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Completion of partial operator matrices

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COMPLETION OF PARTIAL OPERATOR MATRICES

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APPROVAL SHEET

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This thesis is lovingly dedicated to my parents.

ABSTRACT

This work concerns completion problems for partial operator matrices. A partial matrix is an m-by-n array in which some entries are specified and the remaining are unspecified. We allow the entries to be operators acting between corresponding vector spaces (in general, bounded linear operators between Hilbert spaces). Graphs are associated with partial matrices. Chordal graphs and directed graphs with a perfect edge elimination scheme play a key role in our considerations. A specific choice for the unspecified entries is referred to as a completion of the partial matrix. The completion problems studied here involve properties such as: zero-blocks in certain positions of the inverse, positive (semi)definitness, contractivity, or minimum negative inertia for Hermitian operator matrices. Some completion results are generalized to the case of combinatorially nonsymmetric partial matrices. Several applications including a "maximum entropy" result and determinant formulae for matrices with sparse inverses are given.

In Chapter II we treat completion problems involving zero-blocks in the inverse. Our main result deals with partial operator matrices R, for which the directed graph is associated with an oriented tree. We prove that under invertibility conditions on certain principal minors, R admits a unique invertible completion F such that $(F^{-1})_{ij} = 0$ whenever R_{ij} is unspecified.

Chapter III treats positive semidefinite and Hermitian completions. In the case of partial positive operator matrices with a chordal graph, a "maximum entropy" principle is presented, generalizing the maximum determinant result in the scalar case. We obtain a linear fractional transform parametrization for the set of all positive semidefinite completions for a generalized banded partial matrix. We also give an inertia formula for Hermitian operator matrices with sparse inverses.

In Chapter IV prior results are applied to obtain facts about contractive and linearly constrained completion problems. The solution to a general n-by-n "strong-Parrott" type completion problem is the main result. We prove necessary and sufficient conditions for the existence of a solution as well as a cascade transform parametrization for the set of all solutions.

Chapter V extends the results in Chapter II and III to prove determinant formulae for matrices with sparse inverses. Several ideas from graph theory are used. An inheritance principle for chordal graphs is also presented.

COMPLETION OF PARTIAL OPERATOR MARICES

CHAPTER I
INTRODUCTION

This thesis extends, in a variety of ways, the literature on matrix completion problems. Our purpose is to extend several results in the scalar case to operator matrices, as well as to extend some results on positive definite and contractive completion problems to extremal cases. We also generalize some completion results involving zero-blocks in certain positions of the inverse to the case of combinatorially nonsymmetric partial operator matrices. Several applications are given, including a "maximum entropy result" and determinant formulae for matrices with sparse inverses.

An operator matrix completion problem may be described as follows. Given is a partial operator matrix, i.e. a matrix $A = (A_{ij})_{i=1,j=1}^{n\ m} : \oplus_{i=1}^{m}\mathcal{H}_i \rightarrow \oplus_{j=1}^{n}\mathcal{K}_j$ in which some of the entries are specified (bounded linear) operators acting between vector spaces (Hilbert spaces, in our case) and the remaining entries are "unspecified", that is, they may be chosen to be operators acting between the vector spaces belonging to the particular position in the matrix in which they are placed. A specific choice is referred to as a completion of the partial matrix. The completion problems in this thesis involve properties such as: zero-blocks in certain positions of the inverse, positive (semi)definiteness, contractivity and minimum negative inertia.

A partial matrix A is called (combinatorially) symmetric if: all the diagonal entries A_{ii} are specified and A_{ij} is specified if and only if A_{ji} is also specified. In [38] undirected graphs were associated with symmetric partial matrices (see Section 1.3 for the exact definition). In this way it was possible to connect the combinatorial aspects of graph theory with the algebraic side of the matrix completion theory. A special role is played by the chordal graphs, which first came into attention in connection with perfect Gaussian elimination ([59]). Among the first completion problems considered were ones involving banded partial matrices. An $n-by-n$ partial matrix A is called *banded* whenever A_{ij} is specified if and only if $|i-j| \leq m$, m being a fixed integer, $0 \leq m < n$. Banded partial matrices R with block matrix entries and certain invertibility conditions of some principal minors of R have been considered in [24]. Under these conditions, there is a unique invertible completion F of R, such that the factors of the UDL factorization of F^{-1} have zero-blocks outside the band of width m. In Section 2.1 we generalize the above mentioned result of [24] in two directions, first allowing the entries to be linear operators and second, we shall consider the graph of the partial matrix to be chordal. In [43], the authors considered partial matrices R with a chordal graph and the invertibility of certain fully specified pricipal minors of R. They prove the existence of a unique completion F of R such that F^{-1} has zeros in all the positions corresponding to unspecified entries of R. In Section 2.1 a different proof of this latter result is presented, which easily generalizes in Section 2.2 for partial matrices with a nonsymmetric support. Directed graphs are associated with combinatorially nonsymmetric partial matrices (see Section 1.3). In Section 2.2 we consider partial operator matrices R with their directed graph belonging to a certain class and the invertibility condition of some key principal minors of R. Under this circumstances, we prove the existence of a unique invertible completion F of R such that $(F^{-1})_{ij}$ is zero whenever R_{ij} is unspecified. The importance of the "zero in the inverse completions" will be outlined throughout the paper. The applications include fast factorization algorithms, maximum entropy results and determinant formulae.

