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Modifications of the nucleon properties in the nuclear medium

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Modifications of the Nucleon Properties in the Nuclear Medium

A Dissertation

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The Faculty of the Department of Physics

The College of William and Mary

In Partial Fulfillment

Of the Requirements for the Degree of

Doctor of Philosophy

By

Paolo Amore

July 2000
APPROVAL SHEET

This thesis is submitted in partial fulfillment of the requirement for a degree of

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Hampton University and TJNAF

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A. Molinari
University of Turin
To my parents and to my brother
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ABSTRACT

Modifications of the Nucleon Properties in the Nuclear Medium

In this thesis we consider the possible modifications of the properties of the nucleon inside a nucleus. This problem is studied by applying the Wigner-Seitz approximation to different effective field theory models of the nucleon based on the underlying symmetries of QCD. The Wigner-Seitz approximation reduces the complex many-body problem to an effective one-body problem. By following this approach we calculate the static properties of the nucleon in the nuclear medium in both the Skyrme and the chiral quark soliton models. We also use the same chiral quark-soliton model to calculate the modification of the quark distribution functions measured in deep inelastic electron scattering when a nucleon is inside the nucleus.
Chapter 1
Introduction

The topic of this dissertation is the study of the modifications of the properties of the nucleon when surrounded by other nucleons in a nucleus. We may speak of this as the change in the properties of the nucleon at finite baryon (nuclear) density.

In fact, since nucleons are composite particles, made out of quarks and gluons, it has been speculated that in a hot and/or dense system interesting modifications could occur, possibly resulting in extreme conditions in a phase in which color is deconfined. In nuclear matter, where such conditions are not met and nucleons and mesons are still the relevant degrees of freedom, minor modifications to their properties are expected. The study of these modifications, while challenging, would be valuable for improving our understanding of the strong interaction, ordinary nuclei, and of dense nuclear systems.

Unfortunately the theoretical investigation of this problem is not possible using a conventional nuclear physics approach in which nucleons and mesons are treated as elementary particles. Here all modifications in their structure have to be put in by hand, thus limiting the predictive power. On the other hand, since it is not possible to do an analysis based directly on Quantum Chromodynamics (QCD), i.e. the underlying theory of the strong interactions, one needs to resort to models in which the substructure of the nucleon is not neglected, and where it is properly implemented in the presence of a dense medium.

Clearly, the description of a relativistic many-body system, like a nucleus, whose constituents — the nucleons — are themselves composite objects, represents an ex-
tremely challenging task. The solution to this problem, even a partial one, requires the intervention of some approximation[Wal95].

Before describing the approach taken in this work, we first review some of the methods used and developed in the literature. Here the discussion is limited to a brief account of these approaches, outlining the main ideas at the basis of each and referring the interested reader to the relevant literature on the subject.

One of the earliest approaches to this problem can be found in the work of Nyman[Nym70]. In this work a dynamical model of the pion field surrounding the nucleon is employed. This allows one to calculate the low-energy properties of the nucleon directly in terms of the pion field. The nucleon is then "put" in nuclear matter, by modifying the pion field according to the Wigner-Seitz approximation[Wig33]. In this approach, a nucleon is enclosed in a finite volume related to the density of the system, and the boundary conditions on the fields are modified in order to account for the presence of surrounding nucleons. This model yields a quenching of the pion-nucleon coupling constant, as well as of the anomalous magnetic moments and of the axial coupling constant.

Another simple example of such analyses has been carried out in [Jam89, Rip97]; in this approach a system at finite baryon density is described in terms of quarks filling a Fermi sea. This picture is mostly of academic interest, since it completely neglects the binding of quarks inside nucleons and, as a result, it also over-estimates the Pauli repulsion in the system. It does provide a mechanism for the restoration of chiral symmetry at large densities; the quark scalar condensate, which is directly related to the large dynamical masses of the constituent quarks, decreases in this model at finite densities.

Notice that the possibility that chiral symmetry is partially restored at finite density deserves particular attention. In fact, since chiral symmetry is believed to
govern the properties of the light mass hadrons\(^1\), a partial restoration in a dense environment could produce noticeable modifications. Unfortunately, the theoretical analysis of this phenomenon is impaired by the lack of a “microscopic” understanding — i.e. based on QCD — of the spontaneous breaking of chiral symmetry. As a result the theoretical investigations must rely on models.

In particular, the calculations of [Jam89, Rip97] were carried out using the Nambu-Jona Lasinio (NJL) model\([NJL61a, NJL61b]\). This model is based on a chiral invariant non-renormalizable lagrangian, which contains only quarks as elementary degrees of freedom and allows a dynamical realization for the spontaneous breaking of chiral symmetry: large quarks masses are generated in the model in the strong coupling regime.

A different approach is instead used in the Quark-Meson-Coupling model (QMC), first proposed by Guichon [Gui88], and later extended by Thomas and collaborators [Sai94]. Here the nucleon is described as a system of three quarks, bound by a confining force, and interacting with the meson mean fields generated by the surrounding nucleons. In this respect, this model can be considered the natural evolution of the QHD model of Walecka [Ser86] to account for the internal quark structure of the nucleon.

The analysis of Khanna and collaborators [Kha98] falls in a similar line. These authors use the Skyrme model to describe the nucleon. The pion field — the only degree of freedom available in this description — is modified in their approach by the presence of the nuclear medium through the insertion of a pion-polarization term, whose parameters are experimentally determined.

A very original approach has been employed by Manton and Ruback [Man86] in their study of the Skyrme model on the hyper-sphere instead of that on the conven-

\(^{1}\)Chiral models have indeed had a considerable success in describing the nucleon and the light hadrons.
tional three-dimensional space $\mathbb{R}^3$. They find that below a critical radius a phase in which the baryon density (here identified with the topological density) is uniformly distributed in space is favorable. This calculation was later generalized to the Nambu-Jona Lasinio model by Forkel [For95], proving the occurrence of a partial restoration of chiral symmetry.

In other approaches, nuclear matter has been described as a crystal, letting the nucleons sit on a regular lattice [NymTO, Ach85, Kle86, Zha86]. In particular, Klebanov [Kle86] has studied the properties of a crystal of solitons, arranged to form a cubic structure, within the Skyrme model. Serious computational difficulties in properly imposing the Bloch boundary conditions must be faced in these calculations. Moreover, such a description of nuclear matter is clearly reliable only in the limit of large number of colors ($N_c \to \infty$). In fact, the nucleons, which are color singlet objects, are very massive in this regime, and their motion can be reasonably neglected.

In this dissertation the problem of the modification of the nucleon properties in the nuclear medium will be analyzed by using phenomenological models of the nucleon. Two models are investigated, the Skyrme model and the quark soliton model. Each provides a dynamical description of the properties of the nucleon with increasing sophistication. The field equations will be solved in a simple approximation — the Wigner-Seitz approximation — to account for the presence of a medium. This will provide a consistent frame for exploring regimes at finite density, while still retaining a description based on subnucleonic degrees of freedom (pions and/or quarks).

Originally, this method was introduced by Wigner and Seitz [Wig33], who were interested in calculating the binding energy of metallic sodium. Instead of attempting the solution of the Schrödinger equation for the entire crystalline structure, they restricted the problem to the study of an elementary cell, containing a single atom,
and imposed suitable boundary conditions on the wave functions of the electrons, extracted from Bloch's theorem. Remarkably, this approximation yields a good estimate for the binding energy, while greatly simplifying the problem. It effectively maps a complex many-body problem into a 1-body problem.

In this dissertation this method has been applied to the study of the nucleon in a dense environment. The elementary cell contains a single nucleon, and suitable boundary conditions for the fields are applied at the surface of the cell, in order to account for the presence of the other nucleons. The choice of these average boundary conditions is dictated by the physics. The isotropy of the system is taken into account by choosing a spherical unit cell.

By following this approach, one is thus able to provide a dynamical description of a finite density system without having to introduce, in principle, new degrees of freedom and parameters in the model.

In the first part of this dissertation, the Wigner-Seitz approximation is applied to the Skyrme model of the nucleon[Sky61]. The lagrangian for this model reads

\[
L = \frac{f_s^2}{4} \text{tr} \left( \partial_\mu U_0 \partial^\mu U_0^\dagger \right) + \frac{1}{32G^2} \text{tr} \left[ \partial_\mu U_0 U_0^\dagger, \partial_\nu U_0 U_0^\dagger \right]^2 ,
\]

where \( U_0(x) = \exp \left\{ i \vec{\pi} \cdot \vec{\xi}(x) \right\} \) is a SU(2) matrix and \( \xi(x) \) is the chiral angle\(^2\). The only available degree of freedom is the pion. This is related to the chiral angle through the relation \( \tilde{\pi}(x) = f_\pi \xi(x) \). The nucleon (and the \( \Delta \)) emerge as solitonic solutions and correspond to non-trivial topological configurations of the chiral field. A conserved topological current can be found

\[
W_\mu^{(0)} \equiv \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} \text{tr} \left[ U_0^\dagger \partial^\nu U_0 U_0^\dagger \partial^\alpha U_0 U_0^\dagger \partial^\beta U_0 \right]
\]

and used to classify the solutions in terms of the topological charge, or winding number. The conservation of this current, which does not follow from the Noether

\(^2\)The subscript 0 is used here to indicate a free-space solution.
theorem, and therefore is independent of the details of the lagrangian, allows one to identify the latter with the conserved baryon current.

In the medium, the nucleon is here described in the Wigner-Seitz approximation and enclosed in a single cell, of radius $R_{WS}$. In this case, suitable boundary conditions for the chiral angle have to be imposed at the surface of the cell. Physically, the most attractive configuration in a dense system corresponds to a sharing of the pion field between neighbor nucleons. This can be implemented by requiring the “flatness” of the chiral angle at the border, i.e. $\xi'(R_{WS}) = 0$, as discussed also in [Nym70]. However, because of the impossibility of accommodating a unit topological charge in the finite volume with this boundary conditions, the form of the solution must accordingly be modified into

$$U(r) = h(r) \ U_0(r) \quad (1.3)$$

where $h$ depends in principle on the radial distance and is determined by requiring $B = 1$ inside the cell. The main predictions of the model as developed in this thesis are summarized in table (1.1), where the nucleon and $\Delta$ masses are taken as inputs in the model.

These predictions include a swelling of the nucleon, an increase of the nucleon magnetic moment and a consistent quenching of the axial coupling constant. These effects are found to be more contained when a finite mass for the pion is allowed\footnote{In this case, because the pion has a finite range, the nucleon is less sensitive to the presence of neighboring nucleons.}. A binding energy of the correct magnitude is also found, even if at a much too low density. This work is published in [Amo98].

The second part of this thesis concerns the quark-soliton model of the nucleon\footnote{A connection between the models can actually be established for very large solitons, in which case the valence quarks lie very close the the negative energy continuum.}. This is a field theoretical model, which was first introduced by Diakonov and collaborators in the late eighties [Dia88].
Figure 1.1  Chiral angle in the Skyrme model as a function of the distance both in free space (solid line) and in the Wigner-Seitz approximation (dashed line) for $f_\pi = 63.45$ MeV and $G = 5.346$ in the chiral limit.
Table 1.1 Predictions of the Skyrme model in free space and in the Wigner Seitz approximation, using $m_N$ and $m_\Delta$ as inputs. The density at which the in medium results are given (last three columns) can be read off from the fourth column, which provides the corresponding Wigner-Seitz radiiuses.
The lagrangian density for the quark-soliton model reads

\[ \mathcal{L} = \bar{\psi}(x) \left( i\phi - MU_5(x) \right) \psi(x) \]  \hspace{1cm} (1.4)

where \( \psi(x) \) are the quark fields and \( U_5(x) \) is a soliton coupled to the quarks:

\[
\begin{align*}
U_5(x) &= \frac{1 + \gamma_5 U(x)}{2} + \frac{1 - \gamma_5 U^\dagger(x)}{2} \\
U(x) &= \exp \left\{ i \frac{\vec{\tau} \cdot \vec{\tau}(x)}{f_r} \right\}.
\end{align*}
\]

A global color symmetry \( SU(N_c) \) is assumed, \( N_c \) being the number of colors (\( N_c = 3 \) in nature). The quarks carry also flavor indices (\( N_f = 2 \) in the present case) corresponding to the light (u,d) quarks. Notice that the lagrangian (1.4) is invariant under vector and chiral transformations\(^5\). In order to prove this statement, let us decompose the quark spinor into a left and right handed spinors, i.e

\[ \psi = \psi_L + \psi_R, \quad \psi_{L,R} = \frac{1 \pm \gamma_5}{2} \psi, \]

and consider the transformation of the fields

\[
\begin{align*}
\psi_L &\rightarrow L \psi_L, \quad \psi_R \rightarrow R \psi_R \\
U &\rightarrow R U L.
\end{align*}
\]

L and R here are global \( SU(2) \) transformations, i.e.

\[
L = \exp \left\{ i \frac{\vec{\tau} \cdot \vec{\alpha}_L}{2} \right\}, \quad R = \exp \left\{ i \frac{\vec{\tau} \cdot \vec{\alpha}_R}{2} \right\}.
\]

The invariance of the lagrangian becomes manifest when the various terms are decomposed in terms of the left and right components, i.e.\(^6\)

\[
\begin{align*}
\left( \bar{\psi} \gamma^\mu \psi \right)' &= \bar{\psi}_L L^\dagger \gamma^\mu L \psi_L + \bar{\psi}_R R^\dagger \gamma^\mu R \psi_R = \bar{\psi} \gamma^\mu \psi \\
\left( \bar{\psi} U_5 \psi \right)' &= \bar{\psi}_L L^\dagger R U^\dagger R \psi + \bar{\psi}_R R^\dagger R U L^\dagger L \psi_L = \bar{\psi} U_5 \psi.
\end{align*}
\]

\(^5\)These are also symmetries of the QCD lagrangian, if the current quarks are taken to be massless, which is effectively true in the nuclear domain of (u,d) quarks.

\(^6\)Since the transformations are global this results hold also for terms which contain derivatives of the fields.
Notice that, as a result of the vector invariance, degenerate multiplets of mesons are present (for example, $\pi^0$, $\pi^+\pi^-$). This degeneracy is lifted in nature, in part by the electromagnetic interactions and in part because of the difference in the current quark masses $- m_u \neq m_d$.

Moreover, the presence of a large constituent quark mass, $M \approx 350$ MeV, signals that chiral symmetry is spontaneously broken. However, in this model, no mechanism for the spontaneous breakdown of chiral symmetry is specified, and $M$ is therefore taken to be a parameter to be fixed phenomenologically.

Even if the present model cannot be directly derived from the underlying theory of the strong interactions QCD, since this would entail solving the non-perturbative regime of QCD, it is nonetheless reasonable to believe that the lagrangian (1.4) provides a low-energy effective model of QCD. As a matter of fact, it possesses the basic symmetries of the underlying theory — chiral symmetry above all — and it is cast in terms of the relevant degrees of freedom. These are constituent quarks and pions, where the pions are here the Goldstone bosons associated with the spontaneously broken chiral symmetry.

One of the most interesting features of the model is the possibility of allowing stable solitonic configurations of the meson field, in which the valence quarks are bound. Unlike the Skyrme model, where the stability of the solutions is provided by a term introduced ad-hoc, this property here follows directly from the lagrangian (1.4) once the contributions stemming from the vacuum are correctly taken into account. Indeed, a sizeable part of the energy in the model is due the excitation of pairs of virtual quarks out of the Dirac sea.

---

7 The Nambu-Jona Lasinio model, which is closely related to the present model, provides a dynamical mechanism for the spontaneous breaking of chiral symmetry, as did the original $\sigma$ model [Gel60].

8 These are massless and spinless excitations which appear when a continuous global symmetry is spontaneously broken. The proof of this statement is known as Goldstone theorem [Gol61].

9 By soliton it is generally meant a localized solution of the equations of motion.
The field theory defined by Eq. (1.4) contains dynamical quark loops. We refer to the contribution of these loops as the "vacuum energy".

The evaluation of the vacuum energy can be performed either in terms of Feynman diagrams (in fig. 1.2 the lowest order contributions are displayed), or directly in terms of a path integral. In the latter approach one integrates out the quark fields. This is the approach taken in the present case and it essentially corresponds to a calculation of the effective action for the model.

The effective action for this model is thus obtained in the form

\[
\exp \{ i S_{\text{eff}}[U] \} = \int [d\bar{\psi}[d\psi] \exp \left\{ i \int d^4x \ \bar{\psi}(x) \left[ i\partial - MU_5(x) \right] \psi(x) \right\} = [\det (i\partial - MU_5(x))]^{N_\xi}, \quad (1.5)
\]

which, for static configurations, is easily related to the vacuum energy

\[
S_{\text{eff}}[U] = -T \int d^3x \ H_{\text{eff}} = -T E_{\text{vac}}.
\]

A detailed calculation of the vacuum energy is given in chapter 3; here we prefer to focus only on the physics. In order to obtain the total energy of the system (soliton + quarks) one still needs to add to $E_{\text{vac}}$ the energy carried by the valence quarks. This is calculated by finding the eigenfunctions of the quark hamiltonian

\[
H = -i\alpha \cdot \nabla + \beta M \left[ \cos \theta(r) - i\gamma_5 \tau \cdot \hat{r} \sin \theta(r) \right], \quad (1.6)
\]

obtained by considering a spherically symmetric configuration for the pion field, normally known as "hedgehog ansatz":

\[
U(x) = \exp \{ i\tau \cdot \vec{\pi}(x) \} = \exp \{ i\tau \cdot \hat{r} \ \theta(r) \} . \quad (1.7)
\]

The field $\theta(r)$ in this expression is usually referred to as the "chiral angle".

Notice also that a regularization of otherwise divergent quantities stemming from 1-loop diagrams is also required: in general, this introduces a dependence upon a
Figure 1.2  Feynman diagrams corresponding to the lowest order vacuum contributions. The solid and dashed lines represent the quarks and the pions respectively.

cutoff, which is determined by the requirement of reproducing the experimental pion decay constant, i.e.

\[ f_\pi(\Lambda_{\text{cutoff}}) = f_\pi^{(\text{exp.})}. \]

Typically a value \( \Lambda_{\text{cutoff}} \approx 600 \text{ MeV} \) is required.

A qualitative insight on the problem can be gained by performing a variational study of the energy, obtained by choosing an exponential ansatz for the chiral angle, i.e.

\[ \theta(r) = \pi e^{-r/r_0}, \]

where \( r_0 \) is a variational parameter (see fig. (1.3)). For small values of \( r_0 \) the quarks are bound in a narrow potential well, and as a result the bound state tends to be close to the positive energy continuum (of course, if the well is too narrow, the bound state disappears). Given the dynamical nature of the "potential" in this description, i.e. the meson fields, the energy carried by the vacuum in this regime is small. In the opposite regime, for large values of \( r_0 \), most of the energy is now carried by the mesons, while the quarks sit in a tightly bound state. As a result of the combined behaviour of the valence and vacuum contributions a stable solution is found in an intermediate regime, corresponding to \( r_0 \approx 0.6 \text{ fm} \). Of course, this variational analysis is only meant to provide a qualitative estimate: in chapter 3, this solution is
Figure 1.3 Variational estimate of the energy. Total energy (solid line), vacuum (dashed line) and valence (dotted line) energies. $x_0 = Mr_0$. 

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found directly by solving the equations of motion.

The property of being able to bind the quarks in a soliton, without having to introduce other forces, is extremely appealing, because it opens the way to a description of the nucleon in terms of chiral fields\textsuperscript{10}. The absence of a confining mechanism in the model is not crucial, because the quarks are strongly bound by the fields and therefore are effectively "confined" in a region of finite volume. Of course, a different situation would occur when trying to describe excited states of the nucleon: in this case, the finite depth of the potential well would not be sufficient to bind the quarks anymore, and the model could not be considered realistic without the introduction of confining forces.

A self-consistent solution for the quark and soliton fields, reproducing the results in \cite{Dia88}, is displayed in fig. (1.4). In the upper plot the upper and lower component of the quark spinor ($u(r)$ and $v(r)$ respectively) are shown, whereas in the lower plot $\cos \theta(r)$ and $\sin \theta(r)$ (solid and dashed lines) are plotted. The horizontal line marks the location of the bound state, in dimensionless units, i.e. $\epsilon = E_{\text{val}}/M$.

Let us now consider the extension of the model to finite density in the Wigner-Seitz approximation\cite{Wig33}. Within this description, a single nucleon is enclosed in a spherical cell of radius $R_{WS}$. The density of the system is therefore directly related to $R_{WS}$ through the relation $\rho = 3/(4\pi R_{WS}^3)$. In the limit $R_{WS} \to \infty$ the case of a nucleon in free space must therefore be recovered\textsuperscript{11}. In this case the boundary conditions for the fields require

$$u(r), v(r) \to 0 \quad , \quad \theta(r) \to 0 .$$

\textsuperscript{10}Because the hamiltonian does not commute with isospin and total angular momentum when the usual hedgehog configuration for the soliton is chosen, but rather with the sum of the two, i.e. the "grand spin", these solutions are actually a mixture of nucleon and $\Delta$ states. The physical states are then obtained by performing a semiclassical projection over the mean field solutions.

\textsuperscript{11}Such a situation is described in Fig.1.5, where the free space solution for the chiral angle is compared with the similar solution in the Wigner-Seitz approximation with a cell radius $R_{WS} = 5 \text{ fm}$. The solutions are practically indistinguishable.
Figure 1.4 The upper and lower component of the quark spinor (upper plot) in the corresponding meson fields (lower plot). The horizontal line corresponds to the presence of a bound state ($\epsilon = E_{\text{val}}/M \approx 0.42$).
Figure 1.5  Comparison between the chiral angle in free space (solid line) and in the Wigner-Seitz approximation, with $R_{WS} = 5$ fm (dot and dashed lines).
In a finite volume one needs to impose average boundary conditions for the quark and meson fields; these b.c. reflect the presence of surrounding nucleons. Figure 1.6 illustrates schematically the situation at finite density: here two nucleons are separated by a distance $2\ R_{WS}$; notice that the quarks, at least at low density, are effectively confined into small regions (the shaded areas), whereas the pion, which is massless if chiral symmetry is exact, can reach out of these regions.

Since we want to avoid solving the dynamics over all the space, we consider a cell of radius $R_{WS}$ and enforce on the surface of the cell the boundary conditions dictated by the presence of the other nucleons. By looking at the figure it is easy to convince oneself that the appropriate boundary conditions are:

$$\frac{d\rho_B}{dr}(R) = 0 \quad \text{(1.8)}$$
$$\theta(R) = 0 \quad \text{or} \quad \theta'(R) = 0 \quad \text{(1.9)}$$

where $\rho_B(x) = \psi^\dagger \psi$ is the baryon density and

$$\psi = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} u(\tau) \xi \\ v(r) i\vec{\sigma} \cdot \vec{\xi} \end{pmatrix}$$

is the valence quark spinor, normalized to give a unit baryon charge inside the cell,
The reader will notice that, on one hand, the flatness of the baryon density at the surface of the cell is imposed while, on the other hand, two different boundary conditions for the chiral angle are explored. Clearly our physical intuition would suggest that the flatness of the chiral angle at the cell surface should be enforced to account for the presence of the surrounding nucleons, as it is done for the baryon density. However, the choice of a chiral angle vanishing at the cell surface has been preferred in the literature when topological models of the nucleons are considered (see for example [Wus87, Hah87]). Both boundary conditions have been investigated here, finding only a moderate dependence of the solutions.

While we postpone to chapter 3 a detailed discussion of the boundary conditions and of the numerical solution to the problem, we stress a few important points. First of all, we notice that the choice of a spherical cell reflects the isotropy of the nuclear system. Also, if nuclear matter were a crystal instead of a fluid, then long-range correlations would also be present between nucleons; however, these long-range correlations are here "washed-out" by considering only average boundary conditions, which essentially retain only a next-neighbor interaction. In this way, therefore, the nuclear many-body problem is transformed into a much simpler 1-body problem. It is important to realize that the Wigner-Seitz approximation does not require any compromise on the field-theoretical nature of the model: as a matter of fact, unlike in a non-relativistic quantum mechanical approach, where the number of particles is at all times fixed, in this case the "vacuum" displays a truly dynamical behaviour. The virtual quark pairs, which can be excited or absorbed in the Dirac sea, will also feel the presence of a finite density medium. This aspect, of course, can be fully implemented only in a model, such as the chiral soliton model of the nucleon, which

\[ \int_{cell} d^3 x \, \psi^\dagger \psi = 1. \]
allows the calculation of vacuum effects. In this respect, the analysis which has been carried out in this work and was published in [Amo00], represents the first of this kind within the Wigner-Seitz approximation. All the previous analysis have been restricted to consider the valence contributions and have assumed a fixed form for the meson contributions.

Finally we notice that within the method employed here no new degrees of freedom or parameters are needed: the system can be described in terms of the same degrees of freedom as in free space.

It is therefore clear that in this picture the presence of a stable bound state, corresponding to the modified boundary conditions, if found, can be related to the in-medium nucleon (notice however that, as in free space, the hedgehog solutions don't have definite spin or isospin, and therefore they describe a mixture of nucleon and Δ states, rather than a “pure” nucleon). The calculation proceeds in a similar fashion as the one previously illustrated for the free nucleon, with some important differences. For example, the expression (1.5) for the vacuum energy becomes in the Wigner-Seitz approximation:

\[
S_{\text{eff}}[U] = -\frac{i}{2} N_c \text{tr} \left\{ \log \left[ \Box + M^2 + iM\partial U_5(x) \right]_{\text{ws}} - \log \left[ \Box + M^2 \right]_{\text{free}} \right\}
\]

\[\equiv -\frac{i}{2} N_c \text{tr} \log \left[ \frac{\Box + M^2 + iM\partial U_5(x)}{\Box + M^2} \right]_{\text{ws}} \tag{1.10}\]

This expression formally resembles the expression (1.5) for the vacuum energy and can be evaluated by performing a non-local expansion, along the lines which have been already explained in this Introduction. However, unlike in the previous case, where virtual quarks of arbitrary momentum could be excited out of the Dirac sea, here only specific momenta are allowed; this is enforced by expanding the vacuum energy in terms of an orthonormal and complete basis in the Wigner-Seitz sphere. The integrals over momentum, as a result, will be replaced by sums over the eigenmodes in the passage from free space to a cell of finite-volume. Once this expansion is
Figure 1.7 Dependence of the total energy (upper plot) and of the binding energy when the center-of-mass motion is taken into account (lower plot) upon the cell radius.

performed, an expression for the effective action can be finally obtained and used to self-consistently calculate the solutions.

In the upper plot of figure (1.7) we display the total energy of the solution corresponding to the Wigner-Seitz boundary conditions (denoted as type II in chapter 3) as a function of the cell radius. The solid and dashed lines in the plot represent two possible solutions corresponding to the boundary conditions enforced on the problem. Physically, the existence of multiple solutions, which mathematically emerge from the quadratic equation (1.8), reflects the possible formation of a band. This is a common feature in solid state problems, and it is due to the presence of a crystalline, i.e. periodic, structure.

However, the presence of a band structure in nuclear matter cannot be considered
very realistic. As a matter of fact, the simple application of the Wigner-Seitz approximation to the description of nuclear matter yields, already at moderate density — below $R_{WS} \approx 2\,\text{fm}$, a band with a non-negligible width. Unfortunately, such an outcome cannot be accepted acritically. As already said, nuclear matter is not a crystal since no long-range ordered structure is present in it. Moreover, even if such an ordered structure was present, a realistic calculation of the band would still not be possible disregarding the confinement of color, which is not enforced in the model lagrangian. Color confinement would suppress the sharing of quarks between different nucleons, and would therefore favor those levels in the band corresponding to more deeply bound quarks. In the case of figure (1.7), such a configuration corresponds to the solid curve in the upper plot, i.e. the top of the band. Interestingly, once the contributions stemming from the center-of-mass motion (lower plot) are taken into account, the level with the lowest baryon density at the cell boundary turns out to provide also the lowest energy. In the absence of a confining mechanism within the model, this will be considered as the best candidate for the description of the in-medium nucleon. Notice that the solution corresponding to the dashed line describes a swollen nucleon, in which, at finite density, the quarks have a larger probability to be found at the surface of the cell, and therefore to be “shared” by neighbor nucleons.

It is very interesting to notice that, unlike previous applications of the Wigner-Seitz approximation (see, for example, [Hah87]), the model developed here is able to produce a shallow minimum in the binding energy at finite density ($R_{WS} \approx 1.8\,\text{fm}$), as a result of the different behaviour of the vacuum and valence contributions to the energy with respect to the density. This implies that nuclear matter is self-bound in the present approach, in accord with the experiment. We believe that the occurrence of nuclear saturation at too low density is a result of many factors. A first factor is the chiral limit employed in the calculation. In fact, since pions are massless here and
extend to large distances from the center of the nucleon, the presence of a neighbor nucleon will be felt at lower density (i.e. larger $R_{WS}$) than in the case in which pions are massive. A second factor is the lack of confinement in the model. In fact, since the quarks are dynamically bound to the soliton, i.e. a "potential" of finite depth, but not strictly confined in the interior of the nucleon, they are more sensitive to modification of the environment at finite density. Finally, one must realize that the binding energy is obtained through the cancellation of large quantities. For this reason, it is also possible that higher order corrections in the large $N_c$ expansion, which are negligible on the scale of the nucleon mass, could play some role in the determination of the binding energy, whose scale is in the order of few tens of MeV.

Another interesting result of this model is the suppression of the pion decay constant (see fig.(1.8)). This is in agreement with other theoretical investigations which predict an in-medium reduction — typically of the order of $20 - 30\%$ — of the pion and kaon decay constant, $f_\pi$ and $f_K$ (see for example, [Akh89, Ain87]) along with a reduction of the masses of the vector mesons ($\rho$ and $\omega$)[Bro91, Hat92], a broadening of their widths, and more in general an in-medium reduction of the quark and gluon condensates[Gas91].

Generally the experimental detection of such effects, which would provide valuable information on a strongly interacting system at finite density, is very challenging, due to the presence of many-body forces and to the strong final-state interactions. In particular, the study of the decay of vector mesons into dileptons has been suggested as a possible tool to measure in-medium modifications. In this process the final products do not experience strong final-state interactions and they can therefore carry the information from the interior of nuclei almost without distortion. Also, since vector mesons decay directly into dileptons, the change in the invariant mass can be seen in the dilepton invariant mass spectrum. The $\rho$ meson, in particular,
Figure 1.8 Dependence of the pion decay constant upon the density.

offers the advantage of being a broad resonance and therefore of being able to decay while still inside the nucleus. A theoretical analysis of this process can be found in [Fri96, Pos98, Wol99], where a phenomenological meson lagrangian has been used.

Other experiments have also been suggested; these include the subthreshold production of $\bar{p}$ and $K^-$ or the decay $\phi \rightarrow K^+ + K^-$ [Gil99, Fri99], the measurement of $g_A$ inside nuclei, or the study of the $\Delta$ excitation in nuclei. The reader can find a detailed account of the experimental status in [Ko97].

In Chapter 4 and 5 we extend the model calculations of this thesis to include the semiclassical projection of the mean field solution, which is needed in order to obtain solutions of definite spin and isospin. We also give a simple example of the modification of the quark distribution function in nuclear matter.

In table 1.2 are the results concerning the semiclassical projection of the hedgehog solutions in a simpler version of the model, i.e. the non-linear sigma model with
quarks\textsuperscript{12}. The main reason for considering a simpler model is practical: in fact the calculation of the vacuum contributions to the moment of inertia of the soliton is not very satisfactory within the non-local expansion, as pointed out in [Adj95], thus requiring the application of the Kahana-Ripka algorithm [Kah84].

The first three columns refer to the free space: the quantities depending only on the hedgehog solutions and the quantities which depend on the semiclassical projection are given in the first and second columns, respectively; the corresponding experimental values are given in the third column. Finally, in the last column, we give the results obtained by extending the Wigner-Seitz approximation to include the cranked solutions (we consider here a Wigner-Seitz radius of $R_{WS} \approx 1.5 \text{ fm}$). Notice that in the present calculation the first two entries in the table, i.e. the constituent quark mass and the pion decay constant, are parameters. It is interesting to notice that the calculation yields an increase in the $N\Delta$ splitting, as well as in the nucleon and $\Delta$ masses separately. The isoscalar and isovector magnetic moment of the nucleon are also seen to behave in opposite ways: in fact, at finite density, while the first is observed to increase, the second is seen to largely decrease. Notice that the suppression of the isovector magnetic moment $\mu_{t=1}$ is mostly due to the behaviour of its mesonic component, the quark component being rather stable. Finally, the isoscalar and isovector mean square radiuses are also seen to decrease in the medium.

Finally, another interesting application that we consider in this thesis is the calculation of the quark distribution functions in nuclear matter. The latter are introduced in the study of Deep Inelastic Scattering (DIS) processes on the nucleon (see, for example, [Wal95]). These processes are characterized by a large four-momentum transfer ($q^2 \rightarrow \infty$) and a large energy transfer ($\omega = (q \cdot P)/M \rightarrow \infty$). $q_\mu$ here is the four-momentum of the virtual photon exchanged between the leptons and the

\textsuperscript{12}The vacuum effects will not be calculated dynamically in this case, but parametrized in a given mesonic lagrangian
Table 1.2 Semiclassical quantization of the hedgehog in the non-linear $\sigma$ model with quarks. The constituent quark mass and the pion decay constant are taken as input.

<table>
<thead>
<tr>
<th></th>
<th>hedgehog</th>
<th>semiclassical</th>
<th>experimental</th>
<th>$R_{WS} = 1.5$ fm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M^+_q$ (MeV)</td>
<td>400</td>
<td>400</td>
<td>-</td>
<td>400</td>
</tr>
<tr>
<td>$f^+_q$ (MeV)</td>
<td>93</td>
<td>93</td>
<td>93.5</td>
<td>93</td>
</tr>
<tr>
<td>$E_{CM}$ (MeV)</td>
<td>252.8</td>
<td>-</td>
<td>-</td>
<td>312.2</td>
</tr>
<tr>
<td>$m_\Delta - m_N$ (MeV)</td>
<td>-</td>
<td>160.1</td>
<td>193.5</td>
<td>267.1</td>
</tr>
<tr>
<td>$m_N$ (MeV)</td>
<td>-</td>
<td>1261.7</td>
<td>-</td>
<td>1439.7</td>
</tr>
<tr>
<td>$m_\Delta$ (MeV)</td>
<td>-</td>
<td>928.82</td>
<td>938.5</td>
<td>993.9</td>
</tr>
<tr>
<td>$f_{tot}$ (fm)</td>
<td>-</td>
<td>1.85</td>
<td>-</td>
<td>1.1</td>
</tr>
<tr>
<td>$\mu_{I=0}/\mu_N$</td>
<td>-</td>
<td>0.36</td>
<td>0.88</td>
<td>0.6</td>
</tr>
<tr>
<td>$\mu_{I=1}/\mu_N$</td>
<td>5.6</td>
<td>5.6</td>
<td>4.71</td>
<td>2.6</td>
</tr>
<tr>
<td>$\mu_{I=1}/\mu_N$</td>
<td>3.8</td>
<td>-</td>
<td>-</td>
<td>1.1</td>
</tr>
<tr>
<td>$\mu_{I=1}/\mu_N$</td>
<td>1.8</td>
<td>-</td>
<td>-</td>
<td>1.5</td>
</tr>
<tr>
<td>$\sqrt{\langle r^2 \rangle} (\text{fm})$</td>
<td>0.76</td>
<td>0.76</td>
<td>0.72</td>
<td>0.66</td>
</tr>
<tr>
<td>$\sqrt{\langle r^2 \rangle} (\text{fm})$</td>
<td>-</td>
<td>1.38</td>
<td>0.88</td>
<td>0.72</td>
</tr>
</tbody>
</table>

The cross-section for the process in Fig. 1.9 can be shown to have the form

$$
\frac{d^2\sigma}{d \Omega \, d\epsilon'} = \frac{\sigma_{Mott}}{M} \left[ W_2(\omega, q^2_\mu) + 2 W_1(\omega, q^2_\mu) \tan^2 \frac{\theta}{2} \right],
$$

(1.11)

where the Mott cross-section is given by

$$
\sigma_{Mott} = \frac{\alpha^2 \cos^2 \theta/2}{4 \epsilon^2 \sin^4 \theta/2};
$$

(1.12)

and the hadronic tensor is written in terms of $W_{1,2}(\omega, q^2_\mu)$ in the most general form

$$
W^{\mu\nu} = - \left( g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2_\alpha} \right) W_1(\omega, q^2_\mu)
+ \frac{1}{M^2} \left( p^\mu - \frac{q^\mu p}{q^2_\alpha} \right) \left( p^\nu - \frac{q^\nu p}{q^2_\alpha} \right) W_2(\omega, q^2_\mu).
$$

(1.13)

All the information on the structure of the nucleon is contained in the functions $W_1$ and $W_2$. 

