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Catastrophes in the Elmo bumpy torus

Alkesh R. Punjabi

College of William & Mary - Arts & Sciences

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CATASTROPHES IN THE ELMO BUMPY TORUS

A Dissertation
Presented to
The Faculty of the Department of Physics
The College of William and Mary in Virginia

In Partial Fulfillment
Of the Requirements for the Degree of
Doctor of Philosophy

by
Alkesh R. Punjabi
February 1983
APPROVAL SHEET

This dissertation is submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

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Experimentally it is observed that the plasma in Elmo Bumpy Torus (EBT) shows discontinuous changes in the electron line density, electron and ion temperatures and fluctuation levels as the ambient gas pressure or electron cyclotron heating is varied continuously. We use the Point Model of Hedrick et al. for the toroidal core plasma in EBT. The Point Model is not a gradient dynamic system. Hence the Elementary Catastrophe Theory is not directly applicable to the Point Model. Nonetheless, the Point Model equilibria will be shown to exhibit properties which are quite akin to the canonical cusp catastrophe. The ambipolar electric field is taken as a control parameter. When electrons are nonresonant, the equilibrium surfaces show only one fold; but when ions are nonresonant, equilibrium surfaces may show single or multiple folds. In the former (electron) case, qualitative agreement with experiments is quite good. For the case of nonresonant ions, predictions are made as to the possible plasma behavior which, when ICRH heating is sufficiently intense, may be checked by experiments. The nonlinear time evolution of the Point Model equations show that the plasma follows the Delay Convention. This simple model of the EBT plasma exhibits rich structure, i.e., equilibrium surfaces with single or multiple folds, attractors, repellors, appearance and disappearance of folds, reversal of direction on equilibrium trajectories, catastrophes, hysteresis, competition between multiple point attractors, and basins of attractors. This leads us to conclude that a simple model of physical systems governed by external parameters can unravel general, complicated qualitative behavior of the system.
CATASTROPHES IN ELMO BUMPY TORUS
I. INTRODUCTION

Elmo Bumpy Torus (EBT) is one of the alternative concepts in the program for achieving controlled thermonuclear reaction. The distinguishing feature of the Elmo Bumpy Torus is that it is a steady state device. An important observation in the EBT experiments is that the plasma in EBT shows abrupt changes in the plasma density, electron and ion temperatures and fluctuation levels in the plasma density as the experimental parameters of ambient gas pressure or microwave power for electron cyclotron heating are varied smoothly. In Chapter II, we describe these experimental observations in detail and also discuss the plasma confinement properties of EBT. The mathematical theory which deals with such discontinuous changes in the properties of a system when the parameters are changed smoothly is catastrophe theory. In Chapter III, we discuss catastrophe theory and illustrate its use by applying it to some systems. Also, we show how the theory attempts to deal with evolution of system in time as the parameters are changed. In Chapter IV, we set up the point model for the toroidal core plasma in EBT. Point model equations are the conservation laws for particles and energy. They are the simplest way of describing plasma behavior in EBT. Since point model cannot handle velocity dependent processes, we choose to use diffusion coefficients for particles whose diffusion does not depends upon velocity space coordinates (the so called nonresonant particles). To this end, we show that when
the ambipolar electrostatic potential \( \phi \) is negative, all electrons are nonresonant, while some ions are resonant and most ions are nonresonant. When \( \phi > 0 \), all ions are nonresonant, some electrons are resonant and most electrons are nonresonant.

In Chapter V, we solve point model equations for nonresonant electrons with \( e\phi/T_e \) as a parameter. We obtain equilibrium solutions in two cases, (i) when ion cyclotron resonance heating is absent, and (ii) when it is present. For the equilibrium solutions obtained in case (i), we see that the qualitative agreement with experiments is quite good. We analyze equilibrium solutions in detail and discuss their peculiarities. Also, we state the implications of these results for the experimentalists.

In Chapter VI, we repeat the same for point model equations with nonresonant ions. In this case, we see that topology of equilibria have rich structure.

In Chapter VII, we solve the time dependent nonlinear point model equations. A number of interesting features emerge; e.g. competition between attractors and critical dependence of final state on initial state; which can have serious consequences for experiments.

Chapter VIII summarizes the results and states the conclusions of this research work.
II. ELMO BUMPY TORUS

A. EBT - a controlled nuclear fusion device

For 20 years experiments have shown that electron cyclotron heating (ECH) is an efficient and reliable technique for creating and sustaining energetic plasmas. ECH experiments were conducted in both simple and minimum B mirror geometries. The minimum B experiments were directed towards providing a target plasma of sufficient quality (i.e. plasma with sufficiently high value of the ratio $n_e/n_0$ where $n_e$ is number density of charged particles and $n_0$ is number density of neutrals) for effective neutral injection, and the simple mirror experiments were directed towards gaining a general understanding of heating, equilibrium and stability in hot electron plasmas. At Oak Ridge National Laboratory (ORNL), ECH experiments were done in the ELMO device which had the magnetic field configuration of a simple mirror. In 1967, R. A. Dandl and coworkers showed that the plasma in the ELMO device was in the form of an annulus composed of very energetic electrons ($T_e \sim 0.5 - 1$ MeV) with beta values approaching unity, where beta is the ratio of particle pressure to magnetic field pressure. These rings were a few Larmor radii thick and were found to be macroscopically stable when ambient gas pressure was above a critical value. During this period, more refined energy balance calculations made it clear that, even with classical behavior, it would be very difficult to produce net power in a standard simple mirror configuration.
Thus it was highly desirable to utilize in some way the high beta annulus in a controlled fusion device with toroidal configuration.

A simple toroidal system (i.e., a system with magnetic field only in toroidal direction) does not confine single particles because the vertical drifts due to curvature and grad B induce a rapid radial expulsion of the plasma. Several improved toroidal traps have been devised to provide confinement: those using external windings to generate rotational transform, as in stellators;\(^2\) those using internal windings, such a levitrons;\(^3\) those using (internal) plasma currents, as in tokamaks;\(^4\) and those using periodic spatial modulation of the magnetic field without net plasma currents, the so called bumpy torus.\(^5\) In the tokamak and stellator, the curvature and grad B drifts combine with the motion of guiding centers along the field lines to produce confined particle orbits.

A bumpy torus is a toroidal device with all field lines closed and no toroidal currents (i.e., no rotational transform). The machine at ORNL consists of a series of 24 identical, toroidally connected, canted mirrors with mirror ratio 2:1. Steady state plasma is sustained and heated by electron cyclotron heating. There are two principal plasma components in EBT: mirror trapped, high beta, hot electron rings (annuli) and a denser, lower energy toroidal core plasma. The stability of the toroidal plasma is achieved by the diamagnetism of hot electron annuli which transform the bumpy torus into an average minimum B configuration.\(^6\) In an average minimum B configuration, one encounters increasing magnetic field in almost all directions as one proceeds away from the center of configuration. Without such an effect produced by annuli, the toroidal
plasma would be subject to interchange modes. These modes are shown to be unstable in a closed line torus with isotropic pressure and monotonically increasing $\oint d\alpha / B$. Interchange instability occurs in the regions of bad curvature (i.e., convex field curvature in which the magnetic field strength decreases radially outward). However, theory predicts that a hot electron component with beta in excess of 15% should reverse the local gradient in magnetic field intensity, so that the system satisfies Kadomstev's criterion for stability. The local magnetic wells produced by the hot electron rings in EBT thus stabilize the toroidally confined nonaxisymmetric plasma, while the rings themselves are stabilized by the toroidal core plasma. This interplay between the rings and toroidal core plasma is at present a very active area of research and much still remains to be understood. The rings themselves are created in the midplane of each mirror section by external microwave heating at the electron cyclotron frequency and its harmonics. Confinement of the core plasma is achieved by the modification of the vertical drifts by the poloidal grad $B$ and $E \times B$ drifts produced by the bumpiness in the toroidal magnetic field and by the ambipolar electric field. Figure 1 shows a schematic diagram of EBT. Wavy lines are the bumpy magnetic field lines. Figure 2 shows one mirror section with the ring. Table 1 lists some of the relevant machine and plasma parameters for EBT-I.

B. Experimental Modes of Operation in EBT

In this work, we will be concerned with the following interesting experimental observation made on EBT: there exist three distinct, reproducible modes of operation as a function of the two natural experimental
parameters $-p_0$, the ambient gas pressure and $P_m$, the microwave power delivered to the plasma to generate hot electron rings (see Figs. 3a-f).

For fixed microwave power density $P_m$ and sufficiently low filling pressure $p_0$ one encounters the so called mirror or M mode in which the toroidal core plasma behaves somewhat like a series of disconnected simple mirror machines. This mode is characterized by large fluctuations in number density, and so very few equilibrium measurements can be made. The average electron line density is low and electron temperatures are high in toroidal core plasma (Figs. 3b-d). As the filling pressure $p_0$ is increased smoothly ("adiabatically") at fixed microwave power density $P_m$, there occurs a sharp transition in the fluctuation levels as one passes from the M mode to the so called T mode (Fig. 3(c)). In the T mode, which is the usual operating regime of EBT, the plasma is quiescent with electron temperatures $T_e \lesssim 600$ eV and ion temperatures $T_i \lesssim 40$ eV (there is also a small ion component with $T_i \sim 100$ eV), and average electron line density $n_e$ remains nearly constant as $p_0$ is increased (Figs. 3(b) and 3(c)). On further increasing $p_0$ at fixed $P_m$, there occurs a second sharp transition as one passes abruptly from the quiescent T mode to a somewhat noisier but cold C mode in which electron temperature $T_e \sim 20$ eV (the ion temperatures $T_i$ are not measured), average electron line density $n_e$ increases. In this mode, low amplitude fluctuations are observed in the number density (Fig. 3(c)). These sharp transitions in plasma behavior also occur if one decreases the microwave power $P_m$ at fixed filling pressure. $T$ to $C$ transition is somewhat smoother and more gradual compared to $M$ to $T$ transition. These
transitions also show hysteresis. Experimentally, the filling pressure $p_0$ and microwave power density at the transitions are observed to be related by the power law $p_0 \propto P_w^{0.5}$ (see Fig. 3(a)). In the C mode, the ambipolar electrostatic potential $\phi$ is observed to be close to zero (see Fig. 3(f)). The experimentalist usually finds it more convenient to examine these transitions by varying the filling pressure $p_0$ at fixed $P_w$, while theoretically we will find it more convenient to vary $P_w$ at fixed $p_0$.

The occurrence of sudden changes in the properties of a system induced by a small continuous change in the external parameters is commonly called a catastrophe.\(^{10-14}\) In the next chapter we will briefly review catastrophe theory. However, we should point out that the EBT model with which we shall work, the so-called point model,\(^{15}\) is a non-gradient dynamic system so that the powerful classification theorems of elementary catastrophe theory\(^{10-14}\) are not applicable to our EBT equilibria.

Based on experimental evidence and theoretical calculations, the EBT transport is shown to be dominated by neoclassical processes.\(^{16-18}\) Neoclassical transport processes arise due to guiding center drifts associated with field inhomogeneities such as $\text{grad B}$ and curvature drifts (see Appendix C on neoclassical transport theory). In the EBT, electrons are observed to be in collisionless regime, a regime where the collisions are too infrequent to randomise guiding center motions,\(^{19}\) and the electron transport is dominated by those electrons whose poloidal drifts do not vanish,\(^{18,20}\) (the so-called nonresonant electrons). As shown in Chapter IV, when the ambipolar electric field $E$ in the EBT core plasma is radially inward, the poloidal drift of the electrons can not
vanish (i.e., then there are only nonresonant electrons). Similarly for ions when \( E \) is radially outward. While neoclassical effects are typically much larger than the classical effects, classical and neoclassical transport processes are additive.\(^{21}\)
III. CATASTROPHE THEORY

In this chapter, we will discuss the basic ideas of catastrophe theory and what are some of the problems to which it has been applied; in particular an example in the elementary catastrophe theory (a restrictive but powerful subset of catastrophe theory in which the generalized force can be represented by a scalar potential, i.e., a gradient dynamic system) and two examples of application of catastrophe theory to non-gradient dynamic systems.

A. Introduction

The occurrence of sudden changes in the properties of a system induced by a small continuous change in the external parameters is commonly called a catastrophe.\textsuperscript{10-14} Catastrophe theory attempts to deal with such discontinuous behavior.\textsuperscript{12} Thermodynamic phase transitions are examples of such jump phenomena, and a number of other jump phenomena can be observed in nature.

We take the viewpoint of Gilmore\textsuperscript{13} who considers catastrophe theory to be a theory which studies how the qualitative nature of the solutions of equations depend upon the parameters that appear in the equations.\textsuperscript{13}

To make this program concrete, we look for solutions\textsuperscript{13}

\[ \psi_1(t,x; c_x), \psi_2(t,x; c_x) \ldots \] for a system of \( n \) equations defined over a space \( \mathbb{R}^N \) whose coordinates are \( x = (x_1, x_2, \ldots, x_N) \).\textsuperscript{13}

\[ , x_N \textsuperscript{13} \]
The variables $x_j$ and $t$ may conveniently be regarded as space and time coordinates. Since the solutions $\Psi_j$ describe the state of some system, they are called state variables. The equations $F_i = 0$ depend on $k$ parameters $C_\alpha$. Since the parameters $C_\alpha$ may control the qualitative properties of the solutions $\Psi_j$, the $C_\alpha$ are called control parameters.

The problem of determining the solutions of $F_i = 0$ with above form of $F_i$'s is formidable. This problem can be made more tractable by making some simplifying assumptions about $F_i$'s. If $F_i$'s are such that

$$F_i = \frac{d\Psi_i}{dt} - f_i (\Psi_j, C_\alpha) = 0$$

then the system of equations of this type ($F_i = 0$) is called a dynamical system. Further, if

$$F_i = \frac{d\Psi_i}{dt} - f_i (\Psi_j, C_\alpha) = 0$$

then this system of equations is called an autonomous dynamical system.

A few useful and powerful statements can be made about such autonomous dynamical systems which depend on small number of control parameters ($k \leq 4$). When all the functions $f_i$ can be derived as the gradient (with
respect to $\Psi_i$ of some scalar potential function $V(\Psi_j; C_{\alpha})$

$$\rho_i = -\frac{\partial V}{\partial \Psi_i},$$

the resulting system

$$\dot{\Psi} = -\nabla_{\Psi} V$$

is called a gradient dynamic system. A great deal can be said about
gradient dynamic systems.

Of particular interest are the equilibria $d\Psi_i/dt = 0$ of
dynamical systems (which includes autonomous dynamical systems and
gradient dynamic systems). The equilibria $\Psi_j(C_{\alpha})$ of a gradient dynamic
system are defined by the equations

$$\nabla V(\Psi_j; C_{\alpha})/\nabla_{\Psi} = 0.$$

A great many deep, useful and powerful statements can be made
about equilibria of gradient dynamic systems, and how these equilibria
depend on the control parameters $C_{\alpha}$. Elementary catastrophe theory
(ECT) is the study of how the equilibria $\Psi_j(C_{\alpha})$ change as the control
parameters $C_{\alpha}$ change. ECT studies and classifies the types of dis-
continuous behaviors one can find in the equilibria of gradient dynamic
systems as the control parameters are smoothly changed. Thom has shown
that for gradient dynamic equilibria there are at most seven topologically
distinct stable types of discontinuous behaviors that can occur, if the
number of control parameters does not exceed $(i.e., k \leq 4)$. See Table 2.
Equilibria which can not be related to each other by diffeomorphic map-
ping are termed topologically distinct equilibria; a diffeomorphism
being a 1-1 onto continuously differentiable mapping with a continuously differentiable inverse map.

The terminology and methods developed for studying gradient dynamic systems can be used to study the properties of autonomous dynamical systems as well.\(^\text{13}\) In fact, a general program for discussing the qualitative properties of autonomous dynamical systems, akin to the general program of the Elementary Catastrophe Theory, may be formulated.\(^\text{13}\)

It must be emphasized that the range of applicability of catastrophe theory far exceeds gradient dynamic systems. In an essay entitled "The two-fold ways of Catastrophe Theory," Thom has characterized the two ends of the spectrum of theory's applications as the "physical" and the "metaphysical":

"Either, starting from known scientific laws (from Mechanics or Physics) you insert the catastrophe theory formalism (eventually modified) as a result of these laws: this is the physical way. Or, starting from a poorly understood experimental morphology, one postulates 'a priori' the validity of the catastrophe theory formalism, and one tries to reconstruct the underlying dynamic which generates this morphology: this is the metaphysical way. Needless to say, the second way seems to me more promising than the first, if less secure . . ."\(^\text{29}\)

This shows that an important feature of catastrophe theory is that it can be used not only in many different problems but also in many different ways. Saunders has stated that:

"These examples contribute to our understanding of catastrophe theory by serving as relatively straightforward illustrations and also by
showing how the range of applicability of the theory extends far beyond systems with gradient dynamics.\textsuperscript{29}

While Poston and Stewart have stated that:

"... Systems with more elaborate dynamics than a simple minimization or maximization often exhibit behavior reducible to a catastrophe ... Not all dynamic phenomena are driven by the maximizing or minimizing of potential functions, with consequent catastrophe geometry as these functions vary. Many that are not can be related to catastrophe less directly; by the construction of suitable Liapunov functions, as for the Duffing equation (Holmes and Rand); by variational principles as for optics; by stream functions; and so on. Catastrophe theory (in the sense we have used the phrase) is not universally applicable, but the frequent statement 'it applies only to gradient systems' is simply a misunderstanding brought about by the limited range of early examples.

"... Absolutely all of the chaotic behavior, 'strange attractors' (Guckenheimer, Oster and Ipaktchi), 'omega-explosions' and other freakish familiars of the wizards of modern dynamical systems theory can stably occur in this class of models. (Thom's more general use of the therm 'catastrophe' takes these phenomena into account - but then a general theory is lacking; so the practical advantage is small: the philosophical gain may be greater.)\textsuperscript{12}

B. An Application of Catastrophe Theory to a Gradient Dynamic System

(Elementary Catastrophe Theory)

In order to study one example of application of elementary catastrophe theory to a gradient dynamic system, consider the set of s
Equations

$$\frac{dx_i}{dt} = \frac{\partial f_i(x, \alpha)}{\partial x_i}, \quad i = 1, \ldots, s$$ (1)

where the dependent variable vector $$\mathbf{x} = (x_1, x_2, \ldots, x_s)$$ consists of $$s$$ state (or behavior) variables and the control (or external parameters) vector $$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_C)$$ consists of $$C$$ parameters. If

$$\frac{\partial f_i}{\partial x_i} = -\frac{\partial V(x, \alpha)}{\partial x_i}, \quad i = 1, \ldots, s$$ (2)

then eqs. (1)-(2) constitute a gradient dynamic system. The critical points of eq. (1), namely the equilibrium surfaces

$$x_{i0} = x_{i0}(\alpha) \quad i = 1, \ldots, s$$ (3)

are found by solving the $$s$$ simultaneous equations $$f_i(\mathbf{x}; \alpha) = 0$$. Now consider a system having one state variable and two control parameters ($$s = 1, C = 2$$) with the potential function

$$V(x, \alpha_1, \alpha_2) = \frac{1}{4} x^4 + \frac{1}{2} \alpha_1 x^2 + \alpha_2 x$$ (4)

The two dimensional equilibrium surface

$$x_0 = x_0(\alpha_1, \alpha_2)$$

satisfying

$$\frac{\partial V}{\partial x} = x^3 + \alpha_1 x + \alpha_2 = 0$$ (5)

embedded in three dimensional $$(x_0, \alpha_1, \alpha_2)$$ space is shown in Figure 4a. Note the existence of folds in the equilibrium surface. The projection
of the folds onto the control space \((\alpha_1, \alpha_2)\) gives rise to a curve in the control space called the bifurcation set.

