Spin polarization in electron scattering off few-body nuclei

Veljko Dmitrasinovic
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Spin polarization in electron scattering off few-body nuclei

Dmitrašinović, Veljko, Ph.D.
The College of William and Mary, 1990
SPIN POLARIZATION IN
ELECTRON SCATTERING OFF FEW-BODY NUCLEI

A Dissertation
Presented to
The Faculty of the Department of Physics
The College of William and Mary in Virginia

In Partial Fulfillment
Of the Requirements for the Degree of
Doctor of Philosophy

by
Veljko Dmitrašinović
1990
APPROVAL SHEET

This dissertation is submitted in partial fulfillment of

the requirements for the degree of

Doctor of Philosophy

Veljko Dmitrašinović

Approved, November 1990

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IN MEMORIAM PATRIS
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_Ne cede malis_

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ABSTRACT

The purpose of this investigation is twofold: to establish a relativistic formalism for the description of inelastic, two-body, coincidence, electron scattering off few-body nuclei with spins one half or one, and to investigate the role of strangeness-induced weak-neutral-current matrix elements of few-body nuclei and parity-violating admixtures to the nucleon wave function in elastic, parity-violating electron-nucleus scattering.

Helicity formalism was used in the formulation of comprehensive treatments of all single-polarization observables in scalar and pseudo-scalar electro-production off spin one half targets and all single- and double-polarization observables in deuteron two-body photo- and electro-disintegration, from a unified standpoint. A discussion of necessary and sufficient measurements needed for a complete determination of all transition amplitudes is given. Wigner rotation of the recoil polarization coincidence electro-disintegration and electro-production structure functions due to the boost from the centre-of-mass to the lab frame are calculated. Inequalities among polarized spin one half target and unpolarized ejectile as well as unpolarized target and polarized spin one half ejectile coincidence structure functions due to the positivity of the polarized coincidence cross section are derived for arbitrary reactions with spin one half targets and ejectiles.

It is shown how to separate strangeness-induced form factors of the nucleon from elastic, parity-violating electron scattering data. A parity-violating quark-quark potential is derived by a non-relativistic reduction of one-W^±, Z^0-exchange Feynman diagrams. The abnormal parity admixture in the nucleon wave function is calculated using first order perturbation theory in a non-relativistic quark model with harmonic oscillator wave functions and the lowest-lying negative parity nucleon resonances. The contribution of these parity admixtures to the elastic parity-violating electromagnetic current matrix element of the nucleon is evaluated. Special attention is paid to the gauge invariance of the calculation: gauge invariance preserving two-body quark parity-violating electromagnetic current is constructed. It is shown that the elastic parity-violating electromagnetic current matrix element has the correct behaviour in the long-wave-length limit only if all abnormal parity admixtures to the wave function are included. A new definition of the parity-violating, time-reversal preserving elastic electromagnetic current matrix element, which automatically satisfies the threshold theorem, is given. An exact argument, based on closure, about the contribution of all parity admixtures to the elastic parity-violating electromagnetic current matrix element is established. The complete elastic parity-violating electromagnetic current matrix element of the nucleon is shown to vanish identically, to first order in GF, in all non-relativistic quark models with spin and iso-spin independent, local, strong quark-quark potentials.
SPIN POLARIZATION IN

ELECTRON SCATTERING OFF FEW-BODY NUCLEI
CHAPTER I

INTRODUCTION

1.1 Prologue

Nuclear physics is concerned with understanding the structure of the atomic nucleus and the interactions among its constituents. All research up to date has led us to believe that there are three kinds of fundamental interactions in Nature, neglecting Gravity. They are: electromagnetic, weak and the strong nuclear forces. All of them are believed to be describable by the mathematical language of relativistic quantum field theory (QFT). Modern research has found strong indications that the first two mentioned interactions are but two manifestations of one more fundamental “electro-weak” force. Although experimental corroboration of this point of view is still incomplete, this scheme has gained wide acceptance and has become known as the “Standard Model”.

The study of nuclear physics is accomplished by observation of nuclear scattering, production and decay reactions. This thesis is concerned with the theoretical study of one specific kind of nuclear scattering processes: the scattering of fast electrons off nuclear targets. Electrons are believed to be truly elementary particles which feel only two, or just one if one accepts the Standard Model as true, kinds of forces. The method of electro-nuclear scattering has proven to be one of the most precise instruments at our disposal. One of the basic advantages over other similar scattering methods is that it is readily amenable to interpretation: the electromagnetic and the weak forces are much weaker than the strong nuclear force which holds the nucleus together, so that their effects on the nucleus during the scattering process can be treated as small perturbations calculable to a high degree of precision.

Another, not completely independent from the previously mentioned Standard Model, line of modern thinking has lead to a point of view that the individually observable constituents of the nucleus such as the proton and neutron are not its most elementary building blocks, but rather that they are composite objects, too. Such building blocks have become known as the “quarks” and the assumption of
their existence came about as a consequence of certain symmetries in the tables of the observed strongly interacting particles (hadrons). The quark model has had considerable success in predicting properties of newly discovered hadrons, but at a certain price: one had to either give up a well-established theorem in the mathematical structure of the theory relating the spin of the particle to the properties of the wave function under the exchange of two identical particles, or postulate a new internal degree of freedom. This was not the only reason for widespread scepticism about the model: extensive experimental searches for a single (free) quark produced no results despite its clear experimental signature (fractional electric charge). Wide acceptance by the physics community came only after a set of deeply inelastic, electron scattering experiments, were completed at the Stanford Linear Accelerator, which seem to confirm the existence of pointlike constituents of the nucleon. Mathematical discoveries in QFT in the early 70's led to the second option as the solution to the theoretical problems of the quark model, i.e. a new degree of freedom, now called "colour", and a model for its dynamics called quantum chromodynamics (QCD) were postulated. The absence of free quarks, usually called quark confinement, has not been proven to be a consequence of this model, but there are indications that this might be the case. The experimental evidence in favour of QCD is far less conclusive than that for the electro-weak model, also known as the Salam-Weinberg model, yet these models are considered by some to be on the same footing and the name "Standard Model" often encompasses both.

QCD allows reliable predictions only in the very high energy limit, so that the model in its full form is essentially useless for the purposes of nuclear physics, where different methods, sometimes referred to as "old-fashioned", have to be used. These methods are modern versions of meson interaction theories, where it is assumed that the force between two nucleons is caused by the exchange of light bosonic hadrons (or mesons). They are extensively used in low and medium energy nuclear physics and have been found to be able to describe virtually all observed phenomena. Their main drawback, aside from the fact that the hadrons are themselves composite, is the large number of free parameters which give a considerable latitude in predictions of the theory. Such are most of the theories used to predict experiments about to be done at CEBAF.

A quantum mechanical particle has besides its familiar macroscopic properties such as the mass, velocity and electric charge also some intrinsically quantum mechanical properties such as the intrinsic angular momentum, or spin for short. The nucleon has spin $\frac{1}{2}\hbar$, or $\frac{1}{2}$ for short, as does the electron and many nuclei. The spin dependence of nuclear interactions has historically proven to be very interesting and important. But due to various difficulties the spin dependence of electro-nuclear

* These experiments were awarded the Nobel prize in physics in 1990
processes has only been investigated in detail in the last ten years.

The spin is an important ingredient of the weak interactions, too: the symmetry of the world under the inversion of all spatial directions (parity) is broken in these interactions and this is manifested through their spin dependence.

Technological advances of the past two decades, especially the development of a reliable superconducting accelerator cavity, have allowed the construction of a new generation of electron accelerators with high duty factors, thereby enabling an entirely new kind of electron scattering experiments, where a second particle is observed in coincidence with the scattered electron. It is the formalism describing such processes which is the subject of the first part of this thesis.

This thesis can be naturally divided into two parts, done under the directions of two advisors: the first part, done under Prof. F. Gross, is concerned with the formalism of parity conserving, polarized, inelastic, coincidence electron-nuclear scattering. The second part, done under Prof. J. D. Walecka, is concerned with a certain class of corrections, as calculated in a specific quark model, to the parity violating, polarized, elastic electron-nucleon scattering.
1.2 Inelastic Polarized Coincidence Electron Scattering

The advent of new electron accelerator facilities like CEBAF with the potential of having spin polarized beams and targets as well as recoil polarimeters demands a thorough understanding of polarization phenomena, both for the optimal planning of experiments, as well as for their later interpretation. It is therefore important to set up an appropriate relativistic formalism for the description of these degrees of freedom. Together with Prof. Gross, I have formulated a general, relativistic formalism for the description of polarization observables in the coincidence, two-body final state, inelastic electron scattering, and applied it to two specific examples of particular interest to CEBAF: scalar and pseudoscalar electroproduction off spin $1/2$ targets and deuteron two-body electrodisintegration reaction. The general form of the inelastic coincidence electron scattering cross section for arbitrary polarization of the target and/or ejectile, with two particles in the final state, in the “mixed” frame i.e. with electron variables in the laboratory and the hadronic variables in the centre-of-mass frame, is:

\[
\frac{d^5\sigma}{d\Omega' dE' d\Omega} = \frac{\sigma_{M1}}{4\pi M_T} \left\{ \left( \frac{W}{M_T} \right)^2 v_L R_L + v_T R_T \right\}
\]

\[
+ v_{TT} \left[ \cos 2\phi \ R_{TT}^{(I)} + \sin 2\phi \ R_{TT}^{(II)} \right]
\]

\[
+ \left( \frac{W}{M_T} \right) v_{TL} \left[ \cos \phi \ R_{LT}^{(I)} + \sin \phi \ R_{LT}^{(II)} \right] + 2h v_T' R_T'
\]

\[
+ 2h \left( \frac{W}{M_T} \right) v_T' \left[ \cos \phi \ R_{LT}^{(II)} + \sin \phi \ R_{LT}^{(II)} \right]
\]

(1)

where \( h = \pm \frac{1}{2} \) is the helicity of the incoming electron, \( \sigma_M \) is the Mott cross section and

\[
v_L = \left( \frac{Q}{q_L} \right)^4 \quad v_{TT} = -\frac{Q^2}{2q_L^2}
\]

\[
v_{TL} = -\frac{1}{\sqrt{2}} \left( \frac{Q}{q_L} \right)^2 \sqrt{\frac{Q^2}{q_L^2} + \tan^2 \frac{1}{2}\theta} \quad v_T' = -\frac{1}{\sqrt{2}} \left( \frac{Q}{q_L} \right)^2 \tan \frac{1}{2}\theta
\]

\[
v_T = \frac{1}{2} \frac{Q^2}{q_L^2} + \tan \frac{1}{2}\theta \quad v_{TL}' = \tan \frac{1}{2}\theta \sqrt{\frac{Q^2}{q_L^2} + \tan^2 \frac{1}{2}\theta}
\]

where \( \theta \) is the electron scattering angle in the lab frame and the absolute value of the three-momentum transfer vector in the lab frame \( q_L = |q_L| \). The only assumptions entering this result are: one-photon exchange approximation and conserved hadronic current. Parity conservation has not been assumed, yet. The kinematic variables entering this cross section are: the total cm energy of the system.
\[ W = \sqrt{(P + q)^2}, \] the negative four-momentum transfer squared \( Q^2 \), the absolute value of the momentum transfer three-vector in the lab frame \( q_L \), the electron scattering angle \( \theta \), the ejectile opening angle \( \theta_1 \) and the azimuthal angle \( \phi \) (see Fig. 1).

The response functions \( R \)'s are functions of \( W, Q^2 \) and \( \theta_1 \), but not of \( \phi \) as long as the target polarization is specified with respect to the coordinate system \( (x', y', z') \) (see Fig. 1) and recoil polarization is measured with respect to the \( (x'', y'', z'') \) (Fig. 1) coordinate system. Thus, they can be separated by making measurements at different values of \( \phi \) and otherwise identical kinematics.

![Diagram](image)

Fig. 1 Diagram of the geometry of the inelastic electron scattering process showing the electron scattering plane, the ejectile plane and the two coordinate systems in the ejectile plane: \( (x', y', z') \) and \( (x'', y'', z'') \).

All structure functions can be divided into two classes: class I structure functions \( (R_L, R_T) \) besides all other structure functions explicitly denoted as such by the superscript I) are nonzero in unpolarized, parity conserving reactions, while class II structure functions \( (R_{T'}) \) besides all other structure functions explicitly marked II) vanish identically because of constraints imposed by parity. Once the polarization
measurement is allowed, this rule is modified, but it remains true that one half of all possible structure functions vanish because of parity constraints.

Any quantum mechanical scattering reaction is completely specified by a finite number of independent transition matrix elements, also called transition amplitudes. The problem of finding the necessary and sufficient experimental measurements needed for a complete determination of the transition amplitudes of a process can be important for an effective use of polarization in electron scattering. I, together with my advisor F. Gross and T. W. Donnelly, have solved this problem for the two reactions mentioned above: scalar and pseudo-scalar electroproduction off spin 1/2 targets and two-body deuteron electrodisintegration, and published it in (ref. 1,2).

The number of independent transition amplitudes, at a given set of kinematics, is determined by the size of the spin of the particles involved: if a massive particle has spin \( s \), then there are \( (2s + 1) \) different quantum mechanical states which this particle can occupy. They can be distinguished by the value of the projection of the spin on a certain direction in space, which is referred to as the spin quantization axis and can be chosen at will. In what follows, the spin of each particle will be specified with respect to the direction of motion of that particle. This choice of spin quantization axis leads to so called helicity states.

If a reaction involves particles with spins \( s_1, s_2, s_3, s_4 \), then there are \( (2s_1 + 1)(2s_2 + 1)(2s_3 + 1)(2s_4 + 1) \) amplitudes describing the reaction. It turns out that parity relates states with a certain value of helicity to those with the negative helicity value and inverted coordinates. So, if parity is conserved by the interaction, it puts a constraint on the amplitudes and reduces the number of independent ones by half. In the first part of this thesis we will be concerned only with parity conserving parts of these reactions, which can be used for the study of the strong nuclear forces and electromagnetic properties of hadrons.

The inelastic electron scattering is treated in the one-photon exchange approximation, which means that it can be viewed as a binary collision of a virtual photon and the target particle. The virtual photon has three polarization states just like the deuteron, the nucleon has two and the pion one. Knowing this, we can count the number of independent amplitudes appearing in the two reactions mentioned above. For coincidence deuteron electrodisintegration, \( D(e, e'p)n \), we have \( \frac{3 \times 3 \times 2 \times 2}{2} = 18 \) amplitudes, 12 of which are called transverse and appear in reactions involving real photons, the other six, known as the longitudinal amplitudes, appear only in virtual photon reactions. In coincidence pion electroproduction, \( p(e, e'p)\pi \), there are \( \frac{3 \times 3 \times 2 \times 1}{2} = 6 \) independent amplitudes, four of which are transverse and two longitudinal. Each amplitude is a complex function of the kinematic variables i.e. it can be represented by two real functions of the same variables. So we conclude that there are 36 real functions describing \( D(e, e'p)n \) and 12 describing \( p(e, e'p)\pi \) reaction. But
the observables are bilinear products of amplitudes which means that one, overall, phase cannot be determined. This leads us to the final number of independent functions governing these reactions: 35 for \( D(e,e'p)n \) and 11 for \( p(e,e'p)\pi \).

The polarization of an ensemble of particles, such as a particle beam, is best described by a density matrix which is hermitian and has dimensions determined by the size of the spin of the particle. Spin 1/2 particles have \( 2 \times 2 \) density matrices which are fully specified by three parameters usually taken to be the three components of the polarization vector. On the other hand, the deuteron’s polarization density matrix, which is \( 3 \times 3 \), is specified by eight independent quantities: three components of the polarization vector and five components of the symmetric polarization tensor. The observables \( R' \)s are functions of these polarization parameters.

Besides the three components of the polarization vector, we have the unpolarized \((U)\) case. Parity constraints half the number of nonvanishing response functions, hence there are \( \frac{9 \times (1+3)}{2} = 18 \) possible single polarization observables in situations where the polarized particle has spin 1/2, such as the polarized nucleon ejectile, \( p(e,e'p)\pi \) and \( D(e,e'p)n \), as well as polarized spin 1/2 target reactions, \( \bar{p}(e,e'p)\pi \). But, in situations where the polarization of a spin one particle is observed, such as in polarized deuteron target reaction \( \bar{D}(e,e'p)n \), there are \( \frac{9 \times (1+8)+1}{2} = 41 \) single polarization observables. We see that there are more single polarization observables than unknown functions. This gives us hope that we might be able to extract all amplitudes from the experiments and raises the question: which of these measurements are redundant? Detailed investigation shows that no set of single polarization measurements can completely disentangle all amplitudes in either reaction. In order to prove that, we work out all single polarization observables in \( p(e,e'p)\pi \) and all single and double polarization observables in \( D(e,e'p)n \), except the two-ejectile observables which belong to double coincidence reactions and turn out to be redundant for the purpose of separation of amplitudes.

Experience has shown that certain linear combinations of helicity amplitudes, called hybrid amplitudes, greatly simplify the resulting tables of observables. These amplitudes are made of transversity states, whose spin quantization axis is perpendicular to the reaction plane, for all particles in the reaction except the photon. In our case this axis is the \( y' = y'' \) direction (Fig. 1).

We will present the results for \( p(e,e'p)\pi \) reaction in order to illustrate the method. The six independent helicity amplitudes are

\[
\begin{align*}
F_1 &= \langle -\frac{1}{2} |J \cdot e_+| - \frac{1}{2} \rangle \\
F_2 &= \langle -\frac{1}{2} |J \cdot e_+| + \frac{1}{2} \rangle \\
F_3 &= \langle +\frac{1}{2} |J \cdot e_+| - \frac{1}{2} \rangle \\
F_4 &= \langle +\frac{1}{2} |J \cdot e_+| + \frac{1}{2} \rangle \\
F_5 &= \langle +\frac{1}{2} |J \cdot e_o| + \frac{1}{2} \rangle \\
F_6 &= \langle +\frac{1}{2} |J \cdot e_o| - \frac{1}{2} \rangle
\end{align*}
\]
where $J \cdot e_\lambda = J_\mu e^\mu_\lambda$ and the initial and final states are helicity states with helicities specified above, where the nucleon in the final state is particle 1 as defined by Jacob and Wick (ref.10). $J^\mu$ is the hadronic response current defined in the ejectile plane, with Bjorken and Drell conventions (ref.3) and $e^\mu_\pm = (0, \hat{e}_\pm)$, $e^i_0 = (1, 0, 0, 0)$ in cm frame. The connection between the hybrid amplitudes $g_i$ and the helicity amplitudes $F_j$ can be summarized by the matrix relation

$$g_i = \Lambda_{ij} F_j$$

where, for transverse amplitudes $(i,j = 1-4)$ while, for longitudinal amplitudes $(i,j = 5-6)$ and

$$\Lambda_{ij} = \frac{1}{2} \begin{pmatrix}
1 & i & -i & 1 & 0 & 0 \\
1 & -i & i & 1 & 0 & 0 \\
-i & 1 & 1 & i & 0 & 0 \\
i & 1 & 1 & -i & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -2i \\
0 & 0 & 0 & 0 & 2 & 2i
\end{pmatrix}$$

The results are the following two tables of observables. Each structure function appears in the cross section eq.(1) multiplied by its spin polarization vector, e.g.

$$R^{(1)}_{LT} = \sum_{i,j} p_i p_j \kappa^2 R^{(1)}_{LT}(p_i, p_j)$$

where

$$R_{LT}(U, U) = R_{LT}$$
$$R_{LT}(U, P_n) = R_{LT}(n)$$
$$R_{LT}(p_z, U) = R_{LT}(z)$$

and

$$\kappa^2 = \frac{1}{2W} \left( \frac{M}{2\pi} \right)^2$$

Expressions $R^{(1)}_{LT}(n), R^{(1)}_{LT}(z)$ appear in Tables 1, 2.
Table 1 Recoil polarization observables as functions of hybrid amplitudes $g_i$, where $U$ stands for unpolarized and $(P_n, P_x, P_y)$ are the $(y, x, z)$ components, respectively, of the recoil polarization vector as measured in $cm$ frame with respect to the $(\bar{x}', \bar{y}', \bar{z}')$ coordinate system.

<table>
<thead>
<tr>
<th></th>
<th>$P_n$</th>
<th>$P_x$</th>
<th>$P_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_L$</td>
<td>$a_1$</td>
<td>$a_2$</td>
<td></td>
</tr>
<tr>
<td>$R_T$</td>
<td>$b_1$</td>
<td>$b_3$</td>
<td></td>
</tr>
<tr>
<td>$R_{TT}$</td>
<td>$b_2$</td>
<td>$b_4$</td>
<td></td>
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<tr>
<td>$R_{LT}$</td>
<td>$\text{Re}(e_1)$</td>
<td>$\text{Re}(e_2)$</td>
<td></td>
</tr>
<tr>
<td>$R_{LT}^{\prime}$</td>
<td>$\text{Im}(e_1)$</td>
<td>$\text{Im}(e_2)$</td>
<td></td>
</tr>
<tr>
<td>$R_{LT}^{\prime\prime}$</td>
<td></td>
<td></td>
<td>$\text{Re}(f_1)$</td>
</tr>
<tr>
<td>$R_{LT}^{\prime\prime\prime}$</td>
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<td>$\text{Re}(f_1)$</td>
</tr>
<tr>
<td>$R_{TT}$</td>
<td></td>
<td></td>
<td>$\text{Im}(k_1)$</td>
</tr>
<tr>
<td>$R_{TT}^{\prime}$</td>
<td></td>
<td></td>
<td>$\text{Re}(k_1)$</td>
</tr>
</tbody>
</table>

Table 2 Same as Table 1, but for polarized target observables. $(P_z, P_y, P_x)$ are the $(x, y, z)$ components of the target polarization vector as measured in $cm$ frame with respect to the $(\bar{x}', \bar{y}', \bar{z}')$ coordinate system (Fig.1).

where

$$2a_1 = |g_5|^2 + |g_6|^2$$  
$$2a_2 = |g_5|^2 - |g_6|^2$$  
$$c_1 = g_4^* g_1 + g_3^* g_2$$  
$$c_2 = g_4^* g_1 - g_3^* g_2$$
One can see from Tables 1, 2 that the moduli of the amplitudes are readily separable from \( a_i, b_j \), \((i = 1,2) \ (j = 1,\ldots,4)\). So, only the relative phases remain to be determined. It is also clear from Table 1 that one can determine the relative phases among the members of two disjoint classes of amplitudes: \((1,4,5)\) and \((2,3,6)\) from the recoil polarization observables, but not the relative phase between the two classes. This is most easily seen by looking at an example: from the knowledge of \( \text{Re}, \text{Im} \) parts of \( e_i \), we can get \( \text{Re}, \text{Im} \) parts of \( g_i^* g_5 \) and \( g_2^* g_6 \). If we write the amplitudes \( g_i \) in the form: \( |g_i| \exp(i \phi_i) \), we can get \( |g_i| |g_j| \cos(\phi_j - \phi_i) \) and \( |g_i| |g_j| \sin(\phi_j - \phi_i) \) from those two pieces of information. This allows us to determine the relative phase \( (\phi_j - \phi_i) \), as advertised.

Similarly, from Table 2 we see that the amplitudes which can be separated from polarized target observables fall into two different classes: \((1,3,5)\) and \((2,4,6)\). The complete separation of all amplitudes can be accomplished by making a complete separation of relative phases in the pair of classes from one set of experiments and then finding the relative phase between the two classes from the other set of experiments.

But, this is not the most efficient way of finding the complete information: from the knowledge of the real and imaginary parts of a bilinear product of amplitudes we get the product of the two moduli, as well as the relative phase without quadrant ambiguity. Once the bilinear products of moduli are known, we need to know only one modulus squared, in order to set the absolute scale. That can be got from only a couple of measurements: for example, from the knowledge of \( \text{Re} \) and \( \text{Im} \) of \( e_1, e_2 \) and \( d_1, d_2 \), we can get all relative phases within the classes \((1,4,5)\) and \((2,3,6)\), as well as the respective products of moduli. Then, a measurement of \( a_{1,2} \) provides \( |g_5| \) and \( |g_6| \) which fixes the absolute size of all moduli and completes the information accessible by the recoil polarization experiments. All other structure functions in the recoil polarization cross section are redundant. Equivalent information can be

\[
\begin{align*}
b_1 &= |g_1|^2 + |g_2|^2 + |g_3|^2 + |g_4|^2 \\
b_2 &= -|g_1|^2 - |g_2|^2 + |g_3|^2 + |g_4|^2 \\
b_3 &= |g_1|^2 - |g_2|^2 + |g_3|^2 - |g_4|^2 \\
b_4 &= -|g_1|^2 + |g_2|^2 + |g_3|^2 - |g_4|^2 \\
d_1 &= g_1^* g_5 + g_3^* g_6 \\
d_2 &= g_2^* g_5 - g_3^* g_6 \\
e_1 &= g_1^* g_5 + g_2^* g_6 \\
e_2 &= g_1^* g_5 - g_2^* g_6 \\
f_1 &= g_2^* g_5 + g_3^* g_6 \\
f_2 &= g_3^* g_5 - g_2^* g_6 \\
k_1 &= g_1^* g_3 + g_2^* g_4 \\
k_2 &= g_1^* g_3 - g_2^* g_4
\end{align*}
\]
obtained by choosing any other pair from the triplet \( c_i, d_i, e_i \). In order to complete the separation we need the real and imaginary parts of one of the following: \( f_i, k_i \), which can be obtained only from the polarized target experiments. It will provide the remaining relative phase between the two classes of amplitudes. Thus we see that the 11 unknown functions can be disentangled from 14 measurements. Three measurements overdetermine the amplitudes and hence present a double check of the separation procedure.

All of this proves that: a) no double polarization measurements are necessary, b) one single polarization set of experiments is insufficient for the complete separation of \( p(e, e'p) \pi \) amplitudes. The complete results for scalar coincidence electroproduction on spin 1/2 targets are presented in (ref.1).

Analogous analysis of the \( D(e, e'N)N' \) reaction, presented in (ref.2) is substantially more complicated: there are 57 nonvanishing single and 162 double polarization observables for each of the two different ejectiles. The results are: a) no set of single and/or double polarization observables from proton ejectile experiments is sufficient without at least one measurement with neutron recoil polarization and vice versa, b) no triple polarization measurements are necessary.

Certain amount of detailed technical information did not appear in these publications (ref.1,2), as well as some new insights which were gained after the papers were published. It is this work that I will present in the first part of this thesis.

**Wigner Rotation**

The main technical point not discussed in detail in the publications is the appearance of the precession of the spin polarization direction, purely kinematic in origin, due to the Lorentz transformation of the recoil polarization structure functions from the centre of momentum frame, which is the theoretically preferred frame of reference, to the rest frame of the target i.e. the \( \text{lab} \) frame, in which the observables are measured.

The net effect of the Lorentz boost from the \( \text{cm} \) to the \( \text{lab} \) frame on the polarized cross section is; besides the well understood change of certain kinematical factors in the cross section, the Wigner rotation of the recoil polarization observables \( R_{ij}(s) \), \( R_{ij}(l) \) about the \( y' = y'' \) axis through the Wigner angle \( \omega \):

\[
R_{ij}^{LAB}(\alpha, l) = \cos \omega R_{ij}^{cm}(\alpha, l) - \sin \omega R_{ij}^{cm}(\alpha, s) \\
R_{ij}^{LAB}(\alpha, s) = \sin \omega R_{ij}^{cm}(\alpha, l) + \cos \omega R_{ij}^{cm}(\alpha, s)
\]

where \((n, s, l)\) are the \((y'', x'', z'')\) components, respectively, of the recoil polarization vector, \( \alpha \) stands for all other, e.g. target, polarization specifications and \( i, j \) stand
for the photon polarization. The rotation vanishes if the directions of the boost and the motion of the particle coincide, i.e. at \( \theta_1 = 0, \pi \). The polarization component \( P_n \), which is perpendicular to the plane defined by the boost direction and the particle velocity, remains unchanged.

The spin rotation angle \( \omega \), which is always less than the rotation angle \( \theta_1^{cm} - \theta_1^l \) of the momentum three-vector, is given by:

\[
tg \frac{\omega}{2} = \frac{\sinh v_1 \sinh v_2 \sin \theta_1^{cm}}{(1 + \cosh v_1)(1 + \cosh v_2) + \sinh v_1 \sinh v_2 \cos \theta_1^{cm}}
\]

where \( v_1, v_2 \) are the rapidities of the \( cm \to lab \) and \( cm \to rest \) boosts, respectively. The relationship between the rapidities and the more conventional boost parameters \( \gamma, \beta \) is:

\[
\gamma_i = \cosh(v_i), \quad \gamma_i \beta_i = \sinh(v_i)
\]

The boost parameters, for pion electroproduction, are:

\[
\tilde{\beta}_1 = \frac{\bar{q}l}{\sqrt{\bar{q}l^2 + \mu^2}}, \quad \gamma_1 = \sqrt{1 + \left(\frac{qL}{W}\right)^2}
\]

\[
\tilde{\beta}_2 = \frac{\vec{p}_1^{cm}}{E_1^{cm}}, \quad \gamma_2 = \frac{E_1^{cm}}{M}
\]

where

\[
\vec{p}_1^{cm} = \frac{1}{2W} \sqrt{(W^2 - M^2 - \mu^2)^2 - 4M^2\mu^2}
\]

\[
E_1^{cm} = \frac{1}{2W} \sqrt{(W^2 - M^2 - \mu^2)^2 - 4M^2(\mu^2 - W^2)}
\]

and \( M, \mu \) are the nucleon and pion mass respectively. The kinematics of electroproduction is defined by two independent variables: \( W, Q^2 \). One of them can be substituted by the Bjorken variable \( x \):

\[
x = \frac{Q^2}{2 \vec{p} \cdot q}
\]

Hence

\[
Q = \sqrt{\left(\frac{x}{1-x}\right)(W^2 - M^2)}
\]
As a result, we get Fig. 2:

![Graph showing Wigner angle ω as a function of the cm frame opening angle θ_{cm} for pion electroproduction on nucleon at W = 1232 Mev and Bjorken x = 0.5 (solid), x = 0.75 (dashes).](image)

Note that ω is typically less than 10°, while the rotation angle of the three-momentum vector can be over 90°, in standard kinematic situations. So, to lowest approximation, the Wigner rotation is negligible in pion electroproduction at these kinematics, but if one aspires to making high precision polarization measurements, one must include its effects.

Furthermore, we formally prove that the Wigner rotation angle ω is always less than the three-momentum rotation angle θ_{cm} - θ_{1}. For this purpose we use Sommerfeld's (ref.4) observation that Einstein's theorem about the addition of velocities in special relativity is an example of non-Euclidean geometry. This means that if two nonparallel four-velocities are added relativistically to give a third four-velocity, then their spatial parts do not form a planar triangle, but rather they can be put in correspondence with sides of a triangle on the surface of a hyperboloid. The sum of inner angles in such a triangle is less than π and the difference between π and this sum will be called "hyperbolic defect" in analogy with the spherical excess which
appears in spherical trigonometry. The Wigner angle is one of the inner angles of this triangle (see Fig. 3) and Wigner’s contribution was the observation of its relevance to the precession of the spin of a particle when the particle is subjected to the sequence of Lorentz transformations specified by the three four-velocities.

Fig.3 A schematic representation of three relativistic velocity vectors forming a triangle on a hyperboloid. The sum of inner angles is less than π.

Sommerfeld’s observation came as early as 1909 (ref.4), long before spin was discovered, so that the physical significance of his mathematical discovery remained unclear for some time. But once relativistic effects in atomic physics became more appreciated, Sommerfeld recognized that Thomas precession is just a manifestation of this relativistic kinematic phenomenon in atomic physics (ref.5).

The existence of the Wigner angle has been known for a long time and it is routinely taken into account in hadronic scattering physics (ref.6), but this is the first time, to my knowledge, that it has been applied to electron or photon scattering.

Secondly, I present a new insight gained after the appearance of the two papers (ref.1,2): inequalities among completely unpolarized, as well as polarized target-unpolarized ejectile and unpolarized target-polarized ejectile coincidence inelastic electron scattering observables, also known as structure functions, due to the positivity of the polarized coincidence cross sections.
Positivity Inequalities

The general form of the differential cross section, which is strictly a positive or vanishing, positive semi-definite for short, quantity, in the one-photon-exchange approximation can be written as a quadratic form. This means that one may apply mathematical criteria for the positive semidefiniteness of a quadratic form to the response tensor and thus obtain nontrivial inequalities among the response tensor functions. These, in turn, can be uniquely related to the observables $I^T$'s so that the end result is a set of inequalities among them:

$$R_L, R_T \geq 0, \quad R_T \geq |R_T^{(1)}|$$

$$4R_L (R_T - R_T^{(1)}) \geq [(R_L^{(1)})^2 + (R_T^{(1)})^2]$$

The first two inequalities are trivial: they are equivalent to the statement that the sums of squares of amplitudes have to be positive or zero. The next inequality saying that the transverse structure function is larger or equal to the absolute value of the transverse-transverse interference structure function is familiar from model building, where it was empirically observed, but not proven. The fourth result is a non-trivial inequality involving all five structure functions appearing in the unpolarized cross section, which has not been investigated or applied to models, so far. It can be used to set bounds on one of the structure functions if the other four are known. For example, from the knowledge of $R_L, R_T, R_T^{(1)}, R_T^{(1)}$, we can get an upper bound on the size of the so called fifth structure function $R_T^{(4)}$:

$$|R_T^{(4)}| \leq \sqrt{4R_L (R_T - R_T^{(1)}) - (R_T^{(1)})^2}$$

which could be helpful if one did not have a complete dynamical model of all structure functions. These inequalities were first derived in (ref.8), but they remained largely unknown in the nuclear physics community.

These results might prove to be of practical importance as a consistency check of the experimental extraction procedure for the coincidence observables. The completely unpolarized and some of the polarized target inequalities have been known before, but all of the polarized ejectile and some of the polarized target inequalities are new. The derivation of the inequalities is accomplished by writing the most general unpolarized coincidence cross section for arbitrary targets and final states. This is done in the so called general response tensor formalism. These results were first derived, in a slightly different form and only for unpolarized electrons, by de Forest (ref.7). They are completely general and correct for electron scattering coincidence cross sections...
I rederive these very same inequalities using a seemingly different method and thus establish the equivalence of the methods. The second method turns out to be much more economical for deriving new, polarized structure function inequalities. Some of the "polarized target" inequalities have already been written down in (ref.8), but all of the "polarized ejectile" and some of the polarized target inequalities presented here are new. Here, I show only a few of the simplest such inequalities (for more, and a review of the "target" inequalities, see section 2.5):

\[
R_L(U) \geq |R_L(n)|, \quad R_T(U) \geq |R_T(n)|
\]

\[
R_T(U) \geq |R_{TT}^{(1)}(n)|
\]

\[
R_T(U) - R_{TT}^{(1)}(U) \geq |R_T(n) - R_{TT}^{(1)}(n)|
\]

\[
R_T(U) + R_{TT}^{(1)}(U) \geq |R_T(n) + R_{TT}^{(1)}(n)|
\]

\[
\left[R_T(U) - R_T(n)\right]^2 - \left[R_{TT}^{(1)}(U) - R_{TT}^{(1)}(n)\right]^2 \geq
\]

\[
\geq \left[R_T(s) + R_{TT}^{(1)}(l)\right]^2 + \left[R_{TT}^{(1)}(s) - R_T(l)\right]^2
\]

\[
\left[R_T(U) + R_T(n)\right]^2 - \left[R_{TT}^{(1)}(U) + R_{TT}^{(1)}(n)\right]^2 \geq
\]

\[
\geq \left[R_T(s) - R_{TT}^{(1)}(l)\right]^2 + \left[R_{TT}^{(1)}(s) + R_T(l)\right]^2
\]

where \(l, s, n\) in \(R(i)\) stand for the vector polarization component of the ejected nucleon. All of these results are new. These inequalities have to be satisfied by all experimentally measured structure functions and thus might play a useful role as a check of the experimental separation procedure, specifically as a check of radiative corrections. Finally, they could be used to get order of magnitude estimates for certain structure functions which vanish in simple dynamical models, as in the example given above.

It is appropriate to put the new results in this first part of the thesis into some historical perspective. The helicity formalism for spin polarized reactions was introduced in the late 50's by Jacob and Wick (ref.10), in the West, and by Shirokov and collaborators (ref.11) in the Soviet Union. The principal motivation was to provide a unified relativistic formalism for the description of spin polarization of massive and massless particles. An investigation of the crossing properties of helicity amplitudes led Kotanski (ref.12) to the introduction of transversity states and amplitudes.

The need for an unambiguous, high quality set of elastic nucleon-nucleon phase shifts led Puzikov, Ryndin and Smorodinsky (ref.13) below, and Schumacher and
Bethe (ref.14) above the the pion-production threshold, to the first complete amplitude separation analysis for N-N scattering, which was done in a nonrelativistic formalism. The relevance of transversity amplitudes to this problem seems to have been first recognized by Barker, Donnachie and Storrow (ref.15). Most of the subsequent work on the completeness problem was done by Moravcsik and collaborators (see ref.16, where one can find further references to their older publications, as well as to the work of others). The present work (ref.1) on complete analysis of scalar and pseudo-scalar electroproduction was only the second (after ref.15) work of that kind in electro-nuclear physics. Ref.15 is concerned only with photoproduction.

The first application of the helicity formalism to inelastic, coincidence, electron deuteron scattering was made by Renard, Tran Thanh Van and LeBellac (ref.17), who used in the cm frame. The result of the boost of electron variables to the lab frame was first explicitly written down by Walecka and Zucker (ref.18) (see this paper for further references). An early, nonrelativistic, study of polarized coincidence, inelastic electron scattering from the deuteron was done by Fabian and Arenhövel (ref.19), but it suffered from several shortcomings (see ref.2).

A different, but related, approach to polarized, coincidence, inelastic electron scattering was pioneered by Donnelly and Raskin (ref.20). They use general arguments, such as conservation of angular momentum, in order to expand the transition amplitudes in terms of multipoles and then write the cross section as a function of the latter. While this approach is closer to the traditional methods of nonrelativistic nuclear physics it is much more complicated than the helicity amplitude method and it does not allow a simple separation analysis.

Finally, a relativistic formalism for the description of recoil polarization, which can be extended to describe polarized target reactions as well, in coincidence, inelastic, electron scattering based on the most general expansion of the response tensor, much in the vein of de Forest's (ref.7) unpolarized analysis, was developed by Picklesimer and van Orden (ref.9). The final state channels are not specified in this method, so that the separation of amplitudes is not possible.
1.3 Parity Violating Elastic Electron Scattering

In the second part of this thesis I concern myself with a certain class of corrections to the parity-violating, elastic, electron-nucleon scattering.

In order to understand the significance of this contribution let us remember that elastic parity-violating electron-nucleon scattering is an example of the so-called weak neutral current processes. This means that there is no charge transfer from one particle to another, typical of other weak processes. Such neutral current processes are a specific signature of the unified electro-weak model and their experimental discovery was one of the great triumphs of the Standard Model.

Every model within bounds of QFT, such as the Standard Model, has a certain number of free parameters which have to be determined from experiment. Such parameters are usually the masses of fundamental particles and the strengths of their couplings to mediating particles, known as coupling constants. There are two such fundamental coupling constants in the electro-weak part of the Standard Model: the electric charge and the mixing, or Weinberg, angle of the weak hypercharge gauge field and the third component of the weak isospin gauge field.

One of the nonintuitive aspects of renormalizable QFTs is their property of coupling constant dependence on the distance scale, or momentum transfer. Such theories predict a definite pattern of coupling constant change, usually referred to as the evolution, as a function of momentum transfer, which can be experimentally checked.

The value of the electric charge, at low energies can be determined from low energy Compton scattering experiments. The Standard Model's mixing angle is presently best determined from the experimental values of the weak intermediate boson masses. But, those experiments were done at extremely high energies (approximately $100 \times$ higher than those achievable at CEBAF), so that the mixing angle at low energies may appreciably deviate from its high energy value. An independent measurement of the mixing angle at low energies would provide an interesting test of the Standard Model.

Elastic parity-violating electron-nucleon scattering measurement at CEBAF four-momentum transfers, constitutes such an experimental test of the Standard Model which, if sufficiently precise, provides an independent source of one of its two fundamental coupling constants: the mixing angle $\theta_W$. Secondly, it provides a measure of the weak neutral current distributions in the nuclear system which is as important as the electromagnetic current distribution. Nucleons are believed to be built from three permanent, or valence, quarks and an indefinite number of virtual quark-antiquark pairs. These pairs may be of the same kind, or "flavour", as the valence quarks in the nucleon (the $u, d$ quarks), or some other, heavier kind,
such as the strange \( s \), charm \( c \) etc.. The weak neutral current interactions are sensitive to such flavour components in the nucleon wave function and thus allow, at least in principle, their experimental separation. The extracted quantities are functions of flavour current matrix elements of the nucleon, which can be related to the heavy flavour quark-antiquark pair content of the nucleon. At this moment it is believed that of all heavy quarks, only the strange might play an important role in the nucleon. All elastic current matrix elements can be parametrized in terms of functions called form factors. Extraction of such “strange” form factors, from parity violating electron scattering data is discussed below.

In order to observe parity violation in electron scattering, one needs at least one polarized particle. Since polarization of the electrons is much easier to maintain at high beam currents than that of the target, we will be concerned only with that case. A direct measure of the parity-violating part of elastic electron scattering is the ratio of the difference and the sum of the elastic cross sections for electrons polarized parallel and anti-parallel to the beam. This ratio is called the elastic asymmetry:

\[
A = \frac{\sigma_\uparrow - \sigma_\downarrow}{\sigma_\uparrow + \sigma_\downarrow}
\]

Elastic scattering electromagnetic current matrix elements are functions of only one Lorentz invariant variable: the negative four-momentum transfer squared \( Q^2 \). This quantity is a function of the initial electron energy and the electron scattering angle. But, the elastic electron scattering cross section can be a function of two independent variables: \( Q^2 \) and the electron scattering angle. There are infinitely many ways of attaining a certain value of \( Q^2 \): one can continuously vary the initial electron energy and the scattering angle, while keeping \( Q^2 \) fixed. Such a dependence on two independent variables allows a procedure, known as the Rosenbluth separation, which enables the extraction of several Lorentz invariant parts of the elastic cross section, which are just the elastic response functions from section 1.2, by varying the electron scattering angle, at a fixed value of \( Q^2 \). These response functions depend on the “strange” form factors, and thus enable their separation from the experiment.
Separation of Strange Form Factors

Three targets are considered: $^4\text{He}$, proton $p$ and neutron $n$. The assumptions which enter all of the analyses in this section are: Lorentz invariance, field theory, Standard Model, significance of $u$, $d$, $s$ quarks in $p$, $n$, good parity of states, good isospin of states and the one boson exchange approximation.

$^4\text{He}$ is a particularly interesting target because it is scalar and iso-scalar. As a consequence, all iso-vector and all axial and magnetic currents in the problem vanish and only two independent form factors suffice for the full description of its weak neutral current (WNC): the electromagnetic (EM) charge $G_E = F_1^T$ and the "strange" electric $S_E$ form factors of $^4\text{He}$ (all "strange" form factors will be denoted by capital S with appropriate subscript).

The resulting asymmetry, when expanded in powers of $Q^2$, looks like this (ref. 21):

$$A = \frac{G_F Q^2}{4\pi \alpha \sqrt{2}} \left[ 4\sin^2 \theta_W - \frac{\langle r^2 \rangle}{3} Q^2 \right] + O(Q^6)$$

where $Q^2 = -q^2 = -\left( k - k' \right)^2$ is the negative of the four-momentum transfer squared, $G_F \approx 10^{-5} M^{-2}$ is the Fermi weak interaction constant, $\langle r^2 \rangle$ is the "strangeness radius" of the nucleon (or $\langle r^2 \rangle$ of strangeness radius of $^4\text{He}$) and $\theta_W$ is the Weinberg angle. In the derivation of this formula I have used the fact that the strange electric form factor vanishes at $Q^2 = 0$ to lowest order in the weak coupling constant, which is a reflection of the fact that the net strangeness of helium is zero.

In the case of the nucleon, all components of the weak neutral current are present: electric, magnetic, axial; iso-scalar and iso-vector. This allows the asymmetry to take its most complicated form: it contains a longitudinal (electric), a parity conserving (PC) transverse (magnetic current) and a parity-violating (PV) transverse (axial current) term. The nucleon forms an iso-doublet, which means that there are two kinds of the nucleon differing only by the electric charge: proton and neutron.