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In the DIS regime it is observed experimentally that the structure functions of the nucleon display an approximate scaling behavior, becoming functions of only one variable, the Bjorken variable, which is defined as

\[ x_B = -q^2_{\mu}/(2q \cdot P). \]

A successful explanation of this phenomenon is provided by the parton model, in which the DIS is described as the result of the incoherent scattering off the individual constituents of the nucleon, i.e., the quarks. In a frame in which the nucleon moves very fast, the infinite momentum frame, the quark distribution functions simply count the number of quarks of a given species (i.e., of a given flavor or polarization) carrying a fraction \( x_B \) of the nucleon's momentum.

Let us look at Fig. 1.9 and assume that the quark absorbing the virtual photon carries a fraction \( \xi \) of the nucleon's momentum \( (0 \leq \xi \leq 1) \), i.e., that \( k^\mu = \xi P^\mu \). By squaring the quark momentum one has that

\[
k_{\mu}^2 = (\xi P_\mu + q_\mu)^2 = \xi^2 P_\mu^2 + q_\mu^2 + 2\xi q_\mu P_\mu.
\]

This shows that, in the DIS regime, where it is possible to neglect \( P^2_\mu \) and \( k_{\mu}^2 \)

one obtains that \( \xi = x_B \), as anticipated: the Bjorken variable yields the fraction of momentum of the nucleon carried by the quark in the DIS process.

By assuming the scattering from Dirac quarks, carrying a pointlike charge \( q_i |\epsilon| \), it is then possible to write the hadronic tensor for the nucleon, \( W^{\mu\nu} \), as the result of the incoherent sum over all the individual hadronic tensors of the quarks:

\[
W^{\mu\nu} = \sum_i \int d\xi \ D_i(\xi) \ W^{(i)}_{\mu\nu},
\]

where \( D_i(\xi) \) is the quark distribution function, yielding the number of quarks with quantum numbers \( i \) with a fraction \( \xi \) of the momentum of the nucleon. By explicitly calculating \( W^{(i)}_{\mu\nu} \) one is then able to find an expression for the nucleon structure.
functions

\[ F_1(x_B) = 2 W_1 = \sum_i q_i^2 D_i(x_B) \]  \hspace{1cm} (1.16)

\[ F_2(x_B) = \frac{\omega}{M} W_2 = \sum_i q_i^2 x_B D_i(x_B) . \] \hspace{1cm} (1.17)

The experimental DIS structure functions for electron scattering from the nucleon are found to obey these scaling relations.

Experimentally it has also been found that the distribution functions of quarks inside a nucleus are significantly different from those in the nucleon (this was first discovered by the European Muon Collaboration (EMC)[Aub83]). The experimental data on the EMC effect are shown in Fig. 5.3 and Fig. 5.4.

Among the different theoretical scenarios which have been suggested in order to explain this behavior are the nuclear binding energies, the modification of nucleon structure and the presence of multiquark states in nuclei. The reader can find an account of these approaches in [Rob90, Gee95, Pil00].

\textit{Figure 1.9} Kinematics of the Deep Inelastic Scattering.
Notice that the quark distribution functions carry information on the non-perturbative regime of the strong interaction and that such a regime is not presently accessible directly through QCD, but only through models. The quark-soliton model of the nucleon, in particular, is a field theoretical model, which has chiral symmetry built-in and it is formulated in terms of the relevant degrees of freedom, i.e. constituent quarks and pions. It is also important to have a covariant model of the nucleon: this allows one to transform between different frames of reference (i.e. the infinite-momentum frame, where the distributions are naturally introduced, and the nucleon rest frame, where the model is generally solved). Also, the contribution of the “sea” (i.e. the virtual quark pairs in the nucleon) to the quark distribution functions can be calculated in this model; however, we will confine ourselves to discuss only the valence quark contributions.

The main goal of Chapter 5 is to offer a simple picture of the modification of the quark distribution functions in nuclei, in the chiral quark-soliton model of the nucleon, where the Wigner-Seitz calculation of the present thesis describes the modification of the properties of the nucleon in nuclear matter. These should be looked at as preliminary exploratory calculations whose main purpose is to illustrate that it is possible to correlate the modification of the DIS structure functions to the static properties of the nucleon in the nucleus within this covariant chiral quark soliton field theory framework.13

Chapter 5 is organized in two parts. In the first part we rederive some of the results in [Dia96, Dia98], concerning the calculation of the quark distribution function in the chiral quark-soliton model of the nucleon.

13In [Wal95], pag. 453, it is stated: “We have presented two distinct approaches to the nuclear many-body problem; one starts in the deep-inelastic regime where the quark/gluon degrees of freedom are manifest, and the other starts from the nuclear domain where the hadrons are the appropriate degrees of freedom for strong-coupling QCD. . . . These two descriptions have to be tied together. To do this, it is essential to have realistic, relativistic models and calculations that can be taken back and forth between the different frames. This is an essential problem for CEBAF whose goal is to study the transition between these descriptions of the nucleus.”
An expression for the distribution function for the valence quarks has been rederived in this model in the form:

$$D_i(x_B) = \frac{N_c M_N}{4\pi} \int \frac{d^3k}{(2\pi)^3} \sum_{\text{occ.}} \delta(M_N x_B - k - E_n) \cdot \phi_n^i(k)(1 + \gamma^0 \gamma^3)\gamma^0 \Gamma_i \phi_n(k) .$$  

(1.18)

Here $\phi_n(k)$ are the Fourier transforms of the valence quark wave-functions, in the rest frame of the nucleon. These must be calculated self-consistently as previously explained. $E_n$ and $k = (k_x, k_y, k_z)$ are the energy and momentum of the bound quark, respectively. Notice that $\Gamma^i \equiv \gamma^0 (1 \pm \gamma_5) / 2$ allows us to select a quark with a given polarization (in the unpolarized case $\Gamma^i = \gamma^0$). The $z$-axis is here chosen along the direction of the momentum of the nucleon, i.e. $\vec{P}_N = (0, 0, P_N)$ in the infinite momentum frame.

Notice that if a model of the nucleon in nuclear matter is available, this can in principle be used to investigate the modification of the quark distribution functions at finite densities. As an example, we have applied the Wigner-Seitz solutions, previously obtained, to eq. (1.18) and calculated the modification of the distribution functions for a nucleon at rest in nuclear matter. In fig. (1.10) we show the valence quark contribution to the isosinglet distributions $n(x_B) = u(x_B) + d(x_B) - \bar{u}(x_B) - \bar{d}(x_B)$ (left figure) and $x_B n(x_B)$ (right figure). The solid and dashed lines refer to free space and to $R_{WS} = 2$ fm, respectively. Notice that the distribution $n(x_B)$ fulfills the baryon number sum rule:

$$\int_0^1 dx_B \ n(x_B) = N_c .$$  

(1.19)

These calculations of the effect of the Wigner-Seitz approximation on the quark distribution functions are new.

Notice that the effects stemming from the average Fermi motion of the nucleon in nuclear matter have been neglected in obtaining these results. Fig. 1.11 shows that

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Figure 1.10  Valence quark contribution to the isosinglet distributions $n(x_B)$ and $x_B n(x_B)$. The solid and dashed lines refer to free space and to $R_{WS} = 2$ fm, respectively.

Figure 1.11  Ratio of the isosinglet distributions $n(x_B)$ in the Wigner-Seitz approximation and in a free nucleon, as given in the previous Figure.
the quark distribution functions in the Wigner-Seitz approximation are very sensitive to the density: in particular the ratio \( n_{WS}(x_B)/n(x_B) \) is suppressed at small \( x_B \) and grows rapidly at large \( x_B \).\(^{14}\) We believe that these large effects are due to the absence of confining forces in the model which causes the quark wave functions to be very sensitive to the boundary conditions.\(^{15}\)

In the second part of Chapter 5 we discuss the application of the convolution model to the calculation of the quark distribution function. In this approach the probability of finding a quark carrying a given fraction of the nucleon momentum is calculated by summing incoherently over the nuclear constituents (the nucleons). The diagram describing this process is shown in Fig. 1.12.

In the infinite momentum frame one writes:

\[
D_i(x_B) = \int dy \int \frac{d^3P_N}{(2\pi)^3} \delta \left( y - \frac{AP_N^i}{PA} \right) \langle A_v | c_{P_N}^i c_{P_N} | A_v \rangle \\
\cdot \int \frac{d^3k}{(2\pi)^3} \delta \left( x_B - \frac{k^i A}{PA} \right) \langle N_v | a_k^+ a_k | N_v \rangle \\
= \int \frac{dy}{y} \int \frac{d^3P_N}{(2\pi)^3} \delta \left( y - \frac{AP_N^i}{PA} \right) \langle A_v | c_{P_N}^i c_{P_N} | A_v \rangle \overline{D}_i \left( \frac{x_B}{y}, P \right), \quad (1.20)
\]

where \( y \) is the fraction of the momentum of the nucleus, \( P_A \), assumed to be along the \( z \) direction, carried by a nucleon; \( c_{P_N} \) and \( c_{P_N}^+ \) are the annihilation and creation operators for a nucleon with momentum \( P_N \). \( |A_v\rangle \) describes a nucleus moving with velocity \( v \rightarrow 1 \) and \( |N_v\rangle \) describes a nucleon carrying a momentum \( P_N \) in the same frame. Notice that \( \overline{D}_i (x_B, 0) \) is the quark distribution function for a single nucleon calculated in the previous section. Physically eq. (1.20) expresses the assumptions made in the convolution model: the probability of finding a quark carrying a fraction of the momentum of the nucleon (the matrix element \( \langle N_v | a_k^+ a_k | N_v \rangle \) has to be weighted over the probability of finding a nucleon with a given momentum inside the nucleus.

\(^{14}\)Note that at large \( x_B \) one is really considering the tail of the distribution in Fig. 1.10. Small changes in the distributions in this region can produce sizable effects on the ratio considered.

\(^{15}\)Since the soliton does not absolutely confine the quarks, the quarks acquire larger momentum components at finite density, thus leading the behavior shown in the figure.
(the matrix element $\langle A_{\nu}|c_{P_{N}}^{\dagger}c_{P_{N}}|A_{\nu}\rangle$).

In Chapter 5 we show that the distribution function in the *convolution model* can be cast in the manifestly covariant form

$$D_{i}(x_{B}) = -i \int dy \int \frac{d^{4}P}{(2\pi)^{4}} \delta(y - y^{(0)}) \text{Tr} \left[ \not{p} \gamma_{0} \Gamma_{i} S_{F}(P_{0}, \vec{P}, \vec{P}) \right]$$

$$\cdot N_{c} \int \frac{d^{4}k}{(2\pi)^{3}} \sum_{\text{occ.}} \delta(x_{B} - x_{B}^{(0)}) \delta(k_{\mu}n^{\mu}) \bar{\phi}_{n}(k) \not{p} \gamma_{0} \Gamma_{i} \phi_{n}(k), \quad (1.21)$$

where $n_{\mu}$ is a four-vector which, in the infinite momentum frame reads $n^{\mu} = (1, 0, 0, 0)$.

We also define in that frame

$$x_{B}^{(0)} = \frac{k^{2}A}{P_{A}}$$

$$y^{(0)} = \frac{AP_{K}^{\mu}}{P_{A}},$$

i.e. the fraction of momentum of the target carried by the quark and by the nucleon respectively.

Notice that $S_{F}(P_{0}, \vec{P}, \vec{P})$ is the Fourier transform of the Green's function of the
nucleon in presence of static fields. Under a Lorentz boost this transforms as

\[ S_F^{(RF)}(P'_0, \bar{P}'; \bar{P}') = \Lambda^{-1}_{\frac{1}{2}}(v)S_F(P_0, \bar{P}; \bar{P})\Lambda_{\frac{1}{2}}(v). \]  

(1.22)

where \( \Lambda^{-1}_{\frac{1}{2}}(v) \) are the spinorial representation of the Lorentz group. By exploiting the property of traces the \( \Lambda^{-1}_{\frac{1}{2}}(v) \) can be brought to act in eq. 1.21 directly on the \( \gamma \) matrices, thus yielding:

\[ \Lambda^{-1}_{\frac{1}{2}}(v)\gamma^\mu\Lambda_{\frac{1}{2}}(v) = \Lambda^\mu_{\nu}\gamma^\nu n_\mu = \eta', \]  

(1.23)

which is indeed a scalar.

Given the covariant nature of this equation, an expression for the quark distribution functions in different reference frames can be easily obtained. We show, in fact, that under the assumption that the nucleus in its rest frame can be described as a relativistic Fermi gas, the distribution function takes the form:

\[ D_i^{(A)}(x_B) = \Omega \int dy \int \frac{d^3P}{(2\pi)^4} \delta(y - y^{(0)}) \frac{8\pi}{E_P} \frac{P_z}{E_P} \theta(P_F - P) \cdot N_c \int \frac{d^3k}{(2\pi)^3} \delta(x_B - x_B^{(0)}) \frac{m_N}{E_P + P_z} 4\pi \left[ \frac{E_P + P_z}{m_N} (U(k)^2 + V(k)^2) \right. 
\]

- \[ 2U(k)\bar{\nu}(k)\hat{k}_z - 2U(k)V(k) \frac{\hat{k} \cdot \bar{P}}{m_N} \frac{E_P + m_N + P_z}{E_P + m_N} \left. \right], \]  

(1.24)

where \( U(k) \) and \( V(k) \) are the Fourier transforms of the upper and lower components of the valence quark wave functions, calculated in the frame in which the nucleon is at rest. In this expression \( \hat{k} \equiv \bar{k}/k \) is a unit vector along the direction of the quark momentum. \( \bar{P} \) is the momentum of the nucleon. It is straightforward to check that eq. (1.24) fullfills the baryon number sum rule. Notice that the Lorentz transformation from the rest frame of the nucleus to the rest frame of the nucleon, brings an explicit dependence on the momentum of the nucleon in the square brackets. This effect is usually neglected by convoluting over the quark distribution functions in a free nucleon.
An example of the application of the formula in eq. (1.24) is shown in Fig. 1.13. The energy of the nucleon participating to the DIS process is given by (for a discussion on this point see Chapter 5)

\[ P_0 \approx m_N + \delta = m_N + \delta_B - \langle E_F - T \rangle, \tag{1.25} \]

where \( E_F \) is the Fermi energy of the nucleus, i.e. the energy of the last filled level, and \( T \) is the kinetic energy of the nucleon participating to the process.

A separation energy \( \delta \approx -70 \text{ MeV} \), taken from [Die91], is used. The Fermi momentum is assumed to be \( P_F = 270 \text{ MeV/c} \). Notice the effect of the Fermi momentum at large \( x_B \), which increases the ratio \( D_A(x_B)/D(x_B) \).\(^\text{16}\) We have not yet applied the convolution formula to the Wigner-Seitz wave functions given the present unrealistic behavior of the ratio \( D_A(x_B)/D(x_B) \) in the Wigner-Seitz description, even in absence of the Fermi motion. As already pointed out earlier, it is likely that the inclusion of a confining mechanism in this model could improve the behavior of the ratio \( D_A(x_B)/D(x_B) \).

In summary, in the present thesis, we have considered the general problem of studying the modification of the properties of the nucleon in nuclear matter. The Wigner-Seitz approximation has been applied to the Skyrme model of the nucleon and to the chiral quark-soliton model, finding in both cases that nuclear matter is self-bound. We also have extended these calculations to include the semiclassical projection of the mean field solution and thus have obtained solutions with definite spin-isospin. Finally a general expression for the parton distribution functions in nuclei has been obtained, and the modification of the parton distributions in nuclei investigated within the present framework.

\(^\text{16}\)These results essentially reproduce those of other workers in the field. See for example [Kul94].
Figure 1.13 Ratio of the quark distributions using the convolution formula (1.24).
Chapter 2

The Skyrme model in nuclear matter

2.1 Introduction

The Skyrme model has been successfully applied in the past to the description of the nucleon [Adk83, Wal95]. The "standard" lagrangian of the model reads

\[ \mathcal{L} = \frac{f^2}{4} \text{tr} \left( \partial_\mu U_0 \partial^\mu U_0^\dagger \right) + \frac{1}{32G^2} \text{tr} \left[ \partial_\mu U_0 \partial_\nu U_0^\dagger, \partial_\rho U_0 \partial^\rho U_0^\dagger \right]^2 \],

where \( G \) is the "Skyrme parameter". \( U_0 \) is an SU(2) matrix, which can be explicitly written as

\[ U_0 = exp \left\{ i \vec{r} \cdot \vec{\xi}(x) \right\} \]

in terms of the chiral angle field \( \vec{\xi} \), which is related to the pion field by \( \vec{\pi} = f_\pi \vec{\xi} \). As first noticed by Skyrme [Sky61], the lagrangian (2.1) admits stable soliton solutions, which can be classified in terms of a conserved quantity called the topological charge. The latter is then identified with the baryon number and can be calculated directly from the conserved topological current, which reads

\[ W^{(0)}_\mu \equiv \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} \text{tr} \left[ U_0^\dagger \partial^\nu U_0 \ U_0^\dagger \partial^\alpha U_0 \ U_0^\dagger \partial^\beta U_0 \right] \] (2.3)

Notice that \( W^{(0)}_\mu \) is independent of the details of the lagrangian, not being a Noether current.

The spherically symmetric "hedgehog" ansatz

\[ U_0(r) = exp \left\{ i \vec{r} \cdot \vec{\xi}(r) \right\} \]

(2.4)
is normally used to describe the soliton. Solutions with definite spin and isospin are generated by canonically quantizing the rotations of the soliton in space and isospace [Adk83].

This description can now be extended to nuclear matter [Amo98]; in the Wigner-Seitz (WS) approximation [Wig33, Nym70] a single nucleon is contained in a spherical cell of radius $R/2$ and periodic boundary conditions are imposed on the surface of the cell:

$$ \xi'(R/2) = 0 \quad . \quad (2.5) $$

Notice that in [Wus87] the WS approximation has also been studied by imposing a different boundary condition, in which the chiral field is assumed to vanish at a finite distance. This latter condition automatically ensures a unit baryon charge in the volume occupied by a single nucleon. This constraint, however, does not mimic any attraction, and leads therefore to an unsatisfactory physical description of the nuclear system, as found by Wust et al. in [Wus87]. In the present case (2.5) is a local average boundary condition, motivated by the physical assumption that the fields in the nuclear medium are shared between the nucleons. It is assumed that any long-range order averages out and that a particular soliton in the system feels the combined average effect of the other particles.

In order to comply with the requirement of having unit baryon charge in the WS cell one is forced to modify the hedgehog ansatz (2.4). We choose

$$ U(r) = h(r) \ U_0(r) \quad , \quad (2.6) $$

where $h$ depends only on the radial distance $r$. The current then takes the form

$$ W_\mu = h^6(r) \ W_\mu^{(0)} \quad . \quad (2.7) $$

The requirement

$$ B = \int_{cell} W^0(r) \ d^3x = 1 \quad , \quad (2.8) $$

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allows, in the special case in which $h$ is taken to be constant, the determination of $h$ in terms of the cell size as

$$h = \{1 - \left[2 \xi(R/2) - \sin(2 \xi(R/2))\right]/(2\pi)\}^{-1/6}.$$  \hfill (2.9)

As expected, $h$ correctly reduces to 1 for $R \to \infty$. Notice that the ansatz (2.6) preserves the conservation of the current $W^\mu$ and leaves the lagrangian invariant under the chiral transformation $U \to g_L U g_R^1$, where $g_L, g_R$ are global $SU(2)$ matrices. Finally the canonical quantization of the classical solutions is also unaffected by (2.6) due to the spherical symmetry of the solutions.

### 2.2 Chiral symmetry breaking

Chiral symmetry is only an approximate symmetry for the strong interaction: at the level of QCD the presence of small current masses for the u and d quarks explicitly breaks this symmetry. Consider therefore the Skyrme model to be an effective theory of QCD, in which the relevant degrees of freedom are not the quarks but the pions[Wal95]. The inclusion of terms which break the chiral symmetry is then needed. As a well-known result, the pion gets a mass through the usual lagrangian [Adk83, Bah88, Don92]:

$$\mathcal{L}_{\chi SB} = \frac{m_\pi^2 f_\pi^2}{2} \left( \text{tr} \left[ U \right] - 2 \right).$$  \hfill (2.10)

On a purely phenomenological base, however, more general terms are possible. As an example we have considered the lagrangian

$$\mathcal{L}_{\chi SB} = \frac{m_\pi^2 f_\pi^2}{2} \left( \text{tr} \left[ U \right] - 2 \right) \left\{ \alpha + \frac{\beta}{2} \text{tr} \left[ U \right] + \frac{\gamma}{4} \text{tr} \left[ U \right]^2 \right\}$$  \hfill (2.11)

which at the tree level provides the correct pion mass term and the unchanged $\pi\pi$ scattering amplitude, if the coefficients $\alpha$, $\beta$ and $\gamma$ are related by the constraints.
\[ \beta = 2 \left(1 - \alpha\right) \text{ and } \gamma = \alpha - 1. \]

Even if the two lagrangians (2.10) and (2.11) are very similar in the far "tail" of the soliton \((r \to \infty)\), they differ in the intermediate region and in the core of the soliton, where the pion field has more strength. In the next section the numerical results obtained using both lagrangians (2.10) and (2.11) will be given. The lagrangian (2.11) is introduced here only with the purpose of improving the phenomenology of our model, but without any more fundamental justification. Chiral perturbation theory, for example, provides a consistent scheme for obtaining an effective lagrangian describing low energy processes for pions. The Skyrme model, however, lies outside of its range of applicability, due to the fact that at short distances a gradient expansion is not meaningful.

In addition to (2.11) we have also considered a different chiral symmetry breaking term, which can be obtained from the quark-soliton model [Dia88]:

\[ \mathcal{L}_{\chi SB} = \frac{m^2_{\pi} f^2_\pi}{2} \left[ 1 + \frac{N_c}{16\pi^2 f^2_\pi} \text{tr} \left( \partial_\mu U \partial^\mu U^\dagger \right) \right] \left( \text{tr} [U] - 2 \right), \quad (2.12) \]

where \( N_c \) is the number of colors (physically \( N_c = 3 \)) and \( m_\pi \) is the pion mass.

Some effects of this lagrangian will also be discussed in the next section.

### 2.3 Equation of motion

We now assume a static configuration for the chiral angle and therefore neglect the time derivatives in the expression of the Lagrangian density

\[ \mathcal{L} = -\frac{f^2}{4} \text{tr} \left( \partial_i U \partial_i U^\dagger \right) + \frac{1}{32G^2} \text{tr} \left[ \partial_i U U^\dagger, \partial_j U U^\dagger \right]^2 \quad . \quad (2.13) \]

\(^1\)This can be easily checked by expanding in the pion field:

\[ \left( \text{tr} [U] - 2 \right) \left\{ \alpha + \frac{\beta}{2} \text{tr} [U] + \frac{\gamma}{4} \text{tr} [U]^2 \right\} \approx -\left( \alpha + \beta + \gamma \right) \frac{\pi^2}{f^2_\pi} + \frac{\left( \alpha + 7\beta + 13\gamma \right)}{12} \left( \frac{\pi^2}{f^2_\pi} \right)^2 + \ldots . \]

The tree level relations are preserved if the coefficients \( \beta \) and \( \gamma \) are chosen as previously specified.
The modified hedgehog ansatz for the chiral angle (see eq. (2.6), with $h$ given by eq. (2.9))

$$U(r) = h \left[ \cos \xi(r) + i \vec{r} \cdot \vec{r} \sin \xi(r) \right].$$  \hspace{1cm} (2.14)$$
can be then inserted in the equations (2.1) and (2.11), thus obtaining an expression of the Lagrangian density as functional of the chiral angle. As an example we can work out the form of the kinetic term:

$$\mathcal{L}_{\text{kin}} = -\frac{f_\pi^2}{4} \text{tr} \left( \partial_i U \partial_i U^\dagger \right)$$ \hspace{1cm} (2.15)$$

One needs the derivatives

$$\partial_i U = -h \vec{r} \sin \xi(r) \xi'(r) + ih \vec{r} \left[ A_{ij} + B \delta_{ij} \right]$$

$$\partial_i U^\dagger = -h \vec{r} \sin \xi(r) \xi'(r) - ih \vec{r} \left[ A_{ij} + B \delta_{ij} \right]$$

where the following definitions have been made

$$A \equiv \sin \xi(r) - r \cos \xi(r) \xi'(r)$$

$$B \equiv r \cos \xi(r) \xi'(r)$$

$$a_{ij} \equiv \delta_{ij} - \vec{r}^i \vec{r}^j.$$ 

Notice that the tensor $a_{ij}$ fullfills the properties

$$a_{ij} \vec{r}^j = 0,$$  \hspace{1cm} (2.16)$$

$$a_{ij} a_{jl} = a_{il}, \quad a_{ij} \delta_{ij} = a_{ij} a_{ji} = 2.$$ 

Notice also the $a_{ij}$ and $\delta_{ij}$ are both symmetric tensors and therefore they select only the symmetric part in the product $\tau^i \tau^j$. By using the previous expression one has

$$\partial_i U \partial_i U^\dagger = h^2 \sin^2 \xi(r) \xi'(r)^2 + \frac{h^2}{r^2} \left[ 2(A^2 + 2AB) + 3B^2 \right]$$

$$= h^2 \left[ \xi'(r)^2 + \frac{2}{r^2} \sin^2 \xi(r) \right]$$ \hspace{1cm} (2.16)$$

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The calculation of the quartic term — which however involves a bigger dose of algebra — can be performed in a completely analogous fashion. The reader interested in the details can refer, for example, to [Bah88, Wal95].

Finally we can introduce the dimensionless variables
\[ \tilde{x} \equiv \frac{G f_x r}{h^3}, \quad \tilde{\alpha} \equiv \frac{\alpha}{m^2 f^2}, \quad \tilde{\beta} \equiv \frac{\beta}{m^2 f^2}, \quad \tilde{\gamma} = \tilde{\alpha} - 1. \]
and write the Lagrangian density in the final form
\[ \mathcal{L} = \frac{G^2 f^4}{2h^4} \left\{ \xi'(r)(\tilde{x})^2 \left[ 1 + 2 \frac{\sin^2(\xi)}{\tilde{x}^2} \right] + \frac{\sin^2(\xi)}{\tilde{x}^2} \left[ 2 + 2 \frac{\sin^2(\xi)}{\tilde{x}^2} \right] \right\} - \frac{2m^2 f h^4}{G^2 f^2} (h \cos(\xi) - 1) (\tilde{\alpha} + \tilde{\beta} h \cos \xi + \tilde{\gamma} h^2 \cos^2 \xi) \right \}. \tag{2.17} \]

For a static configuration the mass of the soliton is promptly obtained as
\[ M = - \int d^3 x \mathcal{L} = -4\pi \int dr \ r^2 \ \mathcal{L}. \tag{2.18} \]

The Euler-Lagrange equation corresponding to this lagrangian reads
\[ \xi'' (\tilde{x}^2 + 2 \sin^2(\xi)) = - \sin(2\xi) \xi'' - 2 \tilde{x} \xi' + \sin(2\xi) \left( 1 + 2 \frac{\sin^2(\xi)}{\tilde{x}^2} \right) + \frac{h^5 m^2}{G^2 f^2} \tilde{x}^2 \sin \xi (\tilde{\alpha} + \tilde{\beta} + 3h^2 \tilde{\gamma} \cos^2 \xi + 2\tilde{\beta} h \cos \xi - 2\tilde{\gamma} h \cos \xi) \tag{2.19} \]
and must be solved by imposing the boundary conditions:
\[ \xi(0) = \pi, \quad \xi'(R/2) = 0. \tag{2.20} \]

Since the boundary conditions are imposed at different points, a "shooting" method is suitable for the numerical solution of this differential equation. In this approach two solutions with different slope at the origin but having the same value \( \xi(0) = \pi \) are numerically integrated up to the second boundary, \( R/2 \). If the two solutions have
opposite behaviour at \( R/2 \), i.e. if one is positive and the other negative, a solution having a slope at the origin the average of the previous two, is calculated and used to replace the one of the two solution with the same sign at \( R/2 \). This procedure is then iterated until some tolerance is met.

Notice that because of the form of the ansatz for the chiral angle, the expressions for the various nucleon and \( \Delta \) observables can be easily obtained from those in \[\text{Adk83}\], without the need of performing the algebra. These expressions, which have been rederived independently, can be found in the Appendix A.

### 2.4 Numerical results

In fig. (2.1) the chiral angle is displayed as a function of the radial distance, in the chirally symmetric case, both in free space and in the medium. The presence of different boundary conditions can be easily appreciated from this plot: while in the free space the soliton extends to all space (solid line), in the nuclear medium a single nucleon is found in a spherical cell of radius \( R/2 \), corresponding to the condition \( \xi'(R/2) = 0 \).

The results shown in table (2.1) have been obtained numerically by choosing the masses of the nucleon and of the \( \Delta \) in free space as input parameters\(^2\).

Notice that the internucleon distance, and therefore the nuclear density, is a pre-

\(^2\)Since the "hedgehog" solution does not have a definite spin and isospin, states with a definite spin and isospin are projected out by means of a semiclassical quantization[Adk83]. If \( U_0 \) is the static hedgehog solution, then

\[ U(\vec{x},t) = A(t)U_0(\vec{x})A^\dagger(t) \]

obtained by rotating the static solution, is also a finite energy solution. Here \( A(t) \) is \( SU(2) \) time dependent matrix. \( A \) is now promoted to a quantum mechanical variable and the spin and isospin generators, fulfilling the angular momentum commutation relations, are obtained in terms of \( A \). The nucleon and \( \Delta \) masses are obtained to be

\[ m_N = m + \frac{3}{4} \frac{1}{2\overline{7}} \quad , \quad m_\Delta = m + \frac{15}{4} \frac{1}{2\overline{7}} \]

where \( \overline{I} \) is the moment of inertia of the soliton and \( m \) is the static mass. More detail about the semiclassical projection of hedgehog solutions is left to Chapter 4.
Figure 2.1 Chiral angle as a function of the distance both in free space (solid line) and in the Wigner-Seitz approximation (dashed line) for $f_\pi = 63.45\text{MeV}$ and $G = 5.346$ in the chiral limit.
### Table 2.1  Predictions of the Skyrme model in free space and in the Wigner-Seitz approximation, using $m_N$ and $m_\Delta$ as inputs.

<table>
<thead>
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<th>free space</th>
<th>in medium</th>
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</tr>
</thead>
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<td></td>
<td>$m_\pi = 0$</td>
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<td>exp</td>
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<tr>
<td></td>
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<td>$\alpha = 3.$</td>
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<td>$m_\Delta$ (MeV)</td>
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<td>1231.51</td>
<td>1229.97</td>
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<td>$h$</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$R/2$ (fm)</td>
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<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$f_\pi$ (MeV)</td>
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<td>$\mu_{\Delta^{++}}$</td>
<td>3.77</td>
<td>3.97</td>
<td>4.03</td>
</tr>
<tr>
<td>$g_A$</td>
<td>0.63</td>
<td>0.65</td>
<td>0.63</td>
</tr>
<tr>
<td>$\mathcal{E}$ (MeV)</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>
The binding energy for nuclear matter can also be estimated as:

\[ E = m_N^* + \frac{3}{5} k_F^2 - m_N. \]  

The second term in the r.h.s. of (2.21) is the average kinetic energy of a nucleon and \( m_N^* \) is the mass of the nucleon in the medium. Here \( k_F R/2 = (9\pi/8)^{1/3} \). The predicted \( E \) is the correct order of magnitude (the experimental value is \( E = -15.75 \) MeV); better agreement is obtained when chiral symmetry is broken.

Another effect seen in the table is the swelling in the size of the nucleon: the magnitude of the latter depends on whether chiral symmetry is broken or not, ranging from about 10% to 20 – 25% respectively. Sick [Sick85], studying \( y \)-scaling in quasielastic electron-nucleus scattering, has estimated that this effect cannot exceed 3 – 6%.

An increase of the magnetic moment of the nucleon is also observed in the Wigner-Seitz approximation. It is well known that most nuclear magnetic moments fall inside the Schmidt lines, however this does not necessarily imply a quenching of the intrinsic magnetic moment of the nucleon (configuration mixing and exchange currents could be invoked to explain such an effect). Some authors, for example, have theoretically predicted an increase of the intrinsic magnetic moments in terms of the quark structure of the nucleon [Kar84]. Mulders [Muld88] has estimated the ratio \( \mu_N/\mu_N^* \) between the magnetic moment of the nucleon in free space and in a nucleus from the analysis of exclusive electron scattering \( ^6Li(e,e'p) \) to be about 0.85. Our results range from 0.5 (in the chiral limit) to 0.8 (using (2.11)).

It is particularly interesting to notice the quenching of the axial coupling in the
medium, which has been both experimentally observed and theoretically predicted [Tow87]. Once again, chiral symmetry breaking seems to play an important role in determining the quenching of $g_A$.

Finally, a decrease of the mass of the $\Delta$ of about 10% in the medium is also observed in this model. Experimentally, the $\Delta$ peak in inclusive electron scattering from light nuclei at low $Q^2$ is found to be approximately 30 MeV lower in energy [Sea89]. The authors in [Sea89] have also observed that, at higher $Q^2$, a higher invariant mass is found for the $\Delta$.

A similar analysis has also been performed by choosing the lagrangian of (2.12) to break chiral symmetry. In this case, however, the comparison with the chiral invariant results shows no appreciable effect on the nuclear density and on the binding energy.

2.5 Discussion

The Wigner-Seitz approximation applied to the Skyrme model provides an interesting framework for studying the modifications of the nucleon in the nuclear medium. In our description the many-body problem is embodied in the local average boundary condition on the field and essentially reduced to a mean field one-body problem. This picture is intuitively justified by the action of the Pauli principle, which reduces correlations in the nuclear Fermi gas. It is known that the mean field approximation can produce a successful phenomenological description of nuclei and nuclear matter [Wal95]. More complex descriptions of nuclear matter as a static many-soliton system have been studied by many authors (for example, [Kle86], [Cast89] and [Man95]), as already mentioned in the previous section. Klebanov [Kle86], in particular, has studied a configuration in which the nucleons form a crystalline structure with cubic symmetry, showing that this crystal structure is energetically favored due to the tensor force between neighbor nucleons.
Even if the present model is probably suitable only for obtaining a qualitative picture of a nucleon inside the nuclear medium, still it is particularly interesting because it relies only on the use of physically motivated boundary conditions for the chiral field, without resorting to new parameters or new fields. Also, it has been found that chiral symmetry breaking plays an important role in this model, by modifying the shape of the pion field at large distances from the center of the nucleon. Different chiral symmetry breaking contributions have been analyzed and their effects on the nuclear density have been studied. It is found that the internucleon distance can be reduced by about 20% by breaking chiral symmetry in different fashions.

The choice of the physically motivated boundary conditions (2.5) \(^3\), while still predicting too low density, allows one to obtain values for the binding energy not too far away from the experimental value.

Of course a more quantitative description of the system requires one to take into account the effect of the correlations in the Fermi medium, which would amount to going beyond this simple mean field picture. A second limitation, which is also shared by other calculations, comes from neglecting possible in-medium modifications of the intrinsic mesonic properties (for example, \(f_\pi, G\), etc.). In order to account for these modifications, which could signal a partial restoration of chiral symmetry, a more elaborate model is needed.

\(^3\)Note that with \(h = \text{constant}\) one automatically has \(h'(R/2) = 0\)
Chapter 3
The Chiral quark soliton model

3.1 Introduction

It has been argued, in the previous sections, that while nucleon properties can be modified in nuclear matter, such modifications cannot be fully studied in a conventional nuclear physics approach, where nucleons and mesons are considered elementary degrees of freedom and modifications of the nucleon properties have to be put in by hand. In order to address the problem, one has to consider models in which the substructure of the nucleon is not neglected (see, e.g., Refs. [Bah88, Rip97, Wal95]), and properly implement these models in order to account for the presence of a medium. This has been done, for example, using the Skyrme model [Amo98] and applying the Wigner-Seitz approximation to it, as explained in the previous Chapter.

Now we will extend our results to a chiral quark model of the nucleon, which has been developed by Diakonov et al. [Dia88, Dia89] on the basis of the instanton picture of the QCD vacuum. This model provides a low-energy approximation to QCD that incorporates a non-linear representation of the spontaneously broken chiral symmetry. In this framework pions emerge as Goldstone bosons, dynamically generated by the Dirac sea. Vacuum fluctuations (quark loops) are described by an effective action that yields the pion kinetic term, — which is already included at the classical level in the Lagrangian of other chiral models, — and higher order non-local contributions. The model is also Lorentz covariant and has essentially only one free parameter (apart from the regularization scale), namely the constituent quark mass. Although the latter should in principle be momentum dependent, in practice a constant value is
usually chosen, which better reproduces the phenomenological properties. The model has been successfully applied to the description of a variety of nucleon properties [Dia88, Dia89, Wak91, Dia98].