For the potential, eq. (4), the bifurcation set is the curve

\[
4\alpha_1^3 + 2\alpha_2 \alpha_2^2 = 0
\]  

(6)

found from \(\partial V/\partial x = 0 = \partial^2 V/\partial x^2\). Note the appearance of cusp at \((\alpha_1, \alpha_2) = (0,0)\). The region of control space contained within the bifurcation set separatrix is called the bifurcation space. For \((\alpha_1, \alpha_2)\) lying outside the bifurcation space the potential \(V\) has only one minimum corresponding to a unique equilibrium state (i.e. a point attractor for the dynamical system, eqs. (1) and (2)). For \((\alpha_1, \alpha_2)\) lying inside the bifurcation space, the potential \(V\) exhibits two local minima and one local maximum, i.e., the equilibrium surface consists here of three sheets joined together at the respective fold edges, giving rise to three possible equilibria for a given \((\alpha_1, \alpha_2)\). Two of these equilibria are stable and the third is unstable (i.e. there are two point attractors and one point repellor for the dynamical system).

The equilibrium surface shown in Figure 4a for the gradient dynamic system, eqs. (1)-(2), is the equilibrium surface for the canonical cusp catastrophe.

C. Application of Catastrophe Theory to Nongradient Dynamic Systems

Now we will consider two examples of application of catastrophe theory to nongradient dynamic systems.
1. Duffing Equation

The Duffing equation represents the simplest nonlinear oscillator:

\[ \ddot{x} + k \dot{x} + x + ax^3 = F \cos \Omega t, \quad k > 0 \]

We write

\[ \Omega = 1 + \omega \]

and we suppose that \( k, a, \omega \) are all small. Since \( a \) is small, we expect that the solution of the Duffing equation will be close to that of the linear oscillator, so we attempt a solution of the form

\[ x = A \cos(\Omega t - \phi) \]

We substitute this into the differential equation, ignore second order terms and a term in \( \cos 3\Omega t \), and equate the coefficients of \( \cos \Omega t \) and \( \sin \Omega t \) on either side of equation to obtain

\[ \tan \phi = 4k / (3aA^2 - 8\omega) \]

and

\[ A^2 \left( \frac{3}{4} aA^2 - 2\omega \right)^2 = F^2 - k^2 A^2 \]

The second of these is a cubic in \( A^2 \). Now consider \( A^2 \) as state variable and \( a \) and \( \omega \) as control parameters. Then, from the above cubic equation in \( A^2 \), the surface \( A^2 \) as a function of \( a \) and \( \omega \) is as shown in Figure 4b. We can see that this surface looks double cusp catastrophes. The bifurcation curve can be obtained by differentiating the cubic equation twice (with respect to \( A^2 \)) and eliminating \( A^2 \) from the two resulting equations; this gives
\[(a, \omega) = \pm (32k^2 \sqrt{3/2} F^2, k\sqrt{3/2})\]

so these are, in fact, two cusps as bifurcation curves.

The Duffing equation has a number of applications in physics, but its chief significance from our point of view is that it illustrates another way in which catastrophe theory can apply to systems without gradient dynamics.

2. van der Pol equation with large \(k\)

Consider the van der Pol equation with large \(k\):

\[\ddot{x} + k(x^2 - b) \dot{x} + x = 0, \quad k > 0\]

Let the initial values of \(x\) and \(\dot{x}\) be \(x_0\) and \(\dot{x}_0\), respectively, and let

\[z(t) = z_0 - \frac{1}{k} \int_0^t x(\tau) \, d\tau\]

where

\[z_0 = \frac{1}{3} x_0^3 - bx_0 - \dot{x}_0/k\]

Now

\[\ddot{z} = -x/k\]

Then van der Pol equation becomes

\[\ddot{x} + k(x^2 \dot{x} - bx - \dot{z}) = 0\]

Integrating this, we obtain an alternative form of van der Pol equation:

\[\dot{x} = -k\left(\frac{1}{3} x^3 - bx - z\right) \quad \text{"fast equation"}\]

\[\dot{z} = -x/k \quad \text{"slow equation"}\]
We call these "fast" and "slow" equations because with k very large, the rate of change of x is very much greater than that of z. Consequently, z may be considered as a parameter for determining the behavior of x. The equilibria for x are given by the equation
\[ \frac{1}{3} x^3 - bx - z = 0 \]
and since we are treating z as a parameter, the equilibrium surface is that for the canonical cusp catastrophe. The equilibrium surface and the bifurcation curve are shown in Figure 4c.

We interpret the Figure 4c in the following way. Off the surface, the fast equation ensures that the trajectories are very nearly parallel to the x axis. The phase point will therefore move almost directly onto the surface. This makes x vanish, so the system is then governed entirely by the slow equation. If b is positive and constant, the system moves automatically around an orbit (as shown in Figure 4c), exhibiting characteristic sudden jumps and hysteresis.

Thus a nonlinear oscillator to which ECT is not applicable, can still, under certain circumstances, be represented as an elementary catastrophe. The decomposition of systems of differential equations into subsystems with different timescales is a device which is used in many other contexts. It is sometimes called "adiabatic elimination". It is significant that the approximation which we had to make in order to apply catastrophe theory is the same as is sometimes necessary before other methods can be used.

A number of other examples of application of catastrophe theory to nongradient dynamic systems can be found in literature, e.g. caustics;
spiral chaos, Lorentz attractor; Benard instability, Laser Spiking instability, Ruelle-Takens Picture of Turbulence, etc.\textsuperscript{13}

D. Delay and Maxwell Conventions

The study of the time evolution of the equilibria of gradient dynamic systems is outside the scope of elementary catastrophe theory (ECT). Hence to consider the evolution of equilibria as the control parameters are changed, Thom\textsuperscript{10} was forced to introduce some assumptions regarding the behavior of the system when control parameters lie in the bifurcation space and there exist several point attractors (i.e. several potential minima). Thom introduces two conventions into catastrophe theory: the Delay convention and the Maxwell convention, and further refinements have been considered by Gilmore\textsuperscript{22} and by Agarwal and Shenoy.\textsuperscript{23}

In the delay convention one assumes that the system will remain in the same equilibrium state (i.e. same potential minimum), irrespective of the existence or depth of other potential minima until that (first) potential minimum disappears. In the Maxwell convention one assumes that the system will tend to that equilibrium state which yields a global minimum for \( V \) (i.e., the deepest potential well). The relative magnitudes of the rate of change of parameter and the time required to attain equilibrium (i.e. relaxation time) govern as to which convention a system will follow. If the inverse of the time required to reach a local minimum of potential (i.e., \( t^{-1} \) relaxation, local minimum) is large compared to the rate of change of the control parameter (i.e., \( d\xi /dt \)), and if the inverse of time required to reach global minimum (i.e., \( t^{-1} \) relaxation, global
minimum) is small compared to the rate of change of the control parameter, then the delay convention is applicable. However, if the inverse of time required to global minimum is large compared to rate of change of parameter, Maxwell convention is applicable. In short,

$$t^{-1} \text{ relaxation, local minimum } \Rightarrow \frac{d\mathcal{L}}{dt} \Rightarrow t^{-1} \text{ relaxation, global minimum}$$

(Delay Convention)

$$t^{-1} \text{ relaxation, global minimum } \Rightarrow \frac{d\mathcal{L}}{dt}$$

(Maxwell convention)

where $\mathcal{L}'s$ are in dimensionless form.

Let us consider the delay convention as applied to cusp catastrophe potential, eq. (4) and in particular to the behavior of the equilibrium trajectory as we keep $\mathcal{L}_1 = \text{constant}$ and decrease $\mathcal{L}_2$ with $\mathcal{L}_2 > \mathcal{L}_*$ initially. See Figures 4a and 5. For $\mathcal{L}_2 > \mathcal{L}_*$ there is a unique equilibrium which lies on the top sheet of the equilibrium surface and $V$ has a unique minimum. On decreasing $\mathcal{L}_2$ we cut the bifurcation set at $\mathcal{L}_2 = \mathcal{L}_*$ and then enter the bifurcation space ($\mathcal{L}_** < \mathcal{L}_2 < \mathcal{L}_*$) for which there are now three possible equilibria or sheets. The delay convention implies that the equilibrium trajectory will remain on the top sheet of equilibrium surface. When $\mathcal{L}_2 = \mathcal{L}_**$ we reach the fold edge of the top sheet and on decreasing $\mathcal{L}_2$ further the equilibrium surface reverts back to having only a single sheet - the previous minimum of $V$ disappears and now $V$ has a new unique minimum which is located on the bottom sheet. Thus, using the delay convention, a jump or catastrophe
occurs in the equilibrium state at $\alpha_2 = \alpha_\infty$ for $\alpha_2$ decreasing. However, on retracing the $\alpha_2$ curve in control space by increasing $\alpha_2$ from $\alpha_2 < \alpha_\infty$ initially, we will find, on using delay convention, that the equilibrium trajectory will jump from the bottom sheet to top sheet at $\alpha_2 = \alpha_\infty$. This hysteresis effect would not occur if the system followed the Maxwell convention since the trajectory will always seek a global minimum and the jump in the equilibrium trajectory will always occur at $\alpha_2 = 0$ (for $\alpha_1 < 0$) for the potential function $V$, eq. (4).

Now let us apply Delay convention to the $A^2$ surface of Duffing equation (Fig. 4b). When $a = 0$, there is a minimum of $A^2$ (or $A$) at $\omega = 0$. If $a > 0$ and if $a$ is sufficiently large and if we begin with $\omega < 0$ and then increase it, $A^2$ will increase slowly to its maximum and then decrease, but then it will fall suddenly. At the same time there will be a change in phase of the oscillator. If we now slowly decrease $\omega$, there will be eventually another sudden change in amplitude and phase, but at a different point. A similar effect is observed when $a < 0$.

In the case of van der Pol oscillator with large $k$, we have already seen that the system follows Delay convention (see section C).

Thus the discussion in sections B, C, and D show how the terminology and method developed in section B for studying gradient dynamic systems can be used to study the properties of non gradient dynamic systems, as was claimed in section A.

As an example of the Maxwell convention, consider a gas which obeys van der Waal's equation of state

$$\left(p + a/\sqrt{2}\right) (V - b) = RT$$
Here P is the pressure, V is the volume, T is the absolute temperature, R is the gas constant, and a and b are two constants characteristic of the particular gas. If we take volume as the dependent variable, we may write this equation in the form

\[ V^3 - \left( b + \frac{RT}{p} \right)V^2 + \left( \frac{a}{p} \right)V - \frac{ab}{p} = 0 \]

Since this is a cubic equation, the surface it represents is diffeomorphic to the equilibrium surface of the canonical cusp catastrophe (figure 4a). This suggests that we should be able to interpret the behavior of the system in terms of catastrophe theory, taking V as the state variable and P and T as control parameters. If we attempt this, we discover that sudden jumps occur, since there is an abrupt increase in volume when a liquid changes to vapor. But we do not observe hysteresis; water usually boils at the same temperature at which steam condenses. Also, we can predict the volume uniquely if we are given the temperature and pressure.

The reason for this is that the system does not remain in a local potential well until that minimum disappears. Instead, it seeks a global minimum of the thermodynamic potential, with the result that the phase change occurs when the potential is the same for both phases. In terms of Figure 4a, the jumps in either direction occur as the control trajectory crosses a single curve which lies within the cusp. Figure 5a illustrates the delay convention and Figure 5b illustrates Maxwell convention.
IV. THE POINT MODEL FOR EBT

A. Point Model Equations for EBT

In attempting to explain the existence of abrupt transitions in the modes of operation of EBT, we shall consider the point model of Hedrick et al. with and without ion cyclotron resonance heating (ICRH) and study the properties of equilibria as the external parameters are smoothly varied. This point model is a very simple and primitive model for the toroidal plasma in EBT. In this model radial derivatives are approximated by the inverse of an appropriate scale length. The point model equations are simply the conservation laws for particles and energy.

The point model for toroidal core plasma in EBT assume quasi-neutrality of charge, i.e., $n_e \approx n_i \equiv n$ where $n_e$ is number density of electrons and $n_i$ is number density of ions. Charge density $n$ increases as a result of ionization of neutrals caused by collisions between neutrals and electrons. The decrease in charge density $n$ occurs because of diffusion of charged particles. Electron energy is lost as a result of transport processes and collisions with ions. Energy is gained by electrons due to microwave heating (ECH). Ions gain energy as a result of ion cyclotron resonance heating (ICRH). They also gain energy from electrons as a result of electron-ion collisions. Loss of ion energy occurs as a result of transport processes. All these effects can be expressed in the form of equations as follows:
These are point model equations. Here we have assumed quasi-neutrality of charge, \( n_e = n_i \equiv n \). \( n_0 \) is the neutral particle density, which is related to filling pressure \( p_0 \), \( T_e \) is electron temperature, \( T_i \) is ion temperature, \( n_0 \langle \sigma v \rangle_{\text{ion}} \) is the ionization rate for an electron colliding with neutrals, \( P_{\mu} \) is electron cyclotron microwave power absorbed by electrons per unit volume due to electron cyclotron resonance heating, \( P_{ei} \) is the rate of energy transfer per unit volume from electrons to ions, \( n_0 \langle \sigma v \rangle_{\text{cx}} \) is the charge exchange rate for ions colliding with neutrals. The quantities \( \tau_n \), \( \tau_e \) and \( \tau_i \) are lifetimes for particles, electron energy and ion energy respectively due to transport processes such as diffusion and thermal conduction. \( P_i \) is ion cyclotron resonance heating power absorbed per unit volume by ions.

For present purposes, we shall assume that

\[
\tau_n \approx \tau_e \approx \tau_i \equiv \tau
\]

then equations (8) and (9) can be put into more useful form by differentiating the product \( nT \) and using eq. (7). We find

\[
\frac{dT_e}{dt} = \frac{2}{3} \frac{P_{\mu}}{n} - \frac{2}{3} \frac{P_{ei}}{n} - T_e n_0 \langle \sigma v \rangle_{\text{ion}}
\]
\[
\frac{dT_t}{dt} = \frac{2}{3} \frac{P_i}{n} + \frac{2}{3} \frac{P_e}{n} - T_t n_0 \left( <\sigma v>_{\text{ion}} + <\sigma v>_{\text{cl}} \right)
\]

(11)

Now \(t\) consists of two parts:

\[
\frac{1}{\tau} = \frac{1}{\tau_{\text{nc}}} + \frac{1}{\tau_{\text{c}}}
\]

(12)

Subscript \(\text{nc}\) denotes neoclassical transport and \(\text{c}\) denotes classical transport. We will use the neoclassical diffusion rate based on work of Kovrizhnikh and Harris.\(^{15}\) It is given by

\[
\tau_{\text{nc}}^{-1} = \left( \frac{4 D_{\text{nc}}}{a^2} \right) F(E)
\]

(13)

where \(a\) is minor radius of the EBT and \(D_{\text{nc}}\) is neoclassical diffusion coefficient. Since different species diffuse with different rates, local net charge concentrations are built up in a plasma, giving rise to the ambipolar electric fields in such direction as to equalize the loss rates of different species. \(F(E)\) is a function of ambipolar electric field \(E\).

Reference 15 gives the neoclassical diffusion coefficient as:

\[
D_{\text{nc},\alpha} = \frac{a^2}{3} \left( \frac{R_c}{R_T} \right)^2 \frac{\nu_d}{\gamma_d^2} \frac{1}{\left( 1 + \frac{\nu_e^2}{\gamma_e^2} \Omega_{0e}^2 \right)}
\]

(14)

where \(R_c\) is the minor radius of curvature, \(R_T\) is the major radius of the torus, \(\alpha\) is the subscript denoting species (\(\alpha = e\) for electrons, \(\alpha = i\) for ions), \(\nu_\alpha\) is collision frequency (for \(\alpha = e\), \(\nu_\alpha\) is electron-ion collision frequency; for \(\alpha = i\), \(\nu_\alpha\) is ion-ion collision frequency).
\[ \nu_i = \frac{c_1 n_i}{T_i^{3/2}} \]
\[ c_1 = \left( \frac{e^2}{\varepsilon_0} \right)^2 \frac{\ln \Lambda}{2 \varepsilon_0 B \sum m_i} \]
\[ \nu_e = \left( \frac{T_i}{T_e} \right)^{3/2} \left( \frac{m_i}{m_e} \right)^{1/2} \nu_i \]

(15)

\[ \ln \Lambda \] is Coulomb logarithm, and \( m_i \) and \( m_e \) are ion and electron mass respectively. The average poloidal curvature drift frequencies \( \Omega_{0\alpha} \) (see appendix B) are given by

\[ \Omega_{0\alpha} = \frac{T_\alpha}{\left( q_\alpha B R_c \right)} \]

(16)

where \( T_\alpha \) and \( q_\alpha \) are the temperature and charge respectively for species \( \alpha \). \( B \) is magnitude of magnetic field. \( Y_\alpha \) in eq. (14) is given by

\[ Y_\alpha = 1 + \left( q_\alpha E R_c \right) \left( \frac{B}{T_\alpha} \right) \]

(17)

\( \Omega_{0\alpha} \) incorporates grad \( B \) and curvature drifts (see appendix B) through the mirror radius of curvature \( R_c \), while \( E \times B \) drifts are incorporated through \( Y_\alpha \). It is the existence of poloidal drifts which prevent the disastrous radially outward drifts, and thus permit the basic particle confinement of the toroidal plasmas in EBT.

B. Resonant and Nonresonant Particles

Now the \( E \times B \) drift is given by

\[ \vec{v}_{E \times B} = \left( \vec{E} \times \vec{B} \right) / B^2 \]

(18)
which is independent of the sign of charge of the particle, while the gradient and curvature drifts in magnetic field drifts are dependent on the sign of charge of the particle:

\[ \vec{V}_{vB} = \frac{1}{q_{\alpha}} \omega_\perp \frac{\vec{B} \times \vec{V}B}{B^3} \]  

(19)

\[ \vec{V}_c = -2 \omega_{||} \frac{\vec{B} \times \vec{R}_c}{q_{\alpha} \beta c R_c^2} \]  

(20)

\( \omega_\perp \) and \( \omega_{||} \) are the contributions to the kinetic energy of a particle due to the components of its velocity perpendicular to and parallel to the magnetic field respectively.

Thus the poloidal drift of the charged particles in bumpy torus is given by

\[ \vec{V}_{\theta, \alpha} = \vec{V}_{ExB} + \vec{V}_{vB, \alpha} + \vec{V}_c, \alpha \]  

(21)

When the ambipolar electric field is directed radially inward, \( \vec{V}_{ExB} \) is in \( +\theta \) direction for both ions and electrons. For ions, \( \vec{V}_{vB,i} + \vec{V}_{c,i} \) are directed in the \( -\theta \) direction. While for electrons \( \vec{V}_{vB,e} + \vec{V}_{c,e} \) are directed in a \( +\theta \) direction. See Figure 6. Thus when ambipolar electric field is negative in EBT, poloidal drift velocity of different groups of ions can vanish depending upon the location of each individual group in velocity space. Particles are said to be resonant when their poloidal drift vanishes. The transport processes for resonant particles are critically dependent upon their location in velocity space. This means
that fluid equations of motion are somewhat inadequate in this case, since the particle losses are kinetic in origin. The problem of resonant losses in toroidal EBT geometry is not fully settled yet. Experimentally, the dominant loss mechanism is attributed to neoclassical nonresonant loss. Therefore we will use neoclassical transport coefficient for nonresonant particles and specify \( E \) (or \( \phi \), the ambipolar electrostatic potential on the axis with \( \phi = 0 \) at plasma boundary) as a control parameter. This frees us from having to specify a neoclassical loss rate for resonant particles. When the ambipolar electric field is radially inward, we can approximate \( E \) by \( -d\phi/dr \sim \Phi/a \), where we have replaced the spatial derivative \( d/dr \) by \( -(1/a) \). Thus, when \( \phi \) is negative, equations (18) to (21) show that ions are resonant and electrons are nonresonant and vice versa when \( \phi > 0 \). The factor \( F(E) \) in equation (13) is given by

\[
\tau_{nc}^{-1} = \begin{cases} 
\frac{4Dnc_e}{a^2} \left( 1 + \frac{q_e \phi}{T_e} \right), & \phi < 0 \\
\frac{4Dnc_i}{a^2} \left( 1 + \frac{q_i \phi}{T_i} \right), & \phi > 0 
\end{cases}
\]  

(22)

Until now EBT has been operated with minimal amount of ion cyclotron heating (ICRH) so as to increase ion temperature \( (P_i \ll P_\mu) \) and ambipolar electrostatic potentials have been found experimentally to be negative\(^9,18\) (see also Figures 3(e), 3(f)).