The positive definite completions have been first considered in [24]. It was shown that, in the band case, the existence of a positive definite completion is ensured by the obviously necessary condition that the prescribed principal submatrices

$$(A_{ij})_{i,j=k}^{k+m}, k = 1, ..., n-m$$

are positive definite. Futhermore, the authors showed that when these condition are met, there exists a unique positive definite completion which has, as we shall call it, the "maximum entropy principle". A consequence of this result in the positive definite block matrix completion problem is the existence of a unique maximum determinant positive completion. In [38], the authors investigated the boundaries of generalization of [24], and concluded that the existence of a unique maximum determinant completion is ensured as soon as there exists a positive definite completion at all. For this they use the logconcavity of the determinant. As it turns out, the maximum determinant positive definite completion is the unique positive definite completion with the property that its inverse has zeros in all the positions corresponding to the unspecified entries of the initial partial matrix. In Section 3.1, based on an approach in [67] we extend the method of [38] to prove several determinant optimization results in a more general setting. The characterization of the existence of a positive definite completion is in general not only the requirement that all the fully specified principal submatrices are positive definite. In [38] the authors showed that this is only the case when the graph associated with the partial matrix is chordal. In Section 3.1, we show that in the chordal operator case, the maximum entropy completion has in fact a stronger "maximum diagonal" property, and also give an explicit construction of this completion. This generalizes the scalar band case in [24].

Another approach to the positive (semi)definite completion problem might be refered to as the "Schur analysis approach". A complete Schur analysis of $n - by - n$ positive definite operator matrices was established in [16]. The method provides in the band case a parametrization for the set of all solutions. In the scalar matrix case, existence of linear fractional descriptions for the set of all solutions was established in the papers [8] (the nonsingular case) and [9] (the singular case). A different way of deriving a linear fractional parametrization in the positive definite band case was given in [33]. Here it was recognized that the coefficients for the linear fractional map can be read off from the maximum entropy completion. The authors derived this result in an abstract setting, in order to use it in various algebras ([33]- [36]). In Section 3.2 a linear fractional parametrization for the set of all positive semidefinite solutions is presented, generalizing in this way the results in [33]. This latter result was obtained in [7].

In the area of Hermitian completions the main concern is related to inertia possibilities. The inertia of a an n−by−n Hermitian matrix is a triple $i(A) = (i_+(A), i_-(A), i_0(A))$, in which $i_+(A)$, $i_-(A)$ and $i_0(A)$ denote, respectively, the number of positive, negative and zero eigenvalues of A (counting multiplicities). Because of the interlacing inequlities, the number of negative (resp. positive) eigenvalues of any Hermitian completion of an Hermitian partial matrix cannot be less than the number of negative (resp. positive) eigenvalues for any fully specified principal submatrix. In [26], it was shown that in the band case, nonsingular completions exist which do not increase the number of negative eigenvalues - under a nonsingularity assumption on certain specified principal submatrices. Without the nonsingularity assumption, it is possible to complete without increasing the sum of zero and negative eigenvalues ([26]). The results of [26] were generalized for the chordal case in [49]. In Section 3.3, a different proof of this latter result is presented that further allows the entries to be linear operators.

Given an invertible Hermitian matrix with a banded inverse, in [26] a formula for the inertia of the matrix was established in terms of the inertias of certain of its principal minors. The result was extended in [44] for Hermitian matrices with a chordal inverse. In [45], the formula was further generalized for a certain class of Hermitian operator matrices. In Section 3.3 the result is proved in the most general operator setting.

A contractive completion problem concerning a partial matrix A can be transformed into a positive semidefinite completion problem of the partial matrix $\begin{pmatrix} I & A \\ A^* & I \end{pmatrix}$. Using this observation, results on contractive completions can be derived from the results on positive semidefinite completions. Thus, the research on contractive completions developed in parallel with that on positive definite completions. In [47], the patterns for a partial matrix that guarantee the existence of contractive completions provided all the fully specified submatrices of the partial matrix are contractions have been characterized. In Section 4.2 triangular (i.e., when all the lower triangular entries of the matrix are specified) partial matrices are considered which admit contractive completions. In this case, based on the results in Chapter III, an explicit cascade transform description is obtained for the set of all contractive, isometric, co-isometric and unitary completions. Consequently, we recover the results of [9] stating in the scalar matrix case the existence of such a description.

