1991

Completely bootstrapped tokamak

Richard Henry Weening

College of William & Mary - Arts & Sciences

Follow this and additional works at: https://scholarworks.wm.edu/etd

Part of the Plasma and Beam Physics Commons

Recommended Citation


This Dissertation is brought to you for free and open access by the Theses, Dissertations, & Master Projects at W&M ScholarWorks. It has been accepted for inclusion in Dissertations, Theses, and Masters Projects by an authorized administrator of W&M ScholarWorks. For more information, please contact scholarworks@wm.edu.
INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.
Completely bootstrapped tokamak

Weening, Richard Henry, Ph.D.
The College of William and Mary, 1991
COMPLETELY BOOTSTRAPPED TOKAMAK

A Dissertation
Presented to
The Faculty of the Department of Physics
The College of William and Mary in Virginia

In Partial Fulfillment
Of the Requirements for the Degree of
Doctor of Philosophy

by
Richard H. Weening
1991
This dissertation is submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

Richard H. Weening

Approved, November 1991

Allen H. Boozer
Roy L. Champion
Alkesh Punjabi
Department of Mathematics
Hampton University

George T. Rublein
Department of Mathematics

Eugene R. Trapp

George M. Vahala
It is not your obligation to complete the work of perfecting the world, but you are not free to desist from it either.

- RABBI TARFON, Ethics of the Fathers

But all things very clear are as difficult as rare.

- BARUCH SPINOZA, ETHIC

God is inexorable in offering His gifts. He only gave me the stubbornness of a mule. No! - He also gave me a keen sense of smell.

- ALBERT EINSTEIN
TABLE OF CONTENTS

PREFACE ......................................................... vi
ACKNOWLEDGEMENTS ........................................ viii
LIST OF TABLES ................................................... xi
LIST OF FIGURES .................................................. xii
ABSTRACT ........................................................... xiv

CHAPTER I. INTRODUCTION

Section I-A. Thermonuclear Fusion ......................... 2
Section I-B. Magnetic Confinement ....................... 8
Section I-C. Tokamaks ....................................... 16
Section I-D. Tokamak Current ............................ 29

CHAPTER II. GENERALIZED OHM'S LAW

Section II-A. MHD Theory ................................... 35
Section II-B. Kinetic Theory ................................. 44
Section II-C. Fluid Equations ............................... 51

CHAPTER III. MAGNETIC FIELDS

Section III-A. Canonical Coordinates .................... 59
Section III-B. Magnetic Coordinates ..................... 65
Section III-C. Magnetic Diffusion Equation ............ 71

CHAPTER IV. TEARING MODES

Section IV-A. Ideal MHD ....................................... 81
Section IV-B. Magnetic Topology ......................... 91
PREFACE

This is a dissertation about the leading fusion plasma confinement configuration, the tokamak. The problem we examine here is a fundamental one: How can one efficiently maintain the tokamak toroidal current in the steady-state? As will be seen later, this question is of great practical importance if one considers using the tokamak as a fusion reactor. The theoretical tools used to analyze our problem are fairly sophisticated ones. That is, we employ a new theoretical approach based on a magnetic helicity conserving mean-field Ohm's law. Unfortunately, helicity is a very abstract topological concept. Nevertheless, an attempt has been made throughout the work to motivate theoretical arguments by physical rather than abstract considerations.

The main accomplishment of this work is that, for the first time, a self-consistent mean-field helicity transport model for the tokamak has been successfully implemented on the computer. Two kinds of numerical simulations are accomplished: (1) simulations of inductively driven tokamaks and (2) simulations of completely bootstrapped tokamaks. The inductively driven tokamak simulations are compared with experimental results from the present generation of tokamak devices. The completely bootstrapped tokamak simulations are of great interest, since they suggest that an intrinsic tokamak steady-state can be obtained from the bootstrap current amplification effect alone.

The organization of the dissertation is as follows. First, the Introduction has been written so that it is accessible to all readers and places our research within the context of the magnetic fusion program. Chapters II, III, and IV contain materials of a preliminary or background nature which describe the plasma Ohm's law, magnetohydrodynamics (MHD), representations of magnetic fields, tearing modes, and some tokamak physics. Reading these preliminary chapters will be
particularly worthwhile for those uninitiated in plasma theory or those who want an in-depth view of our work. The reader who already possesses some knowledge of plasma physics may wish to move directly to chapters V and VI, which emphasize our mean-field theoretical approach. Chapter VII describes the numerical methods we have developed for the computer simulations which implement our mean-field theory. The results of the simulations are given in two chapters. Chapter VIII gives the results of simulations involving inductively driven tokamaks. Chapter IX analyzes the possibility of the self-maintenance of the tokamak current by performing simulations of completely bootstrapped tokamaks. Finally, some concluding remarks are given in chapter X.
ACKNOWLEDGEMENTS

I would like to acknowledge and thank some of the many wonderful people who have helped make my graduate school career at William and Mary so enjoyable.

First, I want to thank former Physics Department Chairman Roy Champion and Graduate Director Mort Eckhause for coming to my aid during a period of personal crisis. I'll never forget Roy's suggestion that people would get along with each other much better if they would only remove the high frequency components from their interactions.

I'd also like to thank John Bensel, Sylvia Stout, and Bill Vulcan for being good friends, both at and away from the card table. I particularly want to thank Sylvia for the emotional support she gave me during my father's illness.

I had a lot of fun playing volleyball and drinking beer with the A-1 Physics Department volleyball team of Doug Baker, Carl and Margaret Carlson, Eric Cheung, Jon Goetz, Zhi-Wei Lu, and Jennifer Poor. Every volleyball bruise I got I wore with pride.

I very much enjoyed serving with Melanie Liddle of the Department of History in the Graduate Student Association. However, I'm still smarting from those two parking tickets I received from the Campus Police, within five minutes of each other, while waiting for her to show-up so that we could pick-up food together for a GSA event.

I would like to thank Jim Kempton and Bob Waterland for support during the days leading-up to the formation of the Physics Graduate Student Association. I'll certainly have some trouble forgetting the image of a certain Physics graduate student passed out face first on a bush outside the Small Lab right after our first PGSA wine and cheese party.

I'd like to thank Bob Scholnick, Dean of Graduate Studies and Hillel advisor, especially for inviting me to Jewish student events. Nevertheless, I still do not believe in
miracles, since none of those events ever started on time.

The friendship of Zhi-Wei Lu, my office-mate of several years, was very important to me. Besides enabling me to learn much about Chinese culture, the friendship also made it possible for me to learn new and interesting ways to eat large quantities of wantons in short periods of time.

With regard to my research work, let me begin by thanking Tim Williams and Alastair Neil for their help with the MFE-NERSC CRAY operating systems. The puzzled look on Tim's face when I explained why I ran off 40 minutes of CRAY time on the B machine one weekend ("because it was free") is permanently etched in my memory.

I'd also like to thank Tim, Alastair, Qun Yao, Dave Garren, Jay Larson, and other students in the William and Mary plasma pit for many lively discussions about all sorts of things. Of course, Alastair's cartoons of Jon Goetz and Jennifer Poor belong in the National Lampoon Hall of Fame.

My stay at Los Alamos was made much more enjoyable because of the interesting physics conversations I had with Don Baker, my site advisor at the Lab. I would especially like to thank him for introducing me to the work of Jacobson and Moses.

It was a great pleasure to have had Torkil Jensen as my site advisor at General Atomics. In every way, Torkil confirmed what I have heard many times - that the Danish people are among the finest and friendliest people on this planet. The fourth of July barbecue at his family's place in Del Mar, to which he invited me along with my younger brother Fred, was one of the highlights of my stay in San Diego. Barbecuing a turkey and drinking Danish beer on July 4 reminded me so much of my boyhood - and the interesting things that can happen when two cultures merge together.

I also want to thank Roy Champion, Alkesh Punjabi, George Rublein, Gene Tracy, and George Vahala for agreeing to serve on my dissertation committee. I hope everyone enjoyed reading the dissertation, and that I didn't make the thing too long.

ix
Finally and most importantly, I very much want to thank Allen Boozer, my research advisor and Mentor, the person to whom I am most indebted for making my graduate career at William and Mary such a happy one. From always taking the time to discuss all sorts of questions and issues with me, to helping me find a wonderful job at the end, I could not have had a better advisor. His relaxed attitude, good humor, and keen insights made being his student a pleasure, both on a professional and a personal level. I learned so much about physics and the psychology of doing research from Allen, I wouldn't know how to begin listing it all. (Maybe some members of the dissertation committee will disagree.) In any case, more important to me was the great kindness and generosity Allen showed me throughout the entire period that we worked together. In my short life, I have already learned that those are the sorts of things that really matter.

Part of this work was carried out at General Atomics, San Diego, California, with the support of an Associated Western Universities Fellowship. Most of this work was carried out at the College of William and Mary, Williamsburg, Virginia, with the support of the United States Department of Energy under Grant No. DE-FG0584ER53176.
LIST OF TABLES

Table 1. Output parameters from a typical inductively driven tokamak simulation ............................................ 226

Table 2. Steady-state results of several bootstrap simulations .......................................................................... 248

Table 3. Output parameters from a typical bootstrapped tokamak simulation ......................................................... 251
LIST OF FIGURES

Figure 1. Torus with Boozer coordinates .................... 17
Figure 2. The stellarator configuration ...................... 23
Figure 3. The tokamak configuration ........................ 25
Figure 4. The Joint European Torus (JET) .................. 27
Figure 5. Magnetic island surface plot ..................... 101
Figure 6. $l_1$-$q_a$ diagram ...................................... 223
Figure 7. Poloidal magnetic field from a typical inductively driven tokamak simulation ................. 227
Figure 8. Rotational transform from a typical inductively driven tokamak simulation ................ 228
Figure 9. Toroidal current density from a typical inductively driven tokamak simulation ............. 229
Figure 10. MHD activity parameter from a typical inductively driven tokamak simulation ............ 230
Figure 11. Experimental $l_1$-$q_a$ data points from the TFTR and JET tokamaks ......................... 233
Figure 12. Poloidal magnetic field from a typical bootstrapped tokamak simulation .................. 252
Figure 13. Rotational transform from a typical bootstrapped tokamak simulation .................... 253
Figure 14. Current densities from a typical bootstrapped tokamak simulation ............. 254

Figure 15. MHD activity parameter from a typical bootstrapped tokamak simulation ............. 255
ABSTRACT

A fundamental requirement for successful operation of a tokamak is the maintenance of a toroidal electric current within the tokamak plasma itself. Maintaining this internal plasma current can be a very difficult technological problem. In this work, a well-known but non-standard method for maintaining the tokamak current called the bootstrap effect is discussed. The bootstrap effect occurs when a fusion plasma is near thermonuclear conditions, and allows the tokamak to greatly amplify its electric current.

Because the bootstrap effect amplifies but does not create a plasma current, it has long been argued that a completely bootstrapped tokamak is not possible. That is, it has been argued that some fraction of the tokamak current must be created externally and injected into the plasma for a bootstrap amplification to occur. This injection of current is not desirable, however, since current-drive schemes are difficult to implement and are only marginally efficient.

An important but largely unexplored implication of the bootstrap effect is that the effect, by itself, creates hollow (outwardly peaked) tokamak current profiles. Hollow tokamak current profiles are known to lead to tearing modes, which are resistive (non-ideal) magnetohydrodynamic (MHD) plasma instabilities. Although usually characterized as harmful for plasma confinement, it turns out that tearing modes may actually be beneficial for the tokamak bootstrap effect.

In this work, a new theoretical approach based on a helicity conserving mean-field Ohm's law is used to examine the interaction between the bootstrap effect and tearing modes. Magnetic helicity is a topological quantity which is conserved even in turbulent plasmas. Computer simulation results of the mean-field Ohm's law are presented which suggest that a completely bootstrapped tokamak may indeed be possible. In a completely bootstrapped tokamak, the tokamak self-maintains its electric current by amplifying an intrinsic internal plasma current due to the tearing modes.
COMPLETELY BOOTSTRAPPED TOKAMAK
CHAPTER I. INTRODUCTION

Section I-A. Thermonuclear Fusion

If light atomic nuclei are heated to temperatures in excess of $10^8$ degrees Celsius, nuclear kinetic energies become large enough to overcome the Coulomb barrier of the internuclear potential. Intense heating therefore allows atomic nuclei to join together, rearranging themselves in a process called thermonuclear fusion. Thermonuclear fusion is characterized by one very important property - that of an overall reduction in the total mass of the nuclear matter involved in the fusion process. This means that an enormous amount of energy is released in the nuclear rearrangement. The exact amount of energy can be calculated from Einstein's famous mass-energy relation,$^1$

\[ E = \Delta mc^2. \quad (I-A.1) \]

Thermonuclear fusion is quite common, occurring continually in stars such as our Sun.

Laboratory attempts to reproduce thermonuclear fusion have concentrated on the reaction

\[ \text{H}_2 + \text{H}_3 \rightarrow \text{He}_4 + n^1. \]  

(I-A.2)

The deuterium-tritium reaction (I-A.2) is favored over all other thermonuclear processes, due to its high reaction cross-section at low temperatures. From the Einstein relation (I-A.1), it is straightforward to show that the deuterium-tritium process has an energy release of 17.6 MeV per fusion reaction, which represents a millionfold increase over the energy released from a typical fossil fuel chemical reaction. Fortunately, large supplies of nuclear fuels are readily available on or near Earth. That is, an almost limitless supply of the deuterium exists in ordinary sea water. Tritium is radioactive with the relatively short half-life of 12.3 years, so it is in scarce supply. However, using the neutrons from the fusion reactions (I-A.2), tritium can be bred from the element lithium via the fission process

\[ \text{Li}_6 + n^1 \rightarrow \text{H}_3 + \text{He}_4 \]  

(I-A.3)
or

\[ \text{Li}^7 + n^1 \rightarrow \text{H}^3 + \text{He}^4 + n^1. \]  \hspace{1cm} (I-A.4)

Perhaps future fusion research will even allow for the burning of more advanced nuclear fuels, avoiding many radioactivity problems. For example, the reaction

\[ \text{H}^2 + \text{He}^3 \rightarrow \text{He}^4 + \text{H}^1 \]  \hspace{1cm} (I-A.5)

has the very favorable energy release of 18.3 MeV, without any neutrons. Apparently, large quantities of helium-3 are available on the lunar surface.\(^2\) However, the deuterium-helium reaction (I-A.5) requires a temperature an order of magnitude greater than that of (I-A.2).

Because of the environmentally safe, relatively clean burn afforded by nuclear fusion, many countries are actively participating in an international effort to produce sustained controlled thermonuclear fusion reactions. If the effort is successful, the resulting nuclear fires could replace the ordinary chemical fires currently used to generate electrical

power. While the reality of nuclear fusion has been dramatically demonstrated by detonations of hydrogen bombs, the goal of attaining a sustained controlled thermonuclear reaction has not yet been achieved.

The difficulties associated with sustaining a thermonuclear reaction or nuclear fire are really not fundamentally different from those associated with sustaining a chemical reaction or ordinary fire. To illustrate the point, consider the case of carbon-oxygen chemical fires, the kind frequently used at charcoal barbecues. Charcoal fires fundamentally require two things, a fuel supply of carbon charcoals to mix with oxygen and something to get the fire started. If a charcoal fire is started successfully, carbon and oxygen atoms undergo a process of atomic rearrangement, simultaneously releasing both heat and light. However, as everyone knows, a real difficulty in maintaining a barbecue fire occurs if the heat escapes from the barbecue too rapidly, thus not allowing the carbon charcoals to ignite. Even with an ample fuel supply, if the heat escapes from the charcoal fire too rapidly, the carbon-oxygen reaction may not sustain itself (and dinner may be steak tartare).
A very common and simple way to overcome the heat loss difficulty associated with chemical fires is to construct a container which attempts to isolate the chemical reaction from the rest of the world. The container, which is often a metal chamber, confines the fuel and heat of the chemical reaction in a way that allows the ignition process to occur. Simple examples of such containers include the metal pans of barbecue grills and the piston cylinders of internal combustion engines.

As stated above, the situation with respect nuclear fires is not that different. That is, the fundamental problem of confinement is also the key issue associated with sustaining a thermonuclear reaction. Similarly, the construction of a container to confine the fusion reaction is at the heart of most proposed solutions to the heat loss difficulties associated with nuclear fires. In fact, because of the importance of fusion as a possible future energy source for humanity, the thermonuclear confinement problem has become the major research focus of a group of scientists called plasma physicists. Plasma physics is the study of fully or partially ionized gases and a fusion plasma is a fully ionized gas of various light atomic species. The fusion plasmas used
in thermonuclear reactions are fully ionized because the plasma temperatures are great enough so that all individual atoms separate into electrons and positively charged nuclear ions.

Although fusion plasmas are charge neutral in bulk, the fact that the fundamental constituents of a plasma can have an electric charge have some very interesting and important consequences. The most important consequence for the fusion application is due to the fact that charged particles can interact with magnetic fields. That is, since a plasma consists of charged particles and the motion of charged particles represents an electric current, the plasma can interact with a magnetic field via the magnetic force. This is important because the high temperatures necessary for thermonuclear fusion make it impossible to construct fusion plasma containers from the kind of materials used to confine ordinary chemical reactions. Fortunately, the magnetic interaction of a plasma provides another option. That is, due to the interaction of plasma particles with magnetic fields, fusion plasma containers can be constructed from the magnetic field itself.
Section I-B. Magnetic Confinement

How can the magnetic field confine a fusion plasma? To illustrate the method, consider first the situation in which there is no magnetic field. Then the fusion plasma is sitting by itself, localized in some region of space. But this situation will not last for long. This is because there is a natural tendency for the plasma to expand into larger regions of space and cool down. The expansion of the plasma continues until the plasma pressure \( p \) is equalized throughout all space and the expansion force is just \(-\nabla p\).

Consider now the situation in which a magnetic field \( B(x) \) does exist. If \( j \) is the plasma current density, the magnetic force acting on the plasma is just \( j \times B/c \) per unit volume, where \( c \) is the speed of light and we use the Gaussian system of electromagnetic units.\(^3\) Then the plasma pressure gradient can be balanced by the magnetic force, forming a mechanical equilibrium via

\[
\nabla p = j \times B/c. \tag{I-B.1}
\]

An insight into the meaning of the plasma equilibrium (I-B.1) can be gained by combining Ampere's law

$$\nabla \times B = \left(4\pi/c\right)j$$

(I-B.2)

and a vector identity to obtain the expression

$$j \times B/c = -\nabla(B^2/8\pi) + B \cdot \nabla B/4\pi.$$  

(I-B.3)

Then the equilibrium condition (I-B.1) becomes

$$\nabla(p + B^2/8\pi) = B \cdot \nabla B/4\pi.$$  

(I-B.4)

Equation (I-B.4) shows that a plasma pressure gradient can be balanced by a gradient in the magnetic field pressure $B^2/8\pi$. For example, a magnetic field of 5000 gauss (1/2 Tesla) is equivalent to about one atmosphere of plasma pressure. An important plasma parameter is therefore the dimensionless plasma beta,

$$\beta = \frac{8\pi p}{B^2}.$$  

(I-B.5)
The quantity $\beta$ is just the ratio of the plasma and magnetic pressures and measures the efficiency with which a magnetic field confines a plasma. Economic thermonuclear fusion reactors will require $\beta \approx 5\%$. From the equilibrium equation (I-B.4), the gas pressure gradient can also be balanced by the magnetic tension force $\mathbf{B} \cdot \nabla \mathbf{B}/4\pi$. The tension force is due to the bending of magnetic field lines.

The effect of a magnetic field on a plasma is actually somewhat subtle, however, as can be seen from examining the following argument. Suppose, besides being in mechanical equilibrium, that the plasma is also in the state of thermodynamic equilibrium. Then the relative probability $P_r$ of the plasma being in any particular mechanical configuration is given by the famous Boltzmann factor of statistical mechanics,\(^5\)

$$P_r \propto \exp(-E/T),$$  \hspace{1cm} (I-B.6)

where $T$ is the absolute temperature in energy units and $E$ is

the energy of the (nonrotating) mechanical configuration. The magnetic force, however, does not contribute to the mechanical energy $E$. That is, if $e_j$, $x_j$, and $v_j$ are the $j$th plasma particle charge, position, and velocity, then the rate at which the magnetic force does work on the $j$th plasma particle is

$$\frac{dE_j}{dt} = e_j v_j \cdot (v_j \times B(x_j, t))/c = 0.$$  

Equation (I-B.7) implies that the configuration energy $E$ in the Boltzmann factor (I-B.6) is completely independent of the magnetic field $B$. In thermodynamic equilibrium, therefore, the magnetic field has no effect on a plasma whatsoever. This means that the fusion plasmas we are considering may be steady-state plasmas, but they are certainly not plasmas in thermodynamic equilibrium. In fact, the maintenance of a plasma pressure gradient implies entropy production and the lack of thermodynamic equilibrium. Since fusion plasmas are not in the state of thermodynamic equilibrium, an understanding of magnetic confinement must involve a kinetic theory. Unfortunately, the plasma kinetic theory leads to sets of non-linear integro-
differential equations which are exceedingly difficult to solve. It is this non-linear character of the fundamental plasma equations which limits our understanding of magnetic confinement.

Nevertheless, let us examine a very much simplified form of the plasma kinetic theory, a version of the so-called classical transport theory. As previously stated, the expansion of a plasma is proportional to the force \(-\nabla p\). The rate of the expansion, however, is determined by a thermodynamic diffusion coefficient \(D\). The diffusion coefficient \(D\) depends on the detailed nature of the collisions between plasma particles. The important link between collisions and diffusion was demonstrated by Einstein,\(^6\) who used statistical arguments to correlate atomic collision processes with the diffusion of pollen particles, the so-called Brownian motion. The central point of the statistical arguments is that diffusion can be regarded as a random walk. An effective diffusion coefficient can be calculated from the formula

\[
D \approx \frac{L^2}{2\tau},
\]

\(^{(I-B.8)}\)

where $L^2$ and $\tau$ represent the mean square distance and time intervals for a particle between collisions.

To understand plasma magnetic confinement from the perspective of our simple kinetic theory, let us once again suppose that there is no magnetic field. Then the mean free path $\lambda_C$ of a plasma particle is given by

$$\lambda_C = v_T \tau,$$  \hspace{1cm} (I-B.9)

where the thermal speed

$$v_T = (2T/m)^{1/2}.$$

Typical thermonuclear conditions are plasma temperatures on the order of $T \approx 10$ keV and densities $n \approx 10^{20}$ m$^{-3}$. Under these conditions, the plasma mean free path is huge, on the order of several kilometers. The value of $\lambda_C$ is huge because a typical dimension of a plasma confinement device is ordinarily only on the order of a meter.

Consider now the situation in which a magnetic field does exist. As charged particles move in the vicinity of the
magnetic field, the particle motion becomes localized by the magnetic force. That is, plasma particles have a helical motion along magnetic field lines, travelling unimpeded in directions parallel to the magnetic field for a mean distance $\lambda_C$, but with perpendicular excursions to the field lines limited by the gyroradius\(^7\)

$$\rho = \frac{v_\perp}{\omega_C}. \quad (I-B.11)$$

Here, $v_\perp$ is the particle velocity perpendicular to the magnetic field lines and $\omega_C$ is the cyclotron frequency,

$$\omega_C = \frac{eB}{mc}. \quad (I-B.12)$$

Replacing $v_\perp$ in (I-B.11) by $v_T$ (I-B.10) shows that the ion gyroradius $\rho_i$ is greater than the electron gyroradius $\rho_e$ by a factor $\left(\frac{m_i}{m_e}\right)^{1/2}$. The effective plasma diffusion coefficients in the directions parallel and perpendicular to the magnetic field are then

$D_{\parallel} \approx \frac{\lambda_c^2}{2\tau_i}$ \hfill (I-B.13)

and

$D_{\perp} \approx \frac{\rho_i^2}{2\tau_i}$. \hfill (I-B.14)

Under thermonuclear conditions and with typical magnetic fields of several Teslas, the ion gyroradius $\rho_i$ is only on the order of a few millimeters. Therefore, if the field lines remain localized, so that parallel particle motion is not important, a magnetic field can decrease the plasma transport rate by a factor

$$D_{\perp}/D_{\parallel} \approx (\rho_i/\lambda_c)^2 \approx (10^{-3}/10^3)^2 = 10^{-12}. \hfill (I-B.15)$$

Although the simple theory presented in this section only gives a very rough estimate, the result (I-B.15) indicates that the magnetic field can dramatically decrease the rate at which plasma thermodynamic equilibrium is approached. This is how the magnetic field creates confinement for a fusion plasma.
Section I-C. Tokamaks

An important condition for magnetic confinement is that magnetic field lines remain localized. That is, since there is a strong tendency for plasma particles to follow field lines, if the magnetic field lines are not confined, then the plasma is not confined. In this work, we examine a device called the tokamak,\textsuperscript{8,9} which is the most popular magnetic confinement configuration that offers the required field line localization. Tokamak is a Russian acronym for toroidalnaya-kamera-magnitaya, meaning toroidal-chamber-magnetic. A torus (see Fig. 1) is a body which has the topological shape of a doughnut or bagel (depending on one's ethnicity). It turns out that the torus is the only possible shape which meets the required field line localization condition. Any shape other than the torus would allow field lines to enter or leave the region of magnetic confinement. This can be understood from the topological hair theorem, which states that the torus represents the only closed surface in three dimensions on which hair can be combed without crowns or cusps. That is,

Figure 1. Torus with Boozer coordinates. The quantities $\theta$ and $\phi$ represent angles which parameterize the torus in the poloidal and toroidal directions, respectively. The radial coordinate $\psi$ is the toroidal magnetic flux enclosed by a constant $\psi$-surface. The quantity $-\chi$ is the poloidal magnetic flux outside a constant-$\chi$ surface, that is, in the torus hole.
the hair of the hair theorem can be thought of as a representation of the magnetic field lines.

Mathematically, the hair theorem requirement can be derived by forming the vector product of the magnetic field and the equilibrium condition (I-B.1),

\[ B \cdot \nabla p = 0. \quad \text{(I-C.1)} \]

This equation states that a magnetic field line remains in a region with the same plasma pressure. If a plasma equilibrium exists and \(|\nabla p| \neq 0\), then the magnetic field lines lie in a series of nested toroidal pressure surfaces.\(^{10}\) The toroidal pressure surfaces are also called magnetic surfaces. Having a region of good magnetic surfaces within a plasma is a very important condition for successful magnetic confinement.

Although equation (I-C.1) seems very simple, there are really many subtleties associated with the existence of plasma equilibria and magnetic surfaces. For example, experiments on so-called plasma pinches,\(^{11}\) first studied


\(^{11}\)W. H. Bennett, Phys. Rev. 45, 890 (1934).
intensely in the 1950's, suggested that it would be very useful to have a plasma magnetic field with a large toroidal component $B_\varphi$. While very costly from the economic or $\beta$ standpoint of equation (I-B.5), the large toroidal field provides a rigid frame which is extremely useful for maintaining plasma stability. The toroidal magnetic field $B_\varphi$ can certainly satisfy equation (I-C.1). However, the toroidal field alone cannot satisfy the equilibrium condition (I-B.1). It turns out that a sufficient condition for plasma equilibrium is that, besides having a toroidal component $B_\varphi$, the magnetic field also must have a twist or poloidal component $B_\theta$.\textsuperscript{12,13} This condition is due to the fact that toroidal magnetic field configurations have field gradients, causing plasma particles to drift out of the confinement zone.

Let us examine the question of particle drifts in the field $B_\varphi$ more closely. From Ampere's law (I-B.2), a toroidal magnetic field has a field gradient $\nabla B$ pointing in the direction of the center of the torus hole. This field gradient causes a variation in the particle gyroradius (I-B.11). The variation in gyroradius then induces a particle drift across

\textsuperscript{12}L. Spitzer, Jr., Phys. Fluids 1, 253 (1958).
the magnetic field\textsuperscript{14,15}

\[ \mathbf{v}_{\nabla B} = \pm \left( v_{\perp} p / 2 B^2 \right) \mathbf{B} \times \nabla \mathbf{B}, \quad (I-C.2) \]

where the ± refers to the particle charge sign. If the magnetic field \( \mathbf{B}_\phi \) circulates in the counter-clockwise direction, ions drift upward, to the top of the torus, while electrons drift downward, to the torus bottom. This so-called grad-B drift would continue forever, except that the separation of charge creates a downward electric field \( \mathbf{E} \). This causes the grad-B drift (I-C.2) to saturate. The downward electric field \( \mathbf{E} \), however, has very destructive consequences. This is because the electric field alters the particle speed, again causing a variation in the particle gyroradius (I-B.11). This induces a second particle drift\textsuperscript{16}

\[ \mathbf{v}_{\mathbf{E} \times \mathbf{B}} = \left( c / B^2 \right) \mathbf{E} \times \mathbf{B}, \quad (I-C.3) \]

which is independent of the particle charge sign. The

direction of this E×B drift is out of the torus, meaning that the plasma equilibrium and magnetic confinement are lost. The downward electric field due to charge separation is therefore intolerable and a way must be found to eliminate it.

How can the electric field be eliminated? Since plasma particles move rapidly along the field lines, if the field lines are made to move from the bottom to the top of the torus, then plasma particles can also move from the bottom to the top of the torus. A poloidal magnetic field $B_\theta$ can therefore provide a path for plasma particles to short-circuit the electric field. In effect, the twisting of the field lines around a magnetic surface caused by the poloidal magnetic field allows plasma particles to sample the entire toroidal pressure surface. The net result is that the worst particle drift motions average away.

Although there are many specialized magnetic geometries (RFPs, spheromaks, etc.), the required confinement field $B_\theta$ can be created in fundamentally two alternate ways. One possibility is the stellarator concept,\textsuperscript{17} which involves

creating both a large toroidal field $B_T$ and a smaller poloidal field $B_\theta$ using external field coils. The stellarator configuration (see Fig. 2) represents one of the earliest attempts at magnetic confinement, first being pursued by Spitzer$^{18}$ and the Princeton group in the late 1950’s. A major strength of the stellarator geometry is that creating $B_\theta$ from external coils allows for some measure of control of this magnetic field and for steady-state operation of the stellarator device. However, a serious drawback of the configuration is that, due to the twisting of the external field coils required to create $B_\theta$, the stellarator is fully three-dimensional and very difficult to realize in practice. In fact, the initial results of the Princeton experiments indicated poor magnetic confinement and the stellarator project was eventually abandoned. Recently, the stellarator has undergone a kind of Renaissance with active research on the machine occurring in, for example, Germany, Japan, and the United States. The principle reason for the renewed interest in stellarators is that sophisticated computer codes and other technological advances allow for better positioning of the external field coils.

Figure 2. The stellarator configuration. Field coils with a helical twist produce both a large toroidal magnetic field $B_\varphi$ and a small poloidal magnetic field $B_\theta$. Stellarators are difficult to realize in practice because the twisting of the field coils makes the configuration fully three-dimensional.
The alternate concept to the stellarator is that of the tokamak (see Fig. 3), which is the subject of this research. The basic idea of the tokamak was first described in the early 1950s by Tamm and Sakharov,\textsuperscript{19} and independently by Spitzer.\textsuperscript{20} Like the stellarator, the tokamak also has a large $B_\phi$ made from external field coils. However, unlike the stellarator, the smaller confinement field $B_\theta$ is now created internally by passing a toroidal current $I$ through the plasma. This allows, in principle, for an axisymmetric magnetic confinement configuration. Much of the initial experimental work on the tokamak was performed at the Kurchatov Institute in Moscow. During the 1960's, after a series of impressive experimental results,\textsuperscript{21,22} the tokamak began to attain the position which it holds today, that of the leading magnetic confinement configuration.

The modern tokamak is a large, technologically advanced machine. The largest and most sophisticated tokamak built to

\begin{itemize}
  \item L. A. Artsimovich, Nuc. Fusion 12, 215 (1972).
\end{itemize}
Figure 3. The tokamak configuration. Toroidal field coils produce a large toroidal magnetic field $B_\phi$. The smaller poloidal magnetic field $B_\theta$ is created by passing a toroidal plasma current $I$ through the plasma. If the number of toroidal field coils is large, the tokamak becomes a nearly axisymmetric magnetic confinement configuration.
date is the Joint European Torus (JET). JET (see Fig. 4) has a plasma minor radius $a \approx 1.25 \text{ m}$, a major radius $R_0 \approx 3.0 \text{ m}$, toroidal magnetic fields $B_\phi \approx 3.5 \text{ T}$, plasma currents $I \approx 7.1 \text{ MA}$, and can attain global energy confinement times $\tau_E$ greater than a second.\textsuperscript{23} In contrast, the first series of Soviet tokamaks had parameters $a \approx 0.12 \text{ m}$, $R_0 \approx 1.0 \text{ m}$, $B_\phi \approx 2.5 \text{ T}$, $I \approx 0.06 \text{ MA}$, and attained energy confinement times $\tau_E$ on the order of only several milliseconds. The JET tokamak has been designed to approach the thermonuclear plasma regime. Frequently, attainment of the thermonuclear regime is expressed in terms of the Lawson criterion\textsuperscript{24}

$$n\tau_E > 6 \times 10^{19} \text{ m}^{-3}\text{s}.$$  \hspace{1cm} (I-C.4)

The Lawson criterion is a necessary condition for energy "breakeven." The breakeven condition represents the point at which thermal losses from the plasma are balanced by the thermonuclear power produced by fusion reactions. The requirement for plasma ignition is greater than the Lawson criterion (I-C.4) by a factor of 2 or 3.