C. The Point Model

The lifetime due to classical transport is given by
\[
\tau_c^{-1} = \frac{D_c}{a^2} \tag{23}
\]

where \( D_c \) denotes the classical particle diffusion coefficient which is the same for electrons and ions and is given by

\[
D_c = a_e^2 \omega_{ei} \tag{24}
\]

where \( a_e \) is electron gyroradius,

\[
a_e = \frac{1}{\omega_{ce}} \sqrt{\frac{2T_e}{m_e}} \tag{25}
\]

\( \omega_{ei} \) is electron ion collision frequency and \( \omega_{ce} \) is electron cyclotron frequency, given by

\[
\omega_{ce} = \frac{eB}{m_e} \tag{26}
\]

\( \omega_{ei} \) is given by eq. (15). The rate of power transfer from electrons to ions, \( P_{ei} \), is given by

\[
P_{ei} = \frac{z(e/m_e)}{m_i} n (T_e - T_i) \omega_{ei} \tag{27}
\]

From the expression chosen for \( \tau \), it can be seen that we have assumed that each particle of species \( j \), as it escapes by diffusion across the magnetic field, carries with it its energy \( 3/2 \, T \) and that this is the principal energy transport mechanism. We have also assumed that electrons and ions diffuse at the same rate. \( \tau_{nc} \) depends upon ambipolar electric field \( E \) which is not self consistently determined by our use of the point model. We will treat \( E \), or more precisely, \( e \phi/T \), as an external parameter together with filling density \( n_0 \), microwave power \( P_\mu \), and
ICRH power $P_i$. $\langle \sigma \nu \rangle_{\text{ion}}$ and $\langle \sigma \nu \rangle_{\text{ex}}$ are slowly varying functions of temperature.\textsuperscript{25} For simplicity, we will treat them as constants.

Experimentally, electron or ion temperatures in EBT are of order of $100 \text{ eV}$, toroidal plasmas number density $\sim 10^{12} \text{ cm}^{-3}$, and charged particle lifetimes $\sim 10^{-3} \text{ sec}$ for ambient pressures that indicate neutral particle density of order $10^9 \text{ cm}^{-3}$.\textsuperscript{26} Typically a microwave power of $10 \text{ kw}$ is absorbed in a volume of $4 \times 10^5 \text{ cm}^3$. It is convenient to introduce dimensionless EBT units with

\begin{align*}
T_e \text{ and } T_i \text{ in units of } 100 \text{ eV} \\
n \text{ in units of } 10^{12} \text{ cm}^{-3} \\
n_0 \text{ in units of } 10^{11} \text{ cm}^{-3} \\
t \text{ in units of } 10^{-3} \text{ sec.} \\
P_e \text{ and } P_i \text{ in units of } 10 \text{kw/cm}^3.
\end{align*}

For all calculations presented here, we will use the values\textsuperscript{15}

\begin{align*}
B &= 0.65 \text{ Tesla} \\
R_T &= 1.5 \text{ m} \\
R_c &= 1.88 \text{ m} \\
a &= 0.1 \\
\ln \Lambda &= 17 \\
\langle \sigma \nu \rangle_{\text{ion}} \text{ and } \langle \sigma \nu \rangle_{\text{ex}} &= 3.5 \times 10^{-14} \text{ m}^3/\text{s} \\
m_e/m_i &= 1/1836.
\end{align*}

Using these values, we can obtain the expression for $\tau^{-1} = \tau^{-1}_{\text{nc}} + \tau^{-1}_c$ either for nonresonant electrons or ions from eqs. (12) to
(17), (22) and (23) to (26). The expression for $\tau^{-1}$ so obtained can then be used in eq. (7). Equations (7), (10) and (11), in dimensionless (EBT) units then determine the time evolution of the plasma density $n$ and temperatures $T_e$ and $T_i$. In the following chapters, we will study the equilibria and time development of these equations.
V. EQUILIBRIA FOR POINT MODEL WITH ELECTRONS

A. Point Model Equations with Neoclassical Nonresonant Electrons

As mentioned before, in the absence of ICRH \( P_i = 0 \), the ambipolar electrostatic potential \( \phi \) inside the toroidal core plasma has been experimentally found to be negative and the electron transport is dominated by neoclassical nonresonant effects. We thus consider the point model equations for toroidal core plasma with neoclassical nonresonant electrons with negative \( e\phi / T_e \) as a parameter. As mentioned in the last chapter this frees us from having to specify the resonant ion transport coefficient. In this case the point model equations are:

\[
\frac{dn}{dt} = n \left[ n_o \langle \sigma v \rangle_{ion} - \frac{a_1 n / T_e^{3/2}}{\left( 1 + a_2 n^2 / T_e^3 \right)} - a_3 n / T_e^{1/2} \right] \tag{28}
\]

\[
\frac{dT_e}{dt} = c_1 \frac{P_e}{n} - c_2 \frac{n}{T_e^{1/2}} \left( 1 - \frac{T_i}{T_e} \right) - T_e n_o \langle \sigma v \rangle_{ion} \tag{29}
\]

\[
\frac{dT_i}{dt} = c_1 \frac{P_i}{n} + c_2 \frac{n}{T_e^{1/2}} \left( 1 - \frac{T_i}{T_e} \right) - T_i n_o \langle \sigma v \rangle_{ion} + \langle \sigma v \rangle_{ex} \tag{30}
\]

where \( a_1 \) and \( a_2 \) are functions of \( \phi = 19 e \phi / T_e \), given by

\[
a_1 = c_3 \left( 1 - \phi \right) / \left( 1 - c_4 \phi \right)^2
\]

\[
a_2 = c_5 / \left( 1 - c_4 \phi \right)^2
\]
\(a_3, c_1, c_2, c_3, c_4 \text{ and } c_5 \text{ are constants. Experimentally, } \phi \text{ constant = -0.6 EBT units in the T mode.}^{27} \) See Figure 3(e).

In the language of catastrophe theory, \( \mathbf{x} = (n, T_e, T_i) \) is the state or behavior vector and \( \mathbf{z} = (n_0, P, P_i) \) is the control vector.

B. Method for Obtaining Equilibria

Our aim is to determine the equilibrium state vector \( \mathbf{x} \) as a function of control vector \( \mathbf{z} \). To this end, define two new variables \( x \) and \( y \) as

\[
x = n / T_e^{5/2} > 0
\]

\[
y = T_i / T_e > 0
\]

The equilibrium solutions (i.e., \( d/dt \equiv 0 \)), from eq. (28), satisfy

\[
n_0 \langle \sigma v \rangle_{on} - \frac{q_1 x T_e}{(1 + a_2 x^2)} - a_3 x T_e^2 = 0
\]

From eqs. (29) and (30), with \( d/dt \equiv 0 \), we get (note that we have assumed, for simplicity, \( \langle \sigma v \rangle_{on} = \langle \sigma v \rangle_{ex} \), which is a reasonable assumption based on experimental measurements of these cross sections)

\[
T_e = k / x
\]

where

\[
k = n_0 \langle \sigma v \rangle_{on} \left[ R (1 + 2y) - 1 + 2y \right] / 2c_2 (1 - y)
\]
and

\[ R = \frac{P_\mu - P_i}{P_\mu + P_i} \]  

(36)

clearly \(-1 \leq R \leq 1\). Since \(P_\mu > 0\), \(P_i > 0\), \(x = n/n_e^{5/2} > 0\), \(y = T_i/T_e > 0\) and \(R < 1\); one can distinguish four possibilities arising out of eq. (35):

(i) When \(P_i = 0\), then \(R = 1\) and \(0 < y < 1\) (i.e., \(T_i < T_e\))

(ii) When \(P_\mu = P_i\), then \(R = 0\) and \(1/2 < y < 1\) (i.e., \(T_i < T_e\))

(iii) When \(P_\mu > P_i\), then \(0 < R < 1\) and \(1/2 < y < 1\) (i.e., \(T_i < T_e\))

(iv) When \(P_\mu < P_i\), there are two possibilities:

(a) \(1/2 < y < 1\); \(-1/3 < R < 0\) (i.e., \(T_i < T_e\))

(b) \(1 < y < 1/2\); \(-1 < R < -1/3\) (i.e., \(T_i > T_e\)).

This analysis immediately points towards some important features of equilibria. First, when ICRH is absent \((P_i = 0)\) or when electron cyclotron heating is larger than ion cyclotron heating \((P_\mu > P_i)\), ion temperatures \(T_i\) cannot exceed electron temperatures \(T_e\). Even when \(P_\mu = P_i\) or \(P_\mu > P_i\), it is not necessary that \(T_i > T_e\). Only when \(R < -1/3\), \(T_i > T_e\). This asymmetry between \(P_\mu\) and \(P_i\) arises due to asymmetries in the sign of the term \(P_{ei}\) and coefficients of \(T_{e0}\) and \(T_{i0}\) in eqs. (29) and (30) respectively. Physically, this is because ions loose energy due to charge exchange in collisions with neutrals, while electrons do not (see eqs. (8) and (9)). The factor \(1/2\) comes from the choice

\[ \langle \sigma v \rangle_{\text{ion}} = \langle \sigma v \rangle_{\text{cx}} \] so that

\[ \langle \sigma v \rangle_{\text{ion}}/(\langle \sigma v \rangle_{\text{ion}} + \langle \sigma v \rangle_{\text{cx}}) = 1/2. \] In short, unless \((P_i/P_\mu) > 2\), \(T_i\) cannot exceed \(T_e\).
Substituting $T_e = k/x$ in eq. (33) yields a cubic in $x$:

$$B_1 x^3 + B_2 x^2 + B_3 x + B_4 = 0 \quad (37)$$

where the coefficients $B_i$ are

$$B_1 = \frac{\langle \sigma v \rangle_{ion}}{\langle \sigma v \rangle_{ion}} \alpha_2$$
$$B_2 = -k^2 \alpha_2 \alpha_3$$
$$B_3 = \frac{\langle \sigma v \rangle_{ion}}{\langle \sigma v \rangle_{ion}} - k \alpha_1$$
$$B_4 = -k^2 \alpha_3$$

From the numerical solution of the cubic eq. (37), we always find that there is only real positive root of eq. (37) for $x$. Since we require $x > 0$, we can determine (from given $\tilde{\varphi}$, $n_0$ and $y$)

$$T_e = \frac{k}{x} \quad (38)$$
$$n = x T_e^{5/2} \quad (39)$$
$$T_i = \frac{y}{y} T_e \quad (40)$$

$P_{\mu}$ and $P_i$ can then be determined from the equilibrium solution of eqs. (29) and (30).

$$P_{\mu} = \frac{n}{c_1} \left[ c_2 \frac{n}{T_e^{1/2}} (1-y) + T_e n_0 \langle \sigma v \rangle_{ion} \right] \quad (41)$$

and

$$P_i = \frac{n}{c_1} \left[ -c_2 \frac{n}{T_e^{1/2}} (1-y) + T_i n_0 (\langle \sigma v \rangle_{ion} + \langle \sigma v \rangle_{cx}) \right] \quad (42)$$
This determines the equilibrium values of state vector 
\[ \mathbf{x} = (n, T_e, T_i) \] as a function of the control vector 
\[ \mathbf{u} = (\Phi, n_0, P, P_i). \]

C. Equilibria in Absence of ICRH and Their Comparison with Experiments

In this section, the equilibrium solutions of point model equations with neoclassical nonresonant electrons in the absence of ion cyclotron resonance heating (ICRH) will be discussed. In this case, 
\[ P_i = 0 \text{ or } R = 1. \] Experimentally \[ e\Phi/T_e \lesssim -0.6 \] and \[ n_0 \lesssim 0.01 \text{ EBT units}. \]

In Figure 7 we show the behavior of the equilibrium surface log 
\[ T_e \] as a function of control parameters log \[ P_\mu \] and \[ \Phi = e\Phi/T_e \] for fixed background neutral density \( n_0 = 0.01 \). Arrows on equilibrium trajectories indicate the direction of \[ T_i/T_e \] increasing, and the broken curves indicate the unstable sheet. We will also use this convention in later chapters. The equilibrium trajectories for state variables \( (n, T_e, T_i) \) versus the control parameter \( P_\mu \) at fixed \( \Phi = -0.6 \) for various filling densities \( n_0 \) are shown in Figure 8. Note that there is only one fold in the equilibrium trajectories, as in the elementary cusp catastrophe.\(^{28,29}\) Now from Figure 3(b) showing experimental profiles of electron line density \( n_e \) for different values of \( P_\mu \) as a function of filling pressure \( p_0 \), one can see that for increasing \( n_0 \) (related to \( p_0 \) by ideal gas law), \( M \) to \( T \) and \( T \) to \( M \) transitions occur at increasingly higher values of \( P_\mu \). Also, experimentally it is observed that for the \( M \), \( T \) and \( C \) modes of operations, electron temperature \( T_e \) in the \( M \) mode is higher than in \( T \) mode which in turn is higher than \( T_e \) in \( C \) mode (i.e.
Experimentally, the $T \rightarrow C$ transition is quite smooth while the $M \rightarrow T$ transition is abrupt in $n_e l$, and temperatures $T_i$, $T_e$ (see Chapter II).

We can thus identify the top sheet in Figure 7 [$T_i / T_e \rightarrow 0$ as $\mu \rightarrow \infty$] with the experimental $M$ mode and the bottom sheet with $T$ mode which smoothly merges into $C$ mode [$T_i / T_e \rightarrow 1$ as $\mu \rightarrow 0$]. Thus in Figure 8 top sheet in $\log T_e$ vs. $\log P_\mu$, $\log T_i$ vs. $P_\mu$ and bottom sheet in $\log n$ vs. $\log P_\mu$ curves [$T_i / T_e \rightarrow 0$ as $\mu \rightarrow \infty$] can be identified with the experimental $M$ mode; and the bottom sheet in $\log T_e$ vs. $P_\mu$, $\log T_i$ vs. $P_\mu$ and top sheet in $\log n$ vs. $\log P_\mu$ [$T_i / T_e \rightarrow 1$ as $\mu \rightarrow 0$] curves with experimental $T$ mode smoothly merging into $C$ mode.

We will verify later that the time evolution of the point model equilibria follow the delay convention (with hysteresis effects etc.) and that the middle sheet is unstable. This behavior is also seen experimentally and further justifies the sheet identification modes with those of the experiments.

We can linearize the point model equations about the equilibria assuming $e^{\gamma t}$ time dependence. In the case of gradient dynamic systems it can be shown that the growth rates ($\gamma$'s) in the linearized equations of motion are always real; and along the fold curves of the behavior surface one or more of $\gamma$'s is zero. This means that a fold curve separates a stable sheet of behavior surface ($\gamma < 0$) from an unstable sheet ($\gamma > 0$). So the unstable sheet will join stable sheets. This result need not be true for non-gradient dynamic systems, since if $\gamma$ is complex, its real part can pass through zero without $\gamma$ itself passing through zero. Thus
stable and unstable sheets are not necessarily separated by fold curves for non-gradient dynamic systems. However, both time evolution runs (described in Chapter VII) and a linear stability analysis show that the stable sheets and unstable sheets are separated by fold curves for the point model equations.

From Figure 9, on increasing $\mu$ (at fixed $n_0$ and $\phi$), we find an abrupt $T \rightarrow M$ transition at $y = T_i/T_e = 0.75$ with $n$ decreasing, $T_e$ increasing in the transition. For these values of the control parameters, $T_i$ has only a small increase in the transition (compare with Figure 3(b)). On decreasing $\mu$ and using delay convention, hysteresis is observed with the transition from the $M$ mode now occurring at a much lower value of $\mu$ with $T_i/T_e < 0.03$. These transitions and hysteresis have been observed experimentally. As the filling density $n_0$ increases, both $T \rightarrow M$ and $M \rightarrow T$ transitions move to larger $\mu$, in agreement with experiment with $n_0 \propto \mu^{0.33}$ at $T \rightarrow M$ transitions and $n_0 \propto \mu^{0.22}$ for $M \rightarrow T$ transitions from our calculations (from Figure 3(a) we see that experimentally these transitions scale as $p_0 \propto \mu^{0.5}$). It should be noted in passing that we can also interpret the jump from upper sheet (of log $T_e$ vs. log $\mu$ curve, Figure 9(b)) to the bottom sheet as an $M \rightarrow C$ transition rather than an $M \rightarrow T$ transition since after this jump $T_i, T_e \ll 1$ which are the typical experimental $C$ mode temperatures. Figures 7 and 8 show that in $C$ mode $T_e \ll 1$. This means that in $C$ mode, $\phi \ll 1$ (since $e\phi/T_e = \text{constant}$). This is also in good agreement with experiment (see Chapter II and ref. 27). Thus the qualitative agreement with experiments is quite good, even for such a primitive choice of
The results obtained from a detailed examinations of these equilibria are summarized in Table 3 and 4. In Table 3, the starred quantities indicate the values of these quantities at a particular transition. In Table 3, consider the column under the heading $e\phi/T_e$. $e\phi/T_e \downarrow$ means that we will examine the behavior of quantities as $e\phi/T_e$ decreases with $n_0$ fixed. In the column under $e\phi/T_e$, we see that $P_\mu^*$ for $T \to M$ transition. This means that as $e\phi/T_e$ is decreased, the $T \to M$ transition occurs at increasingly higher values of the microwave power $P_\mu$. The Power Law column shows the relationship between $n_0$ and $P_\mu$ at that particular transition.

From Table 3 we see that as $e\phi/T_e$ decreases, the $T \to M$ and $M \to T$ transitions occur at increasingly higher values of $P_\mu$, $T_e$ and $T_i$ at fixed filling pressure $n_0$. But the $T \to M$ transition occurs at lower values of $n$, and the $M \to T$ transition at high values of $n$ as $e\phi/T_e$ is reduced keeping $n_0$ fixed. Both the $T \to M$ and the $M \to T$ transitions occur at increasingly higher values of $P_\mu$, $T_e$, $T_i$ and $n$ as the control parameter $n_0$ is increased keeping $e\phi/T_e$ fixed. Again this behavior agrees with experiment. Values of $n_0$ and $P_\mu$ at which $T \to M$ transition occurs are related by the power law $n_0 \propto P_\mu^{0.33}$. Power law for the $M \to T$ transition is $n_0 \propto P_\mu^{0.22}$. Experimentally, $n_0 \propto P_\mu^{0.5}$ for both transitions (see Figure 3(a)). These results have obvious implications.

To gain high density and temperatures in quiescent T mode, the filling pressure and microwave power should be raised. In the T
mode, the effect of lowered $e\phi/T_e$ is mixed; it results in higher temperatures but lower density. If the plasma is already in $M$ mode, a jump to $T$ mode with high temperatures and density can be made if the filling pressure and microwave power are sufficiently high or $e\phi/T_e$ is sufficiently low.

Table 4 summarizes the discontinuous jumps in the state variables during the two transitions (for small variations in $P_{\nu}$) at fixed values of the control parameters $n_0$ and $\Phi = e\phi/T_e$. It can be seen from this table that in the $M \rightarrow T$ transition $n$ increases while $T_e$ and $T_i$ decrease. While in the $T \rightarrow M$ transition, $n$ will decrease while $T_e$ will increase. However $T_i$ may either increase or decrease depending on the particular choice of the parameters. This occurs because the equilibrium profile for $T_i$ is convoluted (Figure 8 and 9c). Thus it is possible for the ion temperature $T_i$ to decrease in both ($T \rightarrow M$ and $M \rightarrow T$) transitions. This does not happen with the number density $n$ or electron temperature $T_e$.

Tabulations like Table 3 and 4 will be used to summarize results in future chapters and their interpretation will be similar to what we have done here.

D. Equilibria with ICRH

Very recently ion cyclotron resonance heating has been introduced into the EBT experiment at ORNL, but the present level of ICRH power is quite low. Plans are underway to raise ICRH power density to a level comparable to ECH power density. However, as the ICRH power
density becomes comparable to ECH power density, it is uncertain as to what will be the effect on the ambipolar electric field-will it be pointing radially in or radially out, etc. As mentioned earlier, this has important implications for resonant/nonresonant particle transport.