In [29], a $2-by-2$ linearly constrained contractive problem, named the Strong Parrott problem has been considered. The introduction of the Strong Parrott problem was a consequence of questions arising in the theory of contractive intertwining dilation ([64] and [28]). In Section 4.3 we consider a more general $n-by-n$ linearly constrained completion problem. For this latter problem necessary and sufficient conditions are derived for the existence of a contractive solution. In the case the conditions are met we build a solution with several distinguishing properties, named *central completion.* From the central completion a cascade transform parametrization is constructed for the set of all solutions. The results in Section 4.2 and 4.3 appear as an application of the results on positive semidefinite completions in Section 3.2 and follow the paper [7].

Several determinant formulae and inequalities are strictly related to matrix completion results. In [11], a determinant formula for invertible matrices with a chordal nonzero-pattern of their inverse was obtained in terms of the determinants of certain key principle minors. The result led to a formula ([42]) for the maximum over the determinants of all positive definite completions of a partial positive matrix with a chordal graph, generalizing in this way the corresponding results of [25] in the band case. The paper [42] also includes a "Hadamard-Fischer" type inequality for positive definite matrices. As a consequence of the results in Chapter II, we obtain in Section 5.1 a determinant formula for invertible matrices with a chordal nonzero-pattern of the inverse. In Section 5.2 the results are generalized for a certain class of invertible matrices with a nonsymmetric nonzero-pattern of the inverse. As application to our approach, we mention a determinant formula proved in [12] and a counterexample to a problem raised also in [12].

In the process of completion of partial positive matrices with a chordal graph ([38]) the notion of increasing chordal sequences (see Section 1.2 for definition) plays a central role. Several properties of these increasing chordal sequences are pointed out. First, we obtain a parametrization of all positive definite completions of a partial positive matrix with a chordal graph along a fixed increasing chordal sequence. The parameters are complex numbers of modulus less than 1. Then, in Section 5.3, a formula for computing the determinant of each completion in terms of the associated parameters is given. As an application of our results we obtain a proof of an inheritance principle which was conjectured in [48], generalizing a result of [25]. The conjecture was independently solved by different methods in [13]. Finally, we conclude with a stronger version of this inheritance principle.

1.1 Operator-Theoretic Notions

In this section we introduce some notation concerning Hilbert space operators, present the basic notions and prove several preliminary results. Separable complex Hilbert spaces are considered and usually denoted by $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{L}$ (perhaps also with indices). For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the set of all bounded linear operators acting from $\mathcal{H}_1$ to $\mathcal{H}_2$. We shorten $\mathcal{B}(\mathcal{H}, \mathcal{H})$ to $\mathcal{B}(\mathcal{H})$. An *operator matrix* $A = (A_{ij})_{i,j=1}^n$ is a matrix whose A_{ij} entry is in $\mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$, $\mathcal{H}_1, ..., \mathcal{H}_n$ being Hilbert spaces. Let $A \in \mathcal{B}(\mathcal{H})$ and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ... \oplus \mathcal{H}_n$ be a direct sum decomposition of $\mathcal{H}$. Throughout this paper, for an index set $\alpha \subseteq \{1, ..., n\}$, P_α denotes the orthogonal projection of $\mathcal{H}$ onto $\oplus_{j\in\alpha}\mathcal{H}_j$. The above decomposition of $\mathcal{H}$ produces a matrix decomposition $A = (A_{ij})_{i,j=1}^n$, in which $A_{ij} = P_i A|\mathcal{H}_j$.

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an operator matrix with the property that A is invertible. Then, M admits the following factorization:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} \tag{1.1}$$

The operator $D - CA^{-1}B$ is called the *Schur complement* of A in M. The Schur complement first arose in connection with Gaussian elimination on scalar matrices. It will play a key role in our considerations. As a first application, when M is a matrix, the factorization (1.1) implies that

$$detM = detA \times det(D - CA^{-1}B) \tag{1.2}$$

An Hermitian operator $A \in \mathcal{B}(\mathcal{H})$ is said to be *positive definite* (resp. *positive semidefinite*) if $(Ah, h) > 0$ (resp. $(Ah, h) \geq 0$) for any $0 \neq h \in \mathcal{H}$. We will use the notation $A > 0$ (resp. $A \geq 0$) for positive definite (resp. positive semidefinite) operators. If $A \geq 0$ then $A^{1/2}$ is its unique square root with $A^{1/2} \geq 0$.

For a linear operator A, $\mathcal{R}(A)$ denotes its range and $\overline{\mathcal{R}(A)}$ the closure of its range. The kernel of A will be denoted $ker(A)$.