Figure 4. The Joint European Torus (JET). JET is the largest and most sophisticated tokamak built to date. Note the unconventional D-shape of the plasma cross-section. This shaping optimizes tokamak plasma stability against so-called ballooning modes. Ballooning modes are localized pressure-driven magnetohydrodynamic (MHD) instabilities which limit the maximum attainable plasma $\beta$. 
The major and overriding strength of the tokamak concept is that the axisymmetric nature of the device allows for a relatively easy realization of the configuration. Drawbacks of the confinement scheme involve issues related to the tokamak toroidal current \( I \). Of course, the existence of the toroidal current is necessary if the tokamak configuration is to be axisymmetric. The first drawback issue we discuss concerns the distribution of the tokamak current. Since the current \( I \) is set up within the plasma, the toroidal current becomes a source of plasma free energy and the current density \( j_\varphi \) is controlled to a large extent by the tokamak plasma itself. The source of the confinement field \( B_\varphi \) is therefore not under complete control. This means that tokamak plasmas are susceptible to certain current-driven instabilities called tearing modes.\(^{25}\) The tearing modes literally rip apart the poloidal magnetic field \( B_\varphi \). While the tokamak is very resilient to tearing modes, the plasma can undergo a total disruption if sufficient care is not exercised.\(^{26}\) The second drawback issue concerns the steady-state operation of the tokamak. Since the toroidal

current is essential to the tokamak magnetic confinement scheme, steady-state operation requires an efficient method to maintain this current. This issue is an especially critical one for the proposed next generation of tokamaks such as the International Thermonuclear Experimental Reactor (ITER). In fact, the issue of steady-state operation is at the heart of the question of whether the tokamak can be considered as a viable and practical fusion reactor.

Section I-D. Tokamak Current

In this work, we present new methods to analyze the distribution and maintenance of the tokamak toroidal current. The new methods we present are based on the theory of mean magnetic fields. In mean-field theory, the equations for the magnetic field are averaged over small-scale plasma instabilities, such as tearing modes. Because of the averaging procedure, the field lines of the mean magnetic field lie in perfect toroidal magnetic surfaces, thus greatly simplifying the analysis. The complex behavior of tearing modes on

the plasma then reduces to that of a current viscosity\textsuperscript{30} whose form we determine. The current viscosity represents the averaged effect of so-called "cross-field" transport processes in regions of complicated magnetic field line topology\textsuperscript{31,32}

The mean-field theory we employ derives from the helicity conservation theorem\textsuperscript{33,34} a very firm foundation on which to build a theory indeed. The magnetic helicity $K$ is defined as\textsuperscript{35,36}

$$K = \int A \cdot B \, d^3x, \quad (I-D.1)$$

where $A$ is the vector potential. The helicity is a topological quantity related to the intertwining of the poloidal and toroidal magnetic fluxes. Although the application of helicity methods to tokamaks is rather novel, the calculations presented here are still based on a very simple procedure.

\textsuperscript{34}A. H. Boozer, J. Plasma Phys. 35, 133 (1986).
\textsuperscript{35}W. M. Elsasser, Rev. Mod. Phys. 28, 135 (1956).
This procedure rests on the observation that the tokamak plasma is an electrical conductor. In fact, thermonuclear plasmas can have conductivities greater than that of copper. As in the case of a current in a copper wire, the analysis of the tokamak current is based on a single simple equation, Ohm's law.

For the bulk plasma current $I$, Ohm's law is just

$$I = \frac{V}{R_\Omega}.$$

where $V$ is the loop voltage and $R_\Omega$ is the effective plasma resistance. For the current density $j$, the Ohm's law we consider is

$$E \cdot B = \eta (j \cdot B - j_B B) - \nabla [\lambda \nabla (j_B / B)],$$

where $\eta$ is the plasma resistivity and the current viscosity $\lambda$ is a new term, modeling the averaged effect of tearing modes on the plasma current. In addition to the tearing mode term, the mean-field Ohm's law (I-D.3) includes a term $j_B$ to model a tokamak current amplification scheme called the
bootstrap effect.\textsuperscript{37,38} The bootstrap effect is due to a diffusion driven plasma current which is predicted to exist in the thermonuclear plasma regime.\textsuperscript{39} The term "bootstrap effect" refers to the legendary exploits of the 18th century Baron von Münchhausen,\textsuperscript{40} who was said to be able to fly by lifting himself up by his bootstraps. As we shall see, the bootstrap effect and tearing modes are intimately connected together in the tokamak.

In this work, the main question we investigate is whether the tokamak is capable of maintaining its own current via the combination of the bootstrap effect and tearing modes alone. That is, whether the current $I$ of equation (I-D.2) can be maintained with the loop voltage $V=0$ and no external current-drive. This is not ruled out because the bootstrap effect can make the effective plasma resistance $R_Q$ zero (or even negative), thus allowing for a finite tokamak current $I$.\textsuperscript{41} However, a well-known restriction on the bootstrap effect is that it does not create

\begin{itemize}
\item \textsuperscript{38}B. B. Kadomtsev and V. D. Shafranov, Nuc. Fusion Suppl., p. 209 (1972).
\item \textsuperscript{39}A. A. Galeev and R. Z. Sagdeev, Sov. Phys. JETP 26, 233 (1968).
\item \textsuperscript{40}R. E. Raspe, Adventures of Baron von Münchhausen, (London, 1785).
\item \textsuperscript{41}A. H. Boozer, Phys. Fluids 29, 4123 (1986).
\end{itemize}
the poloidal magnetic flux necessary to maintain the tokamak
in a steady-state. Instead, the bootstrap effect pushes
poloidal flux out of regions of high plasma pressure. This
lack of poloidal flux production is precisely why it is
important to consider the combination of the bootstrap effect
and tearing modes together. Because tearing modes can rip
apart the poloidal magnetic field $B_q$, they can create
additional poloidal magnetic flux. Although usually
characterized as harmful for plasma confinement, it turns
out that tearing modes may actually be beneficial for the
tokamak bootstrap effect. An earlier study which did not
include the effect of tearing modes along with the bootstrap
effect concluded that “a completely bootstrapped tokamak is
not possible” and that some source of poloidal flux, such as
external current-drive, is required for a steady-state
tokamak bootstrap effect.\textsuperscript{42} For example, the tokamak
current could be externally seeded by a wave driven plasma
current.\textsuperscript{43} This is somewhat undesirable, however, because
external current drive schemes are difficult to implement
and are only marginally efficient.\textsuperscript{44} Taking into account the

(1971).
\textsuperscript{43}N. J. Fisch, Rev. Mod. Phys. 59, 175 (1987).
combination of the bootstrap effect and tearing modes, however, our analysis shows that a completely bootstrapped tokamak is indeed possible. That is, a completely bootstrapped tokamak, like Baron von Münchausen, may indeed be able to "fly."

\[4^{4}A. \, H. \, Boozer, \, Phys. \, Fluids \, 30, \, 591 \, (1988).\]
CHAPTER II. GENERALIZED OHM'S LAW

Section II-A. MHD Theory

An electrically conducting fluid, such as the molten core of planet Earth or an interstellar nebula, naturally gives rise to a magnetic field. The combined theory of the magnetic field and fluid mechanics has been studied intensely in the latter part of this century and is called hydromagnetics or magnetohydrodynamics (MHD).\textsuperscript{45,46} MHD gives a plasma description based on the hydrodynamic variables density $n(x,t)$, fluid velocity $v(x,t)$, and pressure $p(x,t)$, and therefore only gives an approximate description of plasma behavior. MHD theory does, however, provide a good description of most low-frequency, large-scale plasma phenomena. For this reason, all the calculations of our work are based on some sort of MHD procedure.

The fact that MHD can give a good description of plasma behavior is by no means apriori obvious. In fact, in ordinary

\textsuperscript{46}J. P. Freidberg, Ideal Magnetohydrodynamics, (Plenum Press, New York, 1987).
fluid mechanics, the applicability of the fluid description is contingent on the fluid being sufficiently collisional. In other words, the particles within an individual fluid element must stick together long enough so that the concept of fluid element itself is physically meaningful. The collisional requirement for a fluid is just

\[ \lambda_c \ll a, \]  

(II-A.1)

where \( \lambda_c \) is the mean free path (I-B.9) and \( a \) is a typical macroscopic dimension. Since for thermonuclear plasma conditions \( \lambda_c \approx 1 \, \text{km} \) and \( a \approx 1 \, \text{m} \), it is indeed initially surprising that the MHD description of a plasma is ever valid.

The behavior of a plasma is different than that of an ordinary fluid, however, due to the magnetic field. The magnetic field, in effect, assumes the role played by the collisions of an ordinary fluid, at least for particle motions perpendicular to \( B \). For example, in a uniform magnetic field, perpendicular particle excursions to magnetic field lines are limited by the gyroradius (I-B.11). Since the ion gyroradius \( \rho_i \approx 1 \, \text{mm} \), fusion plasmas satisfy the inequality
\[ \rho_i \ll a, \quad (II-A.2) \]

which shows that \( \rho_i \) can play the role of \( \lambda_c \) in a plasma. This result is sometimes summed-up by saying that the plasma is \( 2/3 \) of a fluid. In fact, the analysis for particle motions parallel to the magnetic field is much more involved, and will not be given here. The conclusion of that analysis, however, is that the most unstable MHD motions are incompressible,\(^{47}\) indicating that the violation of the inequality (II-A.1) for motions parallel to \( B \) will not degrade the reliability of MHD stability predictions.

In the next several sections, we present an MHD derivation of the generalized plasma Ohm's law,

\[ E + \mathbf{v} \times \mathbf{B}/c = \eta j. \quad (II-A.3) \]

This derivation is important because the generalized Ohm's law provides a foundation for the more sophisticated mean-field Ohm's law approach (I-D.3) which we shall adopt later. Before beginning our derivation, let us remark that an appeal

for (II-A.3) is often made in the following way. First, it is assumed that the usual Ohm's law

\[ E' = \eta j' \]  

(II-A.4)

holds in the local rest (primed) frame of the plasma fluid. Since the non-relativistic transformations of electric and magnetic quantities from the laboratory (unprimed) frame are

\[ E' = E + \mathbf{u} \times \mathbf{B} / c, \]  

(II-A.5)

\[ B' = B, \]  

(II-A.6)

and

\[ j' = j - \mathbf{u} \rho_c, \]  

(II-A.7)

the combination of equations (II-A.5)-(II-A.7) yields (II-A.3), where the charge density \( \rho_c = 0 \) in the lab frame. While there is nothing fundamentally "wrong" with this argument, we prefer to give an alternate, more complete derivation of

---

Ohm's law. The derivation is useful because it brings out some of the approximations, limitations, and subtleties inherent in any MHD model.

We begin our discussion of MHD equations by defining the plasma mass density $\rho_m$ and fluid velocity $\mathbf{v}$ from

$$\rho_m = \sum_{\alpha} m_{\alpha} n_{\alpha} \quad \text{(II-A.8)}$$

and

$$\mathbf{v} = \left(1/\rho_m\right) \sum_{\alpha} m_{\alpha} n_{\alpha} \mathbf{v}_{\alpha} \quad \text{(II-A.9)}$$

Here, the sums are taken over the various species of particles, ions and electrons, contained within the plasma. For simplicity, we assume a single species of ions. The first MHD approximation is to neglect the electron mass $m_e$ compared to the ion mass $m_i$. Then (II-A.8) and (II-A.9) become

$$\rho_m = m_i n_i \quad \text{(II-A.10)}$$

---

and

\[ \mathbf{v} = \mathbf{v}_i. \]  

(II-A.11)

For deuterium ions \( \frac{m_e}{m_i} \approx 1/3672 \), so the neglect of the electron inertia is generally a very good approximation.

The plasma charge density \( \rho_C \) and current density \( j \) are similarly defined from

\[ \rho_C = \sum_{\alpha} e_{\alpha} n_{\alpha} \]  

(II-A.12)

and

\[ j = \sum_{\alpha} e_{\alpha} n_{\alpha} \mathbf{v}_\alpha. \]  

(II-A.13)

The second MHD approximation is that the plasma is charge quasi-neutral. The quasi-neutral approximation means that the difference in magnitude of the ion and electron charge densities is much less than than the magnitude of either the ion or the electron charge density itself.
An estimate of the goodness of this approximation can be obtained from Poisson's equation,

$$\nabla \cdot E = 4\pi \rho_C,$$

(II-A.15)

and the electron force balance equation

$$\nabla p_e = -e n_e (E + v_e \times B/c).$$

(II-A.16)

Poisson's equation gives $\rho_C \approx |E|/4\pi a$ and electron force balance gives $en_e |E| \approx p_e/a$ or $|E| \approx T/ea$, where $a$ is a typical plasma dimension. Then $|\rho_C/en_e| \approx (\lambda_d/a)^2$, where

$$\lambda_d = (T/4\pi n_e e^2)^{1/2}$$

(II-A.17)

is called the Debye length. The Debye length is always small compared to the plasma scales of interest and defines the boundary below which the quasi-neutrality assumption is no longer valid. For fusion plasmas, $\lambda_d \approx 10^{-4} \text{ m}$ and $a \approx 1 \text{ m}$, so the quasi-neutrality assumption (II-A.14) is very good. In
fact, because the quasi-neutrality assumption is obeyed in all plasmas of practical interest, when considering low-frequency plasma motions, plasma physicists never solve the Poisson equation (II-A.15) for the electric field $E$. Instead, Poisson's equation is replaced with the neutrality condition

$$n_i = n_e = n.$$  \hfill (II-A.18)

If desired, the self-consistent charge density $\rho_C$ can be determined later from the solution $E$ and Poisson's equation.

Another important approximation made with regard to electromagnetic fields is that all processes are assumed to be sufficiently slow. This assumption is sometimes called the quasi-stationary approximation. The meaning of sufficiently slow can be obtained from the Maxwell equation

$$\nabla \times B = (4\pi/c)j + (1/c)\partial E/\partial t.$$  \hfill (II-A.19)

Taking the divergence of (II-A.19) and recalling (II-A.15) results in the continuity equation for electric charge,

$$\partial \rho_C/\partial t + \nabla \cdot j = 0.$$  \hfill (II-A.20)
The quasi-stationary approximation results from neglecting the term $\partial \rho_e/\partial t$ in the continuity equation. What this means is that the characteristic frequency response $\omega_p$ of the plasma to any charge imbalance must be much faster than any plasma frequency $\omega$ of interest,

$$\omega \ll \omega_p. \quad \text{(II-A.21)}$$

Since the electron inertia is much smaller than the ion inertia, the characteristic frequency response to a charge imbalance is given by $\omega_p \approx v_{Te}/\lambda_d$. From (I-B.9) and (II-A.17), this defines the so-called electron plasma frequency

$$\omega_{pe} = (4\pi n_e e^2/m_e)^{1/2}. \quad \text{(II-A.22)}$$

For thermonuclear parameters, $\omega_{pe} \approx 5 \times 10^{11}$ Hz. The neglect of $\partial \rho_e/\partial t$ in (II-A.20) also implies that we should neglect the displacement current $(1/c)\partial E/\partial t$ in (II-A.19). Neglecting the displacement current in this equation just results in Ampere's law (I-B.2),
\[ \nabla \times B = (4\pi/c)J. \]  
\text{(II-A.23)}

The other Maxwell equations, Faraday's law,

\[ \nabla \times \mathbf{E} = -(1/c)\frac{\partial \mathbf{B}}{\partial t}, \]  
\text{(II-A.24)}

and Gauss's law of magnetism,

\[ \nabla \cdot \mathbf{B} = 0, \]  
\text{(II-A.25)}

are unaffected by the plasma approximations.

Section II-B. Kinetic Theory

In order to continue the derivation of the generalized Ohm's law (II-A.3), we now digress a little from our discussion of MHD theory to discuss kinetic theory, which is the study of non-equilibrium transport processes in dilute gases. The central concept of kinetic theory is the phase space distribution function \( f_\alpha(x,v,t) \). The quantity \( f_\alpha(x,v,t)d^3xd^3v \) defines the average number of \( \alpha \)-species
plasma particles in the phase space volume element $d^3x d^3v$.

The kinetic description of a plasma is therefore more fundamental than the MHD description, since kinetic theory includes more information about the distribution of particle velocities than fluid theory.

The central equation of kinetic theory is called the kinetic equation. The kinetic equation gives the evolution of the distribution function,$^{50}$

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{x}} + (e_\alpha/m_\alpha)(\mathbf{E} + \mathbf{v} \times \mathbf{B}/c) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = C(f_\alpha). \quad (\text{II-B.1})$$

Here, the function $C(f_\alpha)$ represents the effects of particle correlations and is called the collision operator. That is, the collision operator incorporates the averaged effects of "random" phase space processes into the theory. In standard plasma kinetic theory, $C(f_\alpha)$ is the integro-differential Fokker-Planck operator and (II-B.1) is also known as the Fokker-Planck or Landau equation.$^{51}$

Using the distribution function $f_\alpha(x,v,t)$, the hydrodynamic variables density, fluid velocity, and pressure

---


tensor are defined as

\[ n(x,t) = \sum \alpha f_\alpha d^3v, \]  
\[ v(x,t) = \sum \alpha (1/n_\alpha) f_\alpha v d^3v, \]  
\[ p(x,t) = \sum \alpha m_\alpha f_\alpha (v-v_\alpha)(v-v_\alpha) d^3v. \]

Similarly, the self-consistent plasma charge and current densities (II-A.12) and (II-A.13), are defined from

\[ \rho_c = \sum \alpha e_\alpha f_\alpha d^3v \]  
\[ j = \sum \alpha e_\alpha f_\alpha v d^3v. \]

Definitions (II-B.5) and (II-B.6) and the previous discussion about plasma quasi-neutrality indicate that the electric and magnetic fields \( E \) and \( B \) of the Fokker-Planck or kinetic
equation (II-B.1) have been smoothed-out on the scale of the Debye length (II-A.17), a subtle and important point to which we will return later. The large-scale, smoothed-out $E$ and $B$ are the fields described by the Maxwell equations (II-A.23)-(II-A.25). In standard plasma kinetic theory, the collision operator $C(f_\alpha)$ incorporates the averaged effects of random fields on scales smaller than the Debye length.

Without going into detail about the rather complicated Fokker-Planck collision operator, let us remark on just what one means by a collision in a plasma. Typically, collisions refer to some kind of random process in the velocity part of phase space. Collisions are an important concept of kinetic theory and what one means by a plasma collision is a little bit different from the usual hard-sphere Boltzmann collisions of an ordinary charge-neutral gas. This is because plasma particles have an electric charge and the long-range nature of the Coulomb electrostatic force allows particles to interact over distances much greater than those of a hard-sphere gas. This modification means that a plasma particle actually feels many small kicks as it moves through the plasma, rather than the one large punch of an ordinary charge-neutral gas.

---

gas collision. The Fokker-Planck operator is therefore a diffusive, Brownian motion-like, velocity space operator. The subtlety in defining a plasma collision is that even though a plasma particle receives many small kicks, what one means by a plasma collision is that the plasma particle has received enough small kicks, on the average, to equal the net effect of one large punch.

An important result of the plasma Coulomb collision analysis is that the collision frequency $\nu_c$ in a plasma is inversely proportional to the cube of the particle speed,$^{53}$

$$\nu_c \propto v^{-3}. \quad \text{(II-B.7)}$$

The characteristic collision relaxation times in a plasma are therefore proportional to the three-halves power of the absolute temperature. For example, the characteristic relaxation time $\tau_e$ for electron collisions with ions is$^{54}$

$$\tau_e = \frac{(3/4) (m_e/2\pi)^{1/2}}{[n_e^4 \ln(\lambda_d/b)]^{-1}} T_e^{3/2}, \quad \text{(II-B.8)}$$


where the electron temperature $T_e$ is in energy units and the Coulomb logarithm $\ln(\lambda_d/b)$ is a numerical factor with $b$ the impact parameter for 90° scattering of thermal particles. The temperature and density dependences of the Coulomb logarithm are very weak and often ignored in plasma calculations. For thermonuclear parameters, $\ln(\lambda_d/b)$ has a value in the 15-20 range and the electron collision time $\tau_e \approx 0.1 \text{ ms}$. Because thermal ions are slower than thermal electrons, the ion collision time is just $\tau_i = (m_i/m_e)^{1/2}\tau_e$.

The thermonuclear values of the characteristic relaxation times $\tau_e$ and $\tau_i$ allow for a simplification of the pressure tensor $\mathbf{p}$ of equation (II-B.4). As mentioned in the Introduction, a modern tokamak can achieve an energy confinement time $\tau_E \approx 1 \text{ s}$. Then the collision times satisfy the inequalities

$$\tau_e \ll \tau_E \quad \text{(II-B.9)}$$

and

$$\tau_i \ll \tau_E \quad \text{(II-B.10)}$$
This means that the pressure tensor $\mathbf{p}$ has time to relax to the isotropic form

$$\mathbf{p} \approx p \mathbf{I}, \quad (\text{II-B.11})$$

where

$$p = (1/3) \Sigma_{\alpha} m_{\alpha} \int f_{\alpha}(v-v_{\alpha})^2 d^3v \quad (\text{II-B.12})$$

is just the scalar plasma pressure.

Before leaving our digression on kinetic theory, let us remark that kinetic theory presents a great challenge to theoretical plasma physicists. That is, while many useful plasma kinetic concepts have been developed,\textsuperscript{55} the experimental transport rates for such basic processes as electron heat transport in a tokamak are greater than the standard theoretical predictions by as many as two orders of magnitude.\textsuperscript{56} These enhanced experimental loss rates are somewhat sheepishly referred to as "anomalous"

\textsuperscript{55} F. L. Hinton and R. D. Hazeltine, Rev. Mod. Phys. 48, 239 (1976).

\textsuperscript{56} J. Hugill, Nuc. Fusion 23, 331 (1983).
transport.\textsuperscript{57} While an agreed upon theory of anomalous transport does not currently exist, explanations of the enhanced transport rates usually depend on the existence of some kind of turbulent plasma microinstabilities. Microinstabilities occur roughly on the scale of the ion gyroradius $\rho_i$ and are due to the non-Maxwellian character of the phase space distribution function. Clearly, plasma fluid theory is not capable of adequately analyzing the effects of anomalous transport. This is due to the fact that spatial scales on the order of the ion gyroradius are outside the domain of any plasma fluid theory, according to (II-A.2). Anomalous transport therefore requires a full non-linear kinetic treatment, which once again explains our lack of theoretical understanding of plasma confinement.

Section II-C. Fluid Equations

Plasma fluid equations are obtained by taking velocity moments of the kinetic equation (II-B.1). In fact, it is a well-known theorem of probability theory that a probability distribution function can be reconstructed by giving all of the

moments of the probability distribution function. Usually, the plasma fluid equations involve only the first three velocity moments. These three moments then lead to equations for the three fluid variables, \( n(x,t) \), \( u(x,t) \), and \( p(x,t) \). As noted above, the macroscopic plasma variables therefore contain less information about the plasma particle motions than the distribution function \( f_{\alpha}(x,v,t) \).

Here, we will confine our attention to the equations for the two lowest order velocity moments, 1 and \( v \). However, from an examination of the moment equations, it becomes apparent that each order of these equations involves a moment of one higher order. For example, the equation for the plasma density \( n \) will involve the plasma velocity \( u \) and the equation for the plasma velocity \( v \) will involve the plasma pressure \( p \). This means that some assumption as to the form of the highest order plasma variable, say, the pressure \( p \) must be made in order to close the moment equations.

We begin by multiplying the kinetic equation (II-B.1) by 1 and integrating over the ion velocity space. This yields the plasma continuity equation.

\[ \frac{\partial n}{\partial t} + \nabla \cdot (nv) = 0, \quad (II-C.1) \]

where we have used the definitions (II-B.2) and (II-B.3) along with the approximations (II-A.10) and (II-A.11). We have also used the property that the collision operator \( C(f_\alpha) \) conserves particles,

\[ \int C(f_\alpha) d^3v = 0. \quad (II-C.2) \]

Similarly, multiplying the kinetic equation by \( m_\alpha v \), integrating over velocity space, and adding the ion and electron equations yields

\[ \rho_m \frac{dv}{dt} = -\nabla p + j x B/c. \quad (II-C.3) \]

Here, we have used (II-C.1), charge neutrality (II-A.18), and once again have used the approximation of neglecting the electron inertia. Also, if the collisional force \( R_\alpha \) is defined as

\[ e_\alpha n_\alpha R_\alpha = -\int m_\alpha v C(f_\alpha) d^3v, \quad (II-C.4) \]
then by the conservation of total momentum

$$\Sigma_{\alpha} e_{\alpha} \nabla_{\alpha} R_{\alpha} = 0.$$  \hfill (II-C.5)

For sufficiently slow processes, all the inertial forces can be ignored and the dominant forces are the pressure gradient and the magnetic force. Then (II-C.3) becomes

$$\nabla p = j \times B / c,$$  \hfill (II-C.6)

which is just the plasma equilibrium equation (I-B.1).

We are now in a position to give a derivation of the generalized Ohm's law (II-A.3). Ohm's law can be interpreted to represent the force balance of the electron plasma species. Taking the $m_e v$ moment of the electron kinetic equation and neglecting electron inertia yields

$$(E + v_e \times B / c) = -\nabla p_e / e n_e + R_e.$$  \hfill (II-C.7)

Comparing with the electron force balance equation (II-A.16), we see that the Ohm's law (II-C.7) has the additional
collisional force term $R_e$. This is the point to which we alluded in the section on kinetic theory and which we now want to emphasize. That is, the electric and magnetic fields, $E$ and $B$, which appear in the Ohm's law (II-C.7) are averaged fields. In this case, the fields are averaged over effects on the scale of the Debye length. The collisional force term $R_e$ then represents the averaged effects of electrostatic fluctuations on scales smaller than the Debye length. This is a specific case of a subtle but important principle: whenever new averaging procedures are introduced into the plasma fluid description, new terms appear in Ohm's law. The physical and logical status of the plasma Ohm's law is therefore much more involved than that of the Maxwell equations, which are assumed to be universally valid in plasma physics.

From equations (II-A.11) and (II-A.13), we have

$$v_e = v - j/en.$$  \hspace{1cm} (II-C.8)

Combining (II-C.7) and (II-C.8) results in

$$E + v \times B/c = R_e + (1/en)(j \times B/c - \nabla p_e).$$ \hspace{1cm} (II-C.9)
From the equilibrium equation (II-C.6) the Hall term $j \times B/c$ and the diamagnetic term $\nabla p_e$ are comparable. A simple scaling argument then shows that

$$\left| \nabla p_e / en \right| / \left| j \times B/c \right| \approx \rho_i/a. \quad (II-C.10)$$

Using the condition of small ion gyroradius (II-A.2), we ignore the Hall and diamagnetic terms, so that the equation (II-C.9) becomes

$$E + j \times B/c = R_e. \quad (II-C.11)$$

which is essentially the generalized Ohm's law (II-A.3).

To complete the derivation, we need only show the equivalence between the collisional force and the resistive drag term

$$R_e = \eta j. \quad (II-C.12)$$

In order to do this, we employ a relaxation time approximation for the collision operator,
\[ C(f_e) \approx - (f_e - f_M)/\tau_e. \]  

(II-C.13)

where \( \tau_e \) is given by (II-B.8) and \( f_M(x,v,t) \) is the local moving Maxwellian function

\[ f_M = n \left[ \frac{2\pi kT_e}{m_e} \right]^{-3/2} \exp\left\{ -\frac{m_e}{2kT_e} \left[ v - \nu_i \right]^2 \right\}. \]  

(II-C.14)

Combining (II-B.6), (II-C.4), (II-C.13), and (II-C.14) then results in equation (II-C.12) with the resistivity

\[ \eta = \frac{m_e}{n_k} e^2 \tau_e. \]  

(II-C.15)

Note that the expression for the plasma resistivity (II-C.15) is of the same form as that of a typical metal. As remarked in the Introduction, thermonuclear plasmas are highly conducting, with conductivities \( \sigma = \eta^{-1} \) greater than that of copper. Using the expression for the electron collision time \( \tau_e \) (II-B.8), the dominant scaling for the plasma resistivity is

\[ \eta \propto T_e^{-3/2}. \]  

(II-C.16)
A more sophisticated kinetic calculation of the resistivity based on the Fokker-Planck collision operator has been carried out by Spitzer.\textsuperscript{59,60} For a plasma imbedded in a magnetic field, the so-called classical Spitzer resistivity is actually a tensor with

\begin{equation}
\eta_{\parallel} = 0.51 \frac{m_e}{n e^2} \tau_e \tag{II-C.17}
\end{equation}

and

\begin{equation}
\eta_{\perp} = \frac{m_e}{n e^2} \tau_e \tag{II-C.18}
\end{equation}

for directions parallel and perpendicular to the magnetic field.

\textsuperscript{59}L. Spitzer, Jr. and R. Härm, Phys. Rev. 89, 977 (1953).
\textsuperscript{60}L. Spitzer, Jr., Physics of Fully Ionized Gases, (Interscience, New York, 1956), pp. 81-86.
CHAPTER III. MAGNETIC FIELDS

Section III-A. Canonical Coordinates

In the next several sections, we discuss representations of the magnetic field $B(x)$. A fundamental requirement of any representation of the magnetic field is that it must have the divergence-free property (II-A.25). For example, the most common and well-known representation of the magnetic field involves the vector potential $A$,

$$B = \nabla \times A. \quad (III-A.1)$$

Since the divergence of the curl of any well-behaved vector field vanishes, the vector potential representation (III-A.1) insures that the magnetic field obeys (II-A.25). In fact, the vector potential $A$ is a very useful construct and is often employed in plasma calculations.

In this work, however, we will find it more convenient to use the so-called canonical or Hamiltonian representation of the magnetic field.\(^{61}\)

---


59
Because the divergence of crossed gradients vanishes, the Hamiltonian representation (III-A.2) also has the divergence-free property (II-A.25). The Hamiltonian form is not well-known outside the plasma physics community, but it provides a very simple and elegant representation for toroidal magnetic fields. Before commenting on the meaning of the so-called Boozer coordinates $\psi$, $\theta$, and $\phi$, let us consider the motivation for the representation (III-A.2).

A strong incentive for adopting the Hamiltonian form (III-A.2) is that it allows us to use the powerful theoretical methods of Hamiltonian mechanics$^{62,63}$ to describe magnetic fields. A Hamiltonian mechanical system can be described by a set of differential equations,

$$\frac{dq}{dt} = +\frac{\delta H}{\delta p}$$  

(III-A.3)


and

\[
\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad (III-A.4)
\]

where \( q \) and \( p \) are the canonical coordinates and momenta and \( t \) is the time. The function \( H = H(q, p, t) \) is called the Hamiltonian. The phase space \( (q, p) \) structure of a mechanical system is completely described by the Hamiltonian function. From Hamilton's equations (III-A.3) and (III-A.4), it is easy to show that the trajectories of the phase space flow field

\[
V = \left( \frac{dq}{dt}, \frac{dp}{dt} \right) \quad (III-A.5)
\]

of a Hamiltonian mechanical system obey the phase space flow equation

\[
\nabla \cdot V = 0. \quad (III-A.6)
\]

[Here, the operator \( \nabla = (\partial/\partial q, \partial/\partial p) \) is the phase space divergence.] Equation (III-A.6) is known as Liouville's equation and can be used to prove that the phase space flow of a Hamiltonian mechanical system is equivalent to that of
an incompressible fluid.

In order to obtain the connection with Hamiltonian mechanics, note that a magnetic field line can be represented parametrically as a solution of the equation

\[ B = \frac{dx}{dl}. \]  \hspace{1cm} (III-A.7)

Here, the quantity \( l \) is a time-like parameter which increases along the direction of a magnetic field line. Mathematically, the magnetic field line equations (III-A.7) and (II-A.25) are completely identical in form to the phase space trajectory equations (III-A.5) and (III-A.6). Using this mathematical analogy, Gauss's law of magnetism (II-A.25) can be interpreted as a kind of "magnetic Liouville equation." The important point is that the trajectories of the magnetic field, the field lines, have exactly the same structure as phase space particle trajectories. That is, a magnetic field is structurally equivalent to a Hamiltonian mechanical system.

To make this point more explicit, let us return to the interpretation of the Boozer coordinates \( \psi, \theta, \) and \( \varphi \). The interpretation is most easily made by simultaneously referring to the torus (see Fig. 1). First, the quantities \( \theta \) and
φ represent angles which parameterize the torus in the poloidal and the toroidal directions, respectively. Using the toroidal and poloidal area elements,

\[ da_\phi = \nabla \psi d\psi d\theta / (\nabla \psi \times \nabla \theta) \cdot \nabla \psi \]  

(III-A.8)

and

\[ da_\theta = \nabla \theta d\chi d\phi / (\nabla \chi \times \nabla \theta) \cdot \nabla \phi, \]  

(III-A.9)

and the Hamiltonian form (III-A.2), it is straightforward to show that

\[ \psi = \int B \cdot da_\phi \]  

(III-A.10)

and

\[ -\chi = \int B \cdot da_\theta. \]  

(III-A.11)

That is, the radial coordinate \( \psi \) is the toroidal magnetic flux enclosed by a constant-\( \psi \) surface, while \( -\chi \) is the poloidal
magnetic flux outside a constant-\( \chi \) surface. Next, if the canonical form (III-A.2) is coupled to the field line trajectory equation (III-A.7) and the chain rule, it is trivial to show that

\[
\frac{d\theta}{d\psi} = +\frac{\partial \chi}{\partial \psi}
\]

(III-A.12)

and

\[
\frac{d\psi}{d\phi} = -\frac{\partial \chi}{\partial \theta},
\]

(III-A.13)

which are just Hamilton's equations (III-A.3-4) with \( \theta \) the "canonical coordinate," \( \psi \) the "canonical momentum," \( \phi \) the "time," and \( \chi(\psi,\theta,\phi) \) the "Hamiltonian." So a magnetic field is equivalent to a one-dimensional, time-dependent Hamiltonian system, with the negative poloidal flux \( \chi=\chi(\psi,\theta,\phi) \) the magnetic Hamiltonian.

The Hamiltonian nature of the magnetic field has several implications. First, since the magnetic Hamiltonian \( \chi \) contains all topological information about the field, a calculation of \( \chi \) completely characterizes the structure of the magnetic field. Second, the magnetic field is an example
of the simplest possible Hamiltonian system which still contains the complex topological structures of classical mechanics. This is because a one-dimensional, time-independent Hamiltonian system is always integrable. In this sense, the magnetic field can be considered as the prototypical Hamiltonian system. The modern theory of Hamiltonian mechanics, often called Kolmogorov-Arnold-Moser (KAM) theory, and the topology of complex magnetic fields will be discussed further in the chapter on tearing modes.

