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Renormalization group theory technique and subgrid scale closure for fluid and plasma turbulence

Ye Zhou

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Renormalization group theory technique and subgrid scale closure for fluid and plasma turbulence

Zhou, Ye, Ph.D.

The College of William and Mary, 1987
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UMI
RENEGALIZATION GROUP THEORY TECHNIQUE AND
SUBGRID SCALE CLOSURE FOR FLUID AND PLASMA TURBULENCE

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A Dissertation
Presented to
The Faculty of the Department of Physics
The College of William and Mary in Virginia

------------------------

In Partial Fulfillment
Of the Requirements for the Degree of
Doctor of Philosophy

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by

Ye Zhou

December 1967
APPROVAL SHEET

This dissertation is submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Ye Zhou

Approved, December 1987

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Department of Mathematics
THIS DISSERTATION IS DEDICATED

TO

MY WIFE HUIQIN (LILY) AND MY SON DAVID

WITH LOVE
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ACKNOWLEDGEMENTS

First of all, I would like to express my deep appreciation to my supervisor, Professor George Vahala, under whose guidance this investigation was conducted, for his support, encouragement and many invaluable help during the entire course of this work and my whole period of residency at College of William and Mary. He led me into the turbulence and theoretical plasma physics research areas. Whenever I came to a point in my study where I thought I could go no further, he always had plenty of ideas about where else to look for solutions. It is a good fortune for me to have the opportunity to study with him. I also would like to thank Professor Vahala for his useful comments on the first version and on the revision of this dissertation. In particular, his detail editorial commentary has been very helpful in improving this thesis from an earlier draft.

I am very thankful to Professor Boozer and Professor Tracy, for their instructions and help, for serving on my dissertation committee and for their very careful reading of my thesis.

I am grateful to Professor Carlson and Professor Andersen for serving on my dissertation committee and for their very careful reading of my thesis.

We wish to thank Dr. R.H.Kraichnan for helpful suggestions and comments, and Dr. W.D.McComb for
correspondance concerning the \( \kappa \rightarrow 0 \) limit of the eddy viscosity.

I am particularly indebted to Professor William H. Matthaius for kindly offer me a post-doctoral position at Bartol research Institute, University of Delaware.

Thanks are given to Dr. Murshed Hossain, my long time friend, for his collaboration on the turbulence research and many other helpful discussions I have had with him.

I wish to express my gratitude to my parents and all members of my family for their constant support and encouragement.

Finally, I would like to take this chance to thank my wife for her love and patience. This work can never be completed successfully without her support and understanding. I would also like to express my gratitude to my son David, for his contribution to our joy and happiness, for his love, and for his 'understanding' that Dad has to go to office at night and meetings out of town.

To my wife and my son, this dissertation is dedicated.
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ABSTRACT

Renormalization group theory is applied to incompressible three-dimension Navier-Stokes turbulence so as to eliminate unresolvable small scales. The renormalized Navier-Stokes equation includes a triple nonlinearity with the eddy viscosity exhibiting a mild cusp behavior, in qualitative agreement with the test-field model results of Kraichnan. For the cusp behavior to arise, not only is the triple nonlinearity necessary but the effects of pressure must be incorporated in the triple term.

Renormalization group theory is also applied to a model Alfven wave turbulence equation. In particular, the effect of small unresolvable subgrid scales on the large scales is computed. It is found that the removal of the subgrid scales leads to a renormalized response function. (i) This response function can be calculated analytically via the difference renormalization group technique. Strong absorption can occur around the Alfven frequency for sharply peaked subgrid frequency spectra. (ii) With the $\epsilon$-expansion renormalization group approach, the Lorenzian wavenumber spectrum of Chen and Mahajan can be recovered for finite $\epsilon$, but the nonlinear coupling constant still remains small, fully justifying the neglect of higher order nonlinearities introduced by the renormalization group procedure.
RENORMALIZATION GROUP THEORY TECHNIQUE AND
SUBGRID SCALE CLOSURE FOR FLUID AND PLASMA TURBULENCE
CHAPTER I

INTRODUCTION

One can usually isolate physical phenomena into distinctive range of length scales. In general, events distinguished by a great disparity in size have little influence on one another: they do not communicate, and so the phenomena associated with each scale can be treated independently. The success of almost all practical theory in physics depends on isolating some limited range of length scale.

Unfortunately, for some cases, there exist some problems which involve multiple scales. Among them only a few can be solved exactly. Furthermore, many of the best known approximations fail to handle these problems.

One of these is fluid turbulence. The characteristic scale ranges from hurricanes to very small 'dust devils' in atmosphere. Another famous example is critical phenomena.

We shall restrict our attention to fully developed turbulence, which is characterized by a spatio-temporal chaos: there is a hierarchy of flow structures (eddies) whose length scales extend from \( L_0 \), characteristic of the boundary and/or initial conditions, to the dissipative scale \( L_{\text{diss}} \ll L_0 \) where viscous and inertial forces become comparable.

The chaotic aspect of fully developed turbulence can be characterized in several ways. In a given flow-realization the trajectory of each fluid particle is extremely intricate, leading to strong mixing of flow and drastically modified
transport properties compared to those in the laminar state. This usually amounts to an enhancement of the transport coefficients (which are usually called turbulent or eddy transport coefficients). Even when the small scales of such a flow cannot be resolved, these changes in transport properties may have observable consequences. Another characterization is found in the instability of a given realization: a small amount of noise in initial conditions will be amplified and attain a significant level independent of its initial value.

Nonlinear effects occur frequently in fluid turbulence and in plasma physics (for example, in Alfven wave (see Appendix C) turbulence which we will discuss in some detail in this dissertation). Typically one resorts to transform methods in tackling these problems (as well as the linear problems). This then leads to an infinite series of convoluted products, especially for quadratically nonlinear problems. The usual approach in handling these infinite Fourier series (or improper Fourier integrals in the limit of discrete--> continuum) is to invoke an upper limit to the wavenumber space by appealing to some physical argument or, at worst, pleading to a limitation in present (and foreseeable) supercomputers. For example, in Navier-Stokes turbulence one would hope to include all wavenumbers up to the Kolmogorov dissipation wavenumber^2, since molecular viscosity will remove the effects of even higher wavenumbers. Unfortunately, this leads to severe limitations in the
maximum Reynolds number (for definition, see Footnote(3)) that can be studied via direct numerical simulation on current and nearfuture supercomputers. In particular, as the turbulence evolves, higher and higher wavenumber and frequency components of velocity fields become excited, leading to a large range of excited space and time scales. It is well known that in homogeneous turbulence dissipation-scale eddies are of order $R^{3/4}$ times smaller than the energy containing eddies. Here $R$ is a Reynolds number. In order to solve the Navier-Stokes equation (NSE) accurately for such a turbulent flow, it is necessary to retain order $(R^{3/4})^3$ spatial degrees of freedom. Also since the time scale of significant evolution of turbulent flow is of the order of the eddy turnover time (defined in footnote (7)) of the energy-containing eddies, it is necessary of perform $R^{3/4}$ time steps to calculate a significant time evolution of the flow. Even if these calculations require only $O(1)$ arithmetic operations per time step, the requirement for computer storage would be $O(R^{9/4})$, and, for computational work $O(R^3)$. In this case, even a mere doubling of the Reynolds number would require an order-of-magnitude improvement in computer capability. With this kind of operation and storage count, it is not likely that foreseeable advances in computers will allow the full simulation of turbulent flows at Reynolds numbers much larger than $R=O(100-1000)$ already achieved.

The main problem is that there are a large number of degrees of freedom in the Fourier representation. Thus the
the problem resolves itself into the need to eliminate modes, in some statistical sense, in order to bring the reduced number of degrees of freedom within the range of existing (or, envisaged) computers.

Large-eddy simulation (LES)\textsuperscript{10} is a relatively new approach to the calculation of turbulent flows. The basic idea stems from two experimental observations. First, the large-scale structure of turbulent flows varies greatly from flow to flow (e.g., jets vs. boundary layers) and is consequently difficult, if not possible, to model in a general way. Secondly, the small-scale turbulence structures are nearly isotropic, very universal in character\textsuperscript{11}, and hence much more amenable to general modelling. In LES, one actually calculates the large-scale motions in a time-dependent, three-dimensional computation, using for the large-scale field dynamical equations that incorporate simple models for small-scale turbulence. Only part of the turbulence field with scales that are small relative to overall dimensions of the flow field is modelled\textsuperscript{12}, usually by an eddy viscosity coefficient. Such a subgrid scale (SGS) eddy coefficient represents the dissipative effect of motions on scales smaller than the effective grid on the large eddies (defined as those motions adequately represented on the numerical grid\textsuperscript{13}). This is in contrast to phenomenological turbulence modelling, in which all the deviations from the mean velocity profile are modelled.

Eddy viscosity has long been a fruitful concept in
turbulence theory, and its use made possible the computation of turbulent flows at Reynolds numbers too high for full numerical simulation. The basic idea of eddy viscosity is that scales of motion of given size are acted on by smaller scales as if the latter were an augmentation of the equilibrium thermal agitation. However, eddy viscosity is not a new concept. It has been recognized as a convenient way to characterize turbulent flows for a long time. Indeed, the eddy viscosity may possibly be the earliest case of a phenomenological renormalized transport coefficient.

Throughout the history of the study of turbulence, this concept has been highly useful in visualizing and parameterizing turbulent transport processes and the passage of turbulent energy between different scales. With the advent of large-scale computer simulation of flows, eddy viscosity parameters have been used to represent the effects of SGS motions. made pioneering studies of turbulent shear flows using the Smagorinsky eddy viscosity. This SGS eddy viscosity model was originally constructed from dimensional analysis. So-called 'subgrid modelling problems' have been calculations performed at a phenomenological level.

However, in the last decade, a new method called renormalization group theory (RNG) has been introduced for dealing with problems that have phenomena occurring on multiple length scales. Indeed, RNG of phase transitions and critical phenomena represents a major achievement in theoretical physics in the last decade. Since then, the method of RNG
has been applied to a wide variety of problems\textsuperscript{22}, especially, transitions to chaos. Since the problem of onset to chaos has very strong similarity to critical phenomena, RNG has been successfully established the existence of universal behaviour near transition, the results of the RNG approach are in excellent agreement with experiments and numerical simulation. Since there exists a comprehensive review of RNG for critical and chaotic phenomena\textsuperscript{23}, we shall not give a general discussion here.

Despite the obvious differences\textsuperscript{24,25}, detailed analogies have been sought between fully developed three dimension turbulence and critical phenomena, since turbulence exhibits mathematical similarities with critical phenomena\textsuperscript{24}. For example, both have asymptotically universal self-similar behaviour, for turbulence -- the limit of infinite wavenumber, while for critical phenomena -- the limit of zero wavenumber. A suggestive table has been constructed by Rose and Sulem\textsuperscript{1} (see footnote\textsuperscript{26}). Application of RNG to fully developed turbulence was first proposed by Nelkin\textsuperscript{24}. However, the first attempt to really implement the RNG ideas is due to Foster, Nelson and Stephen (FNS)\textsuperscript{27}. They considered the infrared (long wave length) properties of a randomly stirred fluid, and derived velocity correlations generated by NSE with a random force term\textsuperscript{28}. Several other authors have also used RNG analysis of NSE to determine the scaling exponents of similarity spectrum ranges of hydrodynamic turbulence -- in both infrared\textsuperscript{29,30} and ultraviolet( short-
distance, short time) \(^3\) limits, and most recently by Fournier and Frisch\(^5\) and as well as Yakhot and Orszag\(^5,3\). Of course, as Wilson\(^4\) points out that there is no cookbook recipe for applying RNG methods and that it is generally difficult to formulate the RNG method for a new problem\(^3\). Indeed, FNS theory\(^2\) is only valid in the asymptotic limit of \(k \rightarrow 0\).

Yakhot and Orszag theory\(^5,3\) also faces convergence difficulties on the perturbation expansion as they try to recover the Kolmogorov spectrum in the inertial range (see chapter II). See also Kraichnan's early critical review\(^3\) and his recent comments\(^3\) on Yakhot-Orszag theory.

There are, at present, basically two distinct RNG approaches being utilized, each with somewhat different objectives in mind (i) the \(\varepsilon\) - expansion method\(^5,27,32,33\) which we will use for Alfven wave turbulence; (ii) the difference recursion RNG technique. In first approach, it is argued that not only does elimination of the subgrid scales lead to an eddy viscosity but also it should give rise to a random forcing term resulting in the production of turbulent energy. This random forcing is introduced into the basic equation, and a power law wavenumber spectrum is associated with this forcing term. A small parameter, \(0 < \varepsilon \ll 1\), is introduced into the wavenumber forcing spectrum with subsequent perturbation expansion in \(\varepsilon\). This iterative procedure generates a renormalized viscosity coefficient and nonlinear coupling constant. Justification can then be made for dropping the higher order nonlinearities in the vicinity
of the fixed point. To recover a universal spectrum, e.g.,
like the Kolmogorov $k^{-5/3}$ spectrum for Navier-Stokes
turbulence, one must extrapolate from $\epsilon \to 0$ to finite $\epsilon$
-- while at the same time assuming that the higher order
nonlinearities (for finite $\epsilon$) can still be ignored even
though the nonlinear coupling constant is not small for
finite $\epsilon$. However, as we will show in this dissertation,
for Alfvén wave turbulence, the Lorenzian wavenumber spectrum
of Chen and Mahajan\cite{38} can be recovered for finite $\epsilon$, but
the nonlinear coupling constant still remains small, fully
justifying the neglect of higher order nonlinearities
introduced by the RNG procedure\cite{39}.

Another aspect of turbulence that can be investigated by
RNG is subgrid modelling. As we mentioned before, existing
subgrid calculations are based either on phenomenological
arguments or closure. It is easy to understand why subgrid
modeling is an ideal candidate for an RNG approach. It may
be shown that closure-based calculations are roughly
equivalent to doing only one step in a RNG iterative process.
However, a distinctive characteristic of the RNG applied to
subgrid scale modeling is that the statistics of the scales
to be eliminated are determined by the large scales which are
the object of the calculation. This is in contrast with the
application of the RNG to critical phenomena where
statistical properties are explicitly determined by the Gibbs
ensemble\cite{1}. For difference recursion RNG technique, one
proceeds by successive elimination of subgrid wavenumber
sheila which leads to an integro-difference recursion relation for the eddy viscosity (for Navier-Stokes turbulence, for example). In the subgrid range, RNG is then applied to this recursion relation. Now unlike the $\epsilon -$ expansion procedure, both free decay and forced turbulence can be handled but there is no need to introduce a small $\epsilon -$ parameter.

In this thesis, we shall apply the difference recursion RNG technique (originally applied to the linear problem of passive scalar diffusion\textsuperscript{40}) to Navier-Stokes turbulence, taking proper account of symmetries. The renormalized viscosity is calculated and compared to that found by the iterative averaging RNG procedure of McComb\textsuperscript{9,16,41,42} (which we believe is incorrect\textsuperscript{17} see Appendix B), and which is claimed not to introduce triple nonlinearities) as well as to the closure models of Kraichnan\textsuperscript{17}, Chollet and Lesieur\textsuperscript{43}, and recent direct numerical simulation\textsuperscript{44}. We conclude that, the importance of local interactions, which are reflected in the cusp behavior of the eddy viscosity can, to some extent, be incorporated into an RNG analysis by not ignoring the triple nonlinearities generated\textsuperscript{45} (as has not been done till now in previous RNG Navier-Stokes theory).

This difference recursion RNG theory is also applied to a model Alfven wave turbulence equation. It is found that the removal of subgrid scales leads to a renormalized response function $\mathcal{E}$ (for definition, see Appendix C). Assuming Lorentzian wavenumber and frequency spectra, an
analytic solution is obtained for $\mathcal{E}$. $\mathcal{E}$ is complex because of the Alfven continuum. Strong absorption is obtained for sharply peaked subgrid frequency spectra.

One may ask how important are the small scale structures in Alfven wave turbulence if an inverse cascade exists as in two-dimensional Navier-Stokes turbulence. In three-dimensional Navier-Stokes turbulence the energy cascade is to smaller and smaller scales and hence their obvious relevance. However, in the presence of an external magnetic field the usual arguments used to suggest the possibility of inverse cascades no longer apply and there is no longer any a priori reason to expect an inverse cascade of any enhanced transfer to low wavelengths in the spectrum of any particular quantity. Indeed, recent numerical results for forced two-dimensional magnetohydrodynamic (MHD) turbulence in a uniform external magnetic field show that the magnetic spectrum back transfer ceases well before the longest wavelength modes contain the same fraction of the total mean square vector potential as in the case for no external magnetic field. Moreover, the three dominant long wavelength modes do not constitute a resonantly coupled triad, but they are coupled by the smaller-scale turbulence.