For a contraction $G : \mathcal{L} \to \mathcal{K}$, denote $D_G = (I_{\mathcal{L}} - G^*G)^{1/2} : \mathcal{L} \to \mathcal{L}$ the *defect operator* of G and $\mathcal{D}_G = \overline{\mathcal{R}(D_G)}$ *the defect space* of G.

The following is a well known result in Operator Theory (see e.g. [28]).

LEMMA 1.1. *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$ *be an operator matrix. Then, A is positive semidefinite if and only if* $A_{11} \geq 0$, $A_{22} \geq 0$ *and*

$$A_{12} = A_{11}^{1/2} G A_{22}^{1/2}. \tag{1.3}$$

in which $G : \overline{\mathcal{R}(A_{22})} \to \overline{\mathcal{R}(A_{11})}$ *is a contraction.*

Consider the $3 - by - 3$ operator matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{pmatrix}$$

and assume $\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \geq 0$ and $\begin{pmatrix} A_{22} & A_{23} \\ A_{23}^* & A_{33} \end{pmatrix} \geq 0$. Let $G_1 : \overline{\mathcal{R}(A_{22})} \to \overline{\mathcal{R}(A_{11})}$ and $G_2 : \overline{\mathcal{R}(A_{33})} \to \overline{\mathcal{R}(A_{22})}$ be uniquely determined contractions such that $A_{12} = A_{11}^{1/2} G_1 A_{22}^{1/2}$ and $A_{23} = A_{22}^{1/2} G_2 A_{33}^{1/2}$. Then, as proved in [16], $A \geq 0$ if and only if there exists a contraction $G : \mathcal{D}_{G_2} \to \mathcal{D}_{G_1^*}$ such that

$$A_{13} = A_{11}^{1/2}(G_1 G_2 + D_{G_1^*} G D_{G_2}) A_{33}^{1/2}. \tag{1.4}$$

The above leads to a "Schur-type" parametrization of positive semidefinite operator matrices ([16]).

THEOREM 1.2. *There exists a one-to-one correspondence between the set of all positive semidefinite operator matrices* $(A_{ij})_{i,j=1}^n$ *with fixed positive semidefinite block diagonal entries and the set of all upper triangular families of contractions* $\mathcal{G} = \{\Gamma_{ij}\}_{1\leq i\leq j\leq n}$, *in which* $\Gamma_{ii} = I_{\overline{\mathcal{R}(A_{ii})}}$, $i = 1, \ldots n$ *and* $\Gamma_{ij} : \mathcal{D}_{\Gamma_{i+1,j}} \to \mathcal{D}_{\Gamma_{i,j-1}}$ *for* $1 \leq i < j \leq n$.

The family of contractions $\mathcal{G}$ is referred to as the *choice triangle* corresponding to $(A_{ij})_{i,j=1}^n$.

It is known that any positive semidefinite operator matrix A admits the factorization:

$$A = V^*V = W^*W \tag{1.5}$$

in which V is upper triangular and W is lower triangular. We will refer the factorizations (1.5) as the *lower-upper* (respectively *upper-lower) Cholesky factorizations* of A.

In [16], given $A \geq 0$ and $\mathcal{G} = \{\Gamma_{ij}\}_{1\leq i\leq j\leq n}$ its choice triangle, an explicit formula for the Cholesky factors V and W in (1.5) was given. It will be of interest in Chapter III:

$$V : \oplus_{i=1}^n \overline{\mathcal{R}(A_{ii})} \to \overline{\mathcal{R}(A_{11})} \oplus (\oplus_{k=2}^n \mathcal{D}_{\Gamma_{1k}}), \tag{1.6}$$

$$W : \oplus_{i=1}^n \overline{\mathcal{R}(A_{ii})} \to \oplus_{k=1}^{n-1} \mathcal{D}_{\Gamma_{kn}^*} \oplus \overline{\mathcal{R}(A_{nn})} \tag{1.7}$$

having dense range, and their block diagonal entries given by

$$V_{ii} = D_{\Gamma_{1i}} \ldots D_{\Gamma_{i-1,i}} A_{ii}^{1/2} \tag{1.8}$$

and

$$W_{ii} = D_{\Gamma_{in}^*} \ldots D_{\Gamma_{i-1,i}^*} A_{ii}^{1/2}. \tag{1.9}$$

In case the operator matrix A acts on a finite dimensional space, the following formula holds ([16]):

$$det A = \prod_{1\leq i<j\leq n} det D_{\Gamma_{ij}}^2 \tag{1.10}$$

If A is an $n-by-n$ (operator) matrix and $\alpha, \beta \subseteq \{1, ..., n\}$ are index sets, then throughout this paper $A(\alpha|\beta)$ will denote the submatrix of A corresponding to the rows in the set α and columns in the set β. We shorten $A(\alpha|\alpha)$ to $A(\alpha)$.

