\lambda H}}{8} \left( \frac{m_p}{g_p} \eta + \eta^2 + \xi^2 \right) \left( \alpha_\mu^2 + \rho_\mu^2 \right) \\
+ \frac{\delta_{\lambda H}}{2} \left( \eta \xi + 2 \frac{m_p}{g_p} \xi \right) \alpha_\mu \cdot \rho^\mu \right) \tag{A.28} \]

The counterterm lagrangian displayed here is the one corresponding to the undiagonalized QHD-III lagrangian. After the change of variables (A.6), there are extra counterterm interactions that appear. As discussed in Chapter 2, it is easier to renormalize amplitudes in the undiagonalized lagrangian because one preserves current conservation at every step.

The Feynman rules for the counterterm lagrangian can be simply obtained by comparing terms between the QHD-III lagrangian and the counterterm lagrangian. These Feynman rules will only differ by factors that can be determined by inspection. This is true for all of the vertices except for the bilinear term that couples the \( a_1 \) to the pion in the counterterm lagrangian. The Feynman rule for that counterterm interaction is:

\[ a_\mu \quad \cdots \quad \delta_{\mu \nu} \quad \cdots \quad b = -p_\mu \sigma_\nu \delta_{\mu \nu} \delta_{ab} \]

We are now in a position to list the diagrams that contribute to \( \pi \pi \) scattering and the pion decay constant to 1-loop in the linear \( \sigma \)-model and QHD-III. In what
follows, the counterterm interactions will always be superimposed with a cross within a circle as above.
A.3 Diagrams in the linear $\sigma$-model
These twenty diagrams are the only ones that contribute to $\pi\pi$ scattering in the linear $\sigma$-model, including the counterterms.

- The first six diagrams are the irreducible diagrams of the pion 4-point function.
- The next six diagrams are the irreducible diagrams of the $\sigma\pi\pi$ vertex.
- The next four diagrams are the irreducible diagrams of the $\sigma$ propagator.
- The next two diagrams are the 1-point $\sigma$ functions.
- The last two diagrams are the irreducible diagrams of the pion external legs.\(^3\)

In the chiral limit, there will come no contribution from pion tadpoles in dimensional regularization.

Once calculated, these 1-loop diagrams can be combined in the following manner to yield the $\pi\pi$ scattering amplitude:

\[ \mathcal{M}_{\pi\pi\rightarrow\pi\pi} = \]

The sum of all these diagrams, and all possible permutations of the irreducible vertices, yields the $\pi\pi$ scattering amplitude in the chiral limit of the linear $\sigma$-model that was discussed in Chapter 2.

\(^3\)There is another diagram with a $\sigma$ loop that does not contribute to the external leg and which is omitted.
The pion decay constant to 1-loop can also be calculated using the following Feynman rule for the destruction of a pion by the axial current here denoted as $X$:

$$\begin{align*}
p & \\ \text{\hphantom{p}} & \rightarrow X
\end{align*} = \sigma_{\sigma}(1 + \delta_{Z})p_{\mu}$$

The 1-loop diagrams that contribute to pion decay in the linear $\sigma$-model are now given by:

This sum yields the pion decay constant discussed in Chapter 2.

A.4 Diagrams in QHD-III

The $\pi\pi$ scattering amplitude will get corrections due to the exchange of the $\rho$ and $a_1$ vector mesons. There are fifty diagrams that contribute which are divided into four classes:

- $O(g_{\rho}^2)$ diagrams that come strictly from the $\sigma$ pion sector. These diagrams exist because of the change of variables (A.6). $O(g_{\rho}^2)$ interactions come from two places: the gradient of the pion in (A.6), and the rescaling of the pion by $m_{\sigma}/m_{\rho}$. The factor $m_{\sigma} = \sqrt{m_{\rho}^2 + g_{\rho}^2\sigma_0}$ must be expanded to $O(g_{\rho}^2)$ and these terms must be included.
APPENDIX A. DIAGONAL QHD-III LAGRANGIAN

- $a_1$ exchange.
- $\rho$ exchange.
- Counterterms.

These sets of diagrams are listed in turn.
APPENDIX A. \textsc{Diagonal QHD-III Lagrangian}

Sigma pion sector

\begin{center}
\begin{tabular}{c c c}
\includegraphics[width=0.3\textwidth]{sigma_pion_sector_1} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_2} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_3} \\
\includegraphics[width=0.3\textwidth]{sigma_pion_sector_4} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_5} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_6} \\
\includegraphics[width=0.3\textwidth]{sigma_pion_sector_7} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_8} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_9} \\
\includegraphics[width=0.3\textwidth]{sigma_pion_sector_10} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_11} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_12} \\
\includegraphics[width=0.3\textwidth]{sigma_pion_sector_13} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_14} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_15} \\
\includegraphics[width=0.3\textwidth]{sigma_pion_sector_16} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_17} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_18} \\
\includegraphics[width=0.3\textwidth]{sigma_pion_sector_19} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_20} & \includegraphics[width=0.3\textwidth]{sigma_pion_sector_21} \\
\end{tabular}
\end{center}
APPENDIX A. DIAGONAL QHD-III LAGRANGIAN

$a_1$ sector
APPENDIX A. DIAGONAL QHD-III LAGRANGIAN

\[ \rho \text{ sector} \]
Counterterms
Appendix B

Pionic Terms Due to Higgs

When retaining the Goldstone boson $\chi'$ in the model, the following extra terms appear:

$$\Delta \mathcal{L}'' = \frac{g^2}{2}(\eta \alpha_\mu \cdot \partial^\mu \chi' + \zeta \rho_\mu \cdot \chi' - \chi' \cdot \rho_\mu \partial^\mu \eta)$$

$$+ \frac{g^2}{2} \chi' \times \rho_\mu \cdot \partial^\mu \chi' - \frac{g^2 m_H^2}{4 m_\rho} \eta \chi'^2 + \frac{g^2}{8} \chi'^2(a_\mu^2 + \rho_\mu^2)$$

$$- \frac{g^2 m_H^2}{16 m_\rho^2} \chi'^2(\zeta^2 + \eta^2) - \frac{g^2 m_H^2}{32 m_\rho^2} \chi'^4.$$  \hspace{1cm} (B.1)