Section II-B. Magnetic Coordinates

Before considering the case of more complicated magnetic fields, let us focus our attention on the simpler case alluded to in the last section, that of an integrable magnetic field. We begin the discussion by noting that a one-dimensional, time-dependent Hamiltonian system, such as the magnetic field, is completely equivalent to a two-dimensional, time-independent Hamiltonian system. That is, the space of the Boozer coordinates can be extended from one to two

dimensions by treating the angle $\phi$ and the flux $-\chi$ as canonically conjugate variables. This procedure is completely analogous to extending the phase space of a classical mechanical system by treating the time $t$ and the negative energy $-H$ as canonically conjugate variables. If magnetic surfaces exist, the motion of a magnetic field line can then be viewed as motion on a 2-torus (see Fig. 1), with the two canonical pairs $(\psi, \theta)$ and $(-\chi, \phi)$ forming action-angle coordinates.

In the language of plasma physics, the action-angle coordinates of the magnetic field are called magnetic coordinates. In magnetic coordinates, the magnetic Hamiltonian $\chi$ is a function of the toroidal flux $\psi$ alone,\textsuperscript{65,66,67}

$$B = \nabla\psi \times \nabla(\theta/2\pi) - \nabla\chi(\psi) \times \nabla(\phi/2\pi).$$

That is, if magnetic coordinates exist, the constant-$\chi$ surfaces coincide with the constant-$\psi$ surfaces and the magnetic field is said to be integrable. Magnetic coordinates

\textsuperscript{65}M. D. Kruskal and R. M. Kulsrud, Phys. Fluids 1, 265 (1958).
give a transparent meaning to the Boozer coordinate system and the Hamiltonian representation of the magnetic field.

Magnetic coordinates exist if there are magnetic surfaces. As mentioned earlier, a region of good magnetic surfaces is essential for successful toroidal magnetic confinement. Mathematically, a necessary and sufficient condition for the existence of magnetic coordinates is that there exists a function \( f(x) \) such that

\[
\nabla f = 0, \quad (\text{III-B.2})
\]

where \( |\nabla f| \neq 0 \), except on isolated curves. As remarked in the Introduction, if an equilibrium plasma has a pressure gradient, then the pressure function \( p(\psi) \) insures the existence of magnetic surfaces.

From the magnetic coordinate representation (III-B.1), the magnetic field line trajectories are given by the equations

\[
B \cdot \nabla \psi = 0 \quad (\text{III-B.3})
\]

and
where the rotational transform \( \lambda(\psi) \) is defined by

\[
\lambda = \frac{d\chi}{d\psi}. \tag{III-B.5}
\]

If magnetic surfaces exist, therefore, the motion of a field line becomes localized and a twist is induced around toroidal magnetic surfaces according to the equations

\[
\psi = \psi_0 \tag{III-B.6}
\]

and

\[
\theta = \theta_0 + \lambda(\psi)\phi. \tag{III-B.7}
\]

The rate of twisting is given by the rotational transform \( \lambda(\psi) \), which is just the ratio of the number of poloidal circuits completed by a field line to the number of completed toroidal circuits.

The rotational transform profile \( \lambda(\psi) \) is of great
importance for tokamak magnetic confinement because it determines the locations of possible tearing mode instabilities due to unfavorable current density profiles.\textsuperscript{68} Typically, the tokamak $\nu(\psi)$ is a monotonically decreasing function of the toroidal flux with a center value $\nu(\psi=0)\approx 1.0$ and an edge value $\nu(\psi=\pi a^2 B_0)\approx 1/3$. The important point is that $\nu(\psi)$ is a continuous function of $\psi$, taking on both rational and irrational values. If the value of the rotational transform $\nu(\psi_0)$ is irrational, then the field line of equations (III-B.6)-(III-B.7) covers the entire toroidal magnetic surface. If, however, $\nu(\psi_0)$ takes on a rational value, then the field line bites its own tail as it twists around the torus and fails to map out the complete toroidal surface.

The issues raised in the last paragraph are worth reiterating because physics teachers frequently tell their students that magnetic field lines must make closed loops, which is manifest nonsense. In fact, Gauss's law of magnetism (II-A.25) merely states that field lines cannot diverge from points, that is, do not begin or end.\textsuperscript{69} This is


clearly the case for irrational values of rotational transform, even though these field lines do not form closed loops. It is only in highly symmetrical situations that field lines actually form closed loops. Nevertheless, the positions where field lines do form closed loops are extremely important for magnetic confinement because at these positions the field lines cannot, in effect, sample the entire magnetic surface. Because of this effect, the magnetic surfaces with rational values of rotational transform, also called rational surfaces, can cause a weakness in the magnetic confinement structure or even instability. Instabilities can lead to the destruction of magnetic surfaces. Whether an instability actually occurs will be discussed in the chapter on tearing modes.

To close this section, let us remark that, because the rational numbers are dense on the number line, one might conclude that the instabilities due to the rational values of the rotational transform would cause an insuperable problem for magnetic confinement. That is, within any region of the plasma, no matter how small, there are always rational values of the rotational transform. By one of Cantor's famous set-theoretic results, however, the rational numbers form a set of measure zero. That is, there are infinitely more
irrational numbers than rational numbers on the number line. This might lead one to the exact opposite conclusion, that the rational values of rotational transform are of no consequence for magnetic confinement. According to KAM theory, however, neither of these conclusions is correct. These considerations give some indication of the rather subtle physics involved with a complex Hamiltonian system such as the magnetic field.

Section III-C. Magnetic Diffusion Equation

As mentioned in section on canonical coordinates, a calculation of the magnetic Hamiltonian or (negative) poloidal flux function $\chi$ completely characterizes the topological structure of the magnetic field. The calculation of $\chi$ is therefore of great importance to our study of the tokamak current. Eventually, we will calculate $\chi$ by integrating a certain fourth-order, parabolic partial differential equation. In fact, this differential equation is just the mean-field Ohm's law (I-D.3). However, because of the complexity of

---

this equation, we do not yet attempt a complete analysis. Rather, in this section, as a preliminary to that analysis, we present a simpler discussion of the flux evolution based on the generalized Ohm's law (II-A.3). This discussion emphasizes the underlying structure of Ohm's law, which is that of a magnetic diffusion equation.

The only part of the Ohm's law (II-A.3) necessary for the determination of the flux evolution $\frac{d\chi}{dt}$ is the parallel component,

$$E \cdot B = \eta \ j \cdot B.$$  \hfill (III-C.1)

From the parallel Ohm's law (III-C.1) or (I-D.3), it is important to note that the flux evolution only involves the parallel current distribution,

$$\mu = (4\pi/c) j \cdot B / B^2.$$ \hfill (III-C.2)

This means that plasma equilibrium (II-C.6), which depends only on the perpendicular components of the plasma current, is completely separated from the flux evolution. In fact, force balance and the perpendicular part of Ohm's law
determine the quantity $\frac{dx}{dt}$, where $x(\psi, \theta, \varphi)$ are called the transformation equations.\footnote{A. H. Boozer, "Plasma Confinement," in Encyclopedia of Physical Science and Technology, (Academic Press, Orlando, Florida, 1987), Vol. 10, p. 680.} Together, the transformation equations $x(\psi, \theta, \varphi)$ and the magnetic Hamiltonian $\chi(\psi, \theta, \varphi)$ completely specify the magnetic field $B(x)$. In our theory, the transformation equations are assumed fixed, and our interest is focused entirely on the flux evolution.

An important simplification of our analysis of the tokamak current results from replacing the complicated geometry of the tokamak torus with that of a simple periodic cylinder. A periodic cylinder with length $2\pi R_0$ is topologically equivalent to a torus with major radius $R_0$, as can be seen from slicing a torus, unfolding it into a cylinder, and identifying the ends. The transformation equations from the Boozer coordinates $(\psi, \theta, \varphi)$ to the cylindrical coordinates $(r, \theta, z)$ are

\begin{align*}
    r &= (\psi / \pi B_0)^{1/2}, \\
    \theta &= \theta, \\
    z &= \varphi R_0. \\
\end{align*}

(III-C.3)
with $B_0$ the large constant toroidal magnetic field. That is, the radial flux coordinate $\psi$ is represented as the cylinder radius $r$ and the toroidal angle $\varphi$ is equivalent to the longitudinal coordinate $z$.

The physical justification of the applicability of the procedure (III-C.3) to our research rests on the observation that the energy source for tearing modes is the poloidal magnetic field.\(^{72}\) Formally, the cylindrical approximation is realized by performing asymptotic expansions of all physical quantities in terms of the inverse aspect ratio,\(^{73}\)

$$\epsilon = a/R_0, \quad (\text{III-C.4})$$

where $a$ and $R_0$ are the plasma minor and major radius, respectively. Typical tokamak values for the inverse aspect ratio (III-C.4) are in the range of $1/7$ to $1/3$. We therefore order all quantities in terms of the (assumed) small parameter $\epsilon$. First, note that the cylindrical approximation means that all quantities are spatial functions of the radial coordinate $r$ alone. Also, the rotational transform (III-B.4) in


cylindrical coordinates is

\[ \ell(r) = R_0B_\theta/rB_0. \]  

(III-C.5)

Since the magnetic field must satisfy the divergence-free condition (II-A.25) and the rotational transform (III-C.5) is of \( O(1) \), a consistent ordering for the magnetic field is\(^{74}\)

\[ B_z \approx B_0(1 + \varepsilon^2), \]
\[ B_\theta \approx \varepsilon B_0, \]
\[ B_r \approx 0. \]  

(III-C.6)

From Ampere's law (II-A.23) and (III-C.6), the ordering of the current density is

\[ j_z \approx \varepsilon j_0, \]
\[ j_\theta \approx \varepsilon^2 j_0, \]
\[ j_r \approx 0. \]  

(III-C.7)

where \( j_0 = (c/4\pi)(B_0/a) \). The induction electric field is ordered from Faraday's law (II-A.24) and (III-C.6),

\[ E_z \approx \varepsilon E_0, \]
\[ E_\theta \approx \varepsilon^2 E_0, \]
\[ E_r \approx 0, \] \hspace{1cm} (III-C.8)

where \( E_0 = \eta j_0 \).

Combining (III-C.6)-(III-C.8) and the parallel Ohm's law (III-C.1), we find the approximation

\[ E_2 B_0 = \eta \ j_2 B_0, \] \hspace{1cm} (III-C.9)

where \( \eta = \eta(r,t) \) and we have ignored terms of \( O(\varepsilon^2) \) smaller than those kept. The evolution of the flux function \( \chi = \chi(r,t) \) is obtained by incorporating the magnetic coordinate representation (III-B.1) along with the transformation equations (III-C.3) into equation (III-C.9). From Ampere's law (II-A.23), the toroidal current density \( j_z \) is

\[ j_z = \left( \frac{c}{4\pi} \right) \left( \frac{1}{2\pi R_0} \right) \nabla^2 \chi, \] \hspace{1cm} (III-C.10)

where the radial Laplacian operator acts as
\( \nabla^2 \chi = \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r} \frac{\partial \chi}{\partial r}. \) \hspace{1cm} (III-C.11)

From Faraday's law (II-A.24), the toroidal electric field \( E_Z \) is

\[ E_Z = \left( \frac{1}{2\pi R_0} \right) V, \] \hspace{1cm} (III-C.12)

where the quantity

\[ V = \frac{1}{c} \frac{\partial \chi}{\partial t} \] \hspace{1cm} (III-C.13)

is known as the loop voltage. Equations (III-C.9), (III-C.10), (III-C.12), and (III-C.13) then yield

\[ \frac{\partial \chi}{\partial t} = \left( \frac{\eta c^2}{4\pi} \right) \nabla^2 \chi, \] \hspace{1cm} (III-C.14)

which is a magnetic diffusion equation with diffusion coefficient \( \eta c^2/4\pi \). Scaling (III-C.14), we expect that the poloidal magnetic flux will diffuse into the plasma with the characteristic time constant

\[ \tau_\chi = \frac{4\pi a^2}{\eta c^2}. \] \hspace{1cm} (III-C.15)
A typical "skin time" \( \tau_{\chi} \) for thermonuclear plasma parameters is \( \tau_{\chi} \approx 10^3 \) s.

Before leaving this section, let us comment on the role of the loop voltage (III-C.13) during the evolution of a tokamak plasma. In a tokamak, the loop voltage is created by transformer action. That is, a time-varying poloidal magnetic field is created by a set of external field coils in the central hole of the tokamak torus. Typically, a large current is built-up in these field coils by converting electrical or mechanical energy stored in capacitor banks or flywheels into magnetic energy. When this so-called primary current reaches its maximum, the discharge is initiated, and the poloidal magnetic flux begins to decrease. The highly conducting tokamak plasma then acts as a secondary transformer circuit and a toroidal current is induced internally according to Ohm's law (III-C.14) with the consequent Ohmic heating

\[
P_{\text{Ohm}} = \eta j_z^2. \tag{III-C.16}
\]

where we use equations (III-C.7) and again ignore terms of \( O(\varepsilon^2) \) smaller than those kept. This is the standard method
for creating and maintaining the tokamak toroidal current.\textsuperscript{75}

There are several interesting issues to discuss here. First, the exact value of the loop voltage depends on the particular tokamak discharge operation phase. That is, combining equations (III-C.16), (III-C.9), and (II-C.16), the start-up phase of tokamak operation has a heating power density $P_{\text{Ohm}} = E_z^2/\eta \propto T_e^{3/2}$. The initial low temperature or "burn-through" phase of the tokamak discharge therefore requires large loop voltages, on the order of $V \approx 100$ Volts. After this initial stage, the loop voltage settles down to a typical plateau value $V \approx 1$ Volt. Second, it is observed that there is little if any skin effect during the initial stages of tokamak experiments. That is, the poloidal flux and toroidal current penetrate into the interior of the plasma in an anomalously rapid manner, much faster than expected from the typical values of $\tau_X$.\textsuperscript{76,77} The explanation of this effect is apparently due to the behavior of tearing modes and will be discussed further in the chapter on inductively driven tokamaks. Lastly, we mention that if the applied loop voltage

\textsuperscript{76}S. V. Mirnov and I. B. Semenov, Atomnaya Energiya 30, 20 (1971).
(III-C.13) is held constant for a long enough time, the magnetic Hamiltonian becomes

\[ \chi(r,t) = \chi_0(r) + V_0 t. \]  

That is, the loop voltage becomes a constant across the plasma with the value \( V=V_0 \) set by the time-varying poloidal field of the transformer in the torus hole. The inductive loop voltage therefore provides a simple way to create quasi-steady tokamak plasma conditions. Unfortunately, however, the inductive loop voltage \( V \) is only of finite duration, since any real transformer contains a limited amount volt-seconds or swing in poloidal magnetic flux. A tokamak with only inductively driven plasma currents must therefore operate in a pulsed mode, with the length of the pulses determined by the flux capacity of the transformer. The maximum inductively driven tokamak pulse length would be on the order of 1000 seconds.\(^78\) Of course, a tokamak operating in such a pulsed mode does not seem to be a very practical fusion reactor. It is this simple, but fundamental problem which causes us to seek other, non-inductive methods for maintaining the tokamak current.

CHAPTER IV. TEARING MODES

Section IV-A. Ideal MHD

The tearing mode is a plasma instability which arises from the effects of finite plasma resistivity $\eta$. In the next several sections, we discuss subjects which are relevant to understanding the effects of tearing modes in tokamaks. Before we begin our discussion, however, let us recall that the tokamak value of $\eta$ is in some sense small, since the tokamak plasma is an excellent conductor of electricity. In fact, since plasma resistivities are generally very small, the approximation is often made that a plasma is an ideal or perfect electrical conductor. This is particularly true with astrophysical plasmas.\(^7^9\) However, there are many subtleties contained in so-called ideal MHD theories based on the assumption of perfect plasma conductivity. Because of the role played by resistivity in the tokamak tearing mode, the subtleties of ideal MHD theory form the subject matter of this section.

In general, the problem with the assumption of perfect plasma conductivity is that it places an unrealistic constraint on the behavior of the plasma magnetic field. For example, setting the resistivity $\eta=0$ in the magnetic diffusion equation (III-C.14), we find the poloidal flux evolution

$$\frac{\partial \chi}{\partial t} = 0.$$  \hspace{1cm} (IV-A.1)

That is, poloidal flux cannot diffuse into a perfectly conducting plasma and the tokamak magnetic topology is completely independent of time. Even if the poloidal flux content of the plasma is assumed non-zero, so that the tokamak has an initial current profile $j_z(r)$, the ideal evolution equation (IV-A.1) means that the underlying structure of the plasma magnetic field does not alter with time. This is certainly not the experimentally observed behavior of the tokamak magnetic field.$^{80}$

Let us, therefore, examine the ideal MHD evolution of a plasma magnetic field more closely. We begin our discussion by setting the resistivity $\eta=0$ in the generalized Ohm's law

---

Combining the so-called ideal Ohm’s law (IV-A.2) and Faraday’s law (II-A.24), we have the magnetic field evolution

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B).$$  \hspace{1cm} (IV-A.3)

The meaning of ideal evolution equation (IV-A.3) has been discussed by Alfven.\textsuperscript{81,82} In a very famous description of the ideal MHD model, Alfven stated that the magnetic field behaves as if it were “frozen” into the perfectly conducting plasma fluid. This description has great intuitive appeal. That is, if the motion of the plasma fluid is smooth and continuous and the magnetic field is “frozen” into the plasma fluid, then the motion of the magnetic field must also be smooth and continuous. This is why the condition of zero resistivity prevents a plasma magnetic field from altering its underlying topology.


Alfvén's somewhat qualitative "frozen" field description can be translated into the form of a mathematical conservation theorem. The conservation theorem states that the total magnetic flux through any closed contour moving with the ideal MHD fluid stays fixed in time. The proof follows directly from Gauss's theorem,

$$d\int B \cdot dS/dt = \oint \partial B/\partial t \cdot dS + \oint (B \times \mathbf{v}) \cdot d\mathbf{l} + \oint (\nabla \cdot B) \mathbf{v} \cdot dS. \quad (IV-A.4)$$

Here, the first term on the right represents the change in the magnetic field, the second term represents the motion of the closed contour, and the third term represents motion through sources. From the divergence-free condition (II-A.25), the source term vanishes. Then, substituting the ideal MHD evolution equation (IV-A.3) into Gauss's theorem (IV-A.4), we have $d(\int B \cdot dS)/dt = 0$, which shows that the magnetic flux through any closed contour is conserved.

Of course, any real plasma will not be perfectly conducting, so Alfvén's topology conservation theorem does not rigorously hold. That is, the resistivity $\eta \neq 0$. From a kind of perturbation analysis, however, one might argue that small
values of the resistivity $\eta$ can only lead to small changes in the magnetic topology. That is, one might expect that the $\eta \to 0$ limit of resistive MHD is ideal MHD. Although this kind of perturbation analysis is used by physicists in a variety of situations, modern developments in analytical mechanics have shown that this kind of analysis is also often not valid. In fact, the $\eta \to 0$ limit of resistive MHD does not, in general, yield ideal MHD.

While the type of behavior exhibited by resistive MHD in the limit $\eta \to 0$ is subtle, it is certainly not unprecedented. In fact, the zero resistivity limit of MHD is completely analogous to zero viscosity limit of hydrodynamics. That is, if the generalized Ohm's law (II-A.3) is combined with Faraday's law (II-A.24), we have

$$\frac{\partial B}{\partial t} = (\eta c^2/4\pi) \nabla^2 B + \nabla \times (\mathbf{v} \times B). \quad (IV-A.5)$$

Here, we have assumed the resistivity $\eta$ to be constant. Scaling equation (IV-A.5) with $\nabla = (1/a) \nabla', \mathbf{v} = v_0 \mathbf{v}', \quad t = (a/v_0) t', \quad$ and $B = B_0 B'$, and dropping primes, we have

$$\frac{\partial B}{\partial t} = (1/R_m) \nabla^2 B + \nabla \times (\mathbf{v} \times B). \quad (IV-A.6)$$
where dimensionless parameter

\[ R_m = (4\pi/\eta c^2) u_0 a \]  \hfill (IV-A.7)

is called the magnetic Reynolds number.\textsuperscript{83} A typical tokamak magnetic Reynolds number is very large, perhaps greater than \(10^6\). It is also important to note that the magnetic Reynolds number obeys

\[ R_m = \tau_X/\tau_{\text{MHD}}. \]  \hfill (IV-A.8)

where the resistive time \(\tau_X\) is given by (III-C.15) and the ideal MHD time \(\tau_{\text{MHD}}\) is defined by

\[ \tau_{\text{MHD}} = a/\nu_0. \]  \hfill (IV-A.9)

In hydrodynamics, on the other hand, the vorticity \(\omega = \nabla \times v\) obeys the evolution equation\textsuperscript{84,85}

\textsuperscript{84}H. Helmholtz, Crelles J. 55, 25 (1858).
\[ \frac{\partial \omega}{\partial t} = (v/\rho_m) \nabla^2 \omega + \nabla \times (v \times \omega). \tag{IV-A.10} \]

Here, we have assumed the mass density \( \rho_m \) and the fluid viscosity \( v \) both to be constant. Scaling equation (IV-A.10) with \( \nabla = (1/a) \nabla' \), \( v = v_0 \nu' \), \( t = (a/v_0) t' \), and \( \omega = (v_0/a) \omega' \), and dropping primes, we have

\[ \frac{\partial \omega}{\partial t} = \left(1/R_f\right) \nabla^2 \omega + \nabla \times (v \times \omega), \tag{IV-A.11} \]

where the dimensionless parameter

\[ R_f = (\rho_m/v)v_0a \tag{IV-A.12} \]

is the called the fluid Reynolds number. Except for the fact that the vorticity \( \omega \) involves the fluid velocity \( v \), the magnetic field equations (IV-A.5)-(IV-A.7) and vorticity equations (IV-A.10)-(IV-A.12) are exactly the same. Now, it is well-known from both theory and experiment in hydrodynamics that the small fluid viscosity or large fluid Reynolds number limit (\( v \rightarrow 0 \) or \( R_f \rightarrow \infty \)) does not necessarily yield the smooth laminar flow of ideal fluid potential.
theory.\textsuperscript{86} In fact, the famous physicist von Neumann has characterized the $v=0$ ideal potential flow of water as the flow of "dry water." That is, fluid viscosity is an essential property of a fluid and really cannot be neglected. Similarly, the plasma resistivity is an essential property of MHD and really cannot be neglected.

A way to understand the subtle nature of the $\eta \to 0$ or $R_m \to \infty$ magnetic field problem is the following simple model. Suppose initially that the resistivity $\eta$ is very small and that everything is smooth and continuous. Then the magnetic field evolves nearly according to the ideal evolution equation (IV-A.3). Suppose now, however, that a region of turbulence develops in the plasma fluid. If the plasma fluid starts to twist and turn, then the magnetic field must also twist and turn. This is because the magnetic field is "frozen" into the fluid, at least for the short time scale $\tau_{\text{MHD}}$. The important point is that, while the magnetic Hamiltonian $\chi(\psi,\theta,\Phi,t)$ evolves on the slow time scale $\tau_{\chi}$ and can be considered fixed on the time scale $\tau_{\text{MHD}}$, the transformation equations of the Boozer coordinates $x(\psi,\theta,\Phi,t)$ can evolve very rapidly.

so the magnetic field $B(x,t)$ can change spatially on the fast time scale $\tau_{\text{MHD}}$. This is the key to understanding the large magnetic Reynolds number behavior of a magnetic field. That is, a highly jumbled plasma fluid means sharp gradients in the plasma magnetic field $B$. Examining the full magnetic field evolution equation (IV-A.5), we see that even though the resistivity $\eta$ is small in this equation, it is multiplied by $\nabla^2 B$, which is the highest spatial derivative in the magnetic field evolution. Since there are sharp gradients in $B$, the quantity $\nabla^2 B$ can be quite large, even if $\nabla^2 B$ represents the effects of small perturbations. In fact, the product $\eta \nabla^2 B$ usually does not vanish in the limit $\eta \to 0$ and must be kept. This subtle kind of limit is called a singular limit.87

The exact same singular limit occurs in the magnetic diffusion equation (III-C.14). Actually, since the function $\nabla^2 \chi$ is the only spatially varying term in this equation, it should always be kept. However, even if $\nabla^2 \chi$ represents the effects of small perturbations, it can be a very spikey function. The product $\eta \nabla^2 \chi$ can therefore be much larger than one might naively expect and must be kept. From the

relation (III-C.10), this means that the underlying topology of a tokamak magnetic field can be altered in the neighborhood of a spikey perturbation current density $j_z$. Notice that underlying topological changes are always modeled by the highest spatial derivative term in evolution equations, an important point to which we will return later.

We end this section by presenting a simple model of solar flare formation. This astrophysical plasma example is of interest because it provides a rather dramatic illustration of what can happen when a plasma magnetic field undergoes a topology altering transformation. First, from the observation of sunspots, it is known that tubes of magnetic flux are constantly rising upwards in the solar atmosphere. The flux tubes rise because differences in the solar plasma densities inside and outside of the tubes create Archimedean buoyancy forces, thus lifting the tubes to the solar surface. As the flux tubes rise, they become tangled and twisted, and large magnetic field gradients soon begin to appear. In fact, if the magnetic field polarity of neighboring flux tubes is oppositely directed, the field gradients become so steep that strong resistive dissipation (III-C.16) occurs. Then the flux

---

tubes break and reconnect with each other, altering their underlying topology. The altering of the magnetic topology means that previously tangled flux tubes can detangle from each other. A useful analogy is to think of the flux tubes as stretched rubber bands, each with an internal tension force of $B^2/4\pi$, as in equation (I-B.4). The removal of topological constraints allows the rubber band-like flux tubes to further lower their energies by relieving themselves of their internal stresses. The tubes therefore act like sling-shots, throwing their charged particle matter into the solar system. Of course, this is a greatly simplified model of solar flares. Nevertheless, photographs of the solar surface clearly show the sling-shot-like bursts of the solar magnetic field.\textsuperscript{89}

Section IV-B. Magnetic Topology

In this section, we discuss the topological structure of the tokamak magnetic field. The fundamental concepts for our discussion come from KAM theory, which is the ultimate statement of modern Hamiltonian mechanics. KAM theory rigorously describes the conditions under which the topology

of a Hamiltonian system is preserved. In fact, the formal KAM theory is exceedingly mathematical, and the proofs are very difficult to understand.\textsuperscript{90,91,92} Fortunately, however, the physical meaning of the KAM concepts is much easier to comprehend.\textsuperscript{93} Our discussion therefore emphasizes the physical content of KAM theory and its applications to magnetic fields.

The topology of tokamak magnetic field lines can always be placed into one of three distinct categories: (1) toroidal, (2) island, or (3) stochastic. By a toroidal topology, we simply mean that magnetic field lines form a sequence of nested toroidal magnetic surfaces, as described by equations (III-B.6) and (III-B.7). While of great practical importance, toroidal field line topology is therefore rather trivial to understand. The cases of island and stochastic field line topologies, on the other hand, represent more complicated, non-trivial magnetic topologies. We therefore focus our attention on these complicated magnetic fields.

In order to discuss complicated tokamak magnetic field

line topologies, we begin by assuming that we are dealing with a magnetic field which is nearly in the magnetic coordinate form (III-B.1). That is, we assume an optimal field line Hamiltonian $\chi$ of the form

$$\chi(\psi,\theta,\varphi) = \chi_0(\psi) + \chi_1(\psi,\theta,\varphi), \quad (IV-B.1)$$

where $\chi_0$ and $\chi_1$ satisfy the inequality

$$|\chi_1| \ll |\chi_0|. \quad (IV-B.2)$$

In fact, if the tokamak magnetic field could not be placed in this form, it would be of limited interest from the standpoint of magnetic confinement. The real advantage of the Hamiltonian (IV-B.1), however, is that this form allows us to study the properties of field lines using perturbation theory. Of course, from the last section, we expect that the application of perturbation theory to magnetic fields is a rather tricky business. Nevertheless, magnetic perturbation theory is adequate for our needs and, more importantly, its use will help us underscore some of the subtle points of modern Hamiltonian mechanics.
Having broken the magnetic Hamiltonian up into two parts, we proceed to do the same thing to the magnetic field. From the canonical form (III-A.2) and (IV-B.1), we define

\[ B = B_0 + B_1, \]  
(IV-B.3)

with

\[ B_0 = \nabla \psi \times \nabla (\theta/2\pi) - \nabla \chi_0(\psi) \times \nabla (\varphi/2\pi), \]  
(IV-B.4)

and

\[ B_1 = -\nabla \chi_1(\psi, \theta, \varphi) \times \nabla (\varphi/2\pi). \]  
(IV-B.5)

Using the inequality (IV-B.2), the two pieces of the magnetic field obey

\[ |B_1| \ll |B_0|. \]  
(IV-B.6)

From the magnetic coordinate form (III-B.1), the field \( B_0 \) has good magnetic surfaces. We desire, however, to find the
magnetic surfaces, should they exist, of the magnetic field $B$. According to (III-B.2), this means that we must find a function $f(x)$ such that

$$B \cdot \nabla f = 0.$$  \hspace{1cm} (IV-B.7)

Then the magnetic surfaces are the surfaces of constant $f$.

Let us make the expansion

$$f = f_0 + f_1 + \ldots ,$$  \hspace{1cm} (IV-B.8)

where

$$|f_{n+1}| \ll |f_n|$$  \hspace{1cm} (IV-B.9)

and take

$$B_0 \cdot \nabla f_0 = 0.$$  \hspace{1cm} (IV-B.10)

The solution to (IV-B.10) is $f_0 = f_0(\psi)$, where $f_0$ is an arbitrary function of $\psi$. If we now subtract (IV-B.10) from (IV-B.7) and keep only the leading order terms, we find
\( B_0 \cdot \nabla f_1 + B_1 \cdot \nabla f_0 = 0. \)  \hspace{1cm} (IV-B.11)

Equation (IV-B.11) is a so-called magnetic differential equation for the function \( f_1 \).

We solve the magnetic differential equation (IV-B.11). First, since the tokamak torus is periodic in the poloidal angle \( \theta \) and the toroidal angle \( \phi \), the perturbation Hamiltonian \( \chi_1(\psi, \theta, \phi) \) can be expressed as the Fourier series

\[
\chi_1 = \sum \chi_{nm}(\psi) \exp[i(m\theta-n\phi)], \hspace{1cm} (IV-B.12)
\]

where the summation extends over the poloidal and toroidal mode numbers \( m \) and \( n \). A straightforward integration of equation (IV-B.11) then shows that

\[
f_1 = \frac{df_0}{d\psi} \sum \left[ \chi_{nm}/(i-n/m) \right] \exp[i(m\theta-n\phi)], \hspace{1cm} (IV-B.13)
\]

where \( i = d\chi_0/d\psi \), as in equation (III-B.5). Classical perturbation theory therefore leads to the so-called resonant denominators \( (i-n/m) \). Unless we impose a whole set of conditions on the perturbation amplitudes \( \chi_{nm} \), this means
that the perturbation expansion (IV-B.13) diverges near the positions of the resonances $\imath(\psi_0)=n/m$. Classical perturbation theory has, in some sense, thus failed. The reason perturbation theory fails is that the magnetic field lines can be strongly perturbed near rational surfaces and therefore undergo a change in topology. In fact, from the set of equations (IV-B.3)-(IV-B.5), we see that the field $B$ has a $\nabla\psi$ component, while the field $B_0$ of (IV-B.4) has zero radial component. Perturbation theory, which is based on the assumption that the perturbed field lines are, in some sense, close to the unperturbed field lines, has trouble mimicking this topology change. The resonance phenomenon therefore represents a physical as well as a mathematical difficulty with the perturbation expansion.

Fortunately, however, we can use a trick to stave off the problem of resonant denominators to a higher order in the perturbation expansion. To illustrate how this staving off procedure works, consider a perturbation with a single $(n,m)$ Fourier mode. Take

$$f_1 = -(df_0/d\psi) \left[ \chi_{nm}/(1-n/m) \right] \cos(m\theta-n\phi). \quad \text{(IV-B.14)}$$
The procedure we use allows the perturbation theory to converge in the region of space about the resonant surface where $\lambda(\psi_0) = n/m$. Actually, for a single Fourier mode, we could transform to a coordinate system rotating at the resonance frequency and use the methods of secular perturbation theory, thus directly removing the problem of small denominators.\textsuperscript{94} We discuss this kind of procedure later. For the staving off process, we begin by noting that the rotational transform has the expansion about the resonant surface

$$\lambda = (n/m) + \lambda' (\psi - \psi_0) + \ldots.$$  \hspace{1cm} (IV-B.15)

where $\lambda' = (d\lambda/d\psi)_0$. Then, since $f_0$ is an arbitrary function of $\psi$, we are free to choose

$$f_0 = (\psi - \psi_0)^2/2.$$  \hspace{1cm} (IV-B.16)

Combining (IV-B.14)-(IV-B.16), we have

\textsuperscript{94}A. J. Lichtenberg and M. A. Lieberman, Regular and Stochastic Motion, (Springer-Verlag, New York, 1983), pp. 100-107.
at least near the rational surface $\psi = \psi_0$. Using the equations (IV-B.8), (IV-B.16), and (IV-B.17), the new constant of the motion $f$ can therefore be approximated as

$$f = (\psi - \psi_0)^{2/2} - (\chi_{nm}/\ell') \cos(m\theta - n\phi). \quad (IV-B.18)$$

Inverting this equation, we determine the function $\psi(f, \phi, \phi)$,

$$\psi = \psi_0 \pm \left\{ 2[f \pm (\chi_{nm}/\ell') \cos(m\theta - n\phi)] \right\}^{1/2}. \quad (IV-B.19)$$

This solution has unphysical behavior for $f \leq |\chi_{nm}/\ell'|$. We can simplify the solution by defining

$$\Delta_{nm} = |4\chi_{nm}/\ell'|^{1/2} \quad (IV-B.20)$$

and

$$s = \pm \left\{ \left( f + \chi_{nm}/\ell' \right) / (2\chi_{nm}/\ell') \right\}^{1/2}. \quad (IV-B.21)$$

Using a trigonometric identity, we then determine the
function $\psi(s, \theta, \phi)$,
\[ \psi = \psi_0 \pm \Delta_{nm} \{ s^2 - \sin^2[(m\theta-n\phi)/2]\}^{1/2}. \quad \text{(IV-B.22)} \]

The quantity $\Delta_{nm}$ is known as the magnetic island half-width in $\psi$-space, while the dimensionless quantity $s$ labels the perturbed magnetic surfaces. For reasons to be given shortly, surfaces with $|s| > 1$ are said to be outside the magnetic island, while surfaces with $|s| < 1$ are said to be inside the island. An extremely important and remarkable feature of the solution (IV-B.22) is that the characteristic scale of a topology altering magnetic perturbation is proportional to $|X_{nm}|^{1/2}$, not $|X_{nm}|$ as one might naively expect.