A variant of Kotelyanskii's inequality follows from (1.10). For any index sets $\alpha = \{1, ..., m\}, \beta = \{k, ...n\}$ with $1 \leq k < m \leq n$ we have that

$$(1.11) \qquad detA = \prod_{\substack{1 \leq i \leq k-1 \\ m+1 \leq j \leq n}} detD^2_{\Gamma_{ij}} \frac{detA(\alpha)detA(\beta)}{detA(\alpha \cap \beta)}.$$

Since $detD^2_{\Gamma ij} \leq 1$, with equality if and only if $\Gamma_{ij} = 0$, from (1.11) we derive Kotelyanskii's inequality ([50]):

$$(1.12) \qquad detA \leq \frac{detA(\alpha)detA(\beta)}{detA(\alpha \cap \beta)}$$

Equality holds in (1.12) if and only if all the parameters Γ_{ij} for $1 \leq i \leq k-1$, $m+1 \leq j \leq n$ are 0. The inequality (1.12) may be extended to finite families of index sets.

We next present several results concerning Hermitian operators and Hermitian operator matrices that will be used in Chapter III.

The *inertia* of an $n-by-n$ Hermitian matrix A is a triple

$$(i_+(A), i_-(A), i_0(A))$$

in which $i_+(A)$ (resp. $i_-(A)$) is the number of positive (resp. negative) eigenvalues of A (counting multiplicities), and $i_0(A) = n - i_+(A) - i_-(A)$ is the dimension of $ker(A)$.

Given a separable Hilbert space $\mathcal{H}$ and an Hermitian operator $A \in \mathcal{B}(\mathcal{H})$, recall the spectral decomposition E_A of A ([61], Chapter 12). Then, let $\mathcal{H}_A^- = E_A((-\infty, 0))\mathcal{H}$ and $\mathcal{H}_A^+ = E_A((0, \infty))\mathcal{H}_A^+$. It is known that $\mathcal{H}_A^-$ and $\mathcal{H}_A^+$ are closed invariant subspaces for A and we have the direct sum decomposition $\mathcal{H} = \mathcal{H}_A^- \oplus ker(A) \oplus \mathcal{H}_A^+$. Further, let $A^+ = A|\mathcal{H}_A^+$, $A^- = A|\mathcal{H}_A^-$, $i_+(A) = dim\mathcal{H}_A^+$, $i_-(A) = dim\mathcal{H}_A^-$ and $i_0 = dim[ker(A)]$. The last three quantities may be finite or infinite.

In the case the operator C is the compression of A to a closed subspace of $\mathcal{H}$, then $i_\pm(C) \leq i_\pm(A)$ and $i_\pm(C) + i_0(C) \leq i_\pm(A) + i_0(A)$.

LEMMA 1.3. *Let $A \in \mathcal{B}(\mathcal{H})$ be Hermitian and $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Then $i_-(X^*AX) \leq i_-(A)$.*

Proof. It is evident that $\mathcal{K}^-_{X^*AX} = \{k \in \mathcal{K} | Xk \in \mathcal{H}^-_A\}$ and thus $dim\mathcal{K}^-_{X^*AX} \leq dim\mathcal{H}^-_A$. □

We order the set of Hermitian operators in $\mathcal{B}(\mathcal{H})$ by $A \leq B$ (resp. $A < B$) if $B - A \geq 0$ (resp. > 0).

REMARK If $A, B \in \mathcal{B}(\mathcal{H})$ are such that $0 < A \leq B$, then $I \leq A^{-1/2}BA^{-1/2}$ and so $A^{1/2}B^{-1}A^{1/2} \leq I$, which implies that $0 < B^{-1} \leq A^{-1}$.

It is known (see e.g. [14]) that given an invertible operator matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with inverse $\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, then

$$i_0(A_{11}) = i_0(B_{22}). \tag{1.13}$$

Denote by $\sigma(M)$ the spectrum of a linear operator M. If M is Hermitian and $\lambda \in \sigma(M)$ is an isolated point of $\sigma(M)$ (i.e. the set $\{\lambda\}$ is the intersection of $\sigma(M)$ and an open interval), then λ is an eigenvalue of M and the range of $M - \lambda I$ is closed ([23]). We have our first result.

PROPOSITION 1.4. *Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$ be an Hermitian operator matrix such that $i_-(A) + i_0(A) < \infty$ and 0 is an isolated point of $\sigma(A)$. If $0 \in \sigma(A_{11})$, then 0 is an isolated point of $\sigma(A_{11})$. Moreover, if $i_0(A) = 0$ and $A^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}$ then*

$$i_-(A) = i_-(A_{11}) + i_0(A_{11}) + i_0(B_{22}). \tag{1.14}$$

Proof. Since A is a finite rank perturbation of a positive definite operator, its compression A_{11} will be a also. Thus, A_{11} will be a Fredholm operator of index 0 (see [23]), and so 0 is isolated in $\sigma(A_{11})$.