In summary, two topics are addressed in this dissertation that are basic to fluid mechanics and plasma physics: Navier-Stokes turbulence and Alfven wave turbulence. In both instances we look at the effect of small 'unresolvable' subgrid scales on the large scales. The main tools that were
used to study these problems are the $\epsilon$- expansion RNG and the difference recursion RNG.

This thesis is conveniently organized into four major chapters. Each chapter is relatively independent, and can be read separately. In chapter II, we review some properties and related basic theories associated with Navier-Stokes turbulence. Then we shall briefly state some recent developments on $\epsilon$- expansion RNG analysis for Navier-Stokes turbulence --- since we will illustrate this method in detail for Alfven wave turbulence in Chapter III. Universal spectra, e.g. the Kolmogrov $k^{-5/3}$ spectrum for Navier-Stokes turbulence and Chen-Mahajan $k^{-2}$ spectrum for Alfven wave turbulence are recovered by extrapolating the $\epsilon \to 0$ limit to finite $\epsilon$. Chapter IV gives a description of the difference recursion RNG that is used for Navier-Stokes turbulence. In Chapter V, we demonstrate how to derive the renormalized response function for Alfven wave turbulence with the difference recursion RNG. Finally, the conclusion of this dissertation is given by Chapter VI. The logical connection between four major chapters can be illustrated by the following diagram
The Appendix A is devoted to the diagram technique. This is an easy way to handle the symmetries for non-linear problems. Alfven wave turbulence is taken as the example to illustrate how this method works. Of course, this method also works just as well for Navier-Stokes turbulence. Criticisms of McComb's iterative average procedure is given in Appendix B. We review briefly the basic definitions of Alfven wave and response function in Appendix C.
As mentioned in the introduction, the existence of an extremely large number of degrees of freedom interaction, makes turbulence one of the most challenging fields in classical nonlinear physics. The basic problem is to reduce the number degrees of freedom. However, the reduced model must correctly describe the real physical system by taking care of the effects of these modes omitted from the reduced system. To have a better understanding of the RNG techniques, this chapter will first give a self contained review of the statistical theory of fully developed turbulence, especially the properties of the small scale motions. The emphasis will be on a phenomenology discussion following the analysis of Kolmogorov. Finally, the $\epsilon$ - expansion RNG analysis of Navier-Stokes turbulence will be briefly reviewed.

A. STATISTICAL EQUATIONS

We shall restrict our attention to incompressible flows. Accordingly, the velocity field $\mathbf{u}(\mathbf{x},t)$ satisfies the Navier-Stokes equations (NSE)

$$\frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot (\mathbf{u} \mathbf{u}) = - \nabla p + \frac{\partial}{\partial x_i} u_j \frac{\partial}{\partial x_j} u_i$$

(2.1)
where \( \nu \) is the molecular viscosity and \( p \) the pressure. The summation convention of repeated subscripts is invoked. Closure is achieved by invoking the incompressibility condition

\[
\frac{\partial}{\partial x_d} u_d = 0 \quad (2.2)
\]

Using Eq. (2.2), the pressure can be eliminated from (2.1) by taking the divergence of Eq. (2.1). Thus

\[
\nu^2 p = - \frac{\partial}{\partial x_d} \frac{\partial}{\partial x_d} \left( u_d u_d \right) \quad (2.3)
\]

The formal solution of Eq. (2.3) is

\[
p = - \frac{1}{\nu^2} \frac{\partial}{\partial x_d} \frac{\partial}{\partial x_d} \left( u_d u_d \right) \quad (2.4)
\]

After eliminating the pressure from NSE, we can rewrite Eq. (2.1) in the form

\[
\frac{\partial^2 u_d}{\partial t^2} - \nu^2 \nabla^2 u_d = - \frac{1}{\nu^2} \nabla \cdot \left( \mathbb{P}_{\alpha\beta} (\nu) \right) (u_d u_d) \quad (2.5)
\]

with operators

\[
\mathbb{P}_{\alpha\beta} (\nu) = \frac{3}{\nu^2} \delta_{\alpha\beta} \mathbb{P}_{\alpha\beta} (\nu) + \frac{3}{\nu^2} \mathbb{P}_{\alpha\beta} (\nu) \quad (2.6)
\]

\[
\mathbb{P}_{\alpha\beta} (\nu) = \delta_{\alpha\beta} - \frac{1}{\nu^2} \frac{\partial}{\partial x_d} \frac{\partial}{\partial x_d} \left( u_d u_d \right) \quad (2.7)
\]
One introduces Fourier transform

\[ u_{\alpha}(\mathbf{r}) = \int u_{\alpha}(\mathbf{k}) e^{i \mathbf{k} \cdot \mathbf{r}} d\mathbf{k} \]  

(2.8)

to avoid the differential operators, so that Eq. (2.5) becomes

\[ \left( \frac{\partial}{\partial t} + \nu \mathbf{k}^2 \right) u_{\alpha}(\mathbf{k}, t) = -i k_{\alpha} M_{\alpha\beta}(\mathbf{k}) u_{\beta}(\mathbf{j}, t) u_{\gamma}(\mathbf{k} \cdot \mathbf{j}, t) \]  

(2.9)

with

\[ k_{\alpha} u_{\alpha}(\mathbf{k}, t) = 0 \]  

(2.10)

where the transformed operators are

\[ M_{\alpha\beta}(\mathbf{k}) = k_\beta D_{\alpha\beta}(\mathbf{k}) + k_\alpha D_{\alpha\beta}(\mathbf{k}) \]  

(2.11)

where

\[ D_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta / \mathbf{k}^2 \]  

(2.12)

Note the symmetry relation \( M_{\alpha\beta} = M_{\beta\alpha} \) with \( k_{\alpha} M_{\alpha\beta}(\mathbf{k}) = 0 \)

**Kolmogorov Phenomenology**

According to the Kolmogorov's picture\textsuperscript{49,50}, when a fluid is stirred at low wavenumber, \( k_0 = 1/L \), large scale eddies are produced. These eddies evolve to large wavenumbers as a
result of non-linear interactions. The cascade continues until, eventually, the small eddies are dissipated because the dissipative interaction \( v_k^2 \) grows as the square of the wavenumber. As the viscosity becomes smaller or the Reynolds number increases, dissipation acts on higher and higher wavenumbers.

Using dimensional analysis one can show that the energy spectrum must have the following form:

\[
E(k) \propto \bar{\epsilon}^{\frac{4}{3}} k^{-\frac{12}{3}} \Phi \left( \frac{k}{k_*} \right) \quad (2.13)
\]

with dissipation wavenumber \( k_d = \left( \frac{\bar{\epsilon}}{\nu^3} \right) \). Here, \( \bar{\epsilon} \) is the rate of energy dissipation per unit volume. Furthermore, in the so-called inertial range (small eddy wavenumber range) \( k_o << k << k_{\text{diss}} \) dissipation is negligible and is not influenced by the external condition. The energy spectrum in this range, depends only on \( \bar{\epsilon} \) and \( k \). Thus equation (2.13) reduces to

\[
E(k) \propto \bar{\epsilon}^{\frac{4}{3}} k^{-\frac{4}{3}} \quad (2.14)
\]

This is the well known Kolmogorov 5/3 power law for inertial range energy spectrum.

The Kolmogorov spectrum can also be derived from the following dynamical argument. Consider stationary homogeneous isotropic turbulence with energy being injected into the system by a forcing spectrum \( E(k) \) which is peaked
about a wavenumber \( k_\circ \) \((=1/L)\). It is assumed that the flow will evolve into a quasi-stationary self-similar hierarchy

\[
k_n = 2^n k_\circ, \quad n=0,1,2,\ldots \tag{2.15}
\]

The tendency of eddies to generate smaller and smaller eddies makes the word cascade appropriate for the description of energy transfer\(^1\).

\( \Pi(k_n) \) measures the rate at which energy is being transferred out of interval \( k_n < k < 2k_n \) into the interval \( 2k_n < k < 4k_n \). This rate is estimated by the amount of energy in \( k_n < k < 2k_n \) divided by the eddy turnover time\(^7\)

\[
\Pi(k_n) \sim \frac{\mathcal{E}_n}{\tau_n} \tag{2.16}
\]

If the flow is not intermittent\(^6\) and the spectrum is local (see Appendix 1 of the ref. (1)), then

\[
\mathcal{E}_n(t) \sim u_n(t) \tag{2.17}
\]

Now, assuming the existence of an inertial wavenumber range (where injection is absent and dissipation negligible), energy conservation implies that \( \Pi(k_n) \) is a constant in this range (symbolized by \( \bar{\epsilon} \)). We thus have

\[
u_n^{-1} / l_n \sim \bar{\epsilon} \tag{2.18}
\]
This leads to

\[ E_n \sim \varepsilon^{\gamma/2} \ell_n^{\gamma/3} \]  

(2.19)

Notice that \( E_n = \int_{k_m} E(k) \, dk \) which by the localness assumption, is equivalent to

\[ E(k) = c_4 \varepsilon^{\gamma/4} k^{-\gamma/4} \]  

(2.20)

This is the Kolmogorov spectrum.

Measurements have been made in order to check the validity of Kolmogorov spectrum and good agreement has been obtained\(^5\).

One of the characteristics of turbulence is the tendency to universality for scales much smaller than the integral scale \( L \) in flow. High-Reynolds number turbulent flow is characterized by basically three different spatial scale ranges\(^5\):

(i) For wavenumber \( k=0(\,\pi/L\,) \) the energy spectrum is strongly anisotropic and is not universal. The integral scale is reflected in both the geometry of the flow and the physicochemical processes taking place on these large scales.

(ii) At much smaller scales, with wavenumbers satisfying \( \pi/L \ll k \ll k_d \), the velocity fluctuation spectrum \( E(k) \) is nearly universal and is approximately given by the Kolmogorov energy spectrum with the Kolmogorov constant \( c_k=1.3-2.3 \).
(iii) In the dissipation range, \( k > 0( k_d) \), the energy spectrum decreases exponentially with \( k \) due to the molecular viscosity.

Indeed, this is the universality properties of the small eddies and the existence of a Kolmogorov spectrum that makes subgrid scale modeling possible by using RNG analysis.

**C. 4. - EXPANSION RNG ANALYSIS OF NAVIER-STOKES TURBULENCE**

The dynamic RNG, originally developed for critical phenomena, has been extensively used to derive scaling laws in the inertial range. Under some postulated equivalence, Yakhot and Orszag\(^\text{5,33}\), without any experimentally adjustable parameters, find numerical values for important constants of turbulent flows (e.g., the Kolmogorov constant for the inertial range spectrum,....)

Consider the \( d \)-dimensional space-time Fourier-transformed NSE for incompressible flow \( \hat{u}(\hat{k}) = (k, \omega) \)

\[
\hat{u}_k(\hat{k}) = G^0\hat{f}_k(\hat{k}) - \frac{1}{2} \sum_{h} G^0 \mathcal{P}_{k}\mathcal{R}\mathcal{P}(\hat{k}) \int u_m(\hat{k}_m) u_n(\hat{k}-\hat{q}) \frac{d^d f}{(2\pi)^d} \tag{2.21}
\]

where the zero-mean Gaussian random force \( f(k, \omega) \) is determined by its correlation function

\[
\langle f_k(\hat{k}) f_j(\hat{k}') \rangle = (2\pi)^d (2\mathcal{D}_0) k^0 \mathcal{D}_{ij}(\hat{k}) \delta(\hat{k}+\hat{k}') \tag{2.22}
\]

here
\[ \mathcal{G}^o = (-i\omega + \nu_0 k^2)^{-1}, \quad P_{ijk}(k) = 2i M_{ijk}(k) \] (2.23)

\( \lambda_0 \) is a parameter introduced for the use of perturbation analysis. It will eventually be set to unity. The exponent \( y > -2 \). The constant \( D_0 \) determines the intensity of the random force.

The problem Eq. (2.21) -- Eq. (2.23) is formulated on the interval \( 0 < k < \Lambda_\omega \) and \( -\omega < \omega < \omega \), where \( \Lambda_\omega \) is a wavenumber beyond the dissipation wavenumber at which substantial modal excitations cease.

As introduced by Ma and Mazenko\(^{52}\), the dynamic RNG procedure consists of two steps. First, we eliminate from Eq. (2.21) the modes \( u_j^>(i,\omega) \) such that \( \Lambda_\omega < |k| < \Lambda_\omega \). This is done by formally solving the equations for \( u_j^>(i,\omega) \) as a power series in \( \lambda_0 \). The solution, because of the nonlinearities, depends on the remaining modes \( u_j^>(i,\omega) \). These formal solutions are then substituted into the equations for \( u_j^>(i,\omega) \) to eliminate their explicit dependence on \( u_j^>(i,\omega) \). Finally, the reduced set of equations is averaged over that part of the force \( \xi_j^>(i,\omega) \) that acts in the wavenumber shell \( \Lambda_\omega < |k| < \Lambda_\omega \). This redefines the coefficients which enter the reduced equations of the motion.

The second step, in early RNG work\(^{27}\), consists of rescaling space, time, and the remaining velocities and forces in order to make the new set of equations look as much
as possible like the original NSE. However, the calculation will be done hereafter, using a new RNG procedure\textsuperscript{5,32,33} with variable ultraviolet cutoff, which is somewhat simpler than the Wilson-type technique used by FNS\textsuperscript{27} (in particular, no rescaling is needed).

The RNG scale-elimination procedure gives the correction to the bare viscosity \( \nu \), in terms of an effective viscosity which takes into account the effect of the eliminated modes. The result is

\[
\nu(r) = \nu_0 \left[ 1 + A_d \frac{\lambda_v^{\bar{\lambda}_v} \left( e^{\bar{\lambda}_v - 1} \right)}{\varepsilon} \right] \tag{2.24}
\]

where \( \varepsilon = 4 + y - d \), \( A_d = \tilde{A}_d \frac{S_d}{(2\pi)^d} \) and

\[
\tilde{A}_d = \frac{1}{2} \left( \frac{d^2 - d - \bar{\zeta}}{d(d+2)} \right) , \quad S_d = \frac{(2\pi)^d}{\Gamma \left( \frac{d}{2} \right)} \tag{2.25}
\]

The dimensionless expansion parameter \( \bar{\lambda}_v \) is defined as \( \bar{\lambda}_v = \frac{D_o}{\nu^2 A_v} \).

We derive differential equation for \( \nu(r) \) by variation of cutoff

\[
\lambda(r) = \lambda_v e^{-\bar{\lambda}_v}
\]

where

\[
\lambda(r) = \frac{D_o}{(\nu(r) \lambda(r))^\gamma} \tag{2.26}
\]

is the effective Reynolds number.

The solution to Eq. (2.26) is
\( \nu(r) = \nu_0 \left[ 1 + 3A_d \lambda_0^2 \left( \frac{\lambda^2 \epsilon^{\gamma}}{\ell} \right) / \ell \right]^{\lambda/2} \) \hspace{1cm} (2.27)

Substituting Eq. (2.27) into Eq. (2.26) we have

\[ \lambda(r) = \tilde{\lambda}_0 \left( \epsilon / 3A_d \right)^{-\lambda/4} \] \hspace{1cm} (2.28)

In the limit \( r \rightarrow \infty \), the parameter \( \tilde{\lambda}_0 \) given by Eq. (2.28) goes to the fixed point

\[ \tilde{\lambda}_0 = \left( \epsilon / 3A_d \right)^{-\lambda/4} \] \hspace{1cm} (2.29)

and in this limit, \( \nu(r) \) reduces to

\[ \nu(r) = \left( \frac{1}{4} A_d \tilde{D}_0 \right)^{\lambda/4} \] \hspace{1cm} (2.30)

If only modes with wavenumbers larger than \( \Lambda(r) \) are removed by the renormalization, then Eq. (2.30) gives a \( k \)-dependent viscosity in the limit \( r \rightarrow \infty \)

\[ \gamma(k) = \left( \frac{1}{4} A_d \tilde{D}_0 \right)^{\lambda/4} k^{-\lambda/4} \] \hspace{1cm} (2.31)

where we now set \( \lambda_0 = 1 \).

The coefficient \( \tilde{A}_d \) is computed from Eq. (2.25) in the lowest order of \( \epsilon \) expansion (\( \epsilon \rightarrow 0 \)); thus \( \tilde{A}_d = 0.2 \) in the three-dimensional case.