There are three kinds of contributions to the nucleon asymmetry: the electric (E) and magnetic (M) form factors and an axial one in this problem. I use a mixed notation: two of the three form factors will correspond to the overall (Sachs EM or weak) form factors of the two nucleons, such as $G_{E,M}^{p,n}$, $F_A^{p,n}$ (see text for details), whereas the third one will describe exclusively the strange quark contributions and will be denoted by $S_{E,M,A}$ (which are isoscalar in the Standard Model with exact isospin symmetry of states). Then, the proton asymmetry has the following form
(ref.21):

\[ A^p = \left( \frac{G_F Q^2}{4\pi \alpha \sqrt{2}} \right) \left\{ - (1 - 4\sin^2 \theta_W) + \right. \\
+ \left[ \epsilon G_E^p (G_E^p + S_E) + \tau G_M^p (G_M^p + S_M) \right. \\
+ \sqrt{(1 - \epsilon^2)\tau (1 + \tau) G_M^p (F_A^p - F_A^p - S_A)(1 - 4\sin^2 \theta_W)} \right\} \]

\[ \times \{d\sigma^\dagger + d\sigma^\dagger\} \]

where

\[ [d\sigma^\dagger + d\sigma^\dagger] = \epsilon (G_E^p)^2 + \tau (G_M^p)^2 \quad \frac{1}{\epsilon} = 1 + 2(1 + \tau) t g^2 \theta \\
\tau = \frac{Q^2}{4M^2} \quad Q^2 = -q^2 = -(k - k')^2 \geq 0 \]

where \( M \) is the nucleon mass. The asymmetry now depends on two kinematic variables: the negative four-momentum transfer squared \( Q^2 \) and the virtual photon polarization \( \epsilon \) defined above.

A Rosenbluth separation can be made by varying \( \epsilon \) while keeping \( Q^2 \) fixed. This allows us to separate three independent quantities:

- term proportional to \( \epsilon \)
- term proportional to \( \sqrt{(1 - \epsilon^2)\tau (1 + \tau)} \)
- terms independent of \( \epsilon \)

Notice that all but one term: \(-(1 - 4\sin^2 \theta_W)\), within the curly brackets in eq.(2) vanish in the \( Q^2 \to 0 \) limit. This fact allows us to separate four quantities from \( A^p, G_E^p, G_M^p \), if we assume that Weinberg angle remains constant in the measured region of \( Q^2 \):

\[ 1 - 4\sin^2 \theta_W \text{ at } Q^2 = 0 \quad G_E^p + S_E \]
\[ G_M^p + S_M \quad F_A^p - F_A^p - S_A \]

No further information about the "strange" form factors can be obtained without a knowledge of neutron EM and axial iso-vector form factors. The neutron magnetic form factor is fairly well known, as is the axial iso-vector one, which allows the extraction of the strange magnetic and axial form factors. But the neutron electric form factor is very poorly known so we cannot learn anything about the "strange"
electric form factor from this experiment. One can use the $^4He$ experiment as a source of information about $S_E$ at low $Q^2$, which would complete the separation of strange form factors (at least in this kinematic regime).

Free neutrons are unstable under $\beta$ decay, so the deuteron is the simplest source of target neutrons. I assume that the deuteron structure can be taken out* of this problem. Then a completely analogous expression to eq.(2) can be written for the neutron asymmetry: the only differences are the sign of the $1 - 4\sin^2\theta_W$ term up front and the exchange of all proton and neutron form factors $p \leftrightarrow n$. Similarly to the proton case, we can separate four independent quantities. Assuming knowledge of $A^n, G^n_{E,M}$ we can get:

$$1 - 4\sin^2\theta_W \text{ at } Q^2 = 0 , \quad G^n_E(G^n_E + S_E)$$

$$G^n_M + S_M , \quad F^n_A - F^n_P - S_A$$

We see that once again we cannot separate $S_E$ because of our ignorance of $G^n_E$. But if we know both asymmetries and the three EM form factors (see Table 3), we can make a complete separation of all other form factors (extraction of $S_E$ is non-linear).

<table>
<thead>
<tr>
<th>$A^p \quad G^p_M$</th>
<th>$A^n \quad G^n_{E,M}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - 4\sin^2\theta_W$</td>
<td>$1 - 4\sin^2\theta_W$</td>
</tr>
<tr>
<td>$G^n_E + S_E$</td>
<td>$G^n_E(G^n_E + S_E)$</td>
</tr>
<tr>
<td>$G^n_M + S_M$</td>
<td>$G^n_M + S_M$</td>
</tr>
<tr>
<td>$F^n_A - \frac{1}{2}S_A$</td>
<td>$F^n_A + \frac{1}{2}S_A$</td>
</tr>
</tbody>
</table>

Table 3 Summary of separable quantities. The ingredients necessary for the extraction are listed at the top of each column.

Thus, by measuring both asymmetries, one obtains the six quantities displayed in the box below. Among them are a low energy value of the fundamental coupling constant $\sin^2\theta_W$ in the semileptonic (quark-lepton) sector and all three "strange" form factors of the nucleon.

* I intend to work on a relativistic calculation of the contribution of the internal structure of the deuteron to deep inelastic electron scattering processes in the future
One can use the measured $F_A^V$ as a double-check of the experiment, and, as a bonus, one gets the neutron electric form factor which was one of the early motivations for proposing these experiments (ref.22). All of these results were obtained independently of ref.21.

All of the above made statements about the separability of the Weinberg angle and the "strangeness" content of the nucleon from the inelastic parity violating electron-nucleon scattering data, are true only if certain "higher order" corrections to this cross section are small and calculable. There are three basic kinds of such corrections:

- parity violating electromagnetic current matrix element of the nucleon
- isospin violating admixture in the nucleon
- Standard Model radiative corrections

The third item on this list has already been treated in ref.23, but the contribution of the hadronic structure has not been explored in detail, whereas the second topic is completely unknown. In order to understand the origin of the first item on the list, let us remember that the electromagnetic current matrix elements of parity conserving systems are parity conserving themselves.

But, if there is a source of parity violation within the system, then there will be a parity violating electromagnetic current matrix element. In a nucleon, such a source of parity violation is the exchange of $W^\pm$, $Z^0$ bosons between the quarks, which is guaranteed to exist by the Standard Model. Consequently, the nucleon will have a nonvanishing parity violating electromagnetic current matrix element which will contribute to the elastic parity-violating electron-nucleon asymmetry in a way which is very similar to that of the weak neutral current. The order of magnitude of such a contribution is determined by the one scale common to all weak processes, the Fermi weak coupling constant $G_F$, which means that this contribution can be comparable to that of the weak neutral current. Nothing more can be said about this term without a detailed model calculation. Clearly, a large contribution of this kind can make the interpretation of the parity-violating elastic electron-nucleon
scattering very difficult. In the following I will be concerned with the evaluation of this term in a quark model of the nucleon.

Firstly, the relativistic parity-violating quark-quark potential due to one-weak boson-exchange is constructed, then it is reduced to a nonrelativistic form, to be used in the nonrelativistic quark model. Within the framework of asymptotically free QCD, we at least start with the correct S-matrix at short distances. Then the fact that the weak interactions present a very small perturbation of the hadrons is used in order to evaluate the parity-violating admixture to the nucleon wave function. The nonrelativistic perturbation theory is applied, where the nonrelativistic parity-violating potential is the perturbation Hamiltonian, and the lowest lying, observed negative parity nucleon resonances are used as the lowest excited states of the system.

The parity-violating electromagnetic current matrix element of the nucleon is calculated in the approximation of nonrelativistic electromagnetic interaction of point-like quarks and with the lowest lying excited state parity admixture. It is found, however that the threshold behaviour in this calculation does not satisfy a well-established theorem (ref.26). The assumptions of the theorem are checked and one of them is found not to be satisfied in this calculation. This assumption is the conservation of the electromagnetic current. The conservation of the electromagnetic current is restored by adding to its conventional, one-body, part another, two-body, exchange term. A repeated calculation of the parity-violating electromagnetic current matrix element of the nucleon is carried out in the same approximation as stated above. The theorem about the threshold behaviour is still violated.

A procedure known as closure is applied to investigate the role of all negative parity excited states of the nucleon. In the closure calculation, the theorem is satisfied. The moral of the story is that no model retaining only a finite number of admixed states can give the correct parity-violating elastic electromagnetic current matrix element.

All of the above described considerations lead to a deeper understanding of the general structure of the parity violating electromagnetic current matrix element of an arbitrary system with nonvanishing spin: certain parts of the conventional definition of the elastic parity-violating electromagnetic current matrix element are shown to vanish identically. The remaining part, which is the new definition, satisfies the threshold theorem automatically and hence can be used consistently in any approximation.

The new definition of the elastic parity-violating electromagnetic current matrix element is applied to the nucleon in two cases: one, where the parity-violating admixture to the wave function is described by the first excited state alone and
second, where the exact value of the contribution of all excited states is established, using closure. The elastic parity-violating electromagnetic current matrix element is shown to vanish identically in this non-relativistic quark model if all possible admixed states are retained in the calculation. Hence, the relative correction to the weak neutral current asymmetry due to the parity-violating electromagnetic current matrix element of the nucleon vanishes, as well. This second result is shown to hold for arbitrary spin and iso-spin independent local strong interaction potentials between the quarks (where SU(6) symmetry is valid for equal mass quarks). The extension to situations with more complicated quark-quark potentials is an open question.

**Parity Of The Nucleon Wave Function In The Quark Model**

The quark model picture of the nucleon is one of a three quark bound state. The binding is supposedly due to gluon exchange, which is parity conserving, but also not exactly calculable. This fact explains the existence of a large number of quark models. Different models have substantially different wave functions. In most conventional models the ground state (nucleon) is an S state, with, possibly, a certain amount of D wave. The Standard Model allows exchange of W and Z intermediate vector bosons, between quarks, which is parity violating. These interactions, although small, induce a finite opposite parity admixture i.e. P waves, in the nucleon wave function. Besides the two-body potentials, there are two- and three-body forces induced by the W,Z exchange, but they, too, will be neglected, because they are higher order in $G_F$. We will confine ourselves to $u$ and $d$ quarks, since they are the valence quarks in the nucleon and we will work in a simple potential model. The relativistic parity violating one-boson-exchange potential derived from the one W and Z exchange Feynman diagrams, with the assumption that the momentum transfer is much less than the intermediate boson mass, is:

$$V^{PV}(x_1 - x_2) = \frac{-G_F}{\sqrt{2}} \delta(x_1 - x_2) \left[ \mathcal{V}^{+}_{12} I^{+}_{12} + \mathcal{V}^{-}_{12} I^{-}_{12} \right]$$

$$\mathcal{V}^{+}_{12} = (\bar{u}\gamma_\mu \gamma_5 u)^1(\bar{u}\gamma_\mu u)^2 + (1 \leftrightarrow 2)$$

$$\mathcal{V}^{-}_{12} = (\bar{u}\gamma_\mu \gamma_5 u)^1(\bar{u}\gamma_\mu u)^2 - (1 \leftrightarrow 2)$$

$$I^{+}_{12} = \frac{\cos^2 \theta_C}{2} [\tau^+_1 \tau^+_2 + \tau^-_1 \tau^-_2] + \left( \frac{1}{4} - \frac{\sin^2 \theta_W}{2}\right) \tau^+_1 \tau^-_2 - \frac{\sin^2 \theta_W}{12} (\tau^+_1 + \tau^-_1)$$

$$I^{-}_{12} = -\frac{\sin^2 \theta_W}{12} (\tau^+_1 - \tau^-_1)$$

where the potential has been broken up into spin-spatial ($\mathcal{V}^{+}_{12}$) and isospin ($I^{+}_{12}$)
parts, and where ± denotes symmetry under the interchange of indices \((1 \leftrightarrow 2)^*\). We will be working with the non-relativistic quark model because it is the simplest quark model which makes reasonable predictions for the negative parity resonances and it allows an exact separation of the centre of mass motion from the wave functions. This latter fact is very important for a reliable estimate of the electromagnetic transition matrix elements. So, one makes a non-relativistic reduction of the spin-spatial part of the potential and gets:

\[
\begin{align*}
\mathcal{V}^+_{12} &= \left( \frac{1}{2m_q} \right) \left( (\vec{\sigma}_1 - \vec{\sigma}_2) \cdot \left\{ \vec{p}_1 - \vec{p}_2, \delta(\vec{r}_1 - \vec{r}_2) \right\} + i(\vec{\sigma}_1 \times \vec{\sigma}_2) \cdot \left[ \vec{p}_1 - \vec{p}_2, \delta(\vec{r}_1 - \vec{r}_2) \right]\right) \\
\mathcal{V}^-_{12} &= \left( \frac{1}{2m_q} \right) (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \left\{ \vec{p}_1 - \vec{p}_2, \delta(\vec{r}_1 - \vec{r}_2) \right\}
\end{align*}
\]

where the bracketed expressions are the anti-commutators and commutators, respectively. Instead of solving the Schrödinger, or Dirac, equation with this potential, we use first order perturbation theory to evaluate the \(P\) wave content of the nucleon. For this, we need the lowest lying, negative parity, excited states of the nucleon, i.e. the lowest mass \(P\)-wave nucleon resonances. They are (ref.24):

\[
\begin{align*}
\text{N}(1535) \ (J^P = \frac{1^-}{2}), \ SU(6): & \ 70 \text{-plet} \ 28 \\
\text{N}(1650) \ (J^P = \frac{1^-}{2}), \ SU(6): & \ 70 \text{-plet} \ 48 \\
\Delta(1620) \ (J^P = \frac{1^-}{2}), \ SU(6): & \ 70 \text{-plet} \ 210
\end{align*}
\]

Hence the "parity admixed" nucleon state is a superposition of the \(S\)-wave ground state and the \(P\)-wave resonances:

\[
|N\rangle = \left[ \frac{1^+}{2} \ |8^2\rangle + \epsilon_1 \frac{1^-}{2} \ |8^4\rangle + \epsilon_2 \frac{1^-}{2} \ |8^4\rangle + \epsilon_3 \frac{1^-}{2} \ |10^2\rangle \right]
\]

where the parity-violating Hamiltonian

\[
\mathcal{V}^{PV} = \sum_{i<j} \mathcal{V}^{PV}_{ij}
\]

is the pair-wise sum of the parity-violating quark-quark potentials in the nucleon.

\* Note that I inadvertently omitted the \(\mathcal{V}_{12}I_{12}\) term from the potential in (ref.25)
and

\[ \epsilon_1 = \frac{\langle \frac{1}{2} - 8^2 | V^{PV} | \frac{1}{2} + 8^2 \rangle}{E_+ - E'_-} \]
\[ \epsilon_2 = \frac{\langle \frac{1}{2} - 8^4 | V^{PV} | \frac{1}{2} + 8^2 \rangle}{E_+ - E''_-} \]
\[ \epsilon_3 = \frac{\langle \frac{1}{2} - 10^2 | V^{PV} | \frac{1}{2} + 8^2 \rangle}{E_+ - E'''_-} \]

where the final states are the above specified members of the SU(6), negative parity, 70-plet and the initial states are nucleons, or positive parity 56-plet in SU(6). This expansion is valid in both the nonrelativistic and relativistic perturbation theory. The higher mass resonance contributions are, naively, expected to be small due to the increasing energy denominator.*

Now one can calculate these matrix elements and get the analytic expressions for the parity admixtures:

\[ \epsilon_i = \frac{-i}{(E_+ - E'_-)} \left( \frac{G_F I}{m_q R} \right) \left\{ \frac{3}{2} \left[ \cos^2 \theta_C + \frac{1}{2} \left( 1 - 2 \sin^2 \theta_W \right) \right] - \frac{\sin^2 \theta_W}{3\sqrt{2}} \tau_3^N \right\} \]

\[ \epsilon_2 = \frac{2i}{(E_+ - E''_-)} \left( \frac{G_F I}{m_q R} \right) \frac{3}{2} \left[ \cos^2 \theta_C + \frac{1}{2} \left( 1 - 2 \sin^2 \theta_W \right) \right] \]

\[ \epsilon_3 = 0 \]

where

\[ I = \left( \frac{1}{\sqrt{2\pi R}} \right)^3 \]
\[ R^{-2} = m_q \omega \]

where \( \omega \) is the oscillator frequency of the model, which is determined by the charge radius of the proton \( (R = .86 fm) \), and the constituent quark mass is approximately one third of the nucleon's, \( m_q = 340 MeV \).

\[ E_+ = 940 MeV \quad E'_- = 1535 MeV \quad E''_- = 1650 MeV \]

are the masses of the lowest negative parity nucleon resonances. After substitution of the empirical values for \( \cos \theta_C, \sin \theta_W \), we get:

\[ \epsilon_1 = i \left( 1.8 - 0.05 \tau_3^N \right) \times 10^{-8} \]
\[ \epsilon_2 = -i \ 3.0 \times 10^{-8} \]

where \( \tau_3^N \) is the third Pauli matrix operating in the nucleon isospin space.

---

* An obvious improvement on the present non-relativistic calculation would be to use a relativistic quark model, such as the MIT bag model, and not to make the non-relativistic reduction
These results for the parity-violating admixture in the nucleon wave function are a new contribution. But, they have to be taken with a grain of salt because, as is shown, there are substantial contributions to the elastic parity-violating electromagnetic current matrix element, arising from all other high-lying negative parity admixed states in the nucleon. As a matter of fact, one of the more important results of this work is the understanding that one must not abbreviate the perturbation expansion when dealing with elastic parity-violating electromagnetic current matrix elements. It is not even clear if the above given parity admixtures can be a reliable approximation to the exact parity admixture in the evaluation of any parity-violating observable.

Parity-Violating Electromagnetic Current Matrix Element Of The Nucleon In The Non-Relativistic Quark Model

These negative parity pieces in the wave function will generate a parity-violating electromagnetic nucleon current, also known as the anapole. There is only one parity-violating, time-reversal conserving, elastic, conserved electromagnetic current matrix element for spin $1/2$ particles. Its relativistic form is:

$$\langle p' | J_{5\mu}^{EM}(0) | p \rangle = \frac{eG_F}{\sqrt{2}} Q^2 H(Q^2) \bar{u}(p') \left( \gamma_\mu - \frac{i}{q^2} q_\mu \right) \gamma_5 u(p)$$ (3)

The calculation of the form factor $H(q^2)$ is an unexplored subject. I am aware of only a few (ref.27,28) studies of such quantities in atomic nuclei and none at all in nucleons, as long as they are treated as bound states of elementary particles. Consequently, the understanding of model independent features of such matrix elements is scarce.

In the process of calculation I derive two new results concerning the general structure of parity-violating electromagnetic current matrix elements of bound states: the first is a general result about the cancellation of one- and two-body current contributions to the parity-violating electromagnetic current matrix element at the threshold ($q^2 = 0$), which leads to a new definition of the parity-violating electromagnetic current matrix element at nonvanishing values of $q^2$, as mentioned before. This holds for any non-relativistic parity-violating bound state system and sheds some new light on the general structure of parity-violating electromagnetic matrix elements.

Secondly, the elastic parity-violating electromagnetic current matrix element of the nucleon is shown to vanish identically, to first order in $G_F$, in the non-relativistic quark model with contact parity-violating interactions and spin and iso-spin independent, local strong interactions. An exact, closed form expression for the contribution of all excited states of the strongly interacting system to the
one—body parity-violating electromagnetic current matrix element is given, within first order non-relativistic perturbation theory. This exact one—body current matrix element turns out to be the ground state expectation value of a commutator of the parity-violating potential and \(-i\) times the charge dipole operator of the system. This commutator is, at least for the contact interaction considered here, exactly the negative of the two-body current contribution and the total operator whose expectation value is being taken vanishes identically. This result is not an obvious consequence of some deeper symmetry principle, the way the first result was, and it is not even clear to which extent it may be broken by more complicated strong-interaction quark-quark potentials or by relativistic corrections.

The one-body electromagnetic current in this model is the usual non-relativistic expression for the interaction with point-like quarks. The electromagnetic current of all interacting non-relativistic quantum systems, such as this quark model, with a parity-violating interaction derived from the exchange of spin zero and/or spin one bosons, is conserved only if there is a two-body current besides the familiar one-body current. The two-body current appropriate to the two-quark parity violating potential is constructed:

\[
\mathbf{J}^{PV}_{2-\text{b}} = \sum_{i<j} \mathbf{J}^{PV}_{2-\text{b}}(ij)
\]

where

\[
\mathbf{J}^{PV}_{2-\text{b}}(\vec{x}_1, \vec{x}_2, \vec{r}) = \frac{eG_F}{\sqrt{2}} \delta(\vec{r} - \vec{x}_1)\delta(\vec{x}_1 - \vec{x}_2) \left[ \vec{S}_{12} \cdot \vec{r}_{12}^* \right]
\]

and

\[
\vec{S}_{12}^1 = \frac{1}{2m_q} (\vec{\sigma}_1 - \vec{\sigma}_2), \quad I_{12}^1 = \frac{1}{2} (\frac{1}{2} - \sin^2 \theta_W) (\tau_1^1 - \tau_3^2)
\]

\[
\vec{S}_{12}^2 = \frac{1}{2m_q} (\vec{\sigma}_1 + \vec{\sigma}_2), \quad I_{12}^2 = \frac{\sin^2 \theta_W}{6} (1 - \tau_3^1 \tau_3^2)
\]

\[
\vec{S}_{12}^3 = \frac{1}{2m_q} i(\vec{\sigma}_1 \times \vec{\sigma}_2), \quad I_{12}^3 = \frac{i}{2} \cos^2 \theta_C (\vec{\tau}_1 \times \vec{\tau}_2)_3
\]

and is shown to restore the conservation of the total electromagnetic current.

Parity violation appears in the form of parity-violating admixtures in the wave functions and the parity-violating current operator. We work with the exact states and the exact current operator which satisfies the non-relativistic current conservation relation:

\[
\vec{\nabla} \cdot \mathbf{J} = -i [\mathbf{H}, \rho]
\]

, to first order in \(G_F\), where the total Hamiltonian \(\mathbf{H}\) is the sum of the strong, parity-conserving \(\mathbf{H}_0\) and the parity-violating potential \(V_{PV}\).
The transverse electromagnetic current matrix element can be written as a finite sum of transverse multipoles. For parity-violating elastic scattering on spin \(1/2\) particles, discrete symmetries and conservation of angular momentum reduce this sum to one term: the transverse electric dipole. It can be proven, using only current conservation, that this term must vanish at least as \(q^2\) in the long-wavelength limit (ref.26). But, in an explicit calculation with the above given wave function admixtures and a conserved electromagnetic current, this theorem is not satisfied.

As discussed previously, the resolution to this puzzle lies in the fact that an approximation was made when only the first excited state was used in the evaluation of the parity-violating admixture to the wave function. The problematic term in the electromagnetic current matrix element vanishes identically, and the correct threshold behaviour is restored, only if all excited states are kept in the perturbation theoretic expansion. An explicit proof of this statement is provided to first order in \(G_F\). The proof rests on the use of "closure", a procedure where one sums over all intermediate states. This provides a cancellation, at the threshold, of the one-body current term by the two-body current term. These arguments are valid for parity-violating elastic scattering amplitudes of particles with arbitrary spin, larger than 1/2.

The above consideration provides a specific example of a more general mechanism at work: the correct threshold behaviour is preserved to all orders of perturbation theory due to fundamental symmetries of the interaction. One term is proportional to the time derivative of the Coulomb dipole moment, which is known to exist only if both parity and time-reversal are simultaneously violated. In our case time-reversal is conserved so that the term has to vanish identically.

This leads us to a new, general and relativistic definition of the parity-violating electromagnetic current matrix element of any spin \(1/2\) system, as advertised:

\[
\mathcal{J}^5 \cdot \mathcal{I} = q \sqrt{3\pi} \int d\bar{r} (\psi^*_j \mathcal{F} \psi_i) \cdot \bar{r} j_1(qr) Y_{11} (\bar{r})
\]

where \(q = |q|\), \(\psi_j^* \mathcal{F} \psi_i\) is the conserved, elastic parity-violating electromagnetic current matrix element, \(j_1\) is a spherical Bessel function, \(\mathcal{I} = \frac{\alpha}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y)\) and \(Y_{11}\) is a spherical harmonic. Note that this definition satisfies the threshold theorem by construction and in this regard it is an improvement on the transverse electric dipole moment definition of ref.26. Insertion of \(\mathcal{J}^5\), the spatial part of the four-current defined in eq.(3), into this expression allows one to identify and hence calculate \(H(Q^2)\) within the framework of non-relativistic bound state model of a spin 1/2 system. This is the first general result mentioned above.
The above shown formula may seem non-relativistic due to the presence of three-momentum transfer and a three-dimensional scalar product, but if it is interpreted as a definition in a special inertial frame of reference (the so-called Breit frame, where the four-momentum transfer retains only its three-vector part) it is completely invariant. This is the first exact, relativistic definition of the elastic parity-violating electromagnetic current matrix element of composite spin 1/2 systems at arbitrary values of four-momentum transfer with the correct long-wave-length limit behaviour. It coincides, at $Q^2 = 0$ with the definition of the static parity-violating electromagnetic current moment, known as the "anapole", given by Flambaum and Khriplovich (ref.28,29).

![Graph](image)

Fig.4 Absolute value of the parity violating electromagnetic form factor $H$ of the proton, calculated as described in the text, as a function of $\tau = (Q/2M)^2$, where $Q^2$ was taken to be equal to $\bar{q}^2$. Absolute value of the one-body current, lowest excited state admixture, contribution (solid), and the negative of the two-body current ground state matrix element (dashes). The total result is the sum of the two contributions.

In Fig.4 we show the result of admixing the lowest-lying negative parity excited state of the nucleon and retaining the one- and the two-body electromagnetic
current. The threshold theorem is satisfied, just by using our new definition of the elastic parity-violating electromagnetic current matrix element. The result is evidently non-zero. The present calculation is the first calculation of the $q^2$ dependence of the parity-violating electromagnetic form factor of the nucleon (or indeed, to the author's knowledge, of any system at all). We discover (Fig. 4) that this form factor can vanish at a nonzero value of $q^2$, since it is a Fourier transform of a product of an $S$ and a $P$ wave function. This is the first example of an elastic nucleon form factor which could change sign.

An approximate closure method, where the smallest energy denominator is factored out in front of the sum and the completeness of the intermediate states is used, was then applied to estimate the size of all $P$-wave excited state contributions to the one-body current part of the parity-violating electromagnetic form factor, in this specific, nonrelativistic model of the nucleon. This approximate closure estimate, in the given simple harmonic oscillator model, turns out to be the negative value of the ground state expectation value of the two-body current contribution. The parity-violating form factor due to the two-body current is:

$$H_{2-b}(Q^2) = \left[ (1 - 2\sin^2\theta_W) \gamma_3^N + \frac{2}{9} \sin^2\theta_W \right] \times$$

$$\times \sqrt{\frac{3\pi \omega}{8m_q}} \left( \frac{1}{4\pi} \right)^2 \exp \left[ -\frac{q^2}{24m_q\omega} \right]$$

The complete result which equals the sum of the closure estimate of the one-body and the exact two-body current contributions, is zero in this approximation with simple harmonic oscillator wave functions. This is the first indication of the second general result mentioned above.

Observe on Fig. 4 that numerically the absolute value of the sum of all excited state one-body current contributions, which equals the two-body current contribution, is substantially ($\approx 20$ times) smaller than that of the lowest-lying excited state contribution. Thus, we see that there is a strong cancellation among the high-lying excited state contributions to the one-body current part of the parity-violating electromagnetic current matrix element. We would have deceived ourselves if we had accepted the first excited state contribution as the leading term with "small" corrections; the higher excited states play an important role and add up to exactly cancel the two-body current contribution.

Finally, an exact calculation of the contribution of all the high-lying negative parity admixtures can be carried out in this non-relativistic quark model. The proof uses methods similar to those used in the proof of the threshold behaviour of elastic parity-violating electromagnetic current matrix elements of an arbitrary
non-relativistic system bound by spin and iso-spin independent two-body potentials. The elastic matrix element of the parity-violating electromagnetic current can be written as the sum of the one- and the two-body current contributions. The Heisenberg equation of motion for the electric dipole operator \( \tilde{d} = \sum_i e_i \delta (\vec{r} - \vec{r}_i) \vec{r}_i \) due to the unperturbed Hamiltonian \( H_0 \):

\[
\tilde{J}_{1-b} = i \left[ H_0, \tilde{d} \right]
\]

allows us to eliminate the energy denominators appearing in the parity admixtures to the wave functions, and to subsequently use the completeness of the eigenstates of the strong-interaction Hamiltonian to obtain the following exact result:

\[
\tilde{J}_{b} \cdot \hat{e}_+ = q \sqrt{3\pi} \int d\vec{r} j_1 (gr) Y_{11}(\hat{r}) \left( \vec{r} \cdot \langle 0 | \tilde{J}^{PV}_{2-b} - i \left[ V_{PV}, \tilde{d} \right] | 0 \rangle \right)
\]

This formula provides an exact expression for the elastic parity-violating electromagnetic current matrix element in terms of the ground state expectation value of the difference between the two-body parity-violating electromagnetic current operator and \( i \) times the commutator of the parity-violating potential and the charge dipole operator.

This result completely eliminates the need to calculate the parity admixtures to the wave functions and it clearly shows the independence of the result on the details of the excited state spectrum, such as the proximity of the lowest-lying excited P-wave state to the ground state, which used to be considered one of the most important factors determining the size of the one-body current contribution to the elastic parity-violating electromagnetic current matrix element.

Khriplovich (ref.28) made the observation that the parity-violating two-body current, in certain models, can be written as:

\[
\tilde{J}^{PV}_{2-b} = i \left[ V_{PV}, \tilde{d} \right]
\]

This relation is a “complement” of the previously used Heisenberg equation of motion with the unperturbed Hamiltonian \( H_0 \), to the similar equation for the complete current \( \tilde{J} = \tilde{J}_{1-b} + \tilde{J}^{PV}_{2-b} \):

\[
\tilde{J} = \tilde{d} = i \left[ H, \tilde{d} \right]
\]

which is just the time derivative of the electric dipole operator \( \tilde{d} \). In the case of contact parity-violating interactions, such as those employed in the full quark model

\[
\text{Recall that if } \tilde{d} = \sum_i e_i \delta (\vec{r} - \vec{r}_i) \vec{r}_i, \text{ then } \dot{\tilde{d}} = i \left[ H_0, \tilde{d} \right] = \sum_i e_i \delta (\vec{r} - \vec{r}_i) \vec{r}_i \text{ is just the corresponding electromagnetic current with spin, iso-spin and momentum independent two-body potentials in } H_0.
\]
of this thesis, this relation is exactly satisfied and, consequently the elastic parity-violating electromagnetic current matrix element vanishes identically. This result holds to first order in $G_F$. The validity of the above operator equation seems to be a general feature of all models with parity-violating potentials derived by non-relativistic reduction of parity-violating, heavy boson exchange potentials to the first order in $\mathcal{E}_m$, where the exchanged bosons are spin zero or spin one.

This very powerful, and even more surprising, result has the immediate consequence of raising questions about previous calculations of the nuclear elastic parity-violating electromagnetic current matrix elements retaining only a truncated space of negative parity admixtures.

The main conclusions of this work are: a) the elastic parity-violating electromagnetic current matrix element of the nucleon in the non-relativistic quark model vanishes to first order in $G_F$, and b) the elastic electron-proton asymmetry, and the accuracy of the extraction of the "strange" axial form factor from it, are not affected by parity admixture contributions, to first order in $G_F$.

There are still Standard Model radiative corrections of order $\alpha G_F$, which are expected to be of the order of 1%, and there are also the, as yet unknown, isospin admixture contributions.

The general result about the long-wave-length limit of the parity-violating electromagnetic current matrix element has been known for some time (ref.26). The argument leading to this result is based on the well known type of argument used in parity-conserving electro-nuclear physics which goes under the name of Siegert's theorem. But, the explicit implementation in a perturbative calculation is new. The two-body currents presented in this thesis are also a new development, as is the two-quark parity-violating potential.

Different closure methods have been applied to calculations of parity- and time reversal-violating quantities like the electric dipole moment of atoms (ref.28,31), but never in a setting like this. The exact estimate of the one-body current contribution to the elastic parity-violating electromagnetic current matrix element, as well as its exact cancellation by the two-body current contribution, in certain classes of theories, are new results.
CHAPTER II

INELASTIC POLARIZED COINCIDENCE ELECTRON SCATTERING

2.1 Introduction

In view of the imminent completion of new electron accelerator facilities with polarized electron beams, targets and recoil polarimeters it has become important to have a good understanding of the formalism and theory of coincidence measurements in inelastic electron scattering. This prospect induced me and my advisor F. Gross to undertake a thorough investigation of the relativistic formalism describing two such reactions: scalar and pseudoscalar electroproduction off a spin 1/2 target, and electrodisintegration of the deuteron into the two body final state allowing for the polarization of the electron beam, the target and one ejectile. A particularly interesting and important problem for the planning of experiments is the question of the necessary and sufficient measurements for the complete determination of the transition amplitudes. This problem was addressed and solved for the two above mentioned reactions in ref.(1,2) (see Appendices O and P). This work will not be repeated here, only one technical topic omitted from these publications will be elaborated on, and then a new development involving non-trivial inequalities among the coincidence observables will be presented.

The first part of this thesis falls into six sections. Besides the introduction and the summary and conclusions, they are:

1. A review of the elements of the general polarized coincidence inelastic electron scattering cross section in the helicity formalism
2. Spin precession induced by relativistic kinematics
3. The unpolarized coincidence cross section and inequalities among its structure functions
4. Polarized coincidence structure function inequalities

In the first section we review the formalism of ref.2 in order to show the necessity of understanding the Lorentz boost properties of the structure functions, and to set the stage for the derivation of the “positivity inequalities”. In the second section
the Wigner rotations of the recoil polarization coincidence inelastic electron scattering structure functions due to the boost from the centre-of-mass to the lab frame are calculated. In the third section the derivation of the unpolarized coincidence cross section in the so called general response tensor formalism is repeated. Then, the inequalities among unpolarized coincidence structure functions are derived, assuming only that the one-photon-exchange cross section is positive or zero. Then I rederive these very same inequalities using a seemingly different method and thus establish the equivalence of the methods. In the fourth section, the second, "non-canonical", method is used to derive some already known polarized-target inequalities and a set of new polarized-ejectile inequalities.

2.2 The General Polarized Inelastic Coincidence Electron Scattering Cross Section

In this section, we review some general results about the inelastic coincidence electron scattering cross section, derived in ref. 2. This is done in order to set the stage for section 2.3 where some of the details about the Lorentz transformation of the recoil polarization observables from the centre-of-mass to the laboratory, or lab for short, frame, which were omitted in ref. 2, are elaborated, and in order to establish a formalism suitable to the derivation of inequalities among the structure functions, which have to hold if the cross section is to be positive (sections 2.4 and 2.5).

The general form of the inelastic coincidence electron scattering cross section for a polarized target and a polarized ejectile, with two particles in the final state, in the "mixed" frame, that is with electron variables in the lab and the hadronic variables in the cm frame, is:

$$\frac{d^5\sigma}{d\Omega'dE'd\Omega_1} = \frac{\sigma_{MP1}}{4\pi M_T} \left\{ \left( \frac{W}{M_T} \right)^2 v_L R_L + v_T R_T + v_{TT} \left[ \cos 2\phi R_{TT}^{(1)} + \sin 2\phi R_{TT}^{(2)} \right] \right. $$

$$+ \left( \frac{W}{M_T} \right) v_{TL} \left[ \cos \phi R_{LT}^{(1)} + \sin \phi R_{LT}^{(2)} \right] + 2h v_{T} R_{TT} $$

$$+ 2h \left( \frac{W}{M_T} \right) v_{TL} \left[ \cos \phi R_{LT}^{(1)} + \sin \phi R_{LT}^{(2)} \right] \right\} \quad (1)$$

where

$$\sigma_M = \left( \frac{\alpha \cos \frac{1}{2}\theta}{2E \sin^2 \frac{1}{2}\theta} \right)^2. \quad (2)$$

is the Mott cross section, the v's are kinematic factors depending on the electron variables defined below, W is the total energy of the system in the cm frame and \(\phi\)
is the azimuthal ("out-of-plane") angle. The $R$'s are the response functions specified below (see Table 4).

$$v_L = \left(\frac{Q}{q_L}\right)^4$$

$$v_{TL} = -\frac{1}{\sqrt{2}} \left(\frac{Q}{q_L}\right)^2 \sqrt{\frac{Q^2}{q_L^2} + t g^2\frac{1}{2}\theta}$$

$$v_T = \frac{1}{2} Q^2 + t g^2\frac{1}{2}\theta$$

$$v_{TT} = -\frac{Q^2}{2q_L^2}$$

$$v_{TL}' = -\frac{1}{\sqrt{2}} \left(\frac{Q}{q_L}\right)^2 t g\frac{1}{2}\theta$$

$$v_T' = t g\frac{1}{2}\theta \sqrt{\frac{Q^2}{q_L^2} + t g^2\frac{1}{2}\theta}$$

where $\theta$ is the electron scattering angle in the lab frame and the absolute value of the three-momentum transfer vector in the lab frame $q_L = |q_L|$. The only assumptions entering this result are: one-photon exchange approximation and conservation of the hadronic current. Parity conservation has not been assumed, yet. The kinematic variables entering this cross section are: the total cm energy $W$, the negative four-momentum transfer squared $Q^2$, the absolute value of the momentum transfer three-vector in the lab frame $q_L$, the electron scattering angle $\theta$, the ejectile opening angle $\theta_1$ and the azimuthal angle $\phi$ (see Fig. 6).

The response functions $R$'s are functions of $W$, $Q^2$ and $\theta_1$, but not of $\phi$ as long as the target polarization is specified with respect to the coordinate system $(x', y', z')$ (see Fig. 5) and recoil polarization is measured with respect to the $(x'', y'', z'')$ coordinate system. Thus, they can be separated by making measurements at different values of $\phi$ and otherwise identical kinematics.

The general structure of the response functions $R$ in terms of the helicity components of the electromagnetic current matrix elements is given in Table 4 (we use the notation of ref.2).
Geometry of this reaction is depicted on Fig. 5.

All structure functions can be divided into two classes: class I structure functions \( R_L \), \( R_T \) and all other structure functions explicitly denoted as such by the superscript I) are nonzero in unpolarized, parity conserving reactions, while class II structure functions \( R_T \) and all other structure functions explicitly marked II) vanish identically because of constraints imposed by parity. Once the polarization measurement is allowed, this rule is modified, but it remains true that one half of all possible structure functions vanish because of parity constraints.
The sums in the utmost right hand side of Table 4 represent the, as yet unspecified, sums over the spins. These results will be used in sections 2.4 and 2.5 for the derivations of inequalities among response functions due to the positivity of the cross section.

| $R_L$  | $\eta^2 R_{oo}$ | $\kappa^2 \sum |J^o|^2$ |
|--------|-----------------|--------------------------|
| $R_T$  | $R_{++} + R_{--}$ | $\kappa^2 \sum (|J_+|^2 + |J_-|^2)$ |
| $R^{(i)}_{TT}$ | $2ReR_{+-}$ | $2\kappa^2 Re \sum (J_+ J_+^\dagger)$ |
| $R^{(i)}_{TT}$ | $-2ImR_{+-}$ | $-2\kappa^2 Im \sum (J_+ J_+^\dagger)$ |
| $R^{(ii)}_{LT}$ | $2\eta Re(R_{o+} - R_{o-})$ | $2\kappa^2 Re \sum J^o (J_+^\dagger - J_-^\dagger)$ |
| $R^{(ii)}_{LT}$ | $2\eta Im(R_{o+} + R_{o-})$ | $2\kappa^2 Im \sum J^o (J_+^\dagger + J_-^\dagger)$ |
| $R_T$  | $R_{++} - R_{--}$ | $\kappa^2 \sum (|J_+|^2 - |J_-|^2)$ |
| $R^{(ii)}_{LT}$ | $2\eta Re(R_{o+} + R_{o-})$ | $2\kappa^2 Re \sum J^o (J_+^\dagger + J_-^\dagger)$ |
| $R^{(ii)}_{LT}$ | $2\eta Im(R_{o+} - R_{o-})$ | $2\kappa^2 Im \sum J^o (J_+^\dagger - J_-^\dagger)$ |

Table 4 Structure functions $R_{ij}$ expressed in terms of the response tensor components (second column) or helicity amplitudes (third column), where $\kappa^2 = \frac{1}{2W} \left( \frac{M^2}{2M} \right)^2$ for pion electroproduction and deuteron electrodisintegration (see ref.2). The helicity amplitudes are defined in the cm frame in the following way: $J_{\pm} = J \cdot e_\pm = -J \cdot \hat{e}_\pm$ and $J^o$ is the zeroth component of the four-vector $J^\mu$ in the cm frame. The subscript 0 in $R_{ij}$'s in the second column corresponds to covariant helicity zero states (ref.2).

Since the cross section (eq.1) is written in two frames and we can make measurements only in the lab frame, we must know how to Lorentz transform ("boost") this cross section. The boosting of the unpolarized cross section was treated in sufficient detail in ref.2, but the details of the Lorentz transformation properties of the recoil polarization structure functions were left out. The basic effect of the cm $\rightarrow$ lab boost on the recoil polarization structure functions is a Wigner rotation, which is different from the rotation of the momentum three-vector.

Before going into details, let us list a few intuitively clear facts about and explain the importance of these effects in electron scattering. Since the Wigner angle is the rotation angle of the helicity, which is a scalar product of the spin three-vector and the direction of motion of the particle, due to a boost, this rotation has to be about the axis perpendicular to the plane defined by the direction of motion and the boost direction. That automatically implies that the spin component perpendicular to the plane defined by the boost direction and the particle velocity will remain unchanged.
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In the case of electron coincidence scattering the boost in question is the \( cm \rightarrow lab \) boost, which is directed along the \( \vec{q} \) axis. The direction of motion of the ejectile \( \vec{p} \) provides the second vector defining a plane which is nothing but the ejectile plane. From this we conclude that the \( n (y'') \) component of the recoil polarization vector remains unchanged under the boost. The \( l (z'') \) and \( s (x'') \) components, on the other hand, get mixed by this transformation. This mixing angle depends on the scattering angle \( \theta_{CM} \) and reaches its maximum at \( \theta_{CM} \approx \frac{\pi}{2} \), whereas it vanishes at \( \theta_{CM} = 0, \pi \).

Another way of expressing this is to observe that the three-momentum and the spin pseudo-three-vector have different \( lab \rightarrow cm \) solid angle Jacobians:

\[
\frac{d\Omega_{spin}^{cm}}{d\Omega_{spin}^{lab}} \neq \frac{d\Omega_{momentum}^{cm}}{d\Omega_{momentum}^{lab}}
\]

In order to take all the mystery out of the Wigner rotation we will reformulate the above given arguments in yet another language. The Thomas precession, which is the name used for this phenomenon in atomic physics, is a purely kinematic effect. It has nothing to do with the dynamics of the process and will have to show up in all (polarized electron) scattering formalisms where recoil polarization cross section is expressed in a "mixed" frame, whether they are relativistic or not. The origin of the different rotation rates for the spin and the momentum three-vectors is their fundamentally different covariant nature: spin is the spatial part of a \( space-like \) pseudo-vector \( s^\mu \) which is orthogonal to the \( time-like \) four-vector \( p^\mu \) whose spatial part is the momentum.

Although the Wigner rotation is well known in hadronic scattering physics (ref.6), its consequences for photon and electron scattering processes have, to my knowledge, never been explored. We will treat these issues in the next section of this thesis.
2.3 Spin Precession Induced by Relativistic Kinematics

The helicity formalism has one basic disadvantage when compared with the Feynman trace technique: the Lorentz transformations from one reference frame to another are nontrivial to implement. In the Feynman trace approach one obtains a result in the form of a Lorentz invariant (Lorentz scalar) which can be evaluated in an arbitrary frame; in the helicity formalism the result is obtained in one frame and any boost to a different frame has to be done with great care. The transformation properties of the amplitudes, which are highly complex, determine the transformation properties of the observables. These transformation properties are derived in Appendices A and B.

Since our polarized coincidence cross section eq.(2.2.1) is written in the "mixed" frame, we must boost the hadron part to the lab frame in order to be able to compare it with experiment. The polarization three-vector of a particle is normalized to unity only in the rest frame of the particle. Hence, it only makes sense to talk about the polarization of the ejectile in its rest frame. But, there are two ways of getting to that frame from the cm frame: one is to boost directly from the cm to the rest frame, the other is to boost the cm to the lab and then from the lab to the rest frame. As was repeatedly pointed out before, the results differ by an overall rotation of the spin, or polarization vector, direction (for proof see Appendix A), which is usually called Wigner rotation.

In inelastic electron coincidence reactions with recoil polarization measurement this rotation is in the plane defined by the following three three-vectors: \( \vec{q}, \vec{p}_1^l, \vec{p}_1^{cm} \) (Fig. 5). The vector \( \vec{q} \) determines the direction of the \( cm \rightarrow lab \) boost \( \vec{\beta}_1, \vec{p}_1^{cm} \) determines the \( cm \rightarrow rest \) boost vector \( \vec{\beta}_2 \), and finally \( \vec{p}_1^l \) determines the \( lab \rightarrow rest \) boost vector \( \vec{\beta}_3 \) (see Fig. 6).

The overall effect on the observables is a mixing (due to the rotation) of the two recoil polarization observables \( R_{ij}(s), R_{ij}(l) \) lying in the ejectile plane: about the \( y' = y'' \) axis through the Wigner angle \( \omega \):

\[
R^{LAB}_{ij}(\alpha, l) = \cos \omega R^{CM}_{ij}(\alpha, l) - \sin \omega R^{CM}_{ij}(\alpha, s) \\
R^{LAB}_{ij}(\alpha, s) = \sin \omega R^{CM}_{ij}(\alpha, l) + \cos \omega R^{CM}_{ij}(\alpha, s)
\]

where \((n, s, l)\) are the \((y'', x'', z'')\) components, respectively, of the recoil polarization vector, \(\alpha\) stands for all other, e.g. target, polarization specifications and \(i, j\) stand for the photon polarization. The rotation vanishes if the directions of the boost and the motion of the particle coincide, i.e. at \(\theta_1 = 0, \pi\). The polarization component \(P_n\), which is perpendicular to the plane defined by the boost direction and the particle velocity, remains unchanged. Note that the rotated recoil polarization structure
functions are still to be measured with respect to the *cm* ejectile coordinate system \((x'', y'', z'')\) on Fig. 5 and not the *lab* ejectile coordinate system.