The behavior of the perturbed field lines can be better understood by plotting the function $\psi(s, \theta, \phi)$ in the $\psi-\theta$ plane with $\phi=0$ (see Fig. 5). In fact, if we follow a single field line of constant $s$ and continually plot its position whenever $\phi=0$ (mod $2\pi$), then this kind surface tracing plot is known as a Poincare plot or surface of section. From (IV-B.22) with $\phi=0$, we see that perturbed magnetic surfaces outside the magnetic island, $|s| > 1$, are well-defined for all values of $\theta$.

\[95\text{A. H. Boozer, Phys. Fluids 27, 2055 (1984).}\]
Figure 5. Magnetic island surface plot. An m=1 magnetic island is illustrated with magnetic surfaces labeled by the parameter $s$. Note that different parts of the magnetic surfaces inside the magnetic island are labeled by positive and negative values of $s$. The sign of $s$ is determined by whether that part of the perturbed magnetic surface lies inside or outside of the original resonant magnetic surface.
and simply wobble about their unperturbed positions. The topology of field lines with $|s| > 1$ is thus not altered. The magnetic surface with $|s| = 1$ is a special case, being the last surface on which all values of $\theta$ are meaningful. This surface is called the separatrix, because the surface $|s| = 1$ separates two very different regions of field line topology. Setting $|s| = 1$ and $\phi = 0$ in (IV-B.22), we see that the separatrix is divided into $m$ segments, each bounded by a so-called island X-point. The island X-points are also called hyperbolic fixed points. The largest excursions of the separatrix from the unperturbed surface $\psi = \psi_0$ are $\pm \Delta_{nm}$, so the full width of the separatrix region in $\psi$-space is $2 \Delta_{nm}$. This is the reason $\Delta_{nm}$ is called the magnetic island half-width.

Inside the magnetic island, $|s| < 1$, the magnetic topology of the field lines is altered. In order to get a feel for the topological structure of these field lines, we consider the case $|s| \ll 1$ and transform to a new set of coordinates. We define the perturbed dimensionless action

$$\xi = (\psi - \psi_0) / \Delta_{nm}$$  \hspace{1cm} (IV-B.23)
and the helical angle

\[ \Omega = m\theta - n\phi. \]  \hspace{1cm} (IV-B.24)

Then near \( s=0 \), which is called the island O-point, equations (IV-B.22)-(IV-B.24) imply that

\[ s^2 = \xi^2 + (\Omega/2)^2. \]  \hspace{1cm} (IV-B.25)

That is, the magnetic surfaces appear elliptical in the helically rotating \( \xi-\Omega \) coordinates. The island O-points are therefore also called elliptic fixed points. Using the expansion of the rotational transform (IV-B.15), we see that the equations of motion for a field line are

\[ \frac{d\Omega}{d\phi} = m\varrho\Delta_{nm}\xi, \]  \hspace{1cm} (IV-B.26)

and

\[ \frac{d\xi}{d\phi} = -(m\varrho\Delta_{nm}/4) \sin\Omega. \]  \hspace{1cm} (IV-B.27)

Combining these equations, we find
\[ \frac{d^2 \Omega}{d \phi^2} = -\nu_{nm}^2 \sin \Omega. \quad (IV-B.28) \]

where

\[ \nu_{nm} = m \Lambda_{nm}/2. \quad (IV-B.29) \]

The extremely remarkable result (IV-B.28) suggests that, in the helically rotating \( \xi-\Omega \) action-angle coordinates, the field line motion near a resonance has the phase space topology of a pendulum, complete with libration \((|s|<1)\), separatrix \((|s|=1)\), and rotation \((|s|>1)\). In fact, according to KAM theory, this is the generic behavior of any Hamiltonian system near resonance.\(^9^6\) The fundamental problem of Hamiltonian mechanics is therefore the resolution of the motion of coupled non-linear oscillators, indeed a rather deep and satisfying conclusion of the theory.

Inside the magnetic island, we see that the topological structure of a magnetic field line about the island O-point is completely self-similar to the toroidal topology about the

magnetic axis $\psi=0$. That is, as field lines rotate about the magnetic axis $\psi=0$ at a rate given by the rotational transform $\tau(\psi)$, field lines within the magnetic island rotate about the island O-point at a rate given by the frequency $\nu_{nm}(s)$, which is a generalization of the quantity $\nu_{nm}$ defined in (IV-B.29). The frequency $\nu_{nm}(s)$ is therefore called the rotational transform of the island in the helical frame of reference. This self-similar motion of the field lines can continue ad-infinitum. That is, the beating between the various orders of rotational transform can produce islands within islands, and so on. In other words, a whole sequence of higher order resonances can occur. The magnetic field can therefore display a fractal structure.\textsuperscript{97}

Although the island topology we have just discussed is in some sense quite involved, from a magnetic surface point of view, it is still relatively simple. That is, perhaps several magnetic islands exist in the plasma, each with $\psi$-space half widths $\Delta_{nm}$ given by (IV-B.20). These island modes, while complicated, do represent a kind of magnetic surface, and the motion of field lines can still be considered smooth and predictable. However, our discussion of possible field line

topologies is not finished. That is, if the separatrices of neighboring magnetic islands overlap, an entirely different kind of field line topology occurs. When magnetic islands overlap, the constants of the motion or isolating integrals $f$ of equation (IV-B.7) disappear and the magnetic surfaces are completely destroyed. Field lines are then free to wander randomly about the entire region encompassed by the magnetic islands. That is, magnetic field lines stochastically fill volumes and do not lie in surfaces at all. A region in which many magnetic islands overlap is called a strong or globally stochastic region. In globally stochastic regions, the field lines can wander over large radial portions of the plasma, $\delta \psi/\psi \approx O(1)$. Since a globally stochastic region has completely de-localized magnetic field lines, this sort of topology is obviously very bad for magnetic confinement. Even if magnetic islands do not overlap, however, it is important to note that an intrinsic region of stochasticity always exists near the separatrix of a magnetic island. In these so-called resonance layers or regions of weak stochasticity, the random excursion of field lines is limited.

---

to the order of the separatrix width, $δ\psi ≈ 2Δ_{nm}$. These localized regions of stochasticity are unavoidable in tokamaks with magnetic islands.

Because field lines wander randomly in island overlap or separatrix regions, one useful description of stochastic topology relies on a magnetic diffusion coefficient $D_m$. If a stochastic field line is followed for a distance $l$, the magnetic diffusion coefficient is defined so that the mean-square radial displacement of the field line is given by the formula

$$\langle (\Delta r)^2 \rangle = 2D_m l.$$  \hspace{1cm} (IV-B.30)

For the case of many overlapping resonances or strong stochasticity, an actual Brownian motion of the field lines occurs, and the magnetic diffusion coefficient can be calculated from the quasi-linear formula\textsuperscript{100}

$$D_m = \pi R_0 \sum |B_{rnm}|^2 / B_0^2.$$  \hspace{1cm} (IV-B.31)

Here, $B_{\text{rnm}}$ is the radial magnetic field perturbation of the $(n,m)$ Fourier mode,

$$B_{\text{rnm}} = \left(\frac{1}{2\pi R_0}\right) |m\chi_{nm}/r|,$$  \hspace{1cm} (IV-B.32)

and the summation extends over all the overlapping magnetic islands.

The magnetic diffusion coefficient description has been used by Rechester and Rosenbluth\textsuperscript{101} to derive an estimate for "cross-field" electron heat transport in a strongly stochastic magnetic field. The derivation begins with the assumption that an electron follows a stochastic field line for a mean time $\tau_e$ before colliding, where $\tau_e$ is the electron collision time (II-B.8). Before colliding, an electron therefore moves a mean distance $\lambda_c$ along a stochastic field line, where $\lambda_c$ is the mean free path (I-B.9). Now, according to the formula (IV-B.30), if a stochastic field line itself is followed for a distance $\lambda_c$, it wanders radially a mean-square distance

$$\langle(\Delta r)^2\rangle = 2 D_m \lambda_c.$$ \hspace{1cm} (IV-B.33)

Since the electron follows the field line, it also suffers the mean-square radial displacement (IV-B.33) before colliding. After a collision, the electron moves to another field line, and the whole process repeats itself. The process is therefore just a random walk and, according to the diffusion formula (I-B.8), an effective electron thermal diffusivity $\chi_e$ can be obtained by dividing the mean-square displacement (IV-B.33) by the collision time $\tau_e$.

$$\chi_e = \frac{\langle (\Delta r)^2 \rangle}{2\tau_e} = D_m(\lambda_c/\tau_e) = D_m v_{Te},$$  \hspace{1cm} (IV-B.34)

where $v_{Te}$ is the mean electron speed $\langle |v_\parallel| \rangle$ along a stochastic field line, the electron thermal speed (I-B.10). For the case of a globally stochastic magnetic field, the electron thermal diffusivity is therefore given by (IV-B.34), with $D_m$ given by the quasi-linear formula (IV-B.31).

In our work, however, we are more interested in the localized separatrix layer diffusion. We mention again that this diffusion is unavoidable in tokamaks with magnetic islands. A useful estimate for the magnitude of the (n,m) island layer cross-field thermal diffusion coefficient $\chi_e$ has
been given by White,\textsuperscript{102}

\[
\chi_e \approx w_{nm}^2/2\tau_e. \quad \text{(IV-B.35)}
\]

Here, $w_{nm}$ is the (n,m) island width in real space,

\[
w_{nm} = (2\Delta_{nm})(dr/d\psi) = 4|R_0B_{rnm}|/[mB_0(dl/dr)]^{1/2}. \quad \text{(IV-B.36)}
\]

We give a heuristic argument for the formula (IV-B.35). First, we once again assume that we are in the long mean free path thermonuclear regime and that an electron moves a mean distance $\lambda_C$ along a stochastic field line before colliding. Now, consider an effective magnetic diffusion coefficient $D_m$ for the separatrix layer. The effective magnetic diffusion coefficient cannot be so great that it gives a radial diffusion which exceeds the (n,m) island layer thickness $w_{nm}$ itself. From equation (IV-B.33), this means that $D_m$ cannot be much greater than the value $w_{nm}^2/2\lambda_C$. On the other hand, since the tokamak electron mean free path $\lambda_C$ is on the order of kilometers and tokamak magnetic islands are on the order of centimeters, a

\footnote{\textsuperscript{102}R. B. White, Theory of Tokamak Plasmas, (North-Holland, Amsterdam, 1989), pp. 312-316.}
stochastic separatrix magnetic field line of length $\lambda_c$
circuits a magnetic island of size $w_{nm}$ many times. This
means that that $D_m$ cannot be much less than the value
$w_{nm}^2/2\lambda_c$. Combining the two arguments, the effective
magnetic diffusion coefficient for the separatrix layer must
be on the order of

$$D_m \approx w_{nm}^2/2\lambda_c.$$  \hspace{1cm} (IV-B.37)

Substitution of (IV-B.37) into the formula of Rechester and
Rosenbluth (IV-B.34) then yields the random walk estimate of
White (IV-B.35).

Section IV-C. Tokamak Plasma Stability

At the beginning of this chapter, we mentioned that
tearing modes arise from the effects of finite plasma
resistivity. As discussed in section IV-A, finite plasma
resistivity allows a magnetic field to undergo a change in its
topological structure. That is, topologically toroidal magnetic
field lines can tear and reconnect, forming the island chains
of section IV-B. It is this transformation or distortion of the
magnetic field which is called a tearing mode.\textsuperscript{103} Tearing modes are prime examples of MHD plasma instabilities. MHD instabilities are the strongest and most important instabilities in tokamaks. In the MHD model, destabilizing plasma forces can arise from gradients in either the plasma current density or the plasma pressure.\textsuperscript{104} Tearing modes are current-driven resistive plasma instabilities which tend to be unstable when the plasma current profile is either hollow (outwardly peaked) or too steep.\textsuperscript{105} In this section, we address the issue of tokamak plasma stability against the tearing mode.

The issue of tokamak plasma stability against the tearing mode is really a question of energetics. In fact, one of the best known ways of analyzing MHD stability properties utilizes the so-called plasma Energy Principle.\textsuperscript{106} The Energy Principle states that a perturbation which lowers the plasma potential energy.

\[ W = \int (3p/2 + B^2/8\pi) \, d^3x, \]

destroys a plasma equilibrium, thus causing the plasma to spontaneously distort. As we shall see, the question of tearing mode energetics boils down to two competing processes: (1) poloidal flux annihilation, which is destabilizing, and (2) field line bending, which is stabilizing. The stabilizing effect of field line bending can be understood by considering the nature of the (n,m) magnetic island. As we have seen, the (n,m) magnetic island chain is bent into \( m \) poloidal segments. This means that low \( m \) tearing modes require the least amount of field line bending. In order to bend field lines, a tearing mode must overcome the magnetic tension force \( B \cdot \nabla B / 4\pi \) of equation (I-B.3). Fourier modes with low poloidal mode number \( m \) therefore tend to be most susceptible to the tearing mode instability. Of course, the stabilizing effect of field line bending is minimized near the magnetic surface where the rotational transform \( \imath(\psi_0) = n/m \). That is, near the rational surface \( \psi = \psi_0 \), the (n,m) Fourier mode oscillation resonates with the natural oscillation frequency of the magnetic field.
In order to simplify the tokamak stability analysis, we assume that the shape of the tokamak current profile is such that both ideal MHD and plasma pressure modes are MHD stable. That is, we assume that the MHD stability of the tokamak against the current profile is determined by tearing mode considerations alone. In fact, ideal MHD current modes with poloidal mode number \( m > 2 \) are stable for sufficiently narrow, centrally peaked current profiles.\textsuperscript{107,108} The neglect of the ideal MHD effects does not, therefore, place any severe restrictions on the allowable tokamak current profiles. Pressure modes do not place severe restrictions on the shape of the current profile either.\textsuperscript{109} For our stability analysis, the pressure gradient is therefore taken as

\[
\nabla p = 0, \quad (IV-C.2)
\]

which means that the plasma is assumed to be "force-free."

From (IV-C.1) and (IV-C.2), a force-free plasma has the free energy

Combining the force-free condition (IV-C.2), the equilibrium relation (II-C.6), Ampere's law (II-A.23), and the definition of the parallel current distribution (III-C.2), we then generate the force-free plasma equation

\[ \nabla \times B = \mu B. \]  

(IV-C.4)

Equation (IV-C.4) forms the basis for our tokamak stability analysis. Note from the divergence of (IV-C.4) and Gauss's law of magnetism (II-A.25) that

\[ B \cdot \nabla \mu = 0. \]  

(IV-C.5)

This means that a force-free current profile with \( |\nabla \mu| \neq 0 \) implies the existence of a set of good magnetic surfaces, just as a non-vanishing pressure gradient does.

Our tokamak stability analysis is performed by comparing plasma equilibria with the same current distribution \( \mu \). That is, we compare two current distributions, the equilibrium

\[ W = \int (B^2/8\pi) \, d^3x. \]  

(IV-C.3)
current distribution $\mu(\psi)$ and a perturbed current distribution $\mu(\psi_p)$, to determine whether the energy of the plasma equilibrium is lowered or raised by the formation of a magnetic island. The fact that the current distribution function $\mu$ is the same function of the toroidal flux before and after the perturbation is implied by the constancy of the steady-state loop voltage, and holds throughout the plasma, except in a thin resistive layer about the magnetic island. As we have seen in the previous section, the perturbed toroidal flux $\psi_p(x)$ can, however, be a very different function of position than $\psi(x)$. A division of the plasma into two regions, an ideal region away from the rational surface and a thin resistive layer about the rational surface, simply makes it easier for us to perform the tokamak stability analysis. Of course, in the resistive layer, the topology of the magnetic field can be altered with the opening of a magnetic island. The resistive layer magnetic field can therefore change very rapidly. This means that spikey currents can arise in the neighborhood of a magnetic island. In our perturbation theory, which conserves magnetic topology away from the rational surface, the spikey currents associated with a magnetic island are modeled by a single delta function current at the
position of the rational surface. As we shall see, it is the sign of the delta function current which determines the tearing mode stability.

We begin the tokamak stability analysis. First, from the force-free plasma equation (IV-C.4), the unperturbed plasma equilibrium is given by

\[ \nabla \times B_0 = \mu(\psi)B_0. \]  \hfill (IV-C.6)

Similarly, the perturbed equilibrium is given by the equation

\[ \nabla \times (B_0 + B_1) = \mu(\psi_p)(B_0 + B_1). \]  \hfill (IV-C.7)

Here, the equilibrium field \( B_0 \) and the perturbation field \( B_1 \) obey (IV-B.3)-(IV-B.6). We assume two sets of magnetic surfaces,

\[ B_0 \cdot \nabla \psi = 0 \]  \hfill (IV-C.8)

and

\[ (B_0 + B_1) \cdot \nabla \psi_p = 0. \]  \hfill (IV-C.9)
Subtracting (IV-C.8) from (IV-C.9), keeping only the leading order terms, we have

\[ B_0 \cdot \nabla (\psi_p - \psi) + B_1 \cdot \nabla \psi = 0, \tag{IV-C.10} \]

which is a magnetic differential equation for \((\psi_p - \psi)\). Using (IV-B.12), the solution to (IV-C.10) is

\[ \psi_p - \psi = \sum \left[ X_{nm}/(\nu-n/m) \right] \exp[i(m\theta-n\varphi)]. \tag{IV-C.11} \]

Similarly, subtracting (IV-C.6) from (IV-C.7), keeping only the leading order terms, we find

\[ \nabla \times B_1 = [\mu(\psi_p) - \mu(\psi)] B_0 + \mu(\psi) B_1, \tag{IV-C.12} \]

where, to leading order,

\[ \mu(\psi_p) - \mu(\psi) = (d\mu/d\psi)(\psi_p - \psi). \tag{IV-C.13} \]

Combining equations (IV-C.11)-(IV-C.13), then yields the so-
The stability equation for the (n,m) Fourier mode takes the much simpler form

\[ \nabla \perp^2 x_{nm} = \left[ (R_0/r)(d\mu/dr)/(\ell-n/m) \right] x_{nm}. \]  

(IV-C.15)

Here, the perpendicular Laplacian \( \nabla \perp^2 \) acts as

\[ \nabla \perp^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2}. \]  

(IV-C.16)

In the so-called marginal stability equation (IV-C.15), the parallel current distribution \( \mu(r) \) and the rotational

The transform \( \lambda(r) \) are

\[
\lambda = \frac{1}{2\pi r B_0} \nabla^2 \chi_0. \tag{IV-C.17}
\]

and

\[
\lambda = \frac{1}{2\pi r B_0} \frac{d\chi_0}{dr}. \tag{IV-C.18}
\]

In making the cylindrical tokamak approximation, we have again ignored terms of \( O(\epsilon^2) \) smaller than those kept. Note that the solution \( \chi_{nm}(r) \) of the marginal stability equation (IV-C.15) depends on the shape of the tokamak current profile through the \( \frac{d\mu}{dr} \) derivative term. The term containing \( \frac{d\mu}{dr} \) can be destabilizing, and is large near the rational surface \( \lambda(r_{nm}) = n/m \), unless the current gradient itself vanishes there. In the expression (IV-C.16) for the perpendicular Laplacian, note also that we have a term \( m^2/r^2 \). This term is stabilizing, and models the previously discussed effect of field line bending.

We now discuss several important properties of the solution for the Fourier amplitude \( \chi_{nm}(r) \). First, the general solution \( \chi_{nm} \) of the second-order, linear differential equation
(IV-C.15) is a linear combination of two solutions,$^{111}$

$$U_{nm}(r) = r^m g_{nm}(r^2), \quad \text{(IV-C.19)}$$

and

$$V_{nm}(r) = r^{-m} h_{nm}(r^2). \quad \text{(IV-C.20)}$$

We assume an analytic solution for $\chi_{nm}$ near the magnetic axis. This means that the solution $\chi_{nm}$ interior to the rational surface $r=r_{nm}$ must come out of the axis $r=0$ like the function $U_{nm}$ of (IV-C.19). Next, if a conducting wall surrounds the plasma at the position $r=b$, then we have the boundary condition $\chi_{nm}(b)=0$. In our model, however, we assume no conducting wall, that is, we let the conducting wall position go to infinity, $b\to\infty$. This means that the solution $\chi_{nm}$ exterior to the rational surface $r=r_{nm}$ must act like the function $V_{nm}$ of (IV-C.20) as $r\to\infty$. In general, because of the different nature of the solutions (IV-C.19) and (IV-C.20), the interior and exterior solutions for $\chi_{nm}$ will not attach smoothly to each other at the resonant surface.

This phenomenon is due to the fact that we have fixed the magnetic topology away from the rational surface, and let a delta function current sit at the position \( r = r_{nm} \).

Let us calculate the value of this delta function current. The perturbed longitudinal current \( j_{nm} \) is just

\[
    j_{nm} = \frac{c}{4\pi} \left( \frac{1}{2\pi R_0} \right) \nabla_{\perp}^2 \chi_{nm} \cos(m\theta - n\phi). \tag{IV-C.21}
\]

We now model the perpendicular Laplacian function as a delta function spike

\[
    \nabla_{\perp}^2 \chi_{nm} = A \delta(r - r_{nm}). \tag{IV-C.22}
\]

where \( A \) is some amplitude. In order to calculate \( A \), we simply integrate (IV-C.22) over the resonant surface. The result is

\[
    A = \Delta'_{nm} \chi_{nm}. \tag{IV-C.23}
\]

where all quantities are evaluated at the resonant surface \( r = r_{nm} \), with the critical factor \( \Delta'_{nm} \) (delta-prime) defined by

\[
    \Delta'_{nm} = \frac{d\chi_{nm}/dr}{\chi_{nm}}. \tag{IV-C.24}
\]
Here, the jump in the derivative \( \frac{d\chi_{nm}}{dr} \) is given by

\[
\left[ \frac{d\chi_{nm}}{dr} \right] = \lim_{\varepsilon \to 0} \left\{ \frac{d\chi_{nm}}{dr}\bigg|_{r_s+\varepsilon} - \frac{d\chi_{nm}}{dr}\bigg|_{r_s-\varepsilon} \right\}. \quad (IV-C.25)
\]

\( \Delta'_{nm} \) depends only on the shape of the current profile and represents the spikey current created in the vicinity of a magnetic island.

The quantity \( \Delta'_{nm} \) is indeed the critical tearing mode factor because its sign and magnitude determine the stability of tokamaks against tearing modes and the growth rate of magnetic islands. The tearing mode stability can be determined by calculating the energy \( \delta W_{nm} \) released by the opening of a magnetic island. This energy is just

\[
\delta W_{nm} = - \int j_{nm} E_{nm} \, d^3x \, dt, \quad (IV-C.26)
\]

where \( j_{nm} \) is given by (IV-C.21) and the perturbed longitudinal electric field \( E_{nm} \) is

\[
E_{nm} = (1/2\pi R_0) (1/c) (\partial \chi_{nm}/\partial t) \cos(m\theta - n\varphi). \quad (IV-C.27)
\]
Performing the integration, we find

$$\delta W_{nm} = - (2\pi R_0 \cdot \pi r_{nm}^2) (r_{nm} \Delta'_{nm}/m^2) (B_{nm}^2/8\pi), (IV-C.28)$$

which shows that $\Delta'_{nm}>0$ leads to a release of magnetic free energy and a tearing mode instability. Relation (IV-C.28) is called the Furth energy criterion.\(^{112}\)

The actual calculation of $\Delta'_{nm}$ from (IV-C.24) using the marginal stability equation (IV-C.15) is complicated\(^{113}\) and will be discussed further in the chapter on numerical methods. The complications occur near the rational surface $r=r_{nm}$, which is a regular singular point of the marginal stability equation. Let us define the dimensionless coordinate

$$y = (r - r_{nm})/r_{nm} \quad (IV-C.29)$$

and the function

$$\Phi_{nm}(y) = \chi_{nm}(r). \quad (IV-C.30)$$


Then, in the vicinity of the rational surface, the marginal stability equation (IV-C.15) reduces to

\[
d^2 \phi_{nm}/dy^2 = (\kappa/y) \phi_{nm}.
\]  

(IV-C.31)

where the dimensionless coefficient

\[
\kappa = R_0 \left( \frac{d\mu/dr}{dt/dr} \right)_{r=r_{nm}}.
\]

(IV-C.32)

The leading behavior of a Frobenius series solution to equation (IV-C.31) is\(^{114,115}\)

\[
\phi_{nm} = 1 + \kappa \ln|y| + \cdots.
\]

(IV-C.33)

which is a perfectly well-behaved function of \(y\). However, the derivative \(d\phi_{nm}/dy\) of (IV-C.33) diverges at \(y=0\), which is difficult to model on a computer.


Section IV-D. Magnetic Island Growth

The so-called delta-prime analysis discussed in the last section is also important for determining the growth rate of tokamak tearing modes, at least for tearing modes with poloidal mode numbers $m \geq 2$. The $m=1$ tearing mode is a special case, which we discuss later in this section. We begin this section with a discussion of the relationship between $\Delta'_nm$ and the non-linear growth of the tearing modes with $m \geq 2$. We are interested in the non-linear or so-called Rutherford regime\textsuperscript{116} of tearing mode growth because our ultimate goal is to determine the saturated or time-stationary widths $w_{nm}$ of magnetic islands. In fact, the initial linear growth period\textsuperscript{117} of the $m \geq 2$ tearing mode is of negligible extent, and thus can be safely neglected.

We give a heuristic derivation of the tearing mode island growth equation. First, from the stability equation (IV-C.15), we remark that tearing mode growth is driven by the magnetic free energy in the gradient $d\mu/dr$ of the parallel current distribution. The stability factor $\Delta'_nm$, which depends only on the shape of the current profile, therefore

will figure prominently into the island growth equation. In the non-linear Rutherford regime, island growth is limited by resistive diffusion. Then, since the island width \( w_{nm} \) of equation (IV-B.36) is proportional to the square-root of the radial magnetic perturbation \( B_{\text{rnm}} \), the growth of a magnetic island can crudely be modeled by a diffusion equation:\(^{118}\)

\[
\frac{\partial B_{\text{rnm}}}{\partial t} \approx (\eta c^2/4\pi) \frac{\partial^2 B_{\text{rnm}}}{\partial r^2}.
\]

(IV-D.1)

Integrating (IV-D.1) across the magnetic island, assuming that \( B_{\text{rnm}} \) is fairly constant across the island region, we find

\[
B_{\text{rnm}}^{-1} w_{nm} \left( \frac{dB_{\text{rnm}}}{dt} \right) \approx (\eta c^2/4\pi) \Delta'_{nm}.
\]

(IV-D.2)

where \( \Delta'_{nm} \) is given by (IV-C.24) and all quantities are evaluated at \( r=r_{nm} \). Using \( w_{nm} \propto B_{\text{rnm}}^{1/2} \), we then have

\[
\frac{dw_{nm}}{dt} \approx (\eta c^2/4\pi) \Delta'_{nm}.
\]

(IV-D.3)

That is, in the non-linear Rutherford regime, islands grow linearly with time, with a growth rate proportional to \( \Delta'_{nm} \).

which represents the strength of the tearing instability. In fact, a much more accurate calculation of Boozer\textsuperscript{119} which equates the magnetic energy $\delta W_{nm}$ of equation (IV-C.28) with the energy dissipated in a magnetic island shows that

$$\frac{dW_{nm}}{dt} = 1.28 \left( \eta c^2 / 4\pi \right) \Delta'_{nm}.$$  \hspace{1cm} (IV-D.4)

Again, we see that a $\Delta'_{nm} > 0$ indicates growth of a tearing instability. We adopt equation (IV-D.4) as our magnetic island growth model. Note from (IV-D.4) that the condition

$$\Delta'_{nm} = 0$$  \hspace{1cm} (IV-D.5)

implies the saturation of the $(n,m)$ magnetic island growth. Equation (IV-D.5) is also referred to as the condition of marginal tearing mode stability. As we shall see, a typical size $w_{nm}$ for a large $m \geq 2$ tokamak magnetic island is on the order of 10% of the plasma minor radius.

As previously mentioned, the $m=1$ tearing mode is a special case, and differs qualitatively from modes with $m \geq 2$. The difference arises because for $m=1$ there is a marginally

stable internal ideal MHD current mode, whereas all internal kinks with $m \geq 2$ are positively MHD stable, to lowest order in inverse aspect ratio. In order to derive this result, we define the $(n,m)$ radial plasma displacement $\xi_{rnm}$ as

$$\xi_{rnm} = -\chi_{nm}/[2\pi r B_0(l-n/m)].$$  \hspace{1cm} (IV-D.6)

A direct substitution of (IV-D.6) into the marginal stability equation (IV-C.15) shows that a solution $\chi_{n1}$ to this equation can be obtained if we take the $m=1$ trial function

$$\xi_{r1} = \text{constant}, \quad r < r_{n1}$$

$$0, \quad r > r_{n1}. \hspace{1cm} (IV-D.7)$$

The solution $\chi_{n1}$ formed from (IV-D.6) and (IV-D.7) is valid for arbitrary current distribution $\mu$. The vanishing of $\chi_{n1}$ at the rational surface $r_{n1}$ indicates marginal tokamak stability against an $m=1$ ideal kink or shift in the position of the magnetic axis. In the tokamak, the important $m=1$ poloidal mode has toroidal mode number $n=1$. A higher order calculation\textsuperscript{120} in the inverse aspect ratio then shows that a

(1,1) current driven instability exists if the central transform $\lambda_0 > 1$. Equivalently, since the experimenters usually characterize their discharges by the safety factor

$$q = 1/\lambda,$$  \hspace{1cm} (IV-D.8)

a (1,1) tokamak current instability exists unless the central safety factor $q_0 > 1$ (hence the name safety factor).

The limitations imposed by the (1,1) tearing mode have very important implications for tokamak magnetic confinement. First, from (IV-C.17) and (IV-C.18), the central current density $j_0$ is related to the central transform $\lambda_0$ by

$$j_0 = 2 (c/4\pi) (B_0/R_0) \lambda_0.$$  \hspace{1cm} (IV-D.9)

The limitation $\lambda_0 < 1$ and technological constraints limiting the maximum toroidal magnetic field $B_0$ then set an upper limit on the tokamak toroidal current. Theoretical calculations show that this limiting toroidal current can only heat a plasma to several keV.\[121\] This means that it will be quite difficult, perhaps even impossible, to ignite a tokamak

plasma by Ohmic heating (III-C.16) alone. As a result, tokamak reactors will probably require some form of auxiliary heating such as neutral beams or rf power.

Next, Ohmically heated tokamaks have an inherent tendency to channel current into the center of the plasma, thus exciting the (1,1) tearing mode. That is, for a constant loop voltage and central transform $\imath_0 < 1$, we have $P_{\text{Ohm}} \propto T_e^{3/2}$, which means that a hot plasma center becomes even hotter, and more current is thus channelled into the plasma interior. This phenomenon is called the tokamak plasma "thermal instability." When the central current density $j_0$ is so great that the central transform $\imath_0 > 1$, the (1,1) current instability can occur. In fact, if we assume that the central transform is limited to $\imath_0 < 1$ due to the (1,1) tearing mode instability, the plasma Ohmic heating scales as $P_{\text{Ohm}} \propto T_e^{-3/2}$, which explains the difficulty of bringing a tokamak plasma to ignition by Ohmic heating alone.

A very famous explanation of the behavior of the $m=1$ tokamak tearing instability has been given by Kadomtsev.\textsuperscript{122,123} In the Kadomtsev model, the $m=1$ tearing instability is described by:

\begin{equation}
\epsilon_0 \propto \frac{T_e}{T_i}^{\frac{1}{2}} \frac{j_0}{B_0} \ \text{or} \ \frac{T_e}{T_i} \propto \frac{B_0}{j_0},
\end{equation}

where $\epsilon_0$ is the plasma beta (ratio of magnetic energy to internal energy), $T_e$ and $T_i$ are the electron and ion temperature, respectively, $j_0$ is the central current density, and $B_0$ is the magnetic field at the plasma boundary. The plasma beta is a measure of the stability of the plasma, and it is clear from the equation that the plasma becomes more unstable as the magnetic field is increased or the central current density is decreased.

instability is described by a process of flux reconnection. The model begins with the thermal instability, which causes the central safety factor $q_0$ to drop. When $q_0 < 1.0$, the $(1,1)$ tearing mode instability is excited. The $m=1$ tearing process causes the field lines of the interior magnetic field to tear and reconnect on the scale $r_{11}$ of the position of the $m=1$ resonant surface. The plasma current inside the region $r < r_{11}$ where $q < 1$ then becomes redistributed, and the current profile $\mu$ in the plasma center flattens on the scale of $r_{11}$. As the current profile flattens, the central safety factor increases, until a time when $q_0 > 1$. Then the whole process rapidly repeats itself, creating a so-called tokamak "sawtooth oscillation." Numerical simulations$^{124}$ and tokamak experiments$^{125}$ give some credence to the Kadomtsev model of the tokamak sawtooth oscillation. However, some deviation from the Kadomtsev model has recently been reported.$^{126}$ Interestingly, the magnetic reconnection process in the tokamak center described by Kadomtsev is not unlike that of the solar flares discussed in section IV-A.