The choice of \( y = d \) (\( \epsilon = 4 \)) recovers the Kolmogorov scaling in the inertial range. We have
At a fixed point, the coupling parameter Eq. (2.29) can be treated, from the point of view of \( \epsilon \)-expansion, as a small parameter. Thus, in zeroth order, neglecting the nonlinear term in the forced NSE, which is defined on the smaller domain \( 0 < k < \lambda \epsilon^{-r} \), one finds that the velocity field is determined by

\[
\mathcal{U}_j (k) = G (k^* ) \mathcal{F}_j (k) \tag{2.33}
\]

where the renormalized propagator \( G(k) \) is given by

\[
G (k^* ) = \left[ -i \omega + v (r) \, k^* \right]^{-1} \tag{2.34}
\]

The energy spectrum is defined as

\[
E(k) = \frac{1}{2} \frac{S_d}{(2 \pi)^d} \int \frac{k^*}{i} \, \mathcal{T}_r \, \mathcal{V}_{ij} (\vec{k}, \omega) \, d\omega \tag{2.35}
\]

where

\[
\mathcal{V}_{ij} (\vec{k}, \omega) = \frac{1}{(2 \pi)^{2d}} \frac{1}{\delta (\vec{k} - \vec{k}') \delta (\omega - \omega')} \left< \mathcal{U}_i (\vec{k}, \omega) \mathcal{U}_j (\vec{k}', \omega') \right> \tag{2.36}
\]

Eq. (2.35) then leads to

\[
E(k) = 1.186 \, \left( 2 \, D_0 \frac{S_d}{(2 \pi)^d} \right)^{\frac{3}{2}} \, k^{-\frac{3}{2}} \tag{2.37}
\]
From a simple dynamical argument, it has been shown that in inertial range, we have

\[ \gamma(k) = \int \hat{\epsilon}^{\frac{\alpha}{2}} k^{-\frac{\alpha}{2}} \]  

(2.38)

and

\[ E(k) = C_h \hat{\epsilon}^{\frac{\alpha}{2}} k^{-\frac{\alpha}{2}} \]  

(2.39)

where

\[ \frac{N}{C_h^2} = 0.1904 \]  

(2.40)

From Eqs. (2.37) -- Eq. (2.40), \( c_h = 1.617 \).
CHAPTER III

\(-\varepsilon\) - EXPANSION RENORMALIZATION GROUP THEORY FOR ALFVEN WAVE TURBULENCE

We now consider the model Alfven wave turbulence system introduced by Chen and Mahajan\(^38\). They found by numerical simulation, under certain assumptions, that the Alfven wave spectrum obeys a power law \(k^{-2}\). The effect of the subgrid scales on the large supergrid scales for Alfven turbulence can indeed be significant (unlike the case of two-dimensional Navier-Stokes turbulence with its inverse cascade). In this chapter, \(-\varepsilon\) - expansion renormalization group theory is applied to this model Alfven wave turbulence equation. In particular, the effect of small 'unresolvable' subgrid scales on the large scales is computed. It is found that the removal of the subgrid scales leads to a renormalized response function \(\xi\). The Lorenzian wavenumber spectrum of Chen and Mahajan can be recovered for finite \(\varepsilon\), but the nonlinear coupling constant still remains small, fully justifying the neglect of higher order nonlinearities introduced by the renormalization group procedure.

In Sec.A, we briefly review the Chen and Mahajan model for Alfven wave turbulence while in Sec.B we consider the \(-\varepsilon\) - expansion renormalization group procedure. In Sec.C we summarize our conclusions.
A. MODEL ALFVEN EQUATIONS

Chen and Mahajan\textsuperscript{38} derived their model Alfven equations in cylindrical geometry from ideal(cold) magnetohydrodynamics.

\begin{align*}
\frac{\partial \rho_{\text{tot}}}{\partial t} + \mathbf{v} \cdot (\rho_{\text{tot}} \mathbf{u}_{\text{tot}}) &= 0 \quad (3.1) \\
\rho_{\text{tot}} (\frac{\partial \mathbf{u}_{\text{tot}}}{\partial t} + \mathbf{u}_{\text{tot}} \cdot \mathbf{v}) &\mathbf{u}_{\text{tot}} = (\mathbf{v} \times \mathbf{B}_{\text{tot}}) \times \mathbf{B}_{\text{tor}} / 4\pi \quad (3.2) \\
\frac{\partial \mathbf{B}_{\text{tot}}}{\partial t} &= \mathbf{v} \times (\mathbf{u}_{\text{tot}} \times \mathbf{B}_{\text{tot}}) \quad (3.3)
\end{align*}

by assuming an equilibrium with no fluid flow, uniform density $\rho_{eq}$ and constant toroidal magnetic field $B_{\phi} \hat{\mathbf{e}}_{\phi}$ . One can express Eq. (3.1)-(3.3) in terms of the fluctuating fields ($\mathbf{u}, \rho, \mathbf{b}$)

\begin{align*}
\mathbf{u} &= \mathbf{u}_{\text{tot}} \quad (3.4) \\
\rho &= \rho_{\text{tot}} - \rho_{eq} \quad (3.5) \\
\mathbf{b} &= \mathbf{B}_{\text{tot}} - B_{\phi} \hat{\mathbf{e}}_{\phi} \quad (3.6)
\end{align*}

to obtain

\begin{align*}
\frac{\partial \rho}{\partial t} + \rho_{eq} \mathbf{v} \cdot \mathbf{u} &= - (\mathbf{u} \cdot \mathbf{v}) \rho - \rho \mathbf{v} \cdot \mathbf{u} / \rho \quad (3.7) \\
4\pi \rho_{eq} \frac{\partial \mathbf{u}}{\partial t} - B_{\phi} (\frac{\partial \mathbf{b}}{\partial z} - \mathbf{v} b_{\phi}) &= -4\pi \rho \frac{\partial \mathbf{u}}{\partial t} \\
&- 4\pi \rho_{eq} (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} + (\mathbf{b} \cdot \mathbf{v}) \mathbf{b} - \nabla (B_{\phi}^2 / 2) \quad (3.8)
\end{align*}
where all the nonlinear terms are written on the right hand side of the equations. On assuming periodicity in the toroidal direction for a cylinder of length $2\pi R$, the fluctuating fields can be represented

$$\{\tilde{u}, \tilde{p}, \tilde{b}\} = \sum_{m,n,\omega} \{\tilde{u}(r), \tilde{p}(r), \tilde{b}(r)\} \Delta x \rho \left[ i k_\theta \epsilon_\varphi + ik_2^2 + i \omega t\right]$$ (3.10)

where

$$k_2 = n/R, \quad k_\theta = m/r$$ (3.11)

To obtain a simple model for shear Alfven turbulence, Chen and Mahajan introduced the following assumptions: (i) the flow is incompressible, and (ii) $|u_\theta| >> |u_r|$, with the principle nonlinear term coming from the poloidal component of momentum balance, Eq. (3.8). Under these conditions, one finally obtains a quadratically nonlinear equation for $u_\theta$

$$F(k,\omega) U_\theta(k,\omega) = \left(\frac{i\omega}{\epsilon_\varphi}\right) 4\pi \rho_0 \sum k_\omega U_\theta(k-k',\omega-\omega') U_\theta(k',\omega')$$ (3.12)

The Alfvén linear operator $F$ is defined by

$$F(k,\omega) \equiv \omega^2/\epsilon_\varphi - k_2^2$$ (3.13)
Chen and Mahajan\textsuperscript{38} further simplify Eq. (3.12) by restricting \( m = \pm 1 \), so that their basic one-dimensional Alfven model equation becomes

\[ \frac{d^2 u}{dt^2} + i \frac{e B}{V_A} k \omega \sum_{k', \omega'} u(k-k', \omega-\omega') u(k', \omega') \]  \hspace{1cm} (3.15)

where \( u_0 = u \), \( k_z = k \), and \( F(k, \omega) = \omega^2 / V_A - k^2 \).

**B. \( \epsilon \) - EXPANSION RENORMALIZATION GROUP THEORY**

In eliminating the small scales and determining the effect of the small scales on the large scales, one assumes in the \( \epsilon \) -expansion renormalization group technique that there is a random forcing term in Eq. (3.15) which simulates the production of turbulent energy. Thus we consider

\[ F(k, \omega) u(k, \omega) = (\frac{i \omega}{V_A^2}) \sum_{k', \omega'} u(k-k', \omega-\omega') u(k', \omega') + f(k, \omega) \]  \hspace{1cm} (3.16)

where the zero mean stationary Gaussian random force \( f \) is given by the two-point correlation

\[ \langle f(k, \omega) f(k', \omega') \rangle = D_0 k^{-\gamma} \delta(k+k') \delta(\omega+\omega') \]  \hspace{1cm} (3.17)

with

\[ V_A^2 = \frac{B_0^2}{4 \pi \rho_s} = \text{const.} \]  \hspace{1cm} (3.14)
The exponent $y$ will be determined later.

In the (so-called 'infrared') renormalization group procedure we will now eliminate a subgrid wavenumber shell from the ultraviolet cutoff $\Lambda$, to a new cutoff $\Lambda_{-\delta}\Lambda$. To facilitate comparison with the work of Yakhot and Orszag, we introduce the parameter $\ell$ such that

$$\Lambda_{-\delta}\Lambda = \Lambda_0 \Lambda_\ell \delta (-\ell)$$ (3.18)

The velocity in the subgrid shell, $\Lambda_0 \delta^\ell < k < \Lambda_0$, is denoted by $v^*$ while velocity in the supergrid domain, $k < \Lambda_0 \delta^{-\ell}$, is denoted by $v^\ell$. Thus, for the supergrid modes, the Alfvén wave equation becomes

$$u^\ell(k,\omega) = \frac{f^\ell(k,\omega)}{F(k,\omega)} + \left(\frac{i\omega/v_\ell^\ell}{\Lambda_0}\right) F^\ell(k,\omega) \times$$

$$\sum_{k',\omega'} \left[ u^\ell(k',\omega') + u^\ell(k',\omega') \right] \left[ u^\ell(k-k',\omega-\omega') + u^\ell(k-k',\omega-\omega') \right]$$

for $k < \Lambda_0 \delta^{-\ell}$ (3.19)

while in the subgrid shell

$$u^*(k,\omega) = \frac{f^*(k,\omega)}{F(k,\omega)} + \left(\frac{i\omega/v_\ell^*}{\Lambda_0}\right) F^*(k,\omega)$$

$$\times \sum_{k',\omega'} \left[ u^*(k',\omega') + u^*(k',\omega') \right] \left[ u^*(k-k',\omega-\omega') + u^*(k-k',\omega-\omega') \right]$$

for $\Lambda_0 \delta^\ell < k < \Lambda_0$ (3.20)

We now introduce the bare Green function
\[ G_0 = \left[ \mathcal{E}_o \mathcal{F}(k, \omega) \right]^{-1} \]

with the bare response function

\[ \mathcal{E}_o = 1 \]  \hspace{1cm} (3.22)

and the Alfvén singularity has been resolved in \( F(k, \omega) \) by the addition of a small imaginary part (due, for example, to resistivity or Landau damping) proportional to \( \omega \)

\[ F(k, \omega) = \omega^2 / \nu_a^2 - k^2 + i \gamma \omega / \nu_a \]  \hspace{1cm} (3.23)

assuming the damping parameter \( \gamma \) small.

Thus \( F(k, \omega) = F^*(k, -\omega) \) and \( G_o(k, \omega) = G_o^*(k, -\omega) \).

We now introduce the bare vertex parameter \( \lambda_o \) as a perturbation parameter (eventually \( \lambda_o \) is set to unity) so that our model problem is

\[ u^c(k, \omega) = G_0^c(k, \omega) f^c(k, \omega) + \lambda_o \left( i \omega / \nu_a^2 \right) G_0^c(k, \omega) \]

\[ \times \sum_{k', \omega'} \left[ u^c(k-k', \omega-\omega') + u^c(k-k', \omega-\omega') \right] \left[ u^c(k', \omega') + u^c(k', \omega') \right] \]

for \( k < \Lambda_0 \ell^{-1} \) \hspace{1cm} (3.24)

while for the subgrid modes

\[ u^s(k, \omega) = G_0^s(k, \omega) f^s(k, \omega) + \lambda_o \left( i \omega / \nu_a^s \right) G_0^s(k, \omega) \]

\[ \times \sum_{k', \omega'} \left[ u^s(k-k', \omega-\omega') + u^s(k-k', \omega-\omega') \right] \left[ u^s(k', \omega') + u^s(k', \omega') \right] \]

for \( \Lambda_0^{-1} \ell < k < \Lambda_0 \) \hspace{1cm} (3.25)
where $G^>$ is the subgrid and $G^<$ the supergrid Green function. The high $k$-modes in the subgrid shell can now be eliminated by substituting Eq. (3.25) for $u^>$ into Eq. (3.24), and then performing an ensemble average over the subgrid scales

$$
\langle u^<(k,\omega) \rangle = u^<(k,\omega), \quad \langle u^>(k,\omega) \rangle = 0
$$

(3.26)

Thus, for $k < \lambda_0 e^{-\frac{\omega}{\nu_A}}$

$$
[ G_0^<(k,\omega)]^{-1} u^<(k,\omega) = \xi^<(k,\omega) + \lambda_0 \left( \frac{i \omega}{\nu_A} \right) \sum_{k',\omega'} G_0^<(k',\omega') u^<(k-k',\omega-\omega')
$$

$$
- 2 \lambda_0^2 \left( \frac{\omega}{\nu_A^4} \right) \sum_{k',\omega'} \frac{1}{2} u^<(k-k',\omega-\omega') G_0^<(k',\omega') \omega' x
$$

$$
\times u^<(k'-k'',\omega'-\omega'') u^<(k'',\omega'') x
$$

$$
\left( \langle u^<(k-k',\omega-\omega') u^<(k-k'',\omega-\omega'') \rangle - \langle u^<(k-k',\omega-\omega') \rangle \langle u^<(k-k'',\omega-\omega'') \rangle \right)
$$

(3.27)

Thus, using the leading order expression for $u^>$ and the two-point correlation, Eq. (3.17), Eq. (3.27) can be reduced to

$$
\mathcal{C}_0(l) u^<(k,\omega) = F^<(k,\omega) f^<(k,\omega)
$$

$$
+ \lambda_0 \left( \frac{i \omega}{\nu_A^2} \right) F^<(k,\omega) \sum_{k',\omega'} u^<(k',\omega') u^<(k-k',\omega-\omega')
$$

$$
- 2 \lambda_0^2 \left( \frac{\omega}{\nu_A^4} \right) F^<(k,\omega) \sum_{k',\omega'} \frac{1}{2} u^<(k-k',\omega-\omega') G_0^<(k',\omega') \omega' x
$$

$$
\times u^<(k'-k'',\omega'-\omega'') u^<(k'',\omega''') u^<(k'',\omega''') + o(\lambda_0^4)
$$

(3.28)

The renormalized response function $\mathcal{C}_0(l)$ is given by
\[ E_0^{(0)} = E_0^* + \delta E_0^* \tag{3.29} \]

with (taking the discrete \(\rightarrow\) continuum limit)
\[ \delta E_0^* u_s(k, \omega) = 4 \lambda_0^2 \left( \frac{\omega}{\omega_A} \right)^2 \mathcal{F}^{-1}(k, \omega) \int d k' d \omega' dk'' d \omega'' G_0(k, \omega) \times G_0(k-k', \omega-\omega') G_0(k-k'', \omega-\omega'') \langle \hat{f}(k', \omega') \rangle \langle \hat{f}(k'', \omega'') \rangle + O(\lambda_0^3) \]
\[ = 4 \lambda_0^2 \left( \frac{\omega}{\omega_A} \right)^2 \mathcal{F}^{-1}(k, \omega) \int d k' d \omega' G_0(k', \omega') \omega' \]
\[ \mathcal{D}_0 \left( k-k' \right)^{-4} G_0(k-k', \omega-\omega') G_0(k-k', \omega-\omega') + O(\lambda_0^3) \tag{3.30} \]

Thus the renormalized correction to the bare response function is
\[ \delta E_0^* = 4 \lambda_0^2 \left( \frac{\omega}{\omega_A} \right)^2 \mathcal{D}_0 \mathcal{F}^{-1}(k, \omega) \int d k' \left( k-k' \right)^{-4} I(k') \tag{3.31} \]

where
\[ I(k') \equiv \int d \omega' \left| G_0(k-k', \omega-\omega') \right|^2 G_0(k', \omega') \omega' \tag{3.32} \]
on using a property of the Alfven linear operator \( F \), Eq.(3.23).