Furthermore, we formally prove that the Wigner rotation angle \(\omega\) is always less than the three-momentum rotation angle \(\theta^\text{cm}_1 - \theta^\text{l}_1\). For this purpose we use Sommerfeld's (ref. 4) observation that Einstein’s theorem about the addition of velocities in special relativity is an example of non-Euclidean geometry. This means that if two nonparallel four-velocities are added relativistically to give a third four-velocity, then their spatial parts do not form a planar triangle, but rather they can be put in correspondence with sides of a triangle on the surface of a hyperboloid. The sum of inner angles in such a triangle is less than \(\pi\) and the difference between \(\pi\) and this sum will be called the “hyperbolic defect” of the triangle \(\varepsilon\), in analogy with the spherical excess which appears in spherical trigonometry. The Wigner angle is one of the inner angles of this triangle (see Fig. 6) and Wigner’s contribution was the observation of its relevance to the precession of the spin of a particle when the particle is subjected to the sequence of Lorentz transformations specified by the three four-velocities.

![Fig. 6 A schematic representation of three relativistic velocity vectors forming a triangle on a hyperboloid. The sum of inner angles in such a triangle is less than \(\pi\).](image)

The rotation angle \(\omega\) (denoted by \(\theta^\text{w}\) in ref. 2) is always less than the rotation angle of the momentum \(\theta^\text{cm}_1 - \theta^\text{l}_{1ab}\). The hyperbolic defect \(\varepsilon = \pi - (\alpha + \beta + \gamma)\) is determined by (Appendix C):
\[
\sin\left(\frac{\varepsilon}{2}\right) = \frac{\sqrt{1 + 2\cosh(v_1)\cosh(v_2)\cosh(v_3) - \cosh^2(v_1) - \cosh^2(v_2) - \cosh^2(v_3)}}{4\cosh\left(\frac{1}{2}v_1\right)\cosh\left(\frac{1}{2}v_2\right)\cosh\left(\frac{1}{2}v_3\right)}
\]

\[
\cos\left(\frac{\varepsilon}{2}\right) = \frac{1 + \cosh(v_1) + \cosh(v_2) + \cosh(v_3)}{4\cosh\left(\frac{1}{2}v_1\right)\cosh\left(\frac{1}{2}v_2\right)\cosh\left(\frac{1}{2}v_3\right)}
\]

Note that \(\varepsilon \geq 0\) for arbitrary values of \(v_i\). Identifying the angles in the triangle on Fig.7, we see that this inequality is equivalent to the statement that \(\theta^\text{cm}_i - \theta^\text{cm}_1 \geq \omega\), which is what we were trying to prove. An alternative expression for \(\omega\), due to Ritus (ref.34), in terms of two boost parameters and one angle (for proof see Appendix C) is:

\[
tg\frac{\omega}{2} = \frac{\sinh v_1 \sinh v_2 \sinh \theta^\text{cm}_1}{(1 + \cosh v_1)(1 + \cosh v_2) + \sinh v_1 \sinh v_2 \cosh \theta^\text{cm}_1}
\]

where \(v_1, v_2\) are the rapidities of the \(\text{cm} \to \text{lab}\) and \(\text{cm} \to \text{rest}\) boosts, respectively. The relationship between the rapidities and the more conventional boost parameters \(\gamma, \beta\) is:

\[
\gamma_i = \cosh(v_i), \quad \gamma_i \beta_i = \sinh(v_i)
\]

The boost parameters, for pion electroproduction, are:

\[
\bar{\beta}_1 = \frac{\bar{q}_L}{\sqrt{\bar{q}_L^2 + W^2}}, \quad \gamma_1 = \sqrt{1 + \left(\frac{\bar{q}_L}{W}\right)^2}
\]

\[
\bar{\beta}_2 = \frac{\vec{P}_1^\text{cm}}{\vec{E}_1^\text{cm}}, \quad \gamma_2 = \frac{\vec{E}_1^\text{cm}}{\vec{M}}
\]

where

\[
\vec{P}_1^\text{cm} = \frac{1}{2W} \sqrt{(W^2 - M^2 - \mu^2)^2 - 4M^2\mu^2}
\]
\[
\vec{E}_1^\text{cm} = \frac{1}{2W} \sqrt{(W^2 - M^2 - \mu^2)^2 - 4M^2(\mu^2 - W^2)}
\]

and \(M, \mu\) are the nucleon and pion mass respectively. The kinematics of electroproduction is defined by two independent variables: \(W, Q^2\). One of them can be substituted by the Bjorken variable \(x\):

\[
x = \frac{Q^2}{2P \cdot q}
\]

Hence

\[
Q = \sqrt{\left(\frac{x}{1-x}\right)(W^2 - M^2)}
\]
Direct application of eq.(3) together with the kinematics of pion electroproduction yields as a result Fig.7:

![Graph showing Wigner angle Ω as a function of CM frame opening angle θ_{CM}.](image)

Fig. 7 Wigner angle ω (solid) and the lab frame opening angle θ_{LAB}^lab (dashes), as a function of the cm frame opening angle θ_{CM}^cm. The kinematics is specified by the total cm energy of the system W and the Bjorken variable x = 0.5.

Pion electroproduction was chosen on purpose because it is “more relativistic” than the two-body deuterium electrodisintegration; hence the Wigner angle is usually larger in the former than in the latter. It is clear from Fig. 7 that the Wigner angle is rather small at these kinematics, which justifies its customary neglect. But, if one is to carry out high precision nucleon recoil polarization experiments such as those planned at CEBAF, one must take it into account.
2.4 The Unpolarized Coincidence Cross Section and Inequalities Among its Structure Functions

In the case when there is no polarization measurement in the final state, the boosting of the cross section eq.(2.2.1) from the cm to lab frame is straightforward and well described in section II H of ref.2. The general form of the inelastic coincidence electron scattering cross section for arbitrary unpolarized target and arbitrary unpolarized ejectile, and two particles in the final state, in the lab frame is:

$$\frac{d^5\sigma}{d\Omega' dE' d\Omega} = \frac{\sigma M_p}{4\pi M_T} r \left\{ v_L R_L + v_T R_T ight\}$$

$$+ v_{TT} \left[ \cos 2\phi R_{TT}^{(I)} + \sin 2\phi R_{TT}^{(II)} \right]$$

$$+ v_{LT} \left[ \cos \phi R_{LT}^{(I)} + \sin \phi R_{LT}^{(II)} \right] + 2hv_T R_T$$

$$+ 2hv_{LT} \left[ \cos \phi R_{LT}^{(II)} + \sin \phi R_{LT}^{(I)} \right]$$

(1)

where the recoil factor $r$:

$$r = \frac{W}{M_T} \left( 1 + \frac{\nu p_1 - E_1 q \cos \theta_1}{M_T p_1} \right)^{-1}$$

(2)

was evaluated for the final state of the deuteron two-body electrodisintegration. The recoil factor appears only if the final state contains two bodies. The structure functions $R$'s are evaluated in the lab frame using Table 4 and are given in Table 5:

<table>
<thead>
<tr>
<th>$R_L$</th>
<th>$\left( \frac{q}{Q} \right)^2 \left{ - W_1 + \left( \frac{q}{Q} \right)^2 \left[ W_2 + cW_3 + c^2W_4 \right] \right}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_T$</td>
<td>$2W_1 + W_4 \left( \frac{E}{M} \sin \theta_1 \right)^2$</td>
</tr>
<tr>
<td>$R_{TT}^{(I)}$</td>
<td>$-W_4 \left( \frac{E}{M} \sin \theta_1 \right)^2$</td>
</tr>
<tr>
<td>$R_{LT}^{(I)}$</td>
<td>$\sqrt{2} \left( \frac{q}{Q} \right)^2 \left( \frac{E}{M} \sin \theta_1 \right) \left[ W_3 + 2c \left( \frac{E}{M} \right)^2 W_4 \right]$</td>
</tr>
<tr>
<td>$R_{LT}^{(II)}$</td>
<td>$\sqrt{2} \left( \frac{q}{Q} \right)^2 \left( \frac{E}{M} \sin \theta_1 \right) W_5$</td>
</tr>
</tbody>
</table>

Table 5. Unpolarized inelastic coincidence structure functions $R$'s in terms of the general response tensor functions $W_i$.

where

$$c = 1 - \left( \frac{\nu p_1}{q_L E_1} \right) \cos \theta_1^L$$
and $W_i$ ($i = 1 - 5$) are the five independent response functions appearing in the response tensor $W_{\mu \nu}$:

$$W^{\mu \nu} = -W_1 \delta^{\mu \nu} + W_2 \frac{p_T^\mu p_T^\nu}{p_T^2} + W_3 \frac{1}{2p_T \cdot p_1} \left[p_T^\mu \bar{p}_1^\nu + \bar{p}_1^\mu p_T^\nu\right] +$$

$$+ W_3 \frac{i}{2p_T \cdot p_1} \left[p_T^\mu \bar{p}_1^\nu - \bar{p}_1^\mu p_T^\nu\right] + W_6$$

$$+ W_8 \frac{i}{2p_T \cdot p_1} \left[p_T^\mu \bar{p}_1^\nu - \bar{p}_1^\mu p_T^\nu\right]$$

and

$$\bar{g}^{\mu \nu} = g^{\mu \nu} - \frac{q^\mu q^\nu}{q^2}$$

$$\bar{p}^\mu = p^\mu - \frac{p \cdot q}{q^2} q^\mu$$

The cross section eq.(1) was first derived, in different notation and for arbitrary final state, and only for unpolarized electrons, by de Forest (ref. 7). It is completely general and hence can be used to describe any type of the inelastic electron scattering coincidence reaction.

**Positivity Inequalities**

The differential cross section is strictly a positive or vanishing, or positive semidefinite, quantity. Its general form in the one-photon-exchange approximation can be written as: $l_\mu W^{\mu \nu} l_\nu$, where $l_\mu$ is the lepton current, up to insignificant, positive, overall factors. From these two facts alone, one can derive nontrivial inequalities among the structure functions. This was first done in ref. 8. In the following we will repeat the derivation of the unpolarized structure function inequalities using two different methods in order to establish their equivalence. The second method will prove to be much more practical for deriving new, polarized structure function inequalities. Some of the polarized target inequalities have already been given in ref. 8, but all of the polarized ejectile and some of the polarized target inequalities presented here are new.

We could directly proceed to find the positivity conditions, but we spare ourselves some labour by remembering that $W_{\mu \nu}$ has rank three despite being a $4 \times 4$ matrix. This, of course, is a consequence of gauge invariance, which relates the longitudinal and the scalar (zeroth) components of this tensor. The key to simplification is to reduce this four-dimensional matrix to a three-dimensional one by a clever choice of gauge or frame. The solution is to work in the generalized, because this is inelastic scattering, Breit frame where the four-vector $q_\mu$ loses its temporal
component and becomes:

\[ q_B^\nu = (0, q_B) = (0, 0, 0, Q) \]

Then, the current conservation condition:

\[ q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0 \]

turns into

\[ W^3_{B\nu} = W^{\mu,3} = 0 \]

i.e. only the first three rows and columns survive. Now that we have reduced the response tensor to a three-dimensional matrix we can apply the positivity constraint.

The positive semi-definiteness of a quadratic form is mathematically equivalent to the statement that all of its principal minors are positive semi-definite i.e. positive or zero. This is easily proven by diagonalizing the quadratic form and remembering that the minors (determinants) do not change under the change of basis of the linear space.

By finding the relation between the R's in the lab frame and the response tensor \( W_{\mu\nu} \) in the Breit frame, we can express all of the elements of \( W_{\mu\nu} \) in terms of R's. Then a straight-forward application of the inequalities for the principal minors (see Appendix D) yields the results:

\[ R_L, R_T \geq 0, \quad R_T \geq \left| R^{(I)}_{TT} \right| \]

\[ 4R_L (R_T - R^{(I)}_{LT}) \geq \left[ \left( R^{(I)}_{LT} \right)^2 + \left( R^{(I)}_{LT} \right)^2 \right] \]

The first two of these inequalities are trivial: they are equivalent to the statement that the sums of squares of amplitudes have to be positive or zero. The next inequality saying that the transverse structure function is larger or equal to the absolute value of the transverse-transverse interference structure function is familiar from model building, where it was empirically observed, but not proven. The fourth result is a non-trivial inequality involving all five structure functions appearing in the unpolarized cross section, which has not been investigated or applied to models, so far. It can be used to set bounds on one of the structure functions if the other four are known. For example, in the so called impulse approximation the fifth structure function \( R^{(I)}_{LT} \) vanishes identically. But from the knowledge of \( R_L, R_T, R^{(I)}_{LT}, R^{(I)}_{LT} \) we can get an upper bound on its size in a more realistic approximation:

\[ \left| R^{(I)}_{LT} \right| \leq \sqrt{4R_L (R_T - R^{(I)}_{LT}) - \left( R^{(I)}_{LT} \right)^2} \]

These inequalities were first derived in (ref.8), but they remained largely unknown in the nuclear physics community. They might prove to be of practical importance.
as a consistency check of the experimental extraction procedure for the coincidence observables, specifically as a check of radiative corrections.

**Helicity Density Matrix Method**

It is important to realize that the transition to the helicity basis for the photon is just an orthogonal coordinate transformation which does not change the positivity relations. Rewriting eqs.(D.5-9) in the helicity basis we realize that this is just the positivity constraint for the virtual photon density matrix. Density matrices are observables, so they must be represented by Hermitian operators. They must also be positive semi-definite: the elements of the diagonalized density matrix are probabilities of the system to be in a certain pure state, and probabilities have to be positive, or zero.

The second method is based on the observation that all that is necessary is an explicit representation of the density matrix in terms of the observables, i.e. coincidence structure functions. How we arrive at this relationship between the density matrix elements and structure functions is irrelevant. One way is to use a general response tensor for the reaction, such as the one shown at the beginning of this section, and work out all structure functions and density matrix elements in terms of the general response functions. This would provide a relation between the density matrix elements and the observables, as desired.

The other possibility is to use a specific reaction whose density matrix can be unambiguously expressed in terms of its structure functions. This can be done as long as all possible structure functions are represented, i.e. none of them vanish “accidentally”. The structure function inequalities obtained in this way are the same as the inequalities obtained from the general response tensor (for proof see Appendix E). This completes the proof of equivalence of the two methods.

We will use only the second, or density matrix, method for the derivation of the polarized structure function inequalities. Some of the polarized-target inequalities have been derived before (ref.8) using the response tensor method. They were confirmed using the density matrix method.
2.5 Polarized Coincidence Structure Function Inequalities

If another spin degree of freedom, besides photon’s, is available, the density matrix increases its size to a $3(2s+1) \times 3(2s+1)$, where $s$ is the additionally observed spin. Just as in section 2.4, the elements of this density matrix are bilinear products of the electromagnetic transition amplitudes, so that they can be expressed in terms of the inelastic electron scattering polarization observables. The condition for this density matrix to be positive semi-definite leads to a multitude of inequalities among the polarized structure functions.

As explained before, the method which is used to find a relationship between the density matrix elements and the observables is unimportant. The “canonical” method would be to expand the response tensor in all possible covariants consistent with the general principles such as the current conservation, hermiticity and parity conservation. Then one expresses the structure functions $R$’s in terms of the response tensor functions. After expressing the density matrix elements in terms of the same response tensor matrix elements and then inverting the relations between the response tensor matrix elements and the observables, one can express the density matrix elements in terms of the observables. Application of the positivity constraints to the density matrix then automatically leads to inequalities among the structure functions.

Instead of using this rather cumbersome approach, we choose a specific process belonging to the given category, i.e. having the required spin degrees of freedom, and then evaluate its response functions in terms of a finite number of transition amplitudes. By expressing the density matrix elements in terms of these very same transition amplitudes one gets a one-to-one relationship between the density matrix elements and the observables. The only danger in this procedure is in choosing a reaction which does not allow a unique assignment of observables to the density matrix elements. An example of such a case would be taking a completely spinless reaction, in order to determine the unpolarized structure function inequalities: there $R_T = \pm R_{TT}$, where the sign depends on the parity of the reaction, and one has an ambiguity as to which observable to use in various places in the density matrix. The solution to this problem is to use a reaction which does not allow a complete separation of amplitudes from the given set of observables. In this way one obtains the same results as by using the general response tensor method, but with less effort.

In the case of the spin $1/2$ target or recoil polarization measurements an example of such a “sample” reaction is scalar or pseudoscalar electroproduction off a spin $1/2$ target. It was proven in ref.2 that the amplitudes of this reaction cannot be completely separated from only one of these two sets of measurements; thus there is no danger of ambiguity. Secondly, all of the observables for these two cases
have already been expressed in terms of the transition amplitudes in ref.1. There remains only the task of expressing the density matrices in terms of amplitudes (this is accomplished in Appendix F).

Relating the density matrix elements and the observables and proceeding as in section 2.4, we obtain the following inequalities for recoil polarization structure functions:

\[ R_L(U) \geq |R_L(n)|, \quad R_T(U) \geq |R_T(n)| \]  \hspace{1cm} (1)

\[ R_T(U) \geq |R_{TT}(n)| \]

\[ R_T(U) - R_{TT}(U) \geq |R_T(n) - R_{TT}(n)| \]

\[ R_T(U) + R_{TT}(U) \geq |R_T(n) + R_{TT}(n)| \]  \hspace{1cm} (2)

\[ \left[ R_T(U) - R_T(n) \right]^2 - \left[ R_{TT}(U) - R_{TT}(n) \right]^2 \geq \]

\[ \left[ R_T(s) + R_{TT}(l) \right]^2 + \left[ R_{TT}(s) - R_T(l) \right]^2 \]

\[ \left[ R_T(U) + R_T(n) \right]^2 - \left[ R_{TT}(U) + R_{TT}(n) \right]^2 \geq \]

\[ \left[ R_T(s) - R_{TT}(l) \right]^2 + \left[ R_{TT}(s) + R_T(l) \right]^2 \]  \hspace{1cm} (3)

\[ 4 \left[ R_T(U) - R_{TT}(U) - R_T(n) + R_{TT}(n) \right] \left( R_L(U) - R_L(n) \right) \geq \]

\[ \left[ R_L(U) - R_L(n) \right]^2 + \left[ R_{LT}(U) - R_{LT}(n) \right]^2 \]

\[ 4 \left[ R_T(U) - R_{TT}(U) + R_T(n) - R_{TT}(n) \right] \left( R_L(U) + R_L(n) \right) \geq \]

\[ \left[ R_L(U) + R_L(n) \right]^2 + \left[ R_{LT}(U) + R_{LT}(n) \right]^2 \]

\[ 4 \left[ R_T(U) + R_{TT}(U) + R_T(n) + R_{TT}(n) \right] \left( R_L(U) + R_L(n) \right) \geq \]

\[ \left[ R_{LT}(s) + R_{LT}(l) \right]^2 + \left[ R_{LT}(s) - R_{LT}(l) \right]^2 \]

\[ 4 \left[ R_T(U) + R_{TT}(U) - R_T(n) - R_{TT}(n) \right] \left( R_L(U) - R_L(n) \right) \geq \]

\[ \left[ R_{LT}(s) - R_{LT}(l) \right]^2 + \left[ R_{LT}(s) + R_{LT}(l) \right]^2 \]  \hspace{1cm} (4)

where \( s, n, l \) stand for recoil polarization vector components along the \( x''', y''', z''' \) directions (Fig.6), respectively. All of these results are new. For the sake of completeness we give the following target polarization inequalities, some of which have
been derived before in ref.8 using different methods and presented in different notation:

\[ R_L(U) \geq |R_L(y)|, \quad R_T(U) \geq |R_T(y)| \]
\[ R_T(U) \geq |R_{TT}^{(1)}(y)| \]

\[ R_T(U) - R_{TT}^{(1)}(U) \geq |R_T(y) - R_{TT}^{(1)}(y)| \]
\[ R_T(U) + R_{TT}^{(1)}(U) \geq |R_T(y) + R_{TT}^{(1)}(y)| \]

\[ R_T(U) - R_{TT}^{(1)}(U) \geq |R_T(y) - R_{TT}^{(1)}(y)| \]
\[ R_T(U) + R_{TT}^{(1)}(U) \geq |R_T(y) + R_{TT}^{(1)}(y)| \]

\[ \left[ R_T(U) + R_T(y) \right]^2 - \left[ R_{TT}^{(1)}(U) + R_{TT}^{(1)}(y) \right]^2 \geq \]
\[ \geq \left[ R_T(x) - R_{TT}^{(1)}(z) \right]^2 + \left[ R_{TT}^{(1)}(x) + R_T(z) \right]^2 \]
\[ \left[ R_T(U) - R_T(y) \right]^2 - \left[ R_{TT}^{(1)}(U) - R_{TT}^{(1)}(y) \right]^2 \geq \]
\[ \geq \left[ R_T(x) + R_{TT}^{(1)}(z) \right]^2 + \left[ R_{TT}^{(1)}(x) - R_T(z) \right]^2 \]

\[ 4 \left[ R_T(U) - R_{TT}^{(1)}(U) - R_T(y) + R_{TT}^{(1)}(y) \right] \left( R_L(U) - R_L(y) \right) \geq \]
\[ \geq \left[ R_{LT}^{(1)}(U) - R_{LT}^{(1)}(y) \right]^2 + \left[ R_{LT}^{(1)}(U) - R_{LT}^{(1)}(y) \right]^2 \]
\[ 4 \left[ R_T(U) - R_{TT}^{(1)}(U) + R_T(y) - R_{TT}^{(1)}(y) \right] \left( R_L(U) + R_L(y) \right) \geq \]
\[ \geq \left[ R_{LT}^{(1)}(U) + R_{LT}^{(1)}(y) \right]^2 + \left[ R_{LT}^{(1)}(U) + R_{LT}^{(1)}(y) \right]^2 \]
\[ 4 \left[ R_T(U) + R_{TT}^{(1)}(U) + R_T(y) + R_{TT}^{(1)}(y) \right] \left( R_L(U) - R_L(y) \right) \geq \]
\[ \geq \left[ R_{LT}^{(1)}(x) - R_{LT}^{(1)}(z) \right]^2 + \left[ R_{LT}^{(1)}(x) + R_{LT}^{(1)}(z) \right]^2 \]
\[ 4 \left[ R_T(U) + R_{TT}^{(1)}(U) - R_T(y) - R_{TT}^{(1)}(y) \right] \left( R_L(U) + R_L(y) \right) \geq \]
\[ \geq \left[ R_{LT}^{(1)}(x) + R_{LT}^{(1)}(z) \right]^2 + \left[ R_{LT}^{(1)}(x) - R_{LT}^{(1)}(z) \right]^2 \]

where \( x, y, z \) stand for target polarization vector components along the \( x', y', z' \) directions (Fig.5), respectively.
Note that these are the inequalities which can only be obtained from minors of the two lowest orders. The density matrix is $6 \times 6$ dimensional now, so there are inequalities with products of up to six response functions ("6th order") although some of them may turn out to be just products of "lower order" inequalities and hence trivial. The "first order" transverse polarized target structure function inequalities eq.(5) are new, as are the first two of the inequalities in eq.(9). All in all, six of the 11 target inequalities written here, are new.

The "higher order" inequalities will be useless for some time to come, because they involve large numbers of various spin observables which are not likely to be measured soon. The high degree of their nonlinearity makes them quite unlikely to be useful for setting up useful bounds on observables. For these reasons they are omitted.
2.6 Summary and Conclusions

To summarize, in this part of the thesis we have:

- Reviewed the results of the general, polarized, inelastic coincidence electron scattering formalism.

- Presented general results concerning the Wigner rotation of the recoil spin observables in polarized coincidence inelastic electron scattering due to the boost from the cm to the lab frame, and evaluated the Wigner angle for pion electroproduction.

- Derived the most general form of the unpolarized coincidence inelastic electron scattering cross section for arbitrary initial and final states, and established inequalities among its structure functions in two ways, assuming only the one-photon-exchange approximation and the fact that the cross section has to be positive semi-definite.

- Derived five known and 17 new inequalities among the polarized coincidence inelastic electron scattering structure functions, assuming that the polarized target, or the polarized ejectile, has spin 1/2.

The presentation of the Wigner rotation concludes the work on the polarized coincidence inelastic electron scattering formalism. The next step is to actually calculate these amplitudes in various models and compare them with experiments to be done. The study of positivity inequalities has only been begun: no inequalities for reactions with spin one targets have been derived, yet. They may well play an interesting role in constraining the deeply inelastic inclusive polarized target and/or electron scattering structure functions. Experiments of this kind have already been proposed at HERA, and are being prepared at present, so that any additional information could be valuable.
CHAPTER III
ELASTIC PARITY-VIOLATING ELECTRON SCATTERING

3.1 Introduction

In view of the increased interest in parity-violating electron scattering and the prospect of precise experiments (ref.36) it has become necessary to systematically calculate the corrections to the Born amplitude contribution to the elastic, parity-violating, electron-nucleon, or electron-nucleus asymmetry. Among these corrections, the following seem to be the most important:

- the abnormal parity admixtures in the nucleon wave function,
- the isospin admixtures in the nucleon wave function, and
- the Standard Model, one-loop, radiative corrections to the asymmetry.

This part of this thesis is concerned with a model calculation of the first point on this list, in a non-relativistic quark model. The same model allows a consistent calculation of the isospin admixtures in the nucleon wave function. The radiative corrections for this process at $Q^2 = 0$, in the Salam-Weinberg model, have already been treated in ref.23 (for a dissenting view, see the second entry in ref.23). This part of this thesis falls into twelve sections, including the introduction and the summary and conclusions:

1. Separation of strange form factors from parity-violating electron-nucleus asymmetry
2. Parity-violating admixture in the nucleon wave function in a non-relativistic quark model
3. Elastic parity-violating matrix element of the nucleon electromagnetic current and its long-wave-length limit behaviour
4. Conservation of the elastic parity-violating matrix element of the nucleon electromagnetic current
5. A simple model and its parity-violating electromagnetic current matrix element
6. New definition of the parity-violating electromagnetic current matrix element

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7. The parity-violating matrix element of the complete, conserved electromagnetic current of the nucleon induced by the exchange of the $Z^0$ meson

8. The parity-violating matrix element of the complete, conserved electromagnetic current of the nucleon induced by the exchange of the $W^\pm$ mesons

9. An approximate closure estimate of all abnormal parity admixture contributions to the elastic parity-violating electromagnetic current matrix element of the nucleon

10. An exact closure estimate of all abnormal parity admixture contributions to the elastic parity-violating electromagnetic current matrix element of the nucleon

Some of the preliminary results\* pertaining to sections 3.2, 3.3 and 3.4 have already been presented in ref.25.

### 3.2 Separation of Strange Form Factors

Three targets are considered: $^4$He, proton $p$ and neutron $n$. The assumptions which enter all of the analyses in this section are: Lorentz invariance, field theory, Standard Model, significance of $u$, $d$, $s$ quarks in $p$, $n$, good parity of states, good isospin of states and the one boson exchange approximation.

$^4$He is a particularly interesting target because it is scalar and iso-scalar. As a consequence, all iso-vector and all axial and magnetic currents in the problem vanish and only two independent form factors suffice for the full description of its weak neutral current (WNC): the electromagnetic (EM) charge $G_p$ and the "strange" electric $S_E$ form factors of $^4$He (all "strange" form factors will be denoted by capital S with appropriate subscript).

The resulting asymmetry, when expanded in powers of $Q^2$, looks like this (ref.21):

$$A = \frac{d\sigma^1 - d\sigma^1}{d\sigma^1 + d\sigma^1} = \left( \frac{G_F Q^2}{4\pi\alpha^2} \right) \left[ 4\sin^2\theta_W - \frac{(r_S^2)}{3} Q^2 \right] + O(Q^6)$$

where $Q^2 = -q^2 = -(k - k')^2$ is the negative of the four-momentum transfer squared, $G_F \approx 10^{-5} M^{-2}$ is the Fermi weak interaction constant, $M$ is the nucleon mass, $(r_S^2)$ is the "strangeness radius" of the nucleon (or $\frac{1}{4}$ of strangeness radius of $^4$He) and $\theta_W$ is the Weinberg angle. In the derivation of this formula I have used the fact that the strange electric form factor vanishes at $Q^2 = 0$ to lowest order in weak coupling constant, which is a reflection of the fact that the net strangeness of helium is zero.

\* which were incomplete in an essential way due to the neglect of two-body currents and higher excited state admixtures
In the case of the nucleon, all components of the weak neutral current are present: electric, magnetic, axial; iso-scalar and iso-vector. This allows the asymmetry to take its most complicated form: it contains a longitudinal (electric), a parity conserving (PC) transverse (magnetic current) and a parity-violating (PV) transverse (axial current) term. The nucleon forms an iso-doublet, which means that there are two kinds of the nucleon differing only by the electric charge: proton and neutron.

There are three kinds of contributions to the nucleon asymmetry in this problem: the electric (E) and magnetic (M) form factors and an axial (A) one. We use a mixture of the “SU(3)” (ref. 37) and the standard nuclear notation: two of the three form factors correspond to the overall (Sachs EM or weak) form factors of the two nucleons, such as \( G^p_{E,M}, F^p_{E,M} \), whereas the third one will describe exclusively the strange quark contributions and will be denoted by \( S_{E,M,A} \) (which are isoscalar in the Standard Model with exact isospin symmetry of states):

\[
\langle p' | \bar{s} \gamma_\mu s | p \rangle = \bar{u}(p') \left[ S_1 \gamma_\mu + \frac{i}{2M} S_2 \sigma_{\mu\nu} q^\nu \right] u(p)
\]

\[
\langle p' | \bar{s} \gamma_\mu \gamma_5 s | p \rangle = \bar{u}(p') \left[ S_A \gamma_\mu \gamma_5 + S_{PS} \sigma_{\mu\nu} q^\nu \right] u(p)
\]

where

\[
G_E = F_1 - \tau F_2 \quad G_M = F_1 + F_2
\]

\[
\tau = \frac{Q^2}{4M^2} \quad Q^2 = -q^2 = -(k - k')^2 \geq 0
\]

Then, the proton asymmetry has the following form (ref. 21):

\[
\mathcal{A}^p = \left\{ \frac{G_F Q^2}{4\pi\alpha\sqrt{2}} \right\} \left\{ - (1 - 4\sin^2 \theta_W) + \right. \\
\left. + \left[ \epsilon G^p_E (G^p_E + S_E) + \tau G^p_M (G^p_M + S_M) \right] + \sqrt{(1 - \epsilon^2)\tau (1 + \tau)} G^p_M (F^p_A - F^p_A - S_A)(1 - 4\sin^2 \theta_W) \right\} \times |d\sigma^\dagger + d\sigma^\dagger|^{-1}
\]

where

\[
|d\sigma^\dagger + d\sigma^\dagger| = \epsilon (G^p_E)^2 + \tau (G^p_M)^2 \quad \frac{1}{\epsilon} = 1 + 2(1 + \tau) t g^2 \frac{\theta}{2}
\]

The asymmetry now depends on two kinematic variables: the negative four-momentum transfer squared \( Q^2 \) and the virtual photon polarization \( \epsilon \) defined above. A Rosenbluth separation can be made by varying \( \epsilon \) while keeping \( Q^2 \) fixed. This allows us to separate three independent quantities:
- term proportional to $\epsilon$
- term proportional to $\sqrt{(1 - \epsilon^2)(1 + \tau)}$
- terms independent of $\epsilon$

Notice that all but one term, $-(1 - 4\sin^2\theta_W)$, within the curly brackets in eq.(2) vanish in the $Q^2 \to 0$ limit. This fact allows us to separate four quantities from $A^p, G_E^p, G_M^p$, if we assume that Weinberg angle remains constant in the measured region of $Q^2$:

$$1 - 4\sin^2\theta_W \text{ at } Q^2 = 0, \quad G_E^n + S_E$$
$$G_M^n + S_M, \quad F_A^p - F_A^n - S_A$$

No further information about the "strange" form factors can be obtained without a knowledge of neutron electromagnetic and axial iso-vector form factors. The neutron magnetic form factor is fairly well known, as is the axial iso-vector one, which allows the extraction of the strange magnetic and axial form factors. But the neutron electric form factor is very poorly known, so we cannot learn anything about the "strange" electric form factor from this experiment. One can use the $^4$He experiment as a source of information about $S_E$ at low $Q^2$, which would complete the separation of strange form factors (at least in this kinematic regime).

Free neutrons are unstable under $\beta$ decay, so the deuteron is the simplest source of target neutrons. I assume that the deuteron structure can be taken out* of this problem. Then a completely analogous expression to eq.(2) can be written for the neutron asymmetry: the only differences are the sign of the $1 - 4\sin^2\theta_W$ term up front and the exchange of all proton and neutron form factors $p \leftrightarrow n$.

Similarly to the proton case, we can separate four independent quantities. Assuming knowledge of $A^n, G_{E,M}^p$, we can get:

$$1 - 4\sin^2\theta_W \text{ at } Q^2 = 0, \quad G_E^n + S_E$$
$$G_M^n + S_M, \quad F_A^n - F_A - S_A$$

We see that once again we cannot separate $S_E$ because of our ignorance of $G_E^n$. But if we know both asymmetries and the three EM form factors (see Table 6), we can make a complete separation of all other form factors (extraction of $S_E$ is non-linear).

---

* I intend to work on a relativistic calculation of the contribution of the internal structure of the deuteron to deep inelastic electron scattering processes in the future.
Table 6  Summary of separable quantities. The ingredients necessary for the extraction are listed.

Thus, by measuring both asymmetries, one gets a low energy value of the fundamental coupling constant $\sin^2 \theta_W$ in the semileptonic (quark-lepton) sector and all three "strange" form factors of the nucleon.

Six quantities which can be extracted from the two complete experiments are shown in the box below.

<table>
<thead>
<tr>
<th>$A^p$</th>
<th>$G^n_M$</th>
<th>$A^n$</th>
<th>$G^n_{E,M}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - 4\sin^2 \theta_W$</td>
<td>$G^n_E + S_E$</td>
<td>$1 - 4\sin^2 \theta_W$</td>
<td>$G^n_E(G^n_E + S_E)$</td>
</tr>
<tr>
<td>$G^n_M + S_M$</td>
<td>$G^n_M + S_M$</td>
<td>$F^V_A - \frac{1}{2}S_A$</td>
<td>$F^V_A + \frac{1}{2}S_A$</td>
</tr>
</tbody>
</table>

One can use the measured $F^V_A$ as a double-check of the experiment, and, as a bonus, one gets the neutron electric form factor which was one of the early motivations for proposing these experiments (ref.22). All of these results were obtained independently of ref.21.

As mentioned above, all of the above made statements about the separability of the Weinberg angle and the "strangeness" content of the nucleon from the inelastic parity-violating electron-nucleon scattering data, are true only if the "higher order" corrections to this cross section are small and calculable. In the rest of this thesis we will be concerned with the parity-violating electromagnetic current matrix element of the nucleon and its contribution to the parity-violating asymmetry.
3.3 Parity-Violating Admixture In The Nucleon Wave Function

The quark model picture of the nucleon is one of a three-quark bound state. The binding, in QCD, is accomplished by gluon exchange, which is parity-conserving, but its practical implementation varies from model to model. As a consequence, different models have substantially different wave functions. In most conventional models the ground state (nucleon) is an S state, with, possibly, a certain amount of D wave admixture.

But, the Standard Model also allows the exchange of W and Z intermediate vector bosons between quarks, which are parity-violating. These interactions, although small, induce a finite abnormal parity admixture, i.e. P waves, in the nucleon wave function. Besides the two-body potentials, there are two- and three-body forces induced by the W, Z exchange, too, but they will be neglected, because they are higher order in the Fermi weak coupling constant $G_F$. I confine myself to u and d quarks, since they are the valence quarks in the nucleon and will work in a simple potential model. The parity-violating one-boson-exchange potential due to W and Z exchange is:

$$V^{PV}(x_1 - x_2) = \frac{-G_F}{\sqrt{2}} \delta(x_1 - x_2) \left[ V_{12}^- I_{12}^- + V_{12}^+ I_{12}^+ \right]$$

(1)

where the potential has been broken up into spin-spatial ($V_{12}^\pm$) and isospin ($I_{12}^\pm$) parts, and ± denotes symmetry under the interchange of indices (1 ↔ 2)\(^*\).

$$V_{12}^+ = (\bar{u}\gamma_\mu \gamma_5 u)^1(\bar{u}\gamma_\mu u)^2 + (1 \leftrightarrow 2)$$

$$V_{12}^- = (\bar{u}\gamma_\mu \gamma_5 u)^1(\bar{u}\gamma_\mu u)^2 - (1 \leftrightarrow 2)$$

$$I_{12}^+ = \frac{\cos^2 \theta_C}{2} [\tau_1^+ \tau_2^2 + \tau_1^- \tau_2^2] + \left( \frac{1}{4} - \frac{\sin^2 \theta_W}{2} \right) \tau_3^1 \tau_3^2 - \frac{\sin^2 \theta_W}{12} (\tau_3^1 + \tau_3^2)$$

$$I_{12}^- = \frac{-\sin^2 \theta_W}{12} (\tau_3^- - \tau_3^2)$$

Instead of solving the Schrödinger, or Dirac, equation with the potential from eq.(1) added to the “strong” potential, I will take advantage of the weakness of the

\(^*\) Note that I inadvertently omitted the $V^- - I^-$ term in the parity violating potential in ref.25, which resulted in a 3% error in the estimate of the abnormal parity admixture in the nucleon
parity-violating interaction and use first order perturbation theory to evaluate the P wave content of the nucleon. For this we need the lowest lying, negative parity, excited states of the nucleon, i.e. the lowest mass P-wave nucleon resonances.

They are (see Table 1.1 in (ref.24)):

\[
\begin{align*}
N(1535) \ (J^P = \frac{1^-}{2}), \ SU(6): \ 70 \text{-plet}^{28} \\
N(1650) \ (J^P = \frac{1^-}{2}), \ SU(6): \ 70 \text{-plet}^{48} \\
\Delta(1620) \ (J^P = \frac{1^-}{2}), \ SU(6): \ 70 \text{-plet}^{210}
\end{align*}
\]

Hence the “parity admixed” nucleon state is a superposition of the S-wave ground state and the P-wave resonances:

\[
|N\rangle = \frac{1^+}{2} \ 8^2 + \epsilon_1 \frac{1^-}{2} \ 8^2 + \epsilon_2 \frac{1^-}{2} \ 8^4 + \epsilon_3 \frac{1^-}{2} \ 10^2
\]  

(2)

where

\[
V^{PV} = \sum_{i<j} V_{ij}^{PV}
\]

and

\[
\begin{align*}
\epsilon_1 &= \frac{\langle \frac{1^-}{2} 8^2 | V^{PV} | \frac{1^+}{2} 8^2 \rangle}{E_+ - E_-} \\
\epsilon_2 &= \frac{\langle \frac{1^-}{2} 8^4 | V^{PV} | \frac{1^+}{2} 8^2 \rangle}{E_+ - E_-'} \\
\epsilon_3 &= \frac{\langle \frac{1^-}{2} 10^2 | V^{PV} | \frac{1^+}{2} 8^2 \rangle}{E_+ - E_-''}
\end{align*}
\]

This expansion is valid in both non-relativistic and relativistic perturbation theory. The higher mass resonance contributions are expected to be small due to the increasing energy denominator (this popular belief will be proven wrong in section 3.11; the magnitude of the contribution of all parity admixtures to the elastic parity-violating nucleon electromagnetic matrix element is evaluated using closure).
Non-Relativistic Quark Model

The non-relativistic quark model (NRQM) was chosen because of its simplicity and the reasonable predictions that it makes for the negative parity excited states of the nucleon. It also allows an exact separation of the centre of mass motion from the electromagnetic transition matrix elements. In the following I adopt the conventions and the notation of ref.38, unless stated otherwise.

So, one makes a nonrelativistic reduction of the spin-spatial parts of the potential, eq.(1), in the configuration representation and gets:

\[
\begin{align*}
\mathcal{V}_{ij}^+ &= \left( \frac{1}{2m_q} \right) (\vec{\sigma}_1 - \vec{\sigma}_2) \cdot \left\{ \vec{p}_1 - \vec{p}_2, \delta(\vec{r}_1 - \vec{r}_2) \right\} + \\
&\quad + i(\vec{\sigma}_1 \times \vec{\sigma}_2) \cdot \left[ \vec{p}_1 - \vec{p}_2, \delta(\vec{r}_1 - \vec{r}_2) \right] \\
\mathcal{V}_{ij}^- &= \left( \frac{1}{2m_q} \right) (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \left\{ \vec{p}_1 - \vec{p}_2, \delta(\vec{r}_1 - \vec{r}_2) \right\}
\end{align*}
\]

These potentials are used to calculate the matrix elements in eq.(2). The SU(6) structure of the \((70, 1^-)\) P-wave resonance wave functions is:

\[
\begin{align*}
^28 : \quad &\frac{1}{2} \left\{ \phi_\lambda \left( \left[ \chi_\lambda \psi^1_{1m\rho} \right]_M^J - \left[ \chi_\lambda \psi^1_{1m\lambda} \right]_M^J \right) + \phi_\rho \left( \left[ \chi_\lambda \psi^1_{1m\rho} \right]_M^J + \left[ \chi_\lambda \psi^1_{1m\lambda} \right]_M^J \right) \right\} \\
^48 : \quad &\frac{1}{\sqrt{2}} \left\{ \phi_\rho \left[ \chi_S \psi^1_{1m\rho} \right]_M^J + \phi_\lambda \left[ \chi_S \psi^1_{1m\lambda} \right]_M^J \right\} \\
^210 : \quad &\frac{1}{\sqrt{2}} \left\{ \phi_S \left[ \chi_\rho \psi^1_{1m\rho} \right]_M^J + \phi_\lambda \left[ \chi_\rho \psi^1_{1m\lambda} \right]_M^J \right\} \\
^221 : \quad &\frac{1}{\sqrt{2}} \left\{ \phi_A \left[ \chi_\rho \psi^1_{1m\rho} \right]_M^J - \phi_A \left[ \chi_\lambda \psi^1_{1m\rho} \right]_M^J \right\}
\end{align*}
\]

where

\[
\left[ \chi_\Sigma \psi^N_{im\Sigma} \right]_M^J = |J, M = \sum_m \sum_S \langle J, M | J, M | l m, S_S S_3 \rangle \chi_\Sigma(S, S_3) \psi^N_{im\Sigma}
\]

and \(\Sigma = \rho, \lambda\) stands for the two kinds of permutational symmetry and \(\chi_\Sigma(S, S_3)\), \(\psi^N_{im\Sigma}\) are given on p.38 of ref.24 and table 4.17 of ref.38, respectively.

The overall symmetry of the spin-spatial-isospin wave functions under interchange of any two quarks is extensively used in the evaluation of the matrix elements in eq.(2) (for details, see Appendices G, H).
Furthermore, the symmetry under the interchange of the first two indices in the states and operators (potentials) leads to further simplifications in the evaluation of the parity-violating potential matrix elements: the matrix element between the \((70, \ ^210)\) and \((56, \ ^28)\) states vanishes identically, whereas the matrix elements between \((70, \ ^28)\) and \((56, \ ^28)\), as well as between \((70, \ ^48)\) and \((56, \ ^28)\), simplify to the following form:

\[
\langle (70, \ ^28), \ J M \mid V_{12}^+ \mid (56, \ ^28), \ J M \rangle = \frac{1}{2\sqrt{2}} \times \\
\left\{ (\phi_\rho \mid I_{12}^+ \mid \phi_\rho) \langle [x_\lambda \psi_\rho]_M \mid \psi_{12}^+ \mid x_\rho, M \psi_S \rangle + (\phi_\lambda \mid I_{12}^+ \mid \phi_\lambda) \langle [x_\rho \psi_\rho]_M \mid \psi_{12}^+ \mid x_\lambda, M \psi_S \rangle \right\}
\]

\[
\langle (70, \ ^28), \ J M \mid V_{12}^- \mid (56, \ ^28), \ J M \rangle = \frac{1}{2\sqrt{2}} \times \\
\left\{ (\phi_\rho \mid I_{12}^- \mid \phi_\rho) \langle [x_\lambda \psi_\rho]_M \mid \psi_{12}^- \mid x_\rho, M \psi_S \rangle + (\phi_\lambda \mid I_{12}^- \mid \phi_\lambda) \langle [x_\rho \psi_\rho]_M \mid \psi_{12}^- \mid x_\lambda, M \psi_S \rangle \right\}
\]

\[
\langle (70, \ ^48), \ J M \mid V_{12}^+ \mid (56, \ ^28), \ J M \rangle = \frac{1}{2} (\phi_\rho \mid I_{12}^+ \mid \phi_\rho) \langle [x_\rho \psi_\rho]_M \mid \psi_{12}^+ \mid x_\rho, M \psi_S \rangle
\]

The fourth matrix element, between \((70, \ ^21)\) and \((56, \ ^28)\), vanishes because we are working in the “nuclear domain” (i.e. with \(u, \ d\) quarks only) which reduces the SU(6) symmetry group to SU(4), which in turn does not allow an isosinglet irreducible representation constructed from three identical members of an isodoublet.