In this Chapter we apply the Wigner-Seitz approximation to this model, with the purpose of describing the in-medium modifications of a nucleon in a dense environment.

As explained in the previous Chapter, when dealing with the Skyrme model, in this approximation the presence of surrounding nucleons is accounted for by enclosing a single nucleon inside a spherical cell of given radius $R_{ws}$ and by imposing suitable boundary conditions on the fields (quarks and pions).

The figure 3.1 illustrates the situation at a finite density: here two nucleons are separated by a distance $2R_{ws}$ and assumed to be static; at normal density, we expect that the quarks are essentially confined in the shaded areas, being subject to color forces, whereas the meson cloud, mostly made out of pions, will be able to reach out

---

1 Notice that the Wigner-Seitz approximation to the treatment of soliton matter has already been applied to a large class of models of the nucleon structure: a soliton model [Nym70], the Skyrme model [Wus87, Amo98], non-topological soliton models [Rei85, Bir88, Web98], the hybrid soliton model [Ban85, Gle86, Hah87, Web98] and the global color model [Joh96, Joh97, Joh98].

2 The spherical symmetry of the cell reflects the isotropy of nuclear matter; a unit baryon charge inside the cell is obtained by normalization of the quark fields.

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of these regions. Since mesons are color singlet objects, a "sharing" of these fields between neighbor nucleons will not be opposed by the presence of color forces.

In the Wigner-Seitz approximation the problem of fig. 3.1 is restricted to a single cell of radius $R_{WS}$ rather than to all space: clearly, the fields which are able to reach out to the outer regions of the cell must feel more strongly the effect of the other nucleons sitting nearby. By looking at the figure one can convince oneself that the symmetry in the problem imposes the condition that these fields are flat at the cell surface$^3$:

$$\frac{d\rho_B}{dr}(R) = 0 \quad \text{or} \quad \theta(R) = 0,$$

where $\rho_B(x) = \psi^\dagger \psi$ is the baryon density and

$$\psi = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} u(r) \xi \\ v(r) i\vec{\sigma} \cdot \vec{r} \xi \end{pmatrix}$$

is the valence quark spinor, normalized to give a unit baryon charge inside the cell, i.e

$$\int_{\text{cell}} d^3x \psi^\dagger \psi = 1.$$

We notice that the choice of a spherical cell reflects the isotropy of the nuclear system. Also, if nuclear matter were a crystal instead of a fluid, then long-range correlations would also be present between nucleons; however, these long-range correlations are here suppressed by considering only average boundary conditions, which essentially retain only a next-neighbor interaction. The nuclear many-body problem is transformed in this way into a much simpler 1-body problem.

Note that when the Wigner-Seitz approximation is applied to a model containing fermions a band structure of energy levels develops. As a matter of fact equation (3.1)

$^3$The boundary condition $\theta(R) = 0$ for the chiral angle, corresponding to a vanishing pion field at the cell surface, is discussed to make contact with previous calculations performed by other authors.
is quadratic in the quark fields and thus yields two different solutions. These solutions are essentially degenerate at low density, but split up at larger density. However, color confinement, which restricts the quarks in the inner part of the nucleon (the shaded areas of fig. 3.1), effectively keeps these solutions degenerate, by suppressing the “sharing” of color fields between different nucleons\(^4\). We will see in this Chapter that the emergence of non-degenerate quark solutions in our model is an effect of the lack of a color confining mechanism. In the present model the quarks are bound in a well of finite depth — the soliton; at finite density, when the energy of the bound state gets closer to the continuum, the quarks are less confined by the potential. This is clearly a shortcoming of this — and similar — models; while other authors have accepted the idea of dealing with a band structure\[\text{Gle86, Hah87, Bir88, Web98}\], we prefer instead to keep a different attitude and select among the two solutions the one corresponding to lower energy and with a smaller sharing of the quark fields between different nucleons. These points will be discussed in detail in the Sections 3.4 and 3.5.

The Chapter is organized as follows: in Sections 3.2.1 and 3.2.2 we briefly discuss the main features of the quark-soliton model and the approximations employed in its implementation: here the contribution to the energy stemming from the Dirac sea, due to the excitation of quark-antiquark pairs, is obtained by calculating the effective action to one-loop in the quark fluctuations. In Sections 3.2.3–3.4 we introduce the Wigner-Seitz approximation, the appropriate boundary conditions for the fields and show how a few observables can be calculated in this model. A new orthonormal and complete basis in the elementary cell is also obtained, in which physical quantities, such as the vacuum energy, are expressed. In Section 3.5 we present the numerical results, obtained by solving the equations of motion. The contributions stemming from

\(^4\)At large density, where the shaded areas are allowed to overlap, a color deconfined phase is possible.
the center-of-mass motion are also estimated and discussed. Finally, in Section 3.6 we draw our conclusions and discuss possible improvements of the model.

3.2 Nucleonic and nuclear models

3.2.1 Chiral quark-soliton model

The chiral soliton model [Dia88, Dia89, Wak91], which provides a non-linear representation of the $SU(2)_L \times SU(2)_R$ symmetry of QCD, is based on the lagrangian

$$\mathcal{L} = \bar{\psi} [i\gamma - MU_5(x)] \psi,$$

where $\psi$ represents the quark fields, carrying color, flavor and Dirac indices, while $U_5$ is a chiral field defined as

$$U_5(x) = \frac{1 + \gamma_5}{2} U^t(x) + \frac{1 - \gamma_5}{2} U(x) \quad (3.4)$$

$$U(x) = \exp \left[ i\vec{r} \cdot \vec{\theta}(x) \right]. \quad (3.5)$$

The large ($\simeq 350$ MeV) dynamical quark mass $M$ which appears in Eq. (3.3) is the result of the spontaneous breakdown of chiral symmetry, which also accounts for the appearance of massless Nambu-Goldstone pions. In this model the nucleon emerges as a bound state of $N_c$ quarks in a color singlet state, kept together by the chiral mean field. Note that no explicit kinetic energy term for the pion is present in (3.3): actually, the $\psi$ and $U$ fields are not independent and the latter is in the end interpreted as a composite field in the quark-antiquark channel.

Introducing the familiar hedgehog shape of the soliton,

$$U(x) = \exp \{ i\vec{r} \cdot \vec{\theta}(r) \}, \quad (3.6)$$

the quark hamiltonian reads

$$H = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \dot{\psi} - \mathcal{L}$$

$$= -i\vec{\alpha} \cdot \vec{\nabla} + \beta M [\cos \theta(r) - i\gamma_5 \vec{r} \cdot \vec{r} \sin \theta(r)]. \quad (3.7)$$

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However, solutions to the classical field equations derived from (3.7) do not describe nucleon and Δ states, since angular momentum and isospin do not commute with $H$. One defines the so called grand spin, $\vec{K} = \vec{J} + \vec{\tau}/2$, for which $[H, \vec{K}] = 0$, and the quark wave function can be written as

$$\psi = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} u(r) \xi \\ v(r) i\vec{\sigma} \cdot \vec{\tau} \xi \end{pmatrix},$$

where $\xi$ is the grand spin state fulfilling

$$(\vec{\sigma} + \vec{\tau}) \xi = 0$$

and the normalization is

$$4\pi \int_0^\infty dr \ r^2 \bar{\psi}\gamma^0\psi = 1.$$ (3.10)

Good spin and isospin quantum numbers may be obtained in the end in a semi-classical approximation by quantizing the adiabatic rotational motion in isospin space [Adk83, Rip97]. However, in the present Chapter, for the sake of simplicity, we limit ourselves to consider soliton matter. The projection of spin-isospin quantum numbers is discussed in Chapter 4.

The classical solutions are found self-consistently by solving the equations obtained by minimization of the total energy

$$E_{\text{tot}}[\psi, \bar{\psi}, \theta] = N_c E_{\text{val}}[\psi, \bar{\psi}, \theta] + E_{\text{vac}}[\theta],$$

where $E_{\text{val}} \equiv \langle \psi|H|\psi \rangle$ and $E_{\text{vac}}$ are the valence and vacuum part of the energy, respectively. The vacuum energy $E_{\text{vac}}$ incorporates, at the mean field level, one-quark-loop contributions and, formally, can be evaluated through the effective action, which is obtained by considering the following path integral over the quark fields

$$\exp \{iS_{\text{eff}}[U]\} = \int [d\bar{\psi}] [d\psi] \exp \left\{ i \int d^4x \ \bar{\psi}(x) \left[ i\partial - MU_5(x) \right] \psi(x) \right\} = \left[ \det \left( i\partial - MU_5(x) \right) \right]^{N_c}.$$ (3.12)
The latter can be easily cast in a more suitable form by means of simple algebraic manipulations

\[ S_{\text{eff}}[U] = -\frac{i}{2} N_c \left\{ \text{tr} \log \left[ \square + M^2 + iM \Phi U_5(x) \right] - \text{tr} \log \left[ \square + M^2 \right] \right\} , \tag{3.13} \]

where the trace is over Dirac and flavor indices, as well as a functional trace\(^5\). Despite its apparent simplicity, (3.13) is actually a complicated nonlocal object.

Although a local derivative expansion of course is possible, it is of little practical use in this case, since the soliton field turns out to vary significantly over the relevant distance scale \( M^{-1} \) and no stable solutions are found for expansions up to sixth order in derivatives [Ait85]. Kahana and Ripka [Kah84] on the other hand have developed a numerical algorithm to directly evaluate vacuum polarization contributions to soliton observables. This technique has been extended in Ref. [Wak91] to the calculation of nucleon observables, that is, after collective quantization has been applied to project out states of definite spin and isospin.

3.2.2 Effective action up to second order in the soliton field

Another path, to which we shall adhere in the following, has been followed in Refs. [Dia88, Adj92] by expanding (3.13) up to second order in \( M(\Phi U_5) \), obtaining

\[ S^{(2)}_{\text{eff}}[U] \approx \frac{i}{4} N_c \text{tr} \left( x \right) \frac{1}{\square + M^2} \left[ iM \Phi U_5 \right] \frac{1}{\square + M^2} \left[ iM \Phi U_5 \right] \left( x \right) \]

\[ = -\frac{1}{4} N_c \text{tr} \int d^4x d^4x' V(x) K(x, x') V(x') , \tag{3.15} \]

\(^5\)Given an operator \( \hat{O}_i \), its functional trace is given by:

\[ \text{tr} \hat{O}_i = \int d^4x \text{tr}_i \left( x \right) \hat{O}_i \left( x \right) = \text{tr}_i \sum_n \left( n | \hat{O}_i | n \right) , \tag{3.14} \]

where \( i \) refers to the flavor and Dirac indices and \( | n \rangle \) is an eigenstate of the operator \( \hat{O}_i \). The trace over color indices simply yields the factor \( N_c \) in equation (3.13).
where

\[
V(x) = iM \partial U_5
\]

\[
K(x, x') = -iG(x, x')G(x', x)
\]

\[
(\Box + M^2)G(x, x') = \delta^4(x - x').
\]

Note that in the standard derivative expansion the second order action would read

\[
S^{(2)}_{\text{eff}}[U] \approx -\frac{i}{4} N_c M^2 \text{tr} [\partial U_5 \partial U_5] \left(x \left| \frac{1}{\Box + M^2} \frac{1}{\Box + M^2} \right| x \right).
\]

In contrast to (3.17), Eq. (3.15) does not assume a slowly varying soliton field and gives rise to non-local contributions. Furthermore, one can see that it gives a good approximation both for small and large momenta, thus providing an interpolation formula between these regimes [Dia88, Dia89].

In the case of static field configurations the vacuum energy can be directly obtained from the effective action through the relation:

\[
E^{(2)}_{\text{vac}} = -S^{(2)}_{\text{eff}} \int dx^0.
\]

Since the integrand in this formula doesn't depend on the time (because the fields are static) the factor \( \int dx^0 \) in the denominator exactly cancels a similar factor present in \( S^{(2)}_{\text{eff}} \). The sign in the formula occurs because \( \mathcal{H}_{\text{static}} = -\mathcal{L}_{\text{static}} \).

By going to momentum space, one gets

\[
E^{(2)}_{\text{vac}} = \frac{N_c}{4} \int \frac{d\mathbf{q}}{(2\pi)^3} \text{tr}[V(\mathbf{q})V(-\mathbf{q})]K(q),
\]

where

\[
\text{tr}[V(\mathbf{q})V(-\mathbf{q})] = \frac{8M^2}{f^2 \pi} q^2 \phi_0(q)\phi_0(-q) + \phi_i(q)\phi_i(-q),
\]

with

\[
\phi_0(q) = 4\pi f_{\pi} \int_0^\infty dr r^2 j_0(qr)[\cos \theta(r) - 1]
\]

\[
\phi_i(q) = i\mathbf{q}_i 4\pi f_{\pi} \int_0^\infty dr r^2 j_1(qr) \sin \theta(r) \equiv i\mathbf{q}_i \phi(q).
\]
and

\[ K(q) = \int dr \, e^{iq \cdot r} K(r) \]  

\[ K(r) = \frac{1}{8\pi^2} \int_0^\infty dk k^2 \int_0^\infty dk' k'^2 \frac{\pi}{E_k E_{k'}} \frac{C[(k + k')/2]}{E_k + E_k'} j_0(kr) j_0(k'r). \]

In (3.23) \( E_k = \sqrt{k^2 + M^2} \), whereas \( C(k) \) is a regulating function, which will be discussed later.

Although here and in the following we display, for convenience, formulae using a momentum cut-off regularization scheme, we shall also employ the Pauli-Villars regularization, to be discussed later.

\( E^{(2)}_{\text{vac}} \) must contain the meson kinetic energy contribution, which can be seen [Adj92] to correspond to keeping only the \( q = 0 \) term in an expansion of \( K(q) \) in (3.19). This requirement fixes the normalization of \( K \) in such a way that

\[ K(0) \equiv \frac{1}{8\pi^2} \int_0^\infty dk k^2 \frac{C(k)}{E_k^3} = \frac{f_\pi^2}{4N_c M^2}. \]  

(3.24)

Then one finds \( E^{(2)}_{\text{vac}} = E^{\text{kin}} + \bar{E}^{(2)} \), where

\[ E^{\text{kin}} = \frac{1}{4\pi^2} \int_0^\infty dq \, q^4 \left[ \phi_0^2(q) + \phi^2(q) \right] \]

\[ = 2\pi f_\pi^2 \int_0^\infty dr \left[ r^2 \theta'^2 + 2 \sin^2 \theta \right], \]  

(3.25)

with \( \theta' = d\theta/dr \), and

\[ \bar{E}^{(2)} = \frac{1}{4\pi^2} \int_0^\infty dq \, q^4 \left[ \frac{K(q)}{K(0)} - 1 \right] \left[ \phi_0^2(q) + \phi^2(q) \right]. \]  

(3.26)

The propagator \( K(q) \) can be brought (see Appendix C) into the form

\[ K(q) = \frac{1}{8\pi^2 q} \int_0^\infty dk \frac{k}{E_k} \int_{|k-q|}^{k+q} dk' \frac{k'}{E_{k'}} \frac{C[(k + k')/2]}{E_k + E_{k'}}. \]  

(3.27)
3.2.3 Nuclear matter in the Wigner-Seitz approximation

In order to describe nuclear matter we shall employ, as anticipated in the Introduction, the Wigner-Seitz (WS) approximation [Wig33], which amounts to enclosing the fields in a spherically symmetric cell of radius $R$ and imposing suitable boundary conditions. Before discussing our choice of boundary conditions, let us describe the evaluation of the vacuum energy in the WS cell.

We have first to find an orthonormal and complete basis of functions inside the elementary cell. We have chosen a spherical basis, in which the radial dependence is expressed through spherical Bessel functions which have vanishing derivative at the boundary. This is the most useful basis in which to perform the calculation, when flat (zero derivative) boundary conditions for the fields are invoked. More details on the basis are given in the Appendix D. All the quantities involved in the calculation of the vacuum energy turn out to converge quickly using this basis. They also converge rapidly when zero boundary conditions are employed for the fields.

Starting again from the static limit of (3.15) and (3.16), one can introduce the Bessel transform of $K(r, r')$ as

$$K(r, r') = \sum_{l \leq m} \sum_{\alpha \alpha'} Y_{lm}(\hat{r})Y_{lm}^*(\hat{r}') \rho_{\alpha l}(r) \rho_{\alpha' l}(r') K_l(\alpha_l, \alpha'_l), \quad (3.28)$$

where

$$\rho_{\alpha l}(r) \equiv \kappa_{\alpha l} j_l(\alpha_l r/R), \quad \frac{d \rho_{\alpha l}}{dr} \bigg|_{r=R} = 0, \quad (3.29)$$

$\kappa_{\alpha l}$ being a normalization constant. Inserting (3.28) into (3.15), one finds

$$E_{WS}^{(2)} = E_{WS}^{\text{kin}} + E_{WS}^{(2)}, \quad (3.30)$$

where, as in the previous subsection, the kinetic energy contribution, — the one stemming from the local part of $K(r, r') = \delta(r - r') K_0(r) + ..., —$ has been separated.
Indeed, one has

\[ E_{\text{WS}}^{\text{kin}} = 8\pi N_c M^2 \int_0^R dr K_0(r) [r^2 \theta^2(r) + 2\sin^2 \theta(r)], \tag{3.31} \]

with

\[ K_0(r) = \sum_l \sum_{\alpha_l} \frac{2l + 1}{16\pi} \frac{C(\alpha_l/R)}{E_{\alpha_l/R}^3} \rho_{\alpha_l}^2(r) \tag{3.32} \]

and \( E_{\alpha_l/R} = \sqrt{(\alpha_l/R)^2 + M^2} \).

In the limit \( R \to \infty \) one can check that

\[ K_0(r) \to \frac{1}{8\pi^2} \int_0^\infty dk k^2 \frac{C(k)}{E_k^3} \equiv \frac{f_{\pi}^2}{4N_c M^2}. \tag{3.33} \]

As in the free space case, this fixes the normalization of \( K_0(r) \) in such a way that

\[ K_0(0) = \frac{1}{16\pi} \sum_{\alpha_0} \frac{C(\alpha_0/R)}{E_{\alpha_0/R}^3} \rho_{\alpha_0}^2(0) = \frac{f_{\pi}^2}{4N_c M^2}. \tag{3.34} \]

Then, one can write

\[ E_{\text{WS}}^{\text{kin}} = 2\pi f_{\pi}^2 \int_0^R dr \frac{K_0(r)}{K_0(0)} [r^2 \theta^2(r) + 2\sin^2 \theta(r)]. \tag{3.35} \]

As we shall discuss below, one can view

\[ f_{\pi,\text{WS}}^2 = f_{\pi}^2 \frac{K_0(r)}{K_0(0)} \tag{3.36} \]

as an in-medium, \( r \)-dependent pion decay "constant".

On the other hand, the non-local contribution \( \tilde{E}_{\text{WS}}^{(2)} \), — the in-medium extension of (3.26), — can be cast into the following form:

\[
\tilde{E}_{\text{WS}}^{(2)} = \frac{8\pi N_c M}{f_{\pi}^2} \sum_{\alpha \alpha'} \left\{ \frac{1}{3} f_0(\alpha_0) \Delta K_0(\alpha_0, \alpha_0') f_0(\alpha_0') + f_1(\alpha_1) \Delta K_1(\alpha_1, \alpha_1') f_1(\alpha_1') \right. \\
+ \left. \frac{2}{3} f_2(\alpha_2) \Delta K_2(\alpha_2, \alpha_2') f_2(\alpha_2') \right\}, \tag{3.37} \]
where

$$f_0(\alpha_0) = M^{1/2} \int_0^R dr \, r^2 \rho_{\alpha_0}(r) \left[ \cos \theta(r) \theta'(r) + 2 \frac{\sin \theta(r)}{r} \right]$$

$$f_1(\alpha_1) = M^{1/2} \int_0^R dr \, r^2 \rho_{\alpha_1}(r) \sin \theta(r) \theta'(r)$$

$$f_2(\alpha_2) = M^{1/2} \int_0^R dr \, r^2 \rho_{\alpha_2}(r) \left[ \cos \theta(r) \theta'(r) - \frac{\sin \theta(r)}{r} \right]$$

and

$$\Delta K_{\alpha_1}(\alpha, \alpha') = \frac{1}{8\pi^2} \sum_{LL'} (2L + 1)(2L' + 1) \begin{pmatrix} L & L' & l \end{pmatrix} \kappa_{LL'}(\alpha_1, \alpha'_1)$$

$$\kappa_{LL'}(\alpha_1, \alpha'_1) = \sum_{\alpha,L} \frac{\pi}{E_{\alpha_L}} \left[ \frac{C[(\alpha_L + \alpha_{L'})/2]}{E_{\alpha_{L'}}(E_{\alpha_L} + E_{\alpha_{L'}})} - \frac{C(\alpha_L)}{2E_{\alpha_L}^2} \right] \xi(\alpha_1, \alpha_{L}, \alpha_{L'})$$

$$\xi(\alpha_1, \alpha_{L}, \alpha_{L'}) = \int_0^R dr \, r^2 \rho_{\alpha_1}(r) \rho_{\alpha_{L}}(r) \rho_{\alpha_{L'}}(r).$$

One should now notice that the straightforward application of (3.13) to the Wigner-Seitz cell is incomplete, because it does not account for the Casimir energy intrinsically connected with the change of topology, which is present even in the absence of background fields. As a matter of fact one should write:

$$S_{eff}[U] = -\frac{i}{2} N_c \text{tr} \left\{ \log \left[ \Box + M^2 + iM \bar{\Phi} U_5(x) \right]_{\text{WS}} - \log \left[ \Box + M^2 \right]_{\text{free}} \right\}$$

$$\equiv -\frac{i}{2} N_c \text{tr} \log \left[ \frac{\Box + M^2 + iM \bar{\Phi} U_5(x)}{\Box + M^2} \right]_{\text{WS}} + \Delta S_{\text{eff}}^{\text{Casimir}},$$

where

$$\Delta S_{\text{eff}}^{\text{Casimir}} \equiv -\frac{i}{2} N_c \text{tr} \left\{ \log \left[ \Box + M^2 \right]_{\text{WS}} - \log \left[ \Box + M^2 \right]_{\text{free}} \right\}.$$
obtains
\[
\Delta S_{\text{Casimir}}^{\text{eff}} \equiv 4N_c T \left\{ \sum_l \sum_{\alpha_l} (2l + 1) \sqrt{M^2 + \frac{\alpha_l^2}{R^2}} \right\} - \lim_{R \to \infty} \sum_l \sum_{\alpha_l} (2l + 1) \sqrt{M^2 + \frac{\alpha_l^2}{R^2}} .
\]  

(3.46)

The Casimir energy, which is obtained by dividing this expression by the time \(-T\), has now the expected form: it is the difference between the zero-point energy in the finite volume, obtained by filling all the negative energy orbitals in the Dirac sea, and the same expression in free space. As it stands, however, Eq. (3.46) is badly divergent and needs to be regularized. Such a task has been recently carried out for massive fermions, in the context of the MIT bag model, in Ref. [Eli98] by means of the zeta function regularization technique. The calculation requires the introduction of a few renormalization parameters, which cannot be estimated within the model and have to be determined by comparison with some phenomenological properties of the system. Note that such Casimir contribution does not have any dynamical content, because it depends only on the geometry of the cell and not on the fields, and therefore it can only provide a density-dependent shift of the energy. As a result, the Euler-Lagrange equations for the fields are unaffected by this term, although it could affect the position of the energy minimum. For the sake of simplicity, in the present work we will neglect this contribution to the total energy.

3.2.4 Regularization of integrals and sums

In calculating the vacuum contributions to the physical observables one has to deal with the appearance of divergent expressions. In this paper we consider two different regularization schemes, applying a momentum cutoff and using the Pauli-Villars regularization.

In the first case we introduce a regulating function, which suppresses the contri-
bution to integrals and sums at momenta $k \gg \Lambda$, where the scale $\Lambda$ is determined by fitting the pion decay constant in free space (see Eq. (3.24). The regulating function we have chosen has the form

$$C(k) = \frac{1 + 1/e}{\exp \left[ (k^2 - \Lambda^2)/\Lambda^2 \right] + 1}.$$  

(3.47)

In the Pauli-Villars regulating scheme [Kub99], on the other hand, the divergent contributions are eliminated through the subtraction

$$K(x, x') \rightarrow K(x, x') - K^{PV}(x, x'),$$  

(3.48)

where $K(x, x')$ is the propagator previously defined, while in $K^{PV}(x, x')$ the quark mass $M$ has been substituted by the mass scale $M_{PV}$, obtained again by fitting the free space pion decay constant (the analog of Eq. 3.24):

$$f_\pi^2 = \frac{N_c M^2}{2\pi^2} \int dk \, k^2 \left[ \frac{1}{(k^2 - M^2)^{3/2}} - \frac{1}{(k^2 - M_{PV}^2)^{3/2}} \right].$$  

(3.49)

One gets

$$M_{PV} = M \exp \left( \frac{2\pi^2 f_\pi^2}{N_c M^2} \right).$$  

(3.50)

3.2.5 The baryon current

Let us now consider the baryon current. Its valence part is given in terms of the quark fields by

$$B^\mu(x) = \frac{1}{N_c} \bar{\psi}(x) \gamma^\mu \psi(x).$$  

(3.51)

The contribution of the Dirac sea can be calculated by introducing the generating functional

$$W[b] = \exp \left\{ \tilde{S}[b] \right\} = \int \! d\bar{\psi} \! d\psi \exp \left\{ i \int \! d^4 x \left[ \bar{\psi} \left( i \partial - MU_5 \right) \psi - b^\mu B^\mu \right] \right\}$$

$$= \left\{ \det \left[ i \partial - MU_5 - \frac{1}{N_c} \gamma^\mu \right] \right\}^{N_c}.$$  

(3.52)
in which \( \delta^\mu \) is a classical source, coupled to the operator \( B^\mu \).

Notice that the vacuum contributions to the baryon density will be extracted by considering the functional derivative of (3.52) with respect to the source, and by then putting the source to zero:

\[
\langle B^\mu \rangle = i \frac{\delta}{\delta b_\mu} \tilde{S}[b] \bigg|_{b=0} = \int d\tilde{\psi} d\psi \ B^\mu \exp \left\{ i \int d^4x \left[ \bar{\psi} \left( i \frac{\delta}{\delta b} - MU_5 \right) \psi \right] \right\} . \tag{3.53}
\]

When the path integral in (3.53) is evaluated one finds

\[
\langle B^\mu \rangle = N_c \frac{\delta}{\delta b_\mu(x)} \text{tr}_f \text{tr}_d \langle x' \rangle \log \left( \frac{i \phi - MU_5 - \frac{1}{N_c} \gamma^\mu}{i \phi - MU_5} \right) |x' \rangle 
\]

\[
= - \text{tr}_f \text{tr}_d \left[ \gamma^\mu \frac{1}{i \phi - MU_5} \right] .
\]

By integrating over the volume the time component of this expression the total baryon charge is obtained, i.e.

\[
B^0 = - \text{tr}_f \text{tr}_d \int d^3x \gamma^0 (x) \frac{1}{i \phi - MU_5} |x\rangle
\]

\[
= - \text{tr}_f \text{tr}_d \int d^3x \gamma^0 (x) \frac{1}{\Box + M^2 - i M \phi U_5^\dagger} \left( i \phi + MU_5 \right) |x\rangle .
\]

This expression is not particularly enlightening, as it stands. However some insight can be obtained by performing a local expansion in the derivative of the soliton. In this case one obtains

\[
\langle B^\mu(x) \rangle = 4M^4 \epsilon^{\mu\nu\alpha\beta}(x) \frac{1}{(\Box + M^2)^4} |x\rangle \text{tr}_f \left[ U^\dagger(x) \partial_\alpha U(x) \partial_\beta U^\dagger(x) \partial_\gamma U(x) \right]. \tag{3.54}
\]

Since, in free space

\[
\langle x | \frac{1}{(\Box + M^2)^4} | x \rangle = \frac{i}{(4\pi)^2} \frac{1}{6M^4} , \tag{3.55}
\]

one can recognize in eq. (3.54) the expression for the topological current, well known from the Skyrme model. Similarly, in the Wigner-Seitz basis, however, one finds that

\[
\langle x | \frac{1}{(\Box + M^2)^4} | x \rangle = -i \frac{5}{16} \sum_i \sum_{\alpha_i} \frac{2l + 1}{8\pi} \frac{\rho^2_{\alpha_i}(r_j)}{(\alpha^2_i/R^2 + M^2)^{7/2}} , \tag{3.56}
\]
$\alpha_l$ being the modes of the WS basis corresponding to a specific value of $l$.

A different expression for $B^0$ can also be obtained. As a matter of fact, one notices that

$$i\partial - MU_5 = \gamma_0 \left( i\partial_0 + i\vec{\alpha} \cdot \vec{\nabla} - \beta MU_5 \right) = \gamma_0 (i\partial_0 - H).$$

$H$ being the Hamiltonian of the system. Therefore

$$B^0 = -tr_f tr_d \int d^3 x |x| \frac{1}{i\partial_0 - H} |x\rangle = -tr_f tr_d \int d^3 x \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \langle x| \frac{1}{-\omega - H + i\epsilon} |x\rangle$$

$$= tr_f tr_d \int d^3 x \theta(-H).$$

In order to regularize this expression, one still needs to subtract the one corresponding to the absence of the external field ($U = 1$), therefore obtaining:

$$B^0 = tr_f tr_d \int d^3 x \left[ \theta(-H) - \theta(-H_0) \right].$$

As a consequence one see that $B^0$ simply counts the number of levels which become negative in the presence of the background field [Dia88]. This result allows one to make contact with the Skyrme model description of the nucleon, which is recovered in the case of large solitons [Kah84].

3.2.6 Pion decay constant

Let us now consider the axial current. Its valence part is again given in terms of the quarks fields by

$$A^\mu_a(x) = \bar{\psi}(x) \frac{\tau^a}{2} \gamma^\mu \gamma_5 \psi(x). \quad (3.57)$$

We wish to calculate also the vacuum (i.e. from the Dirac sea at the one-quark-loop level) contribution to the axial current: this can be done by defining the generating
functional

\[ W[a] = \int [d\bar{\psi}d\psi] \exp \left\{ i \int d^4x \left[ i\partial \cdot \left( i\partial - MU_3 \right) \psi - a^\mu_\alpha A_\alpha^\mu \right] \right\}, \tag{3.58} \]

where \( a^\mu_\alpha \) are classical axial sources coupled to the quantum axial current; then, as before, the vacuum axial current can be obtained by means of a functional derivative with respect to the source as

\[ A_{a,\text{vac}}^\mu(x) = i \frac{\delta}{\delta a^\mu_\alpha(x)} \ln W[a] \bigg|_{a^\mu_\alpha = 0}. \tag{3.59} \]

Calculation provides

\[ A_{a,\text{vac}}^\mu(x) = \frac{4N_cM^2}{f_\pi^2} \int d^4x' K(x, x') \left[ \partial^\mu \phi_0(x') \phi^\alpha(x) - \partial^\mu \phi^\alpha(x') \phi_0(x) \right]. \tag{3.60} \]

By setting

\[ K(x, x') = K_0(r)\delta^4(x - x') + \Delta K(x, x'), \tag{3.61} \]

where \( K_0(r) \) is given by Eq. (3.32), one is able to write

\[
A^\mu_a(x) = \bar{\psi}(x) \frac{\tau^a}{2} \gamma^\mu \gamma_5 \psi(x) + \frac{4N_cM^2}{f_\pi^2} K_0(r) \left[ \partial^\mu \phi_0(x) \phi^\alpha(x) - \partial^\mu \phi^\alpha(x) \phi_0(x) \right] \\
+ \frac{4N_cM^2}{f_\pi^2} \int d^4x' \Delta K(x, x') \left[ \partial^\mu \phi_0(x') \phi^\alpha(x) - \partial^\mu \phi^\alpha(x') \phi_0(x) \right], \tag{3.62}
\]

in which local and non-local contributions have been separated.

Remembering now that the pion decay constant is defined as

\[ \langle 0 | A^\mu_a(x) | \pi^b(p) \rangle = -ip^\mu f_{\pi} \delta_{a,b} e^{-ip \cdot x}, \tag{3.63} \]

one is able to obtain the in-medium pion decay constant as

\[ f_{\pi,\text{med}}^2(r) = 4N_cM^2 K_0(r). \tag{3.64} \]

This is just expression (3.36), which depends, in a medium, on the radial coordinate.
An estimate of the average value of $f_\pi$ at fixed density can be obtained by calculating the constant value of $f_{\pi,WS}$ that would yield the same pion kinetic energy as the $r$-dependent one (compare Eqs. (3.25) and (3.35)):

$$
(f_\pi)^2 = \frac{\int_0^R dr \int_0^{f_{\pi,WS}(r)} \left[ r^2 \theta^2(r) + \sin^2 \theta(r) \right]}{\int_0^R dr \left[ r^2 \theta^2(r) + \sin^2 \theta(r) \right]}
$$

(3.65)

3.2.7 Axial coupling constant

The axial coupling constant, for a system with a finite pion mass is given by

$$
\frac{1}{2}g_A = \langle p \uparrow | \int dr A_3^\uparrow | p \uparrow \rangle,
$$

(3.66)

with the axial current given by eq. (3.62). Here $|p \uparrow \rangle$ describes a proton with spin up. In the next Chapter we will discuss in detail how to extract solutions of definite spin and isospin out of the hedgehog: the basic idea is to consider rotating solutions and to quantize semiclassically the collective variables associated with the rotation[Adk83, Bro86]. The rotating fields are obtained as

$$
\psi(\vec{x}) = R(t)\psi'(\vec{x}), \quad U(\vec{x}) = R(t)U'(\vec{x})R^\dagger(t),
$$

where $R(t) = R_0e^{x F R 3 t}$ is a time-dependent $SU(2)$ matrix.

The calculation of $g_A$ is particularly simple because it is possible to factorize the hedgehog and the semiclassical contributions, the latter providing only a numerical factor [Adk83, Bro86]. The explicit calculation shows that

$$
g_A = \langle p \uparrow | \left[ \frac{1}{2}tr \left[ R_0 \tau^3 R_0^\dagger \tau^3 \right] \right] | p \uparrow \rangle \int_0^R dr r^2 \left[ -N_c \left( u^2 - \frac{1}{3}v^2 \right) \right] + \frac{8\pi}{3} f_{\pi,WS}^2 \left( \theta' + \frac{\sin(2\theta)}{r} \right) + 2 \int dr A_3^\dagger,
$$

(3.67)

with

$$
\int dr A_3^\dagger = -\frac{16\pi N_c M^2}{3} \sum_{\alpha,\alpha'} \left\{ \bar{f}_0(\alpha_0)\Delta K_0(\alpha_0, \alpha'_0)f_0(\alpha'_0) + \bar{f}_1(\alpha_1)\Delta K_1(\alpha_1, \alpha'_1)f_1(\alpha'_1) \right\}
$$
and
\[
\bar{f}_0(\alpha_0) \equiv M^{3/2} \int_0^R dr \ r^2 \ \rho_{\alpha_0}(r) (\cos \theta(r) - 1)
\]
\[
\bar{f}_1(\alpha_1) \equiv M^{3/2} \int_0^R dr \ r^2 \ \rho_{\alpha_1}(r) \sin \theta(r).
\]

The dependence on the semiclassical variables in eq. (3.67) is contained in \( R_0 \) and the matrix element \( \langle p \uparrow | \frac{1}{2} tr \left[ R_0 \tau^3 R_0^\dagger \tau^3 \right] | p \uparrow \rangle \) simply yields the factor \(-1/3\).

Notice that, in the chiral limit, eq. (3.67) still needs to be multiplied by a factor \( 3/2 \), yielding:
\[
g_A = \int_0^R dr \ r^2 \left[ \frac{N_c}{2} (u^2 - \frac{1}{3} v^2) - \frac{4\pi}{3} f_{\pi,WS}^2 \left( \theta' + \frac{\sin(2\theta)}{r} \right) \right] - \int dr \Delta A_3^x. \quad (3.68)
\]

These are the expressions we shall employ in the next Section to evaluate the in-medium modification of the axial coupling constant.