Since \( k' \) is in the subgrid shell, while \( k \) and \( k'' \) are in the supergrid range, Eq.(3.32) can be readily evaluated
\[ I(k') \approx \left( c/4 \omega \omega_A \right)^2 k^{-2} \mathcal{E}_0^{-3} \tag{3.33} \]

with
Thus, the effect of removing the subgrid shell \((\Lambda^* e^{-\phi}, \Lambda^*)\) is to yield an increment to the response function

\[
\delta \mathcal{E} = \lambda^3 \mathcal{E}_0^{-3} \mathcal{D}_o C \int_{\Lambda^* e^{-\phi}}^\Lambda \frac{dk}{k^{(y+1)}}
\]  

(3.35)

In \(\epsilon\)-expansion renormalization group theory procedure, we now introduce the small parameter \(\epsilon\) by \(\epsilon = -2 + y - d = y + 1\) where \(d\) is the dimensionality of the turbulence equation. Integrating Eq. (3.35),

\[
\delta \mathcal{E}_0 (\epsilon) = \lambda^3 \mathcal{D}_o (\frac{C}{\epsilon}) (\mathcal{E}_0^{-1} \mathcal{E}_0^{-1}) \mathcal{E}_0^{-3} \Lambda^\epsilon
\]  

(3.36)

A new coupling coefficient \(\tilde{\lambda}_0\) is introduced by

\[
\tilde{\lambda}_0^2 = \lambda^3 \mathcal{D}_o / \mathcal{E}_0^4 \Lambda^\epsilon
\]  

(3.37)

so that

\[
\delta \mathcal{E}_0 (\epsilon) = \tilde{\lambda}_0^2 C \epsilon^{-1} (\mathcal{E}_0^{-1} \mathcal{E}_0^{-1}) \mathcal{E}_0
\]  

(3.38)

Thus the renormalized response function, after eliminating the subgrid shell \(\Lambda^* e^{-\phi} < k < \Lambda^*\), is

\[
\mathcal{E}_0 (\epsilon) = \mathcal{E}_0 [1 + \tilde{\lambda}_0^2 C \epsilon^{-1} (\mathcal{E}_0^{-1} \mathcal{E}_0^{-1})]
\]  

(3.39)
I. DIFFERENTIAL-RECURSION RELATIONS

One can now eliminate a finite subgrid wavenumber band by successive elimination of infinitesimal bands. This will result in a renormalized response function $\xi - \xi(l)$ and coupling constant $\lambda - \lambda(l)$. The corresponding differential equation can be obtained by taking the limit $\ell \to 0$ in Eq. (3.36) for the cutoff $\Lambda(l) = \Lambda_0 \ell^{-2}$. We obtain

$$d\xi / dt = \lambda^2 D_0 C \xi^{-3} \Lambda^{-\xi} \Lambda \rho (\ell \xi)$$

(3.40)

so that, since $\xi(0) = 1 = \xi_0$,

$$\xi^4(l) = \xi_0^4 + 4 \lambda^2 D_0 C \xi^{-3} \xi^{-1} (l \xi^{-1})$$

$$= \xi_0^4 \left[ 1 + 4 \lambda^2 D_0 C \xi^{-1} (l \xi^{-1}) \right]$$

(3.41)

Thus, as $\ell \to 0$,

$$\xi^4(l) \approx (4 D_0 C \xi^{-1})^{1/2} \Lambda^{-\xi/4}$$

(3.42)

The coupling constant $\lambda(l)$ is determined from Eq. (3.37)

$$\lambda^2(l) = \lambda_0^2 D_0 \xi^{-4} \Lambda^{-\xi} \ell \xi$$

(3.43)

Moreover, in the infrared limit $\ell \to + \infty$ (with $\xi > 0$), the coupling parameter $\lambda$ converges to a fixed point
\[
\bar{\lambda}^* = \left( \frac{\varepsilon}{4\zeta} \right)^{1/2}
\]

which is indeed small (even for finite \( \varepsilon \)) \( \bar{\lambda}^* \ll 1 \) since \( \zeta \) is large.

2. DETERMINATION OF SPECTRAL EXPONENT \( \gamma \) IN \( \langle \epsilon \rangle = k^{-\gamma} \)

Thus, for small \( \varepsilon \) and since \( \zeta \gg 1 \) (see Eq. (3.34) for supergrid \( \omega \) and weak damping coefficient \( \gamma \))

\[
\tilde{\epsilon}(k) \approx (4D_0 \varepsilon^{1/\gamma})^{1/2} k^{-\epsilon/\gamma}
\]

with the removal of subgrid modes with wavenumbers greater than \( k = \Lambda(k) \). Since the fixed point coupling constant \( \bar{\lambda}^* \ll 1 \), the nonlinear terms in the renormalized Alfven wave equation are small so that the renormalized velocity, to leading order, becomes

\[
u(k, \omega) = G^*(k, \omega) \tilde{f}(k, \omega)
\]

where the renormalized Green function is given by

\[
G^*(k, \omega) = \left[ \tilde{\epsilon}(k) F(k, \omega) \right]^{-1}
\]

Using Eqs. (3.45)-(3.47), (3.17) and (3.23) the energy spectrum, for \( k' \) in the subgrid, is given by
\[ E(k') = (2\pi)^\alpha \int_{-\infty}^{\infty} d\omega' \left< \mathbf{u}(k',\omega') \cdot \mathbf{u}(k''\omega'') \right> / (k' \cdot k'') \delta(\omega' - \omega'') \]

\[ = (2\pi)^\alpha \left[ \frac{(1+y)}{4\pi} \right]^{\frac{\alpha}{2}} k' \frac{1}{2}(1+y) \]

\[ \times D_\alpha k^{-2} \int_{-\omega}^{\omega} d\omega' |c|^2 |E(k',\omega')|^{-2} \]  

(3.48)

Since \( \epsilon = 1+y \). Now the major contribution\(^{27,32} \) to the integration comes from the Alfven singularities in subgrid range (i.e., for frequencies \( \omega' \approx \pm k'c \) ) so that

\[ E(k') = \Phi'(D_\alpha, \alpha) k^{-2} k'^{1-\gamma/2} \]  

(3.49)

for a certain coefficient \( \Phi' \) which we do not consider further here since in Alfven turbulence there is no information available, unlike the case of Navier-Stokes turbulence where the Kolmogorov constant is considered quite well known.

From numerical simulation, Chen and Mahajan\(^ {38} \) have found that the energy spectrum

\[ E(k') \approx k'^{-2} \]  

(3.50)

so that we require (\( d \) is the dimensionality of the turbulence)

\[ \gamma = 1 = d \]  

(3.51)

in the forcing spectrum for the energy spectra to agree (see Eq. (3.49) and Eq. (3.50)). The result, Eq. (3.51), is similar
to that in three-dimensional Navier-Stokes turbulence where $y = d = 3$ is required to recover the Kolmogorov energy spectrum $-5/3$.

3. AN ALTERNATIVE DERIVATION OF THE SPECTRAL EXPONENT $\gamma$

We now present an alternative derivation of the spectral exponent $\gamma$, using the argument of Fournier and Frisch. The energy spectrum at wavenumber $k$ with cutoff wavenumber between the subgrid and supergrid scales, due to the renormalization procedure, can be related to the energy spectrum for the bare problem with $\Lambda_0 \rightarrow +\infty$. Since the forcing $D_0$ is not renormalized

$$E(k; E_p = 1) |_{\Lambda_0 \rightarrow \infty} = \overbar{E}(k; E_{p_0}) |_{\Lambda_0 \rightarrow \infty}$$

(3.52)

provided (see Eq. (3.42))

$$E_{p_0} = (4D_0 C^{-1})^{1/4} \Lambda^{-5/4}$$

(3.53)

Now for the bare problem

$$E(k; 1) |_{\Lambda \rightarrow \infty} = \int d\omega \langle ff \rangle / C^{-1} G_0^{-1}$$

$$= C_0^{-2} \int d\omega D_0 k^{-5/3} (\omega^2 - k^2 + i\omega) (\omega^2 - k^2 - i\omega)$$

on normalizing the frequencies to eliminate the Alfvén velocity $v_A$. Thus
\[ E(k, \omega) \approx k^{-2} / E_0^2 \quad (3.54) \]

From the renormalization group procedure we have

\[ E(k, E(\omega)) \approx k^{-2} / (E(\omega))^2 \]
\[ \approx k^{-2} / k^{-(\epsilon+1)/2} \quad (3.55) \]

Since the Chen-Mahajan spectrum has \( E(k) \approx k^{-2} \), we have from Eq. (3.55)

\[ y = 1 \quad (\epsilon = 1 + y = 2) \]

**C. SUMMARY**

The effect of unresolved subgrid scales on the large scale modes in Alfven turbulence is calculated using the methods of \( \epsilon \)-expansion renormalization group. It is found that the renormalized (complex) response function for the large scales is given by

\[ E(k) \approx \left[ 4 \pi \omega V_\kappa \delta / \gamma (\omega + i \chi)(i + y) \right] k^{\chi} \frac{-\left(\frac{\chi \gamma}{4} \right)}{4} \quad (3.56) \]

where the turbulent energy generated by the subgrid scales is modeled by a random forcing with two-point correlation

\[ \langle f(k, \omega) f(k', \omega') \rangle = D_0 k^{-y} \delta(k + k') \delta(\omega + \omega') \]
The Alfvén singularity is resolved by the introduction of a small damping factor $\gamma$ (which can model either resistive, finite Larmor radius or Landau damping effects). The exponent $\gamma$ is determined by requiring the energy spectrum calculated from the $\epsilon$-expansion renormalization group technique agree with that obtained by the numerical simulation of Chen and Mahajan. This procedure is validated by showing that the nonlinear coupling constant converges to a fixed point

$$\xi^{*} = \left[ \frac{\xi \left( \omega + i \gamma \right)}{4\pi \omega \nu^2} \right]^{1/2}$$

for parameter $\epsilon = 1 + \gamma$. It is found that

$$\gamma = 1$$

This result is analogous to those found for $\epsilon$-expansion renormalization group procedures in three dimensional Navier-Stokes turbulence where the corresponding exponent $\gamma$ is shown to be equal to the dimensionality 3.
CHAPTER IV
RENORMALIZATION GROUP THEORY
FOR THE EDDY VISCOSITY IN SUBGRID MODELING

In this chapter, we shall apply the difference recursion RNG technique (originally applied to the linear problem of passive scalar diffusion\textsuperscript{40}) to Navier-Stokes turbulence, taking proper account of symmetries. The RNG procedure by which the subgrid shells are removed iteratively is outlined in Sec.A. In Sec.B, the renormalized viscosity is calculated numerically and compared both to that found by the iterative averaging RNG procedure of McComb\textsuperscript{9,16,41,42} (see also Appendix B) and to the closure models of Kraichnan\textsuperscript{17} and Chollet and Lesieur\textsuperscript{43}. It is shown that not only is the triple nonlinearity necessary for the RNG eddy viscosity to exhibit a cusp behavior near the subgrid/supergrid wavenumber cutoff, but also the presence of the pressure in the Navier-Stokes equation is required. In Sec.C, a linearized model calculation (following a similar model discussed by Rose\textsuperscript{40}) is introduced to examine the direct effect of the triple nonlinearity in the renormalized Navier-Stokes equation. We summarize our results in Sec.D.

A. NAVIER-STOKES TURBULENCE AND
RENORMALIZATION GROUP PROCEDURE

We consider incompressible turbulence, and the Navier-
Stokes equation in wavenumber space (utilizing the summation
convention over repeated subscripts)

\[
\left( \frac{\partial}{\partial t} + \nu_k \right) u_k (\vec{k}, \tau) = \int \frac{d^3 \vec{k}'}{4\pi} n_{\alpha\beta}(k) u_{\alpha}(\vec{k}', \tau) u_{\beta}(\vec{k}', \tau)
\]

(4.1)

The incompressibility condition

\[
k^\alpha u^\alpha_k (\vec{k}, \tau) = 0
\]

(4.2)

has been employed to eliminate the pressure gradient

\[
\nabla^2 p = \frac{\partial}{\partial \tau} \left( u_\alpha u_\beta \right) / \partial x_\alpha \partial x_\beta
\]

(4.3)

resulting in the quadratic nonlinear coupling coefficient
given by

\[
M_{\alpha\beta\gamma}(k) = k^\alpha D_{\alpha\beta}(k) + k^\gamma D_{\alpha\beta}(k)
\]

(4.4)

where

\[
D_{\alpha\beta}(k) = \delta_{\alpha\beta} - k^\alpha k^\beta / k^2
\]

1. RNG PROCEDURE

In the RNG method, one partitions the unresolvable
subgrid scales into shells, characterized by a scale factor
\( f. 0 < f < 1 \). The spectrum is partitioned by the wavenumber set
\( \{ k^N = f^N k_0, \ldots, k_1 = f^1 k_0, \ldots, k_1 = f^1 k_0, k_0 \} \). \( k_0 \) is
typically chosen to be on the order of the Kolmogorov
dissipation wavenumber\(^2\) while \( k_N \) is the wavenumber which
separates the actual resolvable scales \((k < k_N)\) from the unresolvable scales \((k_N < k < k_0)\). The RNG iterative procedure consists of first eliminating the highest wavenumber subgrid shell \(k_1 < k < k_c\) from Eq. (4.1) to leave a modified Navier-Stokes equation for the remaining 'supergrid' wavenumbers \(k < k_1\). It will be shown that the modifications to the Navier-Stokes equation are (i) a renormalized viscosity coefficient, and (ii) a triple nonlinearity in the fluid velocity. One then proceeds iteratively, removing at the i-th step the subgrid shell \(k_i < k < k_{i-1}\) till one reaches the actual resolvable scales at the N-th step. Since we are dealing with free decay, it is assumed that in the subgrid scales the inertial energy spectrum \(E(k)\) obeys some given power law

\[
E(k) \propto k^{-m}, \quad k_N < |k| < k_0 \quad (4.5)
\]

Theoretically, one can proceed with an arbitrary value for the power law exponent, \(m\), but when we present our numerical results for the renormalized eddy viscosity we shall employ the Kolmogorov exponent \(m=5/3\). For isotropic, stationary turbulence in the subgrid scales, the equal time velocity covariance is

\[
\langle \mathbf{u}_x(k,t) \mathbf{u}_y(k',t) \rangle = D_{xy}(k,k') \delta^{(k+k')} Q(|k|)
\]

\[
(4.6)
\]

where \(Q(|k|)\) is related to the energy spectrum \(E(k)\) by
2. REMOVAL OF THE FIRST SUBGRID SHELL

We now consider the effect of removing the first subgrid shell $k_1 < k < k_0$ from the Navier-Stokes equation (4.1) in the RNG procedure. It is convenient to introduce the notation

$$u_a (\vec{k}, t) \equiv u_a (\vec{k}, t) \quad \text{for} \quad |\vec{k}| < k_1$$

$$\equiv u_a (\vec{k}, t) \quad \text{for} \quad k_1 < |\vec{k}| < k_0$$

and to introduce an ensemble average over the particular subgrid shell modes under consideration

$$\langle u_a^\gamma (\vec{k}, t) \rangle \equiv 0$$

$$\langle u_a^\epsilon (\vec{k}, t) \rangle \equiv u_a^\epsilon (\vec{k}, t)$$

For $k$ in the first subgrid shell, Eq. (4.1) becomes

$$\left( \frac{3}{2t} + \nu_b k^2 \right) u_\alpha^\gamma (\vec{k}, t) = M_{\alpha\beta} (\vec{k}) \int d^3 j \left[ u_\beta^\gamma (\vec{j}, t) + u_\beta^\epsilon (\vec{j}, t) \right]$$

$$\times \left[ u_\gamma (\vec{k-j}, t) + u_\gamma (\vec{k-j}, t) \right] \quad k_1 < |\vec{k}| < k_0$$

while for those $k$ in the first 'supergrid' range,

$$\left( \frac{3}{2t} + \nu_b k^2 \right) u_\alpha^\epsilon (\vec{k}, t) = M_{\alpha\beta} (\vec{k}) \int d^3 j \left[ u_\beta^\gamma (\vec{j}, t) + u_\beta^\epsilon (\vec{j}, t) \right]$$

$$\times \left[ u_\epsilon (\vec{k-j}, t) + u_\epsilon (\vec{k-j}, t) \right] \quad |\vec{k}| < k_1$$
(It should be noted that the right hand sides of Eqs. (4.10) and (4.11) are very different, not only because of the k-range but also in the ranges of j-integrations.)