Assume in addition that A is invertible and $A^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}$. First, we consider the case A_{11} is also invertible, thus $i_0(A_{11}) = i_0(B_{22}) = 0$. The factorization

$$A = \begin{pmatrix} I & 0 \\ A_{12}^* & I \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{12}^* A_{11}^{-1} A_{12} \end{pmatrix} \begin{pmatrix} I & A_{12} \\ 0 & I \end{pmatrix}$$

implies that $i_-(A) = i_-(A_{11}) + i_-(A_{22} - A_{12}^* A_{11}^{-1} A_{12})$ and $B_{22} = (A_{22} - A_{12}^* A_{11}^{-1} A_{12})^{-1}$. Thus, $i_-(A) = i_-(A_{11}) + i_-(B_{22})$. This verifies our assertion in the case in which A_{11} is invertible.

Next we drop the assumption that A_{11} is invertible. Then, since $i_-(B_{22}) + i_0(B_{22}) < \infty$, for sufficiently small $\lambda > 0$, $B_{22} + \lambda I$ is invertible. Denote $B_\lambda = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} + \lambda I \end{pmatrix}$, $A_\lambda = B_\lambda^{-1}$. For suffiently small λ, $i_-(B_\lambda) = i_-(A)$ and $i_-(B_{22} + \lambda I) = i_-(B_{22})$. The above Schur complement remark also implies that $(A_\lambda)_{11}$ is invertible. Let introduce one more notation. Let $(T_\lambda)_{\lambda > 0}$ be a family of Hermitian operators on the Hilbert space $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$. Then $T_\lambda \searrow T$ (resp. $T_\lambda \nearrow T$) means that for any $h \in \mathcal{H}$ the sequence $(T_\lambda h, h)$ converges decreasingly (resp. increasingly) to (Th, h) when $\lambda \searrow 0$. Since $B_\lambda^- \searrow B^-$ and $A_\lambda^- = B_\lambda^{-1}$ when $\lambda \searrow 0$, we have that $(A_\lambda)_{11} \nearrow (A_{11})^-$. As consequence, $i_-((A_\lambda)_{11}) = i_0(A_{11}) + i_-(A_{11}) < \infty$. Since $(A_\lambda)_{11}$ is invertible, we have that:

$$i_-(A) = i_-(A_\lambda) = i_-((A_\lambda)_{11}) + i_-(B_{22} + \lambda I)$$

$$= i_-(A_{11}) + i_0(A_{11}) + i_-(B_{22}),$$

which completes the proof. □

1.2 Graph-Theoretic Notions

For terminology and results concerning graph theory we essentially follow the book [37]. An *undirected graph* is a pair $G = (V, E)$ in which V, the *vertex set*, is a finite set (usually $V = \{1, ..., n\}$), and the *edge set* E is a symmetric binary relation on V. The *adjacency set* of a vertex v is denoted by $Adj(v)$, i.e. $w \in Adj(v)$ if $(v, w) \in E$. Given a subset $A \subseteq V$, define the *subgraph induced* by A by $G_A = (A, E_A)$, in which $E_A = \{(x, y) \in E | x \in A \text{ and } y \in A\}$. The *complete graph* is the graph with the property that every pair of distinct vertices is adjacent. A subset $A \subseteq V$ is a *clique* if the induced graph on A is complete.

A special type of undirected graphs are the bipartite graphs. An undirected graph is called *bipartite* if $V = X + Y$ (the union of two disjoint sets X and Y) and any edge $(i,j) \in E$ has one endpoint in X and the other one in Y.

A *path* $[v_1, ..., v_n]$ is a sequence of vertices such that $(v_j, v_{j+1}) \in E$ for $j = 1, ..., k-1$. A cycle of length $k > 2$ is a path $[v_1, ..., v_k, v_1]$ in which $v_1, ...v_k$ are distinct. A graph G is called *chordal* if every cycle of length greater than 3 possesses a chord, i.e. an edge joining two nonconsecutive vertices of the cycle.

The graphs in Fig.I and Fig.II are chordal,

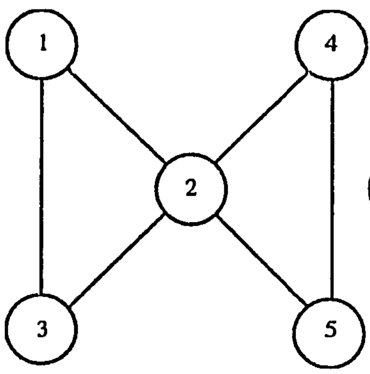

Figure I

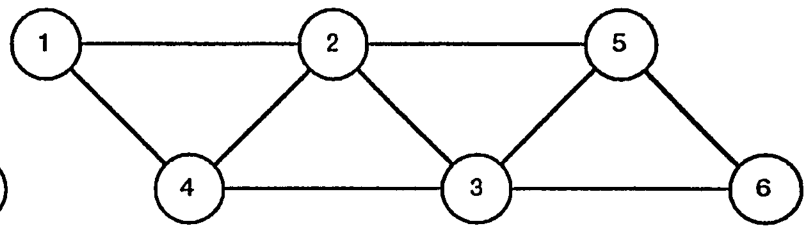

Figure II

while that in Fig. III is not, since $[2,3,4,5]$ is a chordless cycle of length 4.