In this chapter, we shall assume that \( \bar{\phi} = e\phi/T_e \) remains negative throughout the plasma and consider the equilibria for the point model equations (28)-(30) in the presence of ICRH \( (P_i \neq 0) \). Equilibria for the case of ambipolar electric field being positive \( (\bar{\phi} = e\phi/T_e > 0) \) will be considered in the next chapter. The point model cannot handle the change in the sign of ambipolar field inside the plasma.

Now the parameter \( R = (P_e - P_i)/(P_e + P_i) \) is constrained by \(-1 \leq R \leq 1\). When \( R < -1/3 \), ion temperature \( T_i > T_e \) and the rate of energy transfer from electrons to ions becomes negative, i.e. \( P_{ei} < 0 \) in eqs. (29) and (30). Thermodynamically, we have a physical system consisting of two gases—an electron gas and an ion gas. According to point model, the temperature of each gas is uniform throughout its bulk. These gases exchange energy via collisions. When the temperature of the ions is higher than that of the electrons, ions will give energy to electrons, and hence \( P_{ei} < 0 \).

Figure 10 a,b, show typical equilibrium trajectories of the state variables \( n, T_e \) and \( T_i \) as functions of control parameters \( P_e \) and \( P_i \) for fixed values of control parameters \( n_0, \bar{\phi} \) and \( R \) for the case \( T_i < T_e \). [For these curves, \( R = 0.5 \) or \( P_e = 3P_i \).] Figure 11 shows the corresponding equilibrium trajectories for fixed \( n_0, \bar{\phi} \) and \( R \) for the case \( T_i > T_e \). [Here \( R = -0.5 \) or \( P_e = 1/3 P_i \).] We immediately note that
the equilibrium surface has again only one fold. The equilibrium curves
for state variables $n$ vs. $P$ and $n$ vs. $P_i$ are similar whether $T_i > T_e$
of $T_i < T_e$. The same is true for state variable $T_e$. Note that the $T_i$
trajectory is no longer convoluted. However, a major difference is that
the direction of the curves indicating the direction of $T_i/T_e$ increasing
is different in two cases. To see this, let us compare log $T_e$ vs. log $P_i$
curves in Figures 10 a ($T_i > T_e$) and 11a ($T_i < T_e$). In the former,
y = $T_i/T_e \approx 0$ at the beginning of top sheet, and starting from there as
one proceeds along the curve, $y = T_i/T_e$ increases from 0 to 0.05 at the
first fold edge. Here the unstable sheet begins and as one proceeds fur-
ther towards the next fold edge $y = T_i/T_e$ increases from 0.05 to $\approx 1$.
Here the bottom sheet begins and as one advances further along the bottom
sheet $y$ increases and approaches 1 as $P_i \rightarrow 0$. Arrows indicate the direc-
tion of $y = T_i/T_e$ increasing. Now if we examine log $T_e$ vs. log $P_i$
curve in Figure 11a, we see that $y = T_i/T_e$ increases starting from the bottom
sheet as one proceeds towards top sheet. Thus the direction of $y = T_i/T_e$
increasing for the case $T_i < T_e$ is opposite to that for the case $T_i > T_e$.

In the previous section we used experimental observations to
identify the sheets of equilibrium surface with experimental modes of
operation. However, for strong ICRH there are no experimental results
to guide us for this propose. However, there is one signature which
could distinguish the transition from the upper stable sheet to lower
stable sheet from the reverse transition. This signature is the power
law relation between $n_0$ and $P$ at transitions. Figure 12 shows three
equilibrium trajectories at fixed control parameters $n_0$ and $\phi$. 
Figure 12(a) is for the case \( P_i = 0 \) (no ICRH). In this case, we have identified with experiment that the upper sheet is the M mode while the lower sheet is the T mode smoothly merging into C mode.

Figure 12(b) and 12(c) are for the case \( P_i \neq 0 \) for which there are no experimental results. For Figure 12(b), \( T_i < T_e \), and for Figure 12(c), \( T_i > T_e \). In all three cases, we find that for transition from upper to lower sheet, \( n_0 \alpha P_{\mu}^{0.22} \). While for reverse transition, \( n_0 \alpha P_{\mu}^{0.33} \). But from previous sections, the signature of \( M \rightarrow T \) transition is \( n_0 \alpha P_{\mu}^{0.22} \). While the signature of \( T \rightarrow M \) transition is \( n_0 \alpha P_{\mu}^{0.33} \).

Thus, we tentatively identify the upper sheets in \( \log T_e/\log T_i \) vs. \( \log P_{\mu}/\log P_i \) curves in Figures 10 and 11 as the M mode and bottom sheet as T mode smoothly merging into C mode.

Similarly the lower sheets in \( \log n/\log P_{\mu}/\log P_i \) curves in Figure 10 and 11 can be tentatively identified as M mode, and upper sheet as T mode smoothly merging into C mode. However, note that in the M mode \( T_i/T_e \ll 1 \) when \( T_i < T_e \); while \( T_i/T_e > 1 \) in the M mode when \( T_i > T_e \). Again the equilibrium trajectory for the ion temperature \( T_i \) is no longer convoluted as it was for the case \( P_i = 0 \). Table 5, which is similar to Table 3, summarizes the changes in state variables and control parameters \( P_{\mu} \) and \( P_i \) at the transitions as one of the control parameters is changed holding others fixed. Let us illustrate the use of this table. The entry under \( R \uparrow \) for \( T_i > T_e \) against \( P_i^* \) for \( M \rightarrow T \) transition is \( \downarrow \). This means that as the parameter \( R \) is increased for the case \( T_i > T_e \) at fixed \( n_0 \) and \( e\phi/T_e \), \( M \rightarrow T \) transition occurs at successively higher values of \( P_i^* \).
From Table 6 we see that in the $M \rightarrow T$ transition, discontinuous changes in the state variable $n$ are positive (i.e., $n$ increases after the transition), while the changes in state variables $T_e$ and $T_i$ are negative (i.e., they decrease following the transition). These results are reversed in the $T \rightarrow M$ transition. Table 7 shows the projections of the extremities of the equilibrium trajectories on the state variable axes $n$, $T_e$ and $T_i$ and control axes $P_\rho$ and $P_\phi$ as the parameter $R$ approaches the value $-1/3$ from either direction for fixed control parameters $n_0$ and $\Phi$. The results in this table clearly indicate that the extent of the equilibrium trajectories shrink in size as $R$ approaches $-1/3$ from below and from above. Now we have already seen that the direction of the parameter $y = T_i/T_e$ increasing on the equilibrium trajectories when $R > -1/3$ is opposite to its direction when $R < -1/3$. See Figure 13. From these two observations, we tentatively conclude that the equilibrium trajectory will shrink to a point as $R \rightarrow (-1/3)^+$. Thus two unexpected features of equilibria have emerged, namely, reversal of direction of $T_i/T_e$ increasing on the equilibrium trajectories and the trajectories collapse to a point when $y \equiv T_i/T_e \rightarrow 1^+$. Possible implications for future experiments are: If the experimentalists find a negative ambipolar electric field throughout the plasma, the ICRH power density should be at least twice as large as ECH power density in order for $T_i > T_e$. There will still be three modes - $M$, $T$ and $C$. $M \rightarrow T$ and $T \rightarrow M$ transitions will be abrupt in $n$, $T_e$ and $T_i$, while the $T \rightarrow C$ and $C \rightarrow T$ transitions will be smooth (in $n$, $T_e$ and $T_i$). When $T_i < T_e$, starting from low values of $y = T_i/T_e$
as $y = T_i/T_e$ is increased, one will see $M \rightarrow T$ transition. However, when $T_i > T_e$, starting from low value of $y(T_i/T_e = 1)$ as $y$ is increased, one will observe first the smooth and continuous transitions $C \rightarrow T$ and then an abrupt $T \rightarrow M$ transition. On the other hand for $T_i < T_e$, as $y$ is reduced from high to low values, one will see a $T \rightarrow M$ transition; while for $T_i > T_e$, as $y$ is reduced, one will see a $M \rightarrow C$ transition (see Figures 10 and 11). As the ratio of $P_\mu$ to $P_i$ approaches close to 2 ($n_0$ and $\phi$ fixed) the distinction between $M$, $T$ and $C$ modes will disappear and the plasma will achieve a unique, singular equilibrium where the equilibrium trajectory shrinks to a point. This equilibrium may have mathematically interesting properties, however, it may not be desirable physically because it is singular. It could easily be an indication of the inapplicability of the point model equations to handling this situation. Table 5 shows that the parameter $R$ has mixed effect, e.g., for $T_i > T_e$, as $R = (P_\mu - P_i)/(P_\mu + P_i)$ increases, temperatures will rise but the electron/ion density will fall in $T$ mode. Higher filling pressures and higher $e\phi/T_e$ lead to higher temperatures and density in $T$ mode.
VI. EQUILIBRIA FOR POINT MODEL WITH IONS

In this chapter, we will examine the equilibria for the point model for the case in which the ambipolar electric field is observed to be positive due to the presence of sufficiently strong ICRH. Now if \( \phi > 0 \) then the ions become nonresonant while some of the electrons become resonant. Hence, to avoid having to stipulate the resonant electron diffusion coefficient, we will use the ambipolar potential as a control parameter as we did in the previous chapter. We will discuss the point model with neoclassical nonresonant ions using \( e\phi/T_i \) as a parameter. In section (B), equilibria in absence of ICRH will be discussed, even though ions may be resonant when ICRH is absent. In section (C), equilibria in presence of ICRH will be obtained and discussed.

A. Point Model Equations with Neoclassical Nonresonant Ions

The point model equations (7), (10) and (11) with neoclassical nonresonant ions are

\[
\frac{dn}{dt} = n \left[ n_0 \left< \sigma v \right>_{ion} - a_1' \frac{n}{T_i^{3/2}} \frac{T_i}{\left(1 + a_2' n^2 / T_i^2\right)} - a_3 n \frac{T_e}{T_i} \right] \tag{43}
\]

\[
\frac{dT_e}{dt} = c_1\frac{P_e}{n} - c_2 \frac{n}{T_e^{1/2}} \left(1 - \frac{T_i}{T_e}\right) - T_e n_0 \left< \sigma v \right>_{ion} \tag{44}
\]

\[
\frac{dT_i}{dt} = c_1\frac{P_i}{n} + c_2 \frac{n}{T_e^{1/2}} \left(1 - \frac{T_i}{T_e}\right) - T_i n_0 \left< \sigma v \right>_{ion} + \left< \sigma v \right>_{\nu_x} \tag{45}
\]
where \( a_1' \) and \( a_2' \) are functions of parameter \( \Phi = \phi / T_1 \), given by

\[
\begin{align*}
    a_1' &= (1 - \Phi) / (1 - c_4 \Phi)^2 \\
    a_2' &= c_5 / (1 - c_4 \Phi)^2
\end{align*}
\]

\( a_3', c_1', c_2', c_3', c_4', \) and \( c_5' \) are constants.

To obtain equilibrium trajectories for eqs (43)-(45), we define new variables

\[
\begin{align*}
    \chi' &= n / T_i^{5/2} > 0 \\
    \nu' &= T_i / T_e > 0
\end{align*}
\]

The equilibrium version of eq. (43) is

\[
\eta_0 <\nu>/1on - a_1' \chi' T_i \frac{T_i^3}{(1 + a_2' \chi'^2)} - a_3' \chi' \nu'^{1/2} T_i^2 = 0
\]

while the equilibrium solutions of eqs. (44) and (45) satisfy

\[
T_i = \frac{k}{(\chi' \nu'^{3/2})}
\]

where \( k \) is given by eq. (35). Substituting this expression for \( T_i \) into eq. (47), we obtain a cubic equation in \( \chi' \):

\[
B_1' \chi'^3 + B_2' \chi'^2 + B_2' \chi' + B_4' = 0
\]

with coefficients \( B_1', \ldots, B_4' \) given by
Again, we find a unique real positive root $x'$ of eq. (49), so that the state vector $\mathbf{x} = (n, T_e, T_i)$ is given by

$$T_i = k / (x' y^{-3/2})$$ (50)$$T_e = T_i / y$$ (51)$$\eta = x' T_i^{5/2}$$ (52)

while the control parameters, $P_\mu$ and $P_i$, are given by eqs. (41) and (42) respectively. The restrictions on the parameter $R$ and resulting restrictions on $y$ are as before (see section (B), chapter V).

B. Equilibria without ICRH

These equilibria (i.e. equilibria for the point model with neoclassical nonresonant ions in the absence of ICRH) are obtained by solving the cubic equation (49) with parameter $R = 1$. These equilibria are primarily of theoretical interest. Nevertheless, they show some interesting features.

For $0 < \Phi' \equiv e\phi / T_i < 1$, the equilibrium surface has a single fold, the upper and lower sheets are stable, while the middle sheet is unstable. Figure 14 a, b, c show typical equilibrium trajectories for the state variables $(n, T_e, T_i)$ as functions of the control parameter.
at fixed values of $\phi' = 0.6$ and $n_0 = 0.01$. When the control parameter $\phi'$ is increased beyond unity, a new fold begins to emerge in the equilibrium surface. Figure 15 a, b, c show equilibrium trajectories for $\phi' = 1.1$ and $n_0 = 0.01$. On further increasing $\phi'$, this newly emerged fold undergoes strong growth (compare Figs. 15 a,b,c and 16 a,b,c). As before, arrows on these figures indicate direction of $y = T_i/T_e$ increasing. Figures 17 a,b,c,d show the equilibrium trajectories $T_e$ vs. $P_\mu$ in a sequence of $\phi'$ increasing. This figure clearly shows that the new fold is born from the upper state sheet (in $T_e$ vs. $P_\mu$ trajectory), i.e. when $\phi' = 0.9$, there is only one fold; when $\phi' = 1.0$, still there is one fold, but upper sheet now has an indentation; when $\phi' = 1.1$, the indentation in the upper sheet has grown into a fold; when $\phi' = 1.2$, the newly born fold has become fully grown. The values of $y = T_i/T_e$ at the fold edges of the $T_e$ vs. $P_\mu$ trajectory also support this assertion: when $\phi' = 0.9$, $y = T_i/T_e = 0.64$ at the fold edge of the upper sheet (Fig. 17(a)); when $\phi' = 1.0$, $T_i/T_e = 0.7$ (Fig. 17(b)); and when $\phi' = 1.1$, $T_i/T_e = 0.79$ (Fig. 17(c)), but at the fold edge of bottom sheet $T_i/T_e = 1$, for all $\phi'$s.

Another way of looking at this phenomena is: For high $\phi' > 1.2$, the equilibrium surface has two folds; stable and unstable sheets occurring consecutively. As $\phi'$ is decreased, the upper and middle stable sheets merge together and eventually equilibrium trajectory has only one fold. We will call, for $\phi' > 1.2$ the top stable sheet (in the $T_e$ vs. $P_\mu$ trajectory) the 1s mode, the middle stable sheet 2s and the lower stable
sheet 3s. When \( \phi' \leq 1.0 \), 1s and 2s merge together and become degenerate. By considering the range of electron temperatures in the 1s, 2s and 3s modes, we can tentatively identify them as the M, T and C modes respectively. However, we should bear in mind that we do not have any experimental basis for this (unlike the case for non-resonant electrons, discussed in chapter IV). We have employed this identification in labelling the sheets in figures and also in the tables. When the equilibrium surface has one fold, plasma can made a \( T \rightarrow C \) and a \( C \rightarrow T \) transition, depending upon the values of control parameters. This is shown in Figure 18. However, when the equilibrium surface has two folds, the plasma can undergo \( M \rightarrow T, T \rightarrow C, C \rightarrow T \) and \( T \rightarrow M \) transitions as shown in Figure 19 for \( \phi' = 1.5 \). When \( \phi' \approx 1.1 \), however, the plasma now undergoes a \( C \rightarrow M \) transition rather than a \( C \rightarrow T \) transition since for this value of \( \phi' \) the \( T \) mode is very limited in extent, see Figure 20. Note that an \( M \rightarrow C \) transition is not permitted. For \( \phi' < 1 \), the equilibrium trajectory for \( T_1 \) is convoluted and \( T_1 \) decreases during both a \( T \rightarrow C \) and \( C \rightarrow T \) transitions (Figure 14(c)). This does not occur for the state variables \( n \) or \( T_e \) (Figures 14(a),(b)). For \( \phi' > 1 \), equilibrium trajectory for \( T_1 \) is doubly convoluted (Figure 15(c)). Such single or double convolutions in equilibrium trajectories can lead to interesting competitions between the respective point attractors in the time evolution of the state variables towards equilibrium. This will be investigated in the next chapter. Table 8 summarizes the changes in the state variables \( (n, T_e, T_1) \) and the control parameter \( P_{\phi} \) at the transition points as \( \phi' \) is varied keeping \( n_0 \) fixed and vice versa.
For example, an up arrow against $T_i^*$ for $M \rightarrow T$ transition in the column under $e\phi/T_i$ for $1.1 < e\phi/T_i < 2$ means that $M \rightarrow T$ transition occurs at increasingly higher values of $T_i$ as $e\phi/T_i$ is increased keeping $n_0$ fixed for $1.1 < e\phi/T_i < 2$. In this table, blanks appear in the columns for $M \rightarrow T$ and $T \rightarrow M$ transitions when $1 < T$. This is so because, for $1 < T$, $M$ and $T$ modes are degenerate. Table 9 shows the discontinuous jumps in state variables during various possible transitions. For example, a + sign against $\Delta n$ in the column under $T \rightarrow C$ means state variable $n$ increases after a $T \rightarrow C$ transition.

Before we close this section, it should be emphasized that the results presented in this section are principally of theoretical interest only.

C. Equilibria with ICRH

In this section, we investigate the other physically possible equilibria in the presence of ICRH, i.e. equilibria for which the ambipolar electrostatic potential is positive. As regards particle confinement, for ambipolar electric field $E < 0$, the basic confinement time is set by the electrons as we have seen in previous chapters; however, it is believed that if $E > 0$ could be maintained in stable steady state, the basic confinement time would be set by the ion scattering time.$^5$

The equilibrium surfaces, in this case, exhibit multiple folds when the parameter $R$, $R = (P_e - P_i)/(P_e + P_i)$, is close to unity. These folds will degenerate into a single fold as $R$ decreases. In Table 10, we show the range of $y = T_i/T_e$ for a given $R$ and the corresponding ratio of ICRH/ECH.
It is convenient to divide the discussion of this section into two parts: (i) equilibria with multiple folds and, (ii) those equilibria with only a single fold.

(i) Equilibria with multiple folds

Figure 21 shows equilibrium trajectories for the electron temperature $T_e$ as a function of the control parameter $P$ at fixed values of filling number density $n_0 = 0.01$ and potential $\xi' = 0.1$ for three different values of the parameter $R$. Note that for $\xi' = 0.1$, a low ambipolar potential, the equilibrium trajectories exhibit two folds for $R = 0.99$. The middle stable sheet starts shrinking as $R$ decreases and disappears for $R < 0.8$. The top, middle and bottom sheets are stable and will be denoted 1s, 2s and 3s respectively. The unstable sheets are shown dashed in Figure 21 and starting from the top will be labelled 1U and 2U respectively. Now $y = T_1/T_e \sim constant = 1$ at the fold edge joining 3s and 2U for all $R$ in Figure 21. At the fold edge joining 2s and 2U, $y = T_1/T_e$ has values 0.14, 0.36, 0.44, for $R = 0.99, 0.9, 0.8$ respectively. $y$ values at the fold edges joining 1s to 1U and 1U to 2s are quite low (i.e. less than or of the order of 0.01). We interpret these values as follows: at low values of $\xi' = e\phi/T_i \sim 0.1$ and for $R < 0.8$, the equilibrium surface has only one fold whose upper and lower sheets are stable with the middle sheet unstable. As $R$ increases beyond 0.8, the degeneracy in 1s, 2s is broken and the equilibrium surface now develops two folds with stable modes 1s, 2s, 3s and unstable modes 1U, 2U. The ordering of the electron temperatures in the stable modes 1s, 2s, 3s suggests that we can tentatively
identify them as M, T, C modes respectively. This means that when
the equilibrium trajectory has only one fold, the M mode can be con-
sidered to be smoothly merging into the T mode.