Following Rose\textsuperscript{40} and McComb\textsuperscript{9,16,41,42}, we assume that in every realization \( u^\prime \) evolves faster than the supergrid velocity field \( \bar{u} \), so that \( \bar{u}/\partial t \) can be neglected in Eq. (4.10). Thus, from Eq. (4.10)

\[ u^\prime (\omega, t) = \left( \sqrt{\rho_j} \right) M_{\alpha\beta\gamma} (j) \int d^3 j' \left[ u^\prime (j', t) + u^\prime (j, t) \right] \]

\[ \left[ u^\prime (j, t) + u^\prime (j - j', t) \right] \]

for \( k_1 < |k| < k_0 \) \hfill (4.12)

We now substitute Eq. (4.12) into (4.11), taking care of symmetries, and perform the subgrid shell ensemble average of Eq. (4.9) to obtain, for \( |k| < k_1 \)

\[ \left[ \frac{2}{\partial t} + v_1 (k) \left( \frac{\partial^2}{\partial t^2} \right) \right] u^\prime (k, t) = M_{\alpha\beta\gamma} (k) \int d^3 j' u^\prime (j', t) u^\prime (k-j', t) \]

\[ + 2 M_{\alpha\beta\gamma} (k) \int d^3 j' d^3 j' ' \left[ v_0 (k) \right] M_{\alpha\beta\gamma} (j) u^\prime (j', t) u^\prime (k-j', t) \]

\[ \left( \frac{\partial^2}{\partial t^2} \right) \]

Thus the effect of removing the first subgrid shell on the Navier-Stokes can be seen to

(i) renormalize the molecular viscosity to

\[ \nu_1 (k) = \nu_0 + \delta \nu (k) \]

(4.14)

where
\[ \delta v_0 (k) = \frac{1}{2} \int d^4 j \, G \left( \frac{1}{k-j} \right) L_{kj} / 2 v_0 k^2 j^2 \]  \hspace{1cm} (4.15) 

with the coefficient \( L_{kj} \) defined by

\[ L_{kj} = -2 M_{\omega} (k) M_{\nu} (k) D_{\omega} (k) D_{\nu} (k) \]

\[ = \frac{k^2 \mu^2 - k^2 \left( (1+2 \mu^2) / [k^2+j^2-2 k j \mu] \right)}{4} \hspace{1cm} (4.16) \]

\( k,j = k j \mu \), with \( \mu = \cos \theta \). The integration limits in Eq. (4.15) are \( k_1 < |k-j| < k_0 \), and \( k_1 < |j| < k_0 \); and (ii) include a triple nonlinearity \( u^c u^c u^c \). This is a typical biproduct of RNG (see, for example, RNG for the two-dimensional Ising spin Hamiltonian. One finds that after the first spin decimation, not only is there the original nearest-neighbor interaction but also a diagonal nearest-neighbor and four spin-coupling interactions. These new interactions are weaker than the original interaction. Moreover, Wilson\(^{21}\) neglects the four spin-coupling interaction in his RNG and still obtains a very good approximation to the universal critical exponents—although it must be noted that the Ising model is in equilibrium while we are interested in fluid turbulence).

### 3. REMOVAL OF THE N-th SUBGRID SHELL

Thus, after removing the first subgrid shell, the Navier-Stokes equation (4.1) is modified to Eq. (4.13)
where there is now no need for the superscript '<' notation on the velocity field since the wavenumbers are all restricted to $0 < k < k_1$.

To remove the second subgrid shell, we denote the (current) subgrid modes by

$$u_a = u_a^> (k, t) \quad \text{if} \quad k_2 < |k| < k_1 \quad (4.18)$$

and the supergrid modes by

$$u_a = u_a^< (k, t) \quad \text{if} \quad |k| < k_2 \quad (4.19)$$

and proceed as in previous section, but now realize that the triple nonlinearity $uuu$ in Eq.\((4.17)\) will also contribute in the renormalization procedure. We find that the renormalized Navier-Stokes equation is now given by

$$\left[ \frac{3}{t} + \gamma_i (k) k^2 \right] u_a^< (k, t) = M_{a\beta\gamma} (k) \int d^3 j u_a^< \nabla_i j \nabla_\beta j \nabla_\gamma j + 2 \frac{1}{24} \rho M_{a\beta\gamma} (k) \int d^3 j d^3 j' \left[ \gamma_i (j) j' \nabla_{i} j' \nabla_\beta j' \nabla_\gamma j' \right] u_a^< \nabla_\beta j' \nabla_\gamma j'$$

for $k < k_2$, with the eddy viscosity
\[ \nu_2(k) = \nu_1(k) + \delta \nu_1(k) \]  

(4.21)

\[ \delta \nu_2(k) = 2 \sum_{i=0}^{n} \int \frac{d^3 \theta}{\beta} \Theta \left( |\bar{k} - \bar{j}| \right) L_{ij} \left[ \nu_1(\bar{j}) \frac{i^2 k^2}{|k|^2} \right] \]  

(4.22)

In Eq. (4.22), \( k_2 < |k-j| < k_1 \) and \( k_{i+1} < |j| < k_i \), for \( i = 0 \) or 1. Notice that the \( i=1 \) term in Eq. (4.22) is due to the triple nonlinearity in Eq. (4.17).

Proceeding iteratively, it can be seen that after removing the \( (n+1) \)-th subgrid shell the Navier-Stokes equation becomes

\[
\left[ \frac{\partial}{\partial t} + \nu_n(k) k^2 \right] u_\alpha(k,t) = \mathcal{M}_{48}(k) \left[ \int u_\beta(j,t) u_\gamma(j,t) u_\delta(k-j,t) \right] + 2 \mathcal{M}_{48}(k) \sum_{i=0}^{n-1} \int d^3 \theta' d^3 \rho' \mathcal{M}_{58}(\bar{i}) / [\nu_1(\bar{i}) \bar{k}^2] \]

(4.23)

\[ u_\alpha(k) = u_\alpha(\bar{k} - \bar{j}, t) u_\beta(\bar{j}, t) u_\gamma(k-j,t) \]

(4.24)

In the \( i \)-th term of the summation \( \sum_{i} \) the limitations on the \( j \) and \( j' \) integrations are \( |j|, |j-j|, |k-j| < k_{i+1} \); but \( k_{n-1} < |j| < k_{n-i-1}, i = 0, \ldots, n-1 \).

The eddy viscosity recursion relation is given by

\[ \nu_{n+1}(k) = \nu_n(k) + \delta \nu_n(k) \]  

(4.24)

where

\[ \delta \nu_n(k) = 2 \sum_{i=0}^{n} \int \frac{d^3 \theta}{\beta} L_{ij} \Theta \left( |\bar{k} - \bar{j}| \right) \left[ \nu_1(\bar{j}) \frac{i^2 k^2}{|k|^2} \right] \]  

(4.25)
The integration limits in Eq.(4.25) and $k_{n+1} < |k-j| < k_n$
and $k_{i+1} < |j| < k_i$ for $i=0,\ldots,n$.

The RNG transformation for the $(n+1)^{th}$ subgrid shell is

$$k \rightarrow k_{n+1} \tilde{k}$$

(4.26)

with the renormalized viscosity $\nu^*_n$ defined by

$$\nu^*_n(\tilde{k}) = C \frac{m^{n+1}}{k_{n+1}} \nu_n(k_{n+1}\tilde{k})$$

for $\tilde{k} \leq 1$ (4.27)

for some constant $C$. Thus, the renormalized eddy viscosity recursion relation becomes

$$\nu^*_{n+1}(\tilde{k}) = \int \nu^*(\tilde{k}) \left[ \nu^*_n(f_\tilde{k}) + \delta \nu_n(f_\tilde{k}) \right]$$

(4.28)

with

$$\delta \nu^*_n(\tilde{k}) = 2 \sum_{i=0}^{n} \int \nu^*_{n+1}(f_\tilde{k}) \frac{Q(1\tilde{k}^{-1})}{\nu^*_n(f_\tilde{k}) \tilde{k}^2 \delta^2}$$

(4.29)

and integration limits ($\tilde{k} \leq 1$)

$$1 < |\tilde{k}-j| < (1/f), \text{ and } 1 < |j\tilde{k}| < (1/f)$$ (4.30)

It should be noted that the $i=0$ contribution to $\delta \nu^*_n$ in
Eq.(4.29) arises from the usual Navier-Stokes quadratic non-linearity, while $i > 1$ terms arise from the triple non-
ity introduced by the RNG transformations.

**B. RENORMALIZED EDDY VISCOSITY**

The renormalized eddy viscosity is defined as the fixed point \( n \to \infty \) of the recursion relation (4.28) and (4.29). This recursion relation has been solved numerically, and we find that a fixed point exists for each \( k \), and that this fixed point is independent of the initial value of the molecular viscosity \( \nu_0 \) -- as is intuitively expected for the case of strong turbulence.

In Figs. 1 and 2, we plot the \( k \) dependence of the renormalized eddy viscosity (for various choices of the parameter \( f \)) and compare our results with those of McComb\textsuperscript{9,16,41,42} and Kraichnan\textsuperscript{17}, the parameter \( f \) defines the coarseness of the subgrid shell partition. For the finer subgrid partition of \( f=0.7 \) (Fig.1). We see that the RNG eddy viscosity exhibits a mild cusp behavior for \( \tilde{k} \) close to the subgrid/supergrid cutoff -- in qualitative agreement with the test field model of Kraichnan\textsuperscript{17}, the eddy damped quasinormal approximation of Chollet and Lesieur\textsuperscript{43} as well with the recent direct numerical simulation results of Domaradzki et al\textsuperscript{44}, but in contrast to the iterative averaging RNG results of McComb\textsuperscript{42}. 
We will first consider the RNG eddy viscosity. If one totally neglects the contribution of the triple nonlinearity to the eddy viscosity (i.e. retains only the \( i=0 \) term in \( \varepsilon \) in Eq.(4.29)) then the RNG viscosity does not exhibit any cusp behavior. This is shown in Figs.1 and by the solid curves with symbol \( \Delta \). Since the McComb\textsuperscript{42} iterative averaging technique is claimed to result in a renormalized NSE without a triple nonlinearity, the McComb eddy viscosity also does not exhibit any cusp behavior (the dashed curve in Figs.1 and 2). In fact, McComb's recursion relation is essentially Eqs.(4.28)-(4.29) but without the \( 2 \) factor in the \( \mu \) equation (which also has only \( i=0 \) contributions).

However, the appearance of the triple nonlinearity in the eddy viscosity recursion relation is not sufficient for the appearance of the cusp near the supergrid/subgrid cutoff. If the effect of the pressure gradient is dropped from the incompressible NSE it can be shown that no cusp appears even though triple nonlinearities are generated in the RNG procedure. Indeed, with the neglect of the pressure force the problem will reduce to that of advection of a vector field by a solenoidal velocity field\textsuperscript{54} and this is also closely related to the problem originally treated by Rose\textsuperscript{40}. Explicitly, the effect of the pressure can be immediately seen in the nonlinear coupling coefficient \( L_{kj} \), Eq.(4.16).

For the full Navier-Stokes system,
while for the passive advection problem (no pressure term) the coupling coefficient reduces to

$$L_{kj} = \frac{\mathbf{h}_j \cdot (1 - \mu^2) \left( \mathbf{h}_j - \mu \left( \mathbf{h}_j \cdot \mathbf{k} \right) \mathbf{k} \right)}{\left| \mathbf{k} \cdot \mathbf{j} \right|^2} \left( 1 - \frac{1}{\left| \mathbf{k} \cdot \mathbf{j} \right|} \right)$$

Since both $|x-j|$ and $j$ belong to the subgrid shell, the angle $\theta$ is restricted to $\mu = \cos \theta > 0$ for $k$ near the cutoff. This ensures that

$$L_{kj} = \frac{\mathbf{h}_j \cdot (1 - \mu^2)}{\left| \mathbf{k} \cdot \mathbf{j} \right|^2} > 0$$

Moreover, $L_{kj}$ can become negative in certain regions of $j$-space for $k$ near the cutoff. It is this cancellation effect in the two terms of $L_{kj}$ in Eq. (4.31) that causes the cusp behavior in this wavenumber region for Navier-Stokes turbulence. This can be somewhat related to the cancellation effect that Kraichnan finds in his test field eddy viscosity calculation and which leads to the cusp behavior near the wavenumber cutoff.

2. THE RENORMALIZED NAVIER-STOKES EQUATION

Having considered the effect of the triple nonlinearity on the RNG eddy viscosity, $\nu^*(k)$, we now briefly consider the direct effect of the triple nonlinearity on the
final renormalized NSE (on dropping the $\langle \cdot \rangle$ notation)

$$\left[ \frac{\partial u}{\partial t} + \nu^s(k) \nabla^2 u \right] = \mathcal{M}_{\text{AR}}(k) \int d\vec{k} \cdot u_j(t, \vec{x}) u_r(\vec{k} \cdot \vec{x})$$

$$+ 2 \mathcal{M}_{\text{AR}}(k) \sum_{i=0} \int d\vec{k} \cdot d\vec{i} \cdot d\vec{j} \cdot \mathcal{M}_{\text{AR}+\langle \cdot \rangle} \cdot \left[ \nu^s(\vec{k} \cdot \vec{x}) \right]$$

$$\times u_i(\vec{i} \cdot \vec{x}^+ \cdot \vec{j}) u_r(\vec{k} \cdot \vec{j} \cdot \vec{x}^+)$$

(4.34)

Unlike the quadratic nonlinearity in Eq.(4.34), which is energy conserving, the triple nonlinearity is readily shown to be non-energy conserving. In Sec.C, following Rose40, we consider a linearized driven model of Eq.(4.34) to examine the effect of the triple nonlinearity in Eq.(4.34). It is shown that the contribution of the pressure force to the triple nonlinearity will lead to a decay of the velocity field that is slower than the velocity decay if the pressure force was absent. Thus, one can see that the RNG eddy viscosity for the full NSE could exhibit a cusp behavior near the subgrid/supergrid cutoff, while such a cusp behavior need not appear if the pressure force is absent. Indeed, this is just what is found in the full numerical solution of Eqs.(4.27)–(4.29). This also may account for the somewhat weaker RNG cusp behavior found in Fig.1, and for the absence of the cusp for the coarser grid partition parameter of $f=0.6$(Fig.2).

A direct numerical solution of Eq.(4.34) has not been attempted.
C. THE DIRECT EFFECT OF THE TRIPLE NONLINEARITY IN THE RENORMALIZED NAVIER-STOKES EQUATION

In this section, we follow Rose in his calculation for the passive scalar advection to estimate the effect of the triple nonlinearity in the renormalized Navier-Stokes Eq. (4.34).

To be able to proceed analytically, we linearized the NSE by separating the velocity field into an advecting part \( \hat{u} \) and an advected part \( u \) so that

\[
\left( \frac{\partial}{\partial t} + \nabla \cdot \mathbf{u} \right) \mathbf{u} = \int d^3 \mathbf{k} m_{d,r}(\mathbf{k}) u_{A}(\mathbf{k},t) \mathbf{u}_{r}(\mathbf{k},t) \quad (4.35)
\]

where \( \mathbf{u}_{r} \) is a prescribed random variable. We again proceed as in Sec.A to obtain the renormalized equation

\[
\left( \frac{\partial}{\partial t} + \nabla \cdot \mathbf{u} \right) \mathbf{u} = m_{d,r}(\mathbf{k}) \int d^3 \mathbf{k} u_{A}(\mathbf{k},t) \mathbf{u}_{r}(\mathbf{k},t) + m_{d,r}(\mathbf{k}) \int d^3 \mathbf{k} u_{A}(\mathbf{k},t) \mathbf{u}_{r}(\mathbf{k},t) \times \mathbf{g}_{A}, \mathbf{u}_{r}, \mathbf{u}_{r}, \mathbf{u}_{r} \quad (4.36)
\]

Following Rose\textsuperscript{40}, we have also taken the simplifying limit that the partition grid parameter \( f \rightarrow 1 \) and \( k = k_{N} \) is the wavenumber separating the supergrid and subgrid wavenumbers.