CHAPTER V. MEAN-FIELD THEORY

Section V-A. Magnetic Helicity

Consider a plasma magnetic field with a magnetic Hamiltonian \( \chi(\psi, \theta, \phi) \) such as (IV-B.1). Obviously, if one takes into account the effects of tearing modes, a complete topological description of the evolution of the exact magnetic field with the function \( \chi(\psi, \theta, \phi) \) is highly complex. Fortunately, however, the magnetic fields associated with fusion plasmas, such as tokamak magnetic fields, frequently exhibit small-scale MHD fluctuations about some large-scale or mean magnetic field. This is important because the field lines of the large-scale or mean magnetic field can lie in a set of good toroidal magnetic surfaces associated with an integrable Hamiltonian \( \chi_0(\psi) \). Because integrable magnetic field are relatively easy to handle, we are very interested in a set of equations which describe the time evolution of the large-scale or mean magnetic field.\(^{127}\) Of course, the usefulness of such equations will depend upon the simplicity

with which the effects of the small-scale fluctuations on the large-scale fields can be incorporated into the theory and the extent to which the conservation properties of the exact description survive in the mean-field theory. In this chapter, we examine some of the properties of mean magnetic fields.

The most important conservation property of any mean-field theory concerns the plasma magnetic helicity (I-D.1). In fact, a general helicity integral $H$ of the form\(^{128}\)

\[
H = \int X \cdot (\nabla \times X) d^3x \tag{V-A.1}
\]

can be associated with the topological properties of the field lines of any divergence-free vector field $\nabla \times X$. For the divergence-free magnetic field $B$, the field $X$ is replaced by the vector potential $A$ of equation (III-A.1). Then the magnetic helicity $K$ reduces to

\[
K = \int A \cdot (\nabla \times A) d^3x = \int A \cdot B d^3x, \tag{V-A.2}
\]

which is just (I-D.1). Another well-known example of a helicity integral is the fluid helicity

---

\[ I = \int \mathbf{v} \cdot (\nabla \times \mathbf{v}) d^3x = \int \mathbf{v} \cdot \mathbf{\omega} d^3x, \quad (V-A.3) \]

where the vorticity \( \mathbf{\omega} = \nabla \times \mathbf{v} \). The magnetic helicity \( K \) of equation (V-A.2) measures the structural or topological complexity of the magnetic field \( \mathbf{B} \).\(^{129}\)

We illustrate the topological nature of the helicity integral \( K \) by a simple example.\(^{130}\) First, define the magnetic flux \( \Phi \) crossing a simply-connected surface \( S \) as

\[ \Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l}, \quad (V-A.4) \]

where the line integral is along the curve \( C \) which is spanned by the surface \( S \). Now, suppose that the magnetic field \( \mathbf{B} \) vanishes everywhere except inside two singly linked flux tube rings, each with a right-hand orientation, one occupying a volume \( V_1 \) following a closed curved \( C_1 \) and the other occupying a volume \( V_2 \) following a closed curve \( C_2 \). If we use the definition of the flux (V-A.4), the quantity \( \mathbf{B} d^3x \) may be replaced by \( \Phi_1 d\mathbf{l} \) along \( C_1 \) and and by \( \Phi_2 d\mathbf{l} \) along \( C_2 \). From


the definition of the magnetic helicity (V-A.2), this means that for singly linked flux tubes

\[ K_1 = \int_{V_1} A \cdot B \, d^3x = \phi_1 \oint_{C_1} A \cdot dl = \phi_1 \phi_2 \quad (V-A.5) \]

and

\[ K_2 = \int_{V_2} A \cdot B \, d^3x = \phi_2 \oint_{C_2} A \cdot dl = \phi_1 \phi_2. \quad (V-A.6) \]

In fact, if the flux tubes \( \phi_1 \) and \( \phi_2 \) link each other \( N \) times, we have

\[ K_1 = K_2 = \pm N \, \phi_1 \phi_2. \quad (V-A.7) \]

where the \( \pm \) refers to the relative orientation or handedness of the magnetic fluxes. The helicity integrals \( K_1 \) and \( K_2 \) and the total magnetic helicity

\[ K = K_1 + K_2 = \pm 2N \, \phi_1 \phi_2 \quad (V-A.8) \]

are therefore intimately connected to the topological
invariant \( N \). We see that the quantity \( K \) is a measure of the intertwining of the two magnetic fluxes \( \Phi_1 \) and \( \Phi_2 \).

One important, but subtle issue raised by the definition (V-A.2) for the helicity integral concerns the question of gauge invariance. A gauge transformation\(^{131}\) of the vector potential,

\[
A \rightarrow A + \nabla G,
\]

(V-A.9)

where \( G \) is the gauge function, causes a change \( \Delta K \) in the helicity integral

\[
\Delta K = \int B \cdot \nabla G d^3x = \int \nabla \cdot (GB) d^3x = \int_S GB \cdot dS.
\]

(V-A.10)

The helicity integral (V-A.2) will therefore be gauge invariant only if the bounding surface \( S \) is a magnetic surface or if the volume of integration is over all space. Although it will not affect our calculations, a better definition of magnetic helicity turns out to be\(^{132,133}\)


The helicity content $K_0$ is the proper definition of the helicity contained in an arbitrary, time-dependent, bounded region of space, provided the gauge is chosen correctly. Using the relation (III-A.1) and the representation (III-A.2) to obtain the expression for the vector potential

$$A = \psi \nabla(\theta/2\pi) - \chi \nabla(\phi/2\pi),$$

one can show directly from (V-A.11) or, assuming a bounding magnetic surface, from (V-A.2) that

$$K_0 = 2\int(\chi_b - \chi) d\psi d(\theta/2\pi) d(\phi/2\pi),$$

where $-\chi_b$ is the boundary value of the poloidal flux. For a mean-field Hamiltonian of the form $\chi = \chi(\psi)$ alone, the helicity content therefore has the simple form

$$K_0 = 2\int(\chi_b - \chi) d\psi.$$
That is, the helicity represents the intertwining of the poloidal and the toroidal magnetic fluxes.

Magnetic helicity was first introduced into plasma physics within the context of astrophysical applications and ideal MHD theory.\(^{134,135}\) Recall the ideal MHD equation of motion (IV-A.3). From the vector potential representation (III-A.1) and a proper choice of gauge, this equation is equivalent to

\[
\frac{\partial A}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{A}), \tag{V-A.15}
\]

which implies that in ideal MHD

\[(\nabla \times \mathbf{A}) \cdot \frac{\partial \mathbf{A}}{\partial t} = 0. \tag{V-A.16}\]

Using the definition of the helicity (V-A.2), the ideal MHD condition (V-A.16), and integrating by parts, we have

\[
\frac{dK}{dt} = \int_V \mathbf{A} \cdot (\nabla \times \frac{\partial \mathbf{A}}{\partial t}) d^3x
= \int_V (\nabla \times \mathbf{A}) \cdot \frac{\partial \mathbf{A}}{\partial t} d^3x
- \int_S (\mathbf{A} \times \frac{\partial \mathbf{A}}{\partial t}) \cdot d\mathbf{S} = 0, \tag{V-A.17}
\]

\(^{134}\)W. M. Elsasser, Rev. Mod. Phys. 28, 135 (1956).

where the surface integral vanishes if we consider a closed system. That is, the magnetic helicity is an ideal MHD invariant. This is really not surprising, since topological properties of the magnetic field are fixed in ideal MHD, according to Alfvén's theorem (IV-A.4). In fact, Taylor\textsuperscript{136,137} has shown that if a plasma has a set of magnetic surfaces, then the differential ideal MHD constraint (IV-A.3) can be replaced by an infinite set of helicity integrals, one for each magnetic surface of constant $\psi$. The existence of the integral form for the ideal MHD constraint is important because it allows one to treat the motion of magnetic fields from a variational principle. For example, because of plasma stability considerations, we are frequently interested in magnetic fields which possess the minimum value of the magnetic energy (IV-C.3). Using the helicity constraint, we can then minimize the magnetic energy, surface by surface, via the variational principle

\[
\delta [\int \psi (\nabla \times A) \cdot (\nabla \times A) d^3x - \mu(\psi) \int \psi A \cdot (\nabla \times A) d^3x] = 0, \quad (V-A.18)
\]

where $\mu(\psi)$ is the Lagrange multiplier for the magnetic surface $\psi$. Straightforward manipulations of (V-A.18) lead to the force-free plasma equation

$$\nabla \times B = \mu(\psi)B,$$  \hspace{1cm} (V-A.19)

which is just Ampere's law (II-A.23) with $\mu(\psi)$ the parallel current distribution (III-C.2). Again, this is really not surprising, since the magnetic force $j \times B/c$ vanishes for a force-free plasma if the magnetic field produces no more plasma motions.

As was pointed out in section IV-A, however, the ideal MHD constraint is actually not a very realistic one. This point was made explicitly by Taylor\textsuperscript{138} in his famous paper on the Reversed Field Pinch (RFP). The RFP\textsuperscript{139} is a toroidal magnetic confinement configuration in which the magnetic field ratio $B_\psi/B_\theta \approx 0(1)$, thus allowing for efficient, high $\beta$ operation of the RFP device. A major drawback of the RFP configuration, however, is that the relatively small value for the toroidal

magnetic field $B_\psi$ causes the RFP plasma to have a highly turbulent character. Because of the effects finite plasma resistivity and tearing modes, Taylor therefore argued that it made no sense to suggest that an infinite set of helicity constraints exists in the RFP. That is, the many small changes in the optimal RFP magnetic field caused by tearing modes lead to finite changes in the magnetic field topology. However, Taylor also argued that, since the changes in the magnetic field are small, the sum $\int A \cdot B d^3x$ integrated over the entire plasma will almost be unchanged. That is, the effect of the topological changes is merely to redistribute the helicity integrand among the magnetic field lines. Thus the total plasma magnetic helicity $K$ will still be a good RFP invariant.

If we repeat the variational procedure (V-A.18), now minimizing the magnetic energy subject to the single Taylor total helicity constraint, we find

$$\nabla \times B = \mu B,$$  \hspace{1cm} (V-A.20)

where the parallel current distribution $\mu$ is now a constant across the plasma minor radius. So Taylor's helicity theory
leads to the same conclusion as that of the previous chapter on tearing modes. That is, tearing modes allow the plasma to access lower states of magnetic energy by releasing the energy of the magnetic field that is associated with the large-scale gradient of the parallel current, $\partial \mu / \partial \psi$. The formation of magnetic islands and the destruction of magnetic surfaces therefore leads to a flattening of the parallel current profile $\mu(\psi)$.

Because of the highly turbulent nature of the RFP, the minimum energy, constant $\mu$ Taylor state gives a reasonable description of actual RFP discharges. Tokamak plasmas, however, are much more stable to tearing modes than RFP plasmas, and therefore do not relax to the minimum energy Taylor state. In fact, the plasma current distribution $\mu$ in a tokamak is normally far from uniform with a strong peak in the plasma center. Nevertheless, although the tokamak is sufficiently stable to tearing modes that the current profile is generally quite peaked, it is not free of tearing mode activity. As we have previously mentioned, tearing modes tend to arise when the tokamak current profile is hollow or too steep. In the tokamak, the effect of tearing modes is to locally flatten the current profile, reducing the magnetic
energy while holding the helicity almost constant.

Section V-B. Mean-field Ohm’s Law

In this section we discuss Ohm’s law for mean magnetic fields. Our mean-field theory derives from what we will call the helicity conservation theorem. Briefly, the helicity conservation theorem states that if either the plasma magnetic helicity (V-A.2) or the plasma magnetic energy (IV-C.3) is dissipated at an enhanced rate due to a fluctuating or turbulent magnetic field, then the enhancement of the helicity dissipation is much less than the energy dissipation. That is, the helicity is by far the most robust invariant in a turbulent plasma. Because of the great importance of this theorem to our research, we begin this section with a rather detailed proof of it.

First, as is well-known, the internal dissipation of energy dW/dt in a plasma is

\[ \frac{dW}{dt} = -\int E \cdot j \, d^3x. \quad (V-B.1) \]

From (V-A.17) it is also straightforward to show that the internal rate of helicity dissipation $dK/dt$ is

$$dK/dt = -2c\int E \cdot B d^3x. \quad (V-B.2)$$

We assume that the plasma is force-free and define a set of characteristic dissipation rates $dW_c/dt$ and $dK_c/dt$. The characteristic dissipation rates are defined so that they would be the exact dissipation rates (V-B.1) and (V-B.2), if the current in the plasma were $j=(j_j/B)B$, with $(j_j/B)$ a constant. Using the generalized Ohm's law (II-A.3), we find

$$dW_c/dt = -\int n j \cdot j d^3x = -(j_j/B)^2 \int n B^2 d^3x \quad (V-B.3)$$

and

$$dK_c/dt = -2c\int n j \cdot B d^3x = -2c(j_j/B)\int n B^2 d^3x. \quad (V-B.4)$$

Also, the characteristic energy $W_c$ and the helicity $K_c$ are related by
\[ W_C = \frac{1}{8\pi} \int B^2 d^3x = \frac{1}{2c} \int A \cdot j d^3x \]

\[ = \frac{1}{2c} (j_{\parallel}/B) \int A \cdot B d^3x = \frac{1}{2c} (j_{\parallel}/B) K_C. \quad (V-B.5) \]

Equations (V-B.3) and (V-B.4) can therefore be rewritten as

\[ \frac{dW_C}{dt} = -4c^2 \frac{W_C}{K_C} \int \eta B^2 d^3x \quad (V-B.6) \]

and

\[ \frac{dK_C}{dt} = -\frac{K_C}{W_C} \frac{dW_C}{dt}. \quad (V-B.7) \]

Or, combining (V-B.6) and (V-B.7),

\[ \int \eta B^2 d^3x = -\frac{1}{4c^2} (\frac{dK_C}{dt})^2 / (\frac{dW_C}{dt}). \quad (V-B.8) \]

Now, for any two square integrable functions \( f(x) \) and \( g(x) \), the Cauchy-Schwarz inequality\(^{142}\) states that

\[ (\int f^2 d^3x)(\int g^2 d^3x) \geq (\int fg d^3x)^2. \quad (V-B.9) \]

Choosing

\[ f = \eta^{1/2}B \]  \hspace{1cm} (V-B.10)

and

\[ g = \eta^{1/2}j_{\parallel}. \]  \hspace{1cm} (V-B.11)

we have

\[ (\int \eta B^2 d^3x)(\int \eta j^2 d^3x) \geq (\int \eta jB d^3x)^2. \]  \hspace{1cm} (V-B.12)

Finally, combining (V-B.1), (V-B.2), (V-B.8) and (V-B.12), we derive the desired inequality,

\[ (dW/dt)/(dW_C/dt) \geq [(dK/dt)/(dK_C/dt)]^2. \]  \hspace{1cm} (V-B.13)

Note the extreme generality of the result (V-B.13), as it depends only on the force-free assumption and the Cauchy-Schwarz inequality. As mentioned in the Introduction, we are building our mean-field theory on a very firm foundation indeed.
The meaning of the inequality (V-B.13) can be understood as follows. First, if the current has a simple form in the plasma, then the helicity and the magnetic energy evolve on the same characteristic time scale $\tau_\chi$ of equation (III-C.15). However, if the current is highly spikey or turbulent, then the magnetic energy can be dissipated at an enhanced rate on a time scale $\tau_W$, where $\tau_W \ll \tau_\chi$. From the inequality (V-B.13), the enhancement of the helicity dissipation rate due to turbulent fields is then on a time scale $\tau_K$, where

$$\tau_K \geq (\tau_W \tau_\chi)^{1/2} \gg \tau_W.$$ \hfill (V-B.14)

The enhancement of the helicity dissipation is therefore much less than the energy dissipation. From the expressions for the dissipation rates (V-B.3) and (V-B.4), this makes intuitive sense, since a turbulent current density $j$ averages away in the helicity dissipation rate, but a spikey current enhances the energy dissipation rate through the $j^2$ term.

Consider now an Ohm's law for the mean magnetic field. As pointed out in the section on kinetic theory, an averaging procedure can introduce new terms into Ohm's law. The importance of the helicity conservation theorem is that the
requirement that the fluctuating magnetic fields not increase the overall rate of helicity dissipation in the plasma uniquely determines the form of the mean-field Ohm's law. From a detailed argument, Boozer\textsuperscript{143} has shown that helicity conservation requires the mean-field Ohm's law to take the simple form

\[ E + \nu \times B/c = \eta j - (B/B^2) \nabla \cdot [\lambda \nabla (j_||/B)], \] (V-B.15)

where the function \( \lambda(x,t) \) is a new positive parameter which has the form of a current viscosity\textsuperscript{144} and models the effect of topology altering magnetic fluctuations. We emphasize that the field \( B \) of (V-B.15) is now the mean magnetic field. The evolution of the mean-field magnetic Hamiltonian \( \chi = \chi(\psi,t) \) is then given by the parallel part of the mean-field Ohm's law (V-B.15),

\[ E \cdot B = \eta j \cdot B - \nabla \cdot [\lambda \nabla (j_||/B)]. \] (V-B.16)

Combining the parallel Ohm's law (V-B.16) and the internal helicity dissipation rate (V-B.2), we see that a volume

integration of the contribution of the fluctuating fields to
the helicity dissipation rate reduces to a surface integral.

\[ \frac{dK}{dt} = -2c\int_\Omega \nabla \cdot \mathbf{B} d^3x - \int \mathbf{h} \cdot dS, \]  

\[ (V-B.17) \]

where

\[ h = -2c\lambda \nabla (j_\parallel / B) \]  

\[ (V-B.18) \]
is the helicity flux due to the fluctuating fields. In our
mean-field theory, therefore, just as in Taylor's theory,
turbulent magnetic fields have the effect of redistributing
magnetic helicity throughout the plasma. If the turbulent
helicity flux \( h = 0 \) on the plasma boundary, then the total
magnetic helicity of the plasma is conserved. This is the real
significance of the Boozer form for the plasma Ohm's law.

A better understanding of the effect of the new current
viscosity term in Ohm's law \((V-B.16)\) can be obtained by
using the straight cylindrical tokamak approximation of
section III-C. Performing the asymptotic expansions in the
inverse aspect ratio \( \varepsilon = a/R_0 \), we find the fourth order,
parabolic partial differential equation

\[ \frac{\partial \chi}{\partial t} = \left( \frac{\eta c^2}{4 \pi} \right) \nabla^2 \chi - \nabla \cdot \left[ \left( \frac{\lambda c^2}{4 \pi B_0^2} \right) \nabla (\nabla^2 \chi) \right] \]  (V-B.19)

for the evolution of the magnetic poloidal flux \(-\chi(r,t)\). Note that the topology altering term involving the current viscosity \(\lambda(r,t)\) is a \(\nabla^4 \chi\) diffusion of the magnetic flux and is the highest spatial derivative in (V-B.19). Recall from the section on ideal MHD theory that alterations in topology are always modeled by the highest spatial derivatives in evolution equations. From the mean-field Ohm's law (V-B.19), it is straightforward to show that the general effect of the parameter \(\lambda\) is to prevent the large-scale current profile

\[ \mu = \left( \frac{1}{2\pi \tau R_0 B_0} \right) \nabla^2 \chi \]

from changing on a spatial scale shorter than

\[ l^2 = \frac{\lambda}{\eta B_0^2}, \]  (V-B.20)

thus locally flattening \(\mu\) over the scale length \(l\).\(^{145,146}\) As we shall see, the current viscosity \(\lambda\) is large in regions of


significant tearing mode activity. In fact, if $\lambda > \eta B_0^2 a^2$, where $a$ is the plasma minor radius, then the effect of the current viscosity term is to make the parallel current $\mu$ nearly uniform across the entire plasma minor radius. For a highly turbulent plasma our mean-field theory therefore reproduces the Taylor state with constant $\mu$.

Section V-C. Current Viscosity

In this section we present physical arguments for the form of the current viscosity $\lambda(r,t)$, if the small-scale magnetic fluctuations are due to tearing modes. We consider the case where tearing activity leads to the formation of magnetic islands. Two different kinds of arguments are given. First, we give an argument based on the theory of weak turbulence, or quasilinear theory.\textsuperscript{147} Next, we give a detailed kinetic argument utilizing the magnetic diffusion coefficient $D_m$ of section IV-B. As we shall see, both arguments yield the result that the scale length for current flattening $\lambda$ of equation (V-B.20) is on the order of the magnetic island width $w_{nm}$ of equation (IV-B.36), a result

which certainly makes intuitive sense.

Our quasilinear calculation of $\lambda$ is performed by comparing the local energy dissipation rates from the two Ohm's laws (II-A.3) and (V-B.15). Forming the vector product of the mean-field Ohm's law (V-B.15) and the current density $j=(j_\parallel/B)B$, we can show that the mean power $P_{\text{fluc}}$ dissipated by the small-scale magnetic fluctuations is

$$P_{\text{fluc}} = \lambda[\nabla(j_\parallel/B_0)]^2.$$  \hspace{1cm} (V-C.1)

On the other hand, using the generalized Ohm's law (II-A.3), the mean power dissipated by the fluctuations is

$$P_{\text{fluc}} = \eta \langle j^2 \rangle,$$  \hspace{1cm} (V-C.2)

where $\langle j^2 \rangle$ is the mean-square fluctuating field current density. Using (IV-C.15) and (IV-C.21), we can show that

$$\langle j^2 \rangle = \sum_{nm} \xi_{nm}^2 B_0^2 [\nabla(j_\parallel/B_0)]^2.$$  \hspace{1cm} (V-C.3)

where $\xi_{nm}$ is the $(n,m)$ Fourier mode radial plasma density.

displacement (IV-D.6). Then comparing (V-C.1) and (V-C.2), using (V-C.3), we have

\[ \chi = \eta B_0^2 \sum_{nm} \langle \xi_{rnm}^2 \rangle. \]  

(V-C.4)

which was first obtained by Strauss,\textsuperscript{149} in a very different way. From equations (V-B.19) and (V-C.4), the scale length

\[ l^2 = \chi / \eta B_0^2 = \langle \xi_{rnm}^2 \rangle. \]

Of course, since the formula (IV-D.6) for \( \xi_{rnm}^2 \) is only valid away from the rational surface \( \chi(r_{nm}) = n/m \), the relation (V-C.4) is not exactly what we want. The important effects of magnetic fluctuations occur near the rational surfaces, where the topology of the magnetic field can be altered due to the opening of a magnetic island. In fact, the formation of magnetic islands breaks the exact resonance condition \( \chi(r_{nm}) = n/m \), thus allowing for a resolution of the singularity in the function \( \xi_{rnm}^2 \). A crude resolution of the resonant denominator \( (\chi-n/m)^2 \) in \( \xi_{rnm}^2 \) is made by making the replacement\textsuperscript{150}


\[ \xi_{nm}^2 = \frac{\chi_{nm}^2/(2\pi r B_0)^2}{(l-n/m)^2} \rightarrow \frac{\chi_{nm}^2/(2\pi r B_0)^2}{((l-n/m)^2 + \delta_{nm}^2)}, \quad (V-C.5) \]

where the quantity \( \delta_{nm} \) scales as the island width in iota-space,

\[ \delta_{nm} \approx \left| \frac{\partial \varphi}{\partial r} R_0 B_{rnm}/(mB_0) \right|^{1/2}. \quad (V-C.6) \]

Combining (IV-D.6) and (V-C.5), using (IV-B.32), we find the simple scaling

\[
\begin{align*}
l^2 & \approx \frac{R_0^2/m^2 B_0^2 \delta_{nm}^2}{\langle B_{rnm}^2 \rangle} \\
& \approx \left| \frac{R_0 B_{rnm}/mB_0(\partial \varphi/\partial r)}{B_{rnm}} \right| = w_{nm}^2 \quad (V-C.7)
\end{align*}
\]

near the mode rational surface. Note that (V-C.7) implies that the current viscosity \( \lambda \) is proportional to \( |B_{rnm}| \), not \( |B_{rnm}|^2 \), as one might naively expect. This is due to the fact that the characteristic scale of a topology altering magnetic perturbation is proportional to \( |B_{rnm}|^{1/2} \), not \( |B_{rnm}| \), which was already mentioned in the chapter on tearing modes.
A more physical understanding of the current viscosity $\lambda$ can be gained by examining a kinetic theory. The kinetic treatment stems from the pioneering work of Rechester and Rosenbluth$^{151}$ and Jacobson and Moses$^{152}$. These works stress the point that a small, fluctuating radial magnetic field $B_r$ can cause "cross-field" plasma transport in a toroidal confinement system with destroyed magnetic surfaces. In fact, this is a good way of viewing the origin of the $\lambda$ term in the mean-field Ohm's law (V-B.18). That is, in mean-field theory we deal with an integrable magnetic Hamiltonian $\chi(\psi)$ and always have good magnetic surfaces. The theoretical price paid for this simplification, however, is the complication caused by the new current viscosity term. In effect, the current viscosity allows parallel current to radially cross the mean-field magnetic surfaces in regions corresponding to those where the actual magnetic surfaces of the complete magnetic field are destroyed.

Let us examine a kinetic model similar to that of Jacobson and Moses. We begin by assuming that the plasma is a Lorentz gas$^{153}$ of mobile electrons and stationary ions at

$^{153}$L. Spitzer, Jr., Physics of Fully Ionized Gases, (Interscience, New
uniform temperature and density. Recall from the kinetic equation (II-B.1) that the collision operator $C(f_\omega)$ represents the averaged effects of random phase space processes. It is easy to show that, in order to conserve particles, the collision operator must take the form of a divergence of some vector function in the phase space. The usual Fokker-Planck collision operator,\textsuperscript{154} however, which is purely a velocity space divergence, only incorporates the effects of random processes in velocity space. Jacobson and Moses, on the other hand, argued that if the magnetic field were separated into mean and fluctuating parts, then the collision operator should also incorporate the random processes due to the magnetic field. In mean-field theory, therefore, the collision operator also has a coordinate space divergence term. An effective kinetic equation for the cross-field transport of electrons is then

\[-\left(\frac{e}{m_0}\right)E_\parallel \cdot \frac{\partial f}{\partial v_\parallel} = \nabla \cdot (D_m \left| V_\parallel \right| \nabla f), \quad (V-C.8)\]

where we use equation (IV-B.34) to define the diffusion

\footnotesize{York, 1956), p. 83.}
\footnotesize{\textsuperscript{154}M. N. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. 107, 1 (1957).}
coefficient $D_m |v|$

Now, recall from the section on kinetic theory the important principle that new averaging procedures lead to additional terms in Ohm's law. The new cross-field current diffusion term in Ohm's law is obtained by multiplying both sides of (V-C.8) by the moment $-ev_{||} / |v_{||}|$ and integrating over the velocity space. From the Maxwellian distribution function $f_M$ of equation (II-C.14) with $v_i = 0$, the derivative $\partial f / \partial v_{||} = -(m_e v_{||} / T) f_M$. The resulting moment equation can then be written in the simple form\textsuperscript{155}

\begin{equation}
E \cdot B = - \nabla \cdot [\lambda \nabla (j_{||} / B)], \quad (V-C.9)
\end{equation}

where the current viscosity

\begin{equation}
\lambda \approx \eta B_0^2 (2D_m \lambda_C), \quad (V-C.10)
\end{equation}

with the mean free path $\lambda_C$ and the resistivity $\eta$ given by equations (I-B.9) and (II-C.17). For the case of enhanced transport due to magnetic islands, the effective magnetic diffusion coefficient $D_m$ is given by equation (IV-B.37).

Kinetic theory therefore yields the current flattening scale length

\[ l^2 = \lambda/\eta B_0^2 \approx 2D_m \lambda_c \approx w_{nm}^2. \]  \hspace{1cm} (V-C.11)

The results of the quasilinear helicity theory (V-C.7) and the kinetic theory (V-C.11) are therefore in agreement with each other.

It can further be shown that tearing modes create spikey currents and voltages only within the vicinity of magnetic islands.\textsuperscript{156} For the spatial dependence of the current viscosity \( \lambda(r,t) \), we have therefore chosen the simple Gaussian form,

\[ \lambda(r,t) = \sum_{nm} \eta(r_{nm},t) B_0^2 w_{nm}^2 \exp[-4(r-r_{nm})^2/w_{nm}^2], \]  \hspace{1cm} (V-C.12)

where the summation extends over all the magnetic islands in the plasma. The time dependence of the current viscosity \( \lambda(r,t) \) is therefore determined by the resistivity \( \eta(r,t) \), the positions of the resonant surfaces \( r_{nm}(t) \), and the magnetic island widths \( w_{nm}(t) \).

If we are given the form for the plasma resistivity \( \eta(r,t) \), then the procedure outlined by (V-B.19), (V-C.12), (IV-D.4), and (IV-C.15):

\[
\frac{\partial \chi}{\partial t} = \left( \eta c^2 / 4\pi \right) \nabla^2 \chi - \nabla \cdot \left[ \left( \eta c^2 / 4\pi B_0^2 \right) \nabla (\nabla^2 \chi) \right], \quad (V-C.13a)
\]

\[
\chi(r,t) = \sum_{nm} \eta(r_{nm},t) B_0^2 w_{nm}^2 \exp\left[ -4(r - r_{nm})^2 / w_{nm}^2 \right], \quad (V-C.13b)
\]

\[
dw_{nm}/dt = 1.28 \left( \eta c^2 / 4\pi \right) \Delta'_{nm}, \quad (V-C.13c)
\]

and

\[
\nabla_\perp^2 \chi_{nm} = \left[ (R_0/r)(d\mu/dr)/(l-n/m) \right] \chi_{nm}, \quad (V-C.13d)
\]

gives a self-consistent, quasilinear treatment to the problem of tearing modes in an inductively driven tokamak. That is, according to (V-C.13d), if the shape of the current profile causes a tearing instability \( \Delta'_{nm} \) to occur, then a magnetic island begins to grow, at a rate given by (V-C.13c). Following equation (V-C.13b), the island generates a current viscosity \( \chi(r,t) \), which flattens the current profile about the unstable resonant surface, according to the poloidal flux evolution.
equation (V-C.13a). The flattening of the current profile stabilizes the instability, making the procedure self-consistent. Of course, flattening the current profile in one location will steepen it somewhere else, destabilizing other tearing modes. While our mean-field theory does not consider the exact non-linear problem with direct mode-mode couplings, tearing modes are able to interact with each other via the intermediary of the current profile, making the procedure quasi-linear. The set of equations (V-C.13) will be discussed further in the chapters on numerical methods and inductively driven tokamaks.
In this chapter we discuss some of the effects of toroidicity on plasma transport. This is an important subject because collisional transport in a tokamak depends mainly on toroidal effects. That is, in toroidal geometries, the particle and heat diffusion coefficients are much larger than those given by rough estimates such as (I-B.14), which are valid only for a cylinder. The extension of the classical collisional theory to toroidal geometries is called neoclassical transport theory. For our research, we are interested in the neoclassical enhancement of transport in the long mean free path, thermonuclear plasma regime. This regime of low plasma collisionality was first analyzed by Galeev and Sagdeev.\footnote{A. A. Galeev and R. Z. Sagdeev, Sov. Phys. JETP 26, 233 (1968).} In low collisionality regimes, the toroidal enhancement of transport is dominated by the effects of magnetically trapped particles.\footnote{B. B. Kadomtsev and O. P. Pogutse, Nuc. Fusion 11, 67 (1971).} So-called trapped particle effects are also quite common in space plasmas. For example,
the Van Allen radiation belts are composed of charged particles which are trapped by the Earth's magnetic field.\textsuperscript{159} Magnetically trapped particles result from the magnetic mirror effect, which we now briefly discuss.

The bouncing of charged particles in magnetic fields, or the magnetic mirror effect, arises from the existence of the magnetic moment invariant $\mu_m$. As is well-known, if the magnetic field strength $B$ changes slowly within a plasma, then the magnetic moment

$$\mu_m = \frac{mv^2}{2B} \quad (VI-A.1)$$

is an adiabatic invariant for the motion of plasma particles.\textsuperscript{160,161} That is, $d\mu_m/dt=0$ along a plasma particle trajectory, for slowly varying magnetic fields. Replacing the perpendicular kinetic energy $mv^2/2$ with $\mu_m B$ and designating the electric potential by $\phi$, we can then write the particle energy $H$ as

\begin{itemize}
  \item \textsuperscript{160} T. G. Northrop, \textit{The Adiabatic Motion of Charged Particles}, (Interscience, New York, 1963), pp. 41-77.
\end{itemize}
\[ H = m v_{\parallel}^{2}/2 + \mu_m B + e\Phi. \quad (VI-A.2) \]

The energy (VI-A.2) is also a good particle invariant, for time independent fields. For simplicity, suppose that the electric potential \( \Phi = 0 \). If \( B_{\text{max}} \) and \( B_{\text{min}} \) are the maximum and minimum values of the magnetic field strength, then the invariance of the magnetic moment (VI-A.1) and the conservation of the particle energy (VI-A.2) together mean that any particle satisfying the inequality\(^{162}\)

\[ m v_{\parallel}^{2}/2 + \mu_m B_{\text{min}} < \mu_m B_{\text{max}} \quad (VI-A.3) \]

will be trapped between regions of high magnetic field. That is, at or before the position where the field strength \( B = B_{\text{max}} \), a trapped plasma particle has a parallel velocity \( v_{\parallel} = 0 \), and therefore must reverse its sense of motion along the magnetic field, which explains the name magnetic mirror.