Figure 22 shows the equilibrium trajectories $T_e$ vs. $P_1$ at fixed
$\bar{\phi}' = 0.6$ and $n_0 = 0.01$ for two different values of parameter $R = 0.9$
and 0.99. At $R = 0.9$, equilibrium trajectory has one fold with an
indentation in upper stable sheet. As $R$ increases from 0.9 to 0.99, a new
fold is born from the upper stable sheet. However, it is born later than
when the ambipolar potential $\bar{\phi}'$ was lower (i.e. when $\bar{\phi}' = 0.1$, the new
fold is born at $R = 0.8$; while when $\bar{\phi}' = 0.6$, it is born at $R = 0.99$).
The $y = T_1/T_e$ values at the fold edges are indicated in Figure 22. Again,
we tentatively identify the upper, middle and lower stable sheets as M,
T, C modes respectively. The merging of M and T modes is typical
for $0 < \bar{\phi}' \leq 1.0$.

The above results suggest that; if, in the presence of ICRH, the
ambipolar electrostatic potential $\phi$ is found to be positive; the experi-
mentalist may see $M \rightarrow T$, $T \rightarrow C$, $C \rightarrow M$, $T \rightarrow M$ transitions in
$(n, T_e, T_i)$ when $0 < e\phi/T_i \leq 1.0$ and $R \geq 0.8$. The M mode may be
physically less probable since in this mode the ratio $T_e/T_i$ is very large.
As the parameter $R$ decreases (i.e. $R \leq 0.8$) M and T modes merge
smoothly together leaving only the $T \rightarrow C$ and $C \rightarrow T$ transitions in
the state variables.

When $\bar{\phi}' = 1.1$ and $R < 0.9$, the equilibrium trajectory has only
one fold. On increasing $R$ to $R = 0.9$ at fixed $\bar{\phi}'$, the equilibrium tra-
jectory has two folds (Figure 23), while for $R = 0.99$ the equilibrium
trajectory has three folds. These two new folds are born from the upper stable sheet. The \( y = \frac{T_i}{T_e} \) values at the fold edges are shown in Figure 23. As usual, top and bottom sheets are stable; stable and unstable sheets occur alternately. The system can make six different discontinuous transitions when the equilibrium trajectory has three folds. It should be noted that the top stable sheet may be physically inaccessible because \( T_e \gg T_i \) on this sheet.

Figure 24 shows equilibrium trajectories when \( \Phi' \) is increased to \( \Phi' = 1.5 \). As R in increased from 0.9 to 0.99, a stable T mode is born from upper stable sheet. However, the independent M mode may now be physically accessible since \( T_e/T_i \sim 1 \).

Thus the region of small ratio of power densities ICRH to ECH (i.e. when \( P_i \) is about 0.5 to 5.0% of \( P_\mu \)) is full of fold structure and the salient features of equilibria in this region are summarized in Table 11.

(ii) Equilibria with single fold

When the control parameter \( \Phi' \) is not small (i.e. \( \Phi' > 0.1 \)) and the parameter R is not close to unity (i.e. \( R \leq 0.9 \)) the equilibrium surfaces exhibit only one fold. Typical equilibrium trajectories for state variables \( (n, T_e, T_i) \) as functions of control parameters \( P_\mu \) and \( P_i \) at fixed values of \( n_0 = 0.01 \) and \( \Phi' = 0.6 \) are shown in Figures 25 a, b \( (T_e > T_i) \) and 26 a, b \( (T_e < T_i) \). Table 12 is similar to Table 5, except that it is for the point model with ions with ICRH. Table 13 shows the discontinuous changes in the state variables during \( T \rightarrow C \) and \( C \rightarrow T \) transitions.
Figure 27 shows the typical equilibrium profiles for the electron temperature $T_e$ as a function of control parameter $P$ at fixed $n_0 = 0.01$ and $\phi' = 0.6$ when the parameter $R$ is (i) $R < -1/3$ and (ii) $R > -1/3$. As always, the arrows in the trajectory indicate the direction of $y = T_i/T_e$ increasing. This figure shows that the equilibrium surface reverses direction (indicated by $y = T_i/T_e$ increasing) when the parameter $R$ crosses the value of $-1/3$. Table 14 (which is similar to Table 7) indicates that the equilibrium trajectory will collapse to a point as the electron and ion temperatures equalize from either direction. Thus the reversal of direction and collapse to a point of equilibrium trajectories when electron and ion temperatures are equal are two striking features of the point model equilibria with ions with ICRH. These results are very similar to the results obtained when electrons are nonresonant and ICRH is present.
VII. CONVENTION, STABILITY AND INITIAL VALUE PROBLEM
FOR POINT MODEL EQUILIBRIA

So far, in this dissertation, we have investigated the properties and the topological structure of the equilibrium surfaces as the experimental control parameters are smoothly changed. Even though the point model for EBT does not belong to elementary catastrophe theory (since the equations are not of gradient dynamic form) we find that the surfaces are just more complicated versions of the canonical cusp catastrophe. In discussing the various possible transitions, it was tacitly assumed that the transitions followed the delay convention, rather than the Maxwell convention. We now investigate the time evolution of the state variables \((n, T_e, T_i)\) to verify that they do indeed follow the delay convention. Moreover, because of the more complicated fold structure in the equilibrium surfaces we expect the basins of the point attractors (i.e., the region of initial values of state variables, and control parameters for which a particular point of an equilibrium surface develops into a single attractor) in the respective \(M, T, C\) modes to possibly show interesting structure, i.e., the folds in the equilibrium surfaces can, depending on initial conditions, give rise to competition between the various (stable) point attractors on the \(M, T, C\) sheets so that different initial conditions can result in a different final equilibrium state. This has obvious significance to the experimentalist.
who, in EBT and present thinking, would like to work in the $T$ mode because of low fluctuations and quiet stable steady state conditions.

Thus, we now solve the time dependent point model equations (7), (10), (11) for various initial conditions.

A. Time Evolution for Equilibria with One Fold

Figures 28(a), (b) and (c) are typical time evolution plots for the EBT point model equilibria with a single fold. To understand what these figures represent, let us discuss Figure 28(a). The curve with a single fold is the equilibrium profile for electron temperature $T_e$ as a function of the microwave power $P_\mu$ at fixed values of filling pressure $n_0 = 0.01$, ambipolar potential $e\phi / T_e = -0.6$ with $R = (P_\mu - P_i)/(P_\mu + P_i) = -0.5$ (i.e. $T_i > T_e$). The large double arrows on this equilibrium trajectory indicate the direction of $y = T_i / T_e$ increasing. The solid vertical $\uparrow$, $\downarrow$ symbols indicate the initial and final equilibrium values (tip of arrow) of electron temperature $T_e$ in time. For example, the point A shows the value of $T_e$ at time $t = 0$ for fixed values of $P_\mu = 1.0$, $n_0 = 0.01$, $e\phi / T_e = -0.6$ and $R = -0.5$. Now we solve the point model time dependent equations with neoclassical nonresonant electrons (i.e. eqs. (28)-(31)) with above values of $n_0$, $e\phi / T_e$, $P_\mu$ and $P_i$. $P_i$ is determined from eq. (36) for the given values of $R$ and $P_\mu$. The initial value $T_e(0)$ is represented by point A. Initial values $n(0)$ and $T_i(0)$ are also chosen to be close to corresponding equilibrium values (i.e. close to M mode). The solution of these time evolution equations then shows that the system trajectory reaches point B asymptotically in time, i.e. point B represents the final
steady state value of $T_e$. $n$ and $T_i$ also reach their corresponding steady state on $M$ mode. This shows that point B on the equilibrium trajectory is the point attractor for these initial conditions. Now we take the value of $T_e$ corresponding to point B as initial value (and corresponding values of $n$ and $T_i$ also as initial values of $n$ and $T_i$). But now we change the value of microwave power $P_{\mu}$ to the one that corresponds to point C (and accordingly change the value of $P_i$ since we are holding $R$ fixed) and solve the initial value problem once more with these new initial conditions. Thus for these initial conditions the upper sheet (the $M$ mode) is the point attractor set. So long as the upper sheet is available close by to the set of initial conditions, the system trajectories are found to be attracted to it. As soon as the initial point (i.e. point E) is past the fold edge of the upper sheet, the upper sheet ceases to be an attractor and the trajectory is attracted to the bottom sheet which is now the attractor. This jump from point E to point F indicates that the system obeys the delay convention. Note that very small changes in control parameters $P_{\mu}$ and $P_i$ have caused this jump in state variables ($n$, $T_e$, $T_i$). It is found that $n$, $T_e$ and $T_i$ all jump at the same point (i.e. at the same values of $P_{\mu}$ and $P_i$).

Now we are on the lower sheet (which is the $T$ mode). We increase the control parameters $P_{\mu}$ and $P_i$ and observe the time development of ($n$, $T_e$, $T_i$). We find that the lower sheet is the point attractor set. So long as this attractor set is available close to the initial conditions $(n(0), T_e(0), T_i(0))$, the orbits are attracted to it. However, once the initial conditions are such that they lie past the fold edge where lower
sheet ends, the lower sheet is no longer available as an attractor. We find that the orbits jump from initial point G to final equilibrium point H. This catastrophic jump for small continuous changes in control parameters $P_\mu$ and $P_\perp$ is the $T \rightarrow M$ transition. Again indicating that the system obeys the delay convention.

We also observe that if the initial conditions are such that the point lies on middle sheet, it is repelled and is attracted to either upper or lower sheet, depending on whether $M$ or $T$ is the strong attractor. Thus the middle sheet is unstable. Trajectories IJ and KL are two such typical runs.

Our conclusion, then, is that the upper and lower sheets are attractors of orbits, while the middle sheet is a repeller of orbits. The system follows the delay convention and exhibits catastrophic jumps at the fold edges as indicated by our earlier equilibrium analysis. The remaining figures in this chapter are to be interpreted and understood in the spirit of this discussion. Figures 28 (a), (b) and (c) then show that the above conclusions hold whenever the equilibrium surface has only one fold. This is irrespective of the direction of $y = T_i/T_e$ increasing on the equilibrium curves (i.e. whether $T_i > T_e$ or $T_i < T_e$) or whether ICRH is being used or not.

The hysteresis effect is very evident. It is to be noted that in the experiment hysteresis is also observed.

B. Time Evolution for Equilibria with Multiple Folds

Figures 29 a, b show typical results for the nonlinear time evolution runs when equilibrium surfaces have multiple folds which occur
only when the ambipolar electrostatic potential $\phi$ is positive (i.e. when ions are nonresonant). In Figures 29 a, b, the M, T, C modes are joined at fold edges by sheets which by linear theory can be shown to be unstable. This will be verified, of course, by nonlinear time evolution runs. Again we see that the equilibria follow the delay convention and the catastrophic jumps occur at fold edges. Figure 29 (b) shows that for $\tilde{p}' = 1.1$, the C $\rightarrow$ M transition can occur as claimed in the last chapter.

The implications of these results for the EBT experiments are clear: M, T and C modes will be nonlinearly stable. The equilibria shown as dashed curves will be unstable. Thus the toroidal core plasma will be a system exhibiting catastrophic jumps in $(n, T_e, T_i)$ as various experimental parameters $(n_0, P_M, P_i)$ are changed smoothly. The plasma will also exhibit hysteresis.

C. Basins of the Point Attractors

Now we would like to follow the nonlinear time evolution of the system trajectory when one or more of the stable variables are initially on/near different equilibrium surfaces (M, T or C modes).

Figures 30 (a) through (i) show the equilibrium curves for state variables $(n, T_e, T_i)$ as functions of the control parameter $P_M$ at fixed values of $n_0 = 0.01$, $e\tilde{p}/T_i = 0.1$ and $R = 0.9$ for nonresonant neo-classical ions with ICRH. For the particular parameters chosen for this figure, the equilibrium surface has two folds. The upper, lower and middle sheets in $T_e$ or $T_i$ vs. $P_M$ plots are attractors of orbits (i.e. the stable
equilibrium M, C and T modes respectively). Note that in the equilibrium trajectory of n vs. \( P_\mu \), the lower sheet is the stable M mode while the upper sheet is stable C mode. The dashed lines again represent repellor of orbits (the unstable sheets).

In Figure 30 (a), n(0) is chosen to lie on the stable C mode, while \( T_e(0) \) and \( T_i(0) \) are chosen to lie very close to the fold edge of the T mode. The time evolution, using these initial conditions, shows that the final steady state of the system will be the C mode. Now if \( T_e(0) \) and \( T_i(0) \) are kept fixed and n(0) is reduced to lie near the fold edge joining the T mode to the M mode, the final steady state is M mode. See Figure 30 (b). Hence a change in the location of the initial value of the state variable n causes a very significant change in the final equilibrium attractor of the orbits, i.e. by changing n(0) we can finish in M mode with high temperatures for \( T_i, T_e \) rather than in the C mode with \( T_i, T_e \) being quite cool.

In Figure 30 (c), the initial value of the state variable n is located on the repellor joining M and T modes, while \( T_e(0) \) and \( T_i(0) \) are chosen to lie on the C mode. However, the time evolution run shows that the orbits are pulled by another attractor (i.e. M mode). Since \( T_e(0) \) and \( T_i(0) \) are themselves attractors of orbits, there is thus a competition between three attractors; M, T and C modes. Thus for these initial conditions we can say that the M mode is a strong attractor compared to the T and C modes. Figures 30 (d) through (i) provide more examples of such competitions between the attractors. Table 15 summarizes the results of Figures 30. This table shows the location of the
initial values of state variables, which are the competing attractors and which one wins. In Figures 31 we show the actual time evolution of \( n(t) \), \( T_e(t) \), \( T_i(t) \) for the same initial conditions as those chosen in Figures 30 (f) and (e). The solid line corresponds to Figure 30(f), while the dashed line corresponds to Figure 30(e). Figure 31 is a vivid representation of the fact that a small change in the initial state can lead to a drastic change in the final equilibrium state.

In fact, an exhaustive study along above lines can exactly delineate the basin of each attractor and determine the relative strength of attractors when competition exists between multiple attractors. This would completely determine the dynamics of the system.

The above results have important implications for EBT experiments. The choice of the initial conditions \( (n(0), T_e(0), T_i(0)) \) as well as the values of control parameters can be very important as to which mode (M, T or C) the experiment will settle into. One of the experimental modes of operation may be stronger than the others, thus becoming the dominant equilibrium mode.
A number of physical systems show discontinuities in behavior when the external parameters are changed continuously. Such a discontinuity or jump is called a catastrophe. Elementary catastrophe theory (ECT) deals with systems whose governing equations can be expressed as
\[ \frac{dx}{dt} = f(x; \alpha; t) \]
where the state vector \( x \) consists of state variables which describe the behavior of the system, control vector \( \alpha \) consists of the external parameters and \( f \) is expressible as gradient of a scalar (i.e. \( f = \nabla V \) with \( \nabla = \partial / \partial x \)). Such systems are called gradient dynamic systems and ECT applies only to gradient dynamic systems. ECT predicts that there exists at most seven topologically distinct canonical forms of equilibria when the control vector \( \alpha \) is not more than five dimensional. Thom's classification theorem yields a complete list of possible singular behaviors in the equilibrium solutions as one varies the control vector \( \alpha \). This theorem also distinguishes between the structurally stable and unstable parts of equilibrium surfaces \( x_{10} = x_{10}(\alpha) \).

Even if one does not know the potential \( V \) accurately, ECT guarantees that the only singular behavior in \( x_{10} \) that can occur must take one of the seven forms specified by the theorem (fold, cusp, etc.). ECT does not attempt to consider the dynamical behavior of system. Nevertheless Thom introduced two different conventions to be adopted when dealing with the time evolution of systems. If the system seeks the global minimum of
potential, it is said to follow the Maxwell convention. If it seeks local minimum, it is said to obey Delay convention. The delay convention leads to hysteresis, while the Maxwell convention does not.

Experimental observations in the Elmo Bumpy Torus device for controlled thermonuclear fusion shows that the toroidal core plasma shows discontinuous changes in the electron density, electron temperature, ion temperature and also in the fluctuation levels of the electron density as the external parameters are continuously changed. We use the simplest model for toroidal core plasma in EBT – the Point Model of Hedrick et al. to apply catastrophe theory so as to understand the catastrophic behavior. In this model, the velocity space dependent effects are ignored and the ambipolar electric field E is averaged over configuration space. The effects of inhomogeneities in magnetic field are incorporated through neoclassical diffusion coefficient. By considering the ambipolar electric field E as a parameter, we can avoid having to stipulate transport coefficient for resonant particles. We easily verify that EBT point model is not a gradient dynamic system and thus ECT and Thom's powerful classification theorem are inapplicable. Nevertheless, the EBT point model equilibrium surfaces exhibit properties which are characteristic of the canonical cusp catastrophe of ECT – even the stability conditions at the folds, a result that is not obvious. We find that the results explain the experimental findings quite well in a qualitative way. Encouraged by this success, we also study the effects of ion cyclotron resonance heating (ICRH) on the equilibria since ICRH is now on line in EBT. This leads to a number of novel features of equilibria which now become rich in structure and variety.
In general, the equilibrium surfaces are found to be singly or multiply folded with alternating stable and unstable sheets. Whenever possible, attempts are made to identify the stable sheets with experimental modes of operation of EBT. The movement of folds with state variables and control parameters is determined in most common cases. By following the time evolution of the point model, we find that the system obeys the delay convention. Catastrophes in the form of discontinuous changes in the equilibrium values of state variables \((n, T_e, T_i)\) are seen at the fold edges. Changes in the state variables after the catastrophes are determined. The system exhibits hysteresis. The direction of \(T_i/T_e\) on the equilibrium surfaces are seen to reverse as the electron and ion temperatures equalize and there are strong indications that equilibrium trajectories collapse to points at these points of reversal. The choice of initial equilibria as well as values of control parameters are seen to be very important as regards into which mode of operation the plasma will eventually settle. There are some interesting results when there are competitions between multiple point attractors. The implications to experiment of the qualitative properties which follow from the catastrophe analysis of point model and its nonlinear time evolution are indicated.

It is striking that a model (of plasma in EBT) as crude and simple as the one that we have used can furnish so much information and exhibit such rich structure. It not only explains experimental findings but also gives insight into what might happen in future experiments (in the presence of ICRH). On the other hand, a sophisticated model could give more details and more accurate information. Nevertheless, such a
model is not necessary for a general overall qualitative understanding of the behavior of plasma.

The inevitable conclusion this dissertation leads us to is the following: if we encounter a physical system whose behavior is affected by some externally controllable parameters, then it may be very useful to construct a simple model of the system. Such a model can unravel the general, overall qualitative behavior of the system. Once we have gained insight into the topology of the system behavior, a sophisticated model can then be employed to give more details and attain greater accuracy. In particular it seems that this result applies to EBT.
REFERENCES


30. E. G. Harris, ORNL Theory Section Memo #76/49 (May 20, 1976).
APPENDIX A

POINT MODEL EQUATIONS ARE NOT A GRADIENT DYNAMIC SYSTEM

In this appendix we will show that the point model equations are not a gradient dynamic system.

We denote the state vector \( \vec{x} = (n, T_e, T_i) \) as a vector in cartesian coordinates and write the point model equations in the form

\[
\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}, \alpha)
\]  

(A1)

where the control vector

\[ \alpha = (n_0, e\phi/T_e, e\phi/T_i, P^0, P^i). \]

Now the point model equations with neoclassical nonresonant electrons or ions satisfy

\[
\frac{dn}{dt} = \begin{cases} 
\left[ n_0 \langle \sigma v \rangle_{\text{ion}} - \frac{a_1 n / T_e^{3/2}}{(1 + a_2 n^2 / T_e^5)} - \frac{a_3 n / T_e^{1/2}}{T_e^{3/2}} \right] & \text{(electrons)} \\
\left[ n_0 \langle \sigma v \rangle_{\text{ion}} - \frac{a_1' n / T_i^{3/2}}{(1 + a_2' n^2 / T_i^5)} - \frac{a_3 n / T_i^{1/2}}{T_i^{3/2}} \right] & \text{(ions)} 
\end{cases}
\]

(A2)

\[
\frac{dT_e}{dt} = c_1 \frac{P^0}{n} - c_2 \frac{n}{T_e^{1/2}} \left( 1 - \frac{T_i}{T_e} \right) - T_e n_0 \langle \sigma v \rangle_{\text{ion}} 
\]

(A3)

\[
\frac{dT_i}{dt} = c_1 \frac{P^i}{n} + c_2 \frac{n}{T_e^{1/2}} \left( 1 - \frac{T_i}{T_e} \right) - T_i n_0 \langle \sigma v \rangle_{\text{ion}} + \langle \sigma v \rangle_{\text{ion}} 
\]

(A4)
Vector $\vec{F} = (F_x, F_y, F_z)$ can be expressed as a gradient of a scalar if and only if curl $\vec{F}$ vanishes. Now

$$(\nabla \times \vec{F})_x = \frac{\partial F_z}{\partial T_e} - \frac{\partial F_y}{\partial T_i}$$

$$= \frac{3}{2} \frac{n}{T_e^{3/2}} (\frac{T_i}{T_e} - 1)$$

$$= 0 \quad \text{iff} \quad T_i = T_e$$

We now consider the remaining components of curl $\vec{F}$ when $T_i = T_e$.