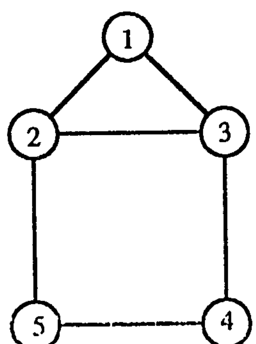

Figure III

tion scheme (or perfect scheme) if each set:

(1.15) $$S_i = \{v_j \in Adj(v_i) | j > i\}$$

is a clique. If a vertex v of G is said to be *simplicial* when $Adj(v)$ is a clique, then σ is a perfect scheme if each v_i is simplicial in the induced graph $G_{\{v_i,...,v_n\}}$. For example, $[1,3,2,4,5]$ and $[4,5,2,1,3]$ are perfect schemes for the graph in Fig.I.

First, a result known as Dirac's Lemma ([21], or Lemma 4.2 in [37]).

LEMMA 1.5. *Every chordal graph has a simplicial vertex, and if G is not a clique, then it has two nonadjacent simplicial vertices.*

The following result ([30], or Theorem 4.1 in [37]) is an algorithmic characterization of chordal graphs.

THEOREM 1.6. *An undirected graph is chordal if and only if it has a perfect scheme. Moreover, any simplicial vertex can start a perfect scheme.*

It is easy to see that $[1,4,2,3,5,6]$ and $[6,1,5,2,3,4]$ are perfect schemes for the graph in Fig.II, thus the graph is chordal. The graph in Fig.III has no perfect schemes, but also is not chordal.

A subset $S \subseteq V$ is called a $u - v$ *vertex separator* for the nonadjacent vertices u and v if the removal of S from the graph separates u and v into distinct connected components. If no proper subset of S contains a $u - v$ separator, then S is a *minimal u-v separator.* Chordality can be characterized in terms of minimal vertex separators (Theorem 4.1 in [37]).

THEOREM 1.7. *An undirected graph is chordal if and only if every minimal vertex separator is a clique.*

For example, the minimal $1 - 6$ separators of the graph in Fig.II are $\{2,4\}$, $\{2,3\}$ and $\{3,5\}$ which are cliques, since the graph is chordal.

The *intersection graph* of a family $\mathcal{F}$ of nonempty sets is obtained by representing each set in $\mathcal{F}$ by a vertex and connecting two vertices by an edge if their corresponding sets intersect. A connected graph with no cycles is called a *tree.* The following represents an important characterization of chordality (Theorem 4.8 in [37]):

THEOREM 1.8. *An undirected graph $G = (V, E)$ is chordal if and only if there exists a tree $T = (K, E)$ whose vertex set is the set of the maximal cliques of G such that each of the induced subgraphs T_{K_v} $(v \in V)$ is connected (and hence a subtree), where K_v consists of those maximal cliques that contain v.*

Let describe more precisely the tree given by the previous theorem. Each vertex of T is a maximal clique of G. Moreover, the tree has the following *intersection property:*

whenever a vertex $v \in V$ is contained in two distinct node sets K and K' of T, then v is contained in any node set lying on the unique path connecting K and K' in T. The tree given by Theorem 1.8 is called a *clique tree* (or briefly *tree*) of the chordal graph G. In general T is not uniquely determined by G. The maximal cliques of the graph in Fig.II are: $K_1 = \{1,2,4\}$, $K_2 = \{2,3,4\}$, $K_3 = \{2,3,5\}$ and $K_4 = \{3,5,6\}$ while

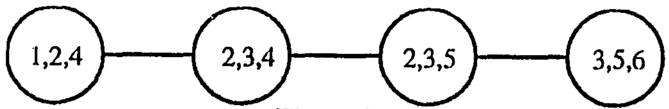

Figure IV

If $T = (\mathcal{E}(T), \mathcal{V}(T))$ is a tree of the chordal graph $G = (V, E)$, then ([13]) the set $\mathcal{S}$ of all minimal vertex separators of G coincides with the set $\{W \cap W' | \{W, W'\} \in \mathcal{E}(T)\}$. For example, $\{\{2,4\},\{2,3\},\{3,5\}\}$ represents the set of minimal vertex separators of the chordal graph in Fig. II.