When $g_\rho \sigma_\rho / m_\pi$ is substituted for the $\chi'$ in Eq. (B.1), the result is:

$$\Delta \mathcal{L}'' = \frac{g^2 \sigma_\rho}{2 m_\mu^2}(\eta \alpha_\mu \cdot \partial^\mu \pi + \zeta \rho_\mu \cdot \pi - \pi \cdot \rho_\mu \partial^\mu \eta - \pi \cdot \rho_\mu \partial^\mu \zeta)$$

$$+ \frac{g^2 \sigma_\rho^2}{2 m_\mu^2} \pi \times \rho_\mu \cdot \partial^\mu \pi - \frac{g^2 \sigma_\rho^2 m_H^2}{4 m_\mu^2 m_\rho} \eta \pi^2 + \frac{g^2 \sigma_\rho^2}{8 m_\mu^2} \pi^2(a_\mu^2 + \rho_\mu^2)$$

$$- \frac{g^2 \sigma_\rho^2 m_H^2}{16 m_\mu^2 m_\rho^2} \pi^2(\zeta^2 + \eta^2) - \frac{g^2 \sigma_\rho^4 m_H^2}{32 m_\mu^2 m_\rho^2} \pi^4.$$  \hspace{1cm} (B.2)

It is seen that the interactions in (B.3) are at least of order $g_\rho^2$. In particular, the terms in the parentheses of the first line of (B.3) are the only ones of $\mathcal{O}(g_\rho^2)$, and
they all involve the Higgs fields; the contributions of these terms can be minimized by making the Higgs fields very heavy.
Appendix C

Effective Action Notes

C.1 Equation for $Z_\mu$

The function containing the unitary logs in the effective action is given by:

$$
\int d^4x d^4y \left\{ M^{\mu\nu}[\Gamma_\mu^i(x)\Gamma_\nu^j(y)] - \frac{i}{4} d_M^2(x-y)[\delta^{ij}(x)\delta^{ij}(y)] \right\}
$$

(C.1)

where $\Gamma_\mu$ is given by Eq. (4.30), $\delta$ is given by Eq. (4.66) and:

$$
d_M(z) = \int d^4l \frac{e^{-il(x-y)}}{l^2 - M^2 + i\epsilon}
$$

(C.2)

$$
M_{\mu\nu}(z) \equiv \frac{i}{2} [\partial_\mu d_M \partial_\nu d_M - d_M \partial_\mu \partial_\nu d_M + g_{\mu\nu} d_M(0) \delta(z)]
$$

(C.3)

$Z_\mu$ was calculated and verified the result obtained by Gasser and Leutwyler.

C.2 Brief discussion on the calculation of the QHD-III effective action

The 1-loop effective action of QHD-III was also calculated by the author, but the results were not displayed because they could not be renormalized unlike the 1-
loop $\pi\pi$ scattering amplitude calculated in QHD-III using Feynman diagrams. The nature of the difficulty in the renormalization of the QHD-III 1-loop effective action is under investigation. The author is currently rederiving the result obtained in an independent fashion from the first calculation. The calculation is very complex, and it is possible that despite all the checks that were used along the way to obtaining a final result, a simple algebraic mistake remains buried in the calculation that prevent its renormalization. The possibility of a conceptual and formal reason for the failure to reproduce the renormalized $\pi\pi$ scattering amplitude derived using a Feynman diagrammatic expansion in QHD-III is also being considered. It is possible that the approach used in the linear $\sigma$-model requires modifications before it can be applied to QHD-III.

One possibility being looked at relates to the basis vectors along which the pion field fluctuates. As discussed in Chapter 4, a representation of the fluctuation basis vectors can lead to terms that are not parity-invariant in a gauge-invariant lagrangian if the basis vectors are not themselves parity eigenstates. In contrast to the linear $\sigma$-model, the gauge fields of QHD-III are dynamical fields and also fluctuate about their classical fields which satisfy their equations of motion. These classical fields can also be expressed in terms of the pion field as discussed in Chapter 4. In the pionic effective action obtained by re-expressing all of the heavy fields in terms of the pion field, there still remains the question of which representation of the fluctuation basis vectors should be used. The substitution of the representation of the basis vectors which are parity eigenstates leads to terms that are not globally invariant under $SU(2)_L \times SU(2)_R$ [an example of which is $(\partial_\mu \epsilon_i \cdot \partial^\mu U)(\epsilon_i \cdot \nabla U)]$. Which representation of the fluctuation basis vectors should be used is a formal issue that
could be at the origin of the difficulties encountered in the derivation of the QHD-III effective action.
Bibliography


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VITA

Gary Marcel Prézeau was born September 12, 1967 in Sherbrooke Québec, Canada. He obtained his BSc in physics from the University of Sherbrooke and won a 4-year NSERC postgraduate fellowship to pursue graduate work. In 1990, he obtained his MSc in physics at the University of Toronto and enrolled in the PhD program of the College of William and Mary in 1991 while working at the Continuous Electron Beam Accelerator Facility (CEBAF). In 1992, he took a leave of absence from the College to perform service in international development. Under the sponsorship of the World University Service of Canada (WUSC), he was a volunteer lecturer in physics at the University of Malawi between 1993 and 1995. In 1995, he returned to the College of William and Mary to complete his PhD.