As we have previously mentioned, in a toroidal magnetic geometry such as the tokamak, the inboard value of the magnetic field is naturally greater than that of the outboard.

side. That is, as discussed in the Introduction, toroidal magnetic fields possess a field gradient $\nabla B$. If we designate $\varepsilon = r/R$ as the local inverse aspect ratio, it can be shown that the field gradient $\nabla B$ causes the trapping of a fraction $\varepsilon^{1/2}$ of plasma particles, each with a typical parallel velocity $v_{\parallel} \approx \varepsilon^{1/2} v_T$. The magnetically trapped plasma particles bounce back and forth in the magnetic mirror at an effective bounce frequency $\omega_b$. The bounce frequency $\omega_b$ is on the order of

$$\omega_b \approx v_{\parallel}/Rq \approx \varepsilon^{1/2} v_T/Rq,$$  \hfill (VI-A.4)

where $q$ is the tokamak safety factor (IV-D.8). If $\tau$ is the particle collision time, then the long mean free path regime is characterized by the inequality

$$\omega_b \gg \nu_{\text{eff}},$$  \hfill (VI-A.5)

where

$$\nu_{\text{eff}} = (\varepsilon \tau)^{-1}$$  \hfill (VI-A.6)

is the effective collision frequency for detrapping. The
inequality (VI-A.5) means that a plasma particle completes many mirror bounces before suffering a collision.

The important difference between magnetically trapped and untrapped, or passing, plasma particles can be understood by examining the different nature of the particle orbits. As mentioned in the Introduction, passing plasma particles are free to make complete transits around the torus, thus averaging away the grad B drift (I-C.2). On the other hand, trapped particles cannot complete full toroidal transits. Instead, trapped particles make small excursions away from their home flux surfaces and bounce repeatedly, tracing out so-called banana orbits.\textsuperscript{163} The typical width $w_B$ of a banana orbit is on the order of

$$w_B \approx \varepsilon^{1/2} q \rho,$$  \hspace{1cm} (VI-A.7)

where $\rho$ is the gyroradius (I-B.11). The banana width (VI-A.7) can be derived from the conservation of the toroidal particle canonical momentum

$$p_\psi = m v_\parallel R - e\chi/c.$$ \hspace{1cm} (VI-A.8)

That is, with the variations $\delta v_\parallel \approx \varepsilon^{1/2} v_T$ and $\delta x \approx R B_\theta w_b$, the variation $\delta p_\theta = 0$ of equation (VI-A.8) yields (VI-A.7). If we now use the general formula (I-B.8) to calculate an effective diffusion coefficient $D_b$ for the $\varepsilon^{1/2}$ fraction of trapped particles, we find

$$D_b \approx \varepsilon^{1/2} (v_{\text{eff}} w_b^2) \approx (\varepsilon^{-3/2} q^2)(\rho^2/\varepsilon), \quad (VI-A.9)$$

which exceeds the classical coefficient (I-B.14) by the potentially large factor $\varepsilon^{-3/2} q^2$. For example, the result of Galeev and Sagdeev for the enhancement of plasma electron radial particle flux in a tokamak may be written as

$$n_e v_r = - (2r/R_0)^{1/2} (R_0/r)^2 q^2 (\rho_e^2/\varepsilon_e) \partial n_e/\partial r, \quad (VI-A.10)$$

where $v_r$ is the radial plasma velocity.

One very important result of neoclassical transport theory concerns the existence of a diffusion driven parallel plasma current in the thermonuclear plasma regime. This diffusion driven or so-called "bootstrap" current is predicted to exist by the very general Onsager symmetry relations of
non-equilibrium thermodynamics. In fact, in their original work, Galeev and Sagdeev gave a simple expression for the bootstrap current of the form

\[ j_b = -c [(2r/R)^{1/2}/B_0] \frac{\partial p}{\partial r}, \]  

(VI-A.11)

where \( p \) is the plasma pressure. We give a heuristic explanation of the bootstrap current density (VI-A.11). First, in the presence of a density gradient \( \partial n/\partial r \), the \( \epsilon^{1/2} \) fraction of trapped electrons carry a current

\[ j_t \approx -e \epsilon^{1/2} (\epsilon^{1/2}v_T) (w_B \partial n/\partial r) \]

\[ \approx -c q \epsilon^{1/2} (T/B) \frac{\partial n}{\partial r}. \]  

(VI-A.12)

However, this is not yet the bootstrap current (VI-A.11). That is, because of the continuity of the electron distribution function in velocity space, or recognizing that the trapped particles interact with the larger fraction of passing particles, entraining them via collisions, the trapped particles transfer their momentum to the passing electrons at the effective collision frequency \( \nu_{\text{eff}} \) of (VI-A.6). The

\[ \text{\footnotesize\cite{164L.Hinton1959}}, \text{\footnotesize\cite{165A.Galeev1978}}. \]
passing electrons then collide with ions at a frequency on the order of $\tau_e^{-1}$. The bootstrap current is therefore

$$j_b \approx (\nu_{\text{eff}} \tau_e) j_t \approx -c \left(\frac{\varepsilon^{1/2}}{B_0}\right) T \frac{\partial n}{\partial r} \tag{VI-A.13}$$

which explains the Galeev-Sagdeev scaling (VI-A.11). It is important to note that recent experiments on the TFTR,$^{166}$ JET,$^{167}$ and JT-60$^{168}$ tokamaks report significant values for the tokamak bootstrap current. The bootstrap current is apparently a very real phenomenon.

Section VI-B. Current Amplification

Since the parallel Ohm's law (III-C.1) represents the parallel force balance on plasma electrons, we can incorporate the bootstrap current term (VI-A.11) into this equation by balancing the forces on the passing electrons.


That is, the bootstrap current term \( j_b \) simply represents the frictional force which the trapped electrons exert on the passing electrons. Let us examine the consequences of the bootstrap current term \( j_b \) in the Ohm's law (VI-B.1). First, consider a plasma pressure profile

\[
p = p_0 \left[ 1 - \left( \frac{r}{a} \right)^2 \right]^{\beta}. \tag{VI-B.2}
\]

That is, the pressure \( p \) is assumed to be a parabola to some positive power \( \beta \). For the discussion, suppose that the poloidal field \( B_\theta \propto r \). From (VI-A.11), we then have

\[
j_b \propto r^{1/2} \left[ 1 - \left( \frac{r}{a} \right)^2 \right]^{\beta-1}. \tag{VI-B.3}
\]

a characteristically hollow (outwardly peaked) bootstrap current profile. Expanding (VI-B.3) about the origin, we see that \( j_b \propto r^{1/2} \) near the magnetic axis. [Actually, since the plasma must be in the plateau,\(^{169}\) not the banana, transport

regime in a small region about the magnetic axis, the non-analytic expression (VI-A.11) does not apply there. The plateau-corrected form\textsuperscript{170} for (VI-A.11) has $j_b \propto r^2$, an analytic function which also vanishes at $r=0$. That is, there is no bootstrap current at the origin.

To appreciate the importance of this last fact, we use the neoclassical transport theory of the last section. Combining equation (VI-A.10) with the approximation

$$j_b = -c (2r/R_0)^{1/2} (T_e/B_0) \partial n_e/\partial r \quad \text{(VI-B.4)}$$

gives

$$\eta j_b = v_r B_\theta /c, \quad \text{(VI-B.5)}$$

an important result first derived by Kadomtsev and Shafranov.\textsuperscript{171} Equation (VI-B.5) indicates that the bootstrap effect is equivalent to a radial outflow of plasma. From the toroidal Ohm's law

$$E_\phi + v_r B_\theta /c = \eta j_\phi, \quad \text{(VI-B.6)}$$

\textsuperscript{170}D. J. Sigmar, Nucl. Fusion 13, 17 (1973).
we see that, in addition to a loop voltage, a radial outflow of plasma can also support the toroidal current. Therefore, consider a steady-state without a loop voltage, $E_\psi=0$. Then the Ohm's law (VI-B.6), along with Ampere's law $j_\psi=(c/4\pi r)d(rB_\theta)/dr$, leads to an equation for the total toroidal current $I(r)$ within a radius $r$,

$$\frac{dl}{dr} = \left[ \nu_F/(\eta c^2/4\pi) \right] I. \quad \text{(VI-B.7)}$$

where $I(r)=rB_\theta c/2$. The very important result (VI-B.7) was first derived by Bickerton, Connor, and Taylor.\textsuperscript{172} Defining

$$\kappa(r) = \int [\nu_F/(\eta c^2/4\pi)] dr, \quad \text{(VI-B.8)}$$

the solution to (VI-B.7) is

$$I(r) = I(0) \exp[\kappa(r)]. \quad \text{(VI-B.9)}$$

Equation (VI-B.9) simply means that the bootstrap effect is a current amplification with $\kappa(r)$ the critical amplification

factor. However, since the current $I(r)$ cannot increase from $I(0)=0$, equation (VI-B.7) also acts as a boundary condition on the bootstrap effect. That is, without some other source of current near the magnetic axis, there is nothing for the bootstrap effect to amplify. The so-called "seed" current near the magnetic axis would apparently have to be driven by some other, non-inductive means. These are the considerations which led Bickerton, et al. to state that "a completely bootstrapped tokamak is not possible," a statement which was already mentioned in the Introduction.

Another, more abstract, but equivalent way of understanding the tokamak bootstrap problem is from a magnetic flux point of view. That is, the poloidal flux content of a plasma is $\chi_b-\chi_0$ with $-\chi_0$ the poloidal flux enclosed by the magnetic axis. In the absence of any inductive drive, the boundary voltage $d\chi_b/dt=0$. Since the bootstrap current on axis $j_b(0)=0$, the usual Ohm's law gives

$$d\chi_0/dt = 2\eta(0)(c/4\pi)(B_0/R_0) I(0) > 0.$$  

(VI-B.10)

If the rotational transform $\iota(r)$ is positive throughout the
plasma, then the plasma has a positive poloidal flux content,

$$\chi_b - \chi_0 = 2\pi B_0 \int \mu(r) r dr > 0.$$  \hspace{1cm} \text{(VI-B.11)}

On the other hand,

$$d(\chi_b - \chi_0)/dt = -d\chi_0/dt < 0,$$  \hspace{1cm} \text{(VI-B.12)}

so a vanishing loop voltage implies that $\chi_b - \chi_0$ must continuously drop. The bootstrap effect cannot, therefore, create poloidal magnetic flux. Instead, the flux is pushed out of regions of high plasma pressure, creating the hollow bootstrap current profile.\textsuperscript{173} It is well-known that a hollow current profile has a very negative effect on tearing mode stability.\textsuperscript{174} We note that hollow tokamak current profiles have been observed recently in JET discharges with pellet injection.\textsuperscript{175}

The fact that the bootstrap effect, by itself, leads to hollow current profiles and that hollow current profiles lead to tearing modes makes one suspect that some analysis of

\textsuperscript{175}W. Kerner (private communication).
tearing modes should also be made in conjunction with the bootstrap effect. Since the effect of tearing modes can be included in Ohm's law via the mean-field theory of chapter V, we analyze the bootstrapped tokamak by combining equations (V-B.16) and (VI-B.1),

\[ \mathbf{E} \cdot \mathbf{B} = \eta (j \cdot \mathbf{B} - j_{\parallel} \mathbf{B}) - \nabla \cdot \lambda \nabla (j_{\parallel} / B), \quad (VI-B.13) \]

which was given in the Introduction as (I-D.3). Using the cylindrical tokamak approximation, equation (VI-B.13) takes the form

\[ \partial \chi / \partial t = \eta [(c^2 / 4\pi) \nabla^2 \chi - 2\pi R_0 c j_{\parallel} B] - \nabla \cdot [(\lambda c^2 / 4\pi B_0^2) \nabla (\nabla^2 \chi)]. \quad (VI-B.14) \]

If we now incorporate the parallel Ohm's law (VI-B.14) into the procedure outlined by (V-C.13), we have a complete set of bootstrapped tokamak equations:

\[ \partial \chi / \partial t = \eta [(c^2 / 4\pi) \nabla^2 \chi - 2\pi R_0 c j_{\parallel} B] - \nabla \cdot [(\lambda c^2 / 4\pi B_0^2) \nabla (\nabla^2 \chi)]. \quad (VI-B.15a) \]

\[ \lambda (r,t) = \sum_{nm} \eta (r_{nm},t) B_0^2 w_{nm}^2 \exp[-4(r-r_{nm})^2/w_{nm}^2], \quad (VI-B.15b) \]
\[
\frac{dW_{nm}}{dt} = 1.28 \left( \eta c^2 / 4\pi \right) \Delta'_{nm},
\]

(VI-B.15c)

and

\[
\nabla_{\perp}^2 \chi_{nm} = \left[ \frac{(R_0/r)(d\mu/dr)/(\bar{\iota}-\bar{n}/m)}{(\bar{\iota}-\bar{n}/m)} \right] \chi_{nm},
\]

(VI-B.15d)

where all physical quantities are defined as in (V-C.13).

As was emphasized in the Introduction, tearing modes can create poloidal magnetic flux, and thus may be very useful for the bootstrap effect. From the point of view of the required bootstrap “seed” current on axis, the current viscosity \( \lambda \) of the mean-field equation (VI-B.14) provides a possible alternative to externally driving the tokamak central current. That is, the tearing mode activity in the tokamak center due to the hollow current profile can be used to transport plasma current to the magnetic axis. This is important because, as mentioned in the Introduction, external current drive schemes are undesirable for steady-state tokamak operation because they are only marginally efficient.\(^{176}\) With the current viscosity induced current, on

the other hand, the possibility exists of intrinsically achieving a tokamak steady-state via the bootstrap effect.\textsuperscript{177} Of course, this type of tokamak operation is somewhat different from the standard inductive one with centrally peaked current profiles. This subject will be discussed further in the chapter on completely bootstrapped tokamaks.

\textsuperscript{177}A. H. Boozer, Phys. Fluids 29, 4123 (1986).
CHAPTER VII. NUMERICAL METHODS

Section VII-A. Scaled Model Equations

This chapter describes some of the numerical methods we have developed in order to solve the set of model equations (V-C.13) for inductively driven tokamaks or the set (VI-B.15) for completely bootstrapped tokamaks. Fundamentally, the numerical procedures used to solve either set of equations differ only slightly from each other. However, because the inductively driven case is somewhat simpler to understand, we initially focus our attention on the numerical methods necessary to solve the set (V-C.13). The modifications of the methods presented in this chapter necessary to solve the bootstrap equations (VI-B.15) will be discussed later in the chapter on completely bootstrapped tokamaks. Actually, ordering our exposition in this way also mimics the true progression of our research.

Let us begin our discussion of (V-C.13) by scaling these equations. First, we make the substitutions
\[ r' = r/a, \] (VII-A.1)

\[ \chi' = \chi/(2\pi a^2 B_0), \] (VII-A.2)

\[ \eta' = \eta/\eta_0, \] (VII-A.3)

\[ \lambda' = \lambda/(\eta_0 a^2 B_0^2), \] (VII-A.4)

and

\[ t' = t/\tau_\chi \] (VII-A.5)

into equations (V-C.13). In the scalings (VII-A.1)-(VII-A.5), \( a \) is the plasma minor radius, \( B_0 \) is the large constant toroidal magnetic field, \( \eta_0 \) is the resistivity on axis, and \( \tau_\chi \) is the skin time (III-C.15) with the resistivity \( \eta_0 \).

\[ \tau_\chi = 4\pi a^2/\eta_0 c^2. \] (VII-A.6)

Dropping the primes from the scaled quantities, the set of equations (V-C.13) becomes Ohm's law,
\[
\frac{\partial \chi}{\partial t} = \eta \nabla^2 \chi - \nabla \cdot [\lambda \nabla (\nabla^2 \chi)], \quad (VII-A.7)
\]

with \(\eta(r,t)\) the scaled plasma resistivity and \(\lambda(r,t)\) the current viscosity,

\[
\lambda(r,t) = \sum_{nm} \eta(r_{nm},t) \omega_{nm}^2 \exp[-4(r-r_{nm})^2/w_{nm}^2], \quad (VII-A.8)
\]

the island growth equation,

\[
\frac{d\omega_{nm}}{dt} = 1.28 \eta(r_{nm},t) \Delta'_{nm}, \quad (VII-A.9)
\]

and the stability equation,

\[
\nabla_\perp^2 \chi_{nm} = [\frac{1}{r^2} \frac{\partial j_z}{\partial r}/(r-n/m)] \chi_{nm}. \quad (VII-A.10)
\]

In (VII-A.7)-(VII-A.10), the scaled tokamak current density is

\[
j_z(r,t) = \nabla^2 \chi, \quad (VII-A.11)
\]

the rotational transform is

\[
n(r,t) = \frac{1}{r} \frac{\partial \chi}{\partial r}, \quad (VII-A.12)
\]
the operator $\nabla_\perp^2$ is

$$\nabla_\perp^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}, \quad (\text{VII-A.13})$$

and the stability factor delta prime is

$$\Delta'_{nm} = \frac{[dX_{nm}/dr]}{X_{nm}}, \quad (\text{VII-A.14})$$

with $[dX_{nm}/dr]$ the jump in the derivative $dX_{nm}/dr$ at the $(n,m)$ rational surface.

Obviously, the simplest equation to solve in our set of scaled equations is the island growth equation (VII-A.9). That is, if at the time $t$ we are given the plasma resistivity $\eta$, the stability factor $\Delta'_{nm}$, and the island width $w_{nm}$, then we can explicitly step equation (VII-A.9) forward in time to obtain an approximation for the island width $w_{nm}$ at the later time $t+\Delta t$. We assume at the initial time $t=0$ that all island widths $w_{nm}=0$. Of course, the width of a magnetic island cannot become negative. This means that should $\Delta'_{nm}$ be negative at some time $t$ for a particular $(n,m)$ mode and
cause \( w_{nm} \) to drop below zero at the time \( t+\Delta t \), then by default we set \( w_{nm}=0 \) at the time \( t+\Delta t \), and that the particular \((n,m)\) mode under consideration does not contribute to the current viscosity sum (VII-A.8).

Unfortunately, both the poloidal flux evolution equation (VII-A.7) and the tearing mode stability equation (VII-A.10) are much more difficult to solve than the island growth equation. The numerical methods used to obtain solutions of these equations forms the subject matter for the remainder of this chapter.

Section VII-B. Poloidal Flux Evolution

Without question, the most difficult and important equation we must solve is (VII-A.7), the mean-field Ohm's law or poloidal flux evolution equation,

\[
\frac{\partial \chi}{\partial t} = \eta \nabla^2 \chi - \nabla \cdot [\lambda \nabla (\nabla^2 \chi)].
\]  

(VII-B.1)

This equation is a fourth-order, parabolic partial differential equation for the poloidal magnetic flux \(-\chi(r,t)\). That is, in addition to the standard \( \nabla^2 \) diffusion of the poloidal
magnetic flux discussed in section III-C, equation (VII-B.1) also includes a $\nabla^4$ diffusion of the flux proportional to the current viscosity $\lambda$. Of course, in regions with no tearing mode activity where the current viscosity $\lambda=0$, the underlying structure of (VII-B.1) remains that of the magnetic diffusion equation (III-C.14). That is, the $\lambda=0$ equation is just

$$\frac{\partial \chi}{\partial t} = \eta \nabla^2 \chi. \quad \text{(VII-B.2)}$$

In the $\lambda=0$ regions, therefore, the shape of the current profile $\nabla^2 \chi$ is determined by the resistivity profile $\eta(r,t)$ and loop voltage $\partial \chi/\partial t$ alone. However, from the discussion of tearing modes in chapter IV, we know that even a small amount of magnetic activity $\lambda$ can significantly alter the properties of the mean-field current density $\nabla^2 \chi$. Mathematically, this is because the current viscosity $\lambda$ multiplies the highest spatial derivative $\nabla^4 \chi$ in the evolution equation (VII-B.1). In regions with tearing mode activity, the viscosity term $\lambda \nabla^4 \chi$ can therefore successfully compete with the resistive evolution term $\eta \nabla^2 \chi$. In fact, comparing these two terms, we see that the critical parameter is
which is just the scaled version of (V-B.20).

In order to solve equation (VII-B.1), we have developed an implicit, finite-difference numerical method similar to the Crank–Nicolson\textsuperscript{178,179} method for solving the diffusion equation (VII-B.2). Because of the implicit nature of our scheme, we expend a rather considerable amount of computational effort in obtaining the numerical solution. That is, in order to evolve our numerical solution forward in time, we must repeatedly solve a certain self-consistent system of linearized equations. The implicitness requirement is important, however, since it insures the numerical stability of the solutions to our finite-difference equations. A numerical instability, in this context, means that small inaccuracies in initial values of the numerical solution can become arbitrarily large as the numerical solution evolves forward in time. Physically, we can understand the

\begin{equation}
l^2 = \lambda / \eta.,
\end{equation}

\text{(VII-B.3)}

stabilizing requirement of implicitness as the need for a specification of relevant boundary conditions at each time step before a satisfactory solution of the numerical problem can be attained.\textsuperscript{180}

The numerical method we have developed is somewhat more involved than the Crank-Nicolson scheme for basically two reasons: (1) the higher order space derivatives of equation (VII-B.1) require rather large computational molecules and (2) the need for an extraordinary high degree of numerical accuracy. Of course, the round-off error due to the finite nature of the CRAY supercomputers used in this work will not seriously effect the numerical accuracy. That is, FORTRAN REAL data elements on the CRAY have approximately 15 decimal digits of precision, which is more than enough. The discretization error due to the replacement of the continuous problem (VII-B.1) by a discrete model, however, is of some concern. Nevertheless, we will show that the numerical method we are using has a global discretization error of only $O(\Delta t^2 + \Delta r^4)$. This high numerical accuracy is required because the stability equation (VII-A.10) requires a knowledge of the current density gradient.

\[ \frac{\partial j_2}{\partial r} = \frac{\partial (\nabla^2 \chi)}{\partial r}. \] This means that we must be able to calculate the derivative \[ \frac{\partial^3 \chi}{\partial r^3} \] to some reasonable accuracy. Numerical differentiation, however, is a very delicate procedure. By definition, the derivative is the limit of a difference quotient, and in the latter we subtract large quantities from each other and divide by a small one. While the computational formulae for derivatives we have developed have truncation errors of \[ O(\Delta r^4) \] or better, the error due to the inaccuracy of the finite-difference approximation \[ \chi_{j,k} \] to \[ \chi(r=j\Delta r, t=k\Delta t) \] is genuinely cause for concern, especially because we must divide this error by a number of \[ O(\Delta r^3) \] in the formula for \[ \frac{\partial^3 \chi}{\partial r^3} \]. Nevertheless, dividing our global finite-difference error to \[ \chi \] by \[ \Delta r^3 \] indicates that our approximation to \[ \frac{\partial j_2}{\partial r} \] will be of \[ O(\Delta r + \Delta t^2/\Delta r^3) \approx O(\Delta r) \], or better, since the errors tend to cancel in the derivatives and we will have \[ \Delta t^2 < \Delta r^4 \].

A further remark about the Ohm's law (VII-B.1) is that this equation is really a non-linear equation for \( \chi \), since in our model the current viscosity \( \lambda \) is actually a feedback of the shape of the current profile \( \nabla^2 \chi \). In our numerical method, however, we will solve a set of implicit linear equations, evolving the numerical solution by solving a
similar set of linear equations at a future time interval, with \( \lambda \) calculated from equations (VII-A.8)-(VII-A.10). One standard method for numerically solving non-linear equations is to iterate the numerical solution a number of times at each time step, thus modeling the non-linearity. Instead, we have chosen a very small time step \( \Delta t = 10^{-5} \), thus solving at an increased number of time intervals, a procedure which is somewhat simpler to implement. In any case, the solutions we produce are steady-state solutions to the model equations (VII-A.7)-(VII-A.10).

Let us begin by rewriting our Ohm's law (VII-B.1) in the symbolic notation

\[
\frac{\partial \chi}{\partial t} = L \chi, \quad (VII-B.4)
\]

where the linear operator \( L \) is given by

\[
L = -\lambda \frac{\partial^4}{\partial r^4} - (2\lambda/r + \partial \lambda/\partial r) \frac{\partial^3}{\partial r^3} + [\lambda/r^2 - (1/r)(\partial \lambda/\partial r) + \eta] \frac{\partial^2}{\partial r^2} + [-\lambda/r^3 + (1/r^2)(\partial \lambda/\partial r) + \eta/r] \partial/\partial r. \quad (VII-B.5)
\]
Note that the symbolic operator (VII-B.5) has the limit

$$\lim_{r \to 0} L = - \left( \frac{8}{3} \right) \lambda \frac{\partial^4}{\partial r^4} + 2 \eta \frac{\partial^2}{\partial r^2}, \quad (VII-B.6)$$

so everything is well-behaved. Suppose that we are given initial values $\chi_{j,k}$ for our finite-difference approximations to $\chi(r=j\Delta r, t=k\Delta t)$. We then wish to calculate an approximation $\chi_{j,k+1}$ for $\chi(r=j\Delta r, t=k\Delta t+\Delta t)$. The Crank-Nicolson method consists of evaluating (VII-B.4) at the time-centered point $t=(k+1/2)\Delta t$. Performing some Taylor series expansions allows us to write

$$(\chi_{j,k+1} - \chi_{j,k})/\Delta t + O(\Delta t^2) = (L\chi_{j,k+1} + L\chi_{j,k})/2 + O(\Delta t^2). \quad (VII-B.7)$$

That is, we center the time derivative and time-average the operator (VII-B.5), an idea originally due to von Neumann.181 From (VII-B.7), we can derive an implicit set of linear equations for the approximations $\chi_{j,k+1}$.

$$\chi_{j,k+1} - (\Delta t/2) L\chi_{j,k+1} = \chi_{j,k} + (\Delta t/2) L\chi_{j,k}. \quad (VII-B.8)$$

---

The solutions to equations (VII-B.8) furnish a set of approximations \( X_{j,k+1} \) which have a local temporal discretization error of \( O(\Delta t^3) \) and are numerically stable.

We now produce formulae for the finite-difference approximations to the spatial derivatives of the symbolic operator (VII-B.5). The generation of these formulae consists largely of algebraic manipulation, as all finite-difference calculations are based upon polynomial interpolation.\(^{182}\) In order to achieve the required numerical accuracy, we work with the sixth-order Lagrange polynomial \( L_6(r) \), which requires the rather large computational molecule of seven spatial grid points. Evaluating (VII-B.5) at the space-centered point \( r = j \Delta r \), we make the approximation

\[
X(r,t) \approx L_6(r) = \sum_i \left[ l_i(r)/l_i(r_i) \right] X_{i,k}. \tag{VII-B.9}
\]

where

\[
l_i(r) = [(r-r_{j-3}) \cdots (r-r_{i-1})(r-r_{i+1}) \cdots (r-r_{j+3}) \tag{VII-B.10}
\]

and the index $i$ ranges from $(j-3)$ to $(j+3)$. Then, using the formula of Leibnitz,

$$\frac{d^n(fg)}{dr^n} = \sum_{m=0}^{n} \frac{n!}{[(n-m)!m!]} \frac{d^mf}{dr^m} \frac{d^{n-m}g}{dr^{n-m}}.$$ \hfill (VII-B.11)

we differentiate (VII-B.9) and perform some algebra to yield the approximations

$$(\partial \chi / \partial r)_{j,k} = (-\chi_{j-3,k} + 9\chi_{j-2,k} - 45\chi_{j-1,k} + 45\chi_{j+1,k}$$
$$- 9\chi_{j+2,k} + \chi_{j+3,k})/60\Delta r$$
$$+ o(\Delta r^6), \hfill (VII-B.12)$$

$$(\partial^2 \chi / \partial r^2)_{j,k} = (2\chi_{j-3,k} - 27\chi_{j-2,k} + 270\chi_{j-1,k} - 490\chi_{j,k}$$
$$+ 270\chi_{j+1,k} - 27\chi_{j+2,k} + 2\chi_{j+3,k})/180\Delta r^2$$
$$+ o(\Delta r^6), \hfill (VII-B.13)$$

$$(\partial^3 \chi / \partial r^3)_{j,k} = (\chi_{j-3,k} - 8\chi_{j-2,k} + 13\chi_{j-1,k} - 13\chi_{j+1,k}$$
$$+ 8\chi_{j+2,k} - \chi_{j+3,k})/8\Delta r^3$$
$$+ o(\Delta r^4), \hfill (VII-B.14)$$

and
\[
(\partial^4 \chi / \partial r^4)_{j,k} = (-X_{j-3,k} + 12X_{j-2,k} - 39X_{j-1,k} + 56X_{j,k} \\
- 39X_{j+1,k} + 12X_{j+2,k} - X_{j+3,k})/6\Delta r^4 \\
+ O(\Delta r^4).
\]

(VII-B.15)

Because the formulae (VII-B.12)-(VII-B.15) are calculated at space-centered points, the various numerical coefficients in these expressions display a high degree of symmetry. Maintaining this symmetry is very important, especially for the reduction of the truncation error in the formulae for the even space derivatives (VII-B.13) and (VII-B.15). Also, by definition of the derivative, the algebraic sum of the coefficients in each of the expressions (VII-B.12)-(VII-B.15) must identically vanish, which offers a simple check on our algebra. In fact, the results (VII-B.12)-(VII-B.15) have each been confirmed by performing Taylor expansions.

Inserting the expressions (VII-B.12)-(VII-B.15) into the symbolic operator (VII-B.5), we must now solve the linear system of N equations (VII-B.8) for the \(X_{j,k+1}\), where \(j=0,1,2,\ldots,N-1\). The usual algorithm of Gaussian elimination requires \(O(N^3)\) operations. This number of calculations is based on the assumption that the matrix of the system is
"full," that is, has few zero elements. However, we have carefully manipulated things so that the matrix of the system (VII-B.8) contains only a narrow band of bandwidth 3+3+1=7 of non-zero elements about the diagonal. The number of required operations of this "banded" system can be shown to be only of $O(N \times 3^2)$. In our simulations, we have chosen the radial grid spacing $\Delta r=10^{-2}$. Then we have $N=100$, and the banded algorithm saves a significant amount of computing power. The actual solution of the system (VII-B.8) is computed with the SLATEC library routine SNBFS. We remark that, since a typical simulation run time is a scaled time interval $\Delta T=1.0$ and costs approximately 40 minutes priority 1.0 CRAY CPU time, as a practical matter, the numerical solution of our problem cannot be attained without the savings gained from the banded algorithm.

We now specify the boundary conditions for (VII-B.1). Near the magnetic axis ($r=0$), we require that the magnetic Hamiltonian $\chi$ be an analytic function of the minor radius $r$. The analyticity requirement simply means that $\chi=\chi(r^2)$. Then we have a symmetry about the magnetic axis.

---

\[ \chi_{-j,k} = \chi_{j,k}. \quad \text{(VII-B.16)} \]

which is relatively simple to implement within the framework of the banded algorithm. At the plasma boundary (r=1), we specify the edge loop voltage (III-C.13). In scaled units, the edge loop voltage is just

\[ \frac{\partial \chi}{\partial t} = V(r=1,t). \quad \text{(VII-B.17)} \]

In actual tokamak experiments, the edge voltage is often preprogrammed or set by a feedback system to maintain a certain value of the total toroidal current I. Integrating equation (VII-B.17), we have the simple Dirichlet boundary condition

\[ \chi_{N,k} = \chi_{N,0} + \int_0^{k\Delta t} V(r=1,t) \, dt. \quad \text{(VII-B.18)} \]

Of course, implementing the boundary condition (VII-B.18) within the framework of the banded algorithm is somewhat difficult, since no numerical values \( \chi_{j,k} \) exist for \( j>N \). This means that derivative formulae (VII-B.12)-(VII-B.15) are no
longer applicable for the equations of (VII-B.8) at the positions j=N-2 and j=N-1. In order to overcome this problem, we adopt the following procedure. First, we construct non-centered, unsymmetric, but still $O(\Delta r^4)$ finite difference approximations for the derivatives at positions j=N-2 and j=N-1. The derivative formulae are constructed by differentiating the sixth-order Lagrange polynomial $L_6(r)$ of (VII-B.9) and a seventh-order Lagrange polynomial $L_7(r)$. The higher order Lagrange polynomial $L_7(r)$ is required for the approximation to $\partial^4 X/\partial r^4$ because we insist that all our derivative formulae be at least $O(\Delta r^4)$, even if they do not display any particular symmetry. Next, we incorporate our $L_6(r)$ and $L_7(r)$ derivative approximations into the symbolic operator (VII-B.5). Finally, we algebraically eliminate all undesirable unknowns from the j=N-2 and j=N-1 equations, reducing these equations to the proper banded form. For completeness, we list the additional derivative formulae we have derived for this procedure. At the position j=N-2,

$$
(\partial X/\partial r)_{N-2,k} = (X_{N-6,k} - 8X_{N-5,k} + 30X_{N-4,k} - 80X_{N-3,k} + 35X_{N-2,k} + 24X_{N-1,k} - 2X_{N,k})/60\Delta r + O(\Delta r^6),
$$

(VII-B.19)
\[
(\partial^2 \chi / \partial r^2)_{N-2,k} = (2 \chi_{N-6,k} - 12 \chi_{N-5,k} + 15 \chi_{N-4,k} \\
+ 200 \chi_{N-3,k} - 420 \chi_{N-2,k} + 228 \chi_{N-1,k} \\
- \frac{13 \chi_{N,k}}{180 \Delta r^2} + O(\Delta r^5)), \quad \text{(VII-B.20)}
\]

\[
(\partial^3 \chi / \partial r^3)_{N-2,k} = (-\chi_{N-6,k} + 8 \chi_{N-5,k} - 29 \chi_{N-4,k} \\
+ 48 \chi_{N-3,k} - 35 \chi_{N-2,k} + 8 \chi_{N-1,k} \\
+ \frac{\chi_{N,k}}{8 \Delta r^3} + O(\Delta r^4)), \quad \text{(VII-B.21)}
\]

and

\[
(\partial^4 \chi / \partial r^4)_{N-2,k} = (\chi_{N-7,k} - 8 \chi_{N-6,k} + 27 \chi_{N-5,k} \\
- 44 \chi_{N-4,k} + 31 \chi_{N-3,k} - 11 \chi_{N-1,k} \\
+ \frac{4 \chi_{N,k}}{6 \Delta r^4} + O(\Delta r^4)). \quad \text{(VII-B.22)}
\]

At the position \(j=N-1\),

\[
(\partial \chi / \partial r)_{N-1,k} = (-2 \chi_{N-6,k} + 15 \chi_{N-5,k} - 50 \chi_{N-4,k} \\
+ 100 \chi_{N-3,k} - 150 \chi_{N-2,k} + 77 \chi_{N-1,k} \\
+ 10 \chi_{N,k}) / 60 \Delta r + O(\Delta r^6), \quad \text{(VII-B.23)}
\]

\[
(\partial^2 \chi / \partial r^2)_{N-1,k} = (-13 \chi_{N-6,k} + 93 \chi_{N-5,k} - 285 \chi_{N-4,k})
\]
Later, we will also need to calculate derivatives at the plasma edge. The required formulae at the position $j=N$ are

$$(\frac{\partial X}{\partial r})_{N,k} = (10X_{N-6,k} - 72X_{N-5,k} + 225X_{N-4,k} - 400X_{N-3,k} + 450X_{N-2,k} - 360X_{N-1,k} + 147X_{N,k})/60\Delta r + O(\Delta r^6), \quad (VII-B.27)$$
\[(\partial^2 \chi / \partial r^2)_{N,k} = (137\chi_{N-6,k} - 972\chi_{N-5,k} + 2970\chi_{N-4,k} - 5080\chi_{N-3,k} + 5265\chi_{N-2,k} - 3132\chi_{N-1,k} + 812\chi_{N,k})/180\Delta r^2 + O(\Delta r^5), \quad (\text{VII-B.28})\]

\[(\partial^3 \chi / \partial r^3)_{N,k} = (15\chi_{N-6,k} - 104\chi_{N-5,k} + 307\chi_{N-4,k} - 496\chi_{N-3,k} + 461\chi_{N-2,k} - 232\chi_{N-1,k} + 49\chi_{N,k})/8\Delta r^3 + O(\Delta r^4), \quad (\text{VII-B.29})\]

and

\[(\partial^4 \chi / \partial r^4)_{N,k} = (-21\chi_{N-7,k} + 164\chi_{N-6,k} - 555\chi_{N-5,k} + 1056\chi_{N-4,k} - 1219\chi_{N-3,k} + 852\chi_{N-2,k} - 333\chi_{N-1,k} + 56\chi_{N,k})/6\Delta r^4 + O(\Delta r^4). \quad (\text{VII-B.30})\]

Note that the algebraic sum of the coefficients in each of the expressions (VII-B.19)-(VII-B.30) vanishes, as it must. The results (VII-B.19)-(VII-B.30) have also each been confirmed by performing Taylor expansions.