First we consider the case of point model equations with neoclassical nonresonant electrons. In this case

$$(\nabla \times \vec{F})_y = \frac{\partial F_x}{\partial T_i} - \frac{\partial F_z}{\partial n}$$

$$= \frac{c_1}{n^2} \frac{P_i}{n}$$

$$= 0 \quad \text{iff} \quad P_i = 0$$

$$(\nabla \times \vec{F})_z = \frac{\partial F_y}{\partial n} - \frac{\partial F_x}{\partial T_e}$$

$$= -\frac{5a_1 a_2 (n^4 / T_e^{15/2})}{(1 + a_2 n^2 / T_e^5)^2} - \frac{3}{2} \frac{a_1 n^2 / T_e^{5/2}}{(1 + a_2 n^2 / T_e^5)}$$

$$- \frac{1}{2} a_3 \frac{n^2}{T_e^{3/2}} - c_1 \frac{P_n}{n}$$

$$\neq 0$$

This means that the point model equations with neoclassical nonresonant electrons are not a gradient dynamic system.
Now let us examine the point model equations with neoclassical nonresonant ions. In this case

\[
(\nabla \times F) y = \frac{\partial F_y}{\partial T_i} - \frac{\partial F_z}{\partial n} = c_1 \frac{p_i}{n^2} + \frac{3}{2} a_1' \left( \frac{n^2}{T_i^{5/2}} \right) + \frac{5a_1' a_2' (n^2/T_i^{15/2})}{(1 + q_2' n^2/T_i^5)^2}
\]

\[
\neq 0
\]

Hence the point model equations with neoclassical nonresonant ions is not a gradient dynamic system.
In this appendix, we will study the motion of charged particles in the magnetic field configuration of the Elmo Bumpy Torus and show what is meant by resonant or nonresonant particle in the context of this motion. The fields required for the confinement of charged particles are quite complicated. Therefore the exact integration of the equations of motion is usually not possible. For this reason, we will treat particle motion by means of approximation techniques; in particular we will use particle drift equations. However, even then the approximate equation itself cannot be integrated and it is still necessary to carry out a further averaging procedure; one must make use of methods based on the existence of a longitudinal invariant.

We start with the nonrelativistic equation of motion of a particle of mass $m$ and charge $e$ in an electric field $\vec{E}$ and a magnetic field $\vec{B}$. We will use c.g.s. units here.

$$m\vec{\dot{V}} = e\vec{E} + \frac{e}{c}\vec{V} \times \vec{B}$$

$(B1)$

$\vec{V}$ is the particle velocity and $C$ is speed of light.

If the particle motion is described in terms of generalized coordinates, $q_i$, then the equation of motion, in Lagrangian form, is
The Lagrangian $\mathcal{L}$ is

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$$  \hspace{1cm} (B2)

The vector potential $\mathbf{A}$ and scalar potential $\Phi$ are related to fields $\mathbf{E}$ and $\mathbf{B}$ by the expressions

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi$$

$$\mathbf{B} = c \text{curl } \mathbf{A}$$  \hspace{1cm} (B4)

The particle equations of motion can also be written in the Hamiltonian form. In that case eq. (B2), the system of three second order equations for the coordinates $q_i$, is replaced by a system of six first order equations for the three coordinates $q_i$ and the three momenta $p_i = \partial \mathcal{L} / \partial \dot{q}_i$:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$  \hspace{1cm} (B5)

The quantity $H(q_i, p_i)$ is the Hamiltonian, which is related to the Lagrangian by

$$H = -\mathcal{L} + \sum_i p_i \dot{q}_i$$  \hspace{1cm} (B6)

In our case

$$H = \frac{1}{2m} (\mathbf{p} - \frac{e}{c} \mathbf{A})^2 + e \Phi$$  \hspace{1cm} (B7)
If the field exhibits axial symmetry, the Lagrangian is independent of aximuthal angle $Q$ in a cylindrical coordinate system, and the angular momentum is conserved:

$$p_Q = \frac{\partial L}{\partial \dot{Q}} = m r^2 \dot{Q} + \frac{e}{c} r A_Q = \text{const.} \quad (B8)$$

If the fields are independent of time, eq. (Bl) allows an integral of energy

$$E = \frac{mv^2}{2} + e \phi = \text{const.} \quad (B9)$$

Since the Lorentz force $e/c \vec{V} \times \vec{B}$ is perpendicular to particle velocity, the magnetic field does no work and therefore the magnetic field does not appear in the energy integral.

In order to eliminate cyclic coordinates $q^{(c)}$, we introduce the Rouse function, defined as

$$R = \mathcal{L} - \sum q^{(c)} \dot{p}^{(c)} \quad (B10)$$

where summation is taken over cyclic coordinates only. In this case, the $q^{(c)}$ in $R$ are eliminated by the integrals $p^{(c)} = \text{constant}$. Introducing the Rouse function corresponds to taking the Hamiltonian form in the cyclic variables and keeping the Lagrangian form in the other coordinates, i.e.

$$\dot{p}^{(c)} = -\frac{\partial R}{\partial q^{(c)}} = 0$$

$$\dot{q}^{(c)} = \frac{\partial R}{\partial p^{(c)}}$$

$$\frac{d}{dt} \frac{\partial R}{\partial q_i} = \frac{\partial R}{\partial q_i} \quad (B11)$$
Now, for the motion of particle in axially symmetric fields, an arbitrary axially symmetric magnetic field can be described by two components of the vector potential, $A_\phi$ and $A_z$. Thus the Lagrangian becomes

$$L = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2 \right) + \frac{e}{c} \dot{z} A_z + \frac{e}{c} r \dot{\phi} A_\phi + e \Phi$$  \hspace{1cm} (B12)

In this case, Rouse function becomes

$$R = L - \dot{\phi} \left( \frac{\partial L}{\partial \dot{\phi}} \right)$$
$$= \frac{m}{2} \left( \dot{r}^2 + \dot{z}^2 \right) + \frac{e}{c} \dot{z} A_z - e \Phi - \dot{u}$$  \hspace{1cm} (B13)

where

$$\dot{u} = \frac{1}{2m} \left( \frac{p_\phi}{r} - \frac{e}{c} A_\phi \right)^2$$
$$p_\phi = \text{const}$$  \hspace{1cm} (B14)

In particular, if $A_z = 0$, there is no $B_\phi$ component and the problem reduces to the motion of a particle in a potential field $u = u^0 + e \Phi$:

$$m \ddot{r} = -\frac{\partial u}{\partial r}$$
$$m \ddot{z} = -\frac{\partial u}{\partial z}$$  \hspace{1cm} (B15)

The existence of energy integral eq. (B9) can be used to estimate the region of possible positions of the particle:

$$e \Phi \leq \mathcal{E} = \frac{m v_0^2}{2} + e \Phi_0$$  \hspace{1cm} (B16)

Here $v_0$ and $\Phi_0$ are the particle speed and potential at the particle location at the initial time.
If the field is constant in time and exhibits axial symmetry, then both energy and angular momentum are conserved. The energy integral can then be written in the form

\[ E = \frac{m}{2} (\dot{r}^2 + \dot{\theta}^2) + u = \text{const.} \]  

(B17)

where \( u \) is given by eq. (B14). We can then estimate the region of possible particle positions:

\[ u \leq \varepsilon \]  

(B18)

If a particle moves in a magnetic field with axial symmetry, described by only one component of the vector potential \( A = A(r) \), then

\[ \frac{p}{q} = \frac{\partial}{\partial t} \dot{r} = m \dot{\theta} + \frac{e}{c} A \dot{z} = \text{const.} \]  

(B19)

From eqs. (B8), (B9) and (B19) we have

\[ \dot{r}^2 = \frac{2}{m} (\varepsilon_\perp - e\Phi) - \frac{1}{m^2} \left( \frac{p_\phi}{r} - \frac{e}{c} A \Phi \right)^2 \]  

(B20)

where

\[ \varepsilon_\perp = \varepsilon - \frac{m \dot{z}^2}{2} \]

The right hand sides of the expressions in (B20) depend on \( r \) only, so the trajectory equations can be reduced to quadratures:

\[ \Phi = \int \dot{\varphi} \frac{dt}{r} \]

\[ t = \int \frac{dr}{r} \]  

(B21)
Equations of Motion in Drift Approximation:

The drift approximation is valid when the Larmor radius of the particle is much smaller than the scale size of any field inhomogeneity and the Larmor frequency is much higher than any characteristic field frequency. This means that the motion of the particle \( \mathbf{R}(t) \) can be represented as a gyration \( \mathbf{S}(t) \) with a slowly varying radius, and a frequency \( \mathbf{r}(t) \), which we call the guiding center:

\[
\mathbf{R}(t) = \mathbf{r}(t) + \mathbf{S}(t)
\]

The equations which describe the averaged motion of the guiding center and the change in Larmor radius \( |\mathbf{S}| \) are called the drift equations.

First consider the case in which the force \( \mathbf{F} = (e\mathbf{E} + f)/m \) is small and the drift velocity is much smaller than the particle velocity

\[
\mathbf{v}_{dr} = \frac{mcF}{eB} \ll \mathbf{v}
\]

This limitation can be removed. Consider the equation of motion of particle:

\[
\dot{\mathbf{v}} = m \mathbf{F} + \frac{e}{c} \mathbf{v} \times \mathbf{B}
\]

where \( \mathbf{F} = 1/m(e\mathbf{E} + f) \) is the force acting per unit mass. Let

\[
\mathbf{\omega} = - \frac{e\mathbf{B}}{mc}
\]

This leads to

\[
\dot{\mathbf{v}} = \mathbf{F} - \frac{1}{c^2} (\mathbf{F} \cdot \mathbf{v}) \mathbf{v} + \mathbf{\omega} \times \mathbf{v}
\]
In order to apply method of averaging \(^{1}\) to eq. (B25), introduce an orthogonal curvilinear coordinate system with unit vectors \(\hat{t}_0, \hat{t}_1\) and \(\hat{t}_2\) which are related to the magnetic field \(\vec{B}\) at a given point:

\[
\begin{align*}
\hat{t}_0 &= \frac{\vec{B}}{B} \\
\hat{t}_1 \times \hat{t}_2 &= \hat{t}_0 \\
\hat{t}_2 \times \hat{t}_0 &= \hat{t}_1 \\
\hat{t}_0 \times \hat{t}_1 &= \hat{t}_2
\end{align*}
\]

(B26)

\[
\vec{v} = v_{\parallel} \hat{t}_0 + v_{\perp} (\hat{t}_1 \cos \theta + \hat{t}_2 \sin \theta)
\]

(B27)

\(v_{\parallel}\) and \(v_{\perp}\) are the velocity components parallel and perpendicular to \(\vec{B}\), \(\theta\) is the phase of gyration around \(\vec{B}\). Then we can obtain equations in the new variables \(r, v_{\parallel}, v_{\perp}\) and \(\theta\):

\[
\begin{align*}
\frac{dr}{dt} &= v_{\parallel} \hat{t}_0 + v_{\perp} (\hat{t}_1 \cos \theta + \hat{t}_2 \sin \theta) \\
\frac{d\theta}{dt} &= \hat{t}_0 \left\{ \vec{F} - \frac{1}{c^2} (\vec{F} \cdot \vec{v}) \vec{v} \right\} + v_{\perp} \hat{t}_0 (\hat{t}_1 \cos \theta + \hat{t}_2 \sin \theta) \\
\frac{dv_{\perp}}{dt} &= (\hat{t}_1 \cos \theta + \hat{t}_2 \sin \theta) \left\{ \vec{F} - \frac{1}{c^2} (\vec{F} \cdot \vec{v}) \vec{v} - v_{\parallel} \hat{t}_0 \right\} \\
\frac{d\theta}{dt} &= \omega - \frac{1}{v_{\perp}} (\hat{t}_1 \sin \theta - \hat{t}_2 \cos \theta) \left\{ \vec{F} - v_{\parallel} \hat{t}_0 - v_{\perp} (\hat{t}_1 \cos \theta + \hat{t}_2 \sin \theta) \right\}
\end{align*}
\]

Now

\[
\dot{\hat{t}}_i = \frac{\partial \hat{t}_i}{\partial t} + v_{\parallel} (\hat{t}_0 \cdot \vec{v}) \hat{t}_i + v_{\perp} [(\hat{t}_1 \cdot \vec{v}) \hat{t}_i \cos \theta + (\hat{t}_2 \cdot \vec{v}) \hat{t}_i \sin \theta]
\]

(B29)
Substituting eq. (B29) in (B28), we obtain the following system of equations:
\[
\frac{d\vec{x}}{dt} = \vec{P}(\vec{x}, t, \theta)
\]
\[
\frac{d\theta}{dt} = \omega(\vec{x}, t) + A(\vec{x}, t, \theta)
\]  
(B30)

which are in the standard form for the method of averaging. ¹

The solution of the system in (B30) by the method of averaging gives the following expression for \( \vec{x} \):
\[
\vec{x} = \vec{\xi}(t) + \frac{1}{\omega} \vec{P}(\vec{\xi}, t, \theta) + \ldots
\]  
(B31)

Here \( \vec{\xi}(t) \) denotes the averaged motion of the particle while the second term, which is periodic in \( \theta \), describes the gyration in the Larmor circle about the guiding center. The average quantity \( \vec{\xi} \) satisfies the equation
\[
\frac{d\vec{\xi}}{dt} = \vec{P} + \left\{ \vec{P} \cdot \nabla \vec{\xi} \right\} \frac{\vec{P}}{\omega} - \vec{A} \frac{\vec{P}}{\omega}
\]  
(B32)

The quantities \( \vec{P}, \vec{\xi} \) and \( \vec{A} \) are defined by
\[
\vec{P} = (\iint_{2\pi}) \int_0^{2\pi} \frac{\vec{P}}{d\theta} \]
\[
\vec{\xi} = \frac{\vec{P} - \vec{F}}{\vec{A} \vec{P}}
\]
(B33)

in which the integration is carried out for fixed values of \( \vec{\xi} \) and \( t \).

Substituting the right hand sides of eq. (B28) in eq. (B32), we obtain following expressions for the averaged quantities \( r, \theta_h \) and \( \theta_\perp \) :
\[
\frac{d\mathbf{r}}{dt} = \mathbf{v}_n \hat{\mathbf{c}}_o + \frac{1}{\omega} \hat{\mathbf{c}}_o \times (\mathbf{F} - \mathbf{v}_n^2 \hat{\mathbf{c}} - \frac{\mathbf{v}_1^2}{2\omega} \nabla \omega)
\]

\[
\frac{dE}{dt} = m \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} + \frac{mv_1^2}{2\omega} \hat{\mathbf{c}}_o \cdot \nabla \mathbf{F} \tag{B34}
\]

\[
\frac{dJ_\perp}{dt} = - \frac{e\mathbf{v}_1^2}{m\omega} (\hat{\mathbf{c}}_o \cdot \nabla \mathbf{F} - \frac{\partial \omega}{\partial t})
\]

Here

\[
E = m(\mathbf{v}_n^2 + \mathbf{v}_1^2)
\]

is the averaged energy and

\[
J_\perp = - \frac{e\mathbf{v}_1^2}{m\omega}
\]

is the so called transverse adiabatic invariant. \( I \) is given by

\[
\mathbf{I} = (\hat{\mathbf{c}}_o \cdot \nabla) \hat{\mathbf{c}}_o = \hat{\mathbf{n}} / R
\]

which is the vector curvature of line of force. If \( \text{curl} \mathbf{f} = 0 \), \( J_\perp \) is conserved and we get

\[
\frac{d\mathbf{r}}{dt} = \mathbf{v}_n \hat{\mathbf{c}}_o + \frac{1}{\omega} \hat{\mathbf{c}}_o \times (\mathbf{F} - \mathbf{v}_n^2 \hat{\mathbf{c}} - \frac{\mathbf{v}_1^2}{2\omega} \nabla \omega)
\]

\[
\frac{dE}{dt} = m \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} + \frac{mv_1^2}{2\omega} \frac{\partial \omega}{\partial t} \tag{B35}
\]

where

\[
\omega = - \omega_B = - \frac{eB}{mc}
\]

We now write the drift equations (B35) for particle motion in an electromagnetic field \( \mathbf{E} \) and \( \mathbf{B} \) assuming \( f = 0 \). We get
The first term in the equation is the motion of guiding center parallel to \( \vec{B} \) field, the second term is the \( \vec{E} \times \vec{B} \) drift, the third term is the curvature drift and the fourth term is the \( \text{grad} \, \vec{B} \) drift. The last three drifts are all perpendicular to \( \vec{B} \) field. When \( \text{curl} \, \vec{B} = 0 \), we get

\[
\frac{d\vec{r}}{dt} = \frac{e}{m} \vec{E} \cdot \frac{d\vec{r}}{dt} + \frac{mc_0 v_{\perp}^2}{2eB} \frac{\partial \vec{B}}{\partial t}
\]

If the field is independent of time, both \( J_\perp \) and \( E + e \Phi \) are constant. If this is the case, we can write

\[
\frac{mv_{\perp}^2}{2} + e \Phi = E_0 = \text{const}
\]

\[
v_{\perp} \frac{1}{B} = J_\perp = \text{const}
\]

Then

\[
v_{\parallel} = (v_{\perp}^2 - J_\perp B)^{1/2}
\]

Also

\[
\text{curl} \left( \frac{m_{\parallel} \vec{B}}{B} \right) = \frac{m_{\parallel}}{B} \text{curl} \left( \frac{\vec{B}}{B} \right) + \nabla \left( \frac{m_{\parallel}}{B} \right) \times \frac{\vec{B}}{B}
\]
Computing $\nabla (mv_{\|})$ and with the help of eqs. (B38) and (B39), eq. (B36) can be written as

$$\frac{d\vec{r}}{dt} = \hat{b}\left\{v_{\|} - \frac{mcv_{\|}^2}{eb} \hat{b} \cdot \nabla \hat{b}\right\} + \frac{cv_{\|}}{b} \text{curl} (mv_{\|} \hat{b})$$  \hspace{1cm} (B40)

where $\hat{b} = \frac{B}{B}$. If $B \cdot \text{curl} B = 0$, the second term of right hand side of eq. (B40) vanishes. Then, with $\vec{B} = \text{curl} \vec{A}$, eq. (B40) becomes

$$\dot{\vec{r}} = \frac{v_{\|}}{B} \text{curl} (\vec{A} + \frac{mcv_{\|}}{eb} \vec{B})$$  \hspace{1cm} (B41)

**Motion of charged particle in axially symmetrical magnetic field**

In this case,

$$B = \text{curl} A$$

$$B_r = -\frac{\partial A \phi}{\partial z}$$

$$B_z = \frac{1}{r} \frac{\partial}{\partial r} (r A \phi)$$

$$A \phi = \frac{r}{2} b(z) + \frac{r^3}{16} b''(z) + \ldots .$$  \hspace{1cm} (B42)

where $b(z)$ is the field at the axis of symmetry $z$. Limiting ourselves to quadratic terms in $r$, the deviation from the $z$ axis, we can write the drift eqs. (B37) in the form

$$\dot{z} = \sqrt{v^2 - J_1 b(z)} \frac{z}{b}$$

$$\dot{r} = -\frac{r}{2} \frac{b'}{b} \frac{z}{b}$$

$$\dot{\phi} = \frac{mc}{2e} \frac{2v^2 - J_1 b}{b^3} \left(-\frac{b b''}{2} + \frac{3b'^2}{4}\right)$$  \hspace{1cm} (B43)
The second of these equations is integrated directly and yields

\[ r^2 b(z) = \text{const} \]

This shows that the particle moves along a surface of rotation formed by the lines of force.