Thus, the problem has been reduced to that of calculating five spin-spatial and four isospin matrix elements. The spin-spatial part of the matrix elements are most easily evaluated in momentum space. Here we follow the conventions and notation of ref.39 except for the notation, and the signs, of the two symmetry classes of the wave function which remain as in the rest of the text. Instead of repeating their formalism, we will simply list the necessary results (details of this calculation are given in Appendix G).

\[
\begin{align*}
\langle [x_\lambda \psi_\rho]_M \mid \psi_{12}^+ \mid x_\rho, M \psi_S \rangle &= \frac{4 \ i \ I}{m_q \ R} \\
\langle [x_\rho \psi_\rho]_M \mid \psi_{12}^+ \mid x_\lambda, M \psi_S \rangle &= 0 \\
\langle [x_\rho \psi_\rho]_M \mid \psi_{12}^- \mid x_\rho, M \psi_S \rangle &= 0 \\
\langle [x_\lambda \psi_\rho]_M \mid \psi_{12}^- \mid x_\lambda, M \psi_S \rangle &= \frac{4 \ i \ I}{\sqrt{3} \ m_q \ R} \\
\langle [x_\rho \psi_\rho]_M \mid \psi_{12}^- \mid x_\rho, M \psi_S \rangle &= -\frac{4 \ i \ I}{m_q \ R} \end{align*}
\]
where \( I = \left( \frac{1}{\sqrt{2\pi R}} \right)^3 \) \( R^{-2} = m_q \omega \), \( J = \frac{1}{2} \) and there is no summation over \( M \).

Using these results, as well as the isospin matrix elements from Appendix H, we get:

\[
\begin{align*}
\varepsilon_1 &= \frac{-i}{(E_+ - E'_-)} \left( \frac{G_F I}{m_q R} \right) \left\{ \frac{3}{2} \left[ \cos^2 \theta_C + \frac{1}{2} \left( 1 - 2 \sin^2 \theta_W \right) \right] - \frac{\sin^2 \theta_W}{3\sqrt{2}} \tau_3^N \right\} \\
\varepsilon_2 &= \frac{2i}{(E_+ - E''_-)} \left( \frac{G_F I}{m_q R} \right) \left\{ \frac{3}{2} \left[ \cos^2 \theta_C + \frac{1}{2} \left( 1 - 2 \sin^2 \theta_W \right) \right] \right\} \\
\varepsilon_3 &= 0
\end{align*}
\]

where

\[
E_+ = 940 \text{ MeV} \quad E'_- = 1535 \text{ MeV} \quad E''_- = 1650 \text{ MeV}
\]

or, after substituting the empirical values for \( \cos \theta_C \), \( \sin \theta_W \) and \( R = .86 \text{ fm} \) from the charge radius of the proton, and taking the \( u \) and \( d \) quark mass to be approximately one third of the nucleon’s, \( m_q = 340 \text{ MeV} \):

\[
\begin{align*}
\varepsilon_1 &= i \left( 1.8 - 0.05 \tau_3^N \right) \times 10^{-8} \\
\varepsilon_2 &= -i \times 3.0 \times 10^{-8}
\end{align*}
\]

The scale of these admixing parameters depends on the following dimensionless combination of four dimensional quantities:

\[
\left( \frac{m_q \omega^2 G_F}{E_+ - E'_-} \right) = -1.5 \times 10^{-7}
\]

which explains their small size. Note, also, that \( \varepsilon_{1,2} \) are not observables (they are factors in the wave functions), so that their absolute phase is not determined. But we know that they are imaginary relative to the \( S \) wave, due to the time-reversal invariance of the two-body potential (note that the sign of \( \varepsilon_2 \) depends on the convention adopted for the \( \psi^\Lambda \) states, which is not universally agreed on (compare with ref.39).
3.4 Elastic Parity-Violating Electromagnetic Current Matrix Element of the Nucleon and its Behaviour in the Long Wave Length Limit

The negative parity pieces in the wave function will generate a parity-violating electromagnetic nucleon current matrix element, also known as the anapole. This elastic scattering matrix element will be determined by using the abnormal parity admixture parameters $\varepsilon_{i,2}$ from the previous section and the electromagnetic transition matrix elements between the nucleon and the $P$ wave resonances as calculated in the non-relativistic impulse approximation.

There is only one allowed parity-violating, time-reversal invariant, electromagnetic, elastic, conserved current matrix element for spin $1/2$ particles. Its relativistic form is:

$$\langle p' | J_{5\mu}^{EM}(0) | p \rangle = \beta \bar{u}(p') \left( \gamma_\mu - \frac{q_\mu}{q^2} q_\mu \right) \gamma_5 u(p)$$  \hspace{1cm} (1)

In the nonrelativistic approximation, where I am working, this matrix element turns into:

$$\tilde{J}_{5}^{EM} = \beta (\vec{\sigma} - (\vec{\sigma} \cdot \hat{q}) \hat{q})$$  \hspace{1cm} (2)

where $\beta$ is an, as yet, undetermined function of $Q^2$. A similar expression, with a factor of $Q^2$, was introduced by Zel’dovich and Perelomov in 1961 (ref.42) and has been used extensively ever since. I do not want to prejudge the long-wave-length behaviour of this matrix element so, for now, I leave it unspecified.

Note that one of the primary arguments for Zel’dovich’s choice of the overall factor was the apparent divergence of the matrix element at the real photon point $Q^2 = 0$. Even though, at first sight, this matrix element seems to be infinite at this kinematic point, a careful analysis shows that it is finite and well-defined. The easiest way to see this is by looking at the individual components of the current matrix element eq.(1). Using the elastic scattering kinematics condition $Q^2 = -q^2 = 2M\nu$ and the Dirac equation for the initial and final states, while remembering that in elastic scattering the $Q^2 \to 0$ limit corresponds to the non-relativistic limit, one obtains eq.(2), which is finite in this limit, and a vanishing parity-violating charge density. Thus, we see that, at least on formal grounds, there is no obstacle for a parity-violating electromagnetic current matrix element which does not vanish at the photon point.

The relation of $\beta$ to the non-relativistic current matrix elements is easily established:

$$\langle p' \uparrow | \hat{e}_+ \cdot \tilde{J}_5^{EM}(0) | p \downarrow \rangle = -\sqrt{2}\beta = -2Re \sum_{i=1}^{2} \epsilon_i A_i(\uparrow, + \downarrow)^*$$  \hspace{1cm} (3)
where

\[ A_1(\uparrow, + \downarrow) = \langle (10^2 8) \uparrow | H_7^+ | (56^2 8) \downarrow \rangle \]

\[ A_2(\uparrow, + \downarrow) = \langle (10^4 8) \uparrow | H_7^+ | (56^2 8) \downarrow \rangle \]

and

\[ H_7^+ = - \sum_{i=1}^{3} \vec{A}_+(i) \cdot \vec{J}_i \]

\[ \vec{J}_i = \frac{e_i}{2m_q} \left[ \left( \vec{p}_i + \vec{p}_i' \right) + i \vec{\sigma}_i \times \left( \vec{p}_i' - \vec{p}_i \right) \right] \]

\[ \vec{A}_+(i) = \hat{e}_+ \exp(i \vec{q} \cdot \vec{r}_i) \quad \hat{e}_+ = \frac{1}{\sqrt{2}} \left( \hat{e}_x + i \hat{e}_y \right) \]

Note that \( A_{1,2} \) are elastic, off-diagonal electromagnetic interaction Hamiltonian matrix elements. All other matrix elements \( A_4(\vec{J}_3, + \vec{J}_3) \), e.g. \( A_4(\uparrow, + \uparrow) \) vanish, thereby confirming the axial vector nature of \( \vec{J}_5 \). They are straightforwardly evaluated using standard non-relativistic quark model methods (see Appendix I). The results are:

\[ A_1 = \frac{ie}{3} \left\{ \tau_3^N \frac{\sqrt{\omega}}{m_q} - \left( \tau_3^N + \frac{1}{2} \right) R \frac{\vec{q}^2}{3m_q} \right\} \exp \left[ -\frac{\vec{q}^2}{6m_q \omega} \right] \]

\[ A_2 = \frac{ie}{36} (1 - \tau_3^N) R \frac{\vec{q}^2}{m_q} \exp \left[ -\frac{\vec{q}^2}{6m_q \omega} \right] \]

where \( \tau_3^N \) is an isospin operator in the nucleon iso-space. Note the way the phases in \( \epsilon_i \) and \( A_i \) cooperate to give a real quantity, which is a confirmation of the conserved time-reversal symmetry of this matrix element.

It is easy to see that in this calculation \( \beta \neq 0 \) at \( Q^2 = 0 \), i.e. this parity-violating electromagnetic current matrix element does not vanish in the long-wave-length (LWL) limit. Such a threshold behaviour is something of a surprise: a general argument of (ref.26) based on the conservation of the electromagnetic current, predicts vanishing at the threshold. We review this argument on the following pages in order to understand the source of failure of the above calculation.
Serot’s Theorem

This is a short review of the argument presented in refs. 26, 27 where gauge invariance was used to prove the vanishing, in the LWL limit, of the elastic transverse electric dipole moment, and hence of the elastic parity-violating, time-reversal invariant electromagnetic current matrix element of any particle with spin.

The matrix element of the electromagnetic Hamiltonian for transverse photon polarizations can be written as a finite sum of transverse electric and transverse magnetic multipoles. In the case of elastic parity-violating scattering off spin $\frac{1}{2}$ targets there are two terms in this sum: transverse electric and transverse magnetic dipole moments. Parity violation appears in two kinds of contributions to these multipoles: one kind comes from the abnormal parity admixtures to the states and the usual, one-body, current operator; the other kind comes from normal parity states and an axial, two-body, current operator. The multipoles are classified using the usual selection rules for off-diagonal multipole matrix elements in the first case and the axial elastic multipole selection rules in the second case. These two cases have the same selection rules. Hence they can be classified as single entities: elastic parity-odd transverse multipoles.

The elastic transverse magnetic and electric dipoles transform in the same way under time-reversal, but have opposite parities. This means that one or the other will vanish, depending on whether it involves a parity-conserving or a parity-violating interaction. The elastic transverse electric dipole moment vanishes identically if parity and time-reversal are good symmetries of the reaction. We are interested in a parity-violating situation here, for which the exact opposite holds. Similarly, the elastic transverse magnetic dipole moment vanishes if time-reversal is a good symmetry of the reaction and parity is violated. This means that the elastic parity-violating transverse electromagnetic Hamiltonian matrix element for $j = \frac{1}{2}$ states is completely determined by a single multipole: transverse electric dipole.

The LWL limit of the transverse electric multipoles moments is well known (ref.13) and happens to be constrained by electromagnetic current conservation, as will be explicitly shown below.
We will work with the exact states, which contain both the parity-even and the parity-odd parts, and the exact current operator, which contains both the vector, one-body, and the axial, two-body, current and satisfies the non-relativistic current conservation relation:

\[ \vec{\nabla} \cdot \vec{J} = -i [H, \rho] \]

The transverse electric multipole is defined as (ref.13):

\[
(f|\hat{T}_{JM}^q|i) = \frac{1}{iq\sqrt{J(J+1)}} \int d\vec{r} \left\{ -\vec{q}^2(\vec{J}_{fi} \cdot \vec{r}) + (\vec{\nabla} \cdot \vec{J}_{fi})[1 + (\vec{r} \cdot \vec{\nabla})] \right\} j_J(qr) Y_{JM}
\]

(5)

Upon using \( i(E_i - E_f) (\psi_f^\ast \rho \psi_i) = \vec{\nabla} \cdot \vec{J}_{fi} \) and remembering that the energy difference is \( E_i - E_f = -\nu = -Q^2/2M \) in the case of elastic scattering, and in the LWL limit \( Q^2 = q^2 \). In the same limit we may use the asymptotic expansion of spherical Bessel functions: \( j_J(qr) \approx \frac{\nu}{r} \) to cancel the \( q \) in the denominator. All of this implies that the elastic, parity-odd, time-reversal conserving, transverse electric dipole goes like \( q^2 \) in the LWL limit.

So, before any definitive conclusions about the correct LWL limit of this matrix element are drawn, one must address the question of current conservation in the preceding calculation. This will be done in section 3.5. Once a conserved current is constructed one would like to see how Serot’s theorem works in practice. This will be shown on a simple model in section 3.6.
3.5 Conservation of the Quark Electromagnetic Current in the Presence of Parity-Violating Quark-Quark Interaction

Since this is a non-relativistic calculation we do not have all four components of a covariant current matrix element at our disposal. In quantum mechanics, the conservation of the electromagnetic current in configuration space is reflected in the fulfillment of the following equation relating the divergence of the three-current and the commutator of the Hamiltonian and the charge density:

$$\vec{\nabla} \cdot \vec{J} = -i\left[H, \rho\right]$$

where $H$ is the total Hamiltonian (kinetic and potential energy) of the interacting system and $\rho$ is the charge density. A large part of the following discussion parallels closely the treatment of so called "meson exchange currents" in parity-conserving nuclear physics (for examples see ref.44).

The divergence of the one-body electromagnetic current equals $(-i)$ times the commutator of the kinetic energy and the one-body charge density. In the case of parity-violating electromagnetic current density the Hamiltonian contains the sum of all two-body parity-violating potentials. The failure of this potential to commute with the one-body charge density implies the existence of a two-body electromagnetic current density which compensates the imbalance in the current conservation equation which exists without it. The two-body charge density is assumed to vanish, and indeed it turns out that in this non-relativistic reduction of the relativistic parity-violating electromagnetic current matrix element, only the one-body charge density is non-zero, to lowest order in the non-relativistic expansion. Thus, the test of conservation of this parity-violating electromagnetic current density is whether or not the parity-violating potential commutes with the one-body charge density:

$$\rho_{1-\delta} = \sum_{i=1}^{3} e_{i}\delta(\vec{r} - \vec{r}_{i})$$

where $e_{i} = \frac{1}{2}(\frac{1}{2} + \tau_{z}^{i})$. There are two possible reasons for the non-commutativity of the parity-violating potential eq.(1.1) and the one-body charge density:

- non-commuting iso-spin factors
- non-commuting spin-spatial factors

In the case of parity-violating quark-quark potential (eq.1.1) both of these possibilities are utilised: the $W^{\pm}$ part of the potential has an isospin operator which does not commute with the charge isospin operator and both the $W^{\pm}$ and the $Z^{0}$ parts of the potential have a spin-spatial operator which does not commute with the delta function in the charge density.
To see this, we use the following identity, which holds if the iso-spin part of the respective operator commutes with its own spin-spatial part (both the charge density and the parity-violating potential satisfy this condition):

\[
\left[ I_{ij}^\pm \mathcal{V}_{ij}^\pm, \epsilon_i \delta(\vec{r} - \vec{r}_i) \right] = \frac{1}{2} \left[ I_{ij}^\pm, \epsilon_i \right] \left[ \mathcal{V}_{ij}^\pm, \delta(\vec{r} - \vec{r}_i) \right] + \frac{1}{2} \left[ I_{ij}^\pm, \epsilon_i \right] \left[ \mathcal{V}_{ij}^\pm, \delta(\vec{r} - \vec{r}_i) \right] 
\]  

The respective (anti-)commutators are:

\[
\begin{align*}
\{ e_i - e_j, I_{ij}^+ \} &= \frac{1}{2} \left( \frac{1}{2} - \sin^2 \theta_W \right) \left( \tau_3^i - \tau_3^j \right) \\
\{ e_i + e_j, I_{ij}^+ \} &= \frac{2}{3} I_{ij}^+ + \frac{1}{2} \left( \frac{1}{2} - \sin^2 \theta_W \right) \left( \tau_3^i + \tau_3^j \right) - \frac{\sin^2 \theta_W}{6} (1 + \tau_3^i \tau_3^j) \\
\{ e_i - e_j, I_{ij}^- \} &= -\frac{1}{6} \sin^2 \theta_W \left( 1 - \tau_3^i \tau_3^j \right) \\
\{ e_i + e_j, I_{ij}^- \} &= \frac{2}{3} I_{ij}^- \\
\left[ e_k, I_{ij}^+ \right] &= \frac{i}{4} \cos^2 \theta_C \left( \vec{r}_i \times \vec{r}_j \right)_3 \left( \delta_{jk} - \delta_{ik} \right) \\
\left[ e_i, I_{ij}^- \right] &= 0
\end{align*}
\]  

whereas, neither the commutators nor the anticommutators of the spin-spatial operators vanish as long as one of the indices \( i, j \) coincides with \( k \), due to the following identities:

\[
\begin{align*}
\left[ \vec{\mathcal{V}}_k, \delta(\vec{r}_i - \vec{r}_j) \right] &= \delta_{ik} \left( \vec{\mathcal{V}}_k \delta(\vec{r}_i - \vec{r}_j) \right) \\
\left\{ \vec{\mathcal{V}}_k, \delta(\vec{r}_i - \vec{r}_j) \right\} &= \delta_{ik} \left( \vec{\mathcal{V}}_k \delta(\vec{r}_i - \vec{r}_j) \right) + 2 \delta(\vec{r}_i - \vec{r}_j) \vec{\mathcal{V}}_k
\end{align*}
\]  

It is clear that the commutator of the parity-violating potential and the one-body charge density does not vanish. We conclude that we must include a two-body, parity-violating electromagnetic current density in order to satisfy the current conservation (gauge invariance) constraint eq.(1). We will construct this "meson exchange current" in the same way such constructions are done in parity conserving electromagnetic nuclear reactions: by non-relativistic reduction of a current conserving (gauge invariant) covariant Feynman amplitude. Such covariant amplitudes are most easily written in momentum space, which leads us to constructing Fourier transforms of two-body operators, such as potentials and two-body currents, in order to obtain the configuration space operators which will be used, together with configuration space wave functions, for evaluation of the required matrix elements. This technical development is presented in Appendices G and L.
3.6 A Simple Model and its Parity-Violating Electromagnetic Current Matrix Element

In order to illustrate some of the concepts introduced in the previous sections and to facilitate the calculations in the full three-body model which still await us, we will construct a simple two-body model with all the previously announced properties.

The Model

The model consists of a two-body bound state of a charged spinless boson (spinless nucleus) and an oppositely charged spin $\frac{1}{2}$ fermion, e.g. an electron. We will let the mass of the boson be much larger than that of the fermion: this step will reduce the following considerations to those of a one-body problem. The binding is accomplished by a simple harmonic oscillator interaction between the two particles (if this interaction were a Coulomb potential, this would be a model of the parity-violating electromagnetic current of an atom). Besides the photon, a massive, neutral, parity-violating vector boson, e.g. $Z^0$, is exchanged between the two particles. The corresponding non-relativistic parity-violating potential is (ref.31, 28):

$$V_{PV}(r) = \frac{G_F Q_W}{4\sqrt{2}m} \cdot \left\{ \vec{p}, \delta(r) \right\}$$

(1)

where $Q_W(Z,N) = -\left[ \left( 4\sin^2 \theta_W - 1 \right) Z + N \right]$ for spinless nuclei in the Standard Model if the fermion is an electron, and $m$ is the fermion mass. This interaction induces a parity-violating admixture in the wave function of the ground state, as well as the wave functions of the excited states:

$$|\text{g.s.}\rangle = |(n' = 0) S\rangle + \sum_{n\geq 1} \frac{(nP)|V_{PV}|(n' = 0) S\rangle}{E_{n'} - E_n} |nP\rangle$$

$$= |(n' = 0) S\rangle + \sum_{n\geq 1} \epsilon_n |nP\rangle$$

(2)

We will evaluate only the lowest energy P wave state ($n=1$) admixture. We use the harmonic oscillator wave functions:

$$R_s = 2 \left( \frac{m\omega}{\pi} \right)^{1/4} \exp \left( -\frac{m\omega r^2}{2} \right)$$

$$R_p = 2 \sqrt{\frac{2(m\omega)^{5/2}}{3\sqrt{\pi}}} \exp \left( -\frac{m\omega r^2}{2} \right)$$

(3)
to get:
\[ \epsilon_1 = \frac{\langle (n = 1)P|V_{PV}|(n' = 0)S \rangle}{E_0 - E_1} \]
\[ = \frac{iG_F}{4\sqrt{2}} \sqrt{\frac{3}{2}} Q_W \left( \frac{m\omega}{\pi^2} \right) \]  
(4)

This abnormal parity admixture induces a parity-violating electromagnetic current which does not vanish in the LWL limit, just like the three-quark electromagnetic elastic parity-violating current matrix element in section 3.4 did not vanish in this limit. The appropriate electromagnetic transition matrix element, at \( q^2 = 0 \), is:
\[ A_1(\uparrow, + \downarrow) = \langle j = 1/2, j_3 = \uparrow; (n = 1, P)| H^{1-b}_+ |j = 1/2, j_3 = \uparrow; (n' = 0, S) \rangle \]
\[ = -ie \sqrt{\frac{\omega}{3m}} \]  
(5)

Using this result, we obtain for the bound state elastic parity-violating electromagnetic current matrix element in the one-body, or impulse, approximation, at \( q^2 = 0 \):
\[ \langle \text{g.s.} \uparrow | \hat{e}_+ \cdot \vec{J} \rangle (0) | \text{g.s.} \downarrow \rangle = -\sqrt{2} \beta \]
\[ = -2Re \left( \epsilon_1^* A_1(\uparrow, + \downarrow) \right) \]
\[ = \frac{eQ_W}{(4\pi)^2} \sqrt{\frac{\omega}{m}} G_F m\omega \]  
(6)

**Current Conservation**

Gauge invariance of this calculation is violated, this time only by the explicit momentum operator dependence of the parity-violating potential (there is no isospin in this problem, and no \( W^\pm \) exchange). This dependence makes the potential-charge density commutator non-zero:
\[ \vec{\nabla} \cdot \vec{J}_{1-b} \neq -i\left[ H, \rho_{1-b} \right] \]  
(7)

where the one-body electromagnetic Hamiltonian and the one-body current are given by eq.(3.4.2). The one-body charge density is \( \rho_{1-b} = e\delta(\vec{R} - \vec{r}) \) and it satisfies:
\[ \vec{\nabla} \cdot \vec{J}_{1-b} = -i\left[ H_0, \rho_{1-b} \right] \]  
(8)

where \( H_0 \) is the parity-conserving Hamiltonian. Clearly we need a two-body, parity-violating, electromagnetic current to satisfy:
\[ \vec{\nabla} \cdot \vec{J}_{2-b} = -i\left[ V_{PV}, \rho_{1-b} \right] \]  
(9)
in order to have a conserved electromagnetic current density. I assume throughout.
this thesis that the two-body charge density vanishes to this order in the non-relativistic expansion; this will shortly be shown to be true.

The Two-Body Current

A two-body parity-violating electromagnetic current is constructed by non-relativistic reduction of the following gauge invariant covariant amplitude:

$$
\frac{f_{\text{fit}}f_{\text{fa}}f_{\text{fix}}}{v y v'} \frac{i}{K - h},
$$

Fig. 8 Feynman diagrams defining the gauge invariant parity-violating four-current used in this simple model (see text).

There are five Feynman diagrams (see Fig. 8): a) two with the photon coupling to the fermion and b) three with the photon coupling to the constituent boson. These two subsets of diagrams are independently gauge invariant (in this sense this
model is simpler than any other to be considered in this thesis). Both subsets are necessary for the non-relativistic current conservation, if the mass of the constituent boson is kept finite. In the limit when the ratio of the constituent boson and fermion masses goes to infinity, the class b) diagrams vanish.

The non-relativistic reduction is accomplished by keeping the negative energy intermediate states in the fermion propagator (as is usual in this sort of construction of meson exchange currents; for tricks and technical details see ref.44); the only two terms which contribute to the lowest order in the expansion in $\frac{P}{m}$ to the two-body three-current density are proportional to:

\[
\bar{u}(p') \gamma S(p' - q) \gamma_0 \gamma_8 u(p_1) = \bar{u}(p') \gamma_0 \gamma_8 S(p_1 + q) \gamma u(p_1) = \frac{i}{4m} \vec{\sigma}
\]

This immediately yields the following expression for the non-relativistic two-body current density:

\[
\vec{J}_{2-b}(\vec{r}, \vec{R}) = \frac{eG_F Q_W}{2\sqrt{2} m} \vec{\sigma} \delta(\vec{R} - \vec{r}) \delta(\vec{r})
\]

This “two-body”, or “contact”, current was known to Flambaum and Khriplovich (ref. 28, 29), who constructed it in their study of parity violation in atoms.

A similar procedure can be followed in the nonrelativistic expansion of the two-body charge density. There are two terms: one involving a fermion amplitude with two $\gamma_0$ multiplied by the zeroth component of the scalar current and another with a fermion amplitude where $\vec{\gamma}$ and $\gamma_5$ are interchanged, where it ($\vec{\gamma}$ is in a scalar product with the constituent boson three-current). Both terms are of $O(\frac{P}{m})$ and hence the whole two-body charge density vanishes to this order, as advertised.

This expression satisfies eq.(9) and hence restores the gauge invariance of the complete current density: $\vec{J} = \vec{J}_{2-b} + \vec{J}_{2-b}$. Thus we have explicitly shown that the gauge invariance of the whole non-relativistic calculation is restored.

The ground state matrix element of the two-body parity-violating electromagnetic Hamiltonian is:

\[
\langle g.s. \uparrow | H_{2-b}^+ | g.s. \downarrow \rangle = \langle g.s. \uparrow | - \vec{J}_{2-b} \cdot \hat{e}_+ e^{i \vec{q} \cdot \vec{r}} | g.s. \downarrow \rangle = \frac{eG_F Q_W}{2\sqrt{2} m} \langle \uparrow | \sigma_+ | \downarrow \rangle \int \frac{d\vec{r}}{4\pi} \left( R_{10} \right)^2 \delta(\vec{r}) e^{i \vec{q} \cdot \vec{r}}
\]

where we see that only the S state contributes. The spatial integral is trivial and completely independent of $| \vec{q} |$, which is a consequence of the contact nature of the interaction. This fact alone is an indication that this term cannot exist on its
own: in configuration space this corresponds to a source of a spatially uniform, constant three-potential $\vec{A}$, which can always be transformed away by a gauge transformation. The result, at $q^2 = 0$ and elsewhere, is:

$$\langle \text{g.s.} \uparrow | H_+^{2-b} | \text{g.s.} \downarrow \rangle = \frac{eQ_W}{2\pi^{3/2}} \sqrt{\frac{\omega}{m}} G_F m \omega$$  \hspace{1cm} (12)$$

which is $2\sqrt{\pi}$ times larger than the absolute value of the one-body part.

**Serot's Argument**

We see that the two contributions to the matrix element differ substantially and the unusual LWL limit behaviour seems to persist. In order to see how Serot's argument works in this specific example, we express the transverse current matrix element in terms of multipoles; in this case, in terms of the transverse electric dipole. Serot's argument ought to be satisfied order by order in perturbation theory, or equivalently, in the expansion in the weak coupling constant $G_F$). We are working to first order in perturbation theory. The critical term, which ought to vanish at least as $q^2$, is proportional to the divergence of the conserved electromagnetic current:

$$\int d\vec{R} \cdot \vec{\nabla} \cdot \vec{J}_{fi} \left[ 1 + \vec{r} \cdot \vec{\nabla} \right] J_1(qR) Y_{11}$$  \hspace{1cm} (13)$$

There are two sources of parity violation, otherwise this matrix element would be identical to zero, to this order in $G_F$: the admixtures in the wave functions and the "contact" current. Let us write them out explicitly:

$$\nabla \cdot \vec{J}_{fi} = \nabla \cdot \langle S|\vec{J}_{2-b}|S \rangle + \sum_{n \neq 0} \left\{ \nabla \cdot \langle S|\vec{J}_{1-b}|P \rangle \epsilon_n + \epsilon^* \nabla \cdot \langle P|\vec{J}_{1-b}|S \rangle \right\}$$

$$= -i\langle S|[V_{PV},\rho]|S \rangle + \sum_{n \neq 0} \left\{ \langle S|[H_0,\rho]|P(n) \rangle \epsilon_n + \epsilon^* \langle P(n)|[H_0,\rho]|S \rangle \right\}$$  \hspace{1cm} (14)$$

(the overall factors are not important in this argument). Now use the definition of the admixture coefficients $\epsilon_n$ (eq.4):

$$\epsilon_n = \frac{\langle P(n)|V_{PV}|S(n') = 0 \rangle}{E_0 - E_n}$$

and the fact that the unperturbed states are eigenfunctions of $H_0$, which implies
that
\[ \langle S| [H_0, \rho]|P(n)\rangle = (E_0 - E_n)\langle S|\rho|P(n)\rangle \]
\[ \langle P(n)| H_0, \rho||S \rangle = (E_n - E_0)\langle P(n)|\rho|S \rangle \]  
(15)

which in turn leads to:

\[ \vec{\nabla} \cdot \vec{J}_{fi} = \vec{\nabla} \cdot \langle S|\vec{J}_{2-b}|S \rangle \]
\[- i \sum_{n \neq 0} (E_0 - E_n)\langle S|\rho|P(n)\rangle \frac{\langle P(n)|V_{PV}|S(n' = 0)\rangle}{E_0 - E_n} \]
\[- i \sum_{n \neq 0} \langle S(n' = 0)|V_{PV}|P(n)\rangle \frac{(E_n - E_0)\langle P(n)|\rho|S \rangle}{E_0 - E_n} \]
\[= -i\langle S|[V_{PV}, \rho]|S \rangle - \]
\[- i \sum_{n \neq 0} \left\{ \langle S|\rho|P(n)\rangle\langle P(n)|V_{PV}|S \rangle - \langle S|V_{PV}|P(n)\rangle\langle P(n)|\rho|S \rangle \right\} \]
\[= -i\langle S|[V_{PV}, \rho]|S \rangle - i \sum_{n} \left\{ \langle S|\rho|n||V_{PV}|S \rangle - \langle S|V_{PV}|n||\rho|S \rangle \right\} \]
(16)

where in the last step the same S-wave expectation value, which happens to be zero because of the "wrong" quantum numbers, was added and subtracted. The (vanishing) contributions of all other partial waves necessary to make this set of states complete were added and their complex conjugates subtracted. All we need to do now is remember that the unperturbed Hamiltonian eigenstates form a complete set, which immediately leads us to the final result:

\[ \vec{\nabla} \cdot \vec{J}_{fi} = -i\langle S|[V_{PV}, \rho]|S \rangle - i\langle S|\rho V_{PV}|S \rangle + i\langle S|V_{PV} \rho|S \rangle = 0 \]  
(17)

This proves that this part of the elastic parity-violating transverse electric dipole, or any other multipole, for that matter, identically vanishes, not just like some power of $q^2$, to this order in $G_F$.

It is now clear why this LWL limit was not obtained in the explicit calculation in this and section 3.4: only the first excited P-wave contribution was included, not all of them. The same proof holds for the three-quark system.

The above shown proof is model independent, so we may set the second term (proportional to the time derivative of the charge dipole moment) in the transverse electric dipole equal to zero, at least to first order in $G_F$. This constitutes a new definition of the elastic parity-violating electromagnetic current matrix element, which satisfies the threshold theorem by construction. In the following section I argue that the new definition is more general than the above, perturbation-theoretic, argument would suggest.
3.7 New Definition of the Elastic Parity-Violating Electromagnetic Matrix Element

We have just proven that the elastic parity-violating electromagnetic current matrix element is equivalent to one part of the transverse electric dipole moment, to first order in \( G_F \). In the following we will argue that this is a general result independent of perturbation theory.

First we observe that the "remaining part" of the transverse electric dipole moment, which vanishes to first order perturbation theory, is related to the time derivative of the elastic Coulomb dipole moment. The Coulomb and the transverse elastic multipoles obey selection rules imposed by parity and time-reversal symmetries of the wave functions and current operators (in other words: interaction). These selection rules can be read off from Tables 7 and 8, where the phases of the elastic multipoles are given in two cases.

The first case (Table 7), where the current operator is a polar vector and the states have no abnormal parity admixtures, or the current operator is an axial vector and the states have abnormal parity admixtures, will be called "normal-parity" case. The second case (Table 8), where the current is an axial vector and the states have no abnormal parity admixtures, or the current operator is a polar vector and the states have abnormal parity admixtures is called "abnormal-parity". Even though the terms comprising each of the two cases may seem very different, they have the same transformation laws. This is not very surprising: the abnormal parity admixtures are induced by the parity-violating two-body potential. The parity-violating axial, two-body current operator is related to the parity-violating two-body potential through the current conservation relation, so that they share transformation properties under discrete symmetries. Hence their contributions to the multipoles can be classified as single entities: the elastic normal- and abnormal-parity multipoles.

The elastic transverse magnetic and electric dipoles transform in the same way under time-reversal, but have opposite parities. This means that one or the other will vanish, depending on whether it is a normal- or abnormal-parity situation. It is easy to see from Table 7 that all normal-parity elastic transverse electric multipoles vanish in time-reversal-invariant electron scattering, whereas Table 8 implies that all abnormal-parity, time-reversal conserving Coulomb and transverse magnetic multipoles vanish. As emphasized before, this means that the elastic abnormal-parity transverse electromagnetic Hamiltonian matrix element for \( j = \frac{1}{2} \) states is completely determined by a single multipole: the transverse electric dipole.
The following parity and time-reversal transformation rules for normal – parity elastic multipoles are taken from Appendix B of (ref.13):

<table>
<thead>
<tr>
<th>Multipole</th>
<th>$M^\text{Coul}_{JM}$</th>
<th>$T^\text{el}_{JM}$</th>
<th>$T^\text{mag}_{JM}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetry</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time reversal</td>
<td>(-1)$^J$</td>
<td>(-1)$^{J+1}$</td>
<td>(-1)$^{J+1}$</td>
</tr>
<tr>
<td>Parity</td>
<td>(-1)$^J$</td>
<td>(-1)$^J$</td>
<td>(-1)$^{J+1}$</td>
</tr>
</tbody>
</table>

Table 7 Phases imposed on normal-parity elastic multipoles due to parity and time reversal in time-reversal conserving scattering.

Next, we present the phases for abnormal – parity elastic multipoles:

<table>
<thead>
<tr>
<th>Multipole</th>
<th>$M^\text{Coul}_{JM}$</th>
<th>$T^\text{el}_{JM}$</th>
<th>$T^\text{mag}_{JM}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetry</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time reversal</td>
<td>(-1)$^J$</td>
<td>(-1)$^{J+1}$</td>
<td>(-1)$^{J+1}$</td>
</tr>
<tr>
<td>Parity</td>
<td>(-1)$^{J+1}$</td>
<td>(-1)$^{J+1}$</td>
<td>(-1)$^J$</td>
</tr>
</tbody>
</table>

Table 8 Phases imposed on abnormal-parity elastic multipoles due to parity and time-reversal in time-reversal conserving scattering. Abnormal-parity multipoles consist of two separate terms with identical parity and time-reversal transformation properties, hence they are classified as single entities.

Furthermore, vanishing of the abnormal-parity Coulomb multipoles implies that the abnormal-parity transverse electric dipole moment has the following form (compare with eq.(3.4.4)):

$$T^\text{el}_{11} = \frac{-q}{i\sqrt{2}} \int d\vec{r} \left( \vec{r} \cdot \vec{J}_{fi} \right) j_1(qr)Y_{11}$$  \hfill (1)
where \( \vec{J}_f \) is the exact current. This result is valid to arbitrary order of perturbation theory.

Hence, we define the elastic parity-violating electromagnetic current matrix element of a spin \( 1/2 \) system this way:

\[
\vec{J}_5 \cdot \hat{e}_+ = \sqrt{3\pi} \ q \ \int d\vec{r} \ (\hat{\vec{r}} \cdot \vec{J}_f) \ j_1(qr)Y_{11} \tag{2}
\]

Using eq.(3.4.2), we can identify \( \beta \). We define the parity-violating electromagnetic form factor \( H(Q^2) \) this way:

\[
\beta = \frac{G_F}{\sqrt{2}} Q^2 H(Q^2) \tag{3}
\]

Let us now apply this definition to the first abnormal parity admixture contribution to the parity-violating electromagnetic form factor of the nucleon:

\[
\frac{G_F}{\sqrt{2}} Q^2 H(Q^2)_{\text{admix}} = -q\sqrt{6\pi} Re \left( \sum_{i=1}^{2} \epsilon_i B_i^* \right)
\]

where \( \epsilon_i \) are first admixture parameters given in eq.(3.3.6). The spin and isospin structure of the matrix elements \( B_i \) is the same as that of \( A_i \) in section 3.4. The only difference between the two calculations is the spatial integral. The results of a straightforward calculation (Appendix J) are

\[
B_1 = \left\{ 1 + \tau_3^N \left[ 13 - \frac{2q^2}{3m_q\omega} \right] \right\} \left( \frac{ie}{3^{3/4}} \right) \frac{1}{12m_q} \frac{q}{\sqrt{\pi m_q\omega}} \exp\left(-\frac{-q^2}{6m_q\omega}\right)
\]

\[
B_2 = (1 - \tau_3^N) \left( \frac{ie}{3^{3/4}} \right) \frac{1}{18m_q} \frac{q}{\sqrt{\pi m_q\omega}} \exp\left(-\frac{-q^2}{6m_q\omega}\right) \tag{4}
\]

where

\[
B_1(\uparrow, + \downarrow) = \int d\vec{r} \ (\langle (70,^28) \uparrow | \vec{J} | (56,^28) \downarrow \rangle \cdot \hat{\vec{r}}) j_1(qr)Y_{11}(\hat{r})
\]

\[
B_2(\uparrow, + \downarrow) = \int d\vec{r} \ (\langle (70,^48) \uparrow | \vec{J} | (56,^28) \downarrow \rangle \cdot \hat{\vec{r}}) j_1(qr)Y_{11}(\hat{r})
\]

Note that \( B_1 \) vanishes at a nonzero value of \( q \), which is typical of Fourier transforms of wave functions with nodes. This is a direct consequence of the fact that this form factor is a Fourier transform of a product of an S and a P wave function. This one-body current, first parity admixture parity-violating form factor will be shown on Fig. 11, together with the results of analogous calculations for the two-body current.
3.8 $Z^0$ Induced Parity-Violating Two-Body Electromagnetic Current and its Matrix Elements

Just as in the simple example in section 3.6, we start the construction of the two-body current by writing down a gauge invariant set of Feynman diagrams. In the case of $Z$ exchange the smallest such set contains four graphs (see Fig.9).

![Feynman diagrams](image)

Fig. 9 Feynman diagrams describing a gauge invariant, covariant parity-violating amplitude due to $Z^0$ exchange.

The four-current defined by these diagrams satisfies:

$$ q^\mu J_\mu = 0 $$ (1)
This can be proven by explicit calculation using the Ward-Takahashi identity for the $\gamma qq$ vertex and the Dirac equation for the external quark lines.

The quark momentum distribution in a nucleon, in this non-relativistic quark model, is a Gaussian with a width determined by the quark mass and the oscillator frequency $\omega$. The dimensional combination $m_q \omega$ is much smaller than $M_Z^2$, hence there is a very small probability that a quark can have momentum $0$ ($M_Z^2$). In the limit when the $Z$ mass is much larger than the exchanged four-momentum, we may neglect the four-momentum dependence of the $Z$ propagator. This fact and a non-relativistic reduction of this current, keeping only the negative energy states in the fermion propagators, lead to the following expression for the two-body $Z$ induced three-current:

$$J_Z^{2-b}(\vec{x}_1, \vec{x}_2, \vec{r}) = \frac{eG_F \delta(\vec{r} - \vec{x}_1)\delta(\vec{x}_1 - \vec{x}_2)}{\sqrt{2}} [\bar{S}_{12}T_{12} + \bar{S}_{12}^2T_{12}^2]$$

where

$$\bar{S}_{12}^1 = \left(\frac{1}{2m_q}\right)(\vec{\sigma}_1 - \vec{\sigma}_2) \quad \bar{S}_{12}^2 = \left(\frac{1}{2m_q}\right)(\vec{\sigma}_1 + \vec{\sigma}_2)$$

$$T_{12}^1 = \frac{1}{2}(1 - \sin^2 \theta_W) (\tau_3^1 - \tau_3^2) \quad T_{12}^2 = \frac{\sin^2 \theta_W}{6} (1 - \tau_3^1 \tau_3^2)$$

Before we proceed with the evaluation of the matrix elements, we ought to check the non-relativistic gauge invariance relation eq.(4.1). This will be done in momentum space in order to avoid complications due to taking derivatives of products of Dirac delta functions. In order to pass to momentum space, we need to know how to take Fourier transforms of few-body operators. This part is done following K. Ohta (ref.46) and is described in Appendix L.

This two-body current satisfies current conservation relation eq.(3.6.9) in conjunction with the two-body potential. Now, we may start calculating the parity-violating transverse electric dipole matrix element as defined in eq.(3.7.2). A straightforward evaluation described in Appendix J leads to the following result:

$$\int d\vec{r} \left( \vec{r} \cdot \langle (56, 2, 8)^\dagger | J_Z^{2-b} | (56, 2, 8)^\perp \rangle \right) j_1(qr)Y_{11}(\hat{r})$$

$$\quad = -\frac{G_F}{\sqrt{2}} \left[ (1 - 2\sin^2 \theta_W) \tau_3^N + \frac{2}{9} \sin^2 \theta_W \right] \sqrt{\frac{\omega}{4m_q (4\pi)^2}} \exp\left[ -\frac{q^2}{24m_q \omega} \right]$$

This expression will not be evaluated numerically until the analogous calculation of the $W$ induced two-body current has been completed.
3.9 $W^\pm$ Induced Parity-Violating Two-Body Electromagnetic Current and its Matrix Elements

In this case there are five diagrams, the "extra" graph being the "W-in-flight" graph (Fig. 10).

![Feynman Diagram](image)

Fig. 10 The fifth Feynman diagram necessary for the gauge invariance of the $W^\pm$ induced parity-violating electromagnetic current.

This graph's contribution is negligible because of the presence of two $W$ propagators, which make it $O(G_F^2)$. Although this diagram plays an important role in preserving the gauge invariance of the covariant amplitude, after the non-relativistic reduction it does not play any role in the non-relativistic gauge invariance relation. The proof of gauge invariance of the covariant amplitude is slightly more complicated than in the $Z$ case, because we now need the spin one boson Ward-Takahashi identity. This identity can be found in ref. 47 and with its application we can prove eq.(3.7.1).

The non-relativistic gauge invariance relation is slightly more complicated, too. There are two sources of gauge invariance breaking (see section 3.5): isospin factors and momentum dependence. It turns out that only one of them contributes to the two-body current and that is the isospin dependence. It has been known for some time (ref. 20) that such "nonlocal-isovector" two-body currents cannot be unambiguously determined from the non-relativistic two-body potential, without a relativistic model for the current operator. This model (the five Feynman diagrams depicted in Fig. 10,11 evaluated in the Salam-Weinberg model) predicts no such isovector, momentum-dependence induced current.
The result is:

\[ J_{W}^{2-\beta}(\vec{x}_1, \vec{x}_2, \vec{r}) = \frac{eG_F}{\sqrt{2}} \delta(\vec{r} - \vec{x}_1) \delta(\vec{x}_1 - \vec{x}_2) \vec{S}_{12}^3 I_{12}^3 \]  

(1)

where

\[ \vec{S}_{12}^3 = \left( \frac{1}{2m_q} \right) i(\vec{\sigma}_1 \times \vec{\sigma}_2) \]

\[ I_{12}^3 = \frac{i}{2} \cos^2 \theta_C (\vec{\tau}_1 \times \vec{\tau}_2)_3 \]

A straightforward evaluation of the spin-spatial matrix elements (Appendix K) tells us that this current's contribution to the nucleon parity-violating electromagnetic form factor is zero. This is a surprise because the charge-changing potential makes a large contribution to the parity admixtures \( \epsilon_{1,2} \) of the nucleon. Now, we plot the parity-violating electromagnetic form factors (Fig. 11).

Fig. 11 Absolute value of the parity violating electromagnetic form factor of the proton as a function of \( \tau = \left( \frac{Q^2}{2M} \right)^2 \), where \( Q^2 \) was taken to be equal to \( q^2 \) and \( M=940 \text{ MeV} \). Solid: absolute value of the one-body current, lowest admixture contribution; dashes: negative two-body current contribution.
The one-body part is positive at first, but then quickly decreases, due to a large negative exponent, and the two-body current contribution takes over just before the one-body part goes through a zero. The zero in the one-body part of the form factor appears because it is, roughly speaking, a Fourier transform of a product of an S and a P wave function, which has a zero in configuration space. The two-body part is negative for a proton (the dashed line is its absolute value) and it dominates at high $q^2$. But, that is the kinematic region where the non-relativistic approximation breaks down and the predictions of this model ought to be taken \textit{cum grano salis}.