The consistent use of finite-difference approximations with spatial errors of \(O(\Delta r^4)\), or better, and the \(O(\Delta t^3)\) local temporal discretization error of (VII-B.8) insures a global
discretization error of $O(\Delta t^2 + \Delta r^4)$, as previously mentioned. A specification of the initial values

$$X_{j,0} = \chi(r=j\Delta r, t=0), \quad (VII-B.31)$$

where $j=0,1,2,...,N$, and the resistivity profile $\eta(r,t)$ then completes our numerical method for the mean-field Ohm's law (VII-B.1).

Section VII-C. Delta Prime Analysis

Having obtained the poloidal magnetic flux or magnetic Hamiltonian function $\chi$, we can calculate all relevant magnetic quantities. That is, we can use the finite-difference approximations $X_{j,k}$ to $\chi(r,t)$ generated by the system (VII-B.8) and the formulae (VII-B.12)-(VII-B.15) and (VII-B.19)-(VII-B.30) to determine approximations for the magnetic derivatives $\partial \chi / \partial r$, $\partial^2 \chi / \partial r^2$, $\partial^3 \chi / \partial r^3$, and $\partial^4 \chi / \partial r^4$. In units of $[(a/R_0) \cdot B_0]$, the poloidal magnetic field is

$$B_\theta = \partial \chi / \partial r. \quad (VII-C.1)$$
As previously mentioned, the rotational transform is

\[ i = \frac{1}{r}(\partial \chi / \partial r), \]  

(VII-C.2)

while its derivatives are

\[ \frac{\partial i}{\partial r} = \frac{1}{r}(\partial^2 \chi / \partial r^2) - \frac{1}{r^2}(\partial \chi / \partial r) \]  

(VII-C.3)

and

\[ \frac{\partial^2 i}{\partial r^2} = \frac{1}{r}(\partial^3 \chi / \partial r^3) - \frac{2}{r^2}(\partial^2 \chi / \partial r^2) + \left( \frac{2}{r^3} \right)(\partial \chi / \partial r). \]  

(VII-C.4)

In units of \([c/(4\pi)(a/R_0)(B_0/a)]\), the toroidal current density is

\[ j_z = \nabla^2 \chi = \partial^2 \chi / \partial r^2 + \frac{1}{r}(\partial \chi / \partial r), \]  

(VII-C.5)

while its derivatives are

\[ \frac{\partial j_z}{\partial r} = \partial^3 \chi / \partial r^3 + \frac{1}{r}(\partial^2 \chi / \partial r^2) - \frac{1}{r^2}(\partial \chi / \partial r) \]  

(VII-C.6)
and

$$\frac{d^2 j_z}{dr^2} = \frac{d^4 \chi}{dr^4} + \frac{1}{r}(\frac{d^3 \chi}{dr^3}) - \frac{2}{r^2}(\frac{d^2 \chi}{dr^2}) + \frac{2}{r^3}(\frac{d \chi}{dr}).$$  \hspace{1cm} (VII-C.7)

From the formulae (VII-C.1)-(VII-C.7), note that the symmetry relation (VII-B.16) guarantees that the poloidal magnetic field on axis \( B_p(r=0)=0 \) and that the derivatives \( \left(\frac{\partial j}{\partial r}\right)_0 = 2 \cdot \left(\frac{\partial ^2 l}{\partial r}\right)_0 = 0 \), so all magnetic functions are well-behaved. Further, note that the central current density and transform are related by \( j_0 = 2 \cdot l_0 = 2 \cdot \left(\frac{d^2 \chi}{dr^2}\right)_0 \). We also remark that, as a practical matter, one should always calculate the approximations for the magnetic derivatives (VII-B.12)-(VII-B.15) and (VII-B.19)-(VII-B.30) prior to constructing the magnetic functions (VII-C.1)-(VII-C.7). That is, one should not attempt to simply add-up all the various \( \chi_{j,k} \)'s in the magnetic functions all at once. Experimentally, we have found that first calculating the magnetic derivatives actually reduces the error in formulae (VII-C.1)-(VII-C.7).

The determination of the mean-field magnetic quantities furnishes us with all the information required to perform the
A stability analysis discussed in chapter IV. We begin by using a linear interpolation formula to locate the positions of the various mode rational surfaces, \( i(r_{nm}) = n/m \). In our simulations, we examine the stability of all poloidal modes with mode numbers \((n/m) < 1.0\) and \( m < 12\), in total 41 poloidal modes. Having determined values for the rotational transform \( i \), the current gradient \( \partial j_z/\partial r \), and the rational surface positions \( r_{nm} \), we solve for the perturbation amplitudes \( X_{nm} \) by integrating (VII-A.10), the marginal stability equation,

\[
\nabla_\perp^2 X_{nm} = \left( r^{-1} \partial j_z/\partial r / (i - n/m) \right) X_{nm}.
\]  

(VII-C.8)

The solution of the stability equation (VII-C.8) is obtained by implementing an \( O(\Delta r^5) \) Runge-Kutta-Nyström integration scheme.\(^{184}\)\(^{185}\) The integration process is greatly accelerated by integrating all 41 poloidal modes simultaneously, using the vectorization property of the CRAY supercomputer.\(^{186}\)

Without going into detail about the Runge-Kutta-Nyström algorithm, let us describe the general integration procedure.


First, we locate the grid point \( i \) such that the resonant surface position \( r_{nm} \) obeys the relation \( r_i < r_{nm} < r_{i+1} \), where the radial position \( r_i = i/N \). Next, using the Runge-Kutta-Nyström algorithm, we obtain an inner computer solution for \( \chi_{nm} \) in the interval \([0, r_i]\), coming out of the axis as \( \chi_{nm} \propto r^m g_{nm}(r^2) \), which is just the analyticity boundary condition. Similarly, we obtain an outer computer solution for \( \chi_{nm} \) in the interval \([r_{i+1}, 1]\) by integrating (VII-C.8) from the plasma edge, enforcing the condition \( \frac{d\chi_{nm}}{dr} = -m\chi_{nm} \) at the position \( r=1 \). That is, at the plasma edge, we have assumed the absence of any stabilizing conducting wall and taken the vacuum boundary condition, \( \chi_{nm} \propto r^{-m} \) for \( r \geq 1 \).

In the interval \([r_i, r_{i+1}]\), computer solutions are not feasible because the perturbation amplitude \( \chi_{nm} \), although perfectly well-defined, has a logarithmic singularity in its derivative \( d\chi_{nm}/dr \) at the rational surface \( r=r_{nm} \). In order to overcome this complication, we develop series solutions about the point \( r=r_{nm} \), a procedure already discussed at the end of section IV-C. The series solutions are then attached to the inner and outer computer solutions at the grid points \( r_i \) and \( r_{i+1} \). The method we use is only slightly different from that found in the classic work of Furth, Rutherford, and
Expanding about \( r = r_{nm} \), recall from section IV-C that the stability equation (VII-C.8) reduces to

\[
d^2 \phi_{nm}/dy^2 = \left( \kappa/y \right) \phi_{nm}, \tag{VII-C.9}
\]

where the function

\[
\phi_{nm}(y) = \chi_{nm}(r), \tag{VII-C.10}
\]

the coordinate

\[
y = (r - r_{nm})/r_{nm}, \tag{VII-C.11}
\]

and we interpolate to obtain the coefficient

\[
\kappa = \left[ \left( \partial j_z/\partial r \right) / \left( \partial \lambda / \partial r \right) \right]_{r = r_{nm}}. \tag{VII-C.12}
\]

Using the method of Frobenius\textsuperscript{188,189} it is straightforward


to show that two independent series solutions to (VII-C.9) are

\[
\Phi_1(y) = 1 + \kappa \ln|y| + (\kappa^2/2) y^2 \ln|y| - (3/4) \kappa^2 y^2 + \ldots
\]  

(VII-C.13)

and

\[
\Phi_2(y) = y + (\kappa/2) y^2 + (\kappa^2/12) y^3 + \ldots
\]  

(VII-C.14)

We now require that the perturbation amplitude \( \Phi_{nm} \) be continuous across \( y=0 \), but allow for a jump in the derivative \( d\Phi_{nm}/dy \) at this point. From equations (IV-C.21)-(IV-C.25), recall that the jump \( [d\Phi_{nm}/dy] \) represents the spikey current distribution created in the vicinity of a magnetic island. The inner and outer series solutions \( \Phi_{in}(y) \) and \( \Phi_{out}(y) \) are then

\[
\Phi_{in}(y) = \Phi_{nm}(0) \Phi_1(y) + A_{in} \Phi_2(y)
\]  

(VII-C.15)

and

\[
\Phi_{out}(y) = \Phi_{nm}(0) \Phi_1(y) + A_{out} \Phi_2(y).
\]  

(VII-C.16)
The constants \( \phi_{nm}(0) \) and \( A_{in} \) of equation (VII-C.15) are obtained by matching the inner computer solution \( \chi_{nm}(r_i) \) and its derivative \( \left( \frac{d\chi_{nm}}{dr} \right)_{r_i} \) to \( \phi_{in}(y) \) and \( \frac{d\phi_{in}}{dy} \) at the position \( y=(r_i-r_{nm})/r_{nm} \). Similarly, the constants \( \phi_{nm}(0) \) and \( A_{out} \) of equation (VII-C.16) are obtained by matching the outer computer solution \( \chi_{nm}(r_{i+1}) \) and its derivative \( \left( \frac{d\chi_{nm}}{dr} \right)_{r_{i+1}} \) to \( \phi_{out}(y) \) and \( \frac{d\phi_{out}}{dy} \) at the position \( y=(r_{i+1}-r_{nm})/r_{nm} \). Note that the constant \( \phi_{nm}(0) \) can be set the same for both relations (VII-C.15) and (VII-C.16) since the equations (VII-C.8) and (VII-C.9) for the perturbation amplitudes \( \chi_{nm} \) and \( \phi_{nm} \) are both linear and homogeneous.

After obtaining the constants \( \phi_{nm}(0), A_{in}, \) and \( A_{out} \), we use the definition (VII-A.14) of \( \Delta'_{nm} \) to calculate that

\[
\Delta'_{nm} = \frac{(A_{out} - A_{in})}{[r_{nm} \cdot \phi_{nm}(0)]}. \tag{VII-C.17}
\]

We see that the logarithmic singularity of the derivative \( \frac{d\phi}{dy} \) cancels itself in \( \Delta'_{nm} \), leaving us with the finite residue (VII-C.17). The entire machinery of our numerical method for (V-C.13) is now in place. That is, the stability factors \( \Delta'_{nm} \) determine the island widths \( w_{nm} \). The island
widths then determine the current viscosity $\lambda$. But the current viscosity feeds back into the current profile via the poloidal flux evolution. Since the $\Delta'_{nm}$ are determined by the shape of the current profile alone, we therefore have a self-consistent, closed physical theory. After initiating our numerical procedure, we simply continue integrating until we reach a steady-state. The realization of a steady-state is quite evident, being characterized by the loop voltage $V=\delta\chi/\delta t$ becoming a spatial constant across the plasma and the stabilization of all excited tearing modes, $\Delta'_{nm}\approx 0$, the marginal stability condition (IV-D.5).
CHAPTER VIII. INDUCTIVELY DRIVEN TOKAMAK

Section VIII-A. Tokamak Temperature Profile

This chapter contains the results of our simulations of inductively driven tokamaks. That is, we now use the set of scaled equations (VII-A.7)-(VII-A.10) to model tokamaks which maintain their toroidal currents via the transformer induced loop voltage of section III-C. These simulations are of interest because the present generation of tokamaks rely primarily on loop voltages for their current maintenance. In section VIII-B, we use the results of our simulations to examine the MHD stable regime of tokamaks which are subject to tearing modes. The MHD-stable regime is examined by plotting the results of our simulations in $l_i$-$q_a$ space.\(^{190}\) The plasma internal inductance $l_i$ and edge safety factor $q_a$ are physical quantities which are related to the toroidal current and are subject to external experimental verification. Before presenting these results, however, we still need to discuss a few more preliminary details.

One important preliminary detail is the fact that we must still specify the form of the plasma resistivity $\eta(r,t)$. Using relation (II-C.16), we see that a specification of the plasma resistivity is roughly equivalent to a specification of the electron temperature, so we initially focus our attention on the function $T_e(r,t)$. We say roughly equivalent because if one includes the neoclassical effects of section VI-A in the calculation of the plasma resistivity, one finds that

$$\eta_{nc} \approx \eta_{cl}/[1 - 2(r/R_0)^{1/2}] .$$

(VIII-A.1)

In any case, from inductively driven tokamak experiments, one can say that the general behavior of the function $T_e$ is rather well-known. That is, by carefully measuring the Thomson scattering of intense laser light\textsuperscript{192,193} or electron cyclotron emission spectra,\textsuperscript{194,195} experimental plasma physicists have determined that the tokamak electron temperature profile generally has a strong peak in the plasma.

\textsuperscript{194}F. Englemann and M. Curatolo, Nucl. Fusion 13, 497 (1973).
center, an effect due to the thermal instability discussed in section IV-D. From a theoretical point of view, we should be able to determine the exact form of the electron temperature profile by solving a heat equation. The heat equation can be derived by taking the $(1/2)m_e v^2$ moment of the kinetic equation (II-B.1) and has the general form\textsuperscript{196}

$$\frac{\partial (3n_e T_e/2)}{\partial t} = \nabla \cdot (\kappa_\perp \nabla T_e) + P_{\text{Ohm}}, \quad (\text{VIII-A.2})$$

where $\kappa_\perp$ is the cross-field electron thermal conductivity and the total Ohmic heating is

$$P_{\text{Ohm}} = n j^2 + \lambda [\nabla (j_\parallel/B)]^2. \quad (\text{VIII-A.3})$$

with the $\lambda$ term representing the effects of fluctuating magnetic fields. Because the Ohmic heating depends on the tokamak current and the current distribution depends on the resistivity or temperature, the heat equation (VIII-A.2) couples to Ohm's law in a complicated way.\textsuperscript{197} However, even without these complications, there exists a very real problem


\textsuperscript{197}M. F. Turner and J. A. Wesson, Nucl. Fusion 22, 1069 (1982).
with the heat equation approach of determining the electron temperature profile. That is, there is no agreed upon theory of electron heat transport with which one can calculate the thermal coefficient $k_\perp$. In fact, as was mentioned at the end of section II-B, so-called anomalous processes dominate the tokamak electron heat transport and, at present, we simply do not fully understand the nature of these processes. Because of this somewhat sorry state of affairs, besides adding an additional term such as (IV-B.35) to $k_\perp$ to represent the enhanced thermal diffusivity in regions containing magnetic islands, perhaps the best we can do in our research is to set the plasma thermal conductivity equal to a constant,

$$k_\perp \approx C \cdot 10^{20} \text{ m}^{-1} \text{s}^{-1},$$  \hspace{1cm} (VIII-A.4)

where $C$ is some number, $1 < C < 5$. The numerical value for the conductivity (VIII-A.4) is chosen to be consistent with the experimentally measured values of Ohmically heated tokamak plasma energy confinement. In fact, if we use the thermal conductivity (VIII-A.4) and scale the heat equation (VIII-A.2),
we find that the energy confinement time $\tau_E$ scales with the line-averaged density $n_e$ like

$$\tau_E \approx (0.5 \times 10^{-20} \text{ m.s}) \overline{n_e} a^2, \quad \text{(VIII-A.5)}$$

which is a commonly used empirical formula called Alcator scaling.\textsuperscript{198} However, this a posteriori approach for obtaining a thermal conductivity $\kappa_\perp$ hardly seems like a justification for introducing the theoretical machinery of (VIII-A.2) in the first place. Another possibility is to assume that the anomalous transport is due to some particular microinstability such as drift waves, a current favorite candidate.\textsuperscript{199} For the drift wave case, it can be shown that one expects a scaling like

$$\kappa_\perp \propto T_e^{3/2}, \quad \text{(VIII-A.6)}$$

so the thermal conductivity is large in the plasma interior. However, many other competing theories have been proposed.


In order to learn something about the plasma physics involved, several attempts were made to couple the tearing mode model (V-C.13) to a heat equation model (VIII-A.2) with a thermal conductivity $\kappa_\perp$ such as (VIII-A.4) or (VIII-A.6). Unfortunately, none of these attempts was really successful in learning anything, except perhaps that this is not a good way to approach the problem, which is maybe significant after all. That is, by adding so many extra theoretical assumptions to our model, it became impossible to disentangle the results of the coupled heat and current equations with any degree of certainty, especially when the numerical code crashed or had other interesting difficulties. We therefore learned the hard way that it is much better to start by building on things which are theoretically understood or have strong experimental support. Attempting the complete or perfect calculation right at the start is a nice idea in principle, but in practice it is really a mild form of insanity. We discovered that a much better procedure is to limit the amount of new assumptions in our physical model, an undoubtedly more modest endeavor which does, however, enable one to more clearly identify the consequences of the
new assumptions.

In our case, it is well-known experimentally that the electron temperature is basically a function which is peaked in the plasma center and vanishingly small at the plasma edge. Since we are not really interested in testing this result, we may as well just use it. For this reason, we have chosen the shape of the resistivity profile to be

\[ \eta = (1 - r^2)^{-\alpha}, \quad (VIII-A.6) \]

the inverse of a parabola to some positive power \( \alpha \). Note that for simplicity we have chosen \( \eta = \eta(r) \), independent of time. Also, we need only give the shape of the resistivity profile, since in the scaled model equations (VII-A.7)-(VII-A.10) the absolute magnitude of the resistivity can be adjusted by scaling the loop voltage. In our simulations, we have chosen resistivity shape parameters \( \alpha \) in the range \([2.0, 4.0]\).

Before leaving the subject of the electron temperature, let us mention that high temperature effects are, nevertheless, of some interest for analyzing the stability of tearing modes. This is because the approximate relation
\[ j_z \propto n^{-1} \propto T_e^{3/2} \]  

(VIII-A.7)

indicates that a flattening of the electron temperature can lead to a flattening of the toroidal current. This current flattening is a stabilizing effect for the tearing modes, if the flattening occurs near a resonant surface \( r_{nm} \). This means that intense local heating, such as electron cyclotron resonance heating, may be a viable procedure for stabilizing tearing modes.\(^{200}\) Although it does not directly involve the electron temperature profile, we also mention that an even more powerful approach to obtaining tearing mode stability would be a so-called active feedback stabilization system in which external magnetic loops exactly match the helicity of unstable tearing modes as they occur.\(^{201}\) While these stabilization techniques are fairly well-known, their implementation might have important implications for the next generation of tokamaks such as ITER.\(^{202}\)


Having chosen the resistivity profile as (VIII-A.6), we initialize our simulations by assuming a quasi-steady state with

\[ j(r,t=0) = \frac{V(t=0)}{\eta} = 2 (1 - r^2)^\alpha, \quad (\text{VIII-A.8}) \]

where we have chosen \( V(t=0) = 2.0 \) so that the central safety factor \( q_0(t=0) = 1.0 \). This quasi-steady assumption is made in order to avoid issues related to anomalous current penetration due to the current viscosity \( \lambda \).\(^{203}\) In principle, as first discussed in section III-C, the relationship between \( \lambda \) and current penetration is an interesting research problem. In practice, however, we have found it to be a complication in carrying out our simulations. The difficulties encountered with the current penetration are related to the higher order tearing modes which tend to form near the plasma edge during the onset of the skin current. A higher order tearing mode occurs when the rotational transform profile is non-monotonic across the plasma minor radius. This means that a rational surface \( r_{nm} \) which satisfies \( \lambda(r_{nm}) = n/m \) can be found at more than one radial position. Higher order tearing

modes are inherently unstable.\textsuperscript{204} Using the initial current density (VIII-A.8), we integrate to get the scaled initial poloidal magnetic field,

\[ B_\theta(r,t=0) = \int_0^r j(r,t=0) r dr = \frac{[1-(1-r^2)^{\alpha+1}]/[(\alpha+1)r]}{r^2}. \] (VIII-A.9)

Then we use (VIII-A.9), relation (VII-C.1), and a better than \( O(\Delta r^6) \) Newton-Cotes\textsuperscript{205} numerical integration formula to determine the initial values \( \chi_{j,0} \).

Another preliminary detail concerns the edge loop voltage boundary condition (VII-B.17). For the edge loop voltage, we have chosen the function

\[ V(r=1,t) = 2.0 + \Delta V [1 - \exp(-10t)]. \] (VIII-A.10)

Since the typical simulation run time in scaled time units is \( \Delta T \approx 1.0 \), using the boundary condition (VIII-A.10), the steady-state loop voltage is just \( V(r=1,t=\Delta T) \approx 2.0 + \Delta V \). The offset voltage \( \Delta V \) has been included in (VIII-A.10) because we wish to examine the effect of increasing the steady-state loop.

\textsuperscript{204}\textsuperscript{H. P. Furth, P. H. Rutherford, and H. Selberg, Phys. Fluids 16, 1054 (1973).}

voltage \( V \) with fixed resistivity profile shape parameters \( \alpha \). This process simulates the effect of progressively driving larger toroidal currents, while maintaining the underlying shape of the resistivity profile. For the various fixed values of \( \alpha \), we have made simulations with \( \Delta V \) in the range \([0.0,2.0]\).

Finally, from Ohm's law (VII-A.7), it is important to note that increasing the steady-state voltage to a value \( V>2.0 \) causes the central safety factor to drop to a value \( q_0<1.0 \). As discussed in section IV-D, a safety factor \( q_0<1.0 \) excites the \( m=1, n=1 \) sawtooth instability. In order to crudely model the time-averaged effect of these sawtooth oscillations, we use the Kadomtsev reconnection theory and treat the region about the magnetic axis as an effective magnetic island, with an island half-width \( r_s(t) \). For the inductively driven tokamak case, we take \( r_s \) as the location \( r_{11} \) of the \((1,1)\) tearing mode singular surface. For the \( m=1 \) contribution to the current viscosity \( \lambda \), we therefore add a term to (VII-A.8) of the form

\[
\lambda_s(r,t) = \eta(0,t) (2r_s)^2 \exp(-r^2/r_s^2), \quad (\text{VIII-A.11})
\]
where the scaled resistivity $\eta(0,t)=\eta(0)=1.0$. To better understand the effect of the current viscosity (VIII-A.11) on the safety factor $q_0$, we substitute the expression for $\lambda_s$ into Ohm's law (VII-A.7). Assuming a steady-state, we find

$$j_0 = V + 2(2r_s)^2 (\partial^2 j/\partial r^2)_0.$$  \hspace{1cm} (VIII-A.12)

Since our inductively driven tokamak current profiles are of the standard, monotonically decreasing sort, the curvature derivative $(\partial^2 j/\partial r^2)_0 < 0$ and the effect of $\lambda_s$ is to decrease the central current density $j_0$. Then, using the relation $q_0 = 2/j_0$, equation (VIII-A.12) shows that $\lambda_s$ will increase $q_0$ as the current density flattens about the axis, an effect which is in accord with the Kadomtsev reconnection model.

Section VIII-B. MHD-Stable Regime of the Tokamak

The MHD-stable regime of the tokamak is examined via the construction of an $l_i$-$q_a$ diagram. Apparently, the $l_i$-$q_a$ approach for analyzing stability is of real significance, as it has recently been used successfully in interpreting the
steady-state experimental data from the large TFTR\textsuperscript{206} and JET\textsuperscript{207} tokamaks. The internal inductance $l_i$ is defined as

$$l_i = \langle B^2 \rangle / B_0^2(a), \quad (\text{VIII-B.1})$$

and is a measure of the shape of the plasma current profile. The formula (VIII-B.1) for $l_i$ follows from the definition of the plasma internal energy inductance $L_i$.

$$L_i l_i^2 / 2 = \int_0^a (B_0^2 / 8\pi) d^3 x, \quad (\text{VIII-B.2})$$

where $I$ is the total toroidal current. That is, a manipulation of (VIII-B.2) using Ampere's law (II-A.23) shows that

$$L_i = (2\pi R_0 / c^2) l_i, \quad (\text{VIII-B.3})$$

so that the dimensionless quantity $l_i$ is actually the internal energy inductance divided by $(2\pi R_0 / c^2)$. From (IV-D.8) and

(II-A.23), the edge safety factor $q_a$ is defined as

$$q_a = r_a^{-1} = c a^2 B_0 / 2 R_0,$$  \hspace{1cm} (VIII-B.4)

and is thus a measure of the total toroidal current $I$.

The major importance of the parameters $l_i$ and $q_a$ is that both of these quantities are subject to external experimental verification and are therefore universally monitored by plasma physicists. The toroidal current $I$ can be measured by placing a Rogowski coil poloidally around the plasma.\textsuperscript{208} Obtaining a value for the internal inductance $l_i$ is more subtle, since it is determined from a series of indirect measurements. First, a measurement of the poloidal magnetic field strength at the plasma edge allows a determination of the Shafranov shift parameter $\Lambda$. The shift parameter $\Lambda$ is defined from the equation\textsuperscript{209}

$$B_p(a, \theta) = B_0(a)[1 + (\Lambda - 1)(a/R_0) \cos \theta].$$ \hspace{1cm} (VIII-B.5)

That is, the edge poloidal magnetic field $B_p(a, \theta)$ includes the


first order toroidal correction to the magnetic field $B_\theta(a)$. From the plasma virial theorem,\textsuperscript{210} it can be shown quite generally that the shift parameter $\Lambda$ obeys

$$\Lambda = l_1/2 + \langle \beta_\theta \rangle.$$  \hspace{1cm} (VIII-B.6)

where the volume averaged or mean poloidal beta $\langle \beta_\theta \rangle$ is defined in a manner similar to that of (I-B.5),

$$\langle \beta_\theta \rangle = 8\pi\langle \rho \rangle / B_\theta^2(a).$$ \hspace{1cm} (VIII-B.7)

The mean poloidal beta $\langle \beta_\theta \rangle$ itself can be determined from a diamagnetic measurement of the excluded magnetic flux,$^\text{211}$

$$\delta \psi = (2\pi l^2 / B_0 c^2) (1 - \langle \beta_\theta \rangle).$$ \hspace{1cm} (VIII-B.8)

That is, $\delta \psi$ represents the total change in the toroidal magnetic flux $\psi$ from its vacuum field value, which can be measured externally using a diamagnetic loop. The measurements of $\Lambda$ and $\langle \beta_\theta \rangle$ allow one to infer the value of


\( l_i \) indirectly from equation (VIII-B.6).

In order to compare the experimental tokamak values of \( l_i \) and \( q_a \) with theoretical ones, we run our computer code over a large range of resistivity profiles (VIII-A.6) and steady-state loop voltages (VIII-A.10). That is, we perform some 40 inductively driven tokamak simulations with resistivity profile shape parameters \( \alpha \) in the range \([2.0,4.0]\) and the steady-state loop voltages \( V \) in the range \([2.0,4.0]\). After each \((\alpha,V)\) simulation locates a steady-state, we calculate values for the internal inductance \( l_i \) and edge safety factor \( q_a \) from the steady-state magnetic profiles. In Fig. 6, we plot the results of our simulations in the \( l_i-q_a \) plane. Each solid curve in this figure represents a curve of constant \( \alpha \), the bottom curve having \( \alpha=2.0 \) and the top curve having \( \alpha=4.0 \). For each solid curve with fixed \( \alpha \), the loop voltage \( V \) increases in the direction from right to left along the curve. In order to delimit the possible range of current profiles, we also plot a dashed curve representing the case of a maximally peaked current profile with \( j_z(r) \) a constant up to \( r/a=(q_0/q_a)^{1/2} \) and surrounded by a vacuum region. From the definition (VIII-B.1), this means that the dashed curve represents the function
Figure 6. $l_1$-$q_a$ diagram. The internal inductance $l_1$ versus the edge safety factor $q_a$ of inductively driven tokamak simulations. Each solid curve represents the results of simulations with fixed resistivity parameter $\alpha$. The value of the parameter $\alpha$ increases in the direction from the bottom to the top of the diagram. For each solid curve with fixed $\alpha$, the loop voltage $V$ increases in the direction from the right to the left along the curve. The dashed curve is the function $\text{Max}(l_1) = (1/2)[1 + 2\ln(q_a/q_0)]$ for $q_0 = 1.0$. Note that the solid simulation curves run into an upper stability wall near the top part of the diagram.
\[ \text{Max}(l_i) = \frac{1}{2} [1 + 2 \ln(q_a/q_0)]. \] (VIII-B.9)

In addition, from a comparison theorem,\textsuperscript{212} it can be shown that if \( q_0 = 1.0 \), then there can be no stability when \( q_a < 2.0 \).

The most outstanding feature of our \( l_i-q_a \) diagram is the result indicating that the resistive modes introduce an upper stability boundary on the possible range of \( l_i \) values, a result which is in agreement with the experiments and also confirms the previous calculations of Cheng, Furth, and Boozer\textsuperscript{213}. It is worth noting, however, that the two sets of calculations, those of Cheng, et al. and our own, have been performed in two very different ways. That is, Cheng, et al. developed a semi-automatic method to computer search the range of stable current profiles, numerically adjusting the current density \( j_z(r) \) in a systematic way to eliminate all unstable modes. Our theoretical model, on the other hand, is clearly more physical, since it includes a definite profile-shaping mechanism through the current viscosity \( \lambda \) of the


mean-field Ohm’s law (V-B.16). However, an advantage of the computer search method, even though it is less physical, is that this method allows one to calculate the stability effects of ideal MHD current modes in a fairly simple way. For this reason, Cheng, et al. were also able to show that the requirement of stability against ideal MHD modes determines a jagged lower stability boundary on the possible range of $l_i$ values, something that we have not attempted to do.

In Table 1, we list the output parameters from a typical inductively driven tokamak simulation. This simulation was constructed with the resistivity shape parameter $\alpha = 2.5$ and the steady-state voltage $V = 2.5$. The profiles of the steady-state poloidal magnetic field, rotational transform, toroidal current density, and MHD activity parameter $l^2 = \lambda / \eta$ from this simulation are also shown in Figs. 7-10. The relative scale of the current density flattening is clearly reflected in the plot of $l^2$, with the strong flattening in the plasma center and edge regions due to the effects of the $m=1$ and $m=2$ modes, respectively.