Differentiation of the first equation with respect to \( t \) yields

\[ \ddot{z} + \frac{J}{2} b'(z) = 0 \]  

(B44)

It follows that the particle can execute oscillatory motion along \( z \) about the minimum of \( b(z) \). These particles are called trapped particles. Besides these trapped particles there are also transmitted particles which are not reflected from the field maxima but continue to move in one direction along \( B \). Axial dependence of bumpy magnetic field is given by

\[ b(z) = b_0 (1 - a \cos \alpha z) \]

\[ \alpha = \frac{2\pi}{L} \]

(B45)

where \( L \) is period and \( \alpha \ll 1 \).

Invoking conservation of \( J_\perp = \frac{v_2}{b} \) at the point where \( v_\perp = 0 \), we find

\[ \cos^2 \chi = \frac{1-a}{1+a} \]

or \( \chi = \gamma_B \sqrt{2a} \) where \( \chi \) is the angle between the normal to \( B \) and \( v \).

Azimuthal Drift

We denote the azimuthal drift angle traversed in one period by \( \Delta \phi \). Then substituting eq. (B45) in eq. (B43) and in eq. (B46)
\[ \varphi = \int \frac{d\varphi}{d\zeta} \, d\zeta \]  

(B46)

we get the following results:

1. For trapped particles, to the first approximation in a

\[ \Delta \varphi_T = -2 \alpha r_L (2a)^{1/2} \left[ E(k) - \frac{1}{2} K(k) \right] \]  

(B47)

where

\[ r_L = \frac{mc\nu}{eb_0} \]

and \( E(k) \) and \( K(k) \) are elliptic integrals with argument \( k^2 = (1-a)/2acot^2 \gamma \).

The quantity \( \Delta \varphi_T \) is the azimuthal displacement per period in the motion of trapped particle (to and from along line of force). This quantity vanishes when \( k^2 = 0.83 \), i.e., when \( \gamma = \gamma_A \approx 0.9(2a)^{1/2} \).

2. For transmitted particles, we take account of terms of order \( a^2 \). Then we calculate the integral in eq. (B46). The resulting expression for \( \Delta \varphi_T \) is a complicated expression in elliptic integrals with argument \( k^2 = 2acot \gamma/(1-a) \). Expanding \( E \) and \( K \) in powers of \( k^2 \), we have

\[ \Delta \varphi_T = \frac{\alpha r_L}{2} \frac{\pi a^2}{8} \frac{1 - 3 \sin^2 \gamma}{\sin^2 \gamma} \]  

(B48)

This expression vanishes when \( \sin \gamma = \sin \gamma_c = 1/\sqrt{3} \) or \( \gamma_c \approx 35^\circ \).

Thus we have the following results:

The particles which just stop at the field maximum have \( \gamma \approx (2a)^{1/2} \). Trapped particles with \( \gamma \approx 0.9 \) (2a)^{1/2} will have zero azimuthal drift, and transmitted particles with \( \gamma \approx 35^\circ \) will also have zero azimuthal drift.
Longitudinal Adiabatic Invariant

Assume that the motion of the guiding center is periodic when drift is neglected, and that the displacement per period (due to drift) is small compared with the characteristic dimensions of the magnetic field; under these conditions, the drift equations allow an additional invariant—the so-called longitudinal adiabatic invariant of the drift equations. An example of this motion occurs in the bumpy torus.

Let us now introduce a curvilinear system of coordinates $\xi_1$, $\xi_2$, $\xi_3$ defined in such a way that the coordinate lines of $\xi_3$ coincide with the lines of force of $\mathbf{B}$. The contravariant components of the unit vectors $\hat{b} = \mathbf{B}/B$ are $b^3 = 1/(g_{33})^{1/2}$, $b^1 = b^2 = 0$.

Limiting ourselves to the case of weak electric field, we can write eq. (B40) in this curvilinear coordinate system

$$\frac{d\xi_1}{dt} = \frac{c\nu_{11}}{eB g_{12}} \left\{ \frac{\partial}{\partial \xi_2} \left( m\nu_{11} b_3 \right) - \frac{\partial}{\partial \xi_3} \left( m\nu_{11} b_2 \right) \right\}$$

$$\frac{d\xi_2}{dt} = \frac{c\nu_{11}}{eB g_{12}} \left\{ \frac{\partial}{\partial \xi_3} \left( m\nu_{11} b_1 \right) - \frac{\partial}{\partial \xi_1} \left( m\nu_{11} b_3 \right) \right\}$$

$$(B49)$$

$$\frac{d\xi_3}{dt} = \frac{\nu_{11}}{\omega_B} \left( \hat{b} \cdot \text{curl} \hat{b} \right) b^3 + \frac{c\nu_{11}}{eB g_{12}} \left\{ \frac{2}{\partial \xi_1} \left( m\nu_{11} b_2 \right) - \frac{\partial}{\partial \xi_2} \left( m\nu_{11} b_1 \right) \right\}$$

Integrating the first two equations with respect to $t$, and making the substitution $\nu_{11} dt \sim d\xi_3/g_{33}^{1/2}$, and taking account of the fact that $\text{div} \mathbf{B} = 0$, we have
We now assume that the motion of guiding center is periodic when the drift is neglected (along the line of force of $\mathbf{B}$). Taking the integral in eqs. (B50) over a period $L$, we have

\[ \Delta \xi_1 = \frac{mc}{eB} \left( \frac{qB}{m} \right)^{1/2} \int \frac{\partial}{\partial \xi_2} \left( m \mathbf{v}_{\perp} \cdot \mathbf{b}_3 \right) - \frac{\partial}{\partial \xi_1} \left( m \mathbf{v}_{\parallel} \cdot \mathbf{b}_2 \right) \frac{L}{2} d\xi_3 \]

\[ \Delta \xi_2 = \frac{mc}{eB} \left( \frac{qB}{m} \right)^{1/2} \int \left\{ \frac{\partial}{\partial \xi_3} \left( m \mathbf{v}_{\perp} \cdot \mathbf{b}_1 \right) - \frac{\partial}{\partial \xi_1} \left( m \mathbf{v}_{\parallel} \cdot \mathbf{b}_3 \right) \frac{L}{2} d\xi_3 \right\} \]

where $d\xi$ is an element of length along the line of force of $\mathbf{B}$.

When the unperturbed motion is oscillatory or takes place along closed lines of force, eq. (B51) gives

\[ \Delta \xi_1 = \frac{mc}{eB} \left( \frac{qB}{m} \right)^{1/2} \frac{\partial J_{\parallel}}{\partial \xi_2} \]

\[ \Delta \xi_2 = -\frac{mc}{eB} \left( \frac{qB}{m} \right)^{1/2} \frac{\partial J_{\parallel}}{\partial \xi_1} \]

These are finite difference equations. If $\Delta \xi_1$ and $\Delta \xi_2$ are small compared to $L$, then to second order in these quantities these difference equations can be replaced by differential equations.
We now multiply the first eq. by $\frac{dx}{\eta n}$, and second by $\frac{dx}{\eta n}$; adding the resulting eqs. and integrating, we obtain

$$J_\parallel (\xi_1, \xi_2) = \oint v_{11} ds = \text{const.} \quad (B54)$$

### Resonant and Nonresonant Particles

Here we will consider the motion of a particle in a weakly bumpy field ($b/b_0 \ll 1$) bent into a torus of large radius. We take a vertical cross section of the bumpy torus. The intersection of this plane with the ring axis determines the origin of our coordinates. $\xi$ and $\theta$ are polar coordinates in this vertical plane. Then eq. (B53) becomes

$$\Delta \theta = \frac{mc}{eB} \frac{\partial J_\parallel}{\partial \xi} \quad (B55)$$

$$\Delta \xi = -\frac{mc}{eB} \frac{\partial J_\parallel}{\partial \theta}$$

If the original straight field is axially symmetric, the longitudinal invariant $J_\parallel$ depends only upon $\xi$ and $J_\perp$. However, if the field is bent into a torus of large radius, then

$$J_\parallel = J_\parallel \left( \frac{\xi}{R}, \frac{x}{R}, J_\perp \right) \quad (B56)$$

where $x = r - R$, $R$ is major radius and $r$ is displacement from origin along $R$. $\frac{\partial J_\parallel}{\partial \xi}$, where $\xi = x/R$, can be expanded in powers of $\xi$. 

To first term of the expansion, we have

$$J_\|^0 (\xi, 0, J_\perp) + \frac{\chi}{R} \frac{\partial}{\partial \xi} J_\|^0 (\xi, 0, J_\perp) = \text{const} \quad (B57)$$

However, here $J_\|^0$ is determined only by the straight field ($R \to \infty$) and the equation $J_\|^0 (\xi, 0, J_\perp) = \text{const}$ is an equation for a family of concentric circles.

We will investigate large and small values of $\xi$ separately in eq. (B57). In the region $\xi = 0$, $J_\| (\xi)$ is expanded in even powers of $\xi$ (since $|B|$ and $B_\phi$ are quadratic in $\xi$) and we get from eq. (B57)

$$\frac{\partial J_\|}{\partial \xi^2} \xi^2 + \frac{1}{2} \frac{\partial^2 J_\|}{\partial (\xi^2)^2} \xi^4 + \ldots + \frac{\chi}{R} \frac{\partial J_\|}{\partial \xi} = \text{const.} \quad (B58)$$

If $\partial J_\|/\partial \xi^2 \neq 0$, neglecting higher terms in expansion we find

$$\frac{\partial J_\|}{\partial \xi^2} \xi^2 + \frac{\chi}{R} \frac{\partial J_\|}{\partial \xi} = \text{const} \quad (B59)$$

which defines a circle whose center is displaced from $\xi = 0$ by an amount

$$\Delta = - \frac{1}{2R} \frac{\partial J_\| / \partial \xi}{\partial J_\| / \partial \xi^2} \quad (B60)$$

Particles obeying eq. (B60) are called nonresonant particles. The case $\partial J_\| / \partial \xi = 0$, which corresponds to zero azimuthal drift, will be called resonance case. In the resonance case, eq. (B58) can be written in the form

$$\left(\frac{\xi}{\lambda}\right)^2 + \frac{\chi}{\lambda} = \text{const} \quad (B61)$$
where

$$\lambda = \frac{2}{R} \left( \frac{\partial J_{\|} / \partial \xi}{(\partial^2 J_{\|} / \partial (\xi^2)^2)^{1/2}} \right)^{1/2}$$

The quantity $\lambda$ is a characteristic dimension that describes the displacement of the trajectory.

We now consider the trajectory when $s$ is much larger than $\Delta$ in the nonresonance case, or much greater than $\lambda$ in the resonance case. Under these conditions, the equation $J_{\|} = \text{const.}$ can be expanded by perturbation methods if we replace $s$ by $s_0 + \delta s$ and expand in powers of $\delta s$. Thus

$$\frac{\partial J_{\|}(s_0)}{\partial s} \delta s + \frac{\partial^2 J_{\|}(s_0)}{\partial s^2} (\delta s)^2 + \ldots + \frac{\chi}{R} \frac{\partial J_{\|}(s_0)}{\partial \xi} + \ldots$$

$$= \frac{x_0}{R} \frac{\partial J_{\|}(s_0)}{\partial \xi}$$

Here $x_0$ and $s_0$ are the coordinates of the line of force on which the particle is located at initial time. If $\frac{\partial J_{\|}(s_0)}{\partial s} \neq 0$, the displacement of particle due to bending of field is proportional to $1/R$; if $\frac{\partial J_{\|}}{\partial s} = 0$ and $\frac{\partial^2 J_{\|}}{\partial s^2} \neq 0$, the displacement is proportional to $1/R^{1/2}$.

References


APPENDIX C

NEOCLASSICAL TRANSPORT THEORY

Binary Coulomb Collisions described by Fokker-Planck collision operator represent an irreducible minimum of dissipation. Transport coefficients derived from such a Fokker-Planck operator are called classical transport coefficients. When experimentally measured value of transport coefficients substantially exceed this Fokker-Planck prediction, the measured transport coefficients are invariably dubbed anomalous.

The classical transport coefficients for plasma transport perpendicular to uniform magnetic field have been computed. It was first pointed out by Galeev and Sagdeev that for small collision frequency these transport coefficients might not even be approximately applicable for transport across a nonuniform magnetic field. This is because the guiding center drifts associated with field inhomogeneities must be taken into account at the microscopic level of Fokker-Planck equation. Transport coefficient computed from the Fokker-Planck collision operator with proper regard to guiding center drifts are called neoclassical transport coefficients. Here we will discuss neoclassical transport from the point of view of the moment equations.

Moment Equations

Let \( f(x, v, t) \) be the distribution function for the particles of mass \( m \) and charge \( e \). Then the Boltzmann equation is

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\[ \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \frac{e}{m} \left( \vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right) \cdot \frac{\partial f}{\partial \vec{v}} = C(f) \]  

(C1)

where \( c(f) \) is the collision operator.

The first moment of eq. (C1) is

\[ m \frac{\partial}{\partial t} (n \vec{V}) + \nabla \cdot \vec{\Pi} - en(\vec{E} + \frac{1}{c} \hat{V} \times \hat{B}) = \vec{F} \]  

(C2)

where \( n \) is the density, \( \vec{V} \) is the mean velocity, \( \vec{\Pi} \) is the stress tensor and \( \vec{F} \) is frictional force; defined as

\[ \vec{\Pi} = \int d^3\vec{v} \ m \vec{V} f \]
\[ \vec{F} = \int d^3\vec{v} \ m \vec{V} C(f) \]  

(C3)

Solving this first moment equation for mean velocity perpendicular to \( \hat{B} \), we get

\[ n \vec{V}_\perp = (m \Omega)^{-1} \hat{n} \times \left[ \nabla \cdot \vec{\Pi} - \vec{F} - en \vec{E} - m \frac{\partial}{\partial t} (n \vec{V}) \right] \]  

(C4)

where \( \Omega = eB/mc \) is gyrofrequency and \( \hat{n} = \hat{B}/B \) is the unit vector along \( \hat{B} \).

Equation (C4) is exact. It is useful when gyroradius is small, i.e.

\[ \mathcal{f} = \frac{v_T}{\Omega} \ll l \]

where \( v_T \) is thermal velocity, \( \mathcal{f} \) is gyroradius and \( l \) is a typical scale length in the problem.
Transport Ordering

Assume that the particles mainly free stream along \( \mathbf{B} \) and temporal changes in macroscopic quantities occur mainly through perpendicular collisional transport such that

\[
\frac{\partial}{\partial t} \sim \nu \frac{\mathbf{l}^2}{\mathbf{L}^2}
\]

where \( \nu \) is a collision frequency associated with the collision operator \( C \). This ordering is called the transport ordering and it leaves out of consideration those processes which occur on timescales faster than eq. (C5). The right hand side of eq. (C4) contains an overall factor of \( \frac{\mathbf{S}}{\mathbf{L}} \) because of the \( \Omega^{-1} \) factor. Therefore to compute perpendicular flux to second order in \( \frac{\mathbf{S}}{\mathbf{L}} \), we need to compute the quantity in the square bracket of eq. (C4) only to first order in \( \frac{\mathbf{S}}{\mathbf{L}} \). Further, since we have assumed transport ordering, we can set \( \Omega^{-1} \nu / \partial \mathbf{t} = \mathcal{O} \left( \frac{\mathbf{S}^3}{\mathbf{L}^3} \right) \). Thus eq. (C4) becomes

\[
\mathbf{nV}_\perp = (m \Omega)^{-1} \mathbf{n} \times \left[ \mathbf{n} \cdot \mathbf{\hat{P}} - \mathbf{F} - en\mathbf{E} \right] + \mathcal{O} \left( \frac{\mathbf{S}^3}{\mathbf{L}^3} \right)
\]

We must also specify the magnitude of the collision frequency \( \nu \). We will restrict our attention to the case

\[
\nu \ll \Omega
\]

However, we still have two regions to distinguish, depending upon the relative size of the collision frequency to the transit frequency \( \omega_T \equiv \frac{\nu T}{\mathbf{L}} \).
**Collisional Regime**

The collisional regime is defined by

\[ \nu \gg \omega_T \]

In this regime, collisions are so frequent that the stress tensor reduces to a scalar pressure

\[ \mathbf{P} = \Pi \mathbf{P} \]

(08)

With this, the expression for \( \mathbf{V}_\perp \) becomes

\[ n \mathbf{V}_\perp = (m \Omega)^{-1} \mathbf{h} \times \left[ \nabla \mathbf{P} - e n \mathbf{E} \right] + n \mathbf{V}_c \]

(09)

where we have neglected terms of order \( \mathbf{s}^3 / \mathbf{l}^3 \) and higher. \( \mathbf{V}_c \) is given by

\[ n \mathbf{V}_c = - (m \Omega)^{-1} \mathbf{h} \times \mathbf{F} \]

(10)

The first two terms in (09) are the diamagnetic and the \( \mathbf{E} \times \mathbf{B} \) drifts. The third term represents classical transport. Since \( \nu \ll \Omega \), collisions are not sufficient to randomise the gyromotion of the particles. This gives rise to a frictional force \( \mathbf{F} \). From eq. (09), one can see that classical transport takes place even in the absence of perpendicular guiding center motion, i.e., even in uniform magnetic field. Thus classical transport is ultimately determined by gyromotion.

**Collisionless Regime**

The collisionless regime is defined by
In this case, collisions do not place any restraint on the stress tensor, but gyrooscillations force the stress tensor to have a diagonal form, i.e.

\[ \vec{\mathbf{P}} = \hat{n} \hat{n} \mathbf{P}_{\parallel} + (\mathbf{I} - \hat{n} \hat{n}) \mathbf{P}_{\perp} + \mathcal{O}(\xi^2/l^2) \]  

(C12)

This reflects the two dimensional smoothing due to dyromotion, while the relation \( \mathbf{I} = \mathbf{I}_p \) reflects the three dimensional randomization effected by collisions.

In the collisionless case, the expression for \( \vec{n} \nabla_1 \) is the same as in the collisional case but with an extra term

\[ \vec{n} \nabla_{NC} = (m \Omega)^{-1} \hat{n} \times \mathbf{V} \left[ \hat{n} \hat{n} \left( \mathbf{P}_{\parallel} - \mathbf{P}_{\perp} \right) + \mathbf{I} \left( \mathbf{P}_{\perp} - \mathbf{P}_{\|} \right) \right] \]  

(C13)

We identify this term with neoclassical transport. Thus neoclassical transport depends upon anisotropy of stress tensor. In collisionless regime, the right side of eq. (C13) can be substantial because collisions are too infrequent to randomize the guiding center motion of particles.