### 3.3 Equations of motion

The equations of motion are found by minimizing the total energy (3.11) with respect to the Dirac fields, \( u \) and \( v \), and to the chiral angle, \( \theta \). It is convenient to write them in a dimensionless form by letting \( q \rightarrow Mq \) and introducing
\[
x = M r, \quad \bar{u}(x) = M^{-3/2} u(r), \quad \bar{v}(x) = M^{-3/2} v(r). \quad (3.69)
\]

\(^{6}\)We postpone the explanation of this factor to Appendix B.
Minimization of $\epsilon_{\text{tot}} \equiv E_{\text{tot}}/M = \epsilon_{\text{val}} + \epsilon_{\text{vac}}^{(2)}$ then yields

\[
\frac{d\tilde{u}}{dx} = -\sin \theta \tilde{u} - (\epsilon_{\text{val}} + \cos \theta)\tilde{v}
\quad (3.70)
\]

\[
\frac{d\tilde{v}}{dx} = -\left(\frac{2}{x} - \sin \theta\right)\tilde{v} + (\epsilon_{\text{val}} - \cos \theta)\tilde{u}
\quad (3.71)
\]

\[
\frac{d^2\theta}{dx^2} + \frac{2}{x} \frac{d\theta}{dx} - \frac{\sin 2\theta}{x^2} + \frac{N_c M^2}{4\pi f_t^2} \left[ (\tilde{u}^2 - \tilde{v}^2) \sin \theta + 2\tilde{u}\tilde{v} \cos \theta \right]
\]

\[
+ \frac{2}{\pi} \sin \theta \int_0^\infty dq q^2 \left[ \frac{K(q)}{K(0)} - 1 \right] j_0(qx)C(q)
\]

\[
- \frac{2}{\pi} \cos \theta \int_0^\infty dq q^2 \left[ \frac{K(q)}{K(0)} - 1 \right] j_1(qx)S(q) = 0,
\quad (3.72)
\]

where

\[
C(q) = \int_0^\infty dx (qx)^2 j_0(qx) [\cos \theta(x) - 1]
\quad (3.73)
\]

\[
S(q) = \int_0^\infty dx (qx)^2 j_1(qx) \sin \theta(x),
\quad (3.74)
\]

and $K(q)$ is still given by (3.27), but now $k, k'$ are dimensionless and $E_k \to \tilde{E}_k = \sqrt{k^2 + 1}$. This is a set of integro-differential equations that has to be solved iteratively as shown, for instance, in Ref. [Adj92].

The same procedure, applied to the system enclosed into the WS cell, yields

\[
\frac{d\tilde{u}}{dx} = -\sin \theta \tilde{u} - (\epsilon_{\text{val}} + \cos \theta)\tilde{v}
\quad (3.75)
\]

\[
\frac{d\tilde{v}}{dx} = -\left(\frac{2}{x} - \sin \theta\right)\tilde{v} + (\epsilon_{\text{val}} - \cos \theta)\tilde{u}
\quad (3.76)
\]

\[
\frac{d^2\theta}{dx^2} + 2 \left[ \frac{1}{x} + \frac{d}{dx} \ln K_0(x) \right] \frac{d\theta}{dx} - \frac{\sin 2\theta}{x^2} + \frac{1}{16\pi K_0(x)} \left[ (\tilde{u}^2 - \tilde{v}^2) \sin \theta + 2\tilde{u}\tilde{v} \cos \theta \right]
\]

\[
+ \sin \theta \frac{\tilde{W}_a(x)}{K_0(x)} + \cos \theta \frac{\tilde{W}_b(x)}{K_0(x)} = 0,
\quad (3.77)
\]
where \( K_0(x) \) is still given by (3.32), but now \( E_{\alpha_1/R} \rightarrow \tilde{E}_{\alpha_1/R} = \sqrt{(\alpha_1/X)^2 + 1} \) and \( \rho_{\alpha_1}(r) \rightarrow \tilde{\rho}_{\alpha_1}(x) = M^{-3/2}\kappa_{\alpha_1}j_i(\alpha_1 x/X) \), having set \( X = MR \). In (3.77) we have also set

\[
\tilde{W}_a(x) = \sum_{\alpha_1,\alpha_1'} \left[ \frac{2}{x} \tilde{\rho}_{\alpha_1}(x) + \frac{d\tilde{\rho}_{\alpha_1}(x)}{dx} \right] \Delta K_1(\alpha_1, \alpha_1') f_1(\alpha_1') \tag{3.78}
\]

\[
\tilde{W}_b(x) = \frac{1}{3} \sum_{\alpha_0,\alpha_0'} \left\{ \frac{d\tilde{\rho}_{\alpha_0}(x)}{dx} \Delta K_0(\alpha_0, \alpha_0') f_0(\alpha_0') \right. \\
\left. + 2 \left[ \frac{3}{x} \tilde{\rho}_{\alpha_2}(x) + \frac{d\tilde{\rho}_{\alpha_2}(x)}{dx} \right] \Delta K_2(\alpha_2, \alpha_2') f_2(\alpha_2') \right\} . \tag{3.79}
\]

### 3.4 Boundary conditions

In the free space problem the boundary conditions on the fields are determined straightforwardly. From inspection of the Dirac equations one has \( \tilde{v}(0) = 0 \) and \( \{\tilde{u}(x), \tilde{v}(x)\} \rightarrow 0, x \rightarrow \infty \); finiteness of the energy requires \( \theta(x) \rightarrow 0 \) when \( x \rightarrow \infty \), whereas by choosing \( \theta(0) = \pi \) one fixes to unity the topological charge associated with the pion field. This is sometimes interpreted as the baryon number, but not in the present model [Dia88], where it is connected to the number of valence levels that are pushed out of the Dirac sea. Baryon number is here fixed by the normalization condition on the quark fields \( \int_0^\infty dx x^2 \tilde{\rho}(x) = 1 \), where \( \tilde{\rho}(x) = \tilde{u}^2(x) + \tilde{v}^2(x) \) represents the (dimensionless) baryon density.

At finite density, suitable boundary conditions have to be imposed on the fields, as discussed in the Introduction of this Chapter. Here we have considered three distinct sets of boundary conditions (sets I, II and III).

In order to make contact with previous calculations, we follow for set I the choice of Ref. [Gle86], where the authors insist in maintaining unit topological charge inside.
the cell:

\[ \theta(0) = \pi \quad , \quad \theta(X) = 0 \]
\[ \bar{\nu}(0) = 0 \quad , \quad \bar{\nu}(X) = 0 \quad ("\text{bottom" of the band}) \quad (3.80) \]
\[ \bar{u}(X) = 0 \quad ("\text{top" of the band}). \]

Note that \( \bar{\nu}(X) = 0 \) implies \( \bar{u}'(X) = 0 \). Recall \( X = MR \) is the dimensionless cell radius.

For set II we choose to impose "flatness" also on the chiral angle, as in Refs. [Nym70, Amo98]. Since in general one has \( \theta(X) \neq 0 \) at the boundary, from inspection of the Dirac equations one sees that \( \bar{\nu}(X) = 0 \) no longer implies \( \bar{u}'(X) = 0 \). We have chosen to impose the physically motivated constraint

\[ \bar{\rho}'(X) = 0, \quad (3.81) \]

that is we require the flatness of the baryon density at the cell boundary. Set II then turns out to be

\[ \theta(0) = \pi \quad , \quad \theta'(X) = 0 \]
\[ \bar{\nu}(0) = 0 \quad , \quad \bar{\nu}(X) = \frac{X \cos \theta(X) \mp \sqrt{X^2 - 2X \sin \theta(X)}}{X \sin \theta(X) - 2} \bar{u}(X) \quad ("\text{bottom"/"top}). \]

For set III we require the flatness of the baryon density together with the requirement of unit topological charge:

\[ \theta(0) = \pi \quad , \quad \theta(X) = 0 \]
\[ \bar{\nu}(0) = 0 \quad , \quad \left\{ \begin{array}{ll} \bar{\nu}(X) = 0 \quad ("\text{bottom")} \\ \bar{\nu}(X) = -X \bar{u}(X) \quad ("\text{top").} \end{array} \right. \quad (3.82) \]

As pointed out earlier, by imposing boundary conditions at finite \( R \), multiple solutions, which become degenerate as \( R \to \infty \) are obtained. This is understandable, since the condition \( \rho_B^\prime(R) = 0 \) is indeed quadratic in the quark field. A band structure
seems therefore to develop at finite density, which explains the notation used in
equations (3.80), (3.81) and (3.82). In our view one cannot accept without question
the presence of a band of quark states. The presence of a band, in fact, would be
affected by confinement, which is absent in the present model. In free space, where the
quarks are deeply bound in the ground state of the chiral fields, this shortcoming is
not crucial (of course, the study of the highly excited states of the nucleon would then
be problematic). In the medium, however, because of the lack of confinement, quarks
exhibit unrealistic long-range correlations. As a result, one observes a relatively large
probability of having a quark sitting at the surface of the WS cell.

In the next Section we shall see that when the density of the medium increases
the quark density tends to be more concentrated in the interior of the bag for the
lowest end of the energy band (lowest when all effects have been included), whereas
the opposite happens on the upper end. As a consequence, quarks sitting in the upper
part of the energy band would be more affected by the confining forces than the ones
in the lower part. Since the confining forces would tend to reduce the quark density
at the boundary, the net result would then be mainly a lowering, — and hence a
narrowing, —of the highest end of the band.
3.5 Results

The numerical results have been obtained by integrating the equations of motion (3.3) for the quarks and the meson fields, using an iterative procedure as, e.g., in Ref. [Adj92] and moving from larger to smaller values of \( R \). For a given value of \( R \), i.e. for a given density, a self-consistent solution has been found by using as initial ansatz the self-consistent chiral profile obtained at the previous value of \( R \). The non-local term in Eq.(3.77) has been switched on adiabatically in order to allow a better convergence\(^7\).

At the smallest density, corresponding to \( R = 5\) fm, an exponential profile \( \theta(r) = \pi \exp(-r/r_0) \) has been used as initial ansatz. A dynamical (constituent) quark mass \( M = 350\) MeV has been assumed in the calculations. This is a value suggested by the phenomenology of the single nucleon [Dia88, Dia89, Wak91], which also turns out to be in the range \((300 - 400\) MeV) where the second order expansion for the effective action works well [Adj92].

The divergences that appear both in free space (in the momentum integrals) and in the medium (in the sums) have been regulated as explained in Sect. 3.2.4, using both a regulating function and the Pauli-Villars regularizations, whose parameters have been fixed by fitting the free space value of the pion decay constant. For \( M = 350\) MeV, the cutoff in the regulating function turns out to be \( \Lambda \cong 500\) MeV, whereas the mass scale \( M_{PV} \) of the Pauli-Villars regularization is given by Eq. (3.50). The two regularization schemes yield qualitatively similar results and in the following we shall display only the outcome from the regulating function approach.

Before discussing the results a comment on the flat basis we have adopted is in order. In fact, this basis contains a zero momentum state, that is, a term constant in space and proportional to \( R^{-3/2} \). When \( R \to 0 \), it would give rise to divergences.

\(^7\)The sums over the modes in the orthonormal and complete basis inside the spherical cell have been restricted to \( l \leq 15 \) and to the first 30 roots \( \alpha_l \), which provides a good degree of accuracy.
As explained in Appendix D, the appearance of such a mode is peculiar of the flat basis: one might introduce bases infinitesimally close to the flat one, in which the zero mode is absent and its strength is distributed among all the other modes. In these alternative bases the effect of the zero mode would be given by an infinite sum of infinitesimal contributions; since we are regularizing the sums, only a finite number of modes enters into the calculation of physical quantities and the contribution to them of the redistributed strength of the zero mode is infinitesimal. Hence, we can cure the divergence simply by dropping the zero mode and we shall henceforth do so.

In order to exhibit the convergence of the WS calculations to the free space results, let us start by comparing, in Fig. 3.2, the chiral angle obtained by solving the equations of motion in free space (solid lines) to the WS solutions corresponding to $R = 5 \text{ fm}$ and obeying the boundary conditions of set II (dashed line) and of sets I and III (dotted line). Note that at such a low density, the top and bottom energy levels practically coincide. In both cases a regulating function has been used to regulate sums and integrals. As expected, the difference between the solutions inside the WS cell and the one in free space is barely noticeable at this density. By looking at Table 3.1 we also notice that the free space energies are recovered with good precision, using any of the boundary conditions.

An important difference between our work and previous calculations employing

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</tbody>
</table>

Table 3.1 Comparison between the energies (in MeV) obtained in free space and in the WS cell using the boundary conditions of sets I-III for $R = 5 \text{ fm}$. Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 3.2 The chiral angle in free space (solid line) and in the WS approximation at $R = 5$ fm, using the boundary conditions of set II (dashed) and of sets I and III (dotted).

Figure 3.3 Total energy of the WS bag (solid) for the "bottom" solution of set I. Also shown are the valence (dotted) and vacuum (dashed c) contributions; of the latter one, we display also the local (dashed a) and non-local (dashed b) components.
Figure 3.4  Total energy of the WS bag. The left panel corresponds to the boundary conditions of set II (solid and dashed lines for the “top” and “bottom” solutions, respectively), whereas the right panel corresponds to the boundary conditions for the “bottom” solutions of sets I and III (dashed) and for the “top” solutions of sets I (dotted) and III (solid).

Figure 3.5  Binding energy obtained by taking out the spurious center-of-mass energy contribution and subtracting the energy at the lowest density (here $R = 5$ fm). The left panel corresponds to the boundary conditions of set II (solid and dashed lines for the “top” and “bottom” solutions, respectively), whereas the right panel corresponds to the boundary conditions for the “bottom” solutions of sets I and III (dashed) and for the “top” solutions of sets I (dotted) and III (solid).
chiral quark models [Web98, Ban85, Gle86, Hah87] is due to the inclusion in our calculations of non-local effects stemming from the vacuum contribution (at the one-quark-loop level). In Fig. 3.3 we display, as a function of $R$, the WS cell total energy (solid line), separated in the valence (dotted line) and vacuum (dashed c line) terms, for the “bottom” solution of set I. Also shown are the separated local, — that is, kinetic, — (dashed a line) and non-local (dashed b line) contributions. The local term displays a behavior similar to the results of, e. g., Ref. [Hah87]: actually, in the chiral quark model employed in that paper, the kinetic meson contribution is present at the classical level, whereas in our case it is dynamically generated from the vacuum. This implies, as we saw in Sect. 3.2, a $r$-dependent pion decay constant (see Eqs. (3.35) and (3.36)); we have however checked that setting $f_{\pi,WS}$ constant in our calculation, the results of Ref. [Hah87] are recovered. On the other hand, the non-local vacuum contribution provides substantial attraction and displays a moderate dependence on $R$; however, it turns out that the valence and kinetic meson terms compensate each other to a large extent yielding a total contribution without any minimum and the non-local term is then instrumental in order to get the (rather shallow) minimum displayed by the solid line.

By looking at Fig. 3.4 one notices that solutions to the equations of motion are no longer found below $R \approx 1.4$ fm. This depends also on the choice of the boundary conditions and in the other cases discussed below we shall see that solutions are found until $R \approx 1.1$ fm. In Ref. [Hah87] solutions have been found till much higher densities ($R \approx 0.4$ fm). In the case of Fig. 2 in that paper, — where a simple Lagrangian containing only terms up to second order in the pion field is used, — this is due to the larger value for the constituent quark mass $M$ chosen in that work (in their notation $M = g f_{\pi} \approx 550$ MeV, where $g$ is the quark-meson coupling constant). We have checked that by dropping the non-local term and by increasing $M$ their results...
can be recovered. Indeed, from Eq. (3.3) one sees that the equations of motion depend on the combination \( X = M R \): by increasing \( M \), one can lower the minimum value of \( R \). However, the authors of Ref. [Hah87] are able to find solutions at higher densities by using more complex Lagrangians containing terms of higher order in the pion field. Following that path in our model would imply going beyond the two-point approximation to the effective action of Sect. 3.2.2.

In Fig. 3.4 we display the WS cell total energy as a function of the cell radius, i.e. of the density. The boundary conditions of set II and of sets I and III have been used in the left and right panels, respectively. In Fig. 3.4 we notice the presence of a very shallow minimum around a density corresponding to \( R \approx 2 \text{ fm} \); this minimum is deeper for the "bottom" solutions.

The occurrence of saturation in nuclear matter cannot however be stated by simply looking at these figures, because here the spurious energy contribution due to the center-of-mass motion has been neglected. This, as a matter of fact, is a well known problem associated with the mean field approximation. An estimate of this effect in the chiral quark soliton model has been obtained in [Pob92], including also the vacuum (mesonic) contributions to the center of mass motion. The findings in that paper, — that valence terms dominate as long as \( E_{\text{val}} \geq 0 \), — make us feel confident in retaining only the latter in our estimate. It reads

\[
E_{\text{CM}} = \frac{\langle P^2 \rangle}{2E_{\text{tot}}} = -\frac{N_c}{2E_{\text{tot}}} \left\{ R^2 \rho'(R) - \int_0^R dr \left[ r^2 (u'^2(r) + v'^2(r)) + 2v^2(r) \right] \right\}. \tag{3.83}
\]

Of course, the validity of this assumption at finite density can only be checked through an explicit calculation of the vacuum terms, which is however beyond the scope of the present analysis.

In Fig. 3.5 we plot the binding energy for the system taking out the spurious contributions stemming from the motion of the center of mass. In order to minimize
Figure 3.6 Dependence of the baryon density upon the cell radius for the “top” (solid) and “bottom” (dashed) solutions of set II.

Figure 3.7 Decomposition of the total energy (solid) into valence (dot) and vacuum (dashed) contributions, for the “top” solutions of set II (left panel) and III (right panel).
the numerical uncertainty, the binding energy has been obtained by subtracting the total energy at the lowest considered density \((R = 5 \text{ fm})\). Interestingly, we find that, when the center-of-mass motion is taken out, a stronger minimum in the energy is found, roughly at the same density as in Fig. 3.4, namely \(R \approx 1.8 \text{ fm}\). Moreover, the boundary conditions of sets I and III provide more binding that those of set II.

Note that in nuclear matter one should have a binding energy of about \(-16 \text{ MeV}\) at \(R \approx 1.1 \text{ fm}\).

We also notice that when the center-of-mass correction term is included the “top” solutions of the various sets of boundary conditions provide more binding than the corresponding “bottom” solutions. The reason is easily understood by looking, for example, at Fig. 3.6. Here we plot the value of the baryon density at the surface of the cell, i.e. \(u^2(R) + v^2(R)\), as a function of the cell radius itself, for the boundary conditions of set II (the solid and dashed lines corresponding to the top and bottom solutions respectively); sets I and III show a similar behavior. Since the top solution corresponds to a configuration in which the quarks are more “compressed” inside the cell, a larger kinetic energy, — and therefore a larger center-of-mass motion, — is associated with it. This problem is of course absent in solid state physics, the original field of application of the Wigner-Seitz approximation, because the electron mass is indeed completely negligible with respect to the mass of the ions, which form the periodic structure. In the present case, even admitting the existence of a periodic structure in nuclear matter, the center-of-mass motion would be negligible only in the large \(N_c\) limit \((N_c \rightarrow \infty)\), given the dependence of the total energy and of the center-of-mass energy on the number of colors, as \(O(N_c)\) and \(O(N_c^0)\) respectively. On the other hand, for \(N_c = 3\), the center-of-mass energy will vary, for each solution inside the band, by an amount comparable with the width of the band itself. Hence, the calculation of a reliable band structure within this model is more delicate.
Figure 3.8 The ratio between the kinetic and potential components of the valence energy as a function of the cell radius, using the "top" boundary conditions of set II (solid) and set III (dashed).

Figure 3.9 Dependence of the pion decay constant on the cell radius for the "top" solution of set II.
In the absence of a confining mechanism within the model, the top solution (solid line) will be considered as the best candidate for the description of the in-medium nucleon. As a matter of fact, the bottom solution, corresponding to the dashed line, describes a swollen nucleon, in which, at finite density, the quarks have a larger probability to be found at the surface of the cell. We believe that color confinement would suppress this "sharing" of quarks between neighbor nucleons, thus sensibly affecting this state.

*It is a second conclusion of the present work that when all effects are included, the solution with lowest energy also has the smallest baryon density (carried by the quarks) at the cell wall.*

In Fig. 3.7 the total energy corresponding to the "top" solutions of set II (left) and III (right) is plotted, and decomposed into the valence (dotted) and vacuum (dashed) contributions. We observe that, at very low densities, the two components bear approximately the same strength, whereas at larger densities, below \( R \approx 2 \) fm, the valence contribution becomes dominant. It is also interesting to display, as a function of \( R \), the ratio between the valence contributions that come from the quark kinetic term in the Lagrangian and from the quark coupling to the mean field. This is done in Fig. 3.8: as expected, this ratio increases with the density.

In Fig. 3.9 the average value of the pion decay constant\(^8\) in the cell, defined in Eq. (3.65), is plotted as a function of the cell radius, for the "top" solution of set II; for all the other sets of solutions the behavior is very similar. The pion decay constant decreases by increasing the density, going in the direction of a partial restoration of chiral symmetry, although the lack of solutions beyond roughly the standard saturation density of nuclear matter, prevents one, — at the present stage of development of the model, — from drawing firmer conclusions. A reduction of \( f_\pi \)

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\(^8\)Notice that \( f_\pi \), as well as the isoscalar mean square radius and the axial coupling constant \( g_A \), which are also evaluated here, are independent of the semiclassical projection.
Figure 3.10 Isoscalar mean square radius, as a function of the cell radius. The solid line corresponds to the "top" solution of set II, whereas the dashed line corresponds to the "bottom" solution of Set III.

Figure 3.11 Chiral angle as a function of the distance from the center of the soliton for $R = 5$ (solid), 2 (dashed) and 1.1 fm (dotted), respectively. The left panel corresponds to the "top" solution of set II, whereas the right one corresponds to the "top" solution of set III.
Figure 3.12 Large and small components of the Dirac spinor as a function of the distance from the center of the soliton for $R = 5$ (solid), 2 (dashed) and 1.1 fm (dotted), respectively. The left panel corresponds to the “top” solution of set II, whereas the right one corresponds to the “top” solution of set III.

Figure 3.13 Baryon density as a function of the distance from the center of the soliton for $R = 5$ (solid), 2 (dashed) and 1.1 fm (dotted), respectively. The left panel corresponds to the “top” solution of set II, whereas the right one corresponds to the “top” solution of set III.
in matter is found both in linear sigma and Nambu-Jona-Lasinio models (see, e. g., Ref. [Bir94] for a list of references).

In Fig. 3.10 the isoscalar mean radius is plotted as a function of the cell radius, for the “top” (solid) and “bottom” (dashed) solutions of set II; the other sets of solutions give very similar results. When all effects are included, the solution with lowest energy and lowest baryon density at the wall implies a shrinking of the isoscalar radius (solid curve in fig. 3.10). We observe, however, that this quantity is extremely sensitive to the choice of “top” or “bottom” boundary conditions, the reason being the same as already noticed when discussing Fig. 3.6: the “bottom” solution in fact corresponds to a configuration in which the quarks are more loosely packed inside the cell. One could in principle define a mean radius averaged over the solutions distributed in the band, assuming, for instance, a uniform distribution. From Fig. 3.10 one would expect such an averaged radius to have a moderate density dependence. However, as already mentioned, we expect confinement to affect the band width, and especially the “bottom” border (actually, the upper end), thus unbalancing the distribution in
favor of the lower curve of Fig. 3.10 and yielding a shrinking of the soliton.

For completeness we display in Figs. 3.11, 3.12 and 3.13, as a function of the distance from the center of the WS cell and for three values of \( R \), the chiral angle, the large and the small components of the Dirac spinor and the baryon density, respectively, using the "top" solutions of sets II and III. Similar results apply for the other solutions.

A quantity which is more sensitive to the choice of boundary conditions is \( g_A \), as one can see in Fig. 3.14. The local mesonic component varies noticeably in the different cases, even at large \( R \). In Table 3.5 we display the value of the different components of \( g_A \) at \( R = 5 \) fm, compared to the results for the free soliton case\(^9\). One clearly sees that the quark and non-local mesonic components have already reached the asymptotic value, whereas the local mesonic term is still higher. The reason for this behavior can be traced back to the slow decay of the chiral angle in the chiral limit \( (\theta(r) \approx 1/r^2) \), which allows sizeable contributions from large distances. Although in principle one could continue the WS calculation at larger radii, one would then need to increase the number of states included in the orthonormal basis employed in the expansion. However, it is likely that, when chiral symmetry is explicitly broken (for example, by an explicit pion mass term in the lagrangian), the exponential decay of the chiral angle would grant a faster convergence and suppress the sensitivity to the boundary conditions. Increasing the density, the mesonic contributions to \( g_A \), both local and non-local, rapidly decrease and most of the strength is now carried by the quarks alone.

---

\(^9\)As noted by the authors of Ref. [Adj92], the two-point approximation to the effective action works for \( g_A \) to a lesser extent than for the energy and the apparent agreement with the experimental value should be regarded as accidental, since the self-consistent calculation (see, e.g., Ref. [Wak91]) gives a smaller value.
Table 3.2 The axial coupling constant $g_A$ in free space and in the Wigner-Seitz approximation at $R = 5$ fm, for the “top” boundary conditions of sets II and III.

<table>
<thead>
<tr>
<th></th>
<th>quark</th>
<th>local mesonic</th>
<th>non-local mesonic</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>free</td>
<td>1.07</td>
<td>0.50</td>
<td>-0.35</td>
<td>1.22</td>
</tr>
<tr>
<td>II</td>
<td>1.06</td>
<td>0.76</td>
<td>-0.34</td>
<td>1.48</td>
</tr>
<tr>
<td>III</td>
<td>1.06</td>
<td>1.03</td>
<td>-0.34</td>
<td>1.76</td>
</tr>
</tbody>
</table>

3.6 Outlook and perspectives

In this Chapter we have applied the Wigner-Seitz approximation to the chiral quark-soliton model of the nucleon. This model, complemented by the WS approximation, provides a simple, yet interesting, framework for studying possible modifications of the nucleon properties in nuclear matter. It is important to remark that we are actually dealing with a parameter-free model, since the only free parameters (the constituent quark mass $M$ and the regularization scale) have been fixed in free space. Note that $M$ could, in principle, be calculated dynamically, as done, e. g., in the Nambu-Jona-Lasinio model [Rip97]; this would require the solution of a gap equation and would probably lead to an in-medium suppression of $M$, similar to what is found for $f_\pi$.\footnote{By increasing the density, only virtual states of higher energy can be excited and, eventually, in a very dense system, only states lying well above the typical momentum cutoff would be available: the quark condensate, $\langle \bar{q}q \rangle$, is thus expected to decrease at finite densities.}

The value we have been using for $M$ (350 MeV) is suggested by free space phenomenology. We have also tried to test the sensitivity to $M$ of our calculations by employing other values found in the literature, $M = 500 - 550$ MeV [Hah87]. While we find results qualitatively very similar to the $M = 350$ MeV case, one should be aware that the two-point approximation to the vacuum contributions works better for masses of the order of 300 – 400 Mev, as found in Ref. [Adj92]. A more careful investigation of this model for larger constituent quark masses might require going
beyond the two-point approximation presently used.

The same method of accounting for in-medium effects has already been applied to a number of microscopic models of the nucleon [Wus87, Amo98, Rei85, Bir88, Web98, Ban85, Gle86, Hah87, Joh96, Joh97, Joh98]. With respect to previous analyses, we have for the first time consistently calculated effects stemming from the Dirac sea, i.e., from excitations of virtual quark-antiquark pairs, including their dependence upon the density of the system.

Vacuum fluctuations manifest themselves in two ways. First, by giving rise to a pion decay constant dependent on the distance from the center of the bag, which in turn modifies the pion kinetic contribution with respect to chiral models where the pion is explicitly included at the classical level [Ban85, Gle86, Hah87, Web98] (see Eqs. (3.35) and (3.36)): the contribution to the energy from this term is however qualitatively similar to the one of previous analyses [Ban85, Gle86, Hah87, Web98]. Second, vacuum fluctuations generate a new non-local, attractive contribution, whose density dependence, albeit moderate, is however relevant in order to bind the system, given the compensation one observes with increasing density between the valence quark and kinetic pion contributions.

Another effect that has usually been neglected in previous calculations is the spurious center-of-mass motion: we have found it to be important not only to give, of course, a more realistic estimate of the energy, but also to determine the relative position of the top and bottom ends of the quark energy band in the medium. These two levels correspond to specific boundary conditions and we have found the role of the boundary conditions generating the upper and lower levels to be exchanged with respect to previous calculations.\(^\text{11}\)

We have also explored the effects stemming from different choices of boundary

\(^{11}\)Note that the intersection of an occupied band with an empty one is often interpreted as the onset of color superconductivity.
conditions. All the cases discussed above display a similar qualitative behavior, although there are definitely quantitative differences, especially for the axial coupling constant, which is however very much affected by the long-range behavior of the chiral field in the (massless pion) chiral limit. Also the two regularization schemes we have adopted turned out to give qualitatively similar results.

We have calculated a few physical quantities, such as $f_\pi$ and $\langle r^2 \rangle_{I=0}$. The pion decay constant has been found to decrease with increasing density, pointing to a partial restoration of chiral symmetry in the medium. The isoscalar mean square radius is seen to decrease at finite density for the physical solution, which corresponds to a configuration in which the quarks are more "compressed" inside the Wigner-Seitz cell.

Unlike previous calculations with chiral models [Ban85, Gle86, Hah87, Web98], we have been able to find binding, but at a density much lower than the standard saturation density of nuclear matter; moreover, solutions of the equations of motion disappear roughly below the latter density. The authors of Ref. [Hah87] have been able to find solutions at high densities, — still keeping realistic values for the constituent quark mass parameter $M$, — by incorporating in their chiral Lagrangian terms of higher order in the pion field. In the present model this would correspond to dropping the two-point approximation to the effective action in evaluating vacuum fluctuations. This is probably the most needed development of the present calculation, — employing, e. g., the numerical algorithm of Ref. [Kah84], — since only in the free case it has been explicitly verified that the two-point approximation works reasonably well [Adj92].

The two-point approximation is not the only one that could affect the search for binding: for instance, as already mentioned, the constituent quark $M$ is in principle density and momentum dependent; although a constant value for $M$ allows for a fair
description of phenomenology in free space, it is not clear whether the same can be assumed in the medium.

Another interesting issue in this connection is related to the absence of confining (color) forces in this model (and similar ones). The lack of confinement, while not essential in the description of a single nucleon, where the quarks are in any case tightly bound by the soliton, can become problematic at non-zero density. In the latter case an unrealistic sharing of quarks between neighboring nucleons can become possible. This is the biggest drawback of the present model. It would be very interesting to implement our calculations in models with confining forces, such as the Color-Dielectric model [Sch86]. However, calculations in this model are at present plagued by insurmountable difficulties when one tries to include the vacuum effects [Rip97] and therefore limited to the inclusion of only the valence contributions.

Another feature that is lacking at the present stage is the projection of good spin-isospin quantum numbers out of the hedgehog state. Since the momentum of inertia is, in general, density-dependent, the projection is likely to affect the position of the minimum in the energy. It is to this topic that we turn in the next Chapter.

Finally, it is worth noticing that the binding in this model (and similar ones) is the result of cancellations between large contributions. Small (on the scale of the total energy) variations in one of this components can have sensitive effects on the saturation curve. The explicit breaking of chiral symmetry due to $m_\pi \neq 0$ should be included. It would also appear to be worthwhile to invest the effort require for a calculation of the Casimir energy in the present model.
Chapter 4
Semiclassical quantization of the hedgehog solutions

4.1 Introduction

In the previous chapter we have discussed a mean-field solution to a chiral model of the nucleon, which contains quarks and pions as explicit degrees of freedom. Because of the lack of invariance under rotations in space or isospace of the hedgehog ansatz used to describe the soliton, particles (the nucleon and the Δ) of definite spin or isospin cannot be obtained. As a result the mean field solutions obtained by solving the Euler-Lagrange equations for the quark and pion field describe a mixture of nucleon and of Δ states and they are found to be eigenstates of the "grand-spin" operator \( \vec{K} = \vec{I} + \vec{S} \). This is clearly a shortcoming of the mean field approach: an other example of this type of problem has been already encountered in the previous Chapter when dealing with the center-of-mass corrections to the soliton mass\(^1\).

In this Chapter we will discuss two main topics: the projection of solutions with definite spin and isospin, corresponding to the observed particles \((N, Δ, \ldots)\), and the extension of this result to the Wigner-Seitz approximation. The extension to the Wigner-Seitz approximation is the original contribution in this Chapter.

We follow here a semiclassical procedure which is described in [Adk83, Bro86]. This technique is equivalent to the "cranking model" introduced by Inglis[Ing54, Ing56] in the description of deformed nuclei. Before considering the application to the model that we are using, it can be useful to see how the technique works in a

\(^1\)The center-of-mass possesses a finite kinetic energy because of the non-vanishing expectation value of \( \langle \vec{P}^2 \rangle \) within a mean field description.
simple non-relativistic quantum-mechanical system. We will follow here the example given in [Ring80], where the reader will find a useful account of the method.

We consider a system described by the single-particle Hamiltonian

$$H(\vec{r}, t) = \frac{\vec{p}^2}{2m} + V(\vec{r}, t), \quad (4.1)$$

where $V(\vec{r}, t)$ is a single-particle (i.e. "mean field") potential acting on the particles. The Schrödinger equation will read

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = H(\vec{r}, t)\psi(\vec{r}, t). \quad (4.2)$$

In spherical coordinates we specify the time dependence of the single-particle potential to be

$$V(\vec{r}, t) = V(r, \theta, \phi - \omega t), \quad (4.3)$$

which is rotating with angular velocity $\omega$ around the $z$ axis.

The single-particle hamiltonian at a time $t$ can be obtained from that at a given time, say $t = 0$, by operating on it with a unitary transformation $U = \exp \left\{ \frac{i}{\hbar} \vec{L} \cdot \vec{\omega} \, t \right\}$:

$$H(\vec{r}, t) = U^{-1} H(\vec{r}, 0) \, U. \quad (4.4)$$

If one introduces the new wave function

$$\tilde{\psi}(\vec{r}, t) = U(t)\psi(\vec{r}, t), \quad (4.5)$$

the Schrödinger equation (4.2) will now read

$$i\hbar \frac{\partial}{\partial t} \tilde{\psi}(\vec{r}, t) = \left[ H(\vec{r}, 0) - i\hbar U \, U^{-1} \right] \tilde{\psi}(\vec{r}, t) \quad (4.6)$$

or

$$i\hbar \frac{\partial}{\partial t} \tilde{\psi}(\vec{r}, t) = \left[ H(\vec{r}, 0) - \vec{L} \cdot \vec{\omega} \right] \tilde{\psi}(\vec{r}, t). \quad (4.7)$$
The interaction potential in eq. (4.7) is now time-independent (i.e. we have now gone to the "body-fixed frame") and stationary states will be found by solving an eigenvalue problem:

\[
\left[ H(\vec{r}, 0) - \vec{L} \cdot \vec{\omega} \right] \tilde{\psi}(\vec{r}) = \tilde{E} \tilde{\psi}(\vec{r}).
\]  

(4.8)

The energy in the original hamiltonian will be now given by

\[
E = \int d^3 x \, \Psi^*(\vec{r}, t) \frac{i\hbar}{\partial t} \Psi(\vec{r}, t) = \int d^3 x \, \tilde{\Psi}^*(\vec{r}, t) \frac{i\hbar}{\partial t} \left[ \mathcal{U} \dot{\mathcal{U}}^{-1} + \frac{\partial}{\partial t} \right] \tilde{\Psi}(\vec{r}, t) \\
= \int d^3 x \, \tilde{\Psi}^*(\vec{r}, t) \left[ \vec{L} \cdot \vec{\omega} + \tilde{E} \right] \tilde{\Psi}(\vec{r}, t),
\]  

(4.9)

where \( \Psi \) and \( \tilde{\Psi} \) are the many-body wave functions for the system of particles and \( \vec{L} \) is now the total angular momentum.

If the rotation is slow, i.e. if \( \omega \ll 1 \), then one can solve (4.8) treating the term \( \vec{L} \cdot \vec{\omega} \) as a perturbation. In this case, the wave functions \( \tilde{\psi} \) can be expanded as

\[
|\tilde{\psi}\rangle = |\psi_0\rangle + \sum_n \frac{\langle n|\vec{L} \cdot \vec{\omega}|\psi_0\rangle}{E_n - E_0} |n\rangle.
\]  

(4.10)

where \( \psi_0 \) are the eigenfunctions of \( H(\vec{r}, 0) \). \( |n\rangle \) is a complete set of eigenstates of \( H(\vec{r}, 0) \). By substitution of this expression in (4.9) one finds the form

\[
E = E_0 + \frac{1}{2} \mathcal{I} \omega^2,
\]  

(4.11)

where \( E_0 \) is the energy of the fixed solution and \( \mathcal{I} \) is the moment of inertia. Eq. (4.11) is the energy of a classical rigid rotor. In order to quantize the rotational spectrum one still needs to quantize the collective motion of the system.