1. EFFECT OF PRESSURE TERM

We now specify the advecting velocity field in
where wavenumber $k'^2 < k^*$. It is convenient to introduce a time-independent source term $\bar{S}$

$$
\bar{S}(\vec{y}) = \bar{S}_0 \left[ \delta(j_x - k'_1) + \delta(j_y + k'_2) \right] \delta(i_y) \delta(j_y)
$$

(4.38)

for some amplitude $\bar{S}_0$ and wavenumber $k'_1 < k^*$. Moreover, we are interested in wavenumber $k'_2$ near the supergrid/subgrid cutoff $k^*$ so that $(k'_1^2 + k'_2^2)^{1/2} > k^*$. Hence the two supergrid modes $u_\lambda(\pm k'_1,0,0)$ are coupled to the other modes only through the triple nonlinearity. Explicitly evaluating Eq.(4.35), we find, after some straightforward algebra,

$$
u_x(k'_1,0,0) = S_{0x} / \hat{G}(k^*) k_1'
$$

(4.39)

$$
u_2(k'_1,0,0) = S_{02} \left[ \hat{G}(k^*) k_1' + \hat{A} k_1'^2 \right]
$$

(4.40)

$$
u_y(k'_1,0,0) = \left\{ \frac{S_{0y} + 2\hat{A} k_1' k_2' / \hat{G}(k_1' + k_2')} {\hat{G}(k_1'^2 + \hat{A} k_1'^2 [1 - 2k_2'/A k_1' + k_2'^2])]} \right\}
$$

(4.41)

where

$$A = \nu^2 / 4 \hat{G}(k^*) (k^*)^{1/2} \left[ k_1'^2 + k_2'^2 \right]^{1/2} > 0
$$

(4.42)
It should be noted that the terms $2\text{Re} \phi_i = \phi_i' (k_i^2)_{1/2}$ and $-\text{Re} \phi_i / (k_i^2)$ in the decay of $u_y$, Eq.(4.39), arise from the pressure effect present in the coupling coefficient $M_{\delta\delta}$. 

2. CASE WHEN PRESSURE TERM YIELDS NO CONTRIBUTION

To examine the overall effect of the pressure in the decay of the source term, Eq.(4.38), we now consider an advecting velocity field with

$$\hat{w}_i (j_i) = \nabla \left[ \delta (j_i - k_i^2) + \delta (j_i + k_i) \right]$$

(4.43)

Since the wavenumber dependence of the source and advecting velocity are parallel, the pressure term in $M_{\delta\delta}$ will have no effect on the decay of $u_y$. Proceeding as before, we readily find now that

$$u_y (k_i, \omega, \omega) = S_{y\omega} (\hat{\nu} (k_i^2) k_i^2)$$

(4.44)

$$u_\delta (k_i, \omega, \omega) = S_{0\omega} (\hat{\nu} (k_i^2) k_i^2 + Ak_i^2)$$

(4.45)

$$u_y (k_i, \omega, \omega) = S_{\omega\omega} (\hat{\nu} (k_i^2) k_i^2 + Ak_i^2)$$

(4.46)

where $A$ is given by Eq.(4.41).

Hence, in the case when the pressure term has no explicit effect in the triple nonlinearity, we find that the decay of
the source is enhanced by the presence of the triple nonlinearity in the renormalized Navier-Stokes: \( \hat{\nabla} k_1'^2 \rightarrow \hat{\nabla} k_1'^2 + A k_1'^2 \). Thus one could argue that the 'pressureless' eddy viscosity near the cutoff \( k^* = k_N \) need not show significant wavenumber cusp dependence since the triple nonlinearity present in the Navier-Stokes equation could itself account for the needed extra dissipation attested to by the Kraichnan test field model.

However, on comparing Eq.(4.40) with (4.46), we see that the pressure effect is to reduce the decay of the source term for wavenumbers near the cutoff \( k^* \). Thus if one is to reproduce the test field eddy viscosity results of Kraichnan, one might expect the presence of cusp-like behavior in the renormalized viscosity, as we have found in Fig.1.

D. SUMMARY

By applying RNG procedures to eliminate the subgrid scales, we have found that the renormalized NSE involves triple nonlinear interactions. Moreover, it has been shown that this triple nonlinearity (with the inclusion of pressure) makes an essential contribution to the renormalized eddy viscosity. Numerical solution of the RNG recursion relation shows that this eddy viscosity now exhibits a cusp-like behavior for wavenumbers near the supergrid/subgrid cutoff. This is in qualitative agreement with Kraichnan's test field model\(^{17}\), with Chollet and Leieur's eddy damped
quasinormal approximation, and with the recent direct numerical simulation results of Domaradzki et al. In Appendix A, it is shown that this triple nonlinearity results from the interaction between subgrid and supergrid velocity fields. This interaction bears some similarity to that needed by Kraichnan to achieve his cusp behavior in the test field eddy viscosity calculation.

It also appears that the McComb eddy viscosity calculation is inconsistent with the previous eddy viscosity results. Indeed, in the iterative technique of McComb, it is claimed that no closure problem arises and no triple nonlinearities are generated. Thus, in McComb's calculation, the eddy viscosity cannot exhibit any cusp behavior near the subgrid/supergrid cutoff. Now the calculations of Kraichnan, Chollet and Lesieur, and Domaradzki et al. are also all based directly on the quadratically nonlinear NSE, but yet all obtain cusp behavior in the viscosity. Now the passive advection of a velocity field will, in the RNG technique, generate triple nonlinearities, but argument, basically given by Rose and in the model calculation in Sec.C, is that the triple nonlinearity in the renormalized NSE will itself contribute to yield extra damping near the wavenumber cutoff. This extra damping, together with the RNG viscosity, can be argued to be somewhat equivalent to the cusp eddy viscosity of Chollet and Lesieur. However in our case the triple nonlinearity in the renormalized NSE contributes less damping than the
similar term for the passive advection problem. This could account for the appearance of the mild cusp behavior in our calculation.
Fig. 1 The scaled renormalized eddy viscosity as a function of the scaled wavenumber for a relatively fine subgrid partition ($f=0.7$). The full curve, exhibiting the cusp behavior for $\tilde{k} = 1$, is the test field model result of Kraichnan, while the curve marked with $\Box$ is our result. The curve marked with $\Delta$ is $\nu^*(k)$ if all triple nonlinear term effects are dropped. McComb's RNG result (which is claimed to be able to avoid the triple nonlinearity) is shown by $---$. Note that the triple nonlinearity in RNG (yielding 2 memory terms in the recursion relation for $\nu^*$ for $\tilde{k} = 1$) is needed to yield the cusp behavior which arises because of local interaction.

Fig. 2 The scaled renormalized eddy viscosity for a coarser subgrid partition ($f=0.6$) chosen so that now only one memory term (arising from the triple nonlinearity) contributes to $\nu^*$. There is no cusp behavior exhibited $---$ curves are labelled as in Fig. 1.
RENORMALIZED EDDY VISCOSITY ($F=0.7$)
RENORMALIZED EDDY VISCOSITY (F = 0.6)
CHAPTER V

SUBGRID SCALE CLOSURE FOR ALFVEN WAVE TURBULENCE

In Secs. A and B we determine the renormalization group differential equation for the response function. The procedure is similar to that of Fournier and Frisch\textsuperscript{32} but we find that the triple nonlinearity does effect the response function differential equation. Assuming Lorentzian wavenumber and frequency spectra we find an analytic solution for the renormalized response function in Sec. C.

A. RNG FOR THE RESPONSE FUNCTION

We assume there exists a sufficiently large wavenumber cutoff in the subgrid region such that the model Alfven problem,

$$ u(k, \omega) = \left( \frac{i \omega}{\nu_A^2} \right) \frac{1}{F(k, \omega)} \sum_{k', \omega'} u(k-k', \omega-\omega') u(k', \omega') (5.1) $$

can be replaced by

$$ \mathcal{E}(\Lambda_0) u(k, \omega) = \left( \frac{i \omega}{\nu_A^2} \right) \frac{1}{F(k, \omega)} \int_{|k'-1, |k-k'| < \Lambda_0} \int_{\omega-\omega'} u(k-k', \omega-\omega') u(k', \omega') \, (5.2) $$

where $\mathcal{E}(\Lambda_0)$ is a response function to be determined such that

$$ \mathcal{E}(\Lambda_0) \to 1 \quad \text{as} \quad \Lambda_0 \to \infty \quad (5.3) $$
1. REMOVAL OF FIRST SUBGRID SHELL

The subgrid scales will be eliminated by peeling away successive differential subgrid shells. In particular, in the first subgrid shell \( \Lambda_1 < k < \Lambda_0 \) with \( \Lambda_1 = \Lambda_0 - \delta \Lambda \) we denote the velocity field

\[
    u(k, \omega) = u^>(k, \omega) \quad \text{for} \quad \Lambda_1 < k < \Lambda_0 \quad (5.4)
\]

while

\[
    u(k, \omega) = u^< (k, \omega) \quad \text{for} \quad k < \Lambda_1 \quad (5.5)
\]

for the supergrid modes. To determine the response function, we first express Eq.(5.2) into the supergrid and subgrid shells:

\[
    \mathcal{E}(\Lambda_0) \ u^< (k, \omega) = \left( \frac{i \omega}{k^2} \right) \frac{\lambda_0}{F(k, \omega)} \int dk' d\omega' \left[ u^>(k', \omega') + u^<(k', \omega') \right] \\
    \times \left[ u^>(k-k', \omega-\omega') + u^<(k-k', \omega-\omega') \right] \\
    \quad \text{for} \quad k < \Lambda_1 \quad (5.6)
\]

and

\[
    \mathcal{E}(\Lambda_0) \ u^>(k, \omega) = \left( \frac{i \omega}{k^2} \right) \frac{\lambda_0}{F(k, \omega)} \int dk' d\omega' \left[ u^<(k', \omega') + u^>(k', \omega') \right] \\
    \times \left[ u^<(k-k', \omega-\omega') + u^>(k-k', \omega-\omega') \right] \\
    \quad \text{for} \quad \Lambda_1 < k < \Lambda_0 \quad (5.7)
\]
For convenience, we have introduced a perturbation parameter $\lambda_0$ into the nonlinear vertex interaction. This parameter will eventually be set to unity. The high k-modes in the subgrid shell are eliminated by substituting Eq. (5.6) for $u$ into Eq. (5.7) and then performing an ensemble over the subgrid scales such that

$$< u(k, \omega) > = u_c(k, \omega) \quad < u^2(k, \omega) > = 0$$  \hspace{1cm} (5.8)

We thus find, working to $O(\lambda_0^2)$, that after removing the first subgrid shell

$$E(\Lambda_0) \ u_c(k, \omega) = \left( \frac{\omega}{\nu_k^2} \right) \frac{1}{E(k, \omega)} \int dk' \delta(k, -k') u_c(k', \omega') u_c(k-k', \omega-\omega')$$

$$- \left( \frac{2\omega}{\nu_k^2} \right) E(k, \omega) \int dk' \delta(k, -k') \int dk'' \delta(k', -k'') \int \omega'' [E(\Lambda_0) \ E^{-1}(k', \omega')]$$

$$\times \ u_c(k-k', \omega-\omega') u^2(k-k', \omega-\omega') \ u_c(k'', \omega'')$$

$$\int \omega'' [E(\Lambda_0) \ E^{-1}(k', \omega')]$$

$$< u^2(k-k', \omega-\omega') > \ u^2(k-k', \omega-\omega')$$

for $k < \Lambda_0$  \hspace{1cm} (5.9)

on setting $\lambda = 1$, since we have assumed isotropic homogeneous subgrid turbulence

$$< u^2(k-k', \omega-\omega') > \ u^2(k-k', \omega-\omega')$$

$$= \delta(\omega-\omega') \delta(k-k') Q_1(k-k') Q_2(\omega-\omega')$$  \hspace{1cm} (5.10)

for certain spectral functions $Q_1$ and $Q_2$. Thus, substi-
tuting Eq. (5.10) into Eq. (5.9) the third term will renormalized the response function \( \mathcal{E}(\Lambda_n) \) to \( \mathcal{E}(\Lambda_0) \), with

\[
\mathcal{E}(\Lambda_n) u^c(k, \omega) = \left( \frac{i \omega}{\nu_A} \right) F^{-1}(k, \omega) \int dk'd\omega' u'(k', \omega') u(k-k', \omega-\omega')
- \left( \frac{2 \omega}{\nu_A} \right) F^{-1}(k, \omega) \int dk'd\omega'dk''\omega'' \omega' \left[ \mathcal{E}(\Lambda_0) \right] F^{-1}(k', \omega')
\]

\[
u^c(k-k', \omega-\omega') \nu^c(k'-k'', \omega''-\omega') \nu^c(k'', \omega')
\]

for \( k < \Lambda_1 = \Lambda_0 - \delta \Lambda \)

\[
\mathcal{E}(\Lambda_0)
\]

is defined by

\[
\mathcal{E}(\Lambda_1) = \mathcal{E}(\Lambda_0) + \delta \mathcal{E}(\Lambda_0)
\]

with

\[
\delta \mathcal{E}(\Lambda_n) = \left( \frac{4 \omega}{\nu_A^2} \right) F^{-1}(k, \omega) \int dk' \int d\omega' \frac{\omega'}{\mathcal{E}(\Lambda_0) F(k', \omega')} Q_1(k-k') Q_2(\omega-\omega')
\]

(5.13)

It should be noted that another effect of the removal of the first subgrid shell is to introduce a cubic nonlinearity—(second term on the right hand side of Eq. (5.11)) into the renormalized Alfven wave equation. This introduction of a new form of coupling is quite standard in the canonical application of RNG approach to Hamiltonian Ising spin problems.
2. REMOVAL OF $n$ SUBGRID SHELLS

We now prepare to remove the second subgrid shell ($\Lambda_2$, $\Lambda_1$), with subgrid modes having $\Lambda_2 < k < \Lambda_1$ while the supergrid modes have $k < \Lambda_1$. The removal of the first subgrid shell has resulted in the renormalized Alfven wave equation, Eq.(5.11),

$$E(\Lambda_1) \psi(k, \omega) = \left( i \omega / c_0^2 \right) \tilde{E}(k, \omega) \int dk' d\omega' \psi(k', \omega') \psi(k - k', \omega - \omega')$$

$$- \frac{2 \omega}{c_0^2} \tilde{E}(k, \omega) \int dk' d\omega' dk'' d\omega'' \omega' \left[ E(\Lambda_0)^2 \right] \tilde{E}(k, \omega) =$$

$$\psi(k - k', \omega - \omega') \psi(k'' - k', \omega'' - \omega') \psi(k'' - k') \psi(k'') \psi(k')$$

$$\tag{5.14}$$

To remove the second subgrid shell, we now denote the subgrid modes by

$$u = u^s(k, \omega), \quad \Lambda_2 < k < \Lambda_1$$

$$\tag{5.15}$$

and the supergrid by

$$u = u^c(k, \omega), \quad k < \Lambda_2$$

$$\tag{5.16}$$

and proceed as in Sec. A part 1, but realize that the triple nonlinear term $uuu$ in Eq.(5.11) will also contribute to the renormalized response function. We find the renormalized Alfven wave equation is now given by
\[ C^n(\Lambda_n) u(k, \omega) = \left( i \omega / \nu^2 \right) F(k, \omega) \int dk' d\omega' u(k', \omega') u(k-k', \omega-\omega') \]

\[ - \left( 2i \omega / \nu^2 \right) F(k, \omega) \sum_{j=0}^{j^*} \left[ C^n(\Lambda_{j}) \right] \int dk' d\omega' d\omega'' \omega' \times \]

\[ F^{-1}(k', \omega') u(k-k', \omega-\omega') u(k', \omega'') u(k', \omega'') \]  

(5.17)

with

\[ C^n(\Lambda_{j}) = \xi(\Lambda_{j}) + \delta C^n(\Lambda_{j}) \]

(5.18)

and

\[ \delta C^n(\Lambda_{j}) = \left( \frac{2i \omega}{\nu^2} \right) F(k, \omega) \sum_{j=0}^{j^*} \int dk' \frac{\Lambda_j}{\xi(\Lambda_{j})} \frac{Q_j(k-k') I(k')}{\xi(\Lambda_{j})} \]

(5.19)

for \( \Lambda_{j} < (k' - k) < \Lambda_{j+1} \), \( I(k') \) is defined by

\[ I(k') = \int \omega \omega' F(k', \omega') Q_{\Lambda}(\omega-\omega') \]

(5.20)

Thus, proceeding iteratively, we find after removing the \((n+1)^{th}\) subgrid shell that the renormalized Alfven wave equation becomes

\[ C^n(\Lambda_{n+1}) u(k, \omega) = \left( i \omega / \nu^2 \right) F(k, \omega) \int dk' d\omega' u(k', \omega') u(k-k', \omega-\omega') \]

\[ - \left( 2i \omega / \nu^2 \right) F(k, \omega) \sum_{j=0}^{j^*} \left[ C^n(\Lambda_{j}) \right] \int dk' d\omega' d\omega'' \omega' \times \]

\[ F^{-1}(k', \omega') u(k-k', \omega-\omega') u(k', \omega'') u(k', \omega'') \]  

(5.21)
with the corresponding response function given by

$$\mathcal{E}_m^{l}(\Lambda_m) = \mathcal{E}_m^{l}(\Lambda_n) + \delta \mathcal{E}_m^{l}(\Lambda_n)$$  \hspace{1cm} (5.22)$$

where

$$\delta \mathcal{E}_m^{l}(\Lambda_n) = (4\omega/\nu_n^*) \mathcal{E}_m^{l}(k,\omega) \sum_{j=0}^{\Lambda_n} \int d\omega' \frac{G_i(k-k') \mathcal{I}(k')}{\mathcal{E}(\Lambda_j)}$$  \hspace{1cm} (5.23)$$

with \( \Lambda_{m+1} < (k' - k) < \Lambda_n \). \( \mathcal{I}(k') \) is given by Eq.(5.20).