**Axisymmetric Systems and Flux Surface Averages**

The equilibrium equations for axisymmetric systems are

\[ \nabla \times \mathbf{B} = c \nabla \mathbf{P} \]

\[ \mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B} \]
If $\phi$ denotes the angle about symmetry axis, $\vec{B}$ can be expressed as
\[ \vec{B} = I(\psi) \nabla \phi + \nabla \phi \times \nabla \psi \] (C14)

where $\psi$ labels the flux surfaces and is related to the vector potential $\vec{A}$ by
\[ \psi = - R A \phi \] (C15)

$R$ is the distance from symmetry axis. The first term in eq. (C14) is the toroidal $\vec{B}$ field and the second term is the poloidal $\vec{B}$ field, i.e.
\[ \vec{B} = \vec{B}_T + \vec{B}_P \]

For a tokamak, the poloidal beta, $\beta_p$ (defined as the ratio of scalar pressure to $B_p^2$); is of order unity while the total beta, $\beta$ (defined as ratio of scalar pressure to $B^2$), is much less than unity. This means that
\[ |\vec{B}_P| \ll |\vec{B}_T| \] (C16)

We will specify field geometry by eq. (C14). We choose coordinates such that $\phi$ is toroidal angle, $\psi$ is the distance from magnetic axis and $\nu$ is the poloidal angle. In this coordinate system the volume element is given by
\[ d^3x = \frac{d\psi d\chi d\phi}{\nabla \nu \cdot \vec{B}} \]

Flux surface average of a quantity is defined as the normalized volume integral of that quantity over an infinitesimal volume between two
neighboring flux surfaces. Flux surface average of a quantity, say $K$, is denoted by $\langle K \rangle$. It can be shown that

$$
\langle K \rangle = 2\pi \int \frac{d\chi}{\chi} \frac{K}{\chi^2}
$$

where

$$
\chi' = 2\pi \int \frac{d\chi}{\chi^2}
$$

The crucial property of flux surface average is

$$
\langle \nabla \cdot \vec{K} \rangle = \frac{1}{\chi'} \frac{\partial \chi'}{\partial \psi} \chi' \langle \vec{K} \cdot \nabla \psi \rangle \quad (C17)
$$

With this discussion in mind, we now return to the first moment equation. Multiply eq. (C2) by $c/e \, R^2 \nabla \phi$ and perform flux surface averaging. The $\nabla \times B$ term then becomes

$$
\langle \nabla \cdot \vec{R}^2 \nabla \phi \cdot \nabla \times \vec{B} \rangle = \langle \nabla \cdot \vec{n} \nabla \times \vec{B} \rangle
$$

This is the flux perpendicular to surface $\psi$. The $\nabla \cdot \vec{P}$ term becomes

$$
\frac{c}{e} \langle R^2 \nabla \phi \cdot \nabla \cdot \vec{P} \rangle = \frac{c}{e} \frac{1}{\chi'} \frac{\partial \chi'}{\partial \psi} \chi' \langle R^2 \nabla \phi \cdot \vec{P} \cdot \nabla \psi \rangle \quad (C18)
$$

Now the right hand side of eq. (C18) involves an off diagonal element for the stress tensor. Comparison with eq. (C12) shows that the stress tensor contributes only in third order in $\nabla \phi$ to the perpendicular velocity. Thus we have

$$
\langle \nabla \cdot \vec{n} \nabla \psi \rangle = - \frac{c}{e} \langle R (F\phi + e \nabla \phi) \rangle + O(\frac{\partial^2 \phi}{\partial \psi^3}) \quad (C19)
$$
Discussion of Equation C19

Equation (C19) has been derived in the collisionless regime assuming small gyroradius and axisymmetry of system. The equation shows that to second order, the transport does not involve the viscosity. It also guarantees ambipolar diffusion. To see this, note that the collisions conserve momentum

\[ \vec{F}_e = - \vec{F}_i \]

Quasineutrality demands that

\[ z |e| n_i = |e| n_e \]

Then eq. (C19) gives

\[ \langle n_e \vec{v}_e \cdot \nabla \psi \rangle = z \langle n_i \vec{v}_i \cdot \nabla \psi \rangle \]

The right hand side of eq. (C19) contains both classical and neoclassical diffusion. To distinguish them, we write

\[ F_\phi = \int d^3 \phi \ m (\vec{v}_\parallel \phi + \vec{v}_\perp \phi ) \cdot c(b) \]

Recall eq. (C10) for the collisional limit

\[ n \vec{v}_c = - (\frac{m \Omega}{\omega})^{-1} \hat{n} \times \vec{F} \]

Comparison of these last two expressions show that \( \vec{v}_\perp \phi \) gives rise to classical diffusion, while \( \vec{v}_\parallel \phi \) gives rise to neoclassical diffusion.
To see how the anisotropy of the stress tensor is contained in eq. (C19), we take the parallel component of the first moment equation. We get

\[ \hat{\eta} \cdot \nabla \cdot \vec{\mathbf{\Pi}} = F_{||} + en E_{||} \]  

(C22)

Comparing this with eq. (C21), we see that \( F_{||} \) is the first term of eq. (C21), corresponding to neoclassical transport. In ideal MHD, \( \vec{\mathbf{\Pi}} = \hat{\Pi} \mathbf{\Pi} \) and eq. (C22) becomes

\[ \hat{\eta} \cdot \nabla \rho = 0 \Rightarrow F_{||} + en E_{||} \]

This means that in ideal MHD, parallel frictional force and electric force precisely cancel. Thus the non-ideal MHD effects associated with neoclassical transport arise from the anisotropy of stress tensor, i.e., neither side of eq. (C22) is zero.

Neoclassical transport is related to the guiding center motion precisely in the same way as classical transport is associated with gyromotion. This is true for asymmetric systems also. Neoclassical transport occurs when collisions are too infrequent to randomize the guiding center motion.

**Conservation Laws**

Energy flux is defined as

\[ \vec{\Phi} = \int d^3v \vec{v} m \vec{v}^2/2 \]
Just as we obtained eq. (C19) for particle flux, we can get the energy flux equation

\[ \langle \vec{\Phi} \cdot \nabla \psi \rangle = -\frac{e}{e} \left< R \left( m_S \phi + \frac{S}{2} p e E \phi \right) \right> \]  

(c23)

where

\[ \vec{S} = \int d^3\psi \left( \vec{v}_|| + \vec{v}_\perp \right) \frac{m \vec{v}^2}{2} C(f) \]  

(c24)

As before, the \( \vec{v}_|| \phi \) term corresponds to neoclassical energy flux while the \( \vec{v}_\perp \phi \) term corresponds to classical flux.

Zeroth and second moment of Boltzmann equation give particle and energy conservation laws.

\[ \left< \frac{\partial n}{\partial t} + \nabla \cdot (n \vec{V}) \right> = 0 \]  

(c25)

\[ \left< \frac{\partial}{\partial t} \frac{3}{2} p_a + \nabla \cdot \vec{F}_a \right> = \delta_{ae} \left< \vec{J} \cdot \vec{E} \right> \]  

\[ \pm \left< Q_{\Delta i} + \vec{V}_i \cdot (\vec{F}_i + e \vec{n}_i \vec{E}) \right> \]  

(c26)

where \( a = i, e; \) \( \delta_{ae} \) is Kronecker delta, the upper sign is for ions, lower sign is for electrons and \( Q_{\Delta i} \) is the collisional energy exchange between particles.

Further, note that in eq. (C25)

\[ \left< \nabla \cdot (n \vec{V}) \right> = \frac{1}{V'} \frac{2}{\partial \psi} \left< \langle n \vec{V} \cdot \nabla \psi \rangle \right> \]

and in eq. (C26)
\[ \langle \nabla \cdot \phi \rangle = \frac{1}{V'} \frac{\partial}{\partial V'} \langle \phi \cdot \nabla \psi \rangle \]

The right hand sides of these equations are given by eqs. (C19) and (C23) respectively.

Summary

The program of neoclassical transport can be summarized as follows: use odd moments of the Boltzmann equation to obtain expressions for energy and particle fluxes in terms of moments of collision operator. Use even moments of Boltzmann equation to determine time evolution of \( n \) and \( p_a \). Magnetic field can be advanced in time using appropriate Maxwell's equation. Thus we have a closed set of equations.

References

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<td><strong>Minor radius</strong></td>
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**Hot electron annulus**

- $n_e$
- $T_e$
- $\beta_{annulus}$

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<td>$n_e$</td>
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<tr>
<td>$T_e$</td>
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<td>$\beta_{annulus}$</td>
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**Toroidal plasma (T mode)**

- $n_e$
- $T_e$
- $T_i$
- $\beta_{max}$
- $n\tau$

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\begin{align*}
R^\uparrow & : n^* \\
T_e^* & : \uparrow, \uparrow, \uparrow, \uparrow, \downarrow, \downarrow \\
T_i^* & : \downarrow, \downarrow, \uparrow, \uparrow, \downarrow, \downarrow \\
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P_i^* & : \downarrow, \downarrow, \uparrow, \uparrow, \downarrow, \downarrow \\
\phi^\uparrow & : \uparrow, \uparrow, \uparrow, \uparrow, \downarrow, \downarrow \\
\end{align*}
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\begin{align*}
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& : n_0^* \alpha_{P,\mu}^*0.22, n_0^* \alpha_{P,\mu}^*0.22 \\
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\end{align*}
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$T \rightarrow C$

|                | $n^*$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ |               |           |
| $T_e^*$        | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ |               |           |
| $P_\mu^*$      | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ |               |           |
| $P_i^*$        | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ |               |           |

$C \rightarrow T$

|                | $n^*$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ |               |           |
| $T_e^*$        | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ |               |           |
| $P_\mu^*$      | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ |               |           |
| $P_i^*$        | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ | $\uparrow$ |               |           |
Table 13

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<tr>
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<td>ΔT_i</td>
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<tr>
<td>---</td>
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Table 15

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<th>( T_1(0) )</th>
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<td>- do -</td>
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<td>( M ) and ( C )</td>
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FIGURE CAPTIONS

1. Schematic representation of EBT.
2. One mirror section of EBT with hot electron ring.
3. M, T, and C modes of operation in EBT.
   a. Microwave power $P_\mu$ vs. ambient gas pressure $p_0$. Solid lines distinguish M, T, and C modes.
   b. Average electron line density $n_e$ vs. filling pressure $p_0$ for different values of microwave power. Broken lines distinguish M, T, and C modes.
   c. Density fluctuations and the line density vs. filling pressure $p_0$ in M, T, and C modes.
   d. Average electron line density vs. pressure $p_0$ in M, T, and C modes.
   e. Constancy of $e\psi/T_e$ in T mode. Plot of $e\psi/T_e$ vs. microwave power $P_\mu$ in quiescent T mode.
   f. Ambipolar electrostatic potential $\phi$ vanishes in C mode. Plot of $\phi$ vs. ambient pressure $p_0$.
4. Equilibrium surface for canonical cusp catastrophe.
5. Examples of delay and Maxwell conventions.
6. Figure showing that when ambipolar electrostatic potential is negative (positive), electrons (ions) are nonresonant (resonant) and ions (electrons) are resonant (nonresonant).
7. Equilibrium surface log $T_e$ as a function of controls log $P_\mu$ and $\phi$ for fixed control $n_0 = 0.01$ in case of point model with electrons in absence of ICRH.
8. Equilibrium trajectories for state variables \((n, T_e, T_i)\) as a function of \(P_{\mu}\) for fixed value of \(\bar{\phi}\) for various filling densities:
   a, \(n_0 = 0.001\); b, \(n_0 = 0.01\); c, \(n_0 = 0.1\) for point model with electrons in absence of ICRH.

9. Catastrophes in EBT in the form of \(M \rightarrow T\) and \(T \rightarrow M\) transitions for point model with electrons in absence of ICRH using delay convention;
   (a) \(\log n\) vs. \(P_{\mu}\), (b) \(\log T_e\) vs. \(P_{\mu}\), (c) \(\log T_i\) vs. \(P_{\mu}\).

10. Equilibrium trajectories as a function of \(P_{\mu}\) and \(P_i\) at fixed \(n_0\), \(\bar{\phi}\) and \(R\) for point model with electrons with ICRH. When \(T_i < T_e\),
    (a) \(n, T_e\) and \(T_i\) vs. \(P_{\mu}\) (b) \(n, T_e, T_i\) vs. \(P_i\).

11. Equilibrium trajectories as a function of \(P_{\mu}\) and \(P_i\), (a) \(n, T_e, T_i\) vs. \(P_{\mu}\), (b) \(n, T_e, T_i\) vs. \(P_i\), \(n_0, \bar{\phi}\) and \(R\) fixed for point model with electrons with ICRH when \(T_i > T_e\).

12. Schematic equilibrium trajectories for point model with electrons showing that power law relation between \(n_0\) and \(P_{\mu}\) can be used as the signature of transitions.

13. Reversal of direction of equilibrium trajectory at \(y = T_i/T_e = 1\) or \(R = -1/3\) for point model with electrons with ICRH.

14. Equilibrium trajectories for state variables \((n, T_e, T_i)\) as functions of control parameter \(P_{\mu}\) for point model with ions without ICRH at fixed \(n_0 = 0.01\) and \(\bar{\phi}' = 0.6\).
    (a) \(n\) vs. \(P_{\mu}\), (b) \(T_e\) vs. \(P_{\mu}\),
    (c) \(T_i\) vs. \(P_{\mu}\).

15. Equilibrium trajectories for state variables \((n, T_e, T_i)\) as functions of control parameter \(P_{\mu}\) at fixed \(n_0 = 0.1\) and \(\bar{\phi}' = 1.1\) for point model with ions without ICRH. (a) \(n\) vs. \(P_{\mu}\), (b) \(T_e\) vs. \(P_{\mu}\),
    (c) \(T_i\) vs. \(P_{\mu}\).
16. Equilibrium trajectories for state variables \((n, T_e, T_i)\) as functions of control parameter \(P_\mu\) at fixed \(n_0 = 0.01\) and \(\phi' = 1.5\) for point model with ions without ICRH. (a) \(n\) vs \(P_\mu\), (b) \(T_e\) vs. \(P_\mu\), (c) \(T_i\) vs. \(P_\mu\).

17. Emergence and development of independent T mode as the control parameter \(\phi'\) increases beyond 1 for point model equilibria with ions without ICRH. (a) \(\phi' = 0.9\), (b) \(\phi' = 1.0\), (c) \(\phi' = 1.1\), (d) \(\phi' = 1.2\).

18. When equilibrium surface has only one fold, \(T \rightarrow C\) and reverse transitions will occur for point model with ions without ICRH.

19. When equilibrium surface has two folds and \(\phi' > 1.1\), \(M \rightarrow T, T \rightarrow C, C \rightarrow T\) and \(T \rightarrow M\) transitions can occur for point model with ions without ICRH.

20. When equilibrium surface has two folds and \(\phi' = 1.1\), \(M \rightarrow T, T \rightarrow C, T \rightarrow M\) and \(C \rightarrow M\) transitions can occur for point model with ions without ICRH. \(C \rightarrow M\) transition is possible only when \(\phi' = 1.1\) and reverse transition is not possible.

21. Equilibrium trajectories for state variable \(T_e\) as a function of control parameter \(P_\mu\) at fixed values of control parameters \(n_0 = 0.01\) and \(\phi' = 0.1\) for three different values of parameter \(R\) for point model with ions with ICRH.

22. Equilibrium trajectories for state variable \(T_e\) as a function of control parameter \(P_i\) at fixed values of \(\phi' = 0.6\) and \(n_0 = 0.01\) for two different values of parameter \(R = 0.9\) and 0.99 for point model with ions with ICRH.
23. Equilibrium trajectories for state variable $T_e$ as a function of control parameter $P^\mu$ at fixed values of $\bar{\phi}' = 1.1$ and $n_0 = 0.01$ for two different values of parameter $R = 0.9$ and $0.99$ for point model with ions with ICRH.

24. Equilibrium trajectories $T_e$ vs. $P^\mu$ at fixed $\bar{\phi}' = 1.5$ and $n_0 = 0.01$ for two values of $R = 0.9$ and $0.99$ for point model with ions with ICRH.

25. Typical equilibrium profiles for state variables $(n, T_e, T_i)$ as functions of control parameters $P^\mu$ and $P_i$ at fixed $\bar{\phi}' = 0.6$ and $n_0 = 0.01$ when $T_i < T_e$ ($R = 0.5$) for point model with ions with ICRH.

(a) $(n, T_e, T_i)$ vs. $P^\mu$, (b) $(n, T_e, T_i)$ vs. $P_i$.

26. Typical equilibrium profiles for state variables $(n, T_e, T_i)$ as functions of control parameters $P^\mu$ and $P_i$ at fixed $\bar{\phi}' = 0.6$ and $n_0 = 0.01$ when $T_i > T_e$ ($R = -0.5$) for point model with ions with ICRH.

(a) $(n, T_e, T_i)$ vs. $P^\mu$, (b) $(n, T_e, T_i)$ vs. $P_i$.

27. Reversal of direction of equilibrium trajectories at $R = -1/3$ for point model with ions with ICRH.

28. Typical time evolution plots for EBT point model equilibria with a single fold. (a) point model with electrons with ICRH, $T_i > T_e$, (b) point model with ions with ICRH $T_i < T_e$, (c) Point model with electron without ICRH.

29. Typical time evolution plots for EBT point model equilibria with multiple folds. (a) point model with ions without ICRH, $\bar{\phi}' = 1.5$, $n_0 = 0.01$, (b) point model with ions without ICRH, $\bar{\phi}' = 1.1$, $n_0 = 0.01$. 
30. Initial value problem for point model equilibria with ions with ICRH at fixed $n_0 = .01$, $e\phi/T_i = 0.1$ and $R = 0.9$.

31. A small change in initial state can drastically change final state.

Solid lines show time development of state variables corresponding to figure 30 (f). Dashed lines show time development of state variables corresponding to figure 30 (e). (a) $n(t)$ vs. $t$, (b) $T_e(t)$ vs. $t$, (c) $T_i(t)$ vs. $t$. 
ELMO
BUMPY TORUS

Fig. 1
Fig. 2
Fig. 3(a)
Fig. 3(b)
Fig. 3(c)
Fig. 3(d)
Fig. 3(e)
Fig. 3(f)
Fig. 5
$e\phi / Te = -0.6 \quad a \quad b \quad c$

Fig. 8
ICRHE

$R = 1.0000$

$NC = 0.01000$

$\Phi = -0.60$

Fig. 9(a)
ICRHE

\[ R = 1.0000 \]
\[ NC = 0.01000 \]
\[ \text{PHI} = -0.60 \]
$R = 1.0000$
$NO = 0.01000$
$PHI = -0.60$

Fig. 9(c)
EICRH

R = 0.5
NC = 0.01
PHI = -0.5

Fig. 10(a)
$R = 0.5$
$N_G = 0.01$
$\Phi = -0.6$

Fig. 10(b)
$E_{ICRH}$

$R = -0.5$

$N_0 = 0.01$

$\phi = -0.6$

Fig. 11(a)
Fig. 11(b)

R = -0.5
NO = 0.01
PHI = -0.6

EICRH

$^{143}$
$R = 1.0000$
$\text{NC} = 0.01000$
$\Phi H = 0.60$
IICRH

R = 1.0000
NO = 0.01000
PHI = 0.60

Fig. 14(b)
IICRH

R = 1.0000
NO = 0.001000
PHI = 0.60
R = 1.0000
NO = 0.01000
PHI = 1.10

Fig. 15(a)
IICRH

R = 1.0000
NG = 0.01000
PHI = 1.10

Fig. 15(b)
R = 1.0000
NO = 0.01000
PHI = 1.10

Fig. 15(c)
IICRH

$R = 1.0000$

$NO = 0.01000$

$PHI = 1.50$

Fig. 16(a)
IICRH

R = 1.0000
NG = 0.01000
PHI = 1.50

Fig. 16(b)
R = 1.0000
NG = 0.01000
PHI = 1.50

Fig. 16(c)
Fig. 17(a)

Fig. 17(b)
R = 1.0000
N0 = 0.01000
PHI = 0.60

Fig. 18
R = 1.0000
NO = 0.01000
PHI = 1.50

Fig. 19
IICRH

R = 1.0000
NO = 0.91000
PHI = 1.10

Fig. 20
Fig. 21
IICRH
NC=0.01000
PHI=0.60

R=0.9000

Y=0.56

Y=1

R=0.9900

Y=0.002

Y=0.006

Y=0.39

Y=1

Fig. 22
Fig. 23
IICRH

NO = 0.01000

PHI = 1.50

R = 0.9000

R = 0.9900

Y = 0.1

Y = 0.18

Y = 1

Y = 1

Fig. 24
Fig. 25(a)
Fig. 25(b)

IICRH

R = 0.5000
NO = 0.01000
PHI = 0.60
Fig. 26(a)

$R = -0.5000$
$N_0 = 0.01000$
$\Phi I = 0.60$
$R = -0.5000$

$NG = 0.01000$

$PHI = 0.60$

Fig. 26(b)
Fig. 28(a)

EICRH

R = -0.5000
NO = 0.01000
PHI = -0.60
IICRH

$R = 0.5000$

$N_0 = 0.01000$

$\phi = 0.60$

Fig. 28(b)
EICRH

R = 1.0000
NO = 0.01000
PHI = -0.60

Fig. 28(c)
Fig. 29(b)
Fig. 30(b)
Fig. 30(g)
Fig. 30(h)
Fig. 31(b)