Neutron: the nucleon parity-violating electromagnetic form factor is almost entirely ($\geq 90\%$) isovector, so that the neutron form factor is negative at low $q^2$ and then it changes sign at $\tau \approx 10^{-1}$ due to a positive two-body current contribution.
3.10 An Approximate Closure Estimate of the Elastic Nucleon Parity-Violating Electromagnetic Current Matrix Element

As was mentioned before, one can try to establish an approximate estimate of the absolute value of the contribution of all excited states to the one-body part of the parity-violating electromagnetic form factor by factoring out the lowest energy denominator and completing the set of states in the sum over intermediate states. Using the fact that this is just a representation of the unit operator, one arrives at a ground state expectation value of the product of the parity-violating potential and the electromagnetic current operator divided by the lowest energy denominator. As will be shown later, this value turns out to be the exact sum of all admixture contributions in models with simple harmonic oscillator wave functions, to first order in $G_F$.

The newly defined operator might be non-hermitian due to non-commutativity of the two basic building blocks. This can arise due to non-commutativity of the isospin factors and/or non-commutativity of the spin-spatial parts. This calls for a symmetrization i.e. taking one half of the anti-commutator of the two operators, which makes the operator-product hermitian. If we remember that both the initial and the final state contain admixtures, we will understand that we must not multiply this anti-commutator by one half, even though it may seem as double-counting. We see that symmetrization is necessary for the correct description of physics and the mathematical consistency of the theory.

In a nonrelativistic quark model calculation like this, there is another point worth discussing before proceeding to calculate: the observable spectrum of nucleon excited states corresponds only to one part of the complete harmonic oscillator spectrum. This is ordinarily explained by invoking a new degree of freedom, called colour, whose dynamics allow only singlet, i.e. completely antisymmetric, states to be observed (this usually goes under the name of confinement). Quarks are postulated to be fermions, so in order to satisfy Pauli's principle we must keep only the completely symmetric spin-spatial-isospin states. But in our sum over all intermediate states, we have included states with other symmetries, or none at all. This would lead us to believe that we are over-counting, and that we must correct for "confinement effects". Fortunately, this is not so: the initial and the final state are completely symmetric, by design. So are the two operators comprising the operator-product considered here, hence allowing only completely symmetric intermediate states. All other states contribute nothing, by symmetry.
Having resolved this issue, we can go on to evaluate this matrix element using this very same symmetry of the problem. The operator-product will consist of nine terms (the potential consists of three terms, as does the current) which can be divided into two classes which are separately symmetric under the exchange of any two indices: $A_{ijk}$, $B_{ijk}$. The first class (A) contains three elements,

$$A = \{V_{12}, J_3\} + \{V_{23}, J_1\} + \{V_{31}, J_2\}$$

schematically denoted by

$$(1,2,3) + (2,3,1) + (3,1,2)$$

and the other (B) contains six terms, in schematic notation:

$$(1,2,1) + (3,1,1) + (1,2,2) + (3,2,2) + (1,3,3) + (2,3,3)$$

The contribution of the first class will turn out to be zero (see Appendix M), so that the whole contribution will come from the second class operator (B). Using the symmetry of the states and the operator, we may write:

$$\langle N | B | N \rangle = 6 \langle N | B_{ijj} | N \rangle$$

where $i \neq j$ are an arbitrary pair of indices. The most convenient choice will be $i = 1$, $j = 2$. A straightforward, but lengthy, calculation described in Appendix M yields the following result for the closure estimate of the one-body current parity-violating form factor:

$$H(q^2)_{1-b} = -H(q^2)_{2-b}$$

The signs of the one- and two-body terms are opposite, in accord with the threshold theorem. This means that the contribution of all admixed states together with the two-body current contribution results in a vanishing matrix element. As will be shown later, the weighting in the closure sum is incorrect, but it still leads to the correct results due to special properties of harmonic oscillator wave functions. This is a very interesting result because it indicates another case of strong cancellation among the higher excited state contributions. It strongly suggests that there might be an analogue of Serot's theorem beyond the threshold region. This subject will be pursued in the next section.
3.11 A General Closure Argument About the Elastic Parity-Violating Electromagnetic Current Matrix Elements

One can do an exact closure calculation, if one can express the one-body electromagnetic current in the form of a commutator of the unperturbed Hamiltonian and some other operator, which would in effect cancel the energy denominator and leave an unweighted closure sum. The one-body convection electromagnetic current can be written as the time derivative of the electric dipole operator \( \vec{d} = \rho(\vec{r})\vec{r} \). Using the Heisenberg equations of motion, we can write:

\[
\vec{J}_{1-b} = i \left[ H_0, \vec{d} \right]
\]

where \( H = H_0 + V_{PV} \), which is exactly the relation we are seeking. The one-body magnetization current is obtained from the same formula if \( H_0 \) includes the magnetic moment Hamiltonian \(-\vec{\mu} \cdot \vec{B}\).

We are working to first order in perturbation theory. As discussed before, the term of interest is proportional to the elastic conserved electromagnetic current matrix element \( \vec{J}_{f'_i} \). There are two sources of parity violation to this order in \( G_F \): the admixtures in the wave functions and the "contact" current. Let us write them out explicitly, using eq.(1):

\[
\vec{J}_{f'_i} = \langle S(0) | \vec{J}_{2-\delta}^{PV} | S(0) \rangle + \sum_{n \neq 0} \left\{ \langle S(n') = 0 | \vec{J}_{1-\delta} | P(n) \rangle \epsilon_n + \epsilon_n^*(P(n)|\vec{J}_{1-\delta}|S(n' = 0)) \right\}
\]  
\[
= \langle S(0) | \vec{J}_{2-\delta}^{PV} | S(0) \rangle + \sum_{n \neq 0} \left\{ \langle S(0)||H_0, \vec{d}||P(n)\rangle \epsilon_n + \epsilon_n^*(P(n)||H_0||S(0)) \right\}
\]

where \( |S(n)\rangle \) denotes the S-wave, \( n^{th} \) eigenstate of the strong Hamiltonian \( H_0 \); the ground state corresponds to \( n = 0 \). Similarly, \( |P(n)\rangle \) denotes the P-wave, \( n^{th} \) eigenstate of the strong Hamiltonian \( H_0 \). In the following we will suppress one or the other of these two indices, when it is obvious from the context what is meant.

Now use the definition of the admixture coefficients \( \epsilon_n \) (section 3.6):

\[
\epsilon_n = \frac{\langle P(n)|V_{PV}|S(n' = 0) \rangle}{E_0 - E_n}
\]

and the fact that the unperturbed states are eigenstates of \( H_0 \), which implies that

\[
\langle S(0)||H_0, \rho||P(n) \rangle = (E_0 - E_n)\langle S(0)||\rho||P(n) \rangle
\]

\[
\langle P(n)||H_0, \rho||S(0) \rangle = (E_n - E_0)\langle P(n)||\rho||S(0) \rangle
\]
which in turn leads to:

\[
\tilde{J}_{fi} = \langle S|\tilde{J}_{2-b}^{PV}|S \rangle \\
+ i \sum_{n \neq 0} \frac{(E_0 - E_n)\langle S(0)|\tilde{d}|P(n)\rangle}{E_0 - E_n} \langle P(n)|V_{PV}|S(0)\rangle \\
+ i \sum_{n \neq 0} \frac{\langle S(0)|V_{PV}|P(n)\rangle}{E_0 - E_n} (E_n - E_0)\langle P(n)|\tilde{d}|S(0)\rangle \\
= \langle S|\tilde{J}_{2-b}^{PV}|S \rangle + \\
+ i \sum_{n \neq 0} \{\langle S(0)|\tilde{d}|P(n)\rangle\langle P(n)|V_{PV}|S(0)\rangle - \langle S(0)|V_{PV}|P(n)\rangle\langle P(n)|\tilde{d}|S(0)\rangle\} \\
= \langle S|\tilde{J}_{2-b}^{PV}|S \rangle + \\
+ i \sum_{n} \{\langle S|\tilde{d}|n\rangle\langle n|V_{PV}|S \rangle - \langle S|V_{PV}|n\rangle\langle n|\tilde{d}|S \rangle\} \\
\]

(4)

where in the last step we have added and subtracted the same S-wave term (zero) and added the (vanishing, because they have the wrong quantum numbers) contributions of all other partial waves necessary to make this set of states complete and subtracted their complex conjugates. All we need to do now is remember that the unperturbed eigenstates form a complete set, which immediately leads us to the final result:

\[
\tilde{J}_{fi} = \langle S(0)|\tilde{J}_{2-b}^{PV} - i[V_{PV}, \tilde{d}]|S(0)\rangle \\
\]

(5)

This provides an exact result for the elastic parity-violating transverse electric dipole, to this order in \( G_F \):

\[
\tilde{J}_6^\gamma \cdot \hat{e}_+ = q\sqrt{3} \pi \int d\hat{r} j_1(qr)Y_{11}(\hat{r}) \left( \tilde{r} \cdot \langle S(0)|\tilde{J}_{2-b}^{PV} - i[V_{PV}, \tilde{d}]|S(0)\rangle \right) \\
\]

(6)

This expression shows that the exact result for the parity-violating electromagnetic current matrix element does not depend on the details of the spectrum of the strong Hamiltonian, such as the proximity of the lowest-lying negative parity excited state, which is a widely held popular belief. It is given in terms of the ground state expectation value of the difference between the two-body parity-violating electromagnetic current operator and \( i \) times the commutator of the parity-violating potential and the electric dipole operator. This result completely eliminates the need to calculate the parity admixtures in the wave functions for the purpose at hand.

It turns out that in the case of contact parity-violating interactions, such as those employed in the “toy” model of section 3.6, or in the full quark model of this
thesis, the relation

\[
\hat{J}^{PV}_{2\rightarrow 0} = i \left[ V_{PV}, \hat{d} \right]
\]

(7)
is exactly satisfied (see Appendix N) and, consequently, the parity-violating electromagnetic current matrix element vanishes, in this approximation.

This fact explains the approximate closure results of the previous section: the dipole operator connects only harmonic oscillator states satisfying \( \Delta E = \hbar \omega \) i.e. only the adjacent states. In the case of the ground state there is only one such state: the first excited state, and it has the correct energy denominator, so the "approximate" result is actually exact.

We suspect that the relation eq.(7) is true even for finite-range parity-violating interactions, but the proof must be provided for each new interaction separately. The most important question is whether the analogous one-body current equation becomes invalid once spin and iso-spin strong potentials are introduced into the Hamiltonian. The higher order relativistic corrections might also spoil the result.
3.12 Summary and Conclusions

To summarize, in this part of this thesis we have presented the results of the following work:

- An analysis of separability of "strange" form factors from elastic, parity-violating electron-nucleus scattering data

- A calculation of the lowest lying abnormal parity admixture to the nucleon wave function

- The elastic parity-violating electromagnetic matrix element of the nucleon induced by the above mentioned admixture

- Check of the conservation of the electromagnetic current in the above calculation

- Construction of a simple model with a conserved electromagnetic current and an explicit demonstration of the cancellation leading to the vanishing threshold behaviour of the parity-violating electromagnetic current matrix element

- A general result about the form of the elastic parity-violating electromagnetic matrix element and an appropriate modification of its definition.

- Construction of the two-body currents associated with the Z and W exchange parity-violating potentials and evaluation of their nucleon expectation values

- We established an approximate closure estimate of the contribution of all parity admixtures to the one-body current term and showed that it is equal and opposite in sign to the two-body current contribution, thus leaving a vanishing parity-violating electromagnetic current matrix element of the nucleon, in this specific non-relativistic quark model with harmonic oscillator wave functions

- We established an exact closure-based sum of the contributions of all parity admixtures to the one-body current term and showed that in all quark models with spin and iso-spin independent local strong quark-quark potentials it is equal and opposite in sign to the two-body current contribution. The resulting total parity-violating electromagnetic current matrix element of the nucleon vanishes, in this specific class of non-relativistic quark models.

The main conclusion of this part of this thesis is that the elastic parity-violating electromagnetic matrix element of the nucleon, in this model, vanishes identically and hence does not affect the extraction of the "strange" form factors of the nucleon from the elastic parity-violating electron-proton asymmetry measurement.
Radiative corrections are expected to be of the order of 1% (ref.23), so that the experimental extraction of "strange" form factors would be thwarted only if they turned out to be of the order of a few % of the $u, d$-induced nucleon form factors. This work has only opened up the subject: there are many other quark models which allow an evaluation of the parity-violating electromagnetic form factor of the nucleon. The MIT bag model seems to deserve attention, because of its relativistic nature.

Secondly, the isospin admixtures in the nucleon and the associated corrections to the parity-violating asymmetry have not been addressed, yet. In short, this is a preliminary contribution to the evaluation of the corrections to the elastic parity-violating electron-nucleon asymmetry in realistic nucleon models.
Appendix A  Lorentz Transformation Properties of Helicity States

In the following we will follow the presentation by Wick (ref.32). The derivation of the Lorentz transformation properties of the helicity states from this reference will be repeated in order to set the stage for further developments.

Let us suppose that in a given reference frame $S$ an observer $O$ sees a particle $A$ in motion with momentum $\vec{p}$ and helicity $\lambda$ i.e. in a state $|\vec{p}\lambda\rangle$. Let $S_I$ be a new reference frame obtained from $S$ by a boost $l$. We want to know how the observer $O_I$ sees the motion of $A$. We apply the transformation law for the states (this is just the generalization of the transformation law for rotations extended to Lorentz boosts):

$$|\vec{p}\lambda\rangle_I = U[l^{-1}]|\vec{p}\lambda\rangle$$  \hspace{1cm} (A.1)

where $U[l]$ is the boost operator for the boost $l$. Let us denote by $\vec{p}'$ the boosted 3-momentum of $A$:

$$p'_\mu \equiv (p_\mu)_{S_I} = \Lambda_{\mu\nu}(l^{-1})p^\nu$$ \hspace{1cm} (A.2)

It is clear that $|\vec{p}\lambda\rangle_{S_I} = |\vec{p}'\lambda\rangle$. The question is: what is the relation relation between $\lambda$ and $\lambda'$? To answer this, we write:

$$U[l^{-1}]|\vec{p}\lambda\rangle = U[l^{-1}]U[h(\vec{p})]|\vec{p}'\lambda\rangle$$ \hspace{1cm} (A.3)

where we used the definition of the helicity states:

$$|\vec{p}\lambda\rangle \equiv |\vec{p},s,s_z = \lambda\rangle = U[h(\vec{p})]|p,s,s_z = \lambda\rangle$$ \hspace{1cm} (A.4)

and

$$p'_\mu = (m, \vec{0})$$

Now, we multiply eq.(A.3) by one, in the following form:

$$U[h(\vec{p}')]U^{-1}[h(\vec{p}')] = 1$$ \hspace{1cm} (A.5)

where:

$$U[l^{-1}]|\vec{p}\lambda\rangle = U[h(\vec{p}')]|\vec{p}'\lambda\rangle$$ \hspace{1cm} (A.6)

Note the absence of prime on $\lambda$. Now we can write eq.(3) in this form:

$$U[l^{-1}]|\vec{p}\lambda\rangle = U[h(\vec{p}')]|\mathcal{R}|p\lambda\rangle$$ \hspace{1cm} (A.7)

where

$$\mathcal{R} = U^{-1}[h(\vec{p}')]U[l^{-1}]U[h(\vec{p})]$$ \hspace{1cm} (A.8)

The crucial observation is that this sequence of transformations is a rotation, no matter what $l$ is. The simplest way to see this is by looking at the effect of
on the four-vector \( \vec{p} \) on the four-vector \( \vec{p}_\mu = (m, \vec{0}) \). We have:

\[
\begin{align*}
    h(\vec{p}) : \vec{p} & \rightarrow p \\
    l^{-1} : p & \rightarrow p' \\
    h^{-1}(\vec{p}') : p' & \rightarrow \vec{p}
\end{align*}
\]

therefore

\[
h^{-1}(\vec{p}')l^{-1}h(\vec{p}) : \vec{p} \rightarrow \vec{p} \tag{A.9}
\]

It is clear that this can be at most a rotation because it takes a particle at rest back to its rest frame: \( \vec{0} \rightarrow \vec{0} \). Let us label this sequence of operations \( r(l, \vec{p}) \):

\[
r(l, \vec{p}) = h^{-1}(\vec{p}')l^{-1}h(\vec{p}) \tag{A.10}
\]

This \( r \) is usually called the Wigner rotation (sometimes also referred to as the Wick helicity rotation). The rest of the evaluation of \( \ket{p\lambda}_l \) is easy, we know what rotations do to states at rest, so:

\[
R|p\lambda\rangle = D^{(s)}_{\lambda'\lambda}[r(l, \vec{p})]|p\lambda\rangle \tag{A.11}
\]

\( D^{(s)}_{\lambda'\lambda} \) are just the Wigner rotation matrices (Wigner d functions, not to be confused with Wigner rotations, the subject of this investigation), so we may insert eq.(A.11) into eq.(A.7):

\[
U[l^{-1}]|p\lambda\rangle = D^{(s)}_{\lambda'\lambda}U[h(\vec{p}')]|p\lambda\rangle
\]

and then into eq.(A.1) to get:

\[
|\vec{p}\lambda\rangle_{S'1} = D^{(s)}_{\lambda'\lambda}U[h(\vec{p}')]|p\lambda\rangle = D^{(s)}_{\lambda'\lambda}|\vec{p}'\lambda\rangle \tag{A.12}
\]

This is the desired relation between states in frames \( S_l \) and \( S \). We have already proven that it is a pure rotation.
Appendix B  Lorentz Transformation Properties of Amplitudes and Observables in the Helicity Formalism

In order to determine the analogous transformation properties of helicity two-body amplitudes we note that the amplitudes are made of direct products of single particle helicity states sandwiching an operator which in our case is a Lorentz scalar. This determines uniquely the transformation law for the amplitudes which is just the direct product of the Wigner d matrices for the initial state particles and the direct product of the inverse (hermitian conjugate) Wigner d matrices for the final state particles:

\[
\langle \lambda_3 \lambda_4 | \lambda_1 \lambda_2 \rangle_{S^i} = D_{\lambda_3 \mu_3}^{(e)}(r_3)D_{\lambda_2 \mu_2}^{(e)}(r_2)D_{\mu_1 \lambda_4}^{(e)}(r_1)(\mu_3 \mu_4 | \mu_1 \mu_2)_{S^i}
\]

Note that each Wigner d matrix has its own, in principle different, rotation angles which depend on the specifics of the reaction such as the masses and velocities of the particles and the direction of the boost.

In order to determine the implications of this transformation law for the observables we must remember their general structure:

\[
\langle O_i \rangle = \text{Tr}[O_i \rho_i]
\]

where \(O_i\) is some spin observable and \(\rho_i\) is the spin density matrix of the \(i^{th}\) particle in the initial state, which is a sum of bilinear products of transition amplitudes:

\[
\rho_i = \sum_\alpha \langle \lambda_i | \alpha \rangle \langle \alpha | \lambda_i \rangle.
\]

Then, all we need is the Lorentz transformation law of the \(i^{th}\) initial particle spin density matrix:

\[
\langle \lambda | \rho | \lambda' \rangle_{S^i} = D_{\lambda \mu}^{(e)}[r]D_{\mu \lambda'}^{(e)}[r]
\]

and an analogous relation for the \(j^{th}\) final state particle where the Wigner d matrices appear with the negative argument. This will be the only case when we will be concerned with, because only the final state helicities are rotated if the boost is collinear the direction of motion of the initial state particles (this will be clear after we derive an explicit formula for the Wigner angle in Appendix C).

The spin density matrix can be expanded in spherical irreducible tensor operators (ITO) \(t_{JM}\) which have well known transformation properties under rotations. The observables are their ensemble expectation values and hence have the same
transformation properties:

\[ t_{JM}^{S_i} = D^{(J)\dagger}_{MM'}[r] t_{JM'}^{S_i} \]  
(B.4)

for initial state particles and:

\[ t_{JM}^{S_i} = D^{(J)}_{MM'}[r] t_{JM'}^{S_i} \]  
(B.5)

for final state particles. This is the principal result of this Appendix. It has immediate consequences for inelastic coincidence electron scattering: the recoil polarization of a spin 1/2 particle constitutes a polarization vector i.e. it can be expressed as an ITO of rank \( J = 1 \). The Wigner \( d \) matrices of order one lead to a transformation law which is just a simple rotation of a vector, as advertised. This result applies to all recoil polarization observables in any formalism (relativistic or non-relativistic) which expresses its results in terms of a “mixed” frame cross section. In other words, they are necessary in Arenhövel’s formalism (ref.9), too.
Appendix C  Evaluation of General Formulae for the Wigner Angle and Application to Inelastic Coincidence Electron Scattering

We are left with the task of evaluating the Euler angles $\alpha, \beta, \gamma$ corresponding to $r(l, \vec{p})$ for various values of $l$. Since we have left $l$ in its most general form, it can be a boost, a rotation or a combination of the two. By taking $l$ to be a rotation one can check that the helicity is a concept invariant under rotations (of course, there is no Wigner rotation in this case). In the case of a pure boost $l$ we can proceed along several different routes leading to different forms (equivalent to each other) of the final result.

The first approach is based on the observation that we need not use the defining representation of the Lorentz group, which is four-dimensional and cumbersome to work with, to evaluate this angle, but may use a lower dimensional one. The simplest nontrivial representation is two-dimensional and it can be completely specified in terms of Pauli matrices (ref.33) (this is the defining representation of $SL(2,\mathbb{C})$ which has the same Lie algebra as the Lorentz group, but different topological properties, in the same vein as the relation between $SO(3)$ and $SU(2)$). The boost elements in this representation are:

$$L(\vec{v}) = \exp(-i\vec{v} \cdot \vec{K}) = \cosh(\frac{\vec{v}}{2}) + \vec{v} \cdot \bar{\sigma}\sinh(\frac{\vec{v}}{2})$$

where $\vec{K} = \frac{1}{2}\sigma$ and the rotations are:

$$R(\vec{\tau}) = \exp(-i\vec{\tau} \cdot \vec{J}) = \cos(\frac{\vec{\tau}}{2}) - i\vec{\tau} \cdot \bar{\sigma}\sin(\frac{\vec{\tau}}{2})$$

where $\vec{J} = \frac{1}{2}\sigma$, and $\bar{\sigma}$ are the Pauli matrices.

An explicit evaluation of eq.(A.10) in this representation:

a) confirms that it is a rotation,

b) requires that $\vec{\tau} \cdot \vec{p} = 0$ for arbitrary $\vec{p}$, which implies that the rotation axis is perpendicular to the plane containing $\vec{p}$, $\vec{p}'$ i.e confirms that the rotation is in the scattering plane (ejectile plane in coincidence electron scattering) which is equivalent to setting the Euler angles $\alpha = \gamma = 0$,

c) gives an explicit formula for the Wigner rotation $r$, where $\omega = \beta$:

$$tg\left(\frac{\omega}{2}\right) = \frac{-\sinh(\frac{1}{2}v_1)\sinh(\frac{1}{2}v_2)\sin(\gamma)}{\cosh(\frac{1}{2}v_1)\cosh(\frac{1}{2}v_2) + \sinh(\frac{1}{2}v_1)\sinh(\frac{1}{2}v_2)\cos(\gamma)}$$
After the use of the half-angle formulae for \( ch, sh \):

\[
2 \cosh^2 \frac{1}{2} \nu = \cosh \nu + 1
\]

\[
2 \cosh \frac{1}{2} \nu \sinh \frac{1}{2} \nu = \sinh \nu
\]

and noticing that \( \gamma = -\theta_{\text{cm}} \), we get Ritus' (ref.34) expression for the Wick rotation angle:

\[
tg \left( \frac{\omega}{2} \right) = \frac{\sinh \nu_1 \sinh \nu_2 \sin \theta_{\text{cm}}}{(1 + \cosh \nu_1)(1 + \cosh \nu_2) + \sinh \nu_1 \sinh \nu_2 \cosh \theta_{\text{cm}}}
\]

Note that the rotation is defined with respect to the coordinate frame defined by the three-momentum of the ejectile in the \( \text{cm} \) frame. The angle \( \omega \), which is the total rotation angle of the spin with respect to the \( \text{cm} \) frame coordinate system, will be referred to as the Wigner angle, or Wick helicity rotation angle. The difference between this "total" rotation angle and the rotation angle of the three-momentum, which is equivalent to the spin precession angle with respect to the three-momentum direction, will be called "hyperbolic defect" of the reaction (sometimes called Wigner angle, too, but not in this thesis). Note that \( \omega \) vanishes for \( \theta_{\text{cm}} = 0, \pi \), as promised, which implies that boosts parallel to the direction of motion of a particle will not induce a Wigner rotation. This is the case for the initial state particles in the \( \text{cm} \rightarrow \text{lab} \) boost, which means that they do not experience a spin rotation, as advertised in Appendix A

This formula will be used for the evaluation of the Wigner angle in electron scattering, where \( \nu_1 \) will be the rapidity of the \( \text{cm} \rightarrow \text{lab} \) boost and \( \nu_2 \) will correspond to the \( \text{cm} \rightarrow \text{rest} \) boost.

The correspondence between the rapidities and the more conventional boost parameters \( \gamma, \beta \) is:

\[
\gamma_i = \cosh(\nu_i), \quad \gamma_i \beta_i = \sinh(\nu_i)
\]

Another way of calculating the Wigner angle is by noticing that the three boosts involved in eq.(A.10) form a vector triangle i.e. they define vector addition of velocities in special relativity. At this point we use an old observation by Sommerfeld (ref.4) that the velocity addition theorem in special relativity corresponds to a noneuclidian triangle on a hyperboloid ("sphere of radius i"). The Wigner angle \( \varepsilon \) is just the hyperbolic defect (difference between \( \pi \) and the sum of inner angles of the triangle) of the velocity triangle formed by the Lorentz boost taking frame one to frame two and the boosts taking the particle to its rest frame from the two frames, respectively. At this point one can use a wide variety of formulae from spherical
trigonometry, appropriately modified to suit hyperbolic trigonometry, to express this angle in terms of any combination of three independent kinematic variables (such as angles, or sides) describing this velocity triangle.

Starting from the hyperbolic cosine theorem (ref.32) for each of the three angles of the ("hyperbolic") triangle:

\[
\cosh(a) = \cosh(b)\cosh(c) - \sinh(b)\sinh(c)\cos(\alpha) \\
\cosh(b) = \cosh(a)\cosh(c) - \sinh(a)\sinh(c)\cos(\beta) \\
\cosh(c) = \cosh(b)\cosh(a) - \sinh(b)\sinh(a)\cos(\gamma)
\]

(C.7)

by simple algebraic manipulation, we get:

\[
\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{\sinh(s-b)\sinh(s-c)}{\sinh(b)\sinh(c)}} \\
\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{\sinh(s)\sinh(s-a)}{\sinh(b)\sinh(c)}} \\
\sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{\sinh(s-a)\sinh(s-c)}{\sinh(a)\sinh(c)}} \\
\cos\left(\frac{\beta}{2}\right) = \sqrt{\frac{\sinh(s)\sinh(s-b)}{\sinh(a)\sinh(c)}} \\
\sin\left(\frac{\gamma}{2}\right) = \sqrt{\frac{\sinh(s-b)\sinh(s-a)}{\sinh(b)\sinh(a)}} \\
\cos\left(\frac{\gamma}{2}\right) = \sqrt{\frac{\sinh(s)\sinh(s-c)}{\sinh(b)\sinh(a)}}
\]

(C.8)

where \( s = \frac{1}{2}(a + b + c) \) and \( a, b, c \) are the hyperbolic arc angles corresponding to the sides of the triangle.

Putting these results into formulae for the sine and cosine of \( \frac{\epsilon}{2} \), where \( \epsilon = \pi - (\alpha + \beta + \gamma) \), we get:

\[
\sin\left(\frac{\epsilon}{2}\right) = \sqrt{1 + 2\cosh(v_1)\cosh(v_2)\cosh(v_3) - \cosh^2(v_1) - \cosh^2(v_2) - \cosh^2(v_3)} \\
4\cosh(\frac{1}{2}v_1)\cosh(\frac{1}{2}v_2)\cosh(\frac{1}{2}v_3) \\
\cos\left(\frac{\epsilon}{2}\right) = \frac{1 + \cosh(v_1) + \cosh(v_2) + \cosh(v_3)}{4\cosh(\frac{1}{2}v_1)\cosh(\frac{1}{2}v_2)\cosh(\frac{1}{2}v_3)}
\]

(C.9)

This formulation is particularly useful in the investigation of the nonrelativistic limit: all \( \cosh(v_i) \to 1 \) in this limit, so we get \( \epsilon = 0 \) i.e. \( \omega = \theta_{cm} - \theta_1 \), where (see Fig.1) \( \alpha = \theta_1 \), \( \beta = \pi - \theta_{cm} \), \( \gamma = \omega \). This is equivalent to the statement that there is no spin precession (Wigner rotation) in the nonrelativistic limit. This formula provides the spin precession angle with respect to the coordinate system defined by the rotated ejectile momentum in the lab frame, i.e. the rotation angle \( \theta_i^f - \theta_i^{f_m} \) of the three-momentum vector is subtracted from \( \omega \).
Note that $\varepsilon \geq 0$ i.e. $(\alpha + \beta + \gamma) \leq \pi$, because both quantities on the left hand sides of eq.(C.9) are positive. This tells us that $\theta_{1m}^c - \theta_{1}^L \geq \omega$, i.e. the spin three-vector rotates in the same direction, but more slowly than the momentum three-vector.

**Application to Inelastic Coincidence Electron Scattering**

I will specialize to pion electroproduction. The three boosts are: *cm to lab*, *cm to rest* and *lab to rest*, respectively. The boost parameters are:

\[
\begin{align*}
\bar{\beta}_1 &= \frac{\bar{q}_L}{\sqrt{\bar{q}_L^2 + W^2}} \quad \gamma_1 = \sqrt{1 + \left(\frac{\bar{q}_L}{W}\right)^2} \\
\bar{\beta}_2 &= \frac{\bar{p}_1^m}{E_1^m} \quad \gamma_2 = \frac{E_1^m}{M} \\
\bar{\beta}_3 &= \frac{\bar{p}_L}{E_1^L} \quad \gamma_3 = \frac{E_1^L}{M}
\end{align*}
\]

(C.10)

The relationship between the *lab* and *cm* frame opening angles is:

\[
ctg(\theta_1^L) = \sqrt{1 + \left(\frac{qL}{W}\right)^2} ctg(\theta^c_{1m}) + \left(\frac{qLE_1^c}{Wp_1^c}\right) \frac{1}{\sin \delta_{1m}^c} 
\]

(C.11)

where

\[
\begin{align*}
p_{1m}^c &= \frac{1}{2W} \sqrt{(W^2 - M^2 - \mu^2)^2 - 4M^2\mu^2} \\
E_1^m &= \frac{1}{2W} \sqrt{(W^2 - M^2 - \mu^2)^2 - 4M^2(\mu^2 - W^2)}
\end{align*}
\]

(C.12)

The kinematics of electroproduction is defined by two independent variables: $W$, $Q^2$. One of them can be substituted by the Bjorken $x$:

\[
Q = \sqrt{\left(\frac{x}{1-x}\right)(W^2 - M_A^2)}
\]

(C.13)

where

\[
x = \frac{Q^2}{2P_T \cdot q} 
\]

(3.C.14)

Direct application of eq.(C.4) leads to the result shown on Fig.2.
Similarly, the three-momentum and the energy of the outgoing nucleons in the cm frame for the deuteron electrodisintegration are given by:

\[ p_{1}^{CM} = \sqrt{\left( \frac{W}{2} - M \right) \left( \frac{W}{2} + M \right)} \]

\[ E_{1}^{CM} = \frac{W}{2} \]  

(3.C.15)

A substitution of eq.(C.15) in eq.(C.4) leads to qualitatively similar, but quantitatively smaller results for similar kinematics (we must stay below the \( \pi \) production threshold in the disintegration, in order to have a two-body final state).
Appendix D Derivation of Unpolarized Positivity Inequalities in the Cartesian Basis

The first step is to work in the Breit frame. We realize that the \( \text{lab} \rightarrow \text{Breit} \) boost is along the the \( z \) axis so that only the zeroth component is influenced. Just as in section 2.2, where the coincidence cross section was written in two (cm and lab) frames, the whole effect of the boost is reduced to multiplicative factors in front of the \( L \) and LT structure functions and a different phase space factor which is unimportant for this purpose. The Breit frame is defined by

\[
q_B' = (0,0,0,q_B)
\]

and in the lab frame, \( P_L' = (M,0) \). The boost transformation is, in matrix form

\[
B_{\text{Breit}} = \begin{pmatrix}
\frac{q_L}{Q} & 0 & 0 & -\nu \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\nu Q & 0 & 0 & \frac{q_L}{Q}
\end{pmatrix}
\]

so that, if \( q_L' = (\nu,0,0,q_L) \), then

\[
q_B = B_{\text{Breit}} q_L = \frac{q_L}{Q} q_L - \frac{\nu}{Q} \nu = Q
\]

The transverse helicity amplitudes are unchanged by these boosts, and \( e^B \) has the same form as in eq.(5) of ref.2 with \( q_B \) and \( \nu_B \) replacing \( q_L \) and \( \nu \) i.e. \( e^B \) in the Breit frame is the unit time-like four-vector:

\[
e^B_0 = (1,0,0,0)
\]

The second simplification used is the rotation of the response tensor about the \( z \)-axis through \( \phi \) which is described in detail in ref.2, so it will not be dwelt on. This transformation does not change the positivity conditions because it is an orthogonal coordinate transformation. The response tensor defined with respect to the rotated coordinate system \( (x',y',z') \) of Fig.6 will be denoted by \( w_{\mu\nu} \).
The relationship between the observables and the response tensor matrix elements in the Breit frame is given in Table 9:

| \( R_L \) | \( \eta^2 R_{oo} \) | \( \eta_B^2 w_{00} \) |
| \( R_T \) | \( R_{++} + R_{--} \) | \( w_{zz} + w_{yy} \) |
| \( R_{TT}^{(I)} \) | \( 2Re R_{+-} \) | \( w_{yy} - w_{zz} \) |
| \( R_{TT}^{(II)} \) | \(-2Im R_{+-} \) | \( 0 \) |
| \( R_{LT}^{(I)} \) | \( 2\eta Re(R_{o+} - R_{o-}) \) | \( 2\sqrt{2}\eta_B Re w_{oz}^B \) |
| \( R_{LT}^{(II)} \) | \( 2\eta Im(R_{o+} + R_{o-}) \) | \( 0 \) |
| \( R_{LT}^{(II)} \) | \( 2\eta Re(R_{o+} + R_{o-}) \) | \( 0 \) |
| \( R_{LT}^{(III)} \) | \( 2\eta Im(R_{o+} - R_{o-}) \) | \( 2\sqrt{2}\eta_B Im w_{oz}^B \) |

Table 9: Structure functions \( R \)'s (first column) as functions of the response tensor components in the helicity basis (second column), where \( \eta_B = \frac{\eta}{Q} \), or as functions of the general response tensor Cartesian components in the Breit frame and defined in the ejectile plane of Fig.5 (third column); all other elements of the general response tensor, e.g. \( w_{xy} \), \( w_{0y} \) are zero.

A direct evaluation of the principal minors of the response tensor yields the following inequalities (all \( w \)'s are in the Breit frame):

\[
\begin{align*}
w_{00}, w_{zz}, w_{yy} & \geq 0 \quad (D.5) \\
\begin{vmatrix} w_{00} & w_{0z} \\ w_{z0} & w_{zx} \end{vmatrix} & \geq 0 \quad (D.6) \\
\begin{vmatrix} w_{00} & 0 \\ 0 & w_{xx} \end{vmatrix} & \geq 0 \quad (D.7) \\
\begin{vmatrix} w_{zz} & 0 \\ 0 & w_{yy} \end{vmatrix} & \geq 0 \quad (D.8) \\
\begin{vmatrix} w_{00} & w_{0x} & 0 \\ w_{z0} & w_{xz} & 0 \\ 0 & 0 & w_{yy} \end{vmatrix} & \geq 0 \quad (D.9)
\end{align*}
\]
which together with the hermiticity of the response tensor \((w_{\mu\nu} = w_{\nu\mu}^*)\) leads to inequalities eq.(4.5): from eq.(D.5) and Table 7, we conclude that

\[
\begin{align*}
R_L &\geq 0 \\
R_T &\geq 0 \\
R_T &+ R_{TT}^{(1)} \geq 0
\end{align*}
\]

while, from eq.(D.6) and Table 7 we obtain:

\[
\begin{align*}
W_{00}w_{xx} &\geq |w_{xx}|^2 \\
4R_L\left[R_T - R_{TT}^{(1)}\right] &\geq \left(R_{LT}^{(1)}\right)^2 + \left(R_{LT}^{(1)}\right)^2
\end{align*}
\]

which completes the proof. All other inequalities emanating from eq.(D.7-9) are equivalent to the above shown.
Appendix E  Positivity Inequalities in the Helicity Basis

Working in the helicity basis, we have the following relationship between the structure functions $R$'s in the $lab$ frame and the rotated response tensor matrix elements in the Breit frame:

$$
\begin{array}{ccc}
R_L & \eta^2 R_{00} & \eta_B^2 w_{0+}^B \\
R_T & R_{++} + R_{--} & 2w_{++} \\
R_{TT}^{(I)} & 2\text{Re}R_{++} & 2w_{+} \\
R_{TT}^{(II)} & -2\text{Im}R_{++} & 0 \\
R_{LT}^{(I)} & 2\eta\text{Re}(R_{0+} - R_{0-}) & 4\eta_B R_{00}^B w_{0+}^B \\
R_{LT}^{(II)} & 2\eta\text{Im}(R_{0+} + R_{0-}) & 0 \\
R_{LT}' & R_{++} - R_{--} & 0 \\
R_{LT}'^{(I)} & 2\eta\text{Re}(R_{0+} + R_{0-}) & 0 \\
R_{LT}'^{(II)} & 2\eta\text{Im}(R_{0+} - R_{0-}) & 4\eta_B\text{Im}w_{0+}^B
\end{array}
$$

Table 10: Structure functions $R$'s (first column) as functions of the response tensor components in the helicity basis (second column) or as functions of the general response tensor components in the helicity basis in the Breit frame and defined in the ejectile plane (third column), where $\eta_B = \frac{Q}{Q'}$.

In the helicity basis the response tensor is:

$$
\begin{pmatrix}
w_{++} & w_{+0} & w_{+-} \\
w_{0+} & w_{00} & w_{0-} \\
w_{-+} & w_{-0} & w_{--}
\end{pmatrix}
$$

(E.1)

and the following inequalities follow from its positive semi-definiteness:

$$
w_{00}, w_{++}, w_{--} \geq 0
$$

(E.2)

$$
\begin{vmatrix}
w_{++} & w_{+0} \\
w_{0+} & w_{00}
\end{vmatrix} \geq 0
$$

(E.3)

$$
\begin{vmatrix}
w_{00} & w_{0-} \\
w_{-0} & w_{--}
\end{vmatrix} \geq 0
$$

(E.4)
Using hermiticity and parity conservation (eq.13, 15 from ref.2), we reduce the number of independent real parameters in the virtual photon density matrix to five: $w_{00}$, $w_{++}$, $w_{+-}$, $Re w_{0+}$, $Im w_{0+}$. Then the inequalities become:

$$\begin{vmatrix} w_{++} & w_{+-} \\ w_{+-} & w_{--} \end{vmatrix} \geq 0 \quad (E.5)$$

$$\begin{vmatrix} w_{++} & w_{+0} & w_{+-} \\ w_{0+} & w_{00} & w_{0-} \\ w_{-+} & w_{-0} & w_{--} \end{vmatrix} \geq 0 \quad (E.6)$$

which leads immediately to the inequalities eq.(2.4.5). This proves the equivalence of the Cartesian general response tensor, and the helicity density matrix methods.
Appendix F Derivation of Polarized Positivity Inequalities

The virtual photon density matrix has to be positive semidefinite:

$$
\begin{pmatrix}
J_+ J_+^\dagger & J_+ J_0^\dagger & J_+ J_-^\dagger \\
J_0 J_+^\dagger & J_0 J_0^\dagger & J_0 J_-^\dagger \\
J_- J_+^\dagger & J_- J_0^\dagger & J_- J_-^\dagger
\end{pmatrix} \geq 0
$$

(F.1)

In the case of reactions with spin, the electromagnetic current matrix elements \( J_a \) become matrices themselves. We are free to apply any similarity transformation to this density matrix because it will not change its positivity properties. This allows us to change the transverse helicity current matrix elements to a new set described below. We are also free to choose any spin quantization axis: a particularly convenient one will turn out to be the normal to the reaction (ejectile) plane. This leads to so called transversity states. I will refer to such amplitudes as the hybrid amplitudes, because the photon will be described by its helicity, while all other particle states will be transversity states. The longitudinal hybrid amplitudes for pseudoscalar electroproduction off a spin \( \frac{1}{2} \) target are defined this way (for details of the method of construction see ref. 2):

$$
J_o = \begin{pmatrix} g_5 & 0 \\ 0 & g_6 \end{pmatrix}
$$

(F.2)

The transverse hybrid amplitudes can be distributed in the following two matrices:

$$
J_s = \frac{1}{2}(J_+ - J_-) = \begin{pmatrix} 0 & g_4 \\ g_3 & 0 \end{pmatrix}
$$

(F.3)

$$
J_a = \frac{1}{2}(J_+ + J_-) = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}
$$

The specific linear combinations which define the \( g \)'s are given in section 1.2, as are the definitions of helicity amplitudes and the tables of observables.

It is straightforward to construct the final state (recoil polarization) density matrix:

$$
\rho_f = \begin{pmatrix} J_a J_a^\dagger & J_a J_o^\dagger & J_a J_-^\dagger \\
J_o J_a^\dagger & J_o J_o^\dagger & J_o J_-^\dagger \\
J_- J_a^\dagger & J_- J_o^\dagger & J_- J_-^\dagger
\end{pmatrix}
$$

(F.4)
and the initial state (target polarization) one:

\[
\rho_i = \begin{pmatrix}
J_a^\dagger J_a & J_a^\dagger J_o & J_a^\dagger J_s \\
J_o^\dagger J_a & J_o^\dagger J_o & 0 \\
J_s^\dagger J_a & 0 & J_s^\dagger J_s
\end{pmatrix}
\] (F.5)

and to express its matrix elements in terms of observables which can be found in ref.1, or in section 1.2 (Tables 1,2). A straightforward application of the positivity conditions for the initial and final state density matrices leads immediately to eq.(2.5.1-9).
Appendix G Evaluation of Parity-Violating Potential Matrix Elements in Momentum Space

The matrix elements of the potential are most easily evaluated in momentum space. The definition of the potential (or any other two-body operator) in momentum representation is:

\[
V(p_i, p_i') = \frac{1}{(2\pi)^3} \int \prod_{i=1}^{2} d\vec{x}_i d\vec{x}_i' \delta(\frac{1}{4} \sum_{j=1}^{2} (\vec{x}_i' + \vec{x}_i)) \times
\exp[-i \sum_{j=1}^{2} (\vec{p}_j' \cdot \vec{z}_j')] V(\vec{z}_i, \vec{z}_i') \exp[i \sum_{j=1}^{2} (\vec{p}_j \cdot \vec{z}_j)]
\]

where the order of factors in the integrand is crucial if the potential is momentum dependent (non-local). As a result of the transformation of the potentials to the momentum space, we get:

\[
V_{12}^+ = \left( \frac{1}{2m_q} \right) \left\{ (\vec{\sigma}_1 - \vec{\sigma}_2) \cdot (\vec{p}_1 - \vec{p}_2 + \vec{p}_1' - \vec{p}_2') + i(\vec{\sigma}_1 \times \vec{\sigma}_2) \cdot (\vec{p}_2 - \vec{p}_1 + \vec{p}_1' + \vec{p}_2') \right\}
\]

\[
V_{12}^- = \left( \frac{1}{2m_q} \right) (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{p}_1 - \vec{p}_2 + \vec{p}_1' - \vec{p}_2')
\]

The inverse Fourier transform is defined this way:

\[
V(\vec{z}_i, \vec{z}_i') = \frac{1}{(2\pi)^6} \int \prod_{i=1}^{2} d\vec{p}_i d\vec{p}_i' V(\vec{p}_i, \vec{p}_i') \times
\delta(\vec{p}_1 + \vec{p}_2 - \vec{p}_1' - \vec{p}_2') \exp[-i \sum_{j=1}^{2} (\vec{p}_j \cdot \vec{z}_j - \vec{p}_j' \cdot \vec{z}_j')] \]

for local potentials we have:

\[
V(\vec{z}_i, \vec{z}_i') = \delta(\vec{z}_1 - \vec{z}_1') \delta(\vec{z}_2 - \vec{z}_2') \int d\vec{k} V(\vec{k}) \exp[i \vec{k} \cdot (\vec{z}_1 - \vec{z}_2)]
\]
The following string of identities will be useful for the derivation of eq.(3.2.5):

\[
J_{mm'} = \frac{1}{2m_q} \sqrt{\frac{3}{2\pi}} (2\pi)^{-3} \int \prod_{i=1}^{3} d\vec{p}_i d\vec{p}_j' \delta(\sum_{j=1}^{3} \vec{p}_j) \delta(\sum_{j=1}^{3} \vec{p}_j') \times \\
\times \tilde{\psi}_m^\dagger(\vec{p}_i') \gamma_{1m'}(\vec{p}_p') \tilde{\psi}_S(\vec{p}_i)
\]

\[
= \frac{\delta_{mm'}}{\sqrt{2m_qR}} (2\pi)^{-3} \int \prod_{i=1}^{3} d\vec{p}_i d\vec{p}_j' \delta(\sum_{j=1}^{3} \vec{p}_j) \delta(\sum_{j=1}^{3} \vec{p}_j') \times \\
\times \tilde{\gamma}_S^\dagger(\vec{p}_i') \tilde{\psi}_S(\vec{p}_i) \delta(\vec{p}_3 - \vec{p}_3')
\]

\[
= \frac{-i}{\sqrt{2m_qR}} \langle \psi_S | \delta(\vec{r}_1 - \vec{r}_2) | \psi_S \rangle \delta_{mm'}
\]

\[
= \frac{-i}{\sqrt{2m_qR}} I \delta_{mm'}
\]

where the momentum space wave functions* are defined in ref.39, specifically:

\[
\tilde{\psi}_m(\vec{p}_i') = i \sqrt{\frac{8\pi}{3}} \gamma_{1m}(\vec{p}_p' R) \tilde{\psi}_S(\vec{p}_i')
\]

\[
\gamma_{1m}(\vec{p}) = |\vec{p}| \gamma_{1m}(\vec{p})
\]

In matrix elements between \( l=1 \) (final) and \( l=0 \) (initial) states, we can write the spin-spatial potentials this way:

\[
\mathcal{V}_{12}^\pm = \frac{1}{2m_q} \sqrt{\frac{4\pi}{3}} \sum_{\lambda} \bar{S}_\lambda^\dagger(\pm) \mathcal{Y}_{1\lambda}(\vec{p}_p') A(\pm)
\]

\[
\mathcal{V}_{12}^\pm = \frac{1}{2m_q} \sqrt{\frac{4\pi}{3}} \sum_{\lambda} \bar{S}_\lambda^\dagger(\pm) \mathcal{Y}_{1\lambda}(\vec{p}_p') A(\pm)
\]

where

\[
\bar{S}(\pm) = (\vec{s}_1 \pm \vec{s}_2), \quad \mathcal{Y}_{1\lambda}(\vec{p}) = |\vec{p}| Y_{1\lambda}
\]

\[
A(+) = \frac{1}{2} (1 - \vec{s}_1 \cdot \vec{s}_2), \quad A(-) = 1
\]

* Note that in (ref.39) there is factor of \( i \) missing in the \( l=1 \) momentum wave functions in their Appendix A, and in all subsequent results, as well as a misprint in their eq.(29): the sign of the exponent of \( ((2\pi)^{3}) \) ought to be negative; both errors propagated into (ref.40) and produced a discrepancy of \( i (2\pi)^{3} \approx 60,000 i \) in their eq.(3.6).
The spin matrix elements are evaluated with the help of the following identities:

\[
\begin{align*}
&\langle 0 0 \mid S_{A} \mid 1 M \rangle = 2 (-1)^{M} \delta_{M,-\lambda} \\
&\langle 0 0 \mid S_{A}^{\dagger} \mid 1 M \rangle = 2 \delta_{M,\lambda} \\
&(1 M \mid S_{A} \mid 0 0) = 2 \delta_{M,\lambda} \\
&(1 M \mid S_{A}^{\dagger} \mid 0 0) = 2 (-1)^{M} \delta_{M,-\lambda}
\end{align*}
\]  

(G.6)

and the wave functions \( \chi_{\Sigma} (\Sigma = \rho, \lambda) \) are given in ref.24, 38 and the Clebsch-Gordan coefficients are from ref.41.