Let $M = (m_{ij})_{i,j=1}^{n}$ be a matrix. The graph $G = (V, E)$ is said to be a graph of the nonzero-pattern of M if $m_{ij} = m_{ji} = 0$ whenever $(i, j) \notin E$. Chordal graphs play an important role in matrix theory in connection with the graph-theoretic description of Gaussian elimination on sparse matrices. Let $G = (V, E)$ be chordal and $\sigma = [v_1, ..., v_n]$ a perfect scheme for G. If G is a graph of the nonzero-pattern of a matrix M, then M can be reduced by perfect Gaussian elimination ([37]). This means that choosing the entries on the positions $(v_1, v_1), ..., (v_n, v_n)$ to act as pivots, M will be reduced to a diagonal matrix without ever changing (even temporarily) a zero entry to a nonzero.

We mention a result of [38]. Given any chordal graph $G = (V, E)$ there exists a *chordal sequence* of G, i.e. a sequence of chordal graphs $G = G_0, G_1, ..., G_t = K_n$ such that each G_j, $j = 1, ..., t$ is obtained by adding exactly one new edge (u_j, v_j) to G_{j-1}. Moreover, given an arbitrary chordal sequence $G = G_0, G_1, ..., G_n = K_n$ of G, each G_j has exactly one maximal clique V_j that is not a clique in G_{j-1}.

A *directed graph* is a pair $H = (V, \mathcal{F})$ in which V, the vertex set, is a finite set (usually $V = \{1, 2, ..., n\}$) and $\mathcal{F}$ is an arbitrary binary relation on V. The basic difference between graphs and directed graphs is that in the case of a directed graph the edge set is not symmetric, so we might have an edge from i to j without having an edge from j to i.

Let $H = (V, \mathcal{F})$ be a directed graph and $y \in V$. Then $Adj^{-1}(y)$ will denote the set $\{x \in V | y \in Adj(x)\}$. An edge $(x, y) \in \mathcal{F}$ is called *bisimplicial* if whenever $z \in Adj(x)$ and $z' \in Adj^{-1}(y)$ it follows that $(z', z) \in \mathcal{F}$. Consider a sequence of edges $\phi = [(x_1, y_1), ..., (x_n, y_n)]$ of H such that $V = \{x_1, ..., x_n\} = \{y_1, ..., y_n\}$. Then ϕ is called a *perfect edge elimination scheme* for H if:

1) (x_1, y_1) is bisimplicial.

2) After removing all edges of the form (x_1, z) and (z', y_1) from H, (x_2, y_2) becomes bisimplicial in the new graph.

3) At step k, $k = 1, ..., n-1$ we remove all remaining edges of the form (x_k, z) and (z', y_k) and in this way (x_{k+1}, y_{k+1}) becomes bisimplicial.

For example, $\phi = [(3,4), (1,1), (2,2), (4,3)]$ is a perfect edge elimination scheme for the directed graph in Fig. V.

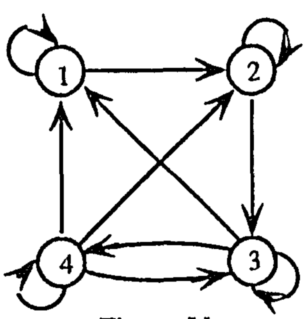

Figure V

Consider a tree $T = (V(T), \mathcal{E}(T))$ such that each node of T is a finite set V_i, $i = 1, ..., m$ and assume that T has the intersection property. Consider on each edge of T an orientation. There are 2^{m-1} distinct orientations on T. Let now $T = (V(T), \mathcal{E}(T))$, $V(T) = \{V_1, ..., V_m\}$ be a tree with the intersection property and D an orientation on $\mathcal{E}(T)$. Then, the directed graph $H = (V, \mathcal{F})$ is said to be allowed by the pair (T, D) if

$$V = \cup_{k=1}^{m} V_k$$

and whenever $(i, j) \in \mathcal{F}$ then either:

i). $\{i,j\} \subseteq V_k$ for some $k = 1, ..., m$

ii). There is an oriented path $(V_{k_1}, V_{k_2}, ..., V_{k_p})$ in D such that $i \in V_{k_1}$ and $j \in V_{k_p}$.

For example, the directed graph in Fig. VI

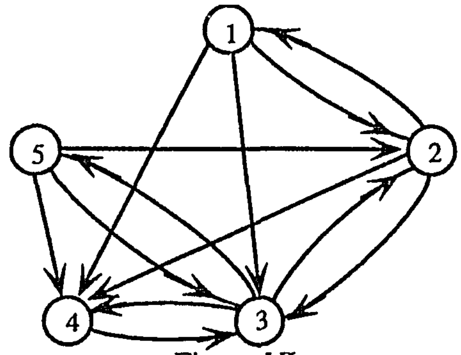

Figure VI

is allowed by the oriented tree in Fig. VII.