**3. DIFFERENTIAL RENORMALIZATION GROUP EQUATION FOR THE RESPONSE FUNCTION**

To indicate explicitly the effect on the response function of the triple nonlinear term in the renormalized Alfven equation, we rewrite Eq. (5.23) in the form

$$\delta \mathcal{E}_m^{l}(\Lambda_n) = (4\omega/\nu_n^*) \mathcal{E}_m^{l}(k,\omega) \left[ \sum_{\Lambda_m}^{\Lambda_n} \int d\omega' \frac{G_i(k-k') \mathcal{I}(k')}{\mathcal{E}(\Lambda_j)} \right] + \int d\omega' G_i(k-k') \mathcal{I}(k') \sum_{j=0}^{\Lambda_n} \int d\omega' \frac{G_i(k-k') \mathcal{I}(k')}{\mathcal{E}(\Lambda_j)}$$  \hspace{1cm} (5.24)$$

The first term on the right hand side of Eq.(5.24) is due to the standard quadratic nonlinear term in the Alfven equation and contributes to the response function for the subgrid shell \( \Lambda_{m+1} < k' < \Lambda_n \). The second term in Eq.(5.24) is the immediate effect of the triple nonlinear term in the Alfven equation and contributes for the subgrid shell \( \Lambda_n < k' < \Lambda_{n-1} \).
The $\Sigma$-term in equation (5.24) is the effect of the earlier subgrid shells of the triple nonlinearity.

Now

$$\Lambda_{n+1} \leq k' - k < \Lambda_n$$  \hspace{1cm} (5.25)

For simplicity, assume an equidistant partitioning of the subgrid range $| \Lambda_{j+1} - \Lambda_j | = \Delta$, for all $j$, and consider the limit at $\Delta \to 0$. We immediately see that the first term in Eq.(5.24) will contribute to the response function provided both Eq.(5.24) and $\Lambda_{n+1} < k < \Lambda_n$ are simultaneously satisfied. This restricts $k$ to

$$k \leq \Delta$$  \hspace{1cm} (5.26)

with $k$ a parameter in Eq.(5.24). The second term in Eq.(5.24) will also contribute since Eq.(5.25), (5.26) and subgrid shell restriction $\Lambda_n < k < \Lambda_{n-1}$ are compatible. However the $\Sigma$-term in Eq.(5.24) comes from non-adjacent subgrid shells from the $(n+1)$-shell. Thus Eq.(5.25), (5.26) and the subgrid shell restriction $\Lambda_{j+1} < k < \Lambda_j$, for $j < n-2$ are incompatible and this 'memory' term does not contribute to the response function. Thus Eq.(5.24) reduces to

$$\mathcal{E}^\Sigma(\Lambda_n) = \frac{(4\pi)^n}{\chi_n^2} \mathcal{F}^\Sigma(k, \omega) \left[ \int_{\Lambda_{n+1}}^{\Lambda_n} dk' \frac{G_1(k'-k)I(k')}{{\mathcal{E}}^\Sigma(\Lambda_n)} + \int_{\Lambda_{n+1}}^{\Lambda_{n-1}} dk' \frac{G_1(k'-k)I(k')}{{\mathcal{E}}^\Sigma(\Lambda_{n-1})} \right]$$  \hspace{1cm} (5.27)
We proceed to the differential limit $\Delta \to 0$ by using the mean-value-theorem on the integrals in Eq.(5.27) and obtain (since $k \to 0$ also, see Eq.(5.26) this is typical for dynamical renormalization group theories).

\[
d \mathcal{E}(\Lambda)/d\Lambda = - \left( \frac{q \omega}{v_a^2} \right) F^{-1}(q, \omega) Q_1(\Lambda) I(\Lambda) \mathcal{E}(\Lambda) \right\}^{-1}
\]

with $\mathcal{E}(\Lambda \to \infty) = 1$ as the initial condition, Eq.(5.3).

Thus

\[
\left[ \mathcal{E}(\Lambda) \right]^3 = 1 + \left( \frac{q \omega}{v_a} \right) F^{-1}(q, \omega) \int_{\Lambda}^{\infty} d\Lambda' Q_1(\Lambda') I(\Lambda')
\]

with

\[
I(\Lambda) = \int_{-\infty}^{\infty} d\omega' \omega' F(\Lambda, \omega') Q_2(\omega - \omega')
\]

where $Q_1(k - k')$ and $Q_2(\omega - \omega')$ are wavenumber and frequency spectra to be prescribed. The renormalized Alfven equation, Eq.(5.21), becomes

\[
\mathcal{E}(\Lambda; \omega) u(k, \omega) = \left( i \omega/4\pi v_a^2 \right) \int dk' d\omega' F^{-1}(k, \omega) u(k, \omega') u(k - k', \omega - \omega')
\]

- $\left[ \frac{q \omega}{(2\pi v_a^2)} \right] F^{-1}(k, \omega) \int_{\Lambda}^{\infty} d\Lambda' \left[ \mathcal{E}(\Lambda'; \omega) \right]^{-1}$

\[
\times \left\{ d k' d\omega' d k'' d\omega'' \omega' F^{-1}(k, \omega') u(k - k', \omega - \omega') u(k - k'', \omega - \omega'') u(k, \omega) \right\}
\]

(5.31)

where we have explicitly exhibited the parametric $\omega -$
dependence in the response function $\mathcal{E}(\lambda; \omega)$ from the subgrid ($\Lambda < k$) wavenumbers.

C. ANALYTIC SOLUTION FOR THE RESPONSE FUNCTION

To find an explicit solution for the response function $\mathcal{E}(\lambda; \omega)$ we must prescribe the frequency and wavenumber spectra -- in particular we chose a Lorentzian frequency spectrum

$$Q_\omega(\omega-\omega') = \frac{1}{4\pi} \left[ (\omega-\omega')^2 + \alpha^2 \right]$$

(5.32)

where $\alpha$ is a parameter determining the width of the Lorentzian, and the wavenumber spectrum of Chen and Mahajan\textsuperscript{38} for the subgrid scales

$$Q_{k,k'} = \frac{1}{4\pi} (k-k')^2$$

(5.33)

With these spectra the integrals in Eqs.(5.29) and (5.30) can be performed analytically. We find, on taking into account the standard resolution of the Alfven singularity $1/F(\lambda, \omega)$ in Eq.(5.30), that

$$\mathcal{E}(\lambda = 1; \omega) = 1 + \left[ \frac{1}{4\pi} \lambda^3 (\omega^3 + \alpha^2) \right] \int Q(\omega+i\alpha)(\omega^3 + \alpha^2)$$

$$+ (\omega+i\alpha)^2 \ln \left[ \frac{(\zeta - \omega - \alpha)}{(\zeta + \omega + \alpha)} \right]$$

$$+ i (\omega+i\alpha)^2 \left[ \pi - \tan^{-1}(\lambda i \omega) - \tan^{-1}(\lambda \omega) \right]$$

(5.34)
The response function $\mathcal{E}(\lambda, \omega)$ becomes complex due to the Alfven singularity.

In Figs. 3 and 4 we show the real part of the response function, $\text{Re} \mathcal{E}(\lambda=1, \omega)$, and the imaginary part, $\text{Im} \mathcal{E}(\lambda=1, \omega)$, as a function of the normalized frequency for various values of the Lorentzian frequency parameter $\alpha$. For a broad Lorentzian frequency spectrum ($\alpha = 1$ in Eq. (5.32)) we find that $\text{Re} \mathcal{E} = 1$ and small 'absorption', $\text{Im} \mathcal{E}$. For peaked frequency frequency profiles, $\alpha = 0.1$, there is considerable variation in both $\text{Re} \mathcal{E}$ and $\text{Im} \mathcal{E}$. Figures 5 and 6 show the behavior of $\text{Re} \mathcal{E}$ and $\text{Im} \mathcal{E}$ for a wider frequency range of $\omega$. As is expected $\text{Re} \mathcal{E} \to 1$ and $\text{Im} \mathcal{E} \to 0$ as $\omega \to \infty$. Moreover a sharp resonance in the renormalized response function occurs for peaked frequency spectra ($\alpha = 0.1$). This absorption, see Fig. 6, occurs at $\omega = 1$ which is just the Alfven frequency.
Fig. 3 The real part of the response function as a function of the normalized frequency $\omega$, for $\omega < 1.2$. $\alpha$ is a parameter determining the width of the subgrid Lorezian frequency spectrum. Sharp Lorezians correspond to small $\alpha$.

Fig. 4 $\text{Im} \mathcal{E}(A=1; \omega)$ as a function of $\omega$, for most frequencies, $\text{Im} \mathcal{E} \ll 1$ indicating negligible absorption.

Fig. 5 The behavior of $\text{Re} \mathcal{E}(A=1; \omega)$ for a large frequency range. As expected, $\text{Re} \mathcal{E} \to 1$ as $\omega \to \infty$. For sharp subgrid Lorezian ($\alpha$ small) there is substantial variation in $\text{Re} \mathcal{E}$ for normalized frequencies $\omega < 4$. This variation rapidly disappears for broad subgrid frequency spectra ($\alpha = 1$).

Fig. 6 The behavior of $\text{Im} \mathcal{E}(A=1; \omega)$ for a large frequency range. As expected, $\text{Im} \mathcal{E} \to 0$ as $\omega \to \infty$. For sharp subgrid spectra ($\alpha = 0.1$) there is a marked resonance peak around the Alfven frequency $\omega = 1$. This absorption is still evident for $\alpha = 0.5$, but it is absent for very broad subgrid spectra ($\alpha = 1$).
FIG. 3

- - - - $\alpha = 0.1$  
- - - - $\alpha = 0.3$
- - - - $\alpha = 0.5$
- - - - $\alpha = 1$

HEAL PART OF RESPONSE FUNCTION
FIG. 4

\[ \text{IMAGINARY PART OF RESPONSE FUNCTION} \]

- - - - $\alpha = 0.1$

- - - - - - $\alpha = 0.3$

- - - - - - - - $\alpha = 0.5$

- - - - - - - - - - $\alpha = 1$
FIG. 5

- - - - $\alpha = 0.1$
- - - - $\alpha = 0.3$
- - - - $\alpha = 0.5$
- - - - $\alpha = 1$
FIG. 6

--- \( \alpha = 0.5 \) --- \( \alpha = 0.3 \) --- \( \alpha = 1 \) --- \( \alpha = 1 \)
CHAPTER VI

CONCLUSION

The major emphasis of this dissertation has been on the effects of numerically unresolvable (subgrid) small scales on the dynamics of the resolvable large scales in fluid and plasma turbulence. One of the problems studied was the Alfvén wave turbulence model constructed by Chen and Mahajan. We first applied the $\epsilon$-expansion RNG theory, a method employed by Wilson in phase transition and critical phenomena studies as well as in the works by Fournier and Frisch, also by Yakhot and Orszag in hydrodynamical turbulence. The basic output of this study is that the wavenumber spectrum does follow the Chen and Mahajan power law found by them by computer simulation. Furthermore, the nonlinear coupling constant still remain small for finite $\epsilon$, fully justifying the neglect of higher order nonlinearities introduced by RNG procedure.

We have formulated an eddy viscosity model for fluid turbulence using RNG method. This method does not use a spectral gap assumption which is an essential ingredient in conventional eddy viscosity models. Difference recursion RNG is applied to incompressible Navier-Stokes turbulence to eliminate unresolvable small scales. As a result,
(i) the molecular viscosity is renormalized, (ii) the renormalized NSE now includes a triple nonlinearity with eddy viscosity exhibiting a mild cusp behavior, in qualitative agreement with the test-field model results of Kraichnan. For the cusp behavior to arise, not only is the triple nonlinearity necessary but also the effects of pressure must be incorporated in the triple term.

Finally, the effects of small 'unresolvable subgrid scales' on the large scales in the Alfven wave turbulence is computed. The removal of the subgrid scales leads to a renormalized response function \( \mathcal{E} \), which can be calculated analytically. Strong absorption can occur around the Alfven frequency.

So far, we have limited ourselves to free decay in both Navier-Stokes and Alfven wave turbulence. The next logical step involves the generalization of the method to forced systems. Instead of the Kolmogorov and Chen-Mahajan spectrums, forcing spectrum will be employed for the similarity range. As we change the driving force, the large scale properties and its transport coefficient will change accordingly. The forced system subgrid modeling is now under consideration.
Here we present some of the details in deriving Eq. (5.9). It is convenient to proceed to a diagrammatic approach. In the removal of the first subgrid shell we denote the supergrid propagator by

\[ \mathcal{C}_s(k,\omega) \left\{ \begin{array}{l} 1 \quad \text{for } k < \Lambda_1 \\ \end{array} \right. \]  \hspace{1cm} (A1)

and the subgrid propagator by

\[ \mathcal{C}_s(k,\omega) \left\{ \begin{array}{l} 1 \quad \text{for } \Lambda_1 < k < \Lambda_0 \\ \end{array} \right. \]  \hspace{1cm} (A2)

The nonlinear (vertex) interaction is denoted by

\[ \left( \frac{i \omega}{\nu} \right) \lambda_0 \int dk' d\omega' \]  \hspace{1cm} (A3)

with \( \lambda_0 \) an ordering parameter, eventually set to unity.

Diagrammatically, Eq. (5.6) is thus represented as

\[ u^c(k,\omega) \left\{ \begin{array}{l} u^c \\ \end{array} \right. + 2 \left\{ \begin{array}{l} u^c \\ \end{array} \right. + \left\{ \begin{array}{l} u^c \\ \end{array} \right. \]  \hspace{1cm} (A4)

while for the subgrid modes
Equation (A5) is substituted into Eq.(A6) and then an average is performed over the subgrid scales, keeping terms only to $O(\varepsilon^2)$.

(i) Term (b) in Eq.(A4)

The effect of term (a) in Eq(A5) on (c) in Eq.(A4) is to produce

\[ 2 \cdot \frac{\partial}{\partial t} \begin{array}{c} \vec{u}^c \\ \vec{u}^c \\ \vec{u}^c \\ \vec{u}^c \end{array} \]

This is the new triple interaction introduced in Eq.(5.23).

The effect of term (b) in Eq.(A5) is to produce

\[ 4 \cdot \begin{array}{c} \vec{u}^c \\ \vec{u}^c \\ \vec{u}^c \end{array} \]

which is zero on subgrid scale averaging, since $\langle u^c \rangle = 0$.

The effect of term (c) in Eq.(A5) yields

\[ 2 \cdot \begin{array}{c} \vec{u}^c \\ \vec{u}^c \end{array} \]

Again, this term is zero on averaging over the homogeneous subgrid scales since the $u$ and $\mu$ are connected by the same
vertex. This can be seen algebraically since for \( j \) in the subgrid shell this term equal, on subgrid averaging,

\[
\int \delta_{ij} \delta_{j'} \delta_{\alpha} \delta_{\alpha'} < u' (i-j, \alpha-\alpha') u' (j', \alpha') > u' (k-j, \omega-\Omega)
\]

\[
= \int \delta_{ij} \delta_{j'} \delta_{\alpha} \delta_{\alpha'} Q_1 (i-j, \alpha-\alpha') \delta (j', \alpha') \delta (k-j, \omega-\Omega)
\]

But \( j \) is in the subgrid scales, so that \( \int \delta j \delta j' = 0 \)

(ii) Term (c) in Eq. (A4)

Working only to \( O(\lambda^2) \), the substitution of (a) in Eq. (A5) yields

\[
\begin{array}{c}
\int \delta_{ij} \delta_{j'} \delta_{\alpha} \delta_{\alpha'} < u' (i-j, \alpha-\alpha') u' (j', \alpha') > u' (k-j, \omega-\Omega)
\end{array}
\]

Under subgrid scale averaging this term is zero since

\( <u'> = 0 \)

On substituting (b) of Eq. (A5) into term (c) of Eq. (A4) we obtain

\[
\begin{array}{c}
\int 4 \delta_{ij} \delta_{j'} \delta_{\alpha} \delta_{\alpha'} \delta (j', \alpha') \delta (k-j, \omega-\Omega) u' (j', \alpha')
\end{array}
\]

which on subgrid scale averaging gives the renormalization of the response function.