Appendix H Evaluation of the Isospin Matrix Elements of the Parity-Violating Potential

The isospin matrix elements are:

\[
\begin{align*}
\langle \phi_{\rho} \mid I_{12}^{\dagger} \mid \phi_{\rho} \rangle &= -\frac{1}{2} \left[ \cos^{2} \theta_{C} + \frac{1}{2} \left( 1 - \sin^{2} \theta_{W} \right) \right] \\
\langle \phi_{\lambda} \mid I_{12}^{\dagger} \mid \phi_{\lambda} \rangle &= -\frac{1}{3} \left\{ \frac{1}{2} \left[ \cos^{2} \theta_{C} + \frac{1}{2} \left( 1 - \sin^{2} \theta_{W} \right) \right] + \frac{\sin^{2} \theta_{W}}{3} \tau_{3}^{N} \right\} \\
\langle \phi_{\rho} \mid I_{12} \mid \phi_{\rho} \rangle &= \langle \phi_{\lambda} \mid I_{12} \mid \phi_{\lambda} \rangle = 0 \\
\langle \phi_{\rho} \mid I_{12} \mid \phi_{\lambda} \rangle &= 0 \\
\langle \phi_{\rho} \mid I_{12} \mid \phi_{\lambda} \rangle &= \frac{\sin^{2} \theta_{W}}{6\sqrt{3}} \tau_{3}^{N}
\end{align*}
\]  

(H.1)  

(H.2)

where \( \tau_{3}^{N} \) is an isospin operator in the nucleon iso-space.
Appendix I Electromagnetic Hamiltonian Matrix Elements Between SU(6) States Belonging to a 56-plet and a 70-plet

Using the SU(6) wave functions for the 70-plet from section 3.2, we can evaluate the necessary electromagnetic transition matrix elements. I use the following property of the nucleon matrix elements to reduce the algebra:

\[ \sum_{i=1}^{3} \langle N' | O_i I_i | N \rangle = 3 \langle N' | O_3 I_3 | N \rangle \quad (I.1) \]

where \( O_i \) is the spin-spatial part of the operator and \( I_i \) is the isospin part. In our case \( I_i = e_i \). Using the symmetry properties of the states and operators under interchange of indices \( 1 \leftrightarrow 2 \) one gets:

\[
\begin{align*}
(70, 2\delta), \quad J M = \frac{1}{2} | H_+^T | (56, 2\delta), \quad J M = -\frac{1}{2} = \frac{-3}{2 \sqrt{2}} \times \\
\times \left\{ (\phi_{\rho} | e_3 | \phi_{\rho}) \left( [\chi_{\rho} \psi_{\lambda}]_\uparrow | O_3 | \chi_{\lambda} \psi_\delta \right) - (\phi_{\lambda} | e_3 | \phi_{\lambda}) \left( [\chi_{\lambda} \psi_{\lambda}]_\uparrow | O_3 | \chi_{\lambda} \psi_\delta \right) \right\} \\
(70, 4\delta), \quad J M = \frac{1}{2} | H_+^T | (56, 2\delta), \quad J M = -\frac{1}{2} = \\
\frac{-3}{2} (\phi_{\lambda} | e_3 | \phi_{\lambda}) \left( [\chi_{\lambda} \psi_{\lambda}]_\uparrow | O_3 | \chi_{\lambda} \psi_\delta \right) \\
\end{align*}
\]

The spin-spatial operator \( O_3 \) can be split up into a “convection current” and a “magnetization current” part:

\[ O_3 = O_3^C + O_3^M \quad (I.3) \]

where

\[
O_3^C = \hat{e}_+ \cdot \left( \vec{F}_3 + \vec{F}_3' \right) \frac{1}{2 m_q} \exp(i \hat{q} \cdot \vec{r}_3) \\
O_3^M = \frac{| \vec{q} |}{\sqrt{2 m_q}} \exp(i \hat{q} \cdot \vec{r}_3) \\
O_3^M = \frac{1}{2} (\sigma_x + i \sigma_y) O_3^M 
\]

The following results are obtained after a straightforward calculation (one may use the results for the electromagnetic transition matrix elements of ref. 45 in this
calculation, but one must keep in mind the difference in conventions):

\[
\langle [\chi_\rho \psi_\lambda]_I \mid O_3 \mid [\chi_{\rho_1} \psi_S]_I \rangle = \frac{-i}{3} \left[ 2 \sqrt{\frac{\omega}{2m_q}} + \sqrt{2m_q^3} \right] \exp \left( \frac{-q^2}{6m_q} \right)
\]

\[
\langle [\chi_\lambda \psi_\lambda]_I \mid O_3 \mid [\chi_{\lambda_1} \psi_S]_I \rangle = \frac{-i}{3} \left[ 2 \sqrt{\frac{\omega}{2m_q}} - \sqrt{\frac{\omega^2}{2m_q^2}} \right] \exp \left( \frac{-q^2}{6m_q} \right) \tag{I.4}
\]

\[
\langle [\chi_S \psi_\lambda]_I \mid O_3 \mid [\chi_{\lambda_1} \psi_S]_I \rangle = \frac{-i}{9} \sqrt{\frac{q^2}{m_q^3}} \exp \left( \frac{-q^2}{6m_q} \right)
\]

The isospin matrix elements of \( e_3 \) are:

\[
(\phi_\rho \mid e_3 \mid \phi_\rho) = \frac{1}{2} \left( \frac{1}{3} + \tau_3^N \right)
\]

\[
(\phi_\lambda \mid e_3 \mid \phi_\lambda) = \frac{1}{6} \left( 1 - \tau_3^N \right) \tag{I.5}
\]

\[
(\phi_\lambda \mid e_3 \mid \phi_\rho) = (\phi_\rho \mid e_3 \mid \phi_\lambda) = 0
\]

Using the above results, one can directly obtain eqn.(3.3.3).
Appendix J  Elastic Parity-Violating Electromagnetic Current Matrix  
Element of the Nucleon Induced by the 70-plet Admixture

The spin spatial matrix elements are calculated using the wave functions and  
current operators defined in Appendix G. The results are:

\[
\int d\vec{r} \left( \vec{r} \cdot \left( \langle \chi_S \psi_\lambda \rangle \overline{\gamma}_3 | \chi_t \psi_S \rangle \right) j_1(qr) Y_{11}(\hat{r}) = \right.
\]
\[
\frac{i}{3^{3/4}} \frac{q}{2m} \frac{1}{\sqrt{\pi m\omega}} \exp \left( \frac{-\tilde{q}^2}{6m\omega} \right)
\]

\[
\int d\vec{r} \left( \vec{r} \cdot \left( \langle \chi_\rho \psi_\lambda \rangle \overline{\gamma}_3 | \chi_t \psi_S \rangle \right) j_1(qr) Y_{11}(\hat{r}) = \right.
\]
\[
\frac{i}{3^{3/4}} \frac{q}{6m} \sqrt{\frac{2}{\pi m\omega}} \exp \left( \frac{-\tilde{q}^2}{6m\omega} \right) \left\{ 5 \left[ 1 - \frac{\tilde{q}^2}{15m\omega} \right] + 2 \right\}
\]

\[
\int d\vec{r} \left( \vec{r} \cdot \left( \langle \chi_\rho \psi_\lambda \rangle \overline{\gamma}_3 | \chi_t \psi_S \rangle \right) j_1(qr) Y_{11}(\hat{r}) = \right.
\]
\[
\frac{i}{3^{3/4}} \frac{q}{6m} \sqrt{\frac{2}{\pi m\omega}} \exp \left( \frac{-\tilde{q}^2}{6m\omega} \right) 5 \left[ 1 - \frac{\tilde{q}^2}{15m\omega} \right]
\]

which, together with eq.(1.2, 5), leads to:

\[
B_1 = 3 \int d\vec{r} \left( \vec{r} \cdot \langle (70,^2 8)^\dagger | e_3 \overline{\gamma}_3 | (56,^2 8)^\dagger \rangle \right) j_1(qr) Y_{11}(\hat{r}) = \right.
\]
\[
\frac{i}{3^{3/4}} \frac{q}{12m} \frac{1}{\sqrt{\pi m\omega}} \exp \left( \frac{-\tilde{q}^2}{6m\omega} \right) \left\{ 1 + \tau_3 \left[ 13 - \frac{2\tilde{q}^2}{3m\omega} \right] \right\}
\]

and

\[
B_2 = 3 \int d\vec{r} \left( \vec{r} \cdot \langle (70,^4 8)^\dagger | e_3 \overline{\gamma}_3 | (56,^2 8)^\dagger \rangle \right) j_1(qr) Y_{11}(\hat{r}) = \right.
\]
\[
\frac{i}{3^{3/4}} \frac{q}{2m} \frac{1}{\sqrt{\pi m\omega}} \exp \left( \frac{-\tilde{q}^2}{6m\omega} \right) \frac{1}{9} (1 - \tau_3^N)
\]
Appendix K Elastic Parity Violating Matrix Elements of the Two-Body Electromagnetic Current

The matrix element of the two-body current is:

$$\int d\vec{r} \left( \mathbf{r} \cdot \langle 56, \uparrow | \mathcal{J}_{12} | 56, \downarrow \rangle \right) j_1(qr) Y_{11}(\hat{r}) = \frac{eG_F}{\sqrt{2}} \sum_{i=1}^{3} \langle 56, \uparrow | I^i_{12} \mathbf{\sigma} \cdot \mathbf{S}^i_{12} | 56, \downarrow \rangle \sqrt{2m\omega} \frac{q}{32\pi^2} \exp \left[ -\frac{q^2}{24m\omega} \right]$$

(K.1)

where I have used:

$$\int_0^{\infty} r^3 j_1(qr) \exp(-\alpha r^2) dr = \frac{q\sqrt{\pi}}{8\alpha^{5/2}} \exp \left( -\frac{q^2}{4\alpha} \right)$$

$$\int_0^{\infty} r^5 j_1(qr) \exp(-\alpha r^2) dr = \frac{5q\sqrt{\pi}}{16\alpha^{3/2}} \left[ 1 - \frac{q^2}{10\alpha} \right] \exp \left( -\frac{q^2}{4\alpha} \right)$$

The spin-isospin matrix elements of the two-body current are:

$$\langle 56, 2 \uparrow | I^2_{12} \mathbf{\sigma} \cdot \mathbf{S}^2_{12} | 56, 2 \downarrow \rangle = -\left( \frac{1}{2m_q} \right) \frac{\sqrt{2}}{3} (1 - 2\sin^2\theta_W) r^N_3$$

$$\langle 56, 2 \uparrow | I^3_{12} \mathbf{\sigma} \cdot \mathbf{S}^3_{12} | 56, 2 \downarrow \rangle = -\left( \frac{1}{2m_q} \right) \frac{2\sqrt{2}}{27} \sin^2\theta_W$$

(K.2)

$$\langle 56, 2 \uparrow | I^3_{12} \mathbf{\sigma} \cdot \mathbf{S}^3_{12} | 56, 2 \downarrow \rangle = 0$$

which immediately leads to eq.(3.8.1).
Appendix L Conservation of the Parity-Violating Electromagnetic Current in Momentum Space

The easiest way of checking gauge invariance is in momentum space, because of the highly singular nature of the two-body current operators. First, we must define the Fourier transforms of few-body operators. This has already been done in Appendix G for the potential, now we do it for current operators.

For a two-body current (three-body in, two-body out), we define the Fourier transform this way:

\[
\tilde{J}(\vec{x}_i, \vec{x}_i', \vec{q}) = \frac{1}{(2\pi)^6} \int \prod_{i=1}^{2} d\vec{p}_i d\vec{p}_i' \delta(\vec{q} + \vec{p}_1 + \vec{p}_2 - \vec{p}_1' - \vec{p}_1') \times \\
\times \tilde{J}(\vec{p}_i, \vec{p}_i') \exp \left[ -i \sum_{j=1}^{2} (\vec{p}_j \cdot \vec{z}_j - \vec{p}_j' \cdot \vec{z}_j') \right]
\]

for local two-body currents we have:

\[
\tilde{J}(\vec{x}_i, \vec{x}_i', \vec{q}) = \delta(\vec{x}_1 - \vec{x}_1') \delta(\vec{x}_2 - \vec{x}_2') \exp \left[ \frac{i}{2} \vec{q} \cdot (\vec{x}_1 + \vec{x}_2) \right] \int d\vec{k} \tilde{J}(\vec{k}) \exp \left[ i \vec{k} \cdot (\vec{x}_1 - \vec{x}_2) \right]
\]

which defines the Fourier transform in the "photon" space:

\[
\tilde{J}(\vec{q}) = \int d\vec{R} \tilde{J}(\vec{R}) \exp(i\vec{q} \cdot \vec{R}); \quad \vec{R} = \frac{1}{2}(\vec{x}_1 + \vec{x}_2)
\]

The inverse Fourier transform is:

\[
\tilde{J}(\vec{p}_i, \vec{p}_i', \vec{q}) = \frac{1}{(2\pi)^3} \int \prod_{i=1}^{2} d\vec{z}_i d\vec{z}_i' \delta\left(\frac{1}{4} \sum_{j=1}^{2} (\vec{z}_j' + \vec{z}_j) \right) \times \\
\times \exp \left[ -i \sum_{j=1}^{2} (\vec{p}_j' \cdot \vec{z}_j') \right] \tilde{J}(\vec{z}_i, \vec{z}_i', \vec{q}) \exp \left[ i \sum_{j=1}^{2} (\vec{p}_j \cdot \vec{z}_j) \right]
\]

(L.2)

where the order of the factors in the integrand is crucial if the current operator depends on momenta i.e. if it is non-local.

Then it is straightforward to prove that the current conservation condition in momentum space:
momentum space:

\[ \vec{q} \cdot \vec{J}_{12} = \left[ \mathcal{V}_{12}, \rho \right] \]

\[ = - \sum_{i=1}^{2} \frac{1}{2} \{ e_i, I_{12} \} \left[ \mathcal{V}_{12}, \text{exp}(i \vec{q} \cdot \vec{z}_i) \right] \]

\[ + \sum_{i=1}^{2} \frac{1}{2} \{ e_i, I_{12} \} \{ \mathcal{V}_{12}, \text{exp}(i \vec{q} \cdot \vec{z}_i) \} \]

\[ \text{(L.3)} \]

can be reduced, with the help of eq.(3.4.4), to the condition that the Fourier transforms of the (anti-)commutators of the spin-spatial parts of the potential and the two-body current satisfy symbolic relations:

\[ \left[ \mathcal{V}_{12}^\pm (\vec{p}_i), \text{exp}(i \vec{q} \cdot \vec{z}_1) \right] = \frac{1}{m} \vec{S}_{12}(\pm) \text{exp}(i \vec{q} \cdot \vec{z}_1) \]

\[ \left[ \mathcal{V}_{12}^\pm (\vec{p}_i), \text{exp}(i \vec{q} \cdot \vec{z}_2) \right] = -\frac{1}{m} \vec{S}_{12}(\pm) \text{exp}(i \vec{q} \cdot \vec{z}_2) \]

\[ \text{(L.4)} \]

where \( \vec{S}_{12}(\pm) \) are given in Appendix G and \( \text{exp}(i \vec{q} \cdot \vec{z}_1) = \text{exp}(i \vec{q} \cdot \vec{z}_2) \), due to the contact nature of the potential. On the other hand, the spin-spatial part of the charge changing two-body current is given by symbolic relation:

\[ \left\{ \mathcal{V}_{12}^\pm (\vec{p}_i), \text{exp}(i \vec{q} \cdot \vec{z}_1) - \text{exp}(i \vec{q} \cdot \vec{z}_2) \right\} = -\frac{1}{m} \vec{q} \cdot \vec{S}_{12} \text{exp}(i \vec{q} \cdot \vec{z}_1) \]

\[ \text{(L.5)} \]

where \( \vec{S}_{12} = \vec{1} (\vec{\sigma}_1 \times \vec{\sigma}_2) \).

Note that these two-body currents could have been obtained in a much easier fashion by making a minimal substitution in the parity violating potential. The only part of the two-body current which is not completely constrained by current conservation has to be iso-vector and it must be induced by the non-local part of the two-body potential (ref.46, 48). This part has to be determined from a relativistic model of the parity violating current, and in our case it happens to vanish. In the case of finite intermediate boson mass, one must apply Ohta’s (ref.46) method of “symmetrization” of the potential.
Appendix M Evaluation of the Closure Matrix Elements

The calculation of the spatial matrix element is the basic task of this appendix. The spin and isospin matrix elements will turn out to be the same as those of a two-body current operator and will be simply taken over from Appendix 4A.

In order to illustrate how this comes about, let us remember that both the potential and the one-body charge density can be written as products of isospin and spin-spatial parts. Then we may use the identity:

\[ \{ A_1 \otimes B_1, A_2 \otimes B_2 \} = \frac{1}{2} \{ A_1, A_2 \} \otimes \{ B_1, B_2 \} \]

Clearly, the iso-spin parts of the Z exchange induced potential commute with the iso-spin parts of the charge operator because the interaction is neutral:

\[ [I_{12}^Z, e_2] = 0 \]

which simplifies the calculation. But, that is not true for the W exchange induced part of the potential, where neither the commutator nor the anti-commutator vanish. In either case we may write:

\[ \{ I_{12}, e_2 \} = \frac{1}{2} \{ I_{12}, e_2 - e_1 \} + \frac{1}{2} \{ I_{12}, e_2 + e_1 \} \]

which produces operators of definite symmetry under the exchange of indices (they are all given in eq.(3.5.4)). This symmetry of the isospin operators, together with the symmetry of the states and the spin operators, produces spin-isospin matrix elements identical to those of the two-body current. The derivation of spatial matrix elements is sketched below.

Firstly we prove that the spatial part of \( A \) vanishes:

\[ \int d\vec{r} ' (\vec{r} \cdot \psi_S^* \{ \vec{j}_3, V_{12} \} \psi_S) j_1(\vec{r}_3) Y_{11}(\vec{r}_3) = 0 \]

\[ \int d\vec{\rho} d\vec{\lambda} ' (\vec{\lambda} \cdot \psi_S^* \vec{\lambda} \cdot \psi_S) j_1(\sqrt{2/3} q\lambda) Y_{11}(\vec{\lambda}) = 0 \]

which automatically implies that the complete matrix element of type A operators vanishes, too.
Secondly, let us prove that the magnetization current contribution to the \( \langle B_{122} \rangle \) is zero:

\[
\int d\vec{r} \left( \vec{r} \cdot \psi_S^* \{ \vec{j}_2^m, V_{12} \} \psi_S \right) j_1(qr_2)Y_{11}(\hat{r}_2) \propto \\
\int d\rho d\tilde{\lambda} \left( \vec{\lambda} \cdot \psi_S^* (\vec{\sigma} \times \vec{q}) \tilde{S} \cdot \vec{\nabla}_\rho (\sqrt{2\rho} \cdot \psi_S \right) j_1(\sqrt{\frac{2}{3}q\lambda})Y_{11}(\hat{\lambda}) \propto \quad (M.5)
\]

\[
\int d\rho \tilde{S} \cdot \vec{\rho} (\sqrt{2\rho}) \int d\tilde{\lambda} \left( (\vec{\sigma} \times \vec{q}) \cdot \psi_S^* \vec{\lambda} \cdot \psi_S \right) j_1(\sqrt{\frac{2}{3}q\lambda})Y_{11}(\hat{\lambda}) = 0
\]

which, by symmetry, implies that all magnetization current contributions to matrix elements of operators of type B vanish.

Finally, the result for the complete closure matrix element:

\[
\langle B_{122} \rangle = \int d\vec{r} \left( \vec{r} \cdot \langle 56, \uparrow | \{ \vec{j}_2^m, V_{12} \} | 56, \downarrow \rangle \right) j_1(qr)Y_{11}(\hat{r}) = \\
= \frac{eG_F}{\sqrt{2}} \sum_{i=1}^{3} \langle 56, \uparrow | I_{12} (\vec{\epsilon}_+ \cdot \vec{S}_{12}^i) | 56, \downarrow \rangle \sqrt{2m\omega} \frac{q\omega}{32\pi^2} \exp \left[ -\frac{q^2}{24m\omega} \right] \quad (M.6)
\]

Note that once eq.(M.6) is divided by the energy denominator \( E - E' = -\omega \), it is exactly the negative of the two-body current contribution eq.(K.1), as advertised.
Appendix N Check of Khriplovich’s Identity for the Parity-Violating Two-Body Current in the Standard Model

We would like to see whether Khriplovich’s identity, eq.(3.11.7):

\[ J_{2-b}^{PV} = i \left[ V_{PV}, \vec{d} \right] \]  

(N.1)

is satisfied by the Standard Model parity-violating non-relativistic quark two-body currents, eq.(3.8.2) and eq.(3.9.1), and the Standard Model parity-violating quark-quark potential eq.(3.3.1). We will use the charge dipole operator \( \vec{d} = \sum_{i=1}^{3} e_i \vec{r}_i \) as well as the identities eq.(3.5.3-5) and eq.(M.1):

\[
\left[ V_{12}^{PV}, \vec{d} \right] = \sum_{k=1}^{3} \left[ V_{12}^{PV}, \vec{r}_k \right] \\
= -\frac{G_F}{\sqrt{2}} \sum_{k=1}^{2} \sum_{\lambda=\pm} \left( \left[ I_{12}^\lambda, \psi_{12}^\lambda, e_k \vec{r}_k \right] \right) \\
= -\frac{G_F}{2\sqrt{2}} \sum_{k=1}^{2} \sum_{\lambda=\pm} \left( \left[ I_{12}^\lambda, e_k \right] \left\{ \psi_{12}^\lambda, \vec{r}_k \right\} + \left\{ I_{12}^\lambda, e_k \right\} \left[ \psi_{12}^\lambda, \vec{r}_k \right] \right) 
\]  

(N.2)

It turns out to be useful to express both terms in the spin-spatial commutators in terms of the CM and the relative coordinates:

\[
\vec{r}_{ij} = \vec{r}_i - \vec{r}_j \\
\vec{R}_{ij} = \frac{1}{2} \left( \vec{r}_i + \vec{r}_j \right) 
\]  

(N.3)

Since the potentials depend only on the relative coordinate, this change of variables simplifies the subsequent calculations. We are immediately lead to:

\[
\vec{d} = \sum_{i=1}^{2} e_i \vec{r}_i \\
= \vec{r}_{12} \frac{1}{2} (e_1 - e_2) + \vec{R}_{12} (e_1 + e_2) 
\]  

(N.4)
which leads to:

\[ \left[ V_{12}^{PV}, \vec{d} \right] = \frac{-G_F}{4\sqrt{2}} \sum_{\lambda=\pm} \left( \left[ I_{12}^\lambda, e_1 - e_2 \right] \left\{ V_{12}^\lambda, \vec{r}_k \right\} + \left\{ I_{12}^\lambda, e_1 - e_2 \right\} \left[ V_{12}^\lambda, \vec{r}_k \right] \right) \]  

(N.5)

This, together with:

\[
\begin{align*}
\{ V_{12}^+, \vec{r}_{12} \} &= 4i \vec{S}_{12}^3 \delta(\vec{r}_{12}) \\
\{ V_{12}^-, \vec{r}_{12} \} &= 0 \\
[ V_{12}^+, \vec{r}_{12} ] &= -4i \vec{S}_{12}^2 \delta(\vec{r}_{12}) \\
[ V_{12}^-, \vec{r}_{12} ] &= -4i \vec{S}_{12}^1 \delta(\vec{r}_{12})
\end{align*}
\]  

(N.6)

thereby leading, together with eq.(3.5.4), to:

\[ \vec{J}_{12} = i \left[ V_{12}^{PV}, \vec{d} \right] \]

\[ = \frac{G_F}{\sqrt{2}} \delta(\vec{r}_{12}) \left( \left[ I_{12}^+, e_1 - e_2 \right] \vec{S}_{12}^3 + \left\{ I_{12}^+, e_1 - e_2 \right\} \vec{S}_{12}^1 - \left\{ I_{12}^-, e_1 - e_2 \right\} \vec{S}_{12}^2 \right) \]  

(N.7)

\[ = \frac{G_F}{\sqrt{2}} \delta(\vec{r}_{12}) \sum_{i=1}^{3} I_{12}^i \vec{S}_{12}^i \]

which is exactly equal to \( \vec{J}_{12} \), as advertised.
Appendix O  Reprint of Reference 1
Complete Measurements of Scalar and Pseudoscalar Electroproduction

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The development of new electron scattering facilities offers, for the first time in the history of electron scattering, the possibility to undertake complete measurements of various inelastic electron scattering processes. We discuss the necessary and sufficient sets of experiments needed to completely determine one of them: scalar (and pseudoscalar) electroproduction on spin 1/2 targets in one photon exchange approximation.

The problem of finding the smallest set of experiments which fully determine the transition amplitudes has so far been discussed for only one electromagnetic process: pseudoscalar photoproduction on spin 1/2 targets (ref. 1). We extend this analysis to the case of electroproduction and show that it applies to certain nuclear electrodisintegration processes.
We treat electroproduction as a binary collision of a virtual photon and a target, thereby keeping the simplicity of a two-body initial state. Neglecting the weak interactions, there are 6 linearly independent helicity amplitudes:

\[
\begin{align*}
    h_1 &= \langle -1/2 \left| \Gamma \right| -1/2 \rangle, \\
    h_2 &= \langle -1/2 \left| \Gamma \right| 1/2 \rangle, \\
    h_3 &= \langle 1/2 \left| \Gamma \right| -1/2 \rangle, \\
    h_4 &= \langle 1/2 \left| \Gamma \right| 1/2 \rangle, \\
    h_5 &= \langle 1/2 \left| \Gamma \right| 1/2 \rangle, \\
    h_6 &= \langle 1/2 \left| \Gamma \right| -1/2 \rangle
\end{align*}
\]

where

\[
h_j = \langle \lambda_f \left| \Gamma \right| \lambda_i \rangle \equiv \epsilon_\mu (\lambda_\gamma) \langle \lambda_f | J^\mu | \lambda_i \rangle
\]

and \(\lambda_f\) and \(\lambda_i\) are the helicities of the final and initial spin 1/2 particles, \(\lambda_\gamma\) the helicity of the photon, and the second outgoing particle has spin zero.

The most general form of the coincidence cross section (with electron variables in LAB frame, and all other (ejectile) variables in the CM frame) with or without observed polarisations is:

\[
\frac{d^5 \sigma}{dc \cdot d\Omega \cdot d\Omega_1} = \sigma_M \left\{ \frac{p_1 \cdot 3 M}{64 \pi^2 M^2} \right\} \left\{ v_L \left( \frac{W}{M} \right)^2 W_L + v_T W_T + \right. \\
+ v_{TL} \left[ W_{LT}^2 \cos^2 \phi + W_{LT}^2 \sin^2 \phi \right] + v_{TT} \left[ W_{TT}^2 \cos^2 \phi + \right. \\
+ W_{TT}^2 \sin^2 \phi \right\} - 2h \left[ v_{T} W_T + v_{TL} \left( \frac{W}{M} \right) \left[ W_{TL}^2 \cos^2 \phi + \right. \\
+ W_{TL}^2 \sin^2 \phi \right] \right\}
\]

where:

\[
\sigma_M = \left( \frac{a \cos \theta/2}{2E \sin^2 \theta/2} \right)^2, \quad h \equiv \text{electron helicity} = \pm 1/2
\]

\[
v_L = \left( \frac{q_1}{q_2} \right)^2, \quad v_T = \frac{1}{2} \left( \frac{q_2}{q_1} \right)^2 + t g^2 \theta/2
\]
The structure functions are functions of initial and/or final polarization vectors in the ejectile plane and therefore may have a $\phi$ dependence of their own. The geometry of the process is depicted in Fig. 1.

The form of these structure functions in terms of bilinear products of helicity amplitudes (h.a. for short) $h_j$, has been determined by Bartl & Majerotto (ref. 2) for the case when the pseudoscalar is observed, with and without polarized target. We generalize this calculation to the cases of either the spin 1/2 particle (of either parity), or the scalar (pseudoscalar) being observed in coincidence, and either polarized target or ejectile polarization measurement.

The transition probability of a binary collision with polarization observed in the initial and/or the final states is, according to (ref. 3):

\[
\text{Transition probability} = \sum_{\lambda, \lambda'} \sum_{\mu, \mu'} \sum_{\lambda_j, \lambda_j'} \langle \mu \Gamma | \lambda, \lambda' \rangle \langle \mu' \Gamma | \lambda_j, \lambda_j' \rangle 
\times \langle \lambda | \rho_1 | \lambda' \rangle \langle \lambda | \rho_1 | \lambda' \rangle \langle \lambda | \rho_1 | \lambda' \rangle 
\]

where

\[
\langle \lambda | \rho_1 | \lambda' \rangle = \text{spin density matrix of the target}
\]

\[
\langle \mu | \rho_2 | \mu' \rangle = \text{spin density matrix of the ejectile}
\]

\[
\langle \lambda_j | \rho_\gamma | \lambda_j' \rangle = \text{spin density matrix of the virtual photon}
\]

All quantities in the above formula (h.a.s and density matrices) are defined with respect to the ejectile plane (see Fig. 1). That eliminates the $\phi$
dependence from the helicity amplitudes and introduces it into density matrices (for both the photon and the spin 1/2 particles; the latter was erroneously omitted by Bartl & Majorotto, yielding thereby wrong \( \phi \) dependence of the polarized target cross section). The virtual photon spin density matrix was calculated following Dombey (ref. 4), with the distinction of being defined in the ejectile plane and in the CM frame (our metric differs, too).

The matrix elements (for unpolarized electrons) are defined by:

\[
\langle \alpha | \rho_{\gamma} | \beta \rangle = \rho_{\alpha \beta} = \rho_{\alpha \beta} e^{i(\beta-\alpha)} \tag{4}
\]

\[
\rho_{++} = \rho_{--} \quad \rho_{+-} = \rho_{-+} 
\]

\[
\rho_{o+} = \rho_{+o} = - \rho_{o-} = - \rho_{-o} 
\tag{5}
\]

where

\[
\rho_{oo}^{\text{CM}} = (2EE' \cos^2 \theta/2 v_L)_L \left( \frac{N}{V} \right)^2 
\]

\[
\rho_{o+}^{\text{CM}} = -(2EE' \cos^2 \theta/2 v_{TL})_L \left( \frac{N}{V} \right) 
\]

\[
\rho_{++}^{\text{CM}} = (2EE' \cos^2 \theta/2 v_T)_L 
\]

\[
\rho_{+o}^{\text{CM}} = (2EE' \cos^2 \theta/2 v_{TT})_L 
\tag{6}
\]

In case of longitudinally polarized electrons we have 2 additional terms:

\[
\rho_{a+} = -\rho_{--} \quad \rho_{a+} = \rho_{a+} = \rho_{a-} = \rho_{a-} e^{i\phi} 
\]

where

\[
\rho_{a+}^{\text{CM}} = 2h (2EE' \cos^2 \theta/2 v_T) 
\tag{7}
\]

\[
\rho_{a+}^{\text{CM}} = 2h (2EE' \cos^2 \theta/2 v_{LT})_L \left( \frac{N}{V} \right) 
\]

The boost to LAB frame has been performed following (ref. 5).
The spin 1/2 density matrices $\rho_1$, $\rho_f$ are:

$$\rho_1 = \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{P})$$
$$\rho_f = (1 + \vec{\sigma} \cdot \vec{P}_R)$$

(8)

where $\vec{P}_R$ is the polarization vector of the ejectile, defined in the ejectile plane. But $\rho_1$ is the density matrix of particle 2 in the initial state which means that its helicity points opposite its spin. This means that vector $\vec{P}$ is not the polarization vector of the target $\vec{P}_T$, but is related to it by a rotation through $\pi$ about the $x'$ axis:

$$P'_x = P_{Tx'}, \quad P'_y = -P_{Ty'}, \quad P'_z = -P_{Tz'}$$

(9)

$\vec{P}_T$ is connected with $\vec{S}_T$, the polarization vector defined in the electron plane, through a rotation:

$$\vec{P}_T = R_{\phi} (\phi) \vec{S}_T$$

(10)

where:

$$R_{\phi} (\phi) = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(11)

By explicit evaluation of the sums in (3), we obtain the expressions for structure functions $W_L$, ..., $W_{RL}$ given in Tables 1 and 2, where $u$ stands for unpolarized and $\vec{P}_{T,R}$ components multiply all entries in the corresponding rows. The parameter $\eta_g$ is defined to be:

$$\eta_g = \pm \pi$$

(12)

where $\pi$ is the parity of the produced particle ($= \pm 1$), and the $+,-$ correspond to the cases of scalar (pseudoscalar) mesons being observed in coincidence, respectively, with the $s = 1/2$ nucleon.

We see that the unpolarized cross section allows separation of 5 linearly independent functions of bi-linear products of helicity amplitudes. Polarized target adds another 13 functions to it, but they are, clearly, related to each
Table 1: Polarized Target

<table>
<thead>
<tr>
<th>( V_L )</th>
<th>( V_T )</th>
<th>( V_{T^c} )</th>
<th>( V_{T^c}^c )</th>
<th>( V_{TL} )</th>
<th>( V_{TL}^c )</th>
<th>( V_{TL}^c )</th>
<th>( V_{TL}^{c2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>21 ( \lambda_L )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>21 ( \lambda_L )</td>
<td>0</td>
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<td>0</td>
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<td>21 ( \lambda_L )</td>
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<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>21 ( \lambda_L )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

where:

- \( A = |b_1|^2 + |b_2|^2 \)
- \( B = \sum_{i=1}^{4} |a_i|^2 \)
- \( c = |b_1|^2 + |b_2|^2 + |b_3|^2 + |b_4|^2 \)
- \( X_1 = h_1 (h_4 - h_2 h_3 - h_2 h_3 - h_2 h_3 - h_2 h_3) \)
- \( X_2 = h_2 (h_4 - h_2 h_3 - h_2 h_3 - h_2 h_3 - h_2 h_3) \)
- \( X_3 = h_3 (h_4 - h_2 h_3 - h_2 h_3 - h_2 h_3 - h_2 h_3) \)
- \( X_4 = h_4 (h_4 - h_2 h_3 - h_2 h_3 - h_2 h_3 - h_2 h_3) \)
- \( Y_1 = h_1^* h_2 + h_2^* h_1 \)
- \( Y_2 = h_2^* h_3 + h_3^* h_2 \)
- \( Y_3 = h_3^* h_4 + h_4^* h_3 \)
- \( Y_4 = h_4^* h_1 + h_1^* h_4 \)
- \( z = b_5 b_6 \)
Table 2: Recoil (ejectile) Polarization

<table>
<thead>
<tr>
<th></th>
<th>$W_L$</th>
<th>$W_T$</th>
<th>$W_{TT}^C$</th>
<th>$W_{TT}^S$</th>
<th>$W_{TL}^C$</th>
<th>$W_{TL}^S$</th>
<th>$W_n^T$</th>
<th>$W_n^C$</th>
<th>$W_n^S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_x^a$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-2\eta_g Y_6^a$</td>
<td>0</td>
<td>2Im$X_8$</td>
<td>2Re $Y_4$</td>
<td>2Re $X_9$</td>
<td>0</td>
</tr>
<tr>
<td>$P_y^a$</td>
<td>$2\eta_g ImZ$</td>
<td>2Im$Y_3$</td>
<td>$-2\eta_g ImY_2$</td>
<td>0</td>
<td>$-2ImX_8$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2Re $X_9$</td>
</tr>
<tr>
<td>$P_z^a$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-2\eta_g ImY_8$</td>
<td>0</td>
<td>2Im$X_7$</td>
<td>D</td>
<td>2Re $X_7$</td>
<td>0</td>
</tr>
</tbody>
</table>

where:

$$D = |h_4|^2 + |h_3|^2 - |h_1|^2 - |h_2|^2$$

$$Y_5 = h_4^* h_3 - h_2^* h_1$$

$$Y_6 = h_3^* h_2 - h_4^* h_1$$

$$X_5 = h_5^* (h_2 + \eta_g h_3) + h_6^* (h_1 - \eta_g h_4)$$

$$X_6 = h_5^* (h_2 - \eta_g h_3) + h_6^* (h_1 + \eta_g h_4)$$

$$X_7 = h_5^* (\eta_g h_1 + h_4) + h_6^* (- \eta_g h_2 + h_3)$$
other. A particularly simple way of finding the linearly independent terms is by using transversity amplitudes (ref. 1) which are defined:

\[
\begin{align*}
    b_1 &= \frac{1}{2} \left[ h_1 + h_4 + i (h_3 - h_2) \right] \\
    b_2 &= \frac{1}{2} \left[ h_1 + h_4 - i (h_3 - h_2) \right] \\
    b_3 &= \frac{1}{2} \left[ h_1 - h_4 - i (h_3 + h_2) \right] \\
    b_4 &= \frac{1}{2} \left[ h_1 - h_4 + i (h_3 + h_2) \right] \\
    b_5 &= \frac{1}{2} \left[ h_5 + i h_6 \right] \\
    b_6 &= \frac{1}{2} \left[ h_5 - i h_6 \right]
\end{align*}
\]  

We choose \( \eta_g = -1 \). It turns out that both polarised target and recoil polarisation can be conveniently summarised on one table 3.

From Table 3 we see that the knowledge of the unpolarized (\( u \)) and one of the polarized (\( P_T \) or \( P_R \)) sets of experiments is insufficient for a complete determination of transversity amplitudes. The moduli of transversity amplitudes are immediately available, as well as the relative phases between members of 2 disjunct groups (e.g. \( (1, 4, 5) \) and \( (2, 3, 6) \) for polarized target, \( (1, 3, 5) \) and \( (2, 4, 6) \) for recoil polarisation). But no phase difference between elements from the two groups can be determined without at least one experiment from the other polarisation set. Thus, we have proven that double polarisation experiments (where polarized electron, polarized target (ejectile) experiments are considered single polarisation measurements) are not necessary for the complete determination of the process, provided that all structure functions from a certain set of measurements can be separated i.e. provided there is an out-of-plane and arbitrary polarisation capability.
### Table 3

<table>
<thead>
<tr>
<th>( W_L )</th>
<th>( W_T )</th>
<th>( W^C_{TT} )</th>
<th>( W^S_{TT} )</th>
<th>( W^C_{TL} )</th>
<th>( W^S_{TL} )</th>
<th>( W_T^C )</th>
<th>( W^S_{TL}^C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u )</td>
<td>( A_1 )</td>
<td>( A_2 )</td>
<td>( A_3 )</td>
<td>( o )</td>
<td>( \text{Re} B_1 )</td>
<td>( o )</td>
<td>( o )</td>
</tr>
<tr>
<td>( p^{(a)}_x )</td>
<td>( o )</td>
<td>( o )</td>
<td>( o )</td>
<td>( \text{Re} Q^{(a)}_1 )</td>
<td>( o )</td>
<td>( -\text{Re} B_2^{(a)} )</td>
<td>( \text{Im} Q^{(a)}_1 )</td>
</tr>
<tr>
<td>( p^{(a)}_y )</td>
<td>( A_4^{(a)} )</td>
<td>( A_5^{(a)} )</td>
<td>( A_6^{(a)} )</td>
<td>( o )</td>
<td>( \text{Re} B_2^{(a)} )</td>
<td>( o )</td>
<td>( o )</td>
</tr>
<tr>
<td>( p^{(a)}_z )</td>
<td>( o )</td>
<td>( o )</td>
<td>( o )</td>
<td>( \text{Im} Q^{(a)}_2 )</td>
<td>( o )</td>
<td>( -\text{Im} B_4^{(a)} )</td>
<td>( -\text{Re} C_2^{(a)} )</td>
</tr>
</tbody>
</table>

Where:

\[
\begin{align*}
A_1 & = \left| b_5 \right|^2 + \left| b_6 \right|^2 ; \quad B_1 = -2 \left( b_5^* b_1 + b_6^* b_2 \right) \\
A_2 & = \left| b_1 \right|^2 + \left| b_2 \right|^2 + \left| b_3 \right|^2 + \left| b_4 \right|^2 ; \quad B_2^T = -B_2^R = 2 \left( b_5^* b_1 - b_6^* b_2 \right) \\
A_3 & = -\left| b_1 \right|^2 - \left| b_2 \right|^2 + \left| b_3 \right|^2 + \left| b_4 \right|^2 ; \quad B_3^T = 2 \left( b_5^* b_4 - b_6^* b_3 \right) \\
& \quad \quad \quad \quad \quad \quad \quad B_3^R = -2 \left( b_5^* b_3 - b_6^* b_4 \right) \\
A_4^T & = -A_4^R = \left| b_6 \right|^2 - \left| b_5 \right|^2 ; \quad B_4^T = -2 \left( b_5^* b_4 + b_6^* b_3 \right) \\
A_5^T & = A_5^R = -\left| b_1 \right|^2 + \left| b_2 \right|^2 - \left| b_3 \right|^2 + \left| b_4 \right|^2 ; \quad B_4^R = 2 \left( b_5^* b_3 + b_6^* b_4 \right) \\
A_6^T & = A_5^R = \left| b_1 \right|^2 + \left| b_2 \right|^2 - \left| b_3 \right|^2 + \left| b_4 \right|^2 \\
C_1^T & = 2 \left( b_3^* b_2 - b_4^* b_1 \right) ; \quad C_1^R = 2 \left( b_3^* b_1 - b_4^* b_2 \right) \\
C_2^T & = -2 \left( b_3^* b_2 + b_4^* b_1 \right) ; \quad C_2^R = 2 \left( b_3^* b_1 + b_4^* b_2 \right)
\end{align*}
\]
The principal example of an $\eta_g = -1$ process is $p(e,e'p)\pi$. The above analysis can be extended to the case $\eta_g = 1$, an example of which is $^3\text{He}(e,e'p)np$ when the np pair in the final state can be approximated by a $1s_0$ state, i.e. for small kinetic energy of the pair.