The lowest edge safety factor $q_a$ or maximum toroidal current $I$ that we have been able to obtain for a stable current profile is $q_a = 2.6$. Cheng, et al. reported finding very
internal inductance, \\
\( l_i \) = 1.368

dedge safety factor, \\
\( q_a \) = 3.073

ergy content in units of \( [(2\pi R_0 \cdot \pi a^2)(a/R_0)^2(B_0^2/8\pi)] \), \\
\( W_0 \) = 0.145

ehelicity content in units of \( [(2\pi a^2 B_0)^2] \), \\
\( K_0 \) = 0.127

magnetic island widths \( w_{nm} \) in units of \([a]\), \\
w_{1,2} = 0.140 \quad w_{2,3} = 0.085 \quad w_{3,4} = 0.042 \\
w_{4,5} = 0.038 \quad w_{5,6} = 0.004

radial position of \( q=1 \) surface in units of \([a]\), \\
r_{1,1} = 0.23

delta primes \( \Delta'_{n,m} \) in units of \([a^{-1}]\), \\
\( \Delta'_{1,3} = -3.71 \quad \Delta'_{4,11} = -21.8 \quad \Delta'_{3,8} = -14.8 \\
\Delta'_{2,5} = -9.10 \quad \Delta'_{4,7} = -16.5 \quad \Delta'_{4,9} = -21.9 \\
\Delta'_{5,11} = -26.9 \quad \Delta'_{1,2} = +1.25 \times 10^{-5} \quad \Delta'_{6,11} = -28.9 \\
\Delta'_{5,9} = -23.4 \quad \Delta'_{4,7} = -17.5 \quad \Delta'_{3,5} = -11.0 \\
\Delta'_{5,8} = -21.9 \quad \Delta'_{7,11} = -31.8 \quad \Delta'_{2,3} = +1.42 \times 10^{-5} \\
\Delta'_{7,10} = -29.8 \quad \Delta'_{5,7} = -18.4 \quad \Delta'_{8,11} = -33.4 \\
\Delta'_{3,4} = +6.09 \times 10^{-6} \quad \Delta'_{7,9} = -25.0 \quad \Delta'_{4,5} = +1.80 \times 10^{-5} \\
\Delta'_{9,11} = -32.0 \quad \Delta'_{5,6} = +1.17 \times 10^{-5} \quad \Delta'_{6,7} = -1.12 \\
\Delta'_{7,8} = -16.5 \quad \Delta'_{8,9} = -23.8 \quad \Delta'_{9,10} = -29.8 \\
\Delta'_{10,11} = -35.6

Table 1. Output parameters from a typical inductively driven tokamak simulation. Listed are the results from the simulation with resistivity shape parameter \( \alpha = 2.5 \) and steady-state loop voltage \( V = 2.5 \).
Figure 7. Poloidal magnetic field from a typical inductively driven tokamak simulation. The function plotted is the steady-state radial profile of the poloidal magnetic field $B_\theta(r) = \frac{\partial\chi}{\partial r}$ in units of $[(a/R_0)B_0]$. 
Figure 8. Rotational transform from a typical inductively driven tokamak simulation. The function plotted is the steady-state radial profile of the rotational transform \( \lambda(r) = r^{-1} \delta \chi / \delta r \). The edge transform is \( \lambda_a = 0.325 \).
Figure 9. Toroidal current density from a typical inductively driven tokamak simulation. The function plotted is the steady-state radial profile of the current density $j_z(r) = \nabla^2 \chi$ in units of $[(a/R_0)(c/4\pi)(B_0/a)]$. 
Figure 10. MHD activity parameter from a typical inductively driven tokamak simulation. The function plotted is the steady-state radial profile of the MHD activity parameter $\ell^2(r) = \lambda / \tau$ in units of [a^2].
specialized stable current profiles with safety factors as low as $q_a=2.0$, but an earlier study by Glasser, Furth, and Rutherford\textsuperscript{214} found $q_a$ limits similar to ours. Physically, our $q_a$ limit is imposed by the action of the $m=2, n=1$ mode near the plasma boundary. As the edge safety factor decreases and the $m=2$ mode nears the plasma boundary, we have observed that the $\lambda$ term attempts to push current towards the plasma edge, which in our model may be thought of as a limiter.\textsuperscript{215} The simulation responds to this action by attempting to hold-on to its current, and a skin current quickly develops near the plasma boundary. Shortly after the onset of the skin current, many tearing modes are excited, the simulation undergoes a disruption, and a large fraction of the current is ejected from the plasma. We purposely use the term disruption to describe this event because of the apparent similarity between this simulated catastrophe and actual disruptions observed in tokamaks.\textsuperscript{216} As briefly mentioned at the end of section I-C, disruptions are a very serious issue for tokamak operation, so this is a subject which might warrant further


study with the mean-field theory.

Finally, let us remark that if one examines the experimental $l_i$-$q_a$ data points from either the TFTR or JET tokamaks (see Fig. 11), one is immediately struck by how well they fit inside the MHD stable region of the $l_i$-$q_a$ space. This phenomenon naturally leads one to speculate about whether there is any special relationship between the MHD stability of tokamaks and the quantities $l_i$ and $q_a$. Actually, in addition to the $l_i$-$q_a$ diagram, there exists another type of plot known as the Hugill diagram\textsuperscript{217} which also seems to give a recognizable pattern to the experimental results. In the Hugill diagram, the inverse edge safety factor $q_a^{-1}$ is plotted against the so-called Murakami parameter\textsuperscript{218} $(\overline{n_e} R_0/B_0)$. The Hugill diagram is very useful in characterizing both low $q_a$ and high $\overline{n_e}$ tokamak disruptions.

Perhaps one way to understand the apparently simple relationship between $l_i$ and $q_a$ is through the principle of "profile consistency." That is, it has been observed\textsuperscript{219,220} that certain families of electron temperature profiles $T_e(r)$

\textsuperscript{218}M. Murakami, J. D. Callen, and L. A. Berry, Nuc. Fusion 16, 347 (1976).
\textsuperscript{220}B. Coppi, Comments Pl. Phys. and Contr. Fusion 5, 261 (1980).
tend to maintain themselves, regardless of considerable variations in the tokamak auxiliary-heating power deposition profile, as long as a specific value of the edge safety factor $q_a$ is prescribed. Because of the close link between the current and temperature profiles indicated by (VIII-A.7), it has been suggested\textsuperscript{221,222} that the consistent maintenance of tokamak temperature profiles is directly related to the requirement that the $j_z(r)$ profiles be stable against both ideal and resistive kink instabilities, that is, against tearing modes. Clearly, as we have seen in our simulations, the tokamak does shape its current profile to maintain stability against the very potent $m=1$ and $m=2$ modes. Even after this initial shaping, however, we observe that the tokamak still has enough resilience left in its current profile to dampen all higher order tearing modes, provided that one does not insist on pushing the device beyond some critical $q_a$ or value of the total toroidal current $I$. In this sense, we can think of the quantities $q_a$ and $I$, and $I$, as moments of some plasma current "distribution function." Apparently, the tokamak has enough

freedom in its current distribution function to have at least two independent moments.

Incidentally, one can show that high values of the total toroidal current $I$ are generally desirable for tokamak operation, since pressure mode stability considerations put a limit on the maximum value of the plasma beta (I-B.5),\textsuperscript{223}

$$\beta_{\text{max}} \approx g \left( \frac{I}{10 a B_0 c} \right). \quad (\text{VIII-B.10})$$

where the so-called Troyon $g$ factor is on the order of $g \approx 3$. On the other hand, a compilation of experimental data from various Ohmically heated tokamaks indicates that the energy confinement time scales roughly as\textsuperscript{224}

$$\tau_E \approx 1.0 \times 10^{-21} \, n_e \, a R_0^2 \, q_a^{1/2}, \quad (\text{VIII-B.11})$$

clearly indicating that the magnetic confinement is degraded with the increasing MHD activity or decreasing edge safety factor $q_a$. In operating the tokamak, therefore, one tries to find a compromise between the opposing demands of the

relationships (VIII-B.10) and (VIII-B.11). Usually, this means taking an edge safety factor on the order of \(q_a \approx 3.0\).

Having observed that the tokamak often has enough resilience to successfully shape its own current profile by means of the current viscosity \(\chi\), we now recall the important helicity conservation theorem of chapter V. That theorem stated that if the tokamak current profile undergoes an alteration due to the current viscosity, then the alteration must lower the scaled magnetic energy

\[
W_0 = 2 \int_0^1 B_0^2 r \, dr, \quad (\text{VIII-B.10})
\]

but must conserve the scaled magnetic helicity

\[
K_0 = 2 \int_0^1 (\chi_b - \chi) r \, dr. \quad (\text{VIII-B.11})
\]

We therefore calculated the steady-state values of both \(W_0\) and \(K_0\) for each of our inductively driven tokamak simulations and made a \(W_0-K_0\) plot of these numbers. The hope was that this plot would somehow shed some light on the nature of the \(l_1-q_a\) stability boundary of Fig. 6. However, we did not ascertain any discernable pattern to this plot.
CHAPTER IX. COMPLETELY BOOTSTRAPPED TOKAMAK

Section IX-A. Tokamak Pressure Profile

This chapter contains the results of our simulations of completely bootstrapped tokamaks. That is, we now use the set of equations (VI-B.15) to model tokamaks which maintain toroidal currents via the bootstrap current amplification scheme discussed in section VI-B. These simulations are of great interest because the steady-state operation of the tokamak requires an efficient, non-inductive method to maintain the toroidal current. In section IX-B, we discuss the steady-state results of several bootstrap simulations. Before presenting these results, however, we once again need to discuss a few preliminary details.

Let us begin our discussion by scaling the set of bootstrap model equations (VI-B.15). First, we make the substitutions (VII-A.1)-(VII-A.6) and

\[ p' = \frac{p}{[(a/R_0)^{3/2}(B_0^2/4\pi)]} \]  

(IX-A.1)
into equations (VI-B.15). Dropping the primes from the scaled quantities, the set of equations (VI-B.15) then becomes Ohm's law.

\[
\frac{\partial \chi}{\partial t} = \eta \left( \nabla^2 \chi - j_b \right) - \nabla \cdot \left[ \lambda \nabla (\nabla^2 \chi) \right],
\]  \hspace{1cm} (IX-A.2)

with the bootstrap current density,

\[
j_b = - \left[ (2r)^{1/2}/B_0 \right] \frac{\partial \rho}{\partial r},
\]  \hspace{1cm} (IX-A.3)

the current viscosity (VII-A.8),

\[
\chi(r,t) = \sum_{nm} \eta(r_{nm},t) w_{nm}^2 \exp[-4(r-r_{nm})^2/w_{nm}^2],
\]  \hspace{1cm} (IX-A.4)

the island growth equation (VII-A.9),

\[
dw_{nm}/dt = 1.28 \eta(r_{nm},t) \Delta'_{nm},
\]  \hspace{1cm} (IX-A.5)

and the stability equation (VII-A.10),

\[
\nabla_{\perp}^2 \chi_{nm} = \left[ (r^{-1} \partial j/\partial r)/(\iota - n/m) \right] \chi_{nm}.
\]  \hspace{1cm} (IX-A.6)
The set of equations (IX-A.2)-(IX-A.6) forms our model for the bootstrapped tokamak simulations.

Solutions to the bootstrap equations (IX-A.2)-(IX-A.6) are obtained by the numerical method outlined in chapter VII, with a minor modification due to the addition of the bootstrap current term (IX-A.3). That is, since the function \( j_b \) has the poloidal magnetic field \( B_\theta = \frac{\partial \chi}{\partial r} \) in its denominator, the mean-field Ohm's law (IX-A.2) is now manifestly a non-linear partial differential equation. We solve this non-linear equation using a simple predictor-corrector method\(^\text{225}\) for the bootstrap term. That is, we take \( B_\theta(r,t+\Delta t) \approx B_\theta(r,t) \) in (IX-A.3), where our time step \( \Delta t = 10^{-5} \) is very small.

We now choose the resistivity and pressure functions for the bootstrapped tokamak simulations. First, for the scaled plasma resistivity profile \( \eta(r,t) \), we once again assume the form (VIII-A.6),

\[
\eta(r) = (1 - r^2)^{-\alpha},
\]

(IX-A.7)

the inverse of a parabola to some positive power \( \alpha \). In the

bootstrap simulations, we have chosen resistivity shape parameters \( \alpha \) in the range \([3.0, 5.0]\).

For the scaled plasma pressure profile \( p(r,t) \), we assume the simple parabolic form (VI-B.2).

\[
p(r,t) = p_0(t) (1 - r^2) \beta, \tag{IX-A.8}
\]

with the pressure profile shape parameter \( \beta \) some positive number. The parabolic form (IX-A.8) is chosen because it is a simple function which is consistent with measurements of the tokamak plasma temperature and density.\(^{226,227}\) The value of the pressure profile shape parameter \( \beta \) is limited by the following considerations. First, recall from (II-C.16) that the plasma resistivity \( \eta \propto T_e^{-3/2} \). Since the partial pressure \( p_e = n_e T_e \), if the electron density profile \( n_e = n_0(1 - r^2)^\nu \), then (IX-A.7) and (IX-A.8) together require that \( \beta \approx (2\alpha/3)^\nu \). Further, we impose the condition that the product \( n_{ji}B \) in Ohm's law (IX-A.2) must be finite at the plasma edge. This condition is \( \beta \geq \alpha + 1 \), which is then equivalent to \( \nu \geq (\alpha/3) + 1 \). In the bootstrap simulations, we have therefore chosen pressure


\(^{227}\)D. Vernon, Optics Communications 10, 95 (1974).
shape parameters \( \delta \) in the range \([5.0, 7.5]\). The central pressure \( p_0(t) \) of (IX-A.8) is taken as

\[
p_0(t) = \Pi_0 \left[ 1 - \exp(-10t) \right].
\]

(IX-A.9)

That is, we initiate our simulations with the bootstrap current term \( j_b \) "turned off." Since the typical bootstrap simulation run time in scaled time units is \( \Delta T \approx 1.0 \), the steady-state value of the tokamak central pressure \( p_0 \) is given by the pressure strength parameter \( \Pi_0 \). The initial condition \( p_0(t=0)=0 \) is imposed purely for numerical convenience. However, if one likes, one can imagine that the gradual increase in the central pressure and bootstrap current simulates the externally controlled injection of particles and heat into the plasma interior. That is, the time evolution described by the pressure function \( p_0(t) \) can be viewed as a crude representation of the process of bringing the tokamak device into the long mean free path, thermonuclear plasma regime discussed in section VI-A.

Having chosen the resistivity profile as (IX-A.7), we initialize the bootstrap simulations by assuming the quasi-
steady initial state (VIII-A.8),

\[ j(r, t=0) = \frac{V(t=0)}{\eta} = 2 \left(1 - r^2\right)^2, \]  

where the initial loop voltage \( V(t=0) = 2.0 \). For the edge loop voltage boundary condition, we again choose the function (VIII-A.10),

\[ V(r=1, t) = 2.0 + \Delta V \left[1 - \exp(-10t)\right], \]  

but now take the offset voltage \( \Delta V = -2.0 \). Then (IX-A.11) can be viewed as representing the gradual decay of the inductive loop voltage. The choice \( \Delta V = -2.0 \) means that in the steady-state there is no loop voltage, \( V = 0 \). That is, we are considering the case of a completely bootstrapped tokamak.

Finally, recall from section VI-B that the bootstrap effect leads to the formation of hollow (outwardly peaked) plasma current profiles. It is not difficult to show that a hollow plasma current profile implies that the tokamak safety factor \( q \) is a non-monotonic function of the minor radius \( r \). That is, a hollow current profile leads to a safety factor profile \( q(r) \) which has a minimum \( \frac{dq}{dr}\bigg|_{r_0=0} = 0 \) at a
radial position $r=r_0$, where the position $r_0<0$. It is well-known\textsuperscript{228,229} that this kind of non-monotonic $q$ profile leads to strong "double" tearing mode activity for all poloidal mode numbers $m$ with rational surfaces within the region $r<r_0$. That is, the hollow plasma current profile caused by the bootstrap effect has the same negative tearing mode stability properties as those of the skin current distribution briefly discussed in section VIII-A. In developing the bootstrap simulations, we therefore consider a model in which there are many small-scale, localized tearing modes associated with the bootstrap current, and assume that the strongest tearing activity occurs within the hollowed region $dq/dr<0$. We crudely approximate the time-averaged effect of this tearing region about the magnetic axis by an effective magnetic island, with an island half-width $r_s(t)$. For the bootstrapped tokamak case, we take $r_s$ as the location $r_0$ of the minimum of the safety factor profile. We therefore model the tearing mode activity in the center of the bootstrapped tokamak by adding a term to the current viscosity (IX-A.4) of the same form as (VIII-A.11).

\textsuperscript{228}S. V. Mirnov and I. B. Semenov, Atomnaya Energiya 30, 20 (1971).
\[ \lambda_s(r,t) = \eta(0,t) (2r_s)^2 \exp(-r^2/r_s^2). \] (IX-A.12)

where the scaled resistivity \( \eta(0,t) = \eta(0) = 1.0 \). To better understand the effect of \( \lambda_s \) in the bootstrap simulations, we substitute the expression (IX-A.12) into Ohm's law (IX-A.2). Recalling from section VI-B the important fact that the bootstrap current density \( j_b \) vanishes on axis, the steady-state tokamak current density on axis is therefore once again given by (VIII-A.12) with \( V=0 \),

\[ j_0 = 2(2r_s)^2 (\partial^2 j / \partial r^2)_0. \] (IX-A.13)

At this point, however, it is very important to note that, unlike inductively driven tokamaks, bootstrapped tokamak current profiles have \( (\partial^2 j / \partial r^2)_0 > 0 \). That is, the plasma current profile is now hollow. Then (IX-A.13) means that the central current density \( j_0 > 0 \). A current viscosity in the tokamak center can therefore maintain a current density \( j_0 \) on axis, an intrinsic "seed" current, which the bootstrap effect can then amplify.
Sec. IX-B. Self-maintenance of the Tokamak Current

Recall that the fundamental Ohm's law (VI-B.13) for the bootstrap effect is

\[ E \cdot B = \eta (j \cdot B - j_B B) - \nabla \cdot [\lambda \nabla (j_\parallel /B)]. \]  
(IX-B.1)

From (IX-B.1) and the definition of the effective plasma resistance \( R_\Omega \), we have

\[ R_\Omega = (\int E \cdot B d^3x) / I \psi = [\int \eta (j \cdot B - j_B B) d^3x] / I \psi, \]  
(IX-B.2)

where the \( \lambda \) term vanishes by (V-B.17). Note from (IX-B.2) that the effective plasma resistance \( R_\Omega \) itself can vanish for an appropriate set of parallel and bootstrap current profiles, \( j_\parallel (r) \) and \( j_B (r) \). As mentioned in the introduction, the vanishing of \( R_\Omega \) would allow for a finite tokamak toroidal current \( I \), even if the loop voltage \( V=0 \). In this section, we show that our numerical code can actually find sets of profiles \( j_\parallel (r) \) and \( j_B (r) \) which make \( R_\Omega = 0 \). That is, we now produce current profiles which are consistent with the self-
maintenance of the tokamak current.

We begin by defining a bootstrap simulation from the relations (IX-A.7)-(IX-A.9). That is, we initiate a bootstrap simulation specifying the resistivity profile shape parameter $\alpha$, the pressure profile shape $\varphi$, and the pressure strength parameter $\Pi_0$. In order to compare one bootstrap simulation with another, after specifying the shape parameters $\alpha$ and $\varphi$, we have varied the pressure strength $\Pi_0$ until the steady-state central safety factor $q_0=1.0$. Actually, the parameter $\Pi_0$ can be chosen independently from $\alpha$ and $\varphi$, so long as the plasma pressure in the tokamak center is sufficiently large to avoid the $m=2$ tearing mode from occurring near the magnetic axis. That is, we observe from our simulations that tearing mode stability considerations require the maintenance of some minimum value of the plasma pressure in the center of a completely bootstrapped tokamak.

The significance of the bootstrap parameters $(\alpha, \varphi, \Pi_0)$ can be understood by calculating the mean poloidal beta (VIII-B.7). Choosing the scaling

$$\langle \beta_\theta \rangle' = \frac{\langle \beta_\theta \rangle}{(R_0/a)^{1/2}}$$

(IX-B.3)
and dropping primes, we find the steady-state value

\[ \langle \beta_\theta \rangle = 2 \left[ \Pi_0 / (\sigma + 1) \right] q_a^2. \]  

(IX-B.4)

where \( \langle \beta_\theta \rangle \) is now expressed in units of \((R_0/a)^{1/2}\). The input parameters \( \alpha, \sigma, \) and \( \Pi_0 \) thus define a bootstrap simulation, and result in the calculation of the internal inductance \( l_i \), the edge safety factor \( q_a \), and the mean poloidal beta \( \langle \beta_\theta \rangle \). That is, our simulations give theoretical predictions for the three physical quantities discussed in section VIII-B which are universally monitored by plasma physicists.

In Table 2, we give the steady-state results of several bootstrap simulations. For a fixed resistivity shape \( \alpha \), we see that the required pressure strength \( \Pi_0 \) decreases with increasing pressure peakedness \( \sigma \), as expected from (IX-A.3). We note that the values of \( \langle \beta_\theta \rangle \) remain relatively constant throughout our simulations, \( \langle \beta_\theta \rangle \approx 1.2 \), in units of \((R_0/a)^{1/2}\). [Of course, the value of this constant depends on the choice made for the numerical factor in the expression for the bootstrap current (VI-A.11), but the value is clearly of \( O(1) \).] Another parameter given in Table II is the ratio \( l_b / l \)
Table 2. Steady-state results of several bootstrap simulations. The input parameters are the resistivity shape $\alpha$, the pressure shape $\sigma$, and the pressure strength $\Pi_0$. Calculated results are the edge safety factor $q_a$, the internal inductance $l_i$, and the mean poloidal beta $\langle \beta_\theta \rangle$ in units of $[(R_0/a)^{1/2}]$. The ratio $I_b/I$ of the total bootstrap current to the total plasma current is also given. The central safety factor $q_0=1.0$. 

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\sigma$</th>
<th>$\Pi_0$</th>
<th>$q_a$</th>
<th>$l_i$</th>
<th>$\langle \beta_\theta \rangle$</th>
<th>$I_b/I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>5.0</td>
<td>0.4310</td>
<td>2.958</td>
<td>1.423</td>
<td>1.257</td>
<td>1.169</td>
</tr>
<tr>
<td>3.0</td>
<td>5.5</td>
<td>0.4030</td>
<td>3.143</td>
<td>1.471</td>
<td>1.225</td>
<td>1.138</td>
</tr>
<tr>
<td>4.0</td>
<td>5.5</td>
<td>0.4020</td>
<td>3.182</td>
<td>1.489</td>
<td>1.252</td>
<td>1.151</td>
</tr>
<tr>
<td>4.0</td>
<td>6.0</td>
<td>0.3780</td>
<td>3.377</td>
<td>1.538</td>
<td>1.232</td>
<td>1.128</td>
</tr>
<tr>
<td>4.0</td>
<td>6.5</td>
<td>0.3575</td>
<td>3.578</td>
<td>1.586</td>
<td>1.221</td>
<td>1.111</td>
</tr>
<tr>
<td>5.0</td>
<td>6.5</td>
<td>0.3565</td>
<td>3.615</td>
<td>1.599</td>
<td>1.243</td>
<td>1.122</td>
</tr>
<tr>
<td>5.0</td>
<td>7.0</td>
<td>0.3385</td>
<td>3.819</td>
<td>1.647</td>
<td>1.235</td>
<td>1.108</td>
</tr>
<tr>
<td>5.0</td>
<td>7.5</td>
<td>0.3220</td>
<td>4.036</td>
<td>1.695</td>
<td>1.234</td>
<td>1.099</td>
</tr>
</tbody>
</table>
of the total bootstrap current to the total plasma current,

\[
\frac{I_b}{I} = \frac{\int j_b(r) \, dr}{\int j_{\parallel}(r) \, dr}. \tag{IX-B.5}
\]

For a fixed resistivity shape \( \alpha \), we find that the ratio \( I_b/I \) also decreases as the pressure peakedness \( \varepsilon \) increases.

In Table 3, we list the output parameters from a typical bootstrapped tokamak simulation. This simulation was constructed with \( \alpha=3.0 \), \( \varepsilon=5.0 \), and \( \Pi_0=0.4310 \). The profiles of the steady-state poloidal magnetic field, rotational transform, bootstrap and parallel plasma current densities, and MHD activity parameter \( \ell^2=\lambda/\eta \) from this simulation are also shown in Figs. 12-15. We see that MHD activity, represented by the profile of \( \ell^2=\lambda/\eta \), has taken the bootstrap current density profile \( j_b(r) \) and shaped it into a parallel plasma current density profile \( j_{\parallel}(r) \) which is MHD marginally stable to tearing modes. The total bootstrap current of this simulation is about 17% greater than the total plasma current.

As in the inductively driven tokamak simulations, if the MHD activity of the \( m=2 \), \( n=1 \) mode is too great near the

\[\text{B. B. Kadomtsev and V. D. Shafranov, Nuc. Fusion Suppl., p. 209 (1972).}\]
plasma edge, a simulation disruption can occur, with a large fraction of the plasma current being ejected from the plasma. In the bootstrap case, we even observe that the direction of the toroidal current can even be altered during a disruption. This is a direct consequence of the non-linear nature of the bootstrap current term (IX-A.3).
| Internal Inductance, Edge Safety Factor, Current Ratio, |
| l_i = 1.423 | q_a = 2.958 | l_b/l = 1.169 |

Mean poloidal beta in units of \([R_0/a]^{1/2}\),
\[\langle \beta_0 \rangle = 1.257\]

Energy content in units of \([(2\pi R_0 \cdot 2 \pi a^2)(a/R_0)^2(B_0^2/8\pi)]\),
\[W_0 = 0.163\]

Helicity content in units of \([(2\pi a^2 B_0^2)]\),
\[K_0 = 0.135\]

Magnetic island widths \(w_{nm}\) in units of \([a]\),
\[w_{1,2} = 0.152 \quad w_{2,3} = 0.111 \quad w_{3,4} = 0.021\]
\[w_{4,5} = 0.043 \quad w_{5,6} = 0.008 \quad w_{6,7} = 0.021\]

Radial position of the \(q\) profile minimum in units of \([a]\),
\[r_0 = 0.25\]

Delta primes \(\Delta'_{n,m}\) in units of \([a^{-1}]\),
\[\Delta'_{4,11} = -22.0 \quad \Delta'_{3,8} = -15.5 \quad \Delta'_{2,5} = -9.16\]
\[\Delta'_{3,7} = -15.8 \quad \Delta'_{4,9} = -21.0 \quad \Delta'_{5,11} = -25.9\]
\[\Delta'_{1,2} = -5.46 \times 10^{-6} \quad \Delta'_{6,11} = -28.1 \quad \Delta'_{5,9} = -23.0\]
\[\Delta'_{4,7} = -17.6 \quad \Delta'_{3,5} = -11.6 \quad \Delta'_{5,8} = -21.3\]
\[\Delta'_{7,11} = -30.5 \quad \Delta'_{2,3} = +2.93 \times 10^{-5} \quad \Delta'_{7,10} = -28.6\]
\[\Delta'_{5,7} = -18.0 \quad \Delta'_{8,11} = -32.1 \quad \Delta'_{3,4} = -2.27 \times 10^{-4}\]
\[\Delta'_{7,9} = -23.9 \quad \Delta'_{4,5} = +1.48 \times 10^{-5} \quad \Delta'_{9,11} = -29.5\]
\[\Delta'_{5,6} = -2.54 \times 10^{-4} \quad \Delta'_{6,7} = -3.43 \times 10^{-4} \quad \Delta'_{7,8} = -0.969\]
\[\Delta'_{8,9} = -4.49 \quad \Delta'_{9,10} = -12.2 \quad \Delta'_{10,11} = -20.1\]

Table 3. Output parameters from a typical bootstrapped tokamak simulation. Listed are the results from the simulation with resistivity shape parameter \(\alpha = 3.0\), pressure shape parameter \(\delta = 5.0\), and pressure strength \(\Pi_0 = 0.4310\).
Figure 12. Poloidal magnetic field from a typical bootstrapped tokamak simulation. The function plotted is the steady-state radial profile of the poloidal magnetic field $B_\theta(r) = \partial \chi / \partial r$ in units of $[(a/R_0)B_0]$.
Figure 13. Rotational transform from a typical bootstrapped tokamak simulation. The function plotted is the steady-state radial profile of the rotational transform \( \lambda(\tau) = r^{-1} \frac{\partial \chi}{\partial \tau} \). Note the hollow character of the profile.
Figure 14. Current densities from a typical bootstrapped tokamak simulation. The functions plotted are the steady-state radial profiles of the current density functions in units of \([(a/R_0)(c/4\pi)(B_0/a)]\). The dashed curve is the bootstrap current density function \(j_b(r) = -[(2r)^{1/2}/B_0] \partial p/\partial r\). The solid curve is the parallel plasma current density function \(j_\| (r) = \nabla^2 \chi\).
Figure 15. MHD activity parameter from a typical bootstrapped tokamak simulation. The function plotted is the steady-state radial profile of the MHD activity parameter \( \ell^2(r) = \lambda / \eta \) in units of \([a^2]\).
CHAPTER X. CONCLUSION

Section X-A. Plasma Theory and Experiment

The results of our mean-field simulations indicate that a completely bootstrapped tokamak is possible. That is, using the mean-field Ohm's law (I-D.3), we have been able to find several steady-state current profiles for completely bootstrapped tokamaks which are marginally stable to tearing modes. An important point to mention, however, is that our bootstrap simulations assume the forms for the resistivity and pressure profiles, and thus we have not attained truly self-consistent solutions. This is an important point because a very critical issue which we have not been able to resolve is whether a completely bootstrapped tokamak can maintain its pressure gradient $\partial p/\partial r$. In fact, the strong desire to resolve this issue was one of the main reasons we spent so much time and effort trying to find a way to include the heat equation (VIII-A.2) in our mean-field model. Of course, considering the large flux of energy produced by the thermonuclear reactions (I-A.2) and the existence of
refuelling mechanisms such as pellet injection\textsuperscript{231} which can deposit large amounts of plasma particles into the tokamak core, it is certainly conceivable that a completely bootstrapped tokamak can maintain its pressure gradient. On the other hand, if the MHD activity in the center of a bootstrapped tokamak plasma is so strong that we lose the pressure gradient, then clearly we will also lose the bootstrap current (VI-A.11).

Actually, what is required to maintain the pressure gradient in a bootstrapped tokamak is that the time-scales for the flattening of the plasma current and the flattening of the plasma pressure in the center of the tokamak be disparate enough so that the pressure gradient can build-up, while the current remains relatively flat. For typical thermonuclear parameters, the ratio of the skin time $\tau_X$ to the global energy confinement time $\tau_E$ is

$$\frac{\tau_X}{\tau_E} \approx 10^3. \quad \text{(X-A.1)}$$

The largeness of the ratio \( X - A.1 \) leads one to be optimistic that the pressure gradient can in fact build up on a time-scale which is much faster than that of the current.

Another way of viewing this very critical issue is to ask whether the MHD activity in the center of a completely bootstrapped tokamak will be any more disastrous than that due to \( m = 1 \) sawtooth oscillations. While the model we have adopted for the MHD activity in the plasma center is very crude, the similarities between the profiles of the MHD parameter \( l^2 = \lambda / \eta B^2 \) for the inductively driven simulation of Fig. 10 and bootstrap simulation of Fig. 15 again lead one to be optimistic that the tokamak is sufficiently resilient to tearing modes that it can maintain its own current via the bootstrap effect.

Incidentally, we note that the MHD activity in the plasma center may actually be helpful in removing the helium ash of the fusion reactions (I-A.2) from the tokamak core. That is, recalling the barbecue analogy to fusion presented in the Introduction, some mechanism must exist to remove the spent fuel from the tokamak plasma interior or the tokamak plasma will begin to choke on itself. On the other hand, one still wants to magnetically confine the helium \( \alpha \)-particles
long enough so that they are able to transfer their thermal energy to the plasma bulk.\textsuperscript{232}

In any case, we now bring this work to a close by making a general observation about the nature of theory and experiment in plasma physics. That is, one major thing we have learned while working on this problem is that a large fraction of what is understood about the physical behavior of the plasma state is a direct consequence of carefully refining and innovating the experimental technique. In many ways, plasma theory is still in its infancy and the theoretical predictions of plasma behavior are often of a very crude sort. The crude nature of plasma theory is well-illustrated by the fact that the best predictors for basic plasma quantities such as the energy confinement time $\tau_E$ are empirical scaling laws such as the Goldston formula (VIII-B.11). Since the scaling laws are developed from large sets of experimental data, these formulae are actually of a purely inductive nature. The plasma scaling laws themselves are perhaps not even that trustworthy. This is because there are many possible dimensionless scaling parameters to consider in plasma physics. While theory\textsuperscript{233} can constrain the possible

types of parameter combinations and experiment can give hints as to which dimensionless parameters are the most important ones to use, there simply is no hard guarantee that when one builds a new tokamak that it will scale according to the experience gained from different sized devices.

Nevertheless, having remarked on the somewhat immature nature of plasma theory, let us end our discussion by pointing out that, ironically, this imperfection means that even rather modest theoretical predictions of plasma behavior can be of great value. That is, when one is truly lost, even a small set of directions can make a big difference. It is in this spirit that we suggest that plasma physicists consider our simulation results,\textsuperscript{234} return to their machines, and search for the completely bootstrapped tokamak.

BIBLIOGRAPHY


VITA
RICHARD HENRY WEENING


Attended grade school classes at the Drexel Hill Elementary School, the Drexel Hill Junior High School, and the Upper Darby Senior High School in Upper Darby, Pennsylvania, receiving a High School Diploma in 1978.

Attended undergraduate school classes at the College of William and Mary in Virginia, the University of Pennsylvania, the University of California-Santa Barbara, and Drexel University, receiving a Bachelor of Science Degree with honors in Physics from William and Mary in 1984.

Attended graduate school classes at the University of California-Berkeley and William and Mary, receiving a Master of Science Degree in Physics from William and Mary in 1986.

Working under the direction of Prof. Allen H. Boozer, completed the requirements for the doctoral degree with a dissertation in Theoretical Plasma Physics, receiving a Doctor of Philosophy Degree from William and Mary in 1991.

Was a Teaching Assistant and Research Assistant at the University of California-Berkeley and William and Mary during the period 1984-1991. Was also an Acting Instructor of Physics at William and Mary in the Summer of 1986.

Spent the Summer of 1988 as a Research Assistant in the Controlled Thermonuclear Reactor Division, Los Alamos National Lab, Los Alamos, New Mexico.

Received an Associated Western Universities Fellowship to work with the Fusion Theory Group, General Atomics, San Diego, California for the Summer of 1989.

Has accepted a post-doctoral position as a Research Physicist at the Centre de Recherches en Physique des Plasmas, Ecole Polytechnique Federale de Lausanne, which is the Swiss member organization of Association EURATOM.