As is usually done in renormalization group theory procedures in fluid turbulence we neglect the effect of substituting (c) in Eq. (A5) into (c) of Eq. (A4). This would have yielded

\[
\begin{array}{c}
\int 2 \delta_{ij} \delta_{j'} \delta_{\alpha} \delta_{\alpha'} \delta (j', \alpha') \delta (k-j, \omega-\Omega) u' (j', \alpha')
\end{array}
\]
The neglect of this term is basically a closure approximation.

An analogous procedure is performed for the removal of the $n^{th}$ subgrid shell.
APPENDIX B

ITERATIVE AVERAGING

McComb\textsuperscript{9,16,41,42} claims that iterative averaging, while equivalent to RNG, leads to the elimination of the subgrid scales without the introduction of the triple nonlinearity in the renormalized Navier-Stokes equation (4.13). In essence, iterative averaging first performs ensemble averaging followed by the necessary substitution followed by ensemble averaging. We believe there is an error in McComb's work and that iterative averaging, in fact, yields no suitable information. To show this we shall revert to McComb's notation. Thus

\begin{equation}
\bar{u}_d(\vec{k},t) = \begin{cases} 
\bar{u}_d^-(\vec{k},t) & \text{for } k \leq k_0 \\
\bar{u}_d^+(\vec{k},t) & \text{for } k > k_0
\end{cases}
\end{equation}

so that the Navier-Stokes equation can be written in detail

\begin{equation}
\frac{3v}{\nu} \sum_{k} \left[ u^a_+(\vec{k},t) \bar{H}(k_0-k) + u^+_d(\vec{k},t) \bar{H}(k-k_0) \right]
= \sum_{\vec{k},(\vec{j})} \int d^3 \bar{\lambda} \left[ u^a_+(\vec{k},t) + u^+_a(\vec{j},t) \right] \left[ u^a_+(\vec{k-j},t) + u^+_a(\vec{k-j},t) \right]
\end{equation}

where $\bar{H}(x)$ is the unit heaviside function: $\bar{H}(x)=1$ for $x>0$, $\bar{H}(x)=0$ for $x<0$. The appropriate wavenumber limitations are not explicitly shown for the right hand of Eq. (B2); $\bar{u}^a(y)$
has $|\vec{y}| < k_o$ while $u^+(y)$ has $|\vec{y}| > k$.

McComb and Shanmugasundaram\textsuperscript{42} immediately perform an ensemble average of Eq. (B2) to obtain

$$\left[ \frac{\partial}{\partial t} + \gamma_6 k^2 \right] u^-_{\alpha}(\vec{k}, t) \equiv (k_0, k) = \mathcal{M}_{\alpha\beta}(k) \left[ \int d^3 \phi \left< u^+_{\beta}(\vec{j}, t) u^-_{\alpha}(\vec{r}, t) \right> \right]$$

$$\quad + \left[ \int d^3 \phi \left< u^-_{\alpha}(\vec{j}, t) u^-_{\alpha}(\vec{r} - \vec{j}, t) \right> \right]$$

(B3)

McComb now subtracts Eq. (B3) from (B2) in an attempt to obtain the evolution equation for $u^+_d(k, t)$. The error arises in not realizing that Eq. (B2) is actually two mutually exclusive evolutionary equations dependent on the value of $k$: one is for $u^+_d(k, t)$ provided $k < k_0$ and the other equation is for $u^+_d(k, t)$ provided $k > k_0$.

Let us consider Eq. (B3) in more detail. Because of the assumption of homogeneous isotropic turbulence we immediately have

$$\left< u^+_{\alpha}(\vec{j}, t) u^-_{\alpha}(\vec{k} - \vec{j}, t) \right> = \mathcal{D}_{\alpha\beta}(j, j') \delta(k)$$

(B4)

(i) $k_d < 0$ ($k < k_0$)

In this case Eqs. (B3) and (B4) yields

$$\left[ \frac{\partial}{\partial t} + \gamma_6 k^2 \right] u^-_{\alpha}(\vec{k}, t) = \mathcal{M}_{\alpha\beta}(k) \left[ \int d^3 \phi \left< u^-_{\beta}(\vec{j}, t) u^-_{\alpha}(\vec{r}, t) \right> \right]$$

(B5)

which appears to be nothing but a straight-forward truncation of the NSE to wavenumbers $|\vec{k}| < k$.
(ii) $k = 0$

Since $M_{k=0}(0) = 0$ we obtain the trivial result $u_0^*(0, t) = \text{constant}$. 
APPENDIX C

ALFVEN WAVE AND RESPONSE FUNCTION

1. ALFVEN WAVE

In ordinary hydrodynamics, apart from surface waves the only small-amplitude waves possible are longitudinal, compressional (sound) waves. These propagate with a velocity $s$ related to the derivative of pressure with respect to density at constant entropy:

$$s' = \left( \frac{\partial p}{\partial \rho} \right)_s$$

If the adiabatic law $p = K \rho^\gamma$ is assumed, $s' = \gamma p_s / \rho_s$, where $\gamma$ is the ratio of specific heats. In magnetohydrodynamics another type of wave motion is possible. It is associated with the transverse motion of lines of magnetic induction. The tension in the lines of force tends to restore them to straight-line form, thereby causing a transverse oscillation. By analogy with ordinary sound waves whose velocity squared is of the order of the hydrostatic pressure divided by the density, we expect that these magnetohydrodynamic waves, called Alfven waves will have a velocity

$$v = \left( \frac{\beta_s}{\rho \mu \rho_o} \right)^{1/2}$$

where $\beta_s / \mu \rho_o$ is the magnetic pressure. This can be seen more formally by solving the MHD equation.
The vacuum wave equation derived from the Maxwell equations are

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{d^2}{dt^2} \right) \left( \frac{\mathbf{E}}{\rho} \right) = 0
\]

The plane wave solutions of these equation are

\[
\left( k^2 - \omega^2/c^2 \right) \left( \frac{\mathbf{E}}{\rho} \right) = 0
\]

which gives rise to the so-called dispersion relation (or response function)

\[
\mathcal{E} = k^2 - \omega^2/c^2 = 0
\]

The Alfven wave model equation is nonlinear and we introduce the response function \( \mathcal{E} \) to replace the role of the dispersion relation. It's initial condition \( \mathcal{E}(\omega \rightarrow \infty) = 1 \) does not change the original equation. However, as we start RNG procedure, the effects of elimination of small subgrid scale will manifest themselves in the renormalized response function.
REFERENCES AND FOOTNOTES

(1) H. Rose and P.-L. Sulem, Journal de Physique 33, 441 (1978)


For a definition of the Kolmogorov dissipation wavenumber, see Chapter II of this thesis.

(3) In fluid mechanics, the ratio $\rho ud/\nu$ of the inertia force to the viscous force $\nu$, where $\rho$ is the fluid density, $u$ is velocity, $d$ is a characteristic length, and $\nu$ is the fluid viscosity. The Reynolds number is significant in design of a model of any system in which the effect of viscosity is important in controlling the velocities or flow pattern. The Reynolds numbers also serve as a criterion in distinguishing different types of fluid motion. In a pipe, for example, laminar flow normally exists at Reynolds numbers less than 2000, and turbulent flow usually occurs at Reynolds numbers above 3000. This standard definition is given by Glenn Murphy in McGraw Hill Encyclopedia of Physics, editor in chief Sybil P. Parker (McGraw-Hill book company, New York 1982). We also would like to mention that most work is for incompressible Navier-Stokes turbulence with $\rho$ constant and this constant can be scaled out.


(6) We have to point out that the discussion here is based on the central assumption that all the fluid is 'active', i.e., that energy dissipation density field is smoothly distributed on the three dimension region. On the contrary, experimental and numerical evidence indicate that spatial intermittency is generally present. See A. C. Monin and A. M. Yaglom, Statistical Fluid Mechanics, MIT press, (Cambridge, MA 1975), Vol. 2, Sec. 25. and A. Craya J. physique Colloq 27 (1976) 21-35 for review. We do not undertake, in this thesis, the task of study of this important subject, rather, we would only like to mention the $\beta$-model of intermittent fully developed turbulence presented by Frisch, Sulem and Nelkin (J. Fluid Mech. 87, 719 (1978)). A maximum principle for determining the intermittency exponent of fully developed steady turbulence was developed by H. Fujisaka and H. Mori (Progress of Theoretical Physics 62 (1), 54 (1979)). Recently, the degree of freedom of intermittent fully developed turbulence has been discussed in detail by G. Paladin and A. Vulpiani (Phys. Rev. A 35 (4), 1971 (1987)).

(7) The eddy-turnover time, $\tau = \frac{l_n}{u_n}$, where $u_n = \left(\frac{|u(x)-u(x+l_n)|}{N}\right)^{\frac{1}{2}}$ is a typical velocity difference across an eddy of size $l_n$. It is the time required for the eddy to be distorted and, in this distortion process, generate smaller eddies. Therefore, it is
associated with energy transfer. See ref. (1).


(11) D.R. Chapman, A.I.A.A. J. 17, 1293 (1979)


(20) Let's filter modes at \( k = k_c \) for Navier-Stokes turbulence, where \( k_c \) stands for the cut-off wavenumber. Then modes \( k < k_c \) are dealt with by direct numerical simulation of NSE, whereas the effect of the subgrid modes (\( k > k_c \)) is represented by an enhanced viscosity acting on the truncated NSE. This approach is LES. However, one has to find an analytical form for
the effective viscosity which represents the drain of energy from modes \( k < k_c \) to modes \( k > k_c \), by inertial transfer. This is known as the 'subgrid modelling problem'. See, for example, ref. (8).

(21) Not by any means complete, the following listed references are direct antecedents of the work of RNG for phase transitions and critical phenomena:
K.G. Wilson, Rev.Mod. Phys. 47, 773 (1975);
M.E. Fisher, Rev.Mod.Phys. 42, 597(1974);
S.-K. Ma, Rev.Mod.Phys. 45, 589(1973); S.-K. Ma Modern Theory of Critical Phenomena (Benjamin, Reading, MA, 1976);

(22) For applications of RNG to high energy physics, see e.g., S. Coleman, in Properties of the Fundamental Interactions, 1971, edited by A. Zichichi (Editrice Compositori, Bologna, Italy, 1972);
For a application to solid state physics (Kondo problem), see K.G. Wilson, Rev.Mod.Phys. 47, 773(1975)


(24) M.Nelkin, Phys.Rev.A 9, 388 (1974);
M.Nelkin, Phys.Rev.A 11, 1737(1975);
(25) It must be noted that critical phenomena is concerned with properties in equilibrium and its properties can be determined by Gibbs ensemble. However, we consider fully developed turbulence to be a statistically steady state far from equilibrium.

(26) See Table 1. In this table (and only in this one), we shall use the inverse Reynolds number $1/R$ and not viscosity to avoid confusion with the standard use of exponent $r$.

(27) D. Foster, D. R. Nelson and M. J. Stephen, Phys. Rev. Lett. 16, 895 (1976);

(28) The use of statistically defined force allows the generation of the statistically steady state. If we take the force to act only at long wavelengths, it is plausible that it sets up an inertial range cascade which, at short distances, is independent of the detail of the force. See for instance, ref. (28). An introduction to modern turbulence theory is given by S. A. Orzag, in Hydrodynamics, edited by R. Balian (Gordon and Breach, New York, 1974)


(35) The renormalization group is not a descriptive theory of nature but a general method for constructing theories. Substantial number of unsolved problems in physics trace their difficulty to a multiplicity of scales. The most promising path to their solution, even if an arduous path, is the further refinement of renormalization group theory. This comment is made by K. G. Wilson in Sci. Am. 241, 158 (August 1979).


(38) C. Y. Chen and S. M. Mahajan, Univ. Texas IFSR #179 (1985)


(40) H. A. Rose, J. Fluid Mech. 81, 719 (1977)


(47) In two dimension (2D) NSE, there are two inviscid constant of motion; namely, the energy and enstrophy (mean square vorticity). Fjortoft (R. Fjortoft, Tellus, 5, 225 (1953)) points out that two constants of motion imply that any transfer of energy to higher wavenumbers must be accompanied by a longer transfer to lower wavenumbers. So in 2D we can not reduce a triad interaction into pair interaction, because pair interaction can not conserve both energy and enstrophy. There can be a range with zero enstrophy and finite energy flux and another range with zero energy and finite enstrophy.

If the energy cascade rate is $\bar{\epsilon}$, which is independent of $k$ (wavenumber), then the enstrophy cascade rate in the same range must have the form $\bar{\eta} = A k^{2} \bar{\epsilon}$ with $A$ independent of $k$. But $A$ itself must be $k$ independent. Therefore $A$ must be equal to zero implying zero enstrophy flux $\bar{\eta} = 0$. That is zero where $\bar{\epsilon}$ is independent of $k$. Similarly energy flux is zero where $\bar{\eta}$ is independent of $k$. Therefore there must be two different inertial ranges for the two quantities to cascade—a dual cascade situation (R. H. Kraichnan, Phys. Fluid, 10, (7) 1417 (1967)).

With a Kolmogorov style dimensional analysis (see
Chap. II of this thesis) we will have the power laws for the omni-directional spectrum function (c is a constant)

\[ E(k) = c \cdot k^{-\frac{\eta}{2}} \quad \text{for energy cascade} \]

and

\[ E(k) = c' \cdot k^{-\frac{\eta}{4}} \quad \text{for enstrophy cascade} \]


(49) A. N. Kolmogorov, J. Fluid. Mech. 13, 82 (1962)


<table>
<thead>
<tr>
<th>Critical phenomena</th>
<th>Turbulence</th>
</tr>
</thead>
<tbody>
<tr>
<td>limit $T - T_c \downarrow 0$</td>
<td>limit $1/R_0 \downarrow 0$</td>
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<tr>
<td>($T_c$ = critical temperature)</td>
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</tr>
<tr>
<td>distance $r$</td>
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<tr>
<td>non-universal small scale fluctuations (characteristic</td>
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<tr>
<td>small scale $\xi$)</td>
<td>(characteristic scale of the energetic eddies</td>
</tr>
<tr>
<td></td>
<td>$L = 1/k_0$).</td>
</tr>
<tr>
<td>correlation length</td>
<td>$k_{diss} = k_0 R_0^\nu$</td>
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<tr>
<td>$\xi = \xi_0 \left( \frac{T - T_c}{T_c} \right)^\nu$</td>
<td></td>
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<tr>
<td>order parameter (magnetization) $M(r)$</td>
<td>Fourier transform of vorticity $\hat{\omega}(k)$</td>
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<tr>
<td>fluctuations of $M$ have an infinite range in</td>
<td>fluctuations of $\hat{\omega}$ have an infinite range in</td>
</tr>
<tr>
<td>configuration space in the limit $T - T_c \downarrow 0$</td>
<td>Fourier space in the limit $1/R_0 \downarrow 0$</td>
</tr>
</tbody>
</table>
spin-spin correlation
function $g(r)$

$$\lim_{r \to r_c} g(r)$$

if $H$ is an external
field which couples to
$M$, then the susceptibility

$$\left. \frac{\delta^2 \mathcal{H}}{\delta H^2} \right|_{H=0} \sim \int_0^\infty r^2 q(r) dr \propto (T-T_c)^{-\eta}$$

g(r) \sim r^{1-\eta} f\left(\frac{r}{R_0}\right)

scaling relation

$\gamma = (z-\eta)\nu$

Fourier transform of
vorticity-vorticity

$$\left\langle \hat{\omega}(k) \hat{\omega}(k') \right\rangle \sim E(k)$$
in three dimensions

limit $E(k)$ exists

$$\gamma \equiv$$

total vorticity

$$\int_0^\infty k^2 E(k) dk \sim \xi R_0$$

This corresponds to a
critical exponent

$\gamma = 1$

Kolmogorov theory

implies that for

$k >> k_0$

$$E(k) \sim \xi^{\frac{1}{2}} k^{-\frac{5}{3}} f\left(\frac{k}{k_{diss}}\right)$$

and $k_{diss} = k_0 R_0^{\frac{1}{3}}$

This corresponds to
critical exponents

$\eta = 2/3$, $\nu = 3/4$

which satisfies the
scaling relation

$\gamma = (z-\eta)\nu$
VITA

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The author was born in Shanghai, China on June 28, 1962. He received a B.S. degree (May, 1982) from the University of Science and Technology of China; he received his M.S. degree (May, 1984) from the College of William and Mary. He received his Ph.D. degree (December, 1987) from the College of William and Mary. He accepted a postdoctoral position at the Bartol Research Institute, University of Delaware.