References

Appendix P  Reprint of Reference 2
Polarization observables in deuteron photodisintegration and electrodisintegration

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(Received 7 June 1989)

A comprehensive relativistic treatment of polarization observables in deuteron photo- and electrodisintegration is presented from a unified standpoint. A discussion of necessary and sufficient measurements needed for a complete determination of all transition amplitudes is given.

I. INTRODUCTION

With the construction of new electron accelerator facilities, new types of experiments which can give more detailed knowledge of electronuclear processes will be feasible. In particular, continuous wave (CW) machines make it possible to do coincidence experiments and determine exclusive cross sections. If, in addition, the polarization of the beam, target, or outgoing ejectile is measured, it may be possible to completely determine all of the helicity amplitudes which contribute to the hadronic current, and in this way place strong constraints on any theoretical calculations.

This prospect has aroused new interest in the formalism and theory of coincidence measurements where the polarization of one or several of the particles is measured. In this paper we derive the coincidence cross section for the $d(e,e'p)n$ reaction in the general case when the incoming electron, deuteron target, and one outgoing nucleon are all polarized. There are 162 observables which describe all possible cases, and they are given as bilinear products of the 18 independent amplitudes which completely describe deuteron electrodisintegration. While some of these results have been given previously in nonrelativistic cases, this is the first time, to our knowledge, that all of these observables have been described in a fully relativistic, unified manner. The formulas, summarized in Tables X–XII, will be useful in subsequent calculations. The formulas for deuteron photodisintegration, $\gamma + d \rightarrow p + n$, which is described by only 12 independent amplitudes, are obtained as a natural by-product of the electrodisintegration results.

Section II contains the derivation of the polarization observables and coincidence cross section. Electrodisintegration is treated in the one-photon exchange approximation as a binary collision of a virtual photon and the deuteron target. The density matrix of the virtual photon is described in terms of the kinematical variables of the electron in the laboratory (lab) system, while the hadronic current can be described in the center-of-mass (c.m.) system of the outgoing nucleon pair, which is a convenient frame in which to present the results for the final state, or to integrate over final-state momenta and convert exclusive cross sections into inclusive cross sections. The density matrices of the deuteron target and the recoiling nucleons are obtained, and their properties under rotations to different coordinate system worked out. First, the structure of the results when expressed in terms of helicity amplitudes is discussed, and then parity conservation is used to simplify (diagonalize) the problem. It is found that the formulas are greatly simplified when a new set of amplitudes is used. These amplitudes, denoted $g_I$, are similar to transversity amplitudes previously introduced into the study of pion photoproduction and discussed by Moravčík and his collaborators. Finally, the modifications in the formulas required if it is desired to express all variables in the lab system are discussed.

The comparative simplicity of the final results makes it possible to discuss the design of experimental programs of measurements which could, at least in principle, lead to a complete determination of the 18 independent complex amplitudes which describe deuteron electrodisintegration. This requires the measurement of at least 35 quantities (since one overall phase can never be determined). This is discussed in some detail in Sec. III, where one strategy for such a program is presented, and it is shown that at least one measurement of a recoil neutron polarization is essential for a program of complete measurements. While such a program may never be carried out, it is still of interest, for planning purposes, to see what kinds of measurements are redundant, and which give truly independent information. These insights can be obtained from the results given in this paper.

This paper contains no dynamical calculations; these are presently under way and will be published elsewhere.

II. FORMALISM

A. The cross-section–electron variables

The Feynman amplitude for the electron scattering process in one-photon exchange approximation, depicted in Fig. 1, is

$$M_F = e^2 q(k') \gamma_{\mu}(k) \frac{D^{\mu\nu}}{q^2} \langle P_1 F_1 | J_\nu | P_T \rangle$$

$$= \frac{e^2}{q^2} j_\mu D^{\mu\nu} j_\nu ,$$

(1)
where \( j^\mu \) and \( J^\nu \) are the electron and hadron currents, respectively, \( q^\mu = k^\mu - k'^\mu \) is the four momentum transferred by the electron, and the photon projection operator is

\[
D^{\mu\nu} g^{\mu\nu} - \frac{q^{\mu}q^{\nu}}{q^2} .
\]

(2)

Since the currents are conserved,

\[
q_{\mu} j^\mu = 0 = q_{\nu} J^\nu ,
\]

(3)

it is customary to drop the \( g^{\mu\nu} q^\nu \) term in \( D^{\mu\nu} \). Alternatively, it is convenient to introduce the expansion

\[
D^{\mu\nu} = \sum_{\lambda, \gamma} (-1)^\gamma e_{\lambda, \gamma}^\mu e_{\lambda, \gamma}^\nu ,
\]

(4)

where the sum is over the photon helicities \( \lambda, \gamma = \pm 0 \), and if the photon momentum is taken to be in the \( +2 \) direction, then in the lab system

\[
e_{\lambda} = \mp \frac{1}{\sqrt{2}} (0, 1, \pm 0) ,
\]

\[
e_{\gamma} = \frac{Q}{Q} (0, 0, \nu , 0) ,
\]

(5)

where

\[
q^\mu = (\nu , 0, 0, q_L)
\]

and \( q^2 = \nu^2 - q_L^2 = -Q^2 < 0 \). Note that the polarization vectors have the following properties:

\[
q_{\mu} e_{\lambda}^{\mu} = 0 ,
\]

\[
Q_{\mu\nu} e_{\lambda}^{\mu} e_{\lambda}^{\nu} = (-1)^\gamma \delta_{\lambda, \lambda'} .
\]

(7)

The expansion (4) is convenient because it separates the scattering amplitude, defined in Eq. (1), into the sum of products of separately covariant currents

\[
M_{ij} = -\sum_{\lambda, \gamma} (-1)^\gamma \frac{e_{\lambda}^\mu e_{\lambda}^{\nu}}{Q^2} (j_{\mu} e_{\lambda}^{\nu})(J_{\nu} e_{\lambda}^{\mu}) .
\]

(8)

In what follows, the electron current will be evaluated in the lab system, and the hadron current in the c.m. system of the outgoing pair of hadrons, with four-momentum \( p_1 \) and \( p_2 \). This is done in order to simplify the description of the hadronic final state; in particular, with this choice it will be possible to integrate easily over the solid angle of the outgoing hadronic pair and reduce the coincidence cross section to the inclusive cross section. To facilitate this, the cross section is written

\[
d\sigma = \frac{\sum_{\lambda, \gamma} (-1)^\gamma}{Q^2} \delta_{\mu, \lambda'} \delta_{\nu, \lambda'} \frac{d^3p_1}{(2\pi)^3} (-1)^\gamma \frac{d^3p_2}{(2\pi)^3} \frac{dE'}{E'} L_{\lambda, \lambda'} W_{\lambda, \lambda'} ,
\]

(9)

where \( E \) and \( E' \gg m \) are the energies of the incoming and outgoing electrons, \( M_T \) the mass of the target, and the electron and hadron density matrices are

\[
W_{\lambda, \lambda'} = \sum_{\text{spins}} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{dE}{E} \frac{dE'}{E'} \frac{2M_T}{\gamma_\lambda \gamma_{\lambda'}}
\]

\[
\times \delta(p_1 + p_2 - p - q) J_{\lambda} e_{\lambda'}^\dagger e_{\lambda'}^* .
\]

(10)

The electron density matrix includes the possibility that the incoming electron has spin polarization \( s^h \) satisfying \( s \cdot k = 0 \), and the hadron density matrix implicitly includes spin projection operators as needed to describe the polarization of the initial deuteron, or the polarization of the recoil neutron or proton, or any combination of these. The structure of \( W_{\lambda, \lambda'} \) will be discussed extensively below—for now we note that it is Lorentz covariant, and hence can be studied in the c.m. system of the outgoing \( np \) system, and that in this system the energy-momentum-conserving \( \delta \) function fixes four of the six integration variables, leaving only the direction of the relative momentum of the final pair \( p = \frac{1}{2}(p_1 - p_2) \) unspecified. The remainder of this part will be devoted to reducing the electron tensor.

Carrying out the trace in Eq. (10) gives

\[
L_{\lambda, \lambda'} = 2\left[ k \cdot e_{\lambda} e_{\lambda'} k' - k' \cdot e_{\lambda'} e_{\lambda} k \right] + 2i\hbar e_{\lambda} e_{\lambda'} \left( k \cdot k' \delta_{\lambda, \lambda'} \right)
\]

\[
= L_{\lambda, \lambda'}^{\mu} + 2\hbar L_{\lambda, \lambda'}^{\mu} ,
\]

(12)

where \( \hbar \) is the helicity of the incoming electron, equal to \( \pm \frac{1}{2} \), and \( L \) is separated into electron-helicity-independent and -dependent terms, \( L^{0} \) and \( L^{\pm} \), respectively. Both parts of \( L \) and \( W \) (see the next section) are Hermitian

\[
L_{\lambda, \lambda'} = W_{\lambda, \lambda'}^{\mu} ,
\]

(13)

The photon polarization vectors (5) satisfy the reflection property

\[
e_{\lambda}^{\dagger} e_{\gamma} = (-1)^\gamma e_{\gamma} e_{\lambda}^{\dagger} .
\]

(14)

from which it follows that

![Feynman diagram](image-url)
Hence there are only six independent density-matrix elements, as given in Table I. These can be readily evaluated in the lab frame using the explicit forms for the polarization vectors given in Eq. (5). Factoring out a common factor

\[ L_{\lambda\lambda} = 4E^2 e^{i\lambda} \theta I_{\lambda\lambda} \]

where \( \theta \) is the scattering angle of the electron in the lab frame, we choose for these six matrix elements

\[ I_{00}, I_{0+}, I_{0-}, I_{+0}, I_{++}, I_{+-} \]


TABLE I. Relations between the photon density matrix elements. Note that there are only six independent elements.

<table>
<thead>
<tr>
<th>Unpolarized electrons</th>
<th>Incoming electron with helicity ( h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_{00} )</td>
<td>( I_{h0} )</td>
</tr>
<tr>
<td>( I_{0+} = I_{0-} = I_{h0} = I_{h0} )</td>
<td>( I_{h0} = 0 )</td>
</tr>
<tr>
<td>( I_{0+} = I_{0-} = I_{h0} = I_{h0} )</td>
<td>( I_{h0} = 0 )</td>
</tr>
<tr>
<td>( I_{0+} = I_{h0} )</td>
<td>( I_{h0} = 0 )</td>
</tr>
<tr>
<td>( I_{0-} = I_{h0} )</td>
<td>( I_{h0} = 0 )</td>
</tr>
</tbody>
</table>

Explicit forms for these matrix elements are given in Table II. Note that they are all real. Using Eq. (9), the relations in Table I, and the hermiticity of \( W_{\lambda\lambda} \), gives an intermediate result for the cross section

\[ d\sigma = \frac{\alpha^2 \cos^4 \theta}{2E \sin^2 \theta} \left( 2 I_{00} \rho_{00} + 2 I_{0+} \rho_{0+} + (W_{++} + W_{--}) + I_{+0} - 2 \text{Re}(W_{++} - W_{00} - 2 \text{Re}(W_{0+} + W_{0-})) \right) \]

To reduce the cross section further, we study the hadron-incidence density matrix \( W_{\lambda\lambda} \) in the next section.

B. The cross-section-hadron variables

The hadronic density matrix was defined in Eq. (11). It is explicitly Lorentz covariant, and can therefore be evaluated in any frame. The kinematics of the lab frame for the entire scattering process are shown in Fig. 2. The struck hadron (which has four-momentum \( p_x \) by convention) emerges at an angle \( \alpha_x \) with respect to the direction of the photon's three momentum, \( q \), and lies in a plane tilted with respect to the electron scattering plane by an angle \( \alpha_0 \) as shown. If \( \alpha \neq 0 \) or \( r \), the exclusive process is referred to as "out of plane."

\[ d\sigma = \frac{\alpha^2 \cos^4 \theta}{2E \sin^2 \theta} \left( 2 I_{00} \rho_{00} + 2 I_{0+} \rho_{0+} + (W_{++} + W_{--}) + I_{+0} - 2 \text{Re}(W_{++} - W_{00} - 2 \text{Re}(W_{0+} + W_{0-})) \right) \]

To reduce the cross section further, we study the hadron-incidence density matrix \( W_{\lambda\lambda} \) in the next section.

\[ d\sigma = \frac{\alpha^2 \cos^4 \theta}{2E \sin^2 \theta} \left( 2 I_{00} \rho_{00} + 2 I_{0+} \rho_{0+} + (W_{++} + W_{--}) + I_{+0} - 2 \text{Re}(W_{++} - W_{00} - 2 \text{Re}(W_{0+} + W_{0-})) \right) \]

where

\[ \sigma_{\lambda\lambda} = \left( \frac{\alpha \cos \frac{1}{2} \theta}{2E \sin \frac{1}{2} \theta} \right)^2 \]

To reduce the cross section further, we study the hadron-incidence density matrix \( W_{\lambda\lambda} \) in the next section.

\[ d\sigma = \frac{\alpha^2 \cos^4 \theta}{2E \sin^2 \theta} \left( 2 I_{00} \rho_{00} + 2 I_{0+} \rho_{0+} + (W_{++} + W_{--}) + I_{+0} - 2 \text{Re}(W_{++} - W_{00} - 2 \text{Re}(W_{0+} + W_{0-})) \right) \]

where

\[ \sigma_{\lambda\lambda} = \left( \frac{\alpha \cos \frac{1}{2} \theta}{2E \sin \frac{1}{2} \theta} \right)^2 \]

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where

\[ \sigma_{\lambda\lambda} = \left( \frac{\alpha \cos \frac{1}{2} \theta}{2E \sin \frac{1}{2} \theta} \right)^2 \]
FIG. 2. Diagram of the electrodisintegration process showing the electron scattering plane, the ejectile plane, and the two coordinate systems in the ejectile plane: \((x',y',z')\) and \((x'',y'',z'')\).

+\(\phi\) about the \(z\) axis. The boost is defined by the requirement that the three momentum of the final state with four momentum \(P = p_1 + p_2\) be brought to zero. If \(W\) is the invariant mass of the final state, then in the lab system

\[
P_{\text{lab}}^{{\mu}} = (W^2 + q_\perp^2)^{1/2}, 0, 0, q_L\]

and in the c.m. system, \(P_{\text{c.m.}}^{{\mu}} = (W, 0)\). Hence the boost transformation is, in matrix form,

\[
B_{\text{c.m.}} = \begin{pmatrix}
    (W^2 + q_\perp^2)^{1/2} & 0 & -q_L \\
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    -q_L & 0 & (W^2 + q_\perp^2)^{1/2} \\
\end{pmatrix}
\]

(20)

This form of the cross section displays the exact \(\phi\) dependence, provided any polarization vectors which enter \(W_{\lambda\lambda'}\) are defined with respect to the \((x'y'z')\) coordinate system.

The next step in obtaining the coincidence cross section is to carry out the integrals in Eq. (11) in the c.m. frame. This gives

\[
W_{\lambda\lambda'} = d\Omega \rho \rho_1 R_{\lambda\lambda'}
\]

(26)

where, if \(\kappa^2 = M_1 M_2/4\pi^2 W\),

\[
R_{\lambda\lambda'} = \kappa^2 \sum_j J_j^* e_\lambda e_\lambda'^* = \kappa^2 \sum_j J_j L_j^*
\]

(27)

where we use the notation \(J_j = J^\lambda e_\lambda\). It is quite common to use current conservation to express the longitudinal hadronic current in terms of its \(J^0\) component. This so that if \(q_{\text{c.m.}}^{{\mu}} = (v_0, 0, 0, q_0)\), then

\[
q_0 = \frac{(W^2 + q_\perp^2)^{1/2}}{W} q_L - \frac{q_L}{W} v = \frac{M_T}{W} q_L.
\]

(21)

The transverse helicity amplitudes are unchanged by the boost, and \(e_\lambda\) has the same form as in Eq. (5) with \(q_0\) and \(v_0\) replacing \(q_L\) and \(v\).

The rotation through \(\theta_f > \pi/2\) about the \(z\) axis leaves the longitudinal polarization vector unchanged, but changes the phase of the transverse components. The transformed polarization vectors, \(e^\prime_\pm\), become

\[
e^\prime_\pm = e^{\mp i\phi} e_\pm
\]

(22)

where \(e_\pm\) are the conventional vectors defined in Eq. (5).

Hence, the overall effect of the transformation from the lab electron plane to the c.m. ejectile plane is to modify \(W_{\lambda\lambda'}\) as follows:

\[
W_{\lambda\lambda'}^* = e^{-i(\lambda - \lambda') W_{\lambda\lambda'}}.
\]

(23)

Inverting this expression gives

\[
W_{\lambda\lambda'} = e^{i(\lambda - \lambda') W_{\lambda\lambda'}}.
\]

(24)

This phase introduces a nontrivial \(\phi\) dependence into the total cross section. Substituting (24) into (17) permits us to extract this \(\phi\) dependence, giving

\[
d\sigma = \frac{dE'dY'}{4\pi M_T} \left[I^0_0(W_0 - W_0') + I^0_+ (W_+ + W_-^-) + I^0_- \cos 2\phi 2 \Re W_+ - - I^0_+ \sin 2\phi 2 \Im W_+ - - - I^0_0 \cos 2\phi 2 \Re W_+ - - + 2h I^0_+ (W_+ - W_-^-) - 2h I^0_0 \cos 2\Re W_0 + + W_0' - - - W_0 + + W_0' - - - W_0 + + W_0' - - - \right].
\]

(25)

This reduction is not frame independent. If used, it should be carried out in the frame in which \(J\) is to be evaluated (the c.m. frame). In this frame

\[
q_0 J^\prime \# = v_0 J^\prime 0 - q_0 J^\prime 3 = 0.
\]

(28)

Hence

\[
J_0 = q_0 J^\prime 0 = \frac{q_0}{Q} J^\prime 0 = \frac{q_0}{Q} J^\prime 0 = \frac{1}{\eta} J^\prime 0
\]

(29)
factor of $1/\eta = [(W/M_T)/(Q/q_L)]$ must accompany each such superscript. Density matrices with these extra factors, $\rho_{\Delta\lambda}$, were defined in Table II. Because this convention has become quite familiar, we will present results in terms of these density matrices, and correct for the factors $1/\eta$ by adding factors of $\eta$ to the corresponding structure functions (see Table III).

Combining Eq. (25), (26), and (27) and using Table II gives us our general form for the coincidence cross section:

$$\frac{d^3\sigma}{d\Omega'dE'd\Omega} = \frac{\eta M_P}{4\pi M_T'} \left[ \begin{array}{c} W/M_T' \end{array} \right]^2 \rho_{L} + \frac{1}{2} \frac{W}{M_T'} \left[ \begin{array}{c} \cos 2\phi \rho_{R} + \sin 2\phi \rho_{R} \end{array} \right].$$

The relationships between the $R$'s introduced in this expression and the covariant $R$'s of Eq. (27) are given in Table III. Our results agree with those previously obtained by Walecka and Zucker.7 However, previous derivations of this cross section given by Arenhovel9 have omitted the factors of $W/M_T$ associated with the transformation (29), although they are included in Ref. 2. (Note that these factors do not occur if the hadronic current is evaluated in the lab frame—see Sec. II H.) Since these derivations treated the hadronic currents nonrelativistically, and these factors are of relativistic origin, it could be argued that they may be neglected. We believe that even if the currents are calculated nonrelativistically, such kinematic factors should be regarded as part of the cross section and should not be neglected. For light targets, such as the deuteron, they are not small. For example, for $Q^2 = (1/\sqrt{2})$ at the quasielastic peak, where $n = Q^2/2M_T = 1$,

$$\frac{W}{M_T'} = \left[ 1 + \frac{Q^2}{M_T'} \left( M_T' - 1 \right) \right]^{1/2} \approx 1 + \frac{Q^2}{2M_T'} = 1.16,$$

which gives an enhancement of approximately 16%. It is very important to treat such factors systematically and consistently.

TABLE III. The $R$'s used in Eq. (30) (left-hand column) are equal to expressions involving $R_{\Delta\lambda}$ [defined in Eq. (27)] given in the center column, or the sums over current operators given in the right-hand column. In the right-hand column, $J_{\Delta} = J'_{\Delta}$ as defined in Eq. (29). Note that, because of the metric tensor, $J_{\Delta} = \pm (1/\sqrt{2}) J_{\Delta\lambda}$, which is opposite in sign from that used in Refs. 3 and 7. Our overall results agree with these references. $R_{\Delta\lambda}$

| $R_L$ | $\eta^2 R_{\Delta\lambda}$ | $\kappa^2 \sum |J|_2$ |
|------|---------------------------|-----------------|
| $R_T$ | $R_{+} + R_{-}$ | $\kappa^2 \sum (|J_+|^2 + |J_-|^2)$ |
| $R_{+}$ | $2 Re R_{+}$ | $2\kappa^2 Re \sum (|J_+|^2)$ |
| $R_{-}$ | $-2 Im R_{+}$ | $-2\kappa^2 Im \sum (|J_+|^2)$ |
| $R_{0}$ | $2 \eta Re (R_{0} - R_{\lambda} - R_{\lambda})$ | $2\kappa^2 Re \sum (|J_0|^2 - |J_\lambda|^2)$ |
| $R_{1}$ | $2\eta Im (R_{0} + R_{\lambda} + R_{\lambda})$ | $2\kappa^2 Im \sum (|J_0|^2 - |J_\lambda|^2)$ |
| $R_{2}$ | $\kappa^2 \sum (|J_0|^2 - |J_\lambda|^2)$ |
| $R_{3}$ | $2\eta Re (R_{0} - R_{\lambda} - R_{\lambda})$ | $2\kappa^2 Re \sum (|J_0|^2 - |J_\lambda|^2)$ |
| $R_{4}$ | $2\eta Im (R_{0} + R_{\lambda} + R_{\lambda})$ | $2\kappa^2 Im \sum (|J_0|^2 - |J_\lambda|^2)$ |

C. The cross section for polarized hadrons—general formulas

The sums over hadron spin states which appear in the spectral functions defined in Table III must still be specified completely. If the hadrons are completely unpolarized, the generic term Eq. (27) is

$$R_{ab} = \kappa^2 \sum J_a J_b,$$

where we have averaged over the polarization states of the deuteron and summed over the polarization states of the two outgoing nucleons. In matrix form, $(\lambda_1 \lambda_2 | J_a | J_b)$ is a 4X3 matrix, and (32) can be written simply as

$$R_{ab} = \frac{\kappa^2}{3} \text{tr}[J_a J_b^*].$$

In general, the target or either of the two hadrons may be polarized, and the polarization can be described using a density matrix $\rho_D$ or $\rho_H$. The generalized expression which allows for this possibility is

$$R_{ab} = 4\kappa^2 \text{tr}[\rho_H J_a J_b^*],$$

where the density matrices for nucleons and deuterons are normalized to $\text{tr}(\rho) = 1$. The remaining task is to describe these density matrices, and to simplify the expressions.

1. Deuteron polarization

The polarization state of the deuteron is described by a spin-1 density matrix. As a consequence of its hermiticity and the normalization condition (trace = 1), this matrix is specified by eight real parameters. In the Cartesian representation these are the three components of the polarization vector $P_T$ and five components of the (symmetric) polarization tensor $P_D$. In the spherical basis they are three rank-1 and five rank-2 tensors: $T_{1M}$ and $T_{2M}$. We follow the notation of Ohlsen.9 These quantities are defined via the (ensemble) average values of the spin-1 operators $S_1$ and $S_{1\lambda}$, where

$$S_0 = \frac{1}{2}(S_{1\lambda} + S_{1\lambda}),$$

In the spherical basis, spanned by $(\xi_+, \xi_0, \xi_-)$ where $\xi_\pm = \mp(1/\sqrt{2})(1, \pm i, 0)$ and $\xi_0 = (0, 0, 1)$, these spin-1 operators assume the following familiar form:
The spin-1 density matrix \( \rho_D \) can be written in the form

\[
\rho_D = \frac{1}{2} \left[ \left( \begin{array}{cccc}
1 & S_{1} & S_{2} & S_{3} \\
\frac{1}{2}S_{1} & 0 & 0 & 0 \\
\frac{1}{2}S_{2} & 0 & 0 & 0 \\
\frac{1}{2}S_{3} & 0 & 0 & 0
\end{array} \right) \right].
\]

The density matrix \( \rho_D \) can also be expanded in terms of spherical irreducible tensor operators of rank 0, 1, and 2:

\[
\rho_D = \frac{1}{2} \sum_{j-0}^{2} \sum_{M=-j}^{j} T_j^M \tau_j^M ,
\]

where

\[
T_j^M = \left( -1 \right)^{M} T_j^{M-M} ,
\]

\[
\tau_j^M = \left( -1 \right)^{M} \tau_j^{M-M} ,
\]

and the relations between the \( \tau_j^M \)s and \( S_j^M \)s (or \( \tau_j^M \)s and \( P_j^M \)s) are given in Table IV. Using these definitions, and Eqs. (35) and (36), we can write the density matrix as

\[
\rho_D = \frac{1}{3} \begin{pmatrix}
1 + \sqrt{3} T_{10} + \frac{1}{\sqrt{2}} T_{20} & -\sqrt{3} (T_{11} + T_{21}) & \sqrt{3} T_{22} \\
-\sqrt{3} (T_{11} + T_{21}) & 1 - \sqrt{2} T_{20} & -\sqrt{3} (T_{11} - T_{22}) \\
\sqrt{3} T_{22} & -\sqrt{3} (T_{11} - T_{21}) & 1 - \sqrt{2} T_{20} + \frac{1}{\sqrt{2}} T_{20}
\end{pmatrix},
\]

where use was made of the symmetry relations (39) for \( T_j^M \). This matrix is with respect to the standard \( \xi_{+}, \xi_{0}, \xi_{-} \) basis, which is appropriate for deuteron polarizations defined with respect to the \( (x',y',z') \) coordinate system shown in Fig. 2 (where the \( z' \) axis is in the direction of the three momentum carried by the photon, \( q \)).

In what follows, we will use the helicity formalism for the hadronic particles. Following the standard conventions, the deuteron will be taken to be particle number 2 in the initial state, and its helicity vectors are therefore obtained from the standard \( \xi_{+} \)’s according to

\[
\xi_{+} = \eta_{d} \xi_{+} = \eta_{d} = A \xi_{+} .
\]

The transformation matrix \( A \) in the spherical basis is

\[
A = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix} = A^t = A^{-1} ,
\]

which gives the relation

\[
\xi_{+} = \xi_{-} A \eta_{d} .
\]

In this new, helicity basis, the density matrix \( \rho_D \) becomes

\[
\rho_D = \frac{1}{3} \begin{pmatrix}
1 - \sqrt{2} T_{10} + \frac{1}{\sqrt{2}} T_{20} & -\sqrt{2} (T_{11} - T_{22}) & \sqrt{3} T_{22} \\
\sqrt{2} (T_{11} - T_{22}) & 1 - \sqrt{2} T_{20} & -\sqrt{2} (T_{11} + T_{22}) \\
\sqrt{3} T_{22} & -\sqrt{2} (T_{11} + T_{22}) & 1 + \sqrt{2} T_{20} + \frac{1}{\sqrt{2}} T_{20}
\end{pmatrix}.
\]


This matrix gives the deuteron polarization observables defined with respect to the primed coordinate system in terms of deuteron helicity states $\mathcal{J}_{10}$ and is the one appropriate to our calculation. Note that the relations (39) insure that this matrix is Hermitian.

2. Nucleon polarization

The density matrices for the outgoing nucleons, normalized so $\text{tr} \rho = 1$, are easily constructed from the familiar spin-$\frac{1}{2}$ projection operators.

$$\rho_p = \frac{1}{2} \left[ \begin{array}{cc} 1 + P_x P_x - P_y P_y & 1 - P_x P_x \\ P_x + P_y & 1 - P_x P_x \end{array} \right].$$

We will choose the proton to be particle 1, and the neutron particle 2 (in the sense of Jacob and Wick), and will define the polarizations with respect to the $(x'',y'',z'')$ system shown in Fig. 2. Later on, we will use the standard notation $s, n$, and $l$ to denote proton polarizations in the $x'', y'', z''$ directions, respectively. The proton is traveling in the $+z''$ direction, and the neutron, in the c.m. of the outgoing pair, in the $-z''$ direction. In this case the matrix in Eq. (45) is already in the correct form to use with both the proton and neutron helicity amplitudes, provided we remember that $P_z = 1$ for the proton corresponds to polarization in the $+z''$ direction, while $P_z = -1$ for the neutron corresponds to polarization in the $-z''$ direction. We will adopt this convention here. If it is desired to express the neutron polarization directly in terms of the $(x'', y'', z'')$ coordinate system, the connection is

$$\overline{X_{\lambda n}} = (-1)^{1/2 - \lambda_e} e^{-i \text{sgn}(\sigma_2/2)} X_{\lambda n}$$

and the formulas we obtain can be converted to this convention by changing $P_z \rightarrow -P_z$ and $P_z \rightarrow -P_z$, while leaving $P_x$ unchanged.

If the proton and neutron helicities states are represented by a four-component vector $V_i$ with components

$$V_1 = |++\rangle, \quad V_2 = |+-\rangle,$$

$$V_3 = |-+\rangle, \quad V_4 = |--\rangle$$

where the first entry is the helicity of the proton, the nucleon density matrix becomes

$$\rho_{NN} = \frac{1}{4} \left[ \begin{array}{cccc} 1 + P_x P_x + P_y P_y & 1 - P_x P_x & (P_x - i P_y)(1 + P_z) & (P_x + i P_y)(1 + P_z) \\ (1 + P_x P_x - P_y P_y) & 1 - P_x P_x & (P_x - i P_y)(1 + P_z) & (P_x - i P_y)(1 + P_z) \\ (P_x + i P_y)(1 + P_z) & (P_x + P_y)(1 + P_z) & (P_x - i P_y)(1 + P_z) & (P_x + i P_y)(1 + P_z) \\ \end{array} \right].$$

where $P$ and $P'$ are the polarizations of the outgoing proton and neutron, respectively, defined with respect to the $(x'', y'', z'')$ coordinate system for the proton, and the $(x'', -y'', -z'')$ coordinate system for the neutron. This is the density matrix we will use. Note that it is Hermitian. We now turn to the specification of the matrix elements of the current.

D. Polarization observables in helicity basis

For electrodisintegration, there are 36 helicity amplitudes, but parity conservation can be used to express 18 of them in terms of the other 18 through the relation

$$\langle -\lambda_{\gamma}, -\lambda_s | J_{-\lambda_{\gamma}} | -\lambda_D \rangle = \eta_{\gamma} (-1)^{1/2} e^{-i \text{sgn}(\sigma_2/2)} \langle \lambda_{\gamma}, \lambda_s | J_{-\lambda_{\gamma}} | \lambda_D \rangle$$

where, for deuteron electrodisintegration

$$\eta_{\gamma} = \eta(-1)^{1/2} e^{-i \text{sgn}(\sigma_2/2)},$$

where $\eta$ is the product of the intrinsic parities of the four particles, and $\gamma$ are the spins. (Note that Ref. 6 contains an error; it is assumed there that $\eta_{\gamma} = -1$ instead of $+1$.) The relation (49) follows from the transformation $\gamma = e^{-i \sigma_2/2}$, which has the same effect as inversion in the $\gamma-z$ plane. The 18 independent helicity amplitudes are labeled using the convention introduced by Renard et al.
\[ F_{1,2} = (\pm i \pm \frac{1}{2}|J\cdot e_+|1), \quad F_{3,4} = (\pm i \pm \frac{1}{2}|J\cdot e_-|0), \]
\[ F_{5,6} = (\pm \frac{1}{2} \pm \frac{1}{2}|J\cdot e_+|1), \quad F_{7,8} = (\pm \frac{1}{2} \mp \frac{1}{2}|J\cdot e_-|1), \]
\[ F_{9,10} = (\pm \frac{1}{2} \pm \frac{1}{2}|J\cdot e_+|0), \quad F_{11,12} = (\pm \frac{1}{2} \mp \frac{1}{2}|J\cdot e_-|0). \]

\[ F_{13,15} = (\pm \frac{1}{2} \pm |J\cdot e_0|\pm |J\cdot e_0|0), \quad F_{16} = (\pm \frac{1}{2} \frac{1}{2}|J\cdot e_0|0), \]
\[ F_{14,18} = (\pm \frac{1}{2} \pm |J\cdot e_0|\pm |J\cdot e_0|0). \]

In matrix form this gives
\[
J_+ = \begin{pmatrix}
F_1 & F_3 & F_5 \\
F_7 & F_9 & F_{11} \\
F_6 & F_{10} & F_{12}
\end{pmatrix},
J_0 = \begin{pmatrix}
F_{13} & F_{14} & F_{15} \\
F_{18} & -F_{17} & F_{16} \\
-F_{15} & F_{14} & -F_{13}
\end{pmatrix},
J_- = \begin{pmatrix}
F_4 & -F_4 & F_2 \\
-F_2 & F_10 & -F_8 \\
F_5 & -F_3 & F_1
\end{pmatrix}
\]

(Note that the \( Y \) parity constraints \( 49 \) take on a simple form in this matrix space. Specifically,
\[
\bar{Y}_4 \bar{Y}_3 = \eta_D (-1)^{J_{z_0}} \bar{Y}_2,
\]
where \( \eta_D = -1 \) is the intrinsic parity of the photon, and the intrinsic parities and phases of the nucleons are incorporated into \( \bar{Y}_4 \) and of the deuteron into \( \bar{Y}_3 \),
\[
\bar{Y}_4 = \eta_p \eta_n \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix},
\bar{Y}_3 = \eta_D \begin{pmatrix}
0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]
where \( \eta_p = \eta_n = \eta_D = 1 \), and \( \bar{Y}_2 = 1 \).

It will be convenient to introduce density matrices which are even or odd under \( Y \) parity. If \( \alpha = \pm 1 \), the combinations
\[
\bar{\rho}^{\alpha, p}_N = \frac{1}{4} (\rho_N + \alpha \bar{\rho}_4 \bar{\rho}_N \bar{\rho}_4),
\bar{\rho}^{\alpha, p}_D = \frac{1}{4} (\rho_D + \alpha \bar{\rho}_3 \bar{\rho}_D \bar{\rho}_3)
\]
transform like
\[
\bar{\rho}^{\alpha, p}_N = \alpha \bar{\rho}_4 \bar{\rho}^{\alpha, p}_N \bar{\rho}_4,
\bar{\rho}^{\alpha, p}_D = \alpha \bar{\rho}_3 \bar{\rho}^{\alpha, p}_D \bar{\rho}_3.
\]

It is straightforward to show, neglecting all terms which depend on the products of \( \rho_N \rho'_N \) (because the case of both nucleons polarized in the final state will not be considered in this paper), that
\[
\bar{\rho}^{+}_N = \frac{1}{4} \begin{pmatrix}
1 & -ip' & -ip & 0 \\
-ip & 1 & 0 & -ip' \\
0 & -ip' & 1 & -ip \\
0 & ip & ip' & 1
\end{pmatrix}
\]
\[
\bar{\rho}^{+}_N = \frac{1}{4} \begin{pmatrix}
P_x + P'_x & P'_x & P_x & 0 \\
P'_x & P_x - P'_z & 0 & P_x \\
P_x & 0 & -P_x + P'_z & P'_x \\
0 & P_x & P'_x & -P_x - P'_x
\end{pmatrix}
\]
Polarization observables in a hybrid basis

The density matrices given in Eqs. (57) and (58) are even or odd under the $Y$ parity transformation, but still have the complexity (number of nonzero elements) of the original density matrices. The reason for this is that these matrices are expressed in terms of the helicity basis, where the axis of quantization is along $z$, the direction of motion. However, since only the $y$ component of spin does not change sign under the $Y$ transformation (none of the components of spin change under parity, but the rotation by $\pi$ around the $y$ axis changes $x \rightarrow -x$ and $z \rightarrow -z$) it is more natural to choose the $y$ axis attached to the particle as the basis for quantization. Amplitudes quantized with respect to the $y$ axis are referred to as transversity amplitudes. The amplitudes introduced here quantize all the hadrons with respect to the $y$ axis, but do not treat the photon in this fashion. For this reason, they will be called hybrid amplitudes. They are linear combinations of helicity amplitudes which greatly simplify the final results.

The density matrices (57), expressed in this hybrid basis, have the components $P_z$ and $P_{-z}$ occupying the locations of $P_y$ and $P_{-y}$ in the original density matrices. Formally, this is accomplished by a rotation by $-\pi/2$ about the $x$ axis, which carries $P_z \rightarrow P_{-y}$ and $P_{-z} \rightarrow -P_x$, leaving $P_x$ unchanged.

The new density matrices $\rho_\gamma^T$ and $\rho_\gamma^T$ simplify the analysis. To see how this works, introduce new $R$ matrices constructed from these density matrices

$$R_{ab}^{\alpha\beta} = 4\pi^2 \text{tr} \left[ \rho_\gamma^T \mathcal{P} \rho_\gamma^T \mathcal{P} \right] .$$

These now have simple symmetry properties, which follow from the $Y$ parity transformations (53) and (56),

$$R_{ab}^{\alpha\beta} = \alpha\beta(-1)^{a+b} R_{a-b}^{\alpha\beta} ,$$

and the original $R_{ab}$ defined in Eq. (34) are linear combinations of the $R_{ab}^{\alpha\beta}$

$$R_{ab} = (R_{ab}^{++} + R_{ab}^{--}) + (R_{ab}^{+-} + R_{ab}^{-+}) .$$

It is now easy to see that the observables given in Table III fall into two classes, depending on how $\rho_\gamma^T$ are paired with $\rho_\gamma^T$. Amplitudes in class I are those observables which occur as the following linear combinations:

$$R_{ab} + (-1)^a R_{-a-b} = 2(R_{ab}^{++} + R_{ab}^{--}) .$$

These are $R_L$, $R_T$, $R_{L'}$, $R_{T'}$, and $R_{L''}$. (This is the origin of the superscript I.) Those in class II are the following:

$$R_{ab} + (-1)^b R_{-a-b} = 2(R_{ab}^{+-} + R_{ab}^{-+})$$

and are $R_T$ and the three interference terms with the superscript (II). The amplitudes which are members of each class are listed in Table V, and the nonzero observables are identified in Table VI. If an observable is in class I, then all entries labeled II in Table VI are zero for that observable. In particular, since the cross section for unpolarized hadrons arises from a $\rho_\gamma^T \rho_\gamma^T$ pairing, only amplitudes in class I can contribute, and the familiar result that the cross section depends on only five structure functions is obtained.

Before we present explicit formulas for the observables, it is convenient to make further simplifications by introducing hybrid amplitudes.

E. Polarization observables in a hybrid basis

The density matrices given in Eqs. (57) and (58) are even or odd under the $Y$ parity transformation, but still have the complexity (number of nonzero elements) of the original density matrices. The reason for this is that these matrices are expressed in terms of the helicity basis, where the axis of quantization is along $z$, the direction of motion. However, since only the $y$ component of spin does not change sign under the $Y$ transformation (none of the components of spin change under parity, but the rotation by $\pi$ around the $y$ axis changes $x \rightarrow -x$ and $z \rightarrow -z$) it is more natural to choose the $y$ axis attached to the particle as the basis for quantization. Amplitudes quantized with respect to the $y$ axis are referred to as transversity amplitudes. The amplitudes introduced here quantize all the hadrons with respect to the $y$ axis, but do not treat the photon in this fashion. For this reason, they will be called hybrid amplitudes. They are linear combinations of helicity amplitudes which greatly simplify the final results.

The density matrices (57), expressed in this hybrid basis, have the components $P_z$ and $P_{-z}$ occupying the locations of $P_y$ and $P_{-y}$ in the original density matrices. Formally, this is accomplished by a rotation by $-\pi/2$ about the $x$ axis, which carries $P_z \rightarrow P_{-y}$ and $P_{-z} \rightarrow -P_x$, leaving $P_x$ unchanged.

The new density matrices, which will be distinguished from (57) and (58) by discarding the tilde, are

$$R_{ab} = (R_{ab}^{++} + R_{ab}^{--}) + (R_{ab}^{+-} + R_{ab}^{-+}) .$$

This is the origin of the superscript I.) Those in class II are the following:

$$R_{ab} + (-1)^b R_{-a-b} = 2(R_{ab}^{+-} + R_{ab}^{-+})$$

and are $R_T$ and the three interference terms with the superscript (II). The amplitudes which are members of each class are listed in Table V, and the nonzero observables are identified in Table VI. If an observable is in class I, then all entries labeled II in Table VI are zero for that observable. In particular, since the cross section for unpolarized hadrons arises from a $\rho_\gamma^T \rho_\gamma^T$ pairing, only amplitudes in class I can contribute, and the familiar result that the cross section depends on only five structure functions is obtained.

Before we present explicit formulas for the observables, it is convenient to make further simplifications by introducing hybrid amplitudes.

<table>
<thead>
<tr>
<th>Class I</th>
<th>Class II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_\gamma^T \rho_\gamma^T$ or $\rho_\gamma^T \rho_\gamma^T$</td>
<td>$\rho_\gamma^T \rho_\gamma^T$ or $\rho_\gamma^T \rho_\gamma^T$</td>
</tr>
<tr>
<td>$R_L$</td>
<td>$R_T$</td>
</tr>
<tr>
<td>$R_T$</td>
<td>$R''_T$</td>
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<td>$R''_T$</td>
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<td>$R''_T$</td>
<td>$R''_T$</td>
</tr>
</tbody>
</table>

TABLE V. The two classes of observables which occur in electrodisintegration. This separation is a consequence of $Y$ parity conservation.
TABLE VI. The nonzero observables in each class. If the entry I occurs, this observable is zero for observables of Class II, and conversely.

<table>
<thead>
<tr>
<th></th>
<th>$T_{10}$</th>
<th>$T_{20}$</th>
<th>$\text{Im}T_{11}$</th>
<th>$\text{Re}T_{11}$</th>
<th>$\text{Im}T_{21}$</th>
<th>$\text{Re}T_{21}$</th>
<th>$\text{Im}T_{22}$</th>
<th>$\text{Re}T_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>I</td>
<td>II</td>
<td>II</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>$P_x$</td>
<td>II</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>I</td>
<td>II</td>
</tr>
<tr>
<td>$P_y$</td>
<td>II</td>
<td>I</td>
<td>I</td>
<td>II</td>
<td>II</td>
<td>I</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>$P_z$</td>
<td>II</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>I</td>
<td>I</td>
</tr>
</tbody>
</table>

$$
\rho_\alpha^+ = \frac{1}{4} \begin{pmatrix} 
1 - P_y - P_y' \\
1 - P_y + P_y' \\
1 + P_y - P_y' \\
1 + P_y + P_y' 
\end{pmatrix}
$$

$$
\rho_\alpha^- = \frac{1}{4} \begin{pmatrix} 
P_x' - iP_x' & P_x - iP_x & 0 \\
P_x' + iP_x' & 0 & 0 \\
P_x + iP_x & 0 & 0 \\
0 & P_x' + iP_x' & 0 
\end{pmatrix}
$$

and

$$
\rho_\beta^+ = \frac{1}{3} \left[ \begin{array}{ccc}
1 + \frac{1}{\sqrt{2}} & T_{20} + \sqrt{2}T_{10} \\
0 & 1 - \sqrt{2}T_{20} \\
\sqrt{3}T_{22} & 0 & 1 + \frac{1}{\sqrt{2}} - \sqrt{3}T_{10} \\
0 & 0 & 0 \\
(T_{11} + T_{21}) & 0 & 0 \\
0 & (T_{11} - T_{21}) & 0
\end{array} \right]
$$

$$
\rho_\beta^- = \frac{-1}{3} \left[ \frac{3}{2} \right]^{1/2} \begin{pmatrix} 
0 & (T_{11} + T_{21}) & 0 \\
0 & (T_{11} - T_{21}) & 0 \\
0 & (T_{11} - T_{21}) & 0
\end{pmatrix}
$$

where the $T_{JM}$ are obtained from $\bar{T}_{JM}$ by the substitution $x \rightarrow y$ and $y \rightarrow z$ (see Table VII). The transformations which achieve these simplifications are

$$
R_\alpha^x(-\pi/2) = \frac{1}{2} \begin{pmatrix}
1 + i\sigma_x & i - \sigma_x \\
i - \sigma_x & 1 + i\sigma_x
\end{pmatrix}
$$

$$
R_\alpha^y(-\pi/2) = \frac{1}{2} \begin{pmatrix}
1 & i\sqrt{2} & -1 \\
i\sqrt{2} & 0 & i\sqrt{2} \\
-1 & i\sqrt{2} & 1
\end{pmatrix}
$$

The hybrid amplitudes are related to the helicity amplitudes by

$$
R_\alpha^x(-\pi/2) \bar{J}_x R_\alpha^y(-\pi/2)^t = \begin{pmatrix} 
S_8 & S_3 & S_{12} \\
S_1 & S_{10} & S_5 \\
S_2 & S_9 & S_6 \\
S_7 & S_4 & S_{11}
\end{pmatrix} = J_+
$$

and

$$
R_\alpha^x(-\pi/2) \bar{J}_x R_\alpha^y(-\pi/2)^t = \begin{pmatrix} 
0 & S_{15} & 0 \\
0 & 0 & S_{17} \\
S_{14} & 0 & S_{18} \\
0 & S_{16} & 0
\end{pmatrix} = J_-
$$

The specific linear combinations which define the $g$'s are

TABLE VII. The deuteron tensor polarization densities in the hybrid basis. The subscripts $x', y', z'$ refer to the $(x', y', z')$ coordinate system shown in Fig. 2.