---

\(^2\)The term \( \vec{L} \cdot \vec{\omega} \) is essentially the Coriolis force felt in the rotating coordinate system.

\(^3\)Rewrite

\[
E = E_0 + \frac{\tilde{E}^2}{2\mathcal{I}}
\]

and then quantize \( \vec{L} \). This produce a solution of given angular momentum from the mean-field solution in the rotating coordinate system, which intrinsically contains all angular momenta.
Now that we have seen how this method works in a non-relativistic quantum-mechanical system, we can generalize the results to a covariant model of the nucleon. We will proceed in close analogy with the previous example. First we notice that the hedgehog solution is invariant only under combined spin-isospin transformations, but not under separate spin or isospin transformations: there exist an infinite number of solutions which are degenerate to the hedgehog solution. Degenerate solutions can be obtained, for example, by picking an ansatz of the form, $\tilde{U}(\vec{x}) = R e^{i\vec{r} \cdot \vec{b}} R^\dagger$, $R$ being a constant SU(2) matrix, thus performing a rotation either in space or isospace.

Consider now a time-dependent isospin rotation of the form\(^4\):

$$R(t) = R_0 e^{\frac{i}{2} \vec{\lambda} \cdot \vec{\lambda} t} ,$$

(4.12)

where $R_0 \equiv b_0 + i\vec{r} \cdot \vec{b}$ is a constant SU(2) matrix and $\vec{\lambda}$ is a constant vector (here $\vec{\lambda} t$ plays the role of the collective coordinate). Note that the rotation (4.12) allows the transition between states with different grand-spin. For example, the state $\xi' = \tau^1 \xi$ corresponds to $(K = 1, K_z = 0)$, if $\xi$ is a grand-spinor with $(K = 0, K_z = 0)$:

$$\vec{K}^2 \tau^1 \xi = [\vec{K}^2, \tau^1] = 2\tau^1 \xi \Rightarrow K = 1.$$

(4.13)

The goal is to find stationary solutions to the Euler-Lagrange equations for the fields in the rotating frame, as it has been done in eq. (4.8). This will lead us to a modified time-independent hedgehog equation.

We write the fields according to:

$$\psi(\vec{x}, t) = R(t) \psi'(\vec{x})$$

(4.14)

$$U(\vec{x}, t) = R(t) U'(\vec{x}) R^\dagger(t) ,$$

(4.15)

where $\psi'$ and $U'$ are the fields in the rotating frame, and look for stationary solutions for $\psi'$ and $U'$. In the end, this "cranking model" procedure will be justified in\(^4\)

\(^4\)We follow here the choice of [Bro86]. A more general form for $R(t)$ was considered in [Adk83].
the present case by showing that the generated solution $\psi(\vec{r}, t)$ is indeed a separate eigenstate of isospin and angular momentum.

Note that the rotational and isorotational spectrum of these solutions will be continuous: angular momentum and isospin will take arbitrarily small values. Therefore one needs to perform a quantization. This is done by promoting $\mathcal{I}R_0$ and $\mathcal{I}\lambda$ to operators with non trivial commutation relations. A discrete spectrum of particles is thus obtained.

Let us consider in detail how this method works, by looking at a particular example. The main goal of the present Chapter is to illustrate the effects of the projection on the Wigner-Seitz solutions. To obtain qualitative, and even semi-quantitative insight into the effects of this projection, we have examined a simplified covariant and chiral invariant model of the nucleon. We take the chiral-invariant lagrangian

$$\mathcal{L} = \bar{\psi}(x) \left( i \gamma - MU_5(x) \right) \psi(x) + \frac{f^2}{4} \text{tr} \left[ \partial_\mu U(x) \partial^\mu U^\dagger(x) \right], \quad (4.16)$$

which differs from (3.3) previously considered by the explicit presence of a meson contribution at the classical level, corresponding to the nonlinear sigma model (NLSM)$^5$. The contributions stemming from the Dirac sea, which would require a more intense numerical effort, will be neglected in the present illustrative case.

We follow the approach and the notation of [Bro86] and write the original hedgehog spinor as

$$\psi_h(x) = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} F(r) \xi \\ G(r) \gamma \cdot \hat{r} \xi \end{pmatrix}. \quad (4.17)$$

The soliton is written as

$$U_5(x) \equiv \frac{\sigma_h + i \vec{r} \cdot \vec{\phi}_h \gamma_5}{f_\pi} \quad (4.18)$$
$$\sigma_h(r) \equiv -f_\pi \cos \theta(r) \quad (4.19)$$
$$\vec{\phi}_h(r) \equiv f_\pi \hat{r} \phi_h(r) = f_\pi \hat{r} \sin \theta(r) \quad (4.20)$$

$^5$The authors in [Bro86] considered the linear sigma model coupled to quarks. The effects of the projection turn out to be very similar in the two models.
where the subscript $h$ is used for the hedgehog solutions. The author has calculated these solutions in the present work by solving numerically the equations of motion for the fields (which are immediately obtained from (4.16)).

In the body-fixed frame, one can solve the Euler-Lagrange equations for the fields and thus find the corresponding stationary solutions. If the rotation is adiabatic (i.e. if $\lambda$ is small), its effect can be treated as a perturbation and the fields can be expanded in $\lambda$ around the hedgehog solutions:

$$
\psi' = \psi_h + \delta \psi \\
\phi' = \phi_h + \delta \phi \\
E' = E_h + \delta E.
$$

In [Bro86] it has been proven that, to first order, $\delta \phi = 0$. As a result, one can consider only the Dirac equation for the quarks, to leading order in the angular velocity, i.e.

$$
\left(-i\alpha \cdot \vec{\nabla} + \beta MU_5 + \frac{\lambda \cdot \vec{r}}{2}\right) \psi' = E' \psi' .
$$

(4.21)

The latter can be written as

$$
\left(-i\alpha \cdot \vec{\nabla} + \beta MU_5 - E\right) \delta \psi = -\frac{\lambda \cdot \vec{r}}{2} \psi ,
$$

(4.22)

by exploiting the properties of the hedgehog spinor, $\delta E = \overline{\psi} \frac{\lambda \cdot \vec{r}}{2} \psi = 0$.

Notice that, because of eq. (4.13), the interaction term in (4.22) represents a grand-spinor with $K = 1$.\(^6\) As a result, $\delta \psi$ must also be a $K = 1$ grand-spinor, which can be expressed in the most general form as:

$$
\delta \psi = \frac{1}{\sqrt{4\pi}} \left( A(r)\lambda \cdot \vec{r} \xi + B(r) \left( \frac{\lambda \cdot \vec{r}}{3} - \lambda \cdot \hat{r} \cdot \vec{\sigma} \right) \xi \right) ,
$$

where $A(r)$, $B(r)$, $C(r)$ and $D(r)$ are unknown functions of the distance from the center of the soliton.

\(^6\)Note that $\psi_h$ has grand-spin 0.
By substituting eq.(4.23) inside eq.(4.22) one obtains the four differential equations:

\[- A' + \frac{2}{3} B' = - \frac{2}{r} B - \frac{M}{f_\pi} \phi_h \left( - A + \frac{2}{3} B \right) - \left( \frac{M}{f_\pi} \sigma_h - E \right) C + \frac{G}{2} \]  
\[A' + \frac{1}{3} B' = - \frac{B}{r} + \frac{M}{f_\pi} \left( A + \frac{1}{3} B \right) - \left( \frac{M}{f_\pi} \sigma_h - E \right) D - \frac{G}{2} \]  
\[D' = \frac{C - D}{r} - \left( \frac{M}{f_\pi} \sigma_h + E \right) \left( A + \frac{1}{3} B \right) - \frac{M}{f_\pi} \phi_h D - \frac{F}{2} \]  
\[C' + D' = \frac{C + D}{r} + \frac{M}{f_\pi} \phi_h (C - D) - \left( \frac{M}{f_\pi} \sigma_h + E \right) B \]  

These results are from [Bro86] and a derivation of these formula is given in the Appendix G. Notice that \( F, G, \sigma_h \) and \( \phi_h \), respectively given in (4.17), (4.19) and (4.20), are the self-consistent hedgehog solutions, obtained by solving the classical equations of motion. Once these solutions are known, then the differential equations (4.24), (4.25), (4.26) and (4.27) can also be solved, by imposing the appropriate boundary conditions (see Appendix G for a discussion of this point.).

We can now turn to consider the expression for the spin and isospin of the rotating solutions. For example, under an infinitesimal isospin transformation (i.e. \( \lambda \ll 1 \)) the change in the fields is given by

\[\delta \psi = - i \frac{\lambda \cdot \vec{r}}{2} \psi \]  
\[\delta U = i \frac{\vec{r} \cdot \left( \lambda \times \vec{\phi}_h \right)}{2} . \]

The corresponding conserved Noether current is obtained from (4.16), which is invariant under isospin transformations, as

\[V^\mu(x) = \frac{\partial L}{\partial \psi} \delta \psi + \frac{\partial L}{\partial U} \delta U + \frac{\partial L}{\partial U^\dagger} \delta U^\dagger . \]

Therefore the integral over space of the time component yields the isospin charge of the rotating solution.
The explicit calculation shows that the expressions for the spin and isospin are\[Bro86\]7

\[
\mathbf{I}^a = -\mathcal{I} \frac{1}{2} \text{tr} \left[ R_0 \mathbf{R}^a R_0 \mathbf{R}^b \right] \lambda^b
\]

\[
\mathbf{J}^k = \mathcal{I} \lambda^k ,
\]

where \( \mathcal{I} \) is the total moment of inertia

\[
\mathcal{I} = \int dr \, r^2 \left\{ N_c \left[ A(r) F(r) + C(r) - \frac{2}{3} D(r) \right] + \frac{8\pi}{3} \phi_0^2(r) \right\} (4.31)
\]

and \( R_0 \) is a constant \( SU(2) \) matrix, previously introduced.

Note that \( \mathbf{I}^a \) and \( \mathbf{J}^k \) must obey the algebra of \( SU(2) \), i.e. the commutation relations

\[
\left[ \mathbf{I}^a, \mathbf{I}^b \right] = i \epsilon^{abc} \mathbf{I}^c \quad (4.32)
\]

\[
\left[ \mathbf{I}^a, \mathbf{J}^k \right] = 0 \quad (4.33)
\]

\[
\left[ \mathbf{J}^l, \mathbf{J}^k \right] = i \epsilon^{ijk} \mathbf{J}^k . \quad (4.34)
\]

This result is achieved by promoting the collective variables to operators obeying the relations

\[
\left[ \lambda^a, \lambda^b \right] = \frac{1}{\mathcal{I}} i \epsilon^{abc} \lambda^c \quad (4.35)
\]

\[
\left[ \lambda^a, R_0 \right] = \frac{1}{\mathcal{I}} R_0 \frac{\tau^a}{2} . \quad (4.36)
\]

One can check that the equations (4.32),(4.33) and (4.34) are then fulfilled with this choice.

An explicit form for the operator \( \vec{\lambda} \) is the following

\[
\lambda^a = \frac{i}{2\mathcal{I}} \left[ b^a \frac{\partial}{\partial b_0} - b_0 \frac{\partial}{\partial b^a} - \epsilon^{abc} b^b \frac{\partial}{\partial b^c} \right] , \quad (4.37)
\]

where we have expressed the rotation matrix \( R(t) \) in the form

\[
R(t) \equiv R_0 e^{\frac{i}{\mathcal{I}} \vec{\tau} \cdot \vec{\lambda}_t} = (b_0 + i \vec{b} \cdot \vec{\tau}) e^{\frac{i}{\mathcal{I}} \vec{\tau} \cdot \vec{\lambda}_t} .
\]

\( ^7 \)The angular momentum is obtained through the energy-momentum tensor in the usual fashion.
The solutions corresponding to a definite spin and isospin, i.e. the nucleon and the Δ, will now be expressed in terms of these collective variables:

\[ |p \uparrow \rangle = \frac{1}{\pi}(b_1 + ib_2) \], \[ |p \downarrow \rangle = -\frac{i}{\pi}(b_0 - ib_3) \]

\[ |n \uparrow \rangle = \frac{i}{\pi}(b_0 + ib_3) \], \[ |n \downarrow \rangle = -\frac{1}{\pi}(b_1 - ib_2) \]

\[ |\Delta^{+\frac{3}{2}} \rangle = \frac{i\sqrt{2}}{\pi}(b_1 + ib_2)^3 \], \[ \ldots \]

Let us consider, for example, \( |p \uparrow \rangle \):

\[ \lambda^a |p \uparrow \rangle = \frac{1}{\pi} \frac{i}{2I} (-\delta_{a1}b_0 - i\delta_{a2}b_0 - \epsilon_{a12}b_1i - \epsilon_{a21}b_2^2) \] (4.38)

Notice that in Appendix H we derive the formula

\[ C_{ab} = \frac{1}{2} \text{tr}_{\text{flavor}} \left[ \tau^a R_0 \tau^b R_0^\dagger \right] = (b_0^2 - \vec{b}^2)\delta_{ab} + 2\vec{b}^a b^b + 2\epsilon_{abc}b_0b^c \] (4.39)

By using this expression it is now easy to check that

\[ I_3 |p \uparrow \rangle = \frac{1}{2} |p \uparrow \rangle \], \[ J^3 |p \uparrow \rangle = \frac{1}{2} |p \uparrow \rangle \]

More detail can be found in Appendix H. Thus in summary, the cranking model procedure does indeed generate the sought-after independent eigenstates of angular momentum and isospin.

### 4.2 Physical observables

Having performed the semiclassical projection over states of definite spin and isospin, one is now able to calculate observables related to the nucleon or to the Δ.

The isoscalar component of the electromagnetic current, in particular, will be simply given by (recall that now \( \psi \rightarrow \psi + \delta\psi \))

\[ B^\mu = \frac{1}{N_c} \bar{\psi} \gamma^\mu \psi \]

\[ B^0 = \frac{1}{4\pi} \left[ F^2(r) + G^2(r) \right] \]

\[ B^i = \frac{1}{4\pi r} \left( 2A(r)G(r) + \frac{2}{3}B(r)G(r) - 2D(r)F(r) \right) (\vec{\lambda} \times \vec{r})^i \]
By using the previous expressions, the isoscalar mean square radius and the isoscalar magnetic moment of the nucleon can be calculated. As a matter of fact, one finds that the isoscalar mean square radius is given by

$$\langle r^2 \rangle_{I=0} = \int dr \ r^4 \ (F^2(r) + G^2(r)), \quad (4.40)$$

where the integral over the collective variables simply yields the normalization integral, i.e. 1. Hence the isoscalar radius is independent of spin and isospin projection, as advertised.

The isoscalar magnetic moment is given by

$$\bar{\mu}_{I=0} = \int d^3r \ [\vec{r} \times \vec{j}]_{I=0} = \frac{1}{2} \int d^3r \ \vec{r} \times \vec{B}. \quad (4.41)$$

Note that $\bar{\mu}_{I=0}$ is an operator in the space of the collective coordinates. For a spin-up proton, in particular, one obtains:

$$\bar{\mu}_{I=0} = \frac{1}{2} \langle p \uparrow | \int d^3r \ \vec{r} \times \vec{B} | p \uparrow \rangle = \langle p \uparrow | \lambda_3 | p \uparrow \rangle \frac{2}{3} \int dr \ r^3 \left[ A(r)G(r) + \frac{1}{3} B(r)G(r) - D(r)F(r) \right]$$

$$= \frac{1}{3I} \int dr \ r^3 \left[ A(r)G(r) + \frac{1}{3} B(r)G(r) - D(r)F(r) \right]. \quad (4.42)$$

Let us now consider the isovector current which is given by

$$V_\mu^a = \frac{1}{2} \bar{\psi} \gamma^\mu \tau^a \psi + \epsilon_{abc} \phi^b \partial_\mu \phi^c. \quad (4.43)$$

The isovector mean square radius is now

$$\langle r^2 \rangle_{I=1} = \frac{1}{I} \int dr \ r^4 \left[ N_c \left( A(r)F(r) + \frac{1}{3} C(r)G(r) \right) \right. $$

$$- \left. \frac{2}{3} D(r)G(r) \right) + \frac{8\pi}{3} \phi^2(r) \right]. \quad (4.44)$$

The isovector magnetic moment of the nucleon will be given by

$$\bar{\mu}_{I=1} = \int d^3r \ \vec{r} \times \vec{V}_3. \quad (4.45)$$
Table 4.1: Semiclassical quantization of the hedgehog in the non-linear $\sigma$ model with quarks (4.16). The constituent quark mass and the pion decay constant are taken as input.

<table>
<thead>
<tr>
<th></th>
<th>hedgehog</th>
<th>semiclassical</th>
<th>experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_q^I$ (MeV)</td>
<td>400</td>
<td>400</td>
<td>-</td>
</tr>
<tr>
<td>$f_\pi^I$ (MeV)</td>
<td>93</td>
<td>93</td>
<td>93.</td>
</tr>
<tr>
<td>$E_{CM}$ (MeV)</td>
<td>252.8</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$m_\Delta - m_N$ (MeV)</td>
<td>-</td>
<td>160.1</td>
<td>193.5</td>
</tr>
<tr>
<td>$\frac{m_N + m_\Delta}{2} + E_{CM}$ (MeV)</td>
<td>-</td>
<td>1261.7</td>
<td>-</td>
</tr>
<tr>
<td>$m_N$ (MeV)</td>
<td>-</td>
<td>928.82</td>
<td>938.5</td>
</tr>
<tr>
<td>$m_\Delta$ (MeV)</td>
<td>-</td>
<td>1088.9</td>
<td>1232</td>
</tr>
<tr>
<td>$T_{tot}$ (fm)</td>
<td>-</td>
<td>1.85</td>
<td>-</td>
</tr>
<tr>
<td>$\mu_{I=0}/\mu_N$</td>
<td>-</td>
<td>0.36</td>
<td>0.88</td>
</tr>
<tr>
<td>$\mu_{I=1}/\mu_N$</td>
<td>5.6</td>
<td>-</td>
<td>4.71</td>
</tr>
<tr>
<td>$\mu_{pion}/\mu_N$</td>
<td>3.8</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\mu_{quark}/\mu_N$</td>
<td>1.8</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\sqrt{\langle r^2 \rangle}_{I=0}$ (fm)</td>
<td>0.76</td>
<td>-</td>
<td>0.72</td>
</tr>
<tr>
<td>$\sqrt{\langle r^2 \rangle}_{I=1}$ (fm)</td>
<td>1.38</td>
<td>0.88</td>
<td></td>
</tr>
</tbody>
</table>

The explicit evaluation of this expression yields

$$\mu_{I=1} = \int dr \ r^2 \left( \frac{2}{9} N_c r G(r) F(r) + \frac{8}{9} \pi \phi^2(r) \right). \quad (4.46)$$

Finally one can calculate the nucleon and $\Delta$ masses. As a matter of fact, the Hamiltonian for the system can be cast in the form

$$\hat{H} = E_0 + \frac{1}{2} T \bar{\lambda} \cdot \lambda = E_0 + \frac{j^2}{2 T}, \quad (4.47)$$

where $E_0$ is the mean-field energy. Notice that the previous expression is still an operator in the space of the collective variables. By taking its expectation value in nucleon or $\Delta$ states, the corresponding nucleon and $\Delta$ masses are expected to be obtained as:

$$m_N = E_0 + \frac{3}{8T}, \quad (4.48)$$

$$m_\Delta = E_0 + \frac{15}{8T}. \quad (4.49)$$
Figure 4.1 Solutions to the cranking equations. \( A(r) \) (solid), \( B(r) \) (dotted), \( C(r) \) (dashed) and \( D(r) \) (dot-dashed).

Figure 4.2 Isoscalar (solid) and isovector (dashed) radial charge densities.
These expressions actually overestimate the nucleon and Δ masses because of the presence of a rotational kinetic energy contribution from the quarks already at the mean field level. This is perfectly analogous to the presence of a spurious center-of-mass motion when the mean field solutions are used. In order to avoid a double counting of the rotational energy one needs to subtract this spurious mean field contribution and write the correct equation

$$m_{N,\Delta} = E_0 + \frac{J(J+1)}{2I} - \frac{(J^2)_{MF}}{2I}.$$ (4.50)

Notice that

$$J^1 = L^i + \frac{\sigma^i}{2},$$

$$J^2 = L^2 + \sigma \cdot L + \frac{3}{4}.$$

It is straightforward to prove that, for the hedgehog spinor,

$$\psi^\dagger L^2 \psi = \frac{1}{2\pi} F^2(r)$$

$$\psi^\dagger \sigma \cdot L \psi = -\frac{1}{2\pi} F^2(r)$$

---

Equations (4.48) and (4.49) hold true for the Skyrme model where quarks are not present.
and therefore

\[ \psi^\dagger \mathcal{J}^2 \psi = \frac{3}{4} \psi^\dagger \psi. \quad (4.51) \]

From this

\[ \langle \mathcal{J}^2 \rangle_{MF} = \frac{3}{4} N_c \quad (4.52) \]

follows and

\[ m_{N,\Delta} = E_0 \mp \frac{3}{4L}. \quad (4.53) \]

In the table (4.1) the numerical results for some nucleon observables are displayed. The constituent quark mass and the pion decay constant are taken as inputs. The second column contains quantities which depend only on the pion field and on the hedgehog spinor, whereas in the third column the solutions which depend also on the cranking corrections are displayed. The results essentially reproduce those of [Bro86].

### 4.3 The Wigner-Seitz approximation

We now want to extend the results of the previous sections to the Wigner-Seitz approximation. This is the new contribution of the present work. As already explained, in this approximation the complex many-body dynamical problem is converted to a single-particle problem, with suitable boundary conditions, which reflect the presence of surrounding nucleons. In the absence of quarks, such as in the Skyrme model, this

\[ \frac{1}{3} [C'(r) - 2D'(r)] = -\frac{2}{3} \frac{C(r) - 2D(r)}{r} + (E + g\sigma_h(r))A(r) + \frac{1}{3} g\phi_h(r)(C(r) + 2D(r)) + \frac{G(r)}{2}. \]

Also note that here \( g \equiv M/f_\pi \).
task is easily achieved since the semiclassical quantization does not introduce new dynamical quantities. This case has been discussed in Chapter 2.

In a model which contains quarks, however, the semi-classical quantization generates new dynamical quantities. These are the cranking corrections to the hedgehog spinor, given in equation (4.23). Appropriate boundary conditions for these quantities have to be imposed. Clearly, four boundary conditions are needed, as one can see by inspection of the equations (4.24),(4.25),(4.26), (4.27). However, the requirement of obtaining solutions which are well behaved at the origin completely determines two of these boundary conditions (see eqns. (G.10) and (G.11)). In free space the remaining two boundary conditions are such to guarantee the exponential decay of the solutions at large distances from the origin.

In a finite volume cell, which contains a single nucleon, we require that the isovector density vanish at the surface of the cell in symmetric nuclear matter. This is done in order to account for the presence of a neighboring nucleon with opposite isospin. This argument would be strictly valid in a 1-dimensional system, where “nuclear matter” would then consist of a linear chain of neutrons and protons. In a real three-dimensional system the configuration is more complex and therefore this boundary condition is meant only to give an average description. A second boundary condition has still to be enforced: we require that the baryon current be continuous at the surface of the WS cell. This entails the condition $B^i(R) = 0$.

It is possible to express the previous conditions explicitly in terms of the cranked spinors as

\[
N_c \left\{ A(R)G(R) + \frac{1}{3} F(R) \left[ C(R) - 2D(R) \right] \right\} + \frac{8\pi}{3} f_\pi^2 \sin^2 \theta(R) = 0 \quad (4.54)
\]

and

\[
\left( A(R) + \frac{1}{3} B(R) \right) F(R) - D(R)G(R) = 0 . \quad (4.55)
\]
Table 4.2  Semiclassical quantization of the hedgehog in the non-linear $\sigma$ model with quarks in the Wigner-Seitz approximation, corresponding to $R_{WS} = 1.5$ fm. These results should be compared with those in column two of Table 4.1.

In table (4.2) we display the numerical results corresponding to the Wigner-Seitz radius $R_{WS} = 1.5$ fm with the above boundary conditions and using the boundary conditions of Set III for the mean field solutions.

Notice that, in the limit of low density, i.e. for large $R$, the correct free-space behaviour of the solutions, i.e. exponential decay at large distances, is obtained. By looking at the table and at the following figures we observe an increase at moderate densities of the $N - \Delta$ splitting and of the nucleon and $\Delta$ masses. Also, the isovector magnetic moment, which is displayed in (4.5) appears to be very sensitive to the density\textsuperscript{10}. A modest reduction of the isoscalar mean square radius is also observed at finite density; the isovector mean square radius, on the other hand, displays a much stronger suppression.

\textsuperscript{10}This is true in particular for the pion contribution, as it can be observed by looking at table (4.1) and to table (4.2).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$M^I_q$ (MeV) & 400 \\
$f^{\uparrow}_\pi$ (MeV) & 93 \\
$E_{CM}$ (MeV) & 312.2 \\
$m_\Delta - m_N$ (MeV) & 267.1 \\
$\frac{m_N - m_\Delta}{2} + E_{CM}$ (MeV) & 1439.7 \\
$m_N$ (MeV) & 993.9 \\
$m_\Delta$ (MeV) & 1261.1 \\
$I_{tot}$ (fm) & 1.1 \\
$\mu_{I=0}/\mu_N$ & 0.6 \\
$\mu_{I=1}/\mu_N$ & 2.6 \\
$\mu_{I=1}^{\pi}$ & 1.1 \\
$\mu_{I=1}^{\text{quark}}/\mu_N$ & 1.5 \\
$\sqrt{\langle r^2 \rangle}_{I=0}$ (fm) & 0.66 \\
$\sqrt{\langle r^2 \rangle}_{I=1}$ (fm) & 0.72 \\
\hline
\end{tabular}
\caption{Semiclassical quantization of the hedgehog in the non-linear $\sigma$ model with quarks in the Wigner-Seitz approximation, corresponding to $R_{WS} = 1.5$ fm.}
\end{table}
Figure 4.4 Radial charge densities for the proton (solid) and neutron (dashed) corresponding to $R_{WS} = 1.5$ fm.

Figure 4.5 Isoscalar (solid) and isovector (dashed) nucleon magnetic moments as a function of the Wigner-Seitz radius.
Figure 4.6 Total (solid) moment of inertia and its decomposition into quark (dashed) and pion (dotted) contributions as a function of Wigner-Seitz radius.

Figure 4.7 Nucleon (solid) and Δ (dashed) masses as a function of the Wigner-Seitz radius.
4.4 Conclusions

In this chapter we have discussed the semi-classical quantization of a chiral model with quarks, which is needed to obtain solutions carrying definite spin and isospin. In order to simplify the numerical calculations, a simpler model than the one considered in the previous chapter has been used: in this model the mesonic contributions are not generated dynamically from the Dirac sea, but are rather inserted at the level of the classical lagrangian. The nonlinear sigma model lagrangian is here chosen to describe the mesons. Within this model we have reproduced the free-space results of [Bro86], which were carried out using the Linear Sigma Model coupled to quarks. The original contribution in this Chapter has been the extension of the model to the Wigner-Seitz approximation: solutions corresponding to different densities (Wigner-Seitz radii) have been obtained numerically. These results suggest that, in the present model, some physical quantities, such as the nucleon mass or the isovector magnetic moment, have a rather strong dependence upon the density. We feel however that this behaviour could be induced by some of the shortcomings of this description (and similar ones). In particular, the absence of a confining mechanism for the quarks, while not crucial in free space, must certainly be relevant at finite density, by allowing an unphysical sharing of quarks between different nucleons. In this respect, it would be interesting to compare the findings in this model with those obtained within models which confine the quarks. In the chiral-bag model, for example, in which the quarks are restricted within a bag of finite size bag, it is likely that, at densities for which the bags do not overlap, only the external (to the bag) pion field would be sensitive to the different boundary conditions, thus probably yielding more moderate effects.\(^{11}\)

A second factor to be taken into account is the behaviour of the vacuum contributions when the hedgehog solutions are quantized semi-classically. In principle, such

\(^{11}\)Moreover, a finite band structure for the quarks would also be avoided at low densities, in this picture.
contributions can be calculated by extending the approach which we have adopted in Chapter 3, i.e. the two-point approximation of [Adj92]. In fact, such a task has been performed by [Adj95]. Unfortunately, the authors in [Adj95] have found out that this approximation does not work very well in this case. The application of the Kahana-Ripka algorithm seems therefore to be needed in order to extract numerical results.
Chapter 5
Quark distribution functions

5.1 Introduction

In this Chapter we discuss the calculation of the quark distribution functions in nuclear matter. The quark distribution functions are introduced in the study of Deep Inelastic Scattering (DIS) processes on the nucleon (see, for example, [Wal95]). These processes are characterized by a large four-momentum transfer \( q_\mu^2 \to \infty \) and a large energy transfer \( \omega = (q \cdot P)/M \to \infty \). \( q_\mu \) is the four-momentum of the virtual photon exchanged between the leptons and the quarks (see Fig. 5.1) and \( P \) and \( M \) are the four-momentum and the rest mass of the hadronic target.

The cross-section for the process in Fig. 5.1 has the form

\[
\frac{d^2\sigma}{d\Omega \, dE'} = \frac{\sigma_{\text{Mott}}}{M} \left[ W_2(\omega, q_\mu^2) + 2W_1(\omega, q_\mu^2) \tan^2 \frac{\theta}{2} \right],
\]

where \( \sigma_{\text{Mott}} \) is the Mott cross-section, given by

\[
\sigma_{\text{Mott}} = \frac{\alpha^2 \cos^2 \theta/2}{4 \, \epsilon^2 \, \sin^4 \theta/2}. \tag{5.2}
\]

The hadronic tensor, which enters in the calculation of the invariant amplitude for this process, can be written in terms of the structure functions \( W_{1,2}(\omega, q_\mu^2) \) in the most general form

\[
W^{\mu\nu} = - \left( g^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q_\mu^2} \right) W_1(\omega, q_\mu^2) + \frac{1}{M^2} \left( p^{\mu} - \frac{q \cdot p}{q_\mu^2} q^{\mu} \right) \left( p^{\nu} - \frac{q \cdot p}{q_\mu^2} q^{\nu} \right) W_2(\omega, q_\mu^2). \tag{5.3}
\]

Note that current conservation imposes \( q_\mu W^{\mu\nu} = 0 \). All the information on the structure of the nucleon is contained in \( W_1 \) and \( W_2 \).
In the DIS regime it is observed experimentally that the structure functions of the nucleon display an approximate scaling behavior and they become functions of only one variable, the Bjorken variable, which is defined as $x_B \equiv -q^2/(2q \cdot P)$.

A successful explanation of this behavior was provided by the parton model\cite{Fey69}, in which the DIS is described as the result of the incoherent scattering off the individual constituents of the nucleon, i.e. the quarks. In a frame in which the nucleon moves very fast, the infinite momentum frame, the quark distribution functions simply count the number of quarks of a given species (i.e. of a given flavor or polarization) carrying a fraction $x_B$ of the nucleon's momentum.

Fig. 5.1 illustrates the kinematics for the deep inelastic scattering of electrons on a nucleon. We assume that the quark absorbing the virtual photon carries a fraction $\xi$ of the nucleon's momentum ($0 \leq \xi \leq 1$), i.e. that $k^\mu = \xi P^\mu$. By squaring the quark momentum one has that

$$
k'_\mu = (\xi P_\mu + q_\mu)^2 = \xi^2 P_\mu^2 + q_\mu^2 + 2\xi q_\mu P^\mu.
$$

(5.4)

\begin{figure}[h]
\centering
\includegraphics{figure5_1}
\caption{Kinematics of the Deep Inelastic Scattering.}
\end{figure}
Therefore, in the DIS regime, where it is possible to neglect $P_\mu^2$ and $k_\mu^2$ one obtains that $\xi = x_B$, as anticipated: the Bjorken variable corresponds to the fraction of momentum of the nucleon carried by the quark in the DIS process.

Under the assumption of scattering from Dirac quarks, each carrying a pointlike charge $q_i |e|$, it is then possible to write the hadronic tensor for the nucleon, $W^{\mu\nu}$, as the result of the incoherent sum over all the individual hadronic tensors of the quarks:

$$W^{\mu\nu} = \sum_i \int d\xi \, D_i(\xi) \, W^{(i)}_{\mu\nu}, \tag{5.5}$$

where $D_i(\xi)$ is the quark distribution function, yielding the number of quarks with quantum numbers $i$ (specifying the flavor and polarization of the quark) with a fraction $\xi$ of the momentum of the nucleon. By explicitly calculating $W^{(i)}_{\mu\nu}$ one is then

---

Figure 5.2 $O(\alpha_s)$ contribution to the DIS process in Fig. 5.1.
Figure 5.3 $\sigma_A/\sigma_D$ for different nuclei as a function of the Bjorken variable $x_B$. The data are taken from [Gom94].

able to find an expression for the nucleon structure functions

$$F_1(x_B) = 2 W_1 = \sum_i q_i^2 D_i(x_B) \quad (5.6)$$

$$F_2(x_B) = \frac{\omega}{M} W_2 = \sum_i q_i^2 x_B D_i(x_B) \quad (5.7)$$

Note that the inclusion of radiative gluonic processes allows to account for the observed scaling violations. Such contributions are calculated through the Altarelli-Parisi equations[Alt77].

Experimentally it has been found that the distribution functions of quarks inside a nucleus are significantly different from those in the nucleon (this was first discovered by the European Muon Collaboration (EMC)[Aub83]). In Fig. 5.3 we display the experimental ratio between the cross-section for deep inelastic scattering off a nucleus and off deuterium (the data are taken from [Gom94]). Similarly, in Fig. 5.4 we display the experimental ratio between $F_2^{(A)}$ and $F_2^{(Deut)}$.

One can see that the behavior of the ratios $\sigma_A/\sigma_D$ and $F_2^{(A)}/F_2^{(D)}$ differs in three
regions: in the region of small $x_B$, i.e. for $x_B \lesssim 0.1$, $\sigma_A/\sigma_D < 1$ (this is usually referred to as "shadowing"); for $0.1 \lesssim x_B \lesssim 0.3$, $\sigma_A/\sigma_D > 1$; finally, in the interval $0.3 \lesssim x_B \lesssim 0.8$ the ratio is significantly reduced with respect to the free case. This latter region should be dominated by the contributions of the valence quarks.

Among the different theoretical scenarios which have been suggested in order to explain this behavior are the nuclear binding energies (see [Kul94]), the modification of nucleon structure (see for example [Naa93]), the presence of multiquark states in nuclei (see [Koe95]) and the exchange of quark in nuclei (see [Hoo87]). The reader will find a detailed discussion of the present theoretical status in [Rob90, Gee95, Pil00].

Notice that the quark distribution functions carry information on the non-perturbative regime of the strong interaction and that such a regime is not presently accessible directly through QCD, but only through models. The quark-soliton model of the nucleon, in particular, is a field theoretical model, which has chiral symmetry built-in.
and it is formulated in terms of the relevant degrees of freedom, i.e. constituent quarks and pions. It is also important to have a covariant model of the nucleon: this allows one to transform between different frames of reference (i.e. the infinite-momentum frame, where the distributions are naturally introduced, and the nucleon rest frame, where the model is generally solved). Also, the contribution of the sea to the quark distribution functions can be calculated in this model and we will confine ourselves to discuss only the valence quark contributions.

The main goal of the present Chapter is to offer a simple picture of the modification of the quark distribution functions in nuclei, in the chiral quark-soliton model of the nucleon, where the Wigner-Seitz calculation of the present Thesis describes the modification of the properties of the nucleon in nuclear matter.

The Chapter is organized in two parts. In the first part we rederive some of the results in [Dia96, Dia98], concerning the calculation of the quark distribution function in the chiral quark-soliton model of the nucleon and apply the Wigner-Seitz approximation to calculate the in-medium modification of the quark distribution functions.

In the second part of the Chapter we discuss the application of the convolution model to the calculation of the quark distribution function. Finally we apply these results to obtain the theoretical ratio $F_2^{(A)}/F_2^{(D)}$.

### 5.2 Distribution function for a free nucleon

In this Section we will derive the quark distribution functions within a chiral and covariant model of the nucleon, which uses constituent quarks and pions as elementary degrees of freedom. The original derivation of the parton distribution functions in this model was performed in [Dia96, Dia97]. Finally, we will apply the results to a simple chiral model of the nucleon (which was used in Chapter 4). Notice that the
The quark distribution functions are by definition the number of quarks and antiquarks whose momentum along the z direction is a fraction $x_B$ of the nucleon momentum $P_N$, in the infinite momentum frame.

In the infinite momentum frame the momentum of the nucleon is given by:

$$P_N = m_N v / \sqrt{1 - v^2} ,$$  \hspace{1cm} (5.8)

where $v \to 1$ is the velocity of the nucleon in this frame, which is assumed to be along the $z$ axis for simplicity.

In terms of the creation and annihilation operators for quarks and antiquarks ($\hat{a}^\dagger$, $\hat{a}$, $\hat{b}^\dagger$ and $\hat{b}$) one can write

$$D_i(x) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{P_N} \delta \left( x_B - \frac{k^z}{P_N} \right) \langle N_v | \hat{a}^\dagger_i (k) \hat{a}_i (k) | N_v \rangle$$  \hspace{1cm} (5.9)

$$\bar{D}_i(x) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{P_N} \delta \left( x_B - \frac{k^z}{P_N} \right) \langle N_v | \hat{b}^\dagger_i (k) \hat{b}_i (k) | N_v \rangle .$$  \hspace{1cm} (5.10)

Here $|N_v\rangle$ describes a nucleon moving with velocity $v$ along the $z$ axis and $i$ are the flavor, spin and color indices of a quark. Note that the matrix elements $\langle N_v | \hat{a}^\dagger_i (k) \hat{a}_i (k) | N_v \rangle$ and $\langle N_v | \hat{b}^\dagger_i (k) \hat{b}_i (k) | N_v \rangle$ clearly yield the probability of finding a quark or antiquark in the nucleon with momentum $k$ and quantum numbers $i$.

Equations (5.9) and (5.10) can also be written in terms of field operators. In the infinite momentum frame one can write

$$\hat{\psi}(\vec{x}, t) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\sqrt{2k_0}} \sum_c \left[ \hat{a}_c (k) e^{-ik \cdot x} u_c (\vec{k}) + \hat{b}_c e^{ik \cdot x} \bar{u}_c (\vec{k}) \right] ,$$

where the quark spinors are normalized according to

$$\bar{u}u = -\bar{v}v = 2M .$$

In order to obtain the combination of creation and annihilation operators of eqn. (5.9) and (5.10) we consider the Fourier transform of the equal time product of the
the fields
\[
\int d^3x_1 d^3x_2 \ e^{-i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} \psi^\dagger (\vec{x}_2, t) \psi (\vec{x}_1, t) = \sum_{c, c'} \frac{1}{2k_0} \left[ \hat{a}_c^\dagger (k) \hat{a}_{c'} (k) u_c^* (k) u_{c'} (k) + \hat{b}_c (-k) \hat{b}_{c'} (-k) u_c^* (-k) u_{c'} (-k) \right. \\
\left. + e^{2ik_0 t} \hat{a}_c^\dagger (-k) b_{c'}^\dagger (-k) u_c^* (-k) u_{c'} (-k) + e^{-2ik_0 t} \hat{b}_c (-k) \hat{a}_{c'} (k) u_c^* (k) u_{c'} (k) \right].
\]

The first term in this expression essentially coincides with the one in eq. (5.9), once the expectation value between nucleon states is calculated. It yields the probability of finding a quark carrying a momentum \( k \), in the infinite momentum frame \((v \to 1)\). The second term in the r.h.s. creates an antiquark with momentum \( k \) and destroys and anti-quark with momentum \(-k\). Finally, the third and fourth term are related to the amplitude of finding a quark-antiquark pair with large and opposite momenta in the infinite momentum frame. The last three terms will contribute to the structure functions of non-leading twist and will be neglected here. By retaining only the first term, one finally obtains

\[
D_i (x_B) = \int \frac{d^3k}{(2\pi)^3} \delta \left( x_B - \frac{k^z}{P_N} \right) \int d^3x_1 d^3x_2 e^{-i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} \\
\cdot \langle N_v | \bar{\psi} (\vec{x}_2, t) \Gamma_i \psi (\vec{x}_1, t) | N_v \rangle \tag{5.11}
\]

\[
\overline{D}_i (x_B) = \int \frac{d^3k}{(2\pi)^3} \delta \left( x_B - \frac{k^z}{P_N} \right) \int d^3x_1 d^3x_2 e^{-i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} \\
\cdot \langle N_v | Tr \left[ \Gamma_i \psi (\vec{x}_2, t) \bar{\psi} (\vec{x}_1, t) \right] | N_v \rangle. \tag{5.12}
\]

Notice that \( \Gamma_i \) are flavor and spin matrices and their form depends on the particular distribution function in which one is interested. In particular, in order to obtain the number of partons polarized along \( v \) this matrix must have the form:

\[
\Gamma_i = \gamma^0 \frac{1 \pm \gamma^5}{2}. \tag{5.13}
\]

It is now possible to express the equations (5.11) and (5.12) in terms of the Green's function for a quark inside the nucleon, which is defined to be (see e.g. [Fet71]):

\[
G_F (x_1, x_2) = -i \langle N_v | T \{ \psi (x_1, t_1) \bar{\psi} (x_2, t_2) \} | N_v \rangle. \tag{5.14}
\]
In the chiral quark-soliton model considered in Chapter 3 or in the non-linear sigma model of Chapter 4 the Green's function fulfills the equation:

$$\left[i\ddot{\phi} - M\vec{U}(x_1, t_1)\right]G_F(x_1, t_1, x_2, t_2) = \delta^4(x_1 - x_2)$$  \hspace{1cm} (5.15)

where

$$\vec{U}(x, t) = U\left(\frac{\vec{x} - \vec{v}t}{\sqrt{1 - \nu^2}}\right)$$

is the soliton in the infinite momentum frame.

By employing the previous relations, the distribution functions can now be directly written in terms of the Green's functions for a quark in the infinite momentum frame as

$$D_1(x_B) = -iN_c\int \frac{d^3k}{(2\pi)^3} \delta\left(x_B - \frac{k^z}{P_N}\right)$$

$$\int d^3x_1 \int d^3x_2 e^{-ik\cdot(x_1-x_2)} Tr\left[\Gamma_iG_F(x_1, t_1, x_2, t_2)\right]_{t_2=t^1_t}$$  \hspace{1cm} (5.16)

$$\overline{D}_1(x_B) = iN_c\int \frac{d^3k}{(2\pi)^3} \delta\left(x_B - \frac{k^z}{P_N}\right)$$

$$\int d^3x_1 \int d^3x_2 e^{-ik\cdot(x_1-x_2)} Tr\left[\Gamma_iG_F(x_1, t_1, x_2, t_2)\right]_{t_1=t^1_t} .$$  \hspace{1cm} (5.17)

The equations (5.16) and (5.17) are not particularly useful, since they are expressed in terms of the Green's function of the quark in the infinite momentum frame, whereas quark model calculations are usually performed in the rest frame of the nucleon. However, given the covariant nature of the model, it is possible to relate quantities in different frames by means of a Lorentz transformation.

We will therefore consider the quark Green's function in the rest frame of the nucleon and then boost to the infinite momentum frame, thus obtaining the appropriate expressions to be used in equations (5.16) and (5.17). In the rest frame of the nucleon
one can write:

$$G_F^{(RF)}(x_1, t_1, x_2, t_2) = -i \left\{ \theta(t_1 - t_2) \sum_{n.o.} \phi_n(x_1) \bar{\phi}_n(x_2) e^{-iE_n(t_1 - t_2)} - \theta(t_2 - t_1) \sum_{\text{occ.}} \phi_n(x_1) \bar{\phi}_n(x_2) e^{-iE_n(t_1 - t_2)} \right\} .$$

(5.18)

Here $\phi_n$ are the quark single-particle wave functions and $E_n$ are energy eigenvalues, which are obtained by solving the Dirac equation:

$$\begin{bmatrix} -i\vec{\alpha} \cdot \vec{\nabla} + \beta M U_5(x) \end{bmatrix} \phi_n(x) = E_n \phi_n(x) .$$

(5.19)

Eq. (5.18) can be also written in the form

$$G_F^{(RF)}(x_1, t_1, x_2, t_2) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left\{ - \sum_{n.o.} \phi_n(x_1) \bar{\phi}_n(x_2) \frac{e^{-i(E_n - \omega)(t_1 - t_2)}}{\omega - i\epsilon} + \sum_{\text{occ.}} \phi_n(x_1) \bar{\phi}_n(x_2) \frac{e^{-i(E_n + \omega)(t_1 - t_2)}}{\omega - i\epsilon} \right\} ,$$

(5.20)

by exploiting the integral representation for the step function, i.e.

$$\theta(x) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\omega \frac{e^{i\omega x}}{\omega - i\epsilon} .$$

It is straightforward to check that eq.(5.20) indeed fullfills eq. (5.15), in which $v$ has been set to 0 (this defines the rest frame of the nucleon). As a matter of fact one obtains in this case

$$i\partial_0 G_F(x_1, t_1, x_2, t_2) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left\{ - \sum_{n.o.} \phi_n(x_1) \bar{\phi}_n(x_2) e^{-i(E_n - \omega)(t_1 - t_2)} \frac{E_n - \omega}{\omega - i\epsilon} + \sum_{\text{occ.}} \phi_n(x_1) \bar{\phi}_n(x_2) e^{-i(E_n + \omega)(t_1 - t_2)} \frac{E_n + \omega}{\omega - i\epsilon} \right\}$$

(5.21)
and

\[
[i\partial_t - MU_5(x_1, t_1)] G_F(x_1, t_1, x_2, t_2) = \gamma^0 (i\partial_0 - H) G_F(x_1, t_1, x_2, t_2)
\]

\[
= \gamma^0 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left\{ -\sum_{n, o.c.} \phi_n(x_1) \bar{\phi}_n(x_2) e^{-i(E_n - \omega)(t_1 - t_2)} \frac{-\omega}{\omega - i\epsilon} \right. \\
+ \sum_{n, o.c.} \phi_n(x_1) \bar{\phi}_n(x_2) e^{-i(E_n + \omega)(t_1 - t_2)} \frac{\omega}{\omega - i\epsilon} \left\}
\]

\[
= \gamma^0 \delta(t_1 - t_2) \sum_{n, o.c.} \phi_n(x_1) \bar{\phi}_n(x_2)
\]

\[
= \gamma^0 \delta(t_1 - t_2) \gamma^0 \delta^3(x_1 - x_2) = \delta^4(x_1 - x_2).
\]

(5.22)

Notice that the last line follows form the completeness of the states.

The Green's function in the infinite momentum frame will now be obtained by performing a Lorentz boost along the \(z\) axis. We use the spinor representation of the Lorentz transformation:

\[
\Lambda_{\frac{1}{2}}(v) = \exp \left( -\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu} \right) = \exp \left( -\frac{i}{2} \omega_{03} \sigma^{03} \right)
\]

(5.23)

where

\[
\omega_{03} = -\omega_{30} = \xi
\]

\[
\tanh \xi = v, \quad \cosh \xi = \frac{1}{\sqrt{1 - v^2}}.
\]

Under the boost the \(\gamma\) matrices transform as

\[
\Lambda_{\frac{1}{2}}^{-1}(v) \gamma^\mu \Lambda_{\frac{1}{2}}(v) = \Lambda^\mu_{\nu} \gamma^\nu
\]

(5.24)

\[
\Lambda_{\frac{1}{2}}^{-1}(v) \gamma_5 \Lambda_{\frac{1}{2}}(v) = \gamma_5.
\]

(5.25)

By using the relations

\[
\sigma_{03} = \frac{i}{2} \{\gamma_0, \gamma_3\} = i\gamma_0 \gamma_3.
\]

\[
(\gamma_0 \gamma_3)^2 = 1
\]

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one obtains the explicit representation for the matrices $\Lambda$:

$$\Lambda_{\frac{1}{2}}(\vec{v}) = \exp\left\{ -\frac{1}{2} \gamma_0 \gamma_3 \xi \right\} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{1}{2} \gamma_0 \gamma_3 \xi \right)^n$$

$$= \cosh \frac{\xi}{2} - \gamma_0 \gamma_3 \sinh \frac{\xi}{2}$$

$$\Lambda^{-1}_{\frac{1}{2}}(\vec{v}) = \gamma_0 \Lambda_{\frac{1}{2}}(\vec{v}) \gamma_0 = \cosh \frac{\xi}{2} + \gamma_0 \gamma_3 \sinh \frac{\xi}{2} .$$

The equations (5.23) and (5.24) can therefore be written explicitly as

$$\Lambda_{\frac{1}{2}}(\vec{v}) \gamma_0 \Lambda^{-1}_{\frac{1}{2}}(\vec{v}) = \cosh \xi \gamma_0 + \sinh \xi \gamma_3$$

$$\Lambda_{\frac{1}{2}}(\vec{v}) \gamma_3 \Lambda^{-1}_{\frac{1}{2}}(\vec{v}) = \cosh \xi \gamma_3 + \sinh \xi \gamma_0 .$$

The quark Green's function in the infinite momentum frame will now be obtained through a Lorentz boost on the Green's function in the nucleon rest frame:

$$G_F^{(RF)}(x'_1, t'_1, x'_2, t'_2) = \Lambda_{\frac{1}{2}}^{-1}(v) G_F(x_1, t_1, x_2, t_2) \Lambda_{\frac{1}{2}}(v) . \quad (5.26)$$

The primed and unprimed coordinates appearing in the above equation are in the rest frame of the nucleon and in the infinite momentum frame of the nucleon, respectively. The two are simply related by a Lorentz transformation:

$$x'_1 = x_1 , \quad x'_2 = x_2$$

$$x'_3 = \frac{x_3 - vt}{\sqrt{1 - v^2}} , \quad t' = \frac{t - vx^3}{\sqrt{1 - v^2}} .$$

The z axis has been taken along the direction of $\vec{v}$.

Finally we are able to write the quark distribution functions in the form

$$D_i(x_B) = -iN_c \lim_{\eta \to 0^+} \int \frac{d^4k'}{(2\pi)^4} e^{ik'\eta} m_N \delta (m_N x_B - k'_z - k'_0) \int d^3x'_1 \int d^3x'_2$$

$$e^{-ik'(x'_1 - x'_2)} Tr \left[ (\gamma^0 + \gamma^3) \gamma_i \Gamma_i G_F^{(RF)}(x'_1, t'_1, x'_2, t'_2) \right] \quad (5.27)$$

$$\bar{D}_i(x_B) = iN_c \lim_{\eta \to 0^+} \int \frac{d^4k'}{(2\pi)^4} e^{-ik'\eta} m_N \delta (m_N x_B - k'_z - k'_0) \int d^3x'_1 \int d^3x'_2$$

$$e^{+ik'(x'_1 - x'_2)} Tr \left[ (\gamma^0 + \gamma^3) \gamma_i \Gamma_i G_F^{(RF)}(x'_1, t'_1, x'_2, t'_2) \right] . \quad (5.28)$$

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These expressions have been obtained by exploiting the transformation properties of (5.16) and (5.17) under a Lorentz boost. In fact one has:

\[
\Lambda_{\frac{1}{2}}(\vec{v}) \Gamma_i \Lambda_{\frac{1}{2}}^{-1}(\vec{v}) = (\cosh \xi + \sinh \xi \gamma_0 \gamma_3) \Gamma_i \approx \frac{1}{\sqrt{1 - v^2}} (1 - \gamma^0 \gamma^3) \Gamma_i,
\]

\[
= \frac{1}{\sqrt{1 - v^2}} (\gamma^0 + \gamma^3) \gamma^0 \Gamma_i
\]

(5.29)

\[
\delta \left( x_B - \frac{k^2}{P_N} \right) = \delta \left( x_B - \frac{k'^2}{m_N} + v E'_n \right) \approx m_N \delta \left( m_N x_B - k'^2 - E'_n \right)
\]

(5.30)

\[
\delta(t_2 - t_1 - \eta) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega(t_2 - t_1 - \eta)}
\]

(5.31)

\[
1|_{t_2=t_1} = \int dt_2 \delta(t_2 - t_1 - \eta).
\]

(5.32)

A factor \( \gamma(v) \) is also obtained from the transformation of the volume element between the two frames (the infinite momentum frame and the rest frame of the nucleon):

\[
d^3x' = \frac{1}{\sqrt{1 - v^2}} d^3x = \gamma(v) d^3x.
\]

(5.33)

We will write now the distribution functions in terms of the Fourier representation for the Green's function:

\[
G_F(x_1', t_1', x_2', t_2') = \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \int \frac{d^3p_1}{(2\pi)^3} \int \frac{d^3p_2}{(2\pi)^3} e^{-i(\omega_1 t_1' - \omega_2 t_2')} e^{i(\vec{p}_1 \cdot \vec{x}_1' - \vec{p}_2 \cdot \vec{x}_2')} G_F(\omega_1, \vec{p}_1, \omega_2, \vec{p}_2).
\]

Since the quarks are in a static field one has

\[
G_F(\omega_1, \vec{p}_1, \omega_2, \vec{p}_2) = 2\pi \delta(\omega_1 - \omega_2) S_F(\omega_1, \vec{p}_1, \vec{p}_2).
\]

(5.34)
By exploiting the Fourier representation of $G_F$ one obtains the expression:

$$D_i(x_B) = -i N_c \int \frac{d^4 \mathbf{k}'}{(2\pi)^4} m_N \delta (m_N x_B - k'_z - k'_0) \int d^3 x'_1 \int d^3 x'_2$$

$$\int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \int \frac{d^3 p_1}{(2\pi)^3} \int \frac{d^3 p_2}{(2\pi)^3} e^{-i(\omega_1 t'_1 - \omega_2 t'_2)} e^{i(p_1 \cdot x'_1 - p_2 \cdot x'_2)}$$

$$\cdot (2\pi) \delta(\omega_1 - \omega_2) e^{i k'(x'_1 - x'_2)} \text{Tr} \left[ (\gamma^0 + \gamma^3) \gamma^0 \Gamma_i S_F(\omega_1, \mathbf{p}_1, \mathbf{p}_2) \right]$$

$$= -i N_c \int \frac{d^4 \mathbf{k}'}{(2\pi)^4} m_N \delta (m_N x_B - k'_z - k'_0)$$

$$\cdot \text{Tr} \left[ (\gamma^0 + \gamma^3) \gamma^0 \Gamma_i S_F(k'_0, k'_1, k'_2) \right]$$

(5.35)

Since it is useful to obtain a representation of the quark distribution function directly in terms of the quark wave functions, we consider the Fourier transform of eq. (5.20), i.e.

$$G_F(k_1, k_2) = \int d^4 x_1 \int d^4 x_2 \ e^{i(k_1 \cdot x_1 - k_2 \cdot x_2)} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}$$

$$\left\{ - \sum_{n.o.} \phi_n(x_1) \overline{\phi}_n(x_2) \frac{e^{-i(E_n - \omega)(t_1 - t_2)}}{\omega - i\epsilon} + \sum_{occ.} \phi_n(x_1) \overline{\phi}_n(x_2) \frac{e^{-i(E_n + \omega)(t_1 - t_2)}}{\omega + i\epsilon} \right\}$$

$$= 2\pi \delta(k^0_1 - k^0_2) \left\{ - \sum_{n.o.} \phi_n(k_1) \overline{\phi}_n(k_2) + \sum_{occ.} \phi^0_n(k_1) \overline{\phi^0_n(k_2)} \right\}$$

(5.36)

which allows one to read off $S_F(k_1, k_2)$.

By substituting this expression in the distribution function, one obtains the form

$$D_i(x_B) = -i N_c m_N \int \frac{d^4 k}{(2\pi)^4} \delta(m_N x_B - k_z - k_0)$$

$$\cdot \text{Tr} \left\{ (\gamma^0 + \gamma^3) \gamma^0 \Gamma_i \left[ \sum_{n.o.} \frac{\phi_n(k) \overline{\phi}_n(k)}{k^0 - E_n + i\epsilon} + \sum_{occ.} \frac{\phi_n(k) \overline{\phi}_n(k)}{k^0 - E_n - i\epsilon} \right] \right\}$$

(5.37)

The integral over $k_0$ can be performed in the complex plane by closing the contour of integration either on the upper or lower half-plane. When this is done, the following
equivalent expressions for the distribution function are obtained:

\[
D_{i}(x_{B}) = N_{c}m_{N} \int \frac{d^{3}k}{(2\pi)^{3}} \sum_{\text{occ}} \delta(m_{N}x_{B} - k_{z} - E_{n}) \cdot \phi_{n}^{i}(k)(1 + \gamma^{0}\gamma^{3})\gamma^{0}\Gamma_{i} \phi_{n}(k) = -N_{c}m_{N} \int \frac{d^{3}k}{(2\pi)^{3}} \sum_{\text{n.o.}} \delta(m_{N}x_{B} - k_{z} - E_{n}) \cdot \phi_{n}^{i}(k)(1 + \gamma^{0}\gamma^{3})\gamma^{0}\Gamma_{i} \phi_{n}(k) .
\] (5.38) (5.39)

Notice that in reaching the form of equations (5.38) and (5.39) no assumption has been made on the model used to describe the nucleon, apart from being covariant.

We now specify these formulas in the case of the chiral soliton model. In this case the wave function of the occupied level reads

\[
\phi_{n}(x) = \frac{1}{\sqrt{4\pi}} \left( \begin{array}{c} u(r) \xi \\ v(r) i\sigma \cdot \hat{r} \xi \end{array} \right).
\] (5.40)

\(u(r)\) and \(v(r)\) are the upper and lower components of the Dirac spinors, introduced in Chapter 3.

The Fourier transform of \(\phi\) is easily calculated to give:

\[
\phi_{n}(k) = \int d^{3}x e^{ik \cdot x} \phi_{n}(x) = \int d^{3}x \sum_{l=0}^{\infty} i^{l} \sqrt{4\pi(2l+1)} j_{l}(kr) Y_{l0}(\theta) \phi_{n}(x) .
\] (5.41)

By expressing the unit vector in terms of spherical harmonics:

\[
\hat{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)
= \sqrt{\frac{2\pi}{3}} \left( Y_{1}^{1}(\Omega) - Y_{1}^{-1}(\Omega), Y_{1}^{1}(\Omega) + Y_{1}^{-1}(\Omega), \sqrt{2}Y_{0}^{0}(\Omega) \right)
\]

one obtains

\[
\phi_{n}(k) = \frac{1}{\sqrt{4\pi}} \int dr \ r^{2} \left( \begin{array}{c} 4\pi j_{0}(kr) u(r) \xi \\ i \ j_{i}(kr) 4\pi \int d\Omega \ Y_{l0}(\Omega) \cdot (0, 0, i \sigma^{3}Y_{l0}^{0}(\Omega)) \ v(r) \xi \end{array} \right) = \sqrt{4\pi} \int dr \ r^{2} \left( \begin{array}{c} j_{0}(kr) u(r) \xi \\ -j_{i}(kr) v(r) \sigma \cdot \hat{k} \xi \end{array} \right) \equiv \sqrt{4\pi} \left( \begin{array}{c} U(k) \xi \\ -V(k) \sigma \cdot \hat{k} \xi \end{array} \right).
\] (5.42)
Moreover, if one is interested only in the isosinglet unpolarized distribution functions, the matrix $\Gamma^i$ takes the simpler form

$$ \langle \Gamma^i \rangle = \gamma^0 $$

and therefore

$$ \phi_n(k) (1 + \gamma^0 \gamma^3) \phi_n(k) = 4\pi \left[ U^2(k) + V^2(k) - 2U(k)V(k)\hat{k}^3 \right]. $$

By substituting this expression into eq.(5.38), one obtains the final form

$$ u(x_B) + d(x_B) = 4\pi N_c m_N \int \frac{d^3k}{(2\pi)^3} \delta(m_N x_B - k_z - E_n) $$

$$ \cdot \left[ U^2(k) + V^2(k) - 2U(k)V(k)\hat{k}^3 \right] $$

$$ = \frac{N_c m_N}{\pi} \int_{-1}^{+1} d\cos \theta_k \int_0^\infty dk \ k \delta \left( \cos \theta_k + \frac{E_n - m_N x_B}{k} \right) $$

$$ \theta \left( 1 - \left| \frac{E_n - m_N x_B}{k} \right| \right) \left[ U^2(k) + V^2(k) - 2U(k)V(k)\hat{k}^3 \right] $$

$$ = \frac{N_c m_N}{\pi} \int_{|E_n - m_N x_B|}^\infty dk \ k \left[ U^2(k) + V^2(k) \right] $$

$$ + 2U(k)V(k) \frac{E_n - m_N x_B}{k} \right]. $$

which is suitable for the numerical evaluation. This coincides with eq. (B.5) in [Dia96].

As an example one can consider a simple harmonic oscillator. The quark wave function in this case reads

$$ \psi(x) = \frac{1}{(\pi R^2)^{3/2}} e^{-\frac{x^2}{2R^2}}. $$

In order to apply eq. (5.38) one needs to use the Fourier transforms, which reads

$$ \tilde{\psi}(k) = 2\sqrt{2} \pi^{3/4} R^{3/2} e^{-\frac{e^2 R^2}{4}}. $$

Eq. (5.38) then yields the following expression for the quark distribution function.
in a free nucleon

\[ D(x_B) = N_c \int \frac{d^3 k}{(2\pi)^3} \delta \left( x_B - \frac{E_n + k_z}{m_N} \right) 8\pi^{3/2} R^3 e^{-k^2 R^2} \]

\[ = \frac{N_c m_N R}{\sqrt{\pi}} e^{-k_{m_n}^2 R^2} \], \quad (5.46) \]

where \( k_{m_n} = |m_N x_B - E_n| \).

It is a simple matter to check that the distribution functions introduced in this Section fulfill the baryon sum rule. This is easily done by integrating eq. (5.38) over \( x_B \) and obtaining

\[ \int_{-1}^{+1} dx_B [u(x_B) + d(x_B)] = \int_{0}^{+1} dx_B [u(x_B) + d(x_B) - \bar{u}(x_B) - \bar{v}(x_B)] = N_c , \]

where the relation between quark and antiquark distribution functions,

\[ \bar{u}(x_B) + \bar{v}(x_B) = -u(-x_B) - d(-x_B) \], \quad (5.47) \]

has been exploited in the second part of the equation.

In order to consistently investigate the modification of the properties of a nucleon in nuclear matter, which is the goal of the present thesis, we have applied the Wigner-Seitz solutions, previously obtained, to eq. (5.43). We have first calculated the modification of the distribution functions for a nucleon at rest in nuclear matter. In fig. (5.5) we show the valence quark contribution to the isosinglet distributions \( n(x_B) = u(x_B) + d(x_B) - \bar{u}(x_B) - \bar{d}(x_B) \) (left figure) and \( x_B n(x_B) \) (right figure). The solid and dashed lines refer to free space and to \( R_{WS} = 2 \text{ fm} \), respectively. Notice that for the numerical calculation we have used the simplified model employed in Chapter 4, and we have ignored the contributions stemming from the Dirac sea, which would considerably complicate the problem. The solid lines in this Figure should be compared with the results of Fig.2 of [Dia96], which was obtained by using the chiral-soliton model for a free, isolated nucleon.
Figure 5.5 Valence quark contribution to the isosinglet distributions $n(x_B)$ and $x_B \, n(x_B)$. The solid and dashed lines refer to free space and to $R_{WS} = 2$ fm, respectively.

The distribution $n(x_B)$ fullfills the baryon number sum rule:

$$
\int_0^1 dx_B \, n(x_B) = N_c.
$$

(5.48)

Notice that the effects stemming from the average Fermi motion of the nucleon in nuclear matter have been neglected in obtaining these results. Fig. 5.6 shows that the quark distribution functions in the Wigner-Seitz approximation are very sensitive to the density: in particular the ratio $n_{WS}(x_B)/n(x_B)$ is suppressed at small $x_B$ and grows rapidly at large $x_B$.\(^1\) We believe that this large effects are due to the absence of confining forces in the model which causes the quark wave functions to be very sensitive to the boundary conditions.\(^2\)

\(^1\)Note that at large $x_B$ one is really considering the tail of the distribution in Fig. 1.10. Small changes in the distributions in this region can produce sizable effects on the ratio considered.

\(^2\)Since the soliton doesn’t confine absolutely the quarks, at finite density the quarks acquire larger momentum components, thus leading the behavior shown in the figure.
Figure 5.6 Ratio of the isosinglet distributions $n(x_B)$ in the Wigner-Seitz approximation and in a free nucleon, as given in the previous Figure

5.3 Quark distribution functions in the nuclear medium

We will now extend the results of the previous Section to the calculation of the quark distribution functions for a nucleon moving in the nuclear medium. In doing so, we will assume that the convolution model holds and that the process shown in Fig. (5.7) dominates. We will then generalize the calculation previously carried out for the free nucleon to this case.

Physically, Fig. (5.7) describes the deep-inelastic scattering off a nucleus as the result of the absorption of a high-energy photon by a quark. The probability of finding a quark will be obtained by summing incoherently over all the nucleons forming the nucleus (assumed to be the only nuclear constituents).
Figure 5.7 Deep Inelastic Scattering on a nucleus in the convolution model.

The distribution function in the infinite momentum frame therefore reads

\[
D_t(x_B) = \int dy \int \frac{d^3P_N}{(2\pi)^3} \delta \left( y - \frac{AP_N^z}{P_A} \right) \langle A_v | c_{P_N}^T c_{P_N} | A_v \rangle \\
\cdot \int \frac{d^3k}{(2\pi)^3} \delta \left( x_B - \frac{k^z A}{P_A} \right) \langle N_v | a_k^T a_k | N_v \rangle \\
= \int \frac{dy}{y} \int \frac{d^3P_N}{(2\pi)^3} \delta \left( y - \frac{AP_N^z}{P_A} \right) \langle A_v | c_{P_N}^T c_{P_N} | A_v \rangle \hat{D}_t \left( \frac{x_B}{y}, P \right),
\]

where \( y \) is the fraction of the momentum of the nucleus, \( P_A \), assumed to be along the \( z \) direction, carried by a nucleon; \( \hat{c}_{P_N} \) and \( \hat{c}_{P_N}^T \) are the annihilation and creation operators for a nucleon with momentum \( P_N \). \( |A_v\rangle \) describes a nucleus moving with velocity \( v \to 1 \) and \( |N_v\rangle \) describes a nucleon carrying a momentum \( P_N \) in the same frame. Notice that \( \hat{D}_t(x_B,0) \) is the quark distribution function for a single nucleon calculated in the previous section. It is important to realize that \( \hat{D}_t(x_B,P) \) is in principle different from the quark distribution function in a free nucleon: the most obvious case corresponds to having intrinsic modifications of the nucleon properties in the nuclear medium (see for example the dashed curves in Fig. 5.5); another
possibility is related to the nucleon being off-shell (as discussed below).

In order to set the notation we will use in the following primed variables to refer to momenta in the rest frame of the nucleus and unprimed one to refer to momenta in the infinite momentum frame of the nucleus, unless otherwise specified.

In particular we will call \( P'_{N} = (P'_N, P'_{Nv}) \) the four-momentum of a nucleon in the rest frame of the nucleus. Similarly \( P'_{A} = (M_A, 0) \) will be the four momentum of the nucleus in the nucleus rest frame.

Here \( \tilde{\gamma}(\tilde{v}) = 1/\sqrt{1 - \tilde{v}^2} \) and \( \tilde{v} \) is a (small) velocity, with transverse components, due to the Fermi motion of the nucleons inside the nucleus. \( M_A \) is the mass of the nucleus.

It is now convenient to express the equation (5.49) using the eq. (5.35) derived in the previous Section

\[
D_i(x_B) = -i \int dy \int \frac{d^4P}{(2\pi)^4} \delta(y - y^{(0)}) Tr \left[ \Gamma_i S_F(P_0, \vec{P}, \vec{P}) \right] \\
\cdot -iN_e \int \frac{d^4k}{(2\pi)^4} \delta(x_B - x_B^{(0)}) Tr \left[ \Gamma_i S_F(k_0, \vec{k}, \vec{k}) \right]. \tag{5.50}
\]

Here \( S_F \) and \( S_F \) are the Green's functions of the nucleon and of the quark in the infinite momentum frame in presence of a static background (see eq. (5.34)). We have also introduced the variables:

\[
x_B^{(0)} = \frac{k^2 A}{P_A}, \quad y^{(0)} = \frac{AP'_{Nv}}{P_A}.
\]

Note that eq. (5.50) is manifestly Lorentz invariant. Since for the moment we are interested in calculating the valence contributions to the quark distribution function we can express this equation directly in term of the wave functions of the valence quarks in the form:

\[
D_i(x_B) = -i \int dy \int \frac{d^4P}{(2\pi)^4} \delta(y - y^{(0)}) Tr \left[ \Gamma_i S_F(P_0, \vec{P}, \vec{P}) \right] \\
\cdot N_e \int \frac{d^3k}{(2\pi)^3} \sum_{occ.} \delta(x_B - x_B^{(0)}) \phi_n^i(k) \gamma^0 \Gamma, \phi_n(k) \tag{5.51}
\]
or

\[ D_i(x_B) = \int dy \int \frac{d^3P}{(2\pi)^3} \sum_{\text{occ.}} \delta(y - y^{(0)}) \Psi_i^\dagger(P) \gamma^0 \Gamma_i \Psi(P) \]

\[ \cdot N_c \int \frac{d^4k}{(2\pi)^4} \sum_{\text{occ.}} \delta(x_B - x_{B}^{(0)}) \phi_i^\dagger(k) \gamma^0 \Gamma_i \phi(k), \quad (5.52) \]

where \( \Psi_n(P) \) are the Fourier transforms of the nucleon wave functions in the infinite momentum frame. \( S_F \) is the Green's function of the nucleon in momentum space in the infinite momentum frame; in the rest frame of the nucleus the latter is defined as

\[ G_{RF}(x_1, t_1, x_2, t_2) = -i(A|T \{ \Psi(x_1, t_1) \overline{\Psi}(x_2, t_2) \}|A) \quad (5.53) \]

and

\[ G_{RF}(x_1, t_1, x_2, t_2) = \int \frac{d\omega}{2\pi} \int \frac{d^3P_1}{(2\pi)^3} \int \frac{d^3P_2}{(2\pi)^3} e^{-i\omega(t_1 - t_2)} e^{i\vec{P}_1 \cdot \vec{x}_1 - i\vec{P}_2 \cdot \vec{x}_2} S_{RF}(\omega, P_1, P_2). \]

Here \(|A\rangle\) describes a nucleus at rest. Clearly, all the details concerning the nuclear structure are contained in \( G_{RF} \).

Since it is preferable to cast eqns. (5.51) and (5.52) in a manifestly covariant form, we introduce the four-vector \( n^\mu \) which in the infinite momentum frame reads \( n^\mu = (1, 0, 0, 0) \) and write

\[ D_i(x_B) = -i \int dy \int \frac{d^4P}{(2\pi)^4} \delta(y - y^{(0)}) Tr \left[ \not\! P \gamma^0 \Gamma_i S_F(P_0, \vec{P}_1, \vec{P}_2) \right] \]

\[ \cdot N_c \int \frac{d^4k}{(2\pi)^4} \sum_{\text{occ.}} \delta(x_B - x_{B}^{(0)}) \delta(k_\mu n^\mu) \phi_i^\dagger(k) \not\! P \gamma^0 \Gamma_i \phi(k). \quad (5.54) \]

It is now straightforward to express (5.54) in a different reference frame. We can write the \( z \) component of the nucleon and quark momenta in the infinite momentum frame by means of a Lorentz transformation as

\[ P^z_N = \gamma(v) \left[ u P^0_N + P^z_N \right] \quad (5.55) \]

\[ k^z = \gamma(v) \left[ u E_n + k^z_n \right] \]

\[ = \gamma(v) \frac{E_n^\prime + k^z_n}{m_N} \left[ m_N + P^\prime + \frac{P^2}{E_P + m_N} \right], \quad (5.56) \]
where \( k'' \) is the quark momentum in the rest frame of the nucleon.

Notice that the \( \delta \) function in eq. (5.54) transforms as

\[
\delta(k_\mu n^\mu) = \frac{1}{n_0^\mu} \delta \left( k_0'' - \frac{\vec{k}'' \cdot \vec{n}''}{n_0''} \right) = \frac{1}{(1 + \vec{v}^3) \gamma(\vec{v}) \gamma(\vec{v})} \delta \left( k_0'' - \frac{\vec{k}'' \cdot \vec{n}''}{n_0''} \right)
\]  

(5.57)

Let us consider now a four vector \( J_\mu \equiv \overline{\phi}_n(k) \gamma^\mu \phi_n(k) \) and write the scalar

\[
J_\mu n^\mu = \overline{\phi}_n(k) \not\!\!\! p \gamma^0 \Gamma_i \phi_n(k)
\]  

(5.58)

where \( \phi_n \) are the valence quark wave function in the infinite momentum frame (\( J^\mu \) is thus the baryon current).

In the rest frame of the nucleon one will have\(^3\)

\[
J_\mu n^\mu = \gamma(\vec{v}) \gamma(\vec{v}) \left[ (1 + \vec{v}^3) J_0 + \gamma(\vec{v}) J_3 + \vec{J} \cdot \vec{v} \left( \frac{1}{1 + \gamma(\vec{v})} \right) \right]
\]

\[
= \gamma(\vec{v}) \left[ \frac{E_p + P_z}{m_N} J_0 + J_3 + \frac{\vec{J} \cdot \vec{P}}{m_N} \frac{E_p + m_N + P_z}{E_p + m_N} \right].
\]  

(5.59)

We have used the relations

\[
\gamma(\vec{v}) = \frac{E_p}{m_N}, \quad \vec{v} = \frac{\vec{P}}{E_p}.
\]  

(5.60)

In the model that we are using one has\(^4\)

\[
J^0 \equiv \overline{\phi}_n(k'') \gamma^0 \phi_n(k'') = 4\pi \left[ U(k)^2 + V(k)^2 \right]
\]

\[
J^i \equiv \overline{\phi}_n(k'') \gamma^i \phi_n(k'') = -8\pi \hat{k}^i U(k)V(k).
\]

and therefore one obtains

\[
J_\mu n^\mu = 4\pi \gamma(\vec{v}) \left[ \frac{E_p + P_z}{m_N} \left( U(k)^2 + V(k)^2 \right) - 2 U(k)V(k) \hat{k}^3 
\right.
\]

\[
- 2 U(k)V(k) \frac{\vec{k} \cdot \vec{P}}{m_N} \frac{E_p + m_N + P_z}{E_p + m_N}. \]

(5.61)

\(^3\)This expression is obtained by performing sequentially two Lorentz boosts: a first boost brings us from the infinite momentum frame, where the nucleus has velocity \( \vec{v} = (0, 0, v) \), to the frame in which the nucleus is at rest, but the nucleons move with velocity \( \vec{v} = (v_x, v_y, v_z) \); a second boost brings us to the rest frame of a given nucleon.

\(^4\)In order to avoid too heavy a notation we will simply call \( \phi \) the wave functions of the quarks in the nucleon rest frame.
No assumption has been made so far on the nuclear content of equation (5.54): to keep the discussion simple we will describe the nucleus using a Fermi Gas model. In this case the Green’s function is given by

\[ S_{RF}(p) = \Omega \left\{ \left( P + m_N \right) \left[ \frac{1}{P^2 - m_N^2} + \frac{i\pi}{E_P} \delta(P_0 - E_P)\theta(P_F - |\vec{P}|) \right] \right\} , \]

where \( \Omega \equiv \frac{3\pi^2}{2P_F}N \) is the nuclear volume.

We also write

\[ x_B^{(0)} = \frac{k^z A}{P_A} = \frac{A}{M_A} \frac{E_n'' + k_z''}{m_N} \left[ m_N + P_z + \frac{P^2_z}{E_P + m_N} \right] \]

\[ y^{(0)} = \frac{A P_{Nz}^2}{P_A} = \frac{A}{M_A} \left[ E_P + P^2 \right] . \]

and thus obtain

\[ D_t(x_B) = -i \int dy \int \frac{d^4 P}{(2\pi)^4} \delta(y - y^{(0)}) \text{Tr} \left[ (\gamma^0 + \gamma^3) S_F(P_0, \vec{P}, \vec{P}) \right] \]

\[ \cdot N_c \int \frac{d^3 k}{(2\pi)^3} \sum_{\text{occ.}} \delta(x_B - x_B^{(0)}) \frac{1}{n_0''} \frac{\phi_n(k)}{\gamma^0 \Gamma_i} \phi_n(k) \]

\[ = \Omega \int dy \int \frac{d^4 P}{(2\pi)^4} \delta(y - y^{(0)}) 8\pi \frac{E_P + P_z}{E_P} \theta(P_F - P) \]

\[ \cdot N_c \int \frac{d^3 k}{(2\pi)^3} \delta(x_B - x_B^{(0)}) \frac{m_N}{E_P + P_z} \frac{4\pi}{m_N} \frac{E_P + P_z}{m_N} \left( U(k)^2 + V(k)^2 \right) \]

\[ - 2U(k)V(k)k^3 - 2U(k)V(k) \frac{k \cdot \vec{P}}{m_N} \frac{E_P + m_N + P_z}{E_P + m_N} \] \tag{5.62}

Notice the presence of the “flux factor” \( \frac{E_P + P_z}{E_P} \) in equation (5.62). In order to obtain a simpler expression one can still exploit the Dirac delta function on \( x_B \) to obtain

\[ k^z \equiv k \cos \bar{\theta} = \left[ \frac{M_A m_N (E_P + m_N)x_B}{A(P_z^2 + (E_P + m_N)(P_z + m_N))} - E_n \right] . \tag{5.63} \]

Also, the quark momentum will be constrained by the requirement that \( |\cos \bar{\theta}| \leq 1 \):

\[ k \leq k_{\text{min}} \equiv \left| \frac{M_A m_N (E_P + m_N)x_B}{A(P_z^2 + (E_P + m_N)(P_z + m_N))} - E_n \right| . \tag{5.64} \]

\[ \text{Since we consider nuclear matter, we assume translational invariance.} \]
We therefore obtain the final expression

\[ D_i(x_B) = \frac{\Omega N_c m_N M_A}{2\pi^4} \int_{-P_F}^{P_F} \int_0^{\sqrt{P_F^2-P_z^2}} dP_z \int_0^{2\pi} d\phi_\perp \frac{E_P + P_z}{E_P} \theta(P_F - P) \]

\[ \cdot \int_{k_{\min}}^{\infty} dk \, k \frac{E_P + m_N}{(E_P + m_N)(P_z + m_N) + P_z^2} \left[ (U(k)^2 + V(k)^2) \right. \]

\[ - 2U(k)V(k) \frac{m_N}{E_P + P_z} - 2U(k)V(k) \frac{E_P + m_N + P_z}{(E_P + m_N)^2} \right) \].

(5.65)

where \( \phi_\perp \) is the angle between \( k_\perp \) and \( P_\perp \) and \( k^2 \) is given by eq.(5.63).

After performing the integration over \( \phi_\perp \) in the above expression one obtains

\[ D_i(x_B) = \frac{\Omega N_c m_N M_A}{\pi^3} \int_{-P_F}^{P_F} \int_0^{\sqrt{P_F^2-P_z^2}} dP_z \int_0^{2\pi} d\phi_\perp \frac{E_P + P_z}{E_P} \theta(P_F - P) \]

\[ \cdot \int_{k_{\min}}^{\infty} dk \, k \frac{E_P + m_N}{(E_P + m_N)(P_z + m_N) + P_z^2} \left[ (U(k)^2 + V(k)^2) \right. \]

\[ - 2U(k)V(k) \frac{m_N}{E_P + P_z} - 2U(k)V(k) \frac{E_P + m_N + P_z}{(E_P + m_N)^2} \right) \].

(5.66)

Notice that the nuclear distribution function fullfills the baryon number sum rule

\[ \int_{-A}^{+A} dx_B \left[ u^{(A)}(x_B) + d^{(A)}(x_B) \right] = N_c \].

(5.67)

This is most easily checked by using eq. (5.54) and the fact the quark wave functions are correctly normalized.

It is important to realize that it was possible to obtain eq. (5.62) and (5.66) because of the covariant nature of the model of the nucleon employed. In a non-covariant model, such as for example the simple harmonic oscillator of eq. (5.44), this would not have been possible. We also notice the presence of kinematical effects on the quark wave functions in eq. (5.66) (inside the square brackets), which reflect the action of the average Fermi motion on the quarks. These effects, which are of the order \( P/m_N \), would be completely absent in a non-relativistic model of the nucleon: as a matter of fact, in this case the lower components of the spinor would be absent and therefore no dependence on \( P \) would be found in the square brackets.
Figure 5.8 Kinematic constraint on the nucleon energy.

Figure 5.9 Mass of the $A - 1$ excited nucleus.
Fig. 5.8 illustrates the kinematical constraints on the energy of the nucleon participating to the DIS. In the laboratory frame the conservation of energy requires that

$$P_0 = M_A - E_{A-1}^*$$  \hspace{1cm} (5.68)

where $M_A$ is the rest mass of the nucleus and $E_{A-1}^*$ is the energy of the residual $A - 1$ nucleus. The latter is produced in an excited state and moves with momentum $-\vec{P}$ by virtue of momentum conservation. For heavy nuclei it is safe to neglect the recoil kinetic energy of the $A - 1$ nucleus and therefore approximate $E_{A-1}^*$ with the rest mass of the excited nucleus, i.e. $E_{A-1}^* \approx M_{A-1}^*$. This quantity can be estimated in a mean field description. By looking at Fig. (5.9) we see that

$$M_{A-1}^* \approx M_A - m_N - \delta_B + \langle E_F - T \rangle,$$  \hspace{1cm} (5.69)

$\langle T \rangle$ being the average kinetic energy of a nucleon in the nucleus and $\delta_B \approx -15.6\text{MeV}$ the binding energy per nucleon. As we see form the Figure, $\delta_B$ is the energy required to bring a nucleon from the highest occupied energy level to the continuum. Notice that in the Figure full and empty circles represent particle and hole states respectively. We can therefore write

$$P_0 \approx m_N + \delta = m_N + \delta_B - \langle E_F - T \rangle.$$  \hspace{1cm} (5.70)

An example of application of this formula is shown in Fig. 5.10. A separation energy $\delta \approx -70\text{ MeV}$, taken from [Die91], is used. The Fermi momentum is assumed to be $P_F = 270\text{ MeV/c}$. Notice the effect of the Fermi at large $x_B$, which increases the ratio $D_A(x_B)/D(x_B)$. These results essentially reproduce those in [Kul94]. We have not applied the convolution formula to the Wigner-Seitz wave functions given the unrealistic behavior of the ratio $D_A(x_B)/D(x_B)$ in the Wigner-Seitz description, even in absence of the Fermi motion, which would further increase the ratio.
In this Chapter we have analyzed the problem of describing the modification of the quark distribution functions in Deep Inelastic Scattering on nuclei. The first part of the Chapter has been devoted to rederive some of the results of [Dia96, Dia98], which were originally obtained in the context of the chiral quark soliton model of the nucleus. We have then used the Wigner-Seitz wave functions to estimate the size of in-medium effects in this approximation. This approach turns out to lead to an unrealistic behavior of the ratio between the in-medium and the free parton distribution function (in particular to a much too rapid growth with $x_B$). In the second part of the Chapter we have applied the convolution model to the description of the parton distribution functions in nuclei. A manifestly covariant expression for the distribution function in the medium has been obtained and numerically evaluated. In a fairly simple picture, corresponding to describing the nucleus as a relativistic Fermi

![Figure 5.10 Ratio of the quark distributions using the convolution formula (5.66).](image-url)
gas of nucleons, we have shown that some of the features of the experimental data can be qualitatively reproduced: in particular the ratio $D^{(A)}(x_B)/D^{(D)}(x_B)$ displays a minimum (at about $x_B \approx 0.3$), followed by a region of growth at larger $x_B$. The latter is understood as an effect of the average Fermi motion of the nucleons in the medium. The convolution formula that we have derived in eq. (5.54), starting from the assumptions stated in eq. (5.49), also takes into account the effects of the Fermi motion on the quark wave function, whereas in the usual convolution model the quark distribution functions in a free nucleon are used.\(^6\) Although the numerical results in Fig. (5.10) are similar to those of other authors, the result in Eq. (5.66) is new.

The study of the EMC effect is, in our opinion, very challenging. We believe that at present there is not yet a fully satisfactory theoretical understanding of this phenomenon: this probably also explains the presence of almost orthogonal approaches to the same problem (for example, the convolution model and models based on dynamical rescaling). It is quite possible that each of these approaches contains a little bit of truth and that only our present inability in solving the non-perturbative regime of QCD keeps us from seeing a more complete picture. A satisfactory understanding of this phenomenon requires, in our view, not only a good understanding of the structure of the nuclei and of the nucleon, which is separately provided to a good extent by the models of the nuclei (shell model, relativistic mean field, ...) and of the nucleon (quark models, soliton models, ...), but also of the interplay between the different degrees of freedom in the problem. In fact these degrees of freedom are not independent. This was a motivation for considering in the present thesis an approximation, the Wigner-Seitz approximation, which aims to describe nuclear and nucleonic properties in term of the same degree of freedom, i.e. quark and pions. Unfortunately, the description of the EMC effect extracted from this approximation

\(^6\)See, for example, eq. (70) of [Kul94]. If one sets the momentum of the nucleon to zero in the second line of equation (5.54), that expression factors in the usual convolution model.
is far from satisfactory. We believe that a more realistic description could be obtained in models of the nucleons in which quarks are confined.

Concerning the approach followed in the second part of the present Chapter, a series of interesting extensions is possible — and worth doing — in our opinion. In order these are: the application to the convolution formula of a more sophisticated model of the nucleus, such as, for example, a relativistic mean field; the calculation of the sea contributions (these contributions are indeed estimated by Diakonov [Dia96] on a free nucleon in the chiral quark-soliton model); finally, the calculation of the quark distribution in nuclei accounting for the possible quark-exchange processes.

Concerning this last issue, we remark that the quark distribution function in a nucleus should be obtained as

\[ D_i(x_B) = \int \frac{d^3k}{(2\pi)^3} \delta \left( x_B - \frac{k^2 A}{P_A} \right) \langle A_v|a_k^i a_k|A_v \rangle , \]

in analogy with the case of a free nucleon.

However, in writing eq. (5.49), we have clearly assumed that the probability of finding a quark with a given momentum \( k \) inside a nucleus is obtained by the product of the probability of finding such a quark in a nucleon with a momentum \( P \) and the probability of finding this nucleon inside the nucleus, summing finally over all the nucleon momenta. In other words, we have made the approximation

\[ \langle A_v|a_k^i a_k|A_v \rangle = \int \frac{d^3P}{(2\pi)^3} \langle A_v|c_P^i c_P|A_v \rangle \langle N_v|a_k^i a_k|N_v \rangle . \tag{5.71} \]

However such an approximation disregards completely the possibility that quarks can be exchanged between nucleons inside the nucleus, while participating to the Deep Inelastic Scattering [Hoo87]. These processes can clearly take place in a nucleus and should therefore be accounted for.
Chapter 6
Conclusions

In this thesis we have considered the problem of studying the modifications of the properties of the nucleon when surrounded by other nucleons in a nucleus. A detailed discussion (that we will not repeat here) of this problem and of the different approaches followed in the literature can be found in the Introduction of the present thesis. In particular, we have applied the Wigner-Seitz approximation to the Skyrme model and to the chiral quark soliton model and we have calculated some of the static properties of the nucleon in a mean field approximation. A discussion of these results can be found in Chapters 2 and 3. In Chapter 4 we have then applied a semiclassical quantization, which is needed in order to extract solutions with definite spin and isospin from the mean fields\(^1\). Finally, in Chapter 5 we have considered the calculation of the quark distribution functions in a field theoretical model of the nucleon (the chiral quark soliton model). The expressions for the quark distribution functions in a free nucleon — which reproduce published results — and the modification inside a nucleus have been obtained.

\(^{1}\)The mean field solution do not carry definite spin and isospin.
Appendix A
The Skyrme model

As a reference for the reader, in table A.1 we give here the explicit expressions for the various observables. The following integrals have been defined:

\[
I_1 = \frac{\pi}{2} \int_0^X \tilde{x}^2 \left\{ (\xi'^2 + 2 \frac{\sin^2 \xi}{\tilde{x}^2}) + \frac{\sin^2 \xi}{\tilde{x}^2} \left( 2\xi'^2 + \frac{\sin^2 \xi}{\tilde{x}^2} \right) \right\}
\]

\[
I_2 = \frac{16\pi}{3} \int_0^X \tilde{x}^2 \sin^2 \xi \left[ 1 + \xi'^2 + \frac{\sin^2 \xi}{\tilde{x}^2} \right]
\]

\[
I_3 = -\frac{8}{\pi} \int_0^X \tilde{x}^2 \sin^2 \xi \xi'
\]

\[
I_4 = \frac{32\pi}{3} \int_0^X \tilde{x}^4 \sin^2 \xi \left[ 1 + \xi'^2 + \frac{\sin^2 \xi}{\tilde{x}^2} \right]
\]

\[
I_5 = -\frac{32}{\pi} \int_0^X \tilde{x}^4 \sin^2 \xi \xi'
\]

\[
I_6 = -4 \int_0^X \tilde{x}^2 \left\{ \xi' + \xi'^2 \frac{\sin 2\xi}{\tilde{x}} + 2 \frac{\sin^2 \xi}{\tilde{x}^2} \xi' + \frac{\sin 2\xi}{\tilde{x}} + \frac{\sin 2\xi}{\tilde{x}^2} \frac{\sin^2 \xi}{\tilde{x}^3} \right\}
\]

where \( \tilde{x} \) is the dimensionless variable previously defined.
<table>
<thead>
<tr>
<th>$M$</th>
<th>$\frac{2I_x h^3}{G} I_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>$\frac{h^{11}}{26^3 f_\pi} I_2$</td>
</tr>
<tr>
<td>$\langle r^2 \rangle_{E,0}$</td>
<td>$\frac{1}{4} \left( \frac{h^3}{G f_\pi} \right)^2 I_3$</td>
</tr>
<tr>
<td>$\langle r^2 \rangle_{E,1}$</td>
<td>$\frac{1}{4} \left( \frac{h^3}{G f_\pi} \right)^2 \left( \frac{h^{11}}{26^3 f_\pi} \right) \frac{I_4}{22}$</td>
</tr>
<tr>
<td>$\langle r^2 \rangle_{M,0}$</td>
<td>$\left( \frac{h^3}{26^3 f_\pi} \right)^2 \frac{I_5}{I_3}$</td>
</tr>
<tr>
<td>$\langle r^2 \rangle_{M,1}$</td>
<td>$\langle r^2 \rangle_{E,1}$</td>
</tr>
<tr>
<td>$\mu_p$</td>
<td>$2m_N \left[ \frac{1}{122} \langle r^2 \rangle_{E,0} + \frac{7}{6} \right]$</td>
</tr>
<tr>
<td>$\mu_n$</td>
<td>$2m_N \left[ \frac{1}{122} \langle r^2 \rangle_{E,0} - \frac{7}{6} \right]$</td>
</tr>
<tr>
<td>$\mu_{\Delta^{++}}$</td>
<td>$2m_N \left[ \frac{1}{122} \langle r^2 \rangle_{E,0} + \frac{37}{10} \right]$</td>
</tr>
<tr>
<td>$\mu_{N\Delta}$</td>
<td>$2m_N \left( \frac{\sqrt{2} I}{6} \right)$</td>
</tr>
<tr>
<td>$g_a^{m_\pi=0}$</td>
<td>$\frac{\pi h^6}{3G^2 f_6}$</td>
</tr>
<tr>
<td>$g_a^{m_\pi \neq 0}$</td>
<td>$\frac{2\pi h^6}{9G^2 f_6}$</td>
</tr>
</tbody>
</table>

**Table A.1** Expression of the various observables in terms of the hedgehog solution.
Appendix B

The axial coupling constant

In this appendix we want to explain the factor 3/2 introduced in equation (3.68). At large distances from the center of the nucleon the axial current (3.62) can be approximated as

\[ A^\mu_a(x) \approx -\partial^\mu \phi^a(x) \phi_0(x), \]  

(B.1)

by neglecting the quark contributions (which are exponentially suppressed) and retaining only the leading term in the pion field. By substituting the hedgehog form in the above equation one obtains

\[ A^i_a(x) \approx - \left\{ \frac{\delta_{ia} - \hat{\mathbf{x}}^i \hat{x}^a}{x} \phi(x) + \hat{x}^a \hat{x}^i \phi'(x) \right\}. \]  

(B.2)

One is now able to write the volume integral of the axial current as

\[ \int d^3x \ A^i_a(x) \approx - \frac{4\pi}{3} \int_{r_a}^\infty dr \frac{d}{dr} \left(r^2 \phi(r)\right) = - \frac{4\pi}{3} \left(r^2 \phi(r)\right)|_{r_a}^\infty, \]  

(B.3)

provided that \( r_a \) is large enough such that eq. (B.2) holds. In this case it is also possible to use in eq. (B.3) the asymptotic form of the pion field, which fulfills the Klein-Gordon equation

\[ [-\nabla^2 + m_\pi^2] \phi^a(x) = 0. \]  

(B.4)

Note that the right hand side of eq. (B.4) clearly vanishes at large \( r \) given the asymptotic behaviour of the quark fields. The solutions to eq. (B.4) are:

\[ \phi^a_{m_\pi=0}(r) = \phi(r) \hat{x}^a = \frac{B}{r^2} \hat{x}^a \]  

(B.5)

\[ \phi^a_{m_\pi \neq 0}(r) = \phi(r) \hat{x}^a = \frac{B}{r^2} e^{-m_\pi r} \left[ 1 + m_\pi r \right] \hat{x}^a, \]  

(B.6)
$B$ being a constant of integration. If one uses these asymptotic forms in eq. (B.3) then the following result is found

$$
\left[ \int d^3x \, A_i^a(x) \right]_{m_\pi=0} = 0 \quad \text{(B.7)}
$$

$$
\left[ \int d^3x \, A_i^a(x) \right]_{m_\pi \neq 0} = \frac{4\pi}{3} Be^{-m_\pi r_a} [1 + m_\pi r_a] \quad \text{(B.8)}
$$

Notice that the two expressions differ in the limit $m_\pi = 0$ as a result of the asymptotic behaviour of the pion field at large distances: as a matter of fact, if the mass of the pion is set to zero before performing the integral, then a finite contribution to the integral is allowed from the surface at $r \to \infty$. The discontinuous behaviour in $m_\pi$ can therefore be removed by eliminating this contribution, which we shall do now.

Following [Adk83] we now use the conservation of the axial current in the chiral limit ($\partial^\mu A_\mu^a = 0$) to write the volume integral (this time over all space) in eq. (3.66) as a surface integral

$$
\int d^3x \, A_i^a(x) = \int d^3x \, \partial_j [x_i A_j^a(x)] = \int dS \, x_i A_j^a \varepsilon j . \quad \text{(B.9)}
$$

By using (B.2) one obtains

$$
\left[ \int d^3x \, A_i^a(x) \right]_{m_\pi=0} = - \lim_{r \to \infty} \int d\Omega \, r^3 \varepsilon^i \varepsilon^j \left\{ \frac{\delta_{ja} - \varepsilon^j \varepsilon^a}{r} \phi(r) + \varepsilon^a \varepsilon^j \phi'(r) \right\}
$$

$$
= - \lim_{r \to \infty} \int d\Omega \, r^3 \varepsilon^i \varepsilon^a \phi'(r) = - \frac{4\pi}{3} \delta_{ia} \lim_{r \to \infty} r^3 \phi'(r)
$$

$$
= \frac{8\pi}{3} B \delta_{ia}. \quad \text{(B.10)}
$$

By using the equations (B.8) and (B.10) we conclude that

$$
\lim_{m_\pi \to 0} \left[ \int d^3x \, A_i^a(x) \right] = \frac{3}{2} \left[ \int d^3x \, A_i^a(x) \right]_{m_\pi=0}, \quad \text{(B.11)}
$$

which is the expected result.
Appendix C

The propagator $K(q)$

We derive here the expression for the propagator $K(q)$, introduced in Sect. 3.2.2. As a first step we perform the angular integrals in Eq. (3.23), getting

$$K(q) = \frac{1}{2\pi^3} \int_0^\infty dk \frac{k^2}{2} \int_0^\infty dk' \frac{k'^2}{2} \frac{1}{E_k E_{k'}} \frac{C[(k + k')/2]}{E_k + E_{k'}} \int_0^\infty dr r^2 j_0(qr) j_0(kr) j_0(k' r).$$  \hspace{1cm} (C.1)

The inner integral can be done analytically [Grad80], yielding

$$\int_0^\infty dr r^2 j_0(qr) j_0(kr) j_0(k' r) = \left(\frac{\pi}{2}\right)^{3/2} \frac{1}{\sqrt{qkk'}} \int_0^\infty dr \sqrt{r} J_{1/2}(qr) J_{1/2}(kr) J_{1/2}(k' r) = \frac{\pi}{4} \frac{\Delta(q, k, k')}{q k k'},$$  \hspace{1cm} (C.2)

where the function $\Delta(x, y, z)$ vanishes whenever it is not possible to build a triangle of sides $x, y$ and $z$, and is equal to 1 otherwise.

One then obtains

$$K(q) = \frac{1}{8\pi^2 q} \int_0^\infty dk \frac{k}{E_k} \int_{|k - q|}^{k + q} dk' \frac{k'}{E_{k'}} \frac{C[(k + k')/2]}{E_k + E_{k'}}.$$.  \hspace{1cm} (C.3)
Appendix D

A basis for flat functions

In free space \((R \to \infty)\) an orthonormal and complete set of states can be chosen as

\[
\Psi_{kltm}(x) = \frac{1}{\pi} e^{ikx} Y_l^m(\Omega) j_l(kr).
\]  

We want to build an orthonormal and complete basis inside a sphere of radius \(R\), having the same form of \((D.1)\). While in that case the momentum could take a continuum set of values, now only a discrete set of momenta will be allowed.

We start considering the equations for the spherical Bessel functions, corresponding to two different momenta, \(k_1 = \alpha/R\) and \(k_2 = \beta/R\):

\[
\frac{d}{dr} \left[ r^2 \frac{d}{dr} j_l \left( \frac{\alpha r}{R} \right) \right] + \left[ \left( \frac{\alpha r}{R} \right)^2 - l(l + 1) \right] j_l \left( \frac{\alpha r}{R} \right) = 0
\]

\[
\frac{d}{dr} \left[ r^2 \frac{d}{dr} j_l \left( \frac{\beta r}{R} \right) \right] + \left[ \left( \frac{\beta r}{R} \right)^2 - l(l + 1) \right] j_l \left( \frac{\beta r}{R} \right) = 0.
\]

Multiplying both expressions by \(j_l(\alpha r/R)\) and \(j_l(\beta r/R)\), respectively, integrating over \(r\) between 0 and \(R\), taking the difference of the two equations and, finally, integrating by parts, one gets

\[
- \int_0^R (\alpha^2 - \beta^2) \frac{r^2}{R^2} j_l \left( \frac{\alpha r}{R} \right) j_l \left( \frac{\beta r}{R} \right) = R [\alpha j_l'(\alpha) j_l(\beta) - \beta j_l'(\beta) j_l(\alpha)]
\]

The above equation states the orthonormality of the elements of the basis, provided that the following condition is met:

\[
\alpha j_l'(\alpha) j_l(\beta) - \beta j_l'(\beta) j_l(\alpha) = 0, \quad \alpha \neq \beta.
\]

The choice of the boundary conditions fulfilled by the elements of the basis will therefore be constrained by this requirement. It is easy to convince oneself that three
different possibilities are available:

\[ j_i(\alpha) = \eta j_i(\alpha), \quad (D.6) \]

where \( \eta \) will be a constant parameter. In the limit \( \eta = 0 \) and \( \eta \rightarrow \infty \) the first two cases are recovered, respectively.

Notice that, in the limit \( R \rightarrow \infty \), — where the modes form a continuum, — all these boundary conditions become equivalent.

We then introduce the normalized functions

\[ \rho_{\alpha_l}(r) \equiv \kappa_{\alpha_l} j_l \left( \frac{\alpha_l r}{R} \right) \]

\[ \kappa_{\alpha_l}^2 = \frac{2}{R^3} \left[ \alpha_l^2 j_l'(\alpha_l)^2 - j_l(\alpha_l) \left( -2\alpha_l j_l'(\alpha_l) + (l(l + 1) - \alpha_l^2) j_l(\alpha_l) \right) \right]^{-1}, \quad (D.7) \]

such that

\[ \int_0^R dr \ r^2 \rho_{\alpha_l}(r) \rho_{\beta_l}(r) = \delta_{\alpha_l \beta_l}. \quad (D.8) \]

The completeness of the basis allows one to write the following representation for the Dirac delta function inside the WS cell:

\[ \sum_{\alpha_l} \rho_{\alpha_l}(r) \rho_{\alpha_l}(r') = \frac{\delta(r - r')}{r^2}. \quad (D.9) \]

It is simple to derive an expression for the Green’s function of the operator \((\Box + M^2)\) in the WS cell. It reads

\[ G(x, x') = \sum_{l,m} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \sum_{\alpha_l} Y_l^m(\Omega) Y_l^{m*}(\Omega') e^{-ik_0(x_0-x_0')} \frac{\rho_{\alpha_l}(r) \rho_{\alpha_l}(r')}{\alpha_l^2/R^2 + M^2 - k_0^2}. \]

The flat basis is obtained, as we said above, in the limit \( \eta \rightarrow 0 \); in this case, however, the lowest energy mode, — corresponding to zero momentum for \( l = 0 \), can only be obtained for a strictly vanishing \( \eta \) (in fact, it corresponds to a negative root of
Eq. (D.6) at finite values of \( \eta \). This means that the strength of this mode has to be redistributed, for finite \( \eta \), over all the other modes. Let us call \( \beta \) and \( \tilde{\beta} \) the modes such that

\[
\tilde{j}_{l}(\beta) = 0, \quad \tilde{j}_{l}(\tilde{\beta}) = \eta j_{l}(\tilde{\beta}).
\] (D.10)

There is a one-to-one correspondence between the modes in the two bases, in such a way that for each \( \beta \neq 0 \) one can define \( \tilde{\beta} = \beta - \epsilon_{\beta} \), where \( \epsilon_{\beta} \ll 1 \) if \( \eta \ll 1 \).

Writing down the transformation from one basis to the other, namely

\[
\rho_{\beta}(r) = \sum_{\tilde{\beta}} c_{\beta\tilde{\beta}} \tilde{\rho}_{\tilde{\beta}}(r),
\] (D.11)

with some algebra one can verify that, for \( l = 0 \) and \( \beta = 0 \), one has

\[
c_{0\tilde{\beta}} = -\frac{\sqrt{6}}{\beta^{2}} \epsilon_{\beta},
\] (D.12)

which indeed proves that the zero mode is now redistributed over all the other modes, in the quasi-flat basis.

Calculations in the two bases are equivalent (to order \( \eta \)) only when considering convergent quantities. Quantities that need regularization are therefore different in the two bases, since only a finite number of modes will contribute. The flat basis is the one we have chosen in the calculations of this paper, since it is the most convenient in order to impose the physically motivated boundary conditions discussed in the text. The zero mode, however, gives rise to divergences when \( R \to 0 \) and one could in principle eliminate this problem by employing a quasi-flat basis infinitesimally close to the flat one. On the other hand, the very fact that for regularized quantities only a finite number of modes is relevant, allows us to keep the flat basis and simply discard the zero mode, since the error introduced in this way will be infinitesimally small.

Finally, we display in Table D.1 a few simple rules that allow one to rewrite in the WS basis quantities expressed in the free basis (D.1).
<table>
<thead>
<tr>
<th>free space</th>
<th>Wigner-Seitz boundary conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_0^\infty dk \ k^2$</td>
<td>$\rightarrow$</td>
</tr>
<tr>
<td>$k$</td>
<td>$\rightarrow$</td>
</tr>
<tr>
<td>$j_l(kr)$</td>
<td>$\rightarrow$</td>
</tr>
</tbody>
</table>

**Table D.1**  Rules for going from free space to the Wigner-Seitz cell.
Appendix E
Derivation of the vacuum energy

In this appendix we want to derive the expression given in eq.(3.35) and (3.37) for the local and non local parts of the vacuum energy. We have seen that the leading term in the expansion of the vacuum energy in the derivatives of the soliton reads

\[ E^{(2)}_{\text{vac}} = \frac{N_c^2}{4} \int d^3x \int d^3y \text{tr}_{\text{f}} \text{tr}_{d} [V(x)V(y)] K(\vec{x}, \vec{y}) \quad \text{(E.1)} \]

This expression needs now to be cast in terms of the elements of an orthonormal and complete basis inside the spherical Wigner-Seitz cell, of radius \( R \).

In particular, by using the definition of \( K(\vec{x}, \vec{y}) \), one finds that

\[ K(\vec{x}, \vec{y}) = \sum_{l,m,l',m'} \sum_{n, n'} \sum_{\Omega, \Omega'} \frac{\pi^3}{4E_{l,m}E_{n, n'}} \left( \rho_{n, n'}(r_x) \rho_{n, n'}(r_y) \right) \]

On the other hand, we can directly write the Bessel-Fourier expansion of \( K(x, y) \) as

\[ K(\vec{x}, \vec{y}) = \sum_{l,m,l',m'} \sum_{n, n'} \sum_{\Omega, \Omega'} Y_{l,m}(\Omega_x)Y_{l',m'}^*(\Omega_y) \rho_{n, n'}(r_x) \rho_{n, n'}(r_y) K_{l,l'}(n, n') \quad \text{(E.3)} \]

which allows the identification

\[ K_{l,l'}(n, n') = \int d^3x d^3y Y_{l,m}(\Omega_x)Y_{l',m'}^*(\Omega_y) \rho_{n, n'}(r_x) \rho_{n, n'}(r_y) K(\vec{x}, \vec{y}) \]

\[ = \int d^3x \int d^3y Y_{l,m}(\Omega_x)Y_{l',m'}^*(\Omega_y) \rho_{n, n'}(r_x) \rho_{n, n'}(r_y) \]

\[ = \frac{2}{\pi^3} \sum_{L,M} \sum_{L',M'} Y_{L,M}(\Omega_x)Y_{L',M'}^*(\Omega_y) Y_{L,M}^*(\Omega_x)Y_{L',M'}(\Omega_y) \]

\[ K_{L,L'}(r_x, r_y) \quad \text{(E.4)} \]
with
\[ K_{L,L'}(r_x, r_y) \equiv \pi^3 \frac{3}{4} \sum_{\alpha_L, \alpha_{L'}} C \left( \frac{\alpha_L + \alpha_{L'}}{2} \right) \frac{\rho_{\alpha_L}(r_x) \rho_{\alpha_L}(r_y)}{E_{\alpha_L} E_{\alpha_{L'}} (E_{\alpha_L} + E_{\alpha_{L'}})} \rho_{\alpha_{L'}}(r_x) \rho_{\alpha_{L'}}(r_y). \]  

Notice that, for simplicity, the above expression has been regularized by means of a regulating function; it is straightforward to use the Pauli-Villars regularization instead, as explained in paragraph 3.2.4.

In the expression of \( K_{L,L'}(n_l, n'_{l'}) \) one can exploit some of the properties of the spherical harmonics, and of the \( 3 - j \) symbols:

\[
\int d\Omega_x Y_{l_1, m_1}(\Omega_x) Y_{l_2, m_2}(\Omega_x) Y_{l_3, m_3}(\Omega_x) = \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix}
\]

\[
Y_{l_1, -m}(\Omega_x) = (-1)^m Y_{l_1, m}(\Omega_x)
\]

\[
\sum_{m_1, m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{1}{2j_3 + 1} \delta_{j_3, j_1} \delta_{m_3, m_1} \delta_{m_2, m_1} \delta(j_1, j_2, j_3)
\]

where \( \delta(j_1, j_2, j_3) = 1 \) if \( j_1, j_2 \) and \( j_3 \) satisfy the triangular condition.

Therefore, after performing the intermediate algebra one obtains

\[
K_{l,l'}(n_l, n'_{l'}) = \frac{\delta_{l,l'} \delta_{m,m'}}{2\pi^4} \sum_{L, L'} (-1)^{l_1 + L + L'} \delta(lL'LL')(2L + 1)(2L' + 1) \begin{pmatrix} L & L' \\ 0 & 0 \end{pmatrix}^2 \int_0^R dr_x r_x^2 \int_0^R dr_y r_y^2 \rho_{n_l}(r_x) \rho_{n'_{l'}}(r_y) K_{L,L'}(r_x, r_y). \]  

(E.6)

Let's now consider the other term in \( E^{(2)}_{\text{vac}} \), i.e.

\[
tr_f tr_d [V(x)V(y)] = -\frac{8M^2}{f_x^2} \left( \partial_\mu \phi_0(x) \partial^\mu \phi_0(y) + \partial_\mu \bar{\phi}(x) \cdot \partial^\mu \bar{\phi}(y) \right)
\]

\[
= \frac{8M^2}{f_x^2} \left( A(r_x, r_y) + B(r_x, r_y)(r_x \cdot r_y) \right.
\]

\[ + \ C(r_x, r_y)(r_x \cdot r_y)^2 \right), \]  

(E.7)
where the following definitions have been introduced

\[
A(r_x, r_y) \equiv \frac{\partial \phi}{\partial r_x} \frac{\phi(r_x)}{r_x} \left( \frac{\partial \phi}{\partial r_y} + \frac{\phi(r_y)}{r_y} \right) - \frac{\partial \phi}{\partial r_x} \frac{\partial \phi}{\partial r_y}
\]

\[
B(r_x, r_y) \equiv \frac{\partial \phi_0}{\partial r_x} \frac{\partial \phi_0}{\partial r_y}
\]

\[
C(r_x, r_y) \equiv \left( \frac{\partial \phi}{\partial r_x} - \frac{\phi(r_x)}{r_x} \right) \left( \frac{\partial \phi}{\partial r_y} - \frac{\phi(r_y)}{r_y} \right)
\]

As before, we expand these terms in the basis and obtain

\[
A(r_x, r_y) = \sum_{\alpha_0, \alpha'_0} \rho_{\alpha_0}(r_x) \rho_{\alpha'_0}(r_y) \left[ a^{(1)}(\alpha_0) a^{(1)}(\alpha'_0) - a^{(2)}(\alpha_0) a^{(2)}(\alpha'_0) \right]
\]

\[
B(r_x, r_y) = \frac{4\pi}{3} \sum_{m} Y_{1,m}(\Omega_x) Y_{1,m}(\Omega_y) \sum_{\alpha_1, \alpha'_1} \rho_{\alpha_1}(r_x) \rho_{\alpha'_1}(r_y) \tilde{b}_{\alpha_1} \tilde{b}_{\alpha'_1}
\]

\[
C(r_x, r_y) = \frac{4\pi}{3} \left\{ \frac{1}{4\pi} \sum_{\alpha_0, \alpha'_0} \rho_{\alpha_0}(r_x) \rho_{\alpha'_0}(r_y) c(\alpha_0) c(\alpha'_0) \right. \\
+ \left. \frac{2}{3} \sum_{m, \alpha_2, \alpha'_2} Y_{2,m}(\Omega_x) Y_{2,m}(\Omega_y) \rho_{\alpha_2}(r_x) \rho_{\alpha'_2}(r_y) c(\alpha_2) c(\alpha'_2) \right\}
\]

where

\[
a^{(1)}(\alpha_0) \equiv \int_0^R dr_x r_x^2 \rho_{\alpha_0}(r_x) \left( \frac{\partial \phi}{\partial r_x} + \frac{\phi(r_x)}{r_x} \right)
\]

\[
a^{(2)}(\alpha_0) \equiv \int_0^R dr_x r_x^2 \rho_{\alpha_0}(r_x) \frac{\partial \phi}{\partial r_x}
\]

\[
\tilde{b}_1(\alpha_0) \equiv \int_0^R dr_x r_x^2 \rho_{\alpha_0}(r_x) \frac{\partial \phi_0}{\partial r_x}
\]

\[
c(\alpha_0) \equiv \int_0^R dr_x r_x^2 \rho_{\alpha_0}(r_x) \left( \frac{\partial \phi}{\partial r_x} - \frac{\phi(r_x)}{r_x} \right)
\]  \hspace{1cm} \text{(E.8)}

Finally one can write

\[
tr_f tr_d [V(x)V(y)] = \frac{8M^2}{f^2} \left\{ \sum_{\alpha_0, \alpha'_0} \rho_{\alpha_0}(r_x) \rho_{\alpha'_0}(r_y) \left[ a^{(1)}(\alpha_0) a^{(1)}(\alpha'_0) - a^{(2)}(\alpha_0) a^{(2)}(\alpha'_0) \right] \\
+ \frac{1}{3} c(\alpha_0) c(\alpha'_0) \right\} + \frac{4\pi}{3} \sum_{m} \sum_{\alpha_1, \alpha'_1} Y_{1,m}(\Omega_x) Y_{1,m}(\Omega_y) \rho_{\alpha_1}(r_x) \rho_{\alpha'_1}(r_y) \tilde{b}_{\alpha_1} \tilde{b}_{\alpha'_1} \\
+ \frac{8\pi}{15} \sum_{m} \sum_{\alpha_2, \alpha'_2} Y_{2,m}(\Omega_x) Y_{2,m}(\Omega_y) \rho_{\alpha_2}(r_x) \rho_{\alpha'_2}(r_y) c_{\alpha_2} c_{\alpha'_2} \right\} \hspace{1cm} \text{(E.9)}
\]
We here introduce the notation

\[ V_0(r_x, r_y) \equiv \sum_{\alpha_0\alpha'_0} \rho_{\alpha_0}(r_x)\rho_{\alpha'_0}(r_y) \left[ a^{(1)}(\alpha_0)a^{(1)}(\alpha'_0) - a^{(2)}(\alpha_0)a^{(2)}(\alpha'_0) + \frac{1}{3}c(\alpha_0)c(\alpha'_0) \right] \]

\[ V_1(r_x, r_y) \equiv \frac{4\pi}{3} \sum_{\alpha_1\alpha'_1} \rho_{\alpha_1}(r_x)\rho_{\alpha'_1}(r_y)b(\alpha_1)b(\alpha'_1) \]

\[ V_2(r_x, r_y) \equiv \frac{8\pi}{15} \sum_{\alpha_2\alpha'_2} \rho_{\alpha_2}(r_x)\rho_{\alpha'_2}(r_y)c(\alpha_2)c(\alpha'_2) \]

and write the vacuum energy as

\[
E^{(2)} = \frac{2N_cM^2}{f^2} \int_0^R dr_x r_x^2 \int_0^R dr_y r_y^2 \sum_{\alpha,\alpha'} \left\{ 4\pi \rho_{\alpha_0}(r_x)\rho_{\alpha'_0}(r_y)V_0(r_x, r_y)K_{0,0}(r_x, r_y) + \sum_m \rho_{\alpha_1}(r_x)\rho_{\alpha'_1}(r_y)V_1(r_x, r_y)K_{1,1}(r_x, r_y) + \sum_m \rho_{\alpha_2}(r_x)\rho_{\alpha'_2}(r_y)V_2(r_x, r_y)K_{2,2}(r_x, r_y) \right\} \quad (E.10)
\]

Now one can observe that

\[
\mathcal{F}_0(\alpha_0, \alpha'_0) \equiv \int_0^R dr_x r_x^2 \int_0^R dr_y r_y^2 \rho_{\alpha_0}(r_x)\rho_{\alpha'_0}(r_y)V_0(r_x, r_y)
= a^{(1)}(\alpha_0)a^{(1)}(\alpha'_0) - a^{(2)}(\alpha_0)a^{(2)}(\alpha'_0) + \frac{1}{3}c(\alpha_0)c(\alpha'_0)
= \int_0^R dr_x r_x^2 \int_0^R dr_y r_y^2 \frac{1}{3} \left[ \frac{\partial \phi}{\partial r_x} \frac{\partial \phi}{\partial r_y} + 2 \left( \frac{\phi}{r_x} \frac{\partial \phi}{\partial r_y} + \frac{\phi}{r_y} \frac{\partial \phi}{\partial r_x} \right) + 4 \frac{\phi(r_x)\phi(r_y)}{r_xr_y} \right]
= \int_0^R dr_x r_x^2 \int_0^R dr_y r_y^2 \rho_{\alpha_0}(r_x)\rho_{\alpha'_0}(r_y) \frac{1}{3} \left( \frac{\partial \phi}{\partial r_x} + 2 \frac{\phi}{r_x} \right) \left( \frac{\partial \phi}{\partial r_y} + 2 \frac{\phi}{r_y} \right)
\]

\[
\mathcal{F}_1(\alpha_1, \alpha'_1) \equiv \frac{1}{4\pi} \int_0^R dr_x r_x^2 \int_0^R dr_y r_y^2 \rho_{\alpha_1}(r_x)\rho_{\alpha'_1}(r_y)V_1(r_x, r_y)
= \frac{1}{3}b(\alpha_1)b(\alpha'_1)
\]

\[
\mathcal{F}_2(\alpha_2, \alpha'_2) \equiv \frac{1}{4\pi} \int_0^R dr_x r_x^2 \int_0^R dr_y r_y^2 \rho_{\alpha_2}(r_x)\rho_{\alpha'_2}(r_y)V_2(r_x, r_y)
= \frac{2}{15}c(\alpha_2)c(\alpha'_2)
\]
and therefore

\[ E^{(2)} = \frac{8\pi NcM^2}{f_\pi^2} \sum_{\alpha,\alpha'} \{ F_0(\alpha_0, \alpha'_0)K_{0,0}(\alpha_0, \alpha'_0) + 3F_1(\alpha_1, \alpha'_1)K_{1,1}(\alpha_1, \alpha'_1) \]

+ \frac{5F_2(\alpha_2, \alpha'_2)K_{2,2}(\alpha_2, \alpha'_2)}

= \frac{8\pi NcM^2}{f_\pi^2} \sum_{\alpha,\alpha'} \left\{ \left[ a^{(1)}(\alpha_0)a^{(1)}(\alpha'_0) - a^{(2)}(\alpha_0)a^{(2)}(\alpha'_0) + \frac{1}{3}c(\alpha_0)c(\alpha'_0) \right] \right\} K_{0,0}(\alpha_0, \alpha'_0)

b(\alpha_1)b(\alpha'_1)K_{1,1}(\alpha_1, \alpha'_1) + \frac{2}{3}c(\alpha_2)c(\alpha'_2)K_{2,2}(\alpha_2, \alpha'_2) \}

\equiv \frac{8\pi NcM^2}{f_\pi^2} \sum_{\alpha,\alpha'} \left\{ \frac{1}{3}f_0(\alpha_0)f_0(\alpha'_0)K_{0,0}(\alpha_0, \alpha'_0) + f_1(\alpha_1)f_1(\alpha'_1)K_{1,1}(\alpha_1, \alpha'_1) \}

+ \frac{2}{3}f_2(\alpha_2)f_2(\alpha'_2)K_{2,2}(\alpha_2, \alpha'_2) \}

where

\[ f_0(\alpha_0) = \int_0^R drz r_2^2 \rho_0(\tau z) \left( \frac{\partial \phi}{\partial r_x} + 2 \frac{\phi}{r_x} \right) \]

\[ f_1(\alpha_1) = \int_0^R drz r_2^2 \rho_1(\tau z) \left( \frac{\partial \phi_0}{\partial r_x} \right) \]

\[ f_2(\alpha_2) = \int_0^R drz r_2^2 \rho_2(\tau z) \left( \frac{\partial \phi}{\partial r_x} - 2 \frac{\phi}{r_x} \right) \]

Notice that, for numerical purposes, it is convenient to cast the above equations for \( f_1 \) and \( f_2 \) in a different form, involving the spherical Bessel function \( j_0 \):

\[ f_1(\alpha_1) = \frac{\kappa_{\alpha_1} R}{\alpha_1} \int_0^R drj_0 \left( \frac{\alpha_1 r}{R} \right) \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi_0}{\partial \tau} \right) \]  \hspace{1cm} (E.11)

\[ f_2(\alpha_2) = \left( \frac{R}{\alpha_2} \right)^2 \kappa_{\alpha_2} \left\{ \alpha_2 j_1(\alpha_2)\phi(R) - j_0(\alpha_2) \left( R^2 \phi''(R) - 2\phi(R) \right) \right. \]

\[ + \int_0^R drj_0 \left( \frac{\alpha_2 r}{R} \right) \left[ r^2 \phi''(r) + 4r \phi''(r) \right] \}} \]  \hspace{1cm} (E.12)

One can now isolate the local part of the vacuum energy. To do so, a representation of the Dirac delta function in terms of the elements of the orthonormal and complete basis is needed:

\[ \delta^{(3)}(x - y) = \sum_{l,m} Y_{l,m}(\Omega_x)Y^*_{l,m}(\Omega_y) \sum_{\alpha_l} \rho_{\alpha_l}(r_x)\rho_{\alpha_l}(r_y) \]  \hspace{1cm} (E.13)
and
\[
K(\vec{x}, \vec{y})|_{\text{local}} = \delta^{(3)}(x - y) \sum_{l,m} \sum_{\alpha_l} \frac{2l + 1}{16\pi} \frac{C(\alpha_l)}{E^3_{\alpha_l}} \rho^2_{\alpha_l}(r_z)
\]
\[
= \delta^{(3)}(x - y) K_{\text{local}}(r_z) .
\]
(E.14)

Thus
\[
K^{\text{local}}_{i,i'}(\alpha_i, \alpha_i') = \int d^3x Y^*_{l,m}((\Omega_x)Y_{l,m'}((\Omega_x)\rho_{\alpha_i}(r_z)\rho_{\alpha_i'}(r_y) \sum_{l,m} \sum_{\alpha_l} \frac{2l + 1}{16\pi} \frac{C(\alpha_l)}{E^3_{\alpha_l}} \rho^2_{\alpha_l}(r_z)
\]
\[
= \delta_{i,i'}\delta_{m,m'} \int_0^R dr_z r_z^2 \rho_{\alpha_i}(r_z)\rho_{\alpha_i'}(r_z)K_{\text{local}}(r_z) .
\]
(E.15)

Notice now that, in the expression for the vacuum energy, I will have terms of the form
\[
\sum_{\alpha_0, \alpha_0'} F_0(\alpha_0, \alpha_0') K^{\text{local}}_{00}(\alpha_0, \alpha_0') = \sum_{\alpha_0, \alpha_0'} \int_0^R dr_z r_z^2 \int_0^R dr_y r_y^2 \{ \ldots \} \rho_{\alpha_0}(r_z)\rho_{\alpha_0'}(r_y)
\]
\[
\int_0^R dr_z r_z^2 \rho_{\alpha_0}(r_z)\rho_{\alpha_0'}(r_z)K_{\text{local}}(r_z)
\]
\[
= \int_0^R dr_z r_z^2 \{ \ldots \}_{r_y = r_z} K_{\text{local}}(r_z) .
\]
(E.16)

By applying the above equation we finally obtain
\[
E^{(2)}_{\text{local}} = \frac{8\pi N_c M^2}{f_\pi^2} \int_0^R dr_z r_z^2 K_{\text{local}}(r_z) \left\{ \frac{1}{3} \left( \frac{\partial \phi}{\partial r_z} + 2 \frac{\phi}{r_z} \right)^2 + \left( \frac{\partial \phi_0}{\partial r_z} \right)^2 + \frac{1}{3} \left( \frac{\partial \phi}{\partial r_z} - \frac{\phi}{r_z} \right)^2 \right\}
\]
\[
= \frac{8\pi N_c M^2}{f_\pi^2} \int_0^R dr_z r_z^2 K_{\text{local}}(r_z) \left[ \left( \frac{\partial \phi}{\partial r_z} \right)^2 + \left( \frac{\partial \phi_0}{\partial r_z} \right)^2 + 2 \left( \frac{\phi(r_z)}{r_z} \right)^2 \right] .
\]
(E.17)

On the other hand, the non-local part of the vacuum energy will be simply given by
\[
E^{(2)}_{\text{non-local}} = \frac{8\pi N_c M^2}{f_\pi^2} \sum_{\alpha, \alpha'} \left\{ \frac{1}{3} f_0(\alpha_0)f_0(\alpha_0')\Delta K_0(\alpha_0, \alpha_0') + f_1(\alpha_1)f_1(\alpha_1')\Delta K_1(\alpha_1, \alpha_1')
\]
\[
+ \frac{2}{3} f_2(\alpha_2)f_2(\alpha_2')\Delta K_2(\alpha_2, \alpha_2') \right\} ,
\]
where

\[ \Delta K_l(\alpha_l, \alpha'_{l'}) = \frac{1}{2\pi^3} \sum_{L,L'} (-1)^{\mu+L+L'}(2L+1)(2L'+1) \left( \begin{array}{ccc} l & L & L' \\ 0 & 0 & 0 \end{array} \right)^2 \]

\[ \int_0^R dr_x r_x^2 \int_0^R dr_y r_y^2 \rho_{n_l}(r_x) \rho_{n'_{l'}}(r_y) \]

\[ \cdot \left( K_{L,L'}(r_x, r_y) - \frac{\delta(r_x - r_y)}{r_x^2} K_{local}(r_x) \right) \]  

(E.18)
Appendix F

Equation of motion for the chiral angle

The equation of motion for the quarks and for the chiral angle can now be obtained by taking the variation of the total energy with respect to each of these fields. For example:

$$\delta E = \frac{\delta E}{\delta \theta} \delta \theta + \frac{\delta E}{\delta \theta'} \delta \theta' = 0$$  \hspace{1cm} (F.1)

where

$$E = E[\theta, \theta'] = \int_0^R drr^2 E[\theta, \theta']$$  \hspace{1cm} (F.2)

Notice that one can also write

$$\delta E = \left[ \frac{\delta E}{\delta \theta} - \partial_r \frac{\delta E}{\delta \theta'} \right] \delta \theta + \int_0^R dr \partial_r \left[ \frac{\partial}{\partial \theta'} (r^2 E) \delta \theta \right]$$  \hspace{1cm} (F.3)

The first term in the r.h.s. of this equation yields the Euler-Lagrange equation for the chiral angle, given the arbitrariness of $\delta \theta$. The second term, however, vanishes only when the value of the chiral angle at the surface of the cell is fixed, i.e. if $\delta \theta(R) = 0$. When this condition is not met, the field must fulfill the so called “natural boundary condition”

$$\frac{\delta E}{\delta \theta'} |_{r=R} = 0 \hspace{1cm} (F.4)$$

Let us now calculate explicitly the Euler-Lagrange equation. In order to calculate the contribution stemming form the non local part of the vacuum energy, eq.(3.37),
we need to consider

\[ f_0(\alpha_0) = \int_0^R drr^2 \rho_{a_0}(r) \left( \frac{\partial \phi}{\partial r} + 2 \frac{\phi(r)}{r} \right) \]

\[ f_1(\alpha_1) = \int_0^R drr^2 \rho_{a_1}(r) \frac{\partial \phi_0}{\partial r} \]

\[ f_2(\alpha_2) = \int_0^R drr^2 \rho_{a_2}(r) \left( \frac{\partial \phi}{\partial r} - \frac{\phi(r)}{r} \right) \]

and calculate

\[ \frac{\partial f_0(\alpha_0)}{\partial \theta(r)} - \frac{\partial \phi_0(r)}{\partial r} \frac{\partial f_0(\alpha_0)}{\partial \theta'(r)} = -\phi_0(r)r^2 \frac{\partial \rho_{a_0}(r)}{\partial r} \]

\[ \frac{\partial f_1(\alpha_1)}{\partial \theta(r)} - \frac{\partial \phi_0(r)}{\partial r} \frac{\partial f_1(\alpha_0)}{\partial \theta'(r)} = \phi(r) \left( 2r \rho_{a_1}(r) + r^2 \frac{\partial \rho_{a_1}(r)}{\partial r} \right) \]

\[ \frac{\partial f_2(\alpha_2)}{\partial \theta(r)} - \frac{\partial \phi_0(r)}{\partial r} \frac{\partial f_2(\alpha_2)}{\partial \theta'(r)} = -\phi_0(r) \left( 3r \rho_{a_2}(r) + r^2 \frac{\partial \rho_{a_2}(r)}{\partial r} \right) \]

As a result one obtains

\[ \frac{\partial E^{(2)}_{\text{non-local}}}{\partial \theta(r)} - \frac{\partial \phi_0(r)}{\partial r} \frac{\partial E^{(2)}_{\text{non-local}}}{\partial \theta'(r)} = \frac{16\pi N_c M^2}{f_\pi} \sum_{\alpha,\alpha'} \left\{ \frac{1}{3} r^2 \frac{\partial \rho_{a_0}(r)}{\partial r} \phi_0(r) f_0(\alpha_0') \Delta K_0(\alpha_0, \alpha_0') \right. \]

\[ + \phi(r) \left( 2r \rho_{a_1}(r) + r^2 \frac{\partial \rho_{a_1}(r)}{\partial r} \right) f_1(\alpha_1') \Delta K_1(\alpha_1, \alpha_1') \]

\[ - \frac{2}{3} \phi_0(r) \left( r^2 \frac{\partial \rho_{a_2}(r)}{\partial r} + 3r \rho_{a_2}(r) \right) f_2(\alpha_2') \Delta K_2(\alpha_2, \alpha_2') \}

Similarly one finds

\[ -\frac{1}{4\pi r^2 F_2^2(r)} \frac{\delta E^{(2)}_{\text{local}}}{\delta \theta(r)} = \frac{\partial^2 \theta(r_z)}{\partial r_z^2} + \frac{2}{r_z} \frac{\partial \theta(r_z)}{\partial r_z} - \frac{\sin(2\theta(r_z))}{r_z^2} + \frac{2}{F_2^2(r_z)} \frac{\partial F_2^2(r_z)}{\partial r_z} \frac{\partial \theta(r_z)}{\partial r_z} \]

\[ -\frac{1}{4\pi r^2 F_2^2(r)} \frac{\delta E^{(2)}_{\text{valence}}}{\delta \theta(r)} = \frac{N_c M}{4\pi r^2 F_2^2(r)} \left\{ \sin \theta(r) \left[ F^2(r) - G^2(r) \right] + 2 F(r) G(r) \cos \theta(r) \right\} \]

The Euler-Lagrange equation will therefore read

\[ \frac{\partial^2 \theta(r_z)}{\partial r_z^2} + \frac{2}{r_z} \frac{\partial \theta(r_z)}{\partial r_z} - \frac{\sin(2\theta(r_z))}{r_z^2} + \frac{2}{F_2^2(r_z)} \frac{\partial F_2^2(r_z)}{\partial r_z} \frac{\partial \theta(r_z)}{\partial r_z} \]

\[ = -\frac{N_c M}{4\pi F_2^2(r_z)} \left\{ \sin \theta(r) \left[ F^2(r) - G^2(r) \right] + 2 F(r) G(r) \cos \theta(r) \right\} - \frac{4N_c M^2}{F_2^2(r_z)} W , \] (F.5)
where the following definition has been introduced:

\[ W = - \frac{1}{f^2} \sum_{\alpha, \alpha'} \left\{ - \frac{1}{3} \frac{\partial \rho_{\alpha_0}(r)}{\partial r} \phi_0(r) f_0(\alpha'_0) \Delta K_0(\alpha_0, \alpha_0') \right. \\
+ \phi(r) \left( \frac{2}{r} \rho_{\alpha_1}(r) + \frac{\partial \rho_{\alpha_1}(r)}{\partial r} \right) f_1(\alpha'_1) \Delta K_1(\alpha_1, \alpha_1') \\
- \left. \frac{2}{3} \phi_0(r) \left( \frac{\partial \rho_{\alpha_2}(r)}{\partial r} + \frac{3}{r} \rho_{\alpha_2}(r) \right) f_2(\alpha'_2) \Delta K_2(\alpha_2, \alpha_2') \right\} \]

(F.6)

Notice that the property

\[ K(\alpha, \alpha') = K(\alpha', \alpha) \]

has been exploited in deriving the previous equation.
Appendix G

Cranking equations

Let us write the cranking correction to the grand-spinor, eq.(4.23), in the form

$$\delta \psi = \frac{1}{\sqrt{4\pi}} \left( \frac{\Gamma_a^i(r) \sigma^i \xi}{(\Gamma_b(r) + \Gamma_a^i(r) \sigma^i)} \right),$$

(G.1)

where

$$\Gamma_a^i(r) \equiv A\lambda^i + B \left( \frac{\lambda^i}{3} - \vec{\lambda} \cdot \vec{\tau}^i \right)$$

(G.2)

$$\Gamma_b(r) \equiv iC \vec{\lambda} \cdot \vec{\tau}$$

(G.3)

$$\Gamma_a^i(r) \equiv -D\epsilon_{ijk} \lambda^j \tau^k$$

(G.4)

and $\xi$ is a grand-spinor with grand-spin 0.

The left side of the Dirac equation for the quarks can now be written as

$$(H' - E) \delta \phi = \frac{1}{\sqrt{4\pi}} \left( \frac{A(r)}{B(r)} \right).$$

(G.5)

by defining

$$A(r) \equiv \left( -\frac{M}{f_{\pi}} \sigma_h - E \right) \Gamma_a^i(r) \sigma^i \xi + \left( -i\vec{\sigma} \cdot \vec{\nabla} - i\frac{M}{f_{\pi}} \vec{\tau} \cdot \vec{\phi}_h \right) (\Gamma_b(r) + \Gamma_a^i(r) \sigma^i) \xi$$

$$B(r) \equiv \left( -i\vec{\sigma} \cdot \vec{\nabla} + i\frac{M}{f_{\pi}} \vec{\tau} \cdot \vec{\phi}_h \right) \Gamma_a^i(r) \sigma^i \xi + \left( \frac{M}{f_{\pi}} \sigma_h - E \right) (\Gamma_b(r) + \Gamma_a^i(r) \sigma^i) \xi.$$

Let us first consider $A(r)$ and use the properties

$$\sigma^i \sigma^j \xi = (\delta_{ij} + i\epsilon_{ijk}\sigma^k) \xi$$

$$\tau^i \sigma^j \xi = (-\delta_{ij} + i\epsilon_{ijk}\sigma^k) \xi.$$

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One is now able to write

\[
A(r) = \left[ -\left( \frac{M}{f_\pi} \sigma_h(r) + E \right) \Gamma_b^i(r) \xi - i \partial_i \Gamma_b^i(r) \xi + i \frac{M}{f_\pi} \hat{r}^i \phi_h \Gamma_b^i(r) \xi \right. \\
+ \epsilon_{ijk} \partial_j \Gamma_c^k(r) \xi + \frac{M}{f_\pi} \epsilon_{ijk} \phi_h^j \Gamma_c^k(r) \xi \left. \right] \sigma^i \\
+ \left( -i \partial_i \Gamma_c^i(r) + i \frac{M}{f_\pi} \phi_h^i \Gamma_c^i(r) \right) \xi .
\] (G.6)

By performing simple algebra, one can prove that

\[
\epsilon_{ijk} \left( \partial_j \Gamma_c^k(r) + \frac{M}{f_\pi} \phi_h^j \Gamma_c^k(r) \right) = -\lambda^i \left( D'(r) + \frac{D(r)}{r} + \frac{M}{f_\pi} \phi h D(r) \right) \\
+ \vec{\lambda} \cdot \hat{r} \cdot \vec{\sigma} \left( D'(r) - \frac{D(r)}{r} + \frac{M}{f_\pi} \phi h D(r) \right)
\]

and that

\[
\partial_i \Gamma_b^i(r) = i \lambda^i \frac{C(r)}{r} + \vec{\lambda} \cdot \hat{r} \cdot \vec{\sigma} \left( C'(r) - \frac{C(r)}{r} \right) \\
\partial_i \Gamma_c^i(r) = 0 \\
\Gamma_c^i(r) \hat{r}^i = 0 .
\]

Finally one obtains

\[
A(r) = \left[ -\left( \frac{M}{f_\pi} \sigma_h(r) + E \right) \left( A(r) + \frac{B(r)}{3} \right) + \frac{C(r)}{r} \right. \\
- \left( D'(r) + \frac{D(r)}{r} + \frac{M}{f_\pi} \phi_h D(r) \right) \left. \right] \vec{\lambda} \cdot \vec{\sigma} \xi \\
+ \left( \frac{M}{f_\pi} \sigma_h(r) + E \right) \left( B(r) + \frac{C(r)}{r} - \frac{M}{f_\pi} \phi_h C(r) \right) \\
+ \left( D'(r) - \frac{D(r)}{r} + \frac{M}{f_\pi} \phi_h D(r) \right) \vec{\lambda} \cdot \hat{r} \cdot \vec{\sigma} \cdot \xi .
\]

Similarly, one must consider \( B \) and write, after elementary algebra,

\[
B(r) = -i \partial_i \Gamma_a^i(r) \xi - \epsilon_{ijk} \partial_j \Gamma_a^i(r) \sigma^k \xi - i \frac{M}{f_\pi} \phi_h^i(r) \Gamma_a^i(r) \xi \\
+ \frac{M}{f_\pi} \phi_h^i(r) \Gamma_a^i(r) \epsilon_{ijk} \sigma^k \xi + \left( \frac{M}{f_\pi} \sigma_h(r) - E \right) \left( \Gamma_b^i(r) + \Gamma_c^i(r) \sigma^i \right) \xi . \quad (G.7)
\]
In this case it is convenient to use the properties
\[\Gamma_\alpha^i(r)\bar{\psi} = \left( A(r) - \frac{2}{3} B(r) \right) \bar{\lambda} \cdot \hat{r} \]
\[\partial_t \Gamma_\alpha^i(r) = \left( A'(r) - \frac{2}{3} B'(r) - \frac{2}{r} B(r) \right) \bar{\lambda} \cdot \hat{r} \]
\[\epsilon_{ijk} \partial_j \Gamma_\alpha^i(r) = \epsilon_{ijk} \left[ \lambda^i \hat{r}^j (A'(r) + \frac{B'(r)}{3}) - \frac{B(r)}{r} \lambda^i \hat{r}^j \right] \]
\[\epsilon_{ijk} \phi_h^j \Gamma_\alpha^i(r) = \epsilon_{ijk} \lambda^i \hat{r}^j \phi_h \left( A(r) + \frac{B(r)}{3} \right) . \]

The final form for \( B \) is therefore
\[B(r) = i \bar{\lambda} \cdot \hat{r} \left[ - \left( A'(r) - \frac{2}{3} B'(r) - \frac{2}{r} B(r) \right) - \frac{M}{\bar{f}_\pi} \phi_h(r) \left( A(r) - \frac{2}{3} B(r) \right) \right] + \left( \frac{M}{\bar{f}_\pi} \sigma_h(r) - E \right) C(r) \] \[+ \epsilon_{ijk} \lambda^i \hat{r}^j \sigma^k \left[ - A'(r) - \frac{B'(r)}{3} - \frac{B(r)}{r} \right] \]
\[+ \frac{M}{\bar{f}_\pi} \phi_h(r) \left( A(r) + \frac{B(r)}{3} \right) - \left( \frac{M}{\bar{f}_\pi} \sigma_h(r) - E \right) D(r) \] \[\xi \]

The r.h.s. of the Dirac equation is easily checked to be
\[-\frac{\bar{\lambda} \cdot \hat{r}}{2} \psi = \frac{1}{\sqrt{4 \pi}} \left( \frac{\bar{\chi}_\alpha}{2} F(r) \xi \right) \left( i \bar{\lambda} \cdot \hat{r} - \epsilon_{ijk} \hat{r}^i \omega^j \sigma^k \right) \]
\[(G.8)\]

As a result the equations for the cranking spinor are obtained.

Following [Adj95] we introduce the dimensionless quantities of eq.(3.69) and the new variables
\[(a, b, c, d) \equiv \frac{1}{\sqrt{M}} (A, B, C, D) \]
\[(G.9)\]

and obtain
\[\frac{da(x)}{dx} = \frac{\sin \theta(x)}{3} \left( a(x) + \frac{4}{3} b(x) \right) - \frac{\epsilon}{3} \left( c(x) - 2d(x) \right) \]
\[- \frac{\cos \theta(x)}{3} \left( c(x) - 2d(x) \right) - \frac{\bar{v}(x)}{2} \]
\[\frac{db(x)}{dx} = - \frac{3}{x} b(x) + \sin \theta(x) \left( 2a(x) - \frac{b(x)}{3} \right) + (\epsilon + \cos \theta(x)) \left( c(x) + d(x) \right) \]
\[\frac{dc(x)}{dx} = \frac{2}{x} d(x) + \sin \theta(x) c(x) + (\epsilon - \cos \theta(x)) \left( a(x) - \frac{2}{3} b(x) \right) + \frac{u(x)}{2} \]
\[\frac{dd(x)}{dx} = \frac{c(x) - d(x)}{x} - \sin \theta(x) d(x) - (\epsilon - \cos \theta(x)) \left( a(x) + \frac{b(x)}{3} \right) - \frac{u(x)}{2} \]
Notice that the boundary conditions for these equations must grant the regularity in the origin, \( x = 0 \), and the exponential damping of \( a, b, c \) and \( d \) at large distances. By considering the equations in the proximity of the origin one obtains

\[
\begin{align*}
    a(0) & \approx K_1 \\
    b(0) & \approx K_2 x^2 \\
    c(0) & \approx K_3 x \\
    d(0) & \approx K_4 x ,
\end{align*}
\]

where \( K_i \) are constant to be determined. It is easy to see that these constants are not completely independent. As a matter of fact, one can see that

\[
\begin{align*}
    b''(0) & \approx -2 \theta'(0) K_1 - 3 K_2 + (\epsilon - 1)(K_3 + K_4) \quad \text{(G.10)} \\
    d'(0) & \approx -\frac{\ddot{u}(0)}{2} + K_1(1 + \epsilon) + K_3 - K_4 . \quad \text{(G.11)}
\end{align*}
\]
Appendix H

Nucleon and \( \Delta \) states

In Chapter 4 we have given the solutions corresponding to a definite spin and isospin. These have been expressed in terms of the collective variables associated with the rotation of a soliton in space or isospace:

\[
|p \uparrow\rangle = \frac{1}{\pi} (b_1 + i b_2), \quad |p \downarrow\rangle = -i \frac{1}{\pi} (b_0 - i b_3)
\]

\[
|n \uparrow\rangle = i \frac{1}{\pi} (b_0 + i b_3), \quad |n \downarrow\rangle = -\frac{1}{\pi} (b_1 - i b_2)
\]

\[
|\Delta^{++} \frac{3}{2}\rangle = i \sqrt{\frac{3}{2}} (b_1 + i b_2)^3, \quad \ldots
\]

For example, the wave function for a proton with spin up will be obtained as the product of the hedgehog wave function and of the state \(|p \uparrow\rangle\):

\[
\Psi_{p\uparrow}(x) = \psi(x) \ | p\uparrow\rangle . \quad (H.1)
\]

Let us check that these states are correctly normalized. We start by defining the constant \(SU(2)\) matrix

\[
R_0 \equiv b_0 + i \tau \cdot \vec{b}, \quad (H.2)
\]

with

\[
R_0 \ R_0^\dagger = b_0^2 + \vec{b}^2 = 1 \quad (H.3)
\]

defining the surface of a 3-sphere in the space of the collective variables \((b_0, \vec{b})\). For
this reason it is useful to parametrize the collective variables in terms of angles as

\[ b_0 = \cos \chi \]  
\[ b_1 = \sin \chi \sin \theta \cos \phi \]  
\[ b_2 = \sin \chi \sin \theta \sin \phi \]  
\[ b_3 = \sin \chi \cos \theta \]  

and write the volume element in this space as

\[
\int d\Omega_3 = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin \theta \int_0^{\pi} d\chi \sin^2 \chi .
\]  

Consider for example the normalization integral for \(|p^{\uparrow}\rangle\):

\[
\langle p^{\uparrow}|p^{\uparrow}\rangle = \frac{1}{\pi^2} \int d\Omega_3 \left( b_1^2 + b_2^2 \right) \\
= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\pi} d\chi \sin^4 \chi \sin^3 \theta = 1 .
\]  

A similar results holds also for the other states.

We can now consider the quantity:

\[
C_{ab} \equiv \frac{1}{2} \text{tr}_{\text{flavor}} \left[ \tau^a R_0 \tau^b R_0^\dagger \right] \\
= \frac{1}{2} \text{tr}_{\text{flavor}} \left[ \tau^a \left( b_0 + i \vec{r} \cdot \vec{b} \right) \tau^b \left( b_0 - i \vec{r} \cdot \vec{b} \right) \right] \\
= \left( b_0^2 - \vec{b}^2 \right) \delta_{ab} + 2b^a b^b + 2\epsilon_{abc} b_0 b^c ,
\]  

which has been used already in Chapter 3 in the discussion of the axial coupling constant. We can calculate the matrix element between nucleon states as:

\[
\langle p^{\uparrow}|C_{ab}|p^{\uparrow}\rangle = \frac{1}{\pi^2} \int d\Omega_3 \left( b_1^2 + b_2^2 \right) \left[ \left( b_0^2 - \vec{b}^2 \right) \delta_{ab} + 2b^a b^b + 2\epsilon_{abc} b_0 b^c \right] \\
= \frac{1}{\pi^2} \int d\Omega_3 \left( b_1^2 + b_2^2 \right) \left[ \left( b_0^2 - \vec{b}^2 \right) \delta_{ab} + 2b^a b^b \delta_{ab} \right] \\
= -\frac{1}{3} \delta_{a3} \delta_{b3} ,
\]  

which is the result which has been used in Chapter 3.
Similarly one obtains for the other states:

\[
\langle p \downarrow | C_{ab} | p \downarrow \rangle = \frac{1}{3} \delta_{a3} \delta_{b3}
\]

\[
\langle n \uparrow | C_{ab} | n \uparrow \rangle = \frac{1}{3} \delta_{a3} \delta_{b3}
\]

\[
\langle n \downarrow | C_{ab} | n \downarrow \rangle = -\frac{1}{3} \delta_{a3} \delta_{b3}.
\]
BIBLIOGRAPHY


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