The hybrid amplitudes are related to the helicity amplitudes by

$$
R_\alpha^x(-\pi/2) \bar{J}_x R_\alpha^y(-\pi/2)^t = \begin{pmatrix} 
S_8 & S_3 & S_{12} \\
S_1 & S_{10} & S_5 \\
S_2 & S_9 & S_6 \\
S_7 & S_4 & S_{11}
\end{pmatrix} = J_+
$$

and

$$
R_\alpha^x(-\pi/2) \bar{J}_x R_\alpha^y(-\pi/2)^t = \begin{pmatrix} 
0 & S_{15} & 0 \\
0 & 0 & S_{17} \\
S_{14} & 0 & S_{18} \\
0 & S_{16} & 0
\end{pmatrix} = J_-
$$

The specific linear combinations which define the $g$'s are
The $Y$ parity transformations take on a very simple form in this hybrid basis. They become

$$Y_4 = R_2^Y(-\pi/2)P_4 R_2^Y(-\pi/2)^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$Y_3 = R_2^Y(-\pi/2)P_3 R_2^Y(-\pi/2)^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.\quad (68)$$

The simplicity of the $J_0$ component of the current, Eq. (67) above, follows directly from the parity transformation, and the $J_-$ current is obtained from Eq. (53), which also takes on a simple form:

$$J_0 = \frac{1}{2}(J_+ - J_-) = \begin{pmatrix} 0 & g_3 & 0 & 0 \\ g_1 & 0 & g_5 & 0 \\ g_2 & 0 & g_6 & 0 \\ 0 & g_4 & 0 & 0 \end{pmatrix},$$

$$J_- = \frac{1}{2}(J_+ + J_-) = \begin{pmatrix} 0 & 0 & g_8 & 0 \\ 0 & 0 & 0 & g_9 \\ 0 & g_8 & 0 & 0 \\ 0 & g_9 & 0 & 0 \end{pmatrix}.\quad (70)$$

This relation suggests separation of the $J_\pm$ currents into pieces even or odd under the $Y$ transformation

$$J_\pm = \begin{pmatrix} 0 & g_3 & 0 & 0 \\ g_1 & 0 & g_5 & 0 \\ g_2 & 0 & g_6 & 0 \\ 0 & g_4 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & g_8 & 0 & 0 \\ 0 & 0 & 0 & g_9 \\ 0 & g_8 & 0 & 0 \\ 0 & g_9 & 0 & 0 \end{pmatrix}.$$

The structure functions of Table III, when expressed in terms of these amplitudes, take on a beautifully simple and symmetric form, given in Table VIII.

Examination of Table VIII shows that the nine structure functions divide into three groups of 2, each of which give the real and imaginary parts of sums over products of two different currents, and three functions related to squares of each of the currents. Table IX shows the patterns in a symbolic way.

It is also possible to reduce all the sums to only one generic sum, which we take to be $\Sigma J_i J_j$. The specific results for the two observables which depend on this sum are presented in Table X as real and imaginary parts of sums over the bilinear products $g_i^* g_{i+12}$, where $i$ runs from 1 to 6. The results for the moduli $|J_0|^2$ and $|J_1|^2$ can be obtained from this sum by substituting for $g_i^* g_{i+12}$ the combinations $|g_i|^2$ and $|g_i+12|^2$, as shown in Table XI. The observables involving $|J_0|^2$ can be obtained by observing that the $4 \times 4$ matrix, which can be written in $2 \times 2$ block form

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

maps $J_a$ into a symmetric form with $g_i \rightarrow g_{i+6}$

$$SJ_a = \begin{pmatrix} 0 & g_9 \\ g_7 & 0 \\ g_8 & 0 \\ 0 & g_{10} \end{pmatrix}.\quad (72)$$

However, $S$ changes the sign of $P_x$ and $P_z$ in the proton density matrices as follows:

$$S \rho^P S = \begin{pmatrix} 1+P_y & 0 \\ 0 & 1-P_y \end{pmatrix},$$

$$S \rho^P S = \begin{pmatrix} 0 & P_z + i P_x \\ P_z - i P_x & 0 \end{pmatrix}.\quad (73)$$

Hence $\Sigma |J_a|^2$ can be obtained from $\Sigma |J_i|^2$ by substituting for $|g_i|^2$ and $|g_{i+12}|^2$ and changing the sign of $P_x$ and $P_z$. This substitution and extra phase $(i)$ is indicated in Table XI. Finally, the four sums of type II involving a product of a symmetric and an antisymmetric current can be handled by using (72) and noting that a single $S$ operating on the proton polarization density matrix converts the forms of $\rho^P$ into each other:

$$\rho^P S = \begin{pmatrix} 0 & P_x + i P_z \\ P_z - i P_x & 0 \end{pmatrix}.$$

### Table VIII

<table>
<thead>
<tr>
<th>$J_a$</th>
<th>$J_0$</th>
<th>$J_1$</th>
<th>$J_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_0$</td>
<td>$R_{J_0}$</td>
<td>$R_{J_0}$</td>
<td>$R_{J_0}$</td>
</tr>
<tr>
<td>$J_1$</td>
<td>$R_{J_1}$</td>
<td>$R_{J_1}$</td>
<td>$R_{J_1}$</td>
</tr>
<tr>
<td>$J_2$</td>
<td>$R_{J_2}$</td>
<td>$R_{J_2}$</td>
<td>$R_{J_2}$</td>
</tr>
</tbody>
</table>

### Table IX

Symbolic representation of the content of the nine observables given in Table VIII. The $i$th diagonal element of the array is proportional to the square of the modulus of the $i$th current component $|J_i|^2$, and the $(i,j)$ elements are proportional to the products $J_i J_j$. If the element is to the upper right of the diagonal it is proportional to $\text{Re}(J_i J_j)$; elements to the lower left of the diagonal are proportional to $\text{Im}(J_i J_j)$.
$P_n \approx P_x + \frac{i}{2}P_x, \quad 0 \quad 0$

$S_P^+ = \begin{pmatrix} P_x + iP_x & 0 \\ 0 & P_x - iP_x \end{pmatrix}$

(74)

Hence the sums of type II have the same general structure as those of type I but with $(1, P_x, P_x, iP_x) \rightarrow (P_x, -iP_x, 1, -P_x)$. These observables are given in Table XII. The structure of the results are different because of the transformations (74), but can be expressed in terms of the same sets of amplitudes (the $a - f$ defined in Table X) with the substitutions indicated.

Finally, the neutron observables can be obtained from

**TABLE X.** The observables which depend on the sum $\sum_i j_s j_p$. The quantity $A(P_i, P_j)$ is given in the table; only proton polarizations are considered here, and $x''=l$, $y''=n, x''=s$. To construct the observable it is necessary to multiply the result by the factor shown at the top of each block; for example, $R_{T'}(P_i, T) = \frac{1}{2} \eta \epsilon^2 \text{Re}(a_i - b_i)$. In order to determine any given product, for example, $g^*_s g^*_{11}$, it is necessary to measure the set of all quantities which contain this product, the three $b_i$'s, for example. The quantities break naturally into different disjoint sets which are labelled with the same lower case letter $a, b, c, d, e, f, g$. The reader can easily find the patterns after a short study.

<table>
<thead>
<tr>
<th>$U$</th>
<th>$P_s$</th>
<th>$P_i$</th>
<th>$P_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{T_{10}}$</td>
<td>$\text{Re}(a_1 + b_1)$</td>
<td>$\text{Re}(a_1 - b_1)$</td>
<td>$\text{Re}(a_1 + b_1)$</td>
</tr>
<tr>
<td>$\sqrt{2}T_{10}$</td>
<td>$\text{Re}(a_2 + b_2)$</td>
<td>$\text{Re}(a_1 - b_1)$</td>
<td>$\text{Re}(a_2 - b_2)$</td>
</tr>
<tr>
<td>$\sqrt{2}T_{20}$</td>
<td>$\text{Re}(a_3 + b_3)$</td>
<td>$\text{Re}(a_1 - b_1)$</td>
<td>$\text{Re}(a_3 - b_3)$</td>
</tr>
<tr>
<td>$\sqrt{3}X_{22}$</td>
<td>$\text{Re}(c_1 + d_1)$</td>
<td>$\text{Re}(c_1 - d_1)$</td>
<td>$\text{Re}(c_1 + d_1)$</td>
</tr>
<tr>
<td>$\sqrt{3}X_{12}$</td>
<td>$-\text{Im}(c_2 + d_2)$</td>
<td>$-\text{Im}(c_2 - d_2)$</td>
<td>$-\text{Im}(c_2 + d_2)$</td>
</tr>
</tbody>
</table>

**TABLE XI.** The quantities $A(P_j, T)$ are given in the table; only neutron polarizations are considered here, and $x''=l$, $y''=n, x''=s$. To construct the observable it is necessary to multiply the result by the factor shown at the top of each block; for example, $R_{T''}(P_i, T') = \frac{1}{2} \eta \epsilon^2 \text{Im}(a_i - b_i)$. In order to determine any given product, for example, $g^*_s g^*_n$, it is necessary to measure the set of all quantities which contain this product, the three $b_j$'s, for example. The quantities break naturally into different disjoint sets which are labelled with the same lower case letter $a, b, c, d, e, f, g$. The reader can easily find the patterns after a short study.

<table>
<thead>
<tr>
<th>$U$</th>
<th>$P_s$</th>
<th>$P_i$</th>
<th>$P_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{T_{10}}$</td>
<td>$\text{Im}(a_1 + b_1)$</td>
<td>$\text{Im}(a_1 - b_1)$</td>
<td>$\text{Im}(a_1 + b_1)$</td>
</tr>
<tr>
<td>$\sqrt{2}T_{10}$</td>
<td>$\text{Im}(a_2 + b_2)$</td>
<td>$\text{Im}(a_1 - b_1)$</td>
<td>$\text{Im}(a_2 - b_2)$</td>
</tr>
<tr>
<td>$\sqrt{2}T_{20}$</td>
<td>$\text{Im}(a_3 + b_3)$</td>
<td>$\text{Im}(a_1 - b_1)$</td>
<td>$\text{Im}(a_3 - b_3)$</td>
</tr>
<tr>
<td>$\sqrt{3}X_{22}$</td>
<td>$\text{Im}(c_1 + d_1)$</td>
<td>$\text{Im}(c_1 - d_1)$</td>
<td>$\text{Im}(c_1 + d_1)$</td>
</tr>
<tr>
<td>$\sqrt{3}X_{12}$</td>
<td>$-\text{Re}(c_2 + d_2)$</td>
<td>$-\text{Re}(c_2 - d_2)$</td>
<td>$-\text{Re}(c_2 + d_2)$</td>
</tr>
</tbody>
</table>

\[a_1 = g^*_s g^*_{14} + g^*_s g^*_{15} + g^*_s g^*_{16} + g^*_s g^*_{17}\]
\[a_2 = g^*_s g^*_{18} - g^*_s g^*_{11}\]
\[a_3 = g^*_s g^*_{14} - 2g^*_s g^*_{16} + g^*_s g^*_{18}\]
\[b_1 = g^*_s g^*_{11} + g^*_s g^*_{15} + g^*_s g^*_{17}\]
\[b_2 = g^*_s g^*_{18} - g^*_s g^*_{11}\]
\[b_3 = g^*_s g^*_{14} - 2g^*_s g^*_{16} + g^*_s g^*_{18}\]
\[c_1 = g^*_s g^*_{18} + g^*_s g^*_{14}\]
\[c_2 = g^*_s g^*_{16} - g^*_s g^*_{18}\]
\[d_1 = g^*_s g^*_{17} + g^*_s g^*_{13}\]
\[d_2 = g^*_s g^*_{11} - g^*_s g^*_{15}\]

\[e_1 = (g^*_s + g^*_s) g^*_{16} + g^*_s (g^*_{18} + g^*_{14})\]
\[e_2 = (g^*_s - g^*_s) g^*_{16} + g^*_s (g^*_{18} - g^*_{14})\]
\[e_3 = (g^*_s - g^*_s) g^*_{18} - g^*_s (g^*_{18} - g^*_{14})\]
\[e_4 = (g^*_s + g^*_s) g^*_{16} - g^*_s (g^*_{18} + g^*_{14})\]

\[f_1 = (g^*_s + g^*_s) g^*_{15} + g^*_s (g^*_{17} + g^*_{13})\]
\[f_2 = (g^*_s - g^*_s) g^*_{15} + g^*_s (g^*_{17} - g^*_{13})\]
\[f_3 = (g^*_s - g^*_s) g^*_{17} - g^*_s (g^*_{17} - g^*_{13})\]
\[f_4 = (g^*_s + g^*_s) g^*_{15} - g^*_s (g^*_{17} + g^*_{13})\]
TABLE XI. The observables which depend on $|g_i|^2$, $|g_i|^2$, and $|g_i|^2$. Again the quantity $A(P, T)$ is given in the table. Sets of quantities which depend on moduli $|g_i|^2$ are denoted by capitalized italic letters. Otherwise, the notation is the same as in Table X. For the observables $R_{L}^t+R_{L}'^{||}$, the relations are $R_L(P_1, T)=\frac{4}{3} |\alpha|^2 A(P, T)$ with all $g_i \to g_{i+4}$ and the phase $\epsilon=1$. For $R_L$, the relations are $R_L(P_1, T)=\frac{4}{3} |\alpha|^2 A(P, T)$ with all $g_i \to g_{i+4}$ and $\epsilon=1$.

<table>
<thead>
<tr>
<th>$U$</th>
<th>$P_1$</th>
<th>$P_1$</th>
<th>$P_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{L}^{-e}$</td>
<td>$R_{L}^{-e}(P_1, T)=\frac{4}{3}</td>
<td>\alpha</td>
<td>^2 A(P, T)$, $\epsilon=1$</td>
</tr>
<tr>
<td>$\sqrt{1/2} T_{10}$</td>
<td>$A_1+B_1$</td>
<td>$A_1+B_1$</td>
<td>$A_1+B_1$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{2}} T_{20}$</td>
<td>$A_1+B_1$</td>
<td>$A_1+B_1$</td>
<td>$A_1+B_1$</td>
</tr>
<tr>
<td>$\sqrt{3} \text{Re} T_{11}$</td>
<td>$\text{Re}(c+d)$</td>
<td>$\text{Re}(c+d)$</td>
<td>$\text{Re}(c+d)$</td>
</tr>
<tr>
<td>$\sqrt{3} \text{Im} T_{11}$</td>
<td>$-\text{Im}(c+d)$</td>
<td>$-\text{Im}(c+d)$</td>
<td>$-\text{Im}(c+d)$</td>
</tr>
</tbody>
</table>

TABLE XII. The observables of type II. The notation is the same as in Table X. For $R_L$ and $R_L'^{||}$ use the $a_s$'s $f_s$ of Table X with $g_i \to g_{i+4}$. For $R_L'^{||}$, use the $a_s$'s $f_s$ of Table X with unchanged $g_i \to g_{i+4}$.

<table>
<thead>
<tr>
<th>$U$</th>
<th>$P_1$</th>
<th>$P_1$</th>
<th>$P_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_L'^{</td>
<td></td>
<td>}$</td>
<td>$R_L'^{</td>
</tr>
<tr>
<td>$\sqrt{1/2} T_{10}$</td>
<td>$\text{Re}(a_1+b_1)$</td>
<td>$\text{Im}(a_1-b_1)$</td>
<td>$\text{Re}(a_1+b_1)$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{2}} T_{20}$</td>
<td>$\text{Re}(a_1+b_1)$</td>
<td>$\text{Im}(a_1-b_1)$</td>
<td>$\text{Re}(a_1+b_1)$</td>
</tr>
<tr>
<td>$\sqrt{3} \text{Re} T_{11}$</td>
<td>$-\text{Re}(e_1+f_1)$</td>
<td>$-\text{Re}(e_1-f_1)$</td>
<td>$-\text{Re}(e_1-f_1)$</td>
</tr>
<tr>
<td>$\sqrt{3} \text{Im} T_{11}$</td>
<td>$\text{Im}(e_1+f_1)$</td>
<td>$\text{Im}(e_1-f_1)$</td>
<td>$\text{Im}(e_1-f_1)$</td>
</tr>
</tbody>
</table>

$A_1=|g_1|^2+|g_3|^2+|g_5|^2$, $B_1=|g_2|^2+|g_3|^2+|g_5|^2$, $e_1=2g_2 g_4+2g_3 g_6$, $f_1=2g_2 g_4-2g_3 g_6$.
the proton observables by noting that the matrix $B$

\[
B = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

transforms neutron density matrices into proton-like density matrices

\[
B \begin{pmatrix} 1 - \sigma_x P'_y \\ 1 - \sigma_y P'_x \end{pmatrix} = \begin{pmatrix} 1 - P'_y \\ 1 + P'_y \end{pmatrix},
\]

while its only effect on the symmetric current is to interchange $g_1 \leftrightarrow g_2$ and $g_9 \leftrightarrow g_{10}$:

\[
B \begin{pmatrix} \sigma_x P'_x + \sigma_y P'_y \\ 0 \\ \sigma_x P'_x + \sigma_y P'_y \end{pmatrix} = \begin{pmatrix} P'_x - iP'_y \\ 0 \\ P'_x + iP'_y \end{pmatrix},
\]

and on the antisymmetric current it interchanges $g_8 \leftrightarrow g_{10}$:

\[
B = \begin{pmatrix}
0 & g_3 & 0 & 0 \\
g_3 & 0 & g_5 & 0 \\
g_2 & 0 & g_6 & 0 \\
g_4 & 0 & 0 & g_4
\end{pmatrix}
\]

and the nonzero terms in each sum for hybrid amplitudes are given in Tables X-XII. In identifying specific terms, we will suppress the $U$ labels, and adopt other simplifying notation as illustrated below:

\[
R_L^U(U,U) = R_L^T(U,U),
\]

\[
R_L^U(P_n,U) = R_L^T(P_n,U),
\]

\[
R_L^U(U,\sqrt{3} \text{Im} T_{22}) = R_L^T(\text{Im} T_{22}),
\]

\[
R_L^U(P_n,\sqrt{3} \text{Im} T_{22}) = R_L^T(n,\text{Im} T_{22}).
\]

Note that, because classes I and II are disjoint, the subscripts on $R_L^{(1)(1)}$ have been suppressed in the expansions (79); the arguments of $R_L^U(P_j,T_i)$ uniquely identify to which expansion it belongs. This notation forces a pairing of $R_L^{(1)}$ and $R_L^{(1)}$, but as shown in Tables VII, IX, and X-XII, the natural pairing is between $R_L^{(1)}$ and $R_L^{(1)}$, for example. For the special case of the amplitudes $R_L^{(1)}$, we will sometimes use the notation $R_L^T$ for the expansion coefficients in Eq. (79).

Hence, all neutron observables have a structure identical to the corresponding proton observable except that for $J$, we must interchange $g_1 \leftrightarrow g_2$ and $g_9 \leftrightarrow g_{10}$, for $J_y$ we interchange $g_8 \leftrightarrow g_{10}$, and for $J_0$ we interchange $g_{11} \leftrightarrow g_{14}$ and $g_{17} \leftrightarrow g_{18}$.

The calculation of the 18 observables which go with each of the nine structure functions can now be done by hand. The results have been presented in Tables X-XII. [These results were also confirmed using the symbolic manipulation program (SMP).] The patterns outlined above can be readily seen in the final results. These tables, together with Eq. (30) for the cross section, are the principal results of this paper.

It may sometimes be necessary to identify a particular observable in the tables. The structure functions will be labeled and identified by the following expansion, given for the pair $R_L^{(1)}$ and $R_L^{(1)}$ as an example:

\[
R_L^{(1)} = \sum_{i,j} T_i P_j R_L^T(P_j,T_i),
\]

\[
R_L^{(1)} = \sum_{i,j} T_i P_j R_L^T(P_j,T_i),
\]

where

\[
T_i = \left\{ U, \sqrt{3} T_{10}, \frac{1}{\sqrt{2}} T_{20}, \sqrt{3} \text{Im} T_{22}, \sqrt{3} \text{Re} T_{22}, \sqrt{3} \text{Re} T_{11}, \sqrt{3} ^2 \text{Im} T_{11}, \sqrt{3} ^2 \text{Re} T_{21}, \sqrt{3} ^2 \text{Im} T_{21} \right\},
\]

\[
P_j = \{ U, P_n, P_T, P_J \},
\]

and the differential cross section for photodisintegration can be obtained quickly from our previous work. The most general polarization state of a real photon can be written as a linear combination of its two helicity states,

\[
|a|^2 + |b|^2 = 1.
\]

Then the differential cross section in the c.m. system is

\[
d\sigma = -\frac{e^2}{4\nu_0 W} p(t_\gamma W_{\gamma K}),
\]

where $|a|^2 + |b|^2 = 1$. Then the differential cross section in the c.m. system is

\[
d\sigma = -\frac{e^2}{4\nu_0 W} p(t_\gamma W_{\gamma K}),
\]

where the hadron density was defined in Eq. (11), $\nu_0$ is the c.m. energy of the photon (note that $\nu_0 = q_0$), and the photon density matrix is

\[
p(t_\gamma W_{\gamma K}) = \frac{|a|^2}{b a^*} \cdot
\]

Using the hermiticity of $W_{\gamma K}$, and reducing in the c.m. as done in Eq. (26), gives immediately
\[
\begin{align*}
\frac{d\sigma}{d\Omega_t} &= e^2 \left[ \frac{p_1}{8\pi W} \right] [R_T + (|d|^2 - |b|^2)R_T'] \\
&\quad + 2 \text{Re}(ab^*)R_{T}^{(i)} \\
&\quad + 2 \text{Im}(ab^*)R_{T}^{(i)}], 
\end{align*}
\]

where the \(R's\) are the same structure functions discussed in the preceding sections, except that they are evaluated at the real photon point \(q^2 = 0\).

**G. Inclusive Cross Sections**

Inclusive electrodisintegration cross sections can be readily obtained by integrating over the solid angle \(d\Omega_t\), and summing over all polarizations of the outgoing nucleons. We will present formulas with and without deuteron polarization.

Special care must be taken with the deuteron polarization. Recall that the results given in Tables X-XII were for deuteron polarizations defined in the *ejectile* plane, rotated through the angle \(\phi\) with respect to the *electron* scattering plane. When integrating over \(\phi\) to obtain inclusive cross sections, care must be taken to express the deuteron polarizations with respect to the electron scattering plane, which is fixed. To avoid any confusion, the polarizations defined with respect to the electron scattering plane will be denoted by \(T_{ij}\) to distinguish them from \(T_{ij}\), the polarizations defined in the ejectile plane.

To express the polarizations \(T_{ij}^{(0)}\) in the ejectile plane, we must rotate the coordinate axes through angle \(\phi\) about the \(z\) axis, which is equivalent to an active rotation of the polarization quantities \(T_{ij}^{(0)}\) through angle \(-\phi\). This is the same transformation we carried out on the virtual photon polarization vector in Sec. II B. However, we have chosen to define the deuteron \(T_{ij}\) with respect to the \(y\) axis, and this makes the transformation laws more complex than that obtained in Eq. (22) for the photon. The correct transformation laws are

\[
\text{Re} T_{11} = \cos \phi \text{Re} T_{11}^{(0)} - \sin \phi \left[ \frac{T_{10}^{(0)}}{\sqrt{2}} \right],
\]

\[
T_{10} = \sin \phi \text{Re} T_{11}^{(0)} + \cos \phi \left[ \frac{T_{10}^{(0)}}{\sqrt{2}} \right],
\]

\[
\text{Im} T_{11} = \text{Im} T_{11}^{(0)},
\]

\[
\text{Re} T_{22} + \frac{1}{\sqrt{6}} T_{20} = \text{Re} T_{22}^{(0)} + \frac{1}{\sqrt{6}} T_{20}^{(0)},
\]

\[
\text{Re} T_{22} - \sqrt{\frac{2}{3}} T_{30} = \cos 2\phi (\text{Re} T_{22}^{(0)} - \sqrt{\frac{2}{3}} T_{30}^{(0)}) - \sin 2\phi (2 \text{Re} T_{21}^{(0)}),
\]

\[
2 \text{Re} T_{21} = \sin 2\phi (\text{Re} T_{21}^{(0)} - \sqrt{\frac{2}{3}} T_{30}^{(0)}) + \cos 2\phi (2 \text{Re} T_{21}^{(0)}),
\]

\[
\text{Im} T_{22} = \cos \phi \text{Im} T_{22}^{(0)} - \sin \phi \text{Im} T_{22}^{(0)},
\]

\[
\text{Im} T_{21} = \sin \phi \text{Im} T_{22}^{(0)} + \cos \phi \text{Im} T_{22}^{(0)}.
\]

These give the deuteron polarization parameters in the *ejectile* plane (those used in Tables X-XII) in terms of polarization parameters in the *electron* scattering plane. These transformations *must* be used when calculating inclusive cross sections, where the deuteron target polarization is *fixed* as we integrate over \(\phi\). For exclusive measurements the deuteron polarization could be oriented with respect to the electron scattering plane, in which case the results in Tables X-XII could be used directly. However, if a number of \(\phi\) angles are measured simultaneously with a fixed deuteron polarization, as would be the case with the STAR spectrometer proposed for use at CEBAF by the Illinois group, it would also be necessary to use the transformations (86) in order to predict the correct \(\phi\) dependence of the cross sections.

Using the transformations (86) it is straightforward to carry out the \(\phi\) integrations analytically. The integrations over \(\theta_1\) cannot be carried out unless the structure functions are known; we will denote these integrations by

\[
\bar{R}_M(T_B) = \frac{1}{2} \int_0^\pi \sin \theta_1 d\theta_1 R_M(T_B).
\]

Using this notation, the most general inclusive cross section can be obtained from Eq. (30). It becomes

\[
\frac{d^3\sigma}{d\Omega dE'} = \sigma_M \left[ \begin{array}{c} W \\ M_T \end{array} \right] \left[ \begin{array}{c} u_L [W_L + P_{ax}(0)W_L(P_{ax})] + v_T [W_T + P_{ax}(0)W_T(P_{ax})] + v_{TT}(P_{ax} - P_{ax})W_{TT} \\ W \end{array} \right]
\]

\[
+ \left[ \begin{array}{c} W \\ M_T \end{array} \right] \left[ \begin{array}{c} u_T [P_T W_{LT}(P_T)] + P_{ax}(0)W_{LT}(P_{ax}] + 2h v_T P_{ax} W_T \\ W \end{array} \right]
\]

\[
+ 2h \left[ \begin{array}{c} W \\ M_T \end{array} \right] \left[ \begin{array}{c} u_L [P_{ax}(0)W_{LT}(P_{ax}) + P_{ax}(0)W_{LT}(P_{ax})] \\ W \end{array} \right],
\]
where the deuteron polarizations $T_{ij}^{(2)}$ have been expressed in Cartesian form to facilitate interpretation of the results, and the inclusive structure functions $W$ are given in terms of the $R$'s of Eq. (87) by

$$W_L = \lambda \bar{R}_L,$$
$$W_T = \lambda \bar{R}_T,$$

$$W_L(P_a) = -\frac{1}{4} \left[ \bar{R}_{LT}(T_{20}) - \frac{1}{\sqrt{2}} \bar{R}_{LT}(T_{11}) \right],$$

$$W_T(P_a) = \frac{3}{4} \left[ \bar{R}_{LT}(T_{11}) + \frac{1}{\sqrt{2}} \bar{R}_{LT}(T_{12}) \right],$$

$$W_L(T_0) = \frac{3}{4} \left[ \bar{R}_{LT}(T_{10}) - \frac{1}{\sqrt{2}} \bar{R}_{LT}(T_{11}) \right],$$

$$W_T(T_0) = \frac{3}{2} \left[ \bar{R}_{LT}(T_{11}) + \frac{1}{\sqrt{2}} \bar{R}_{LT}(T_{12}) \right],$$

where $\lambda = \frac{p_1}{M_T}$. It has been known for some time that time-reversal invariance implies that $W_L(P_a) = 0$.

If the deuteron target is unpolarized, the cross section reduces to the familiar form

$$W_{\sigma L} = -\frac{1}{4} W_{\sigma T},$$

(91)

$$W_{\sigma T} = \frac{1}{3} \left[ \bar{R}_{LT}(T_{20}) - 3 \bar{R}_{LT}(T_{21}) \right].$$

(92)

H. The cross sections and observables in lab frame

In the preceding sections, results for the hadronic currents were presented in the c.m. frame because of the convenience in obtaining inclusive cross sections. However, all observables are measured in the lab frame, so it is necessary to completely clarify the relationship between these two frames.

A review of the derivations given in the preceding sections shows the following differences between the results presented so far (in which the electron variables are in the lab frame and the hadron variables are in the c.m. frame) and those in which all variables are in the lab frame.

(i) There is no boost operator Eq. (20). The primary effect of this is to remove the $W/\bar{R}$ factors from the $J^{0}$ components of the current, defined in Eq. (29).

(ii) The integration in Eq. (11) must be carried out in the lab frame. This introduces the recoil factor discussed below.

(iii) The polarization quantities must be transformed to the lab frame.

The effect of items (i) and (ii) is to introduce a recoil factor and a modified form for the cross section. The integration over the hadronic variables gives, in place of Eq. (26),

$$W_{\sigma L} = d \Omega_L d p_L r \eta_{\lambda \sigma},$$

(92)

where

$$r = \frac{W}{M_T} \left[ \frac{1}{1 + \frac{\nu_{\perp}}{M_T^2} \left( E_{20} - E_{10} \cos \theta_1 \right)} \right].$$

(93)

These changes give the following result for the cross section, Eq. (30), in the lab:

$$\frac{d^2 \sigma}{d \Omega dE} = \sigma_{\sigma L} (W_{\sigma L} + 2 W_{\sigma T}, \tan^2 \theta/2),$$

(90)

$$\frac{d^2 \sigma}{d \Omega dE} = \frac{\sigma_{\sigma L}}{4 \pi M_T} \left[ v_{1R} R_L + v_{1T} R_T + v_{TT} \left[ \cos 2 \phi R^{(1)}_{LT} + \sin 2 \phi R^{(1)}_{LT} \right] + v_{LT} \left[ \cos \phi R^{(1)}_{LT} + \sin \phi R^{(1)}_{LT} \right] \right]$$

$$+ 2h v_{1R} R_T + 2h v_{1T} \left[ \cos \phi R^{(1)}_{LT} + \sin \phi R^{(1)}_{LT} \right] \right],$$

(94)

where the structure functions $R$ are to be evaluated in the lab frame with $\eta = q_L / Q$. Alternatively, if we work only with the covariant form of the longitudinal current, $J - \epsilon_0$, the cross section can be written in the following alternative form:

$$\frac{d^2 \sigma}{d \Omega dE} = \frac{\sigma_{\sigma L}}{4 \pi M_T} \frac{Q^2}{q_L} \left[ R_L + s_T R_T - \frac{1}{2} \left[ \cos 2 \phi R^{(1)}_{LT} + \sin 2 \phi R^{(1)}_{LT} \right] + s_{LT} \left[ \cos \phi R^{(1)}_{LT} + \sin \phi R^{(1)}_{LT} \right] \right]$$

$$+ 2h s_T R_T + 2h s_{LT} \left[ \cos \phi R^{(1)}_{LT} + \sin \phi R^{(1)}_{LT} \right],$$

(95)
where the $\hat{R}$'s in Eq. (95) are now covariant, and are obtained from the covariant $R_{LT}$ of Eq. (27) by setting $\eta = 1$ in the middle column of Table III, and the $s_i$ are

\begin{align*}
  s_T &= \frac{1}{2} + s^2, \\
  s_{LT} &= -\frac{1}{\sqrt{2}}(1 + s^2)^{1/2}, \\
  s_T' &= \frac{s(1 + s^2)}{\sqrt{2}}, \\
  s_{LT}' &= -\frac{1}{\sqrt{2}} s^2,
\end{align*}

where $s = (q_L/Q)\tan\frac{1}{2}\theta$. The quantity $d\Sigma$ is

\begin{align*}
  d\Sigma|_{\text{c.m.}} &= p_1 d\Omega_1, \\
  d\Sigma|_{\text{lab.}} &= p_2 d\Omega_{LT}.
\end{align*}

Hence the advantage of Eq. (95) is that it uses covariant structure functions $\hat{R}$, and combines the c.m. and lab results into one formula—the only difference being the choice of $s_i$ given in Eq. (97).

Finally, we address the question of how the hadronic spin variables are affected by the choice of frame for the hadronic current, item (iii) above. First, note that the boost (20) will not affect the deuteron helicity, since it is colinear with the deuteron momentum. However, since the momenta of the final-state nucleons are in general not colinear with the boost, the boost will introduce a precession (a Wigner rotation) of the nucleon spins through an angle $\theta_0$ about the axis $\hat{q} \times \hat{p}$, which is perpendicular to the plane defined by the direction of the boost (q) and the three momentum of the particle (p). This plane is just the ejectile plane defined in Fig. 2, and hence the boost will rotate the nucleon spins in this plane and therefore mix the $l$ and $s$ components of nucleon polarization. The amplitudes in the third and fourth columns of Tables X—XII will mix. The sizes of these terms will be frame dependent, but the structure of the results given in Tables X—XII will also hold for observables in the lab system. Hence the general conclusions of this paper (including the discussion of Sec. III) do not depend on which frame is used for the hadronic variables; only dynamical calculations of these quantities depend on the frame. The details of the Wigner rotation will be presented elsewhere.

We now turn to a discussion of possible programs of complete measurements.

### III. COMPLETE SEPARATIONS

#### A. Introduction

Complete determination of photonic processes has been attempted in several different ways. We have adopted the method of Barker et al. (for criticism of other approaches see this reference). We looked for a set of new amplitudes which made the separation obvious. The hybrid amplitudes presented in the preceding section accomplished this, just as in pion photoproduction. As we saw in Tables X—XII the structure functions became particularly simple when expressed in terms of the $g_i$. They become real and imaginary parts of bilinear combinations of $g_i$'s. This form makes it comparatively easy to study what measurements are necessary in order to completely and uniquely determine all the complex amplitudes, as will be shown below.

Consider a simple example of the electrodisintegration of a spinless nucleus into two spinless fragments $[^{12}\text{He}(e,e')^{4}\text{He}$, for example]. There are only two independent helicity amplitudes, $J_0 = |J_0|e^{i\phi_0}$ and $J_+ = |J_+|e^{i\phi_+}$, and only five nonvanishing structure functions (only four of which are different):

\begin{align*}
  R_L &= \eta^2|J_0|^2, \\
  R_T &= 2\eta^2|J_+|^2, \\
  R_{LT}^{(1)} &= 2\eta^2\text{Re}(J_0 J_0^{\ast}), \\
  R_{LT}^{(2)} &= 4\eta^2\text{Re}(J_0 J_+^{\ast}), \\
  R_{LT}^{(3)} &= 4\eta^2\text{Im}(J_0 J_+^{\ast}),
\end{align*}

but there are only three independent observables: $|J_0|^2$, $|J_+|^2$, $\phi_+ = (\phi_0 - \phi_+)$. The overall (absolute) phase is not an observable.

If we measure $R_L$, $R_T$, and one of the $R_{LT}$, we still cannot determine $\phi_+$ unambiguously. For example, a measurement of $R_{LT}$ will determine $\cos \phi_+$, which gives $\phi_+$ in the first or fourth quadrant if it is positive, and in the second or third quadrant if it is negative. A similar ambiguity results if only $R_{LT}$, which determines $\sin \phi_+$, is measured. To completely remove the ambiguity both of the "paired" amplitudes ($R_{LT}$ and $R_{LT}$) must be measured. But this measurement will also give the product $|J_0||J_+|$, requiring the measurement of only one other modulus to completely and unambiguously determine the three independent observables. This last measurement can be either $R_L$ or $R_T = -R_{LT}$. The program requiring the fewest number of measurements is therefore one in which "paired" amplitudes, together with a few "moduli," are measured.

Unfortunately, measurements of "paired amplitudes" will generally be more difficult, as they usually involve polarization observables. The best experimental program may well be to measure "moduli" if they are easy, and "paired" amplitudes as needed to eliminate ambiguities.

#### B. Overview of separation strategies

We begin the discussion of separation strategies by seeing what can be learned from the "paired" observables $(R_{LT}^{(1)}, R_{LT}^{(2)})$, $(R_{LT}^{(2)}, R_{LT}^{(3)})$, and $(R_T, R_{LT}^{(3)})$ detailed in Tables X and XII. If all the observables in Table X were measured, the complex amplitudes $a_j$, $b_j$, $c_j$, $d_j$, $e_j$, and $f_j$ would all be determined uniquely, and hence the products of the $g_i$'s which they contain. There are 18 different products of $g_i$'s which are so determined, and these fall into two separate classes, as shown in the two left-hand columns of Table XIII. No product of an amplitude from class A times an amplitude from class B is present (and of course there are no products of the first six amplitudes with each other, nor the last six with each other). Hence, there are terms like $g_7^* g_{13}$ and $g_7^* g_{16}$, but no term like $g_7^* g_{14}$. From $g_7^* g_{13}$ and $g_7^* g_{16}$, one can determine...


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Determined, which could be taken to be $g_{1 L}$ and $g_{2 L}$ (if the relative size of $|g_{1 L}|$ and $|g_{2 L}|$ can be deduced, and the classes of amplitudes are displayed in the four rightmost columns of Table XII. The classes of amplitudes are displayed in the four rightmost columns of Table XII. The nine structure functions can depend on the phase difference between all amplitudes in each class can be determined, but the overall phase difference between the two classes remains undetermined. Also, two moduli cannot be determined. Hence, of the 23 possible observables associated with the amplitudes which contribute to Table X, only 20 can be determined by the 36 possible measurements. Three cannot be determined.

The same analysis works for the other paired amplitudes $(R_{1 L}^{(2)}, R_{1 L}^{(3)})$ and $(R_{T L}^{(2)}, R_{T L}^{(3)})$ of Table XII. The classes of amplitudes are displayed in the four rightmost columns of Table XIII. Note that the combination of all of these measurements still leaves the same three observables undetermined. The two classes of amplitudes are enlarged, but there is no mixing which would determine the missing moduli or phase. The expanded classes are

$$A = \{g_{1}g_{8}g_{9}g_{10}g_{11}g_{12}g_{13}g_{14}g_{15}\}$$

$$B = \{g_{2}g_{8}g_{9}g_{10}g_{11}g_{12}g_{13}g_{14}g_{15}\}$$

Furthermore, the additional measurements are less efficient in adding new information. If all of the products of amplitudes from a given "paired" set were known, for example, the measurements of the next "paired" set would add only 12 new observables, and measurements of the last pair would add no new information. This shows that care must be taken to plan experimental programs optimally.

The missing moduli, one from class $A$ and one from class $B$, can be determined by two measurements from Table XI. A convenient choice would be two of the unpolarized observables $R_{L}$ and $R_{T}$. However, no measurement involving any combination of polarized electron beam, polarized deuteron target, or polarized recoil proton can determine the last phase. For this, at least one measurement of neutron recoil polarization is needed. Recall that the neutron observables are identical to the proton observables except for the interchange of $g_{1} \leftrightarrow g_{2}$, $R_{1 L} \leftrightarrow R_{2 L}$, $R_{1 T} \leftrightarrow R_{2 T}$, and $g_{17} \leftrightarrow g_{18}$. This mixes up the classes in Eq. (99), making it possible to determine the last relative phase. However, only measurements which depend on this phase can be used for this purpose. It turns out that only the observables which depend on $R_{1 L}$ and $R_{2 L}$ are sensitive to this interchange. Our discussion has shown that at least one neutron recoil polarization measurement is essential to a complete determination of the 18 deuteron electrodisintegration hybrid amplitudes, and that this must involve measuring $P_{1 L}$ or $P_{2 L}$. If this measurement is to be made with an unpolarized deuteron target, then the quantities must be chosen from $R_{L}(s')$, $R_{L}(l')$, $R_{T}(s')$, $R_{T}(l')$, $R_{LT}(s')$, $R_{LT}(l')$, or $R_{TT}(l')$, where the prime on the $s$ or $l$ argument signifies a neutron polarization measurement.

We have focused on proton polarization measurements in this section, assuming that such measurements would be easier to carry out experimentally. One can readily see, however, that there is a correspondence between proton and neutron polarization measurements, and the best strategy could well be to use a number of both.

At this point a number of questions arise concerning the "best" methods for separating the 18 amplitudes. We believe that we have provided sufficient detail for the reader to work out his own preferred strategy, and will not pursue discussion of the subject here.

IV. CONCLUSIONS

This paper analyzes the differential cross section for the process $d(e,e'p)n$ in which the scattered electron and the recoil proton are measured in coincidence. The cross section can be expressed in terms of nine (unknown) hadronic structure functions defined in Table III. If the hadronic structure functions are expressed in terms of variables defined in the center of mass (c.m.) of the outgoing hadrons, the relevant formula is Eq. (30). If lab variables are used, the expression given in Eq. (94) is appropriate. Equation (95) gives a formula convenient in either frame. Each of the nine structure functions can depend on the polarization of the deuteron target, and/or the polarization of the recoil proton. If all possible combinations of polarizations are considered, each of the nine structure functions depend on 18 independent observables, for a total of $9 \times 18 = 162$ observables. The dependence of these observables on the 18 independent complex helicity amplitudes.
amplitudes which completely describe deuteron electrodeint
zation is given in Tables X-XII. The 18 hybrid amplitudes $g_i$ are linear combinations of the 18 helicity amplitudes, given explicitly in the Appendix, or in general terms in Eqs. (66) and (67).

Even though there are 162 observables in Tables X-XII, it is shown in Sec. III that these are not sufficient to determine all of the 35 real functions implied by the existence of the 18 complex helicity amplitudes. One phase, relating the two classes of amplitudes given in Eq. (99), can only be obtained by measuring the polarization of the outgoing neutron. Dynamical calculations of these observables are planned for future work.

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APPENDIX

The connection, Eqs. (66) and (67), between the hybrid amplitudes $g_i$ defined in Eq. (51) can be summarized by the matrix relations

$$g_i = \Lambda_{ij} F_j,$$

where, for transverse amplitudes $(i,j=1-12)$,

$$\Lambda_{ij} = \frac{1}{4} \begin{pmatrix}
  1 & i & \sqrt{2} & i & i & -\sqrt{2} & i & 1 & 1 & -1 & -i & 1 \\
  1 & i & \sqrt{2} & i & i & 1 & -i & 0 & 0 & 0 & 0 & 0 \\
 -i\sqrt{2} & i\sqrt{2} & 0 & 0 & -i\sqrt{2} & i\sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & \sqrt{2} & \sqrt{2} \\
 i\sqrt{2} & -i\sqrt{2} & 0 & 0 & i\sqrt{2} & -i\sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & \sqrt{2} & \sqrt{2} \\
 -i & -i & \sqrt{2} & \sqrt{2} & i & i & -1 & -i & 1 & 1 & -\sqrt{2} & i \\
 -i & -i & \sqrt{2} & \sqrt{2} & i & i & 1 & 1 & i & \sqrt{2} & \sqrt{2} & -i \\
 \sqrt{2} & \sqrt{2} & 0 & 0 & \sqrt{2} & \sqrt{2} & i\sqrt{2} & -i\sqrt{2} & 0 & 0 & \sqrt{2} & -i\sqrt{2} \\
 \sqrt{2} & \sqrt{2} & 0 & 0 & \sqrt{2} & \sqrt{2} & -i\sqrt{2} & i\sqrt{2} & 0 & 0 & -i\sqrt{2} & i\sqrt{2} \\
 -1 & 1 & i\sqrt{2} & -i\sqrt{2} & -1 & 1 & -i & -i & \sqrt{2} & \sqrt{2} & -i & i \\
 -1 & 1 & -\sqrt{2} & i\sqrt{2} & -1 & 1 & -i & -i & \sqrt{2} & \sqrt{2} & -i & i
\end{pmatrix}$$

while, for longitudinal amplitudes $(i,j=13-18)$,

$$\Lambda_{ij} = \frac{1}{2} \begin{pmatrix}
  i & \sqrt{2} & i & 1 \\
  i & \sqrt{2} & i & -1 \\
 -i\sqrt{2} & 0 & -i\sqrt{2} & \sqrt{2} & 0 & \sqrt{2} \\
 i\sqrt{2} & 0 & i\sqrt{2} & \sqrt{2} & 0 & -\sqrt{2} \\
 -i & \sqrt{2} & i & 1 & i\sqrt{2} & -1 \\
 -i & \sqrt{2} & i & -1 & -i\sqrt{2} & 1
\end{pmatrix}$$


11 Although the phase $\eta_\nu$ is not explicitly stated in Ref. 6, it can be unambiguously determined from their result for the proton recoil polarization observables. The change of sign of $\eta_\nu$ induces the following change in the observables: $R_{11}^{(2)} \rightarrow -R_{11}^{(2)}, R_{11}^{(3)} \rightarrow -R_{11}^{(3)}$. An explicit calculation of these observables shows that Renard et al. used $\eta_\nu = -1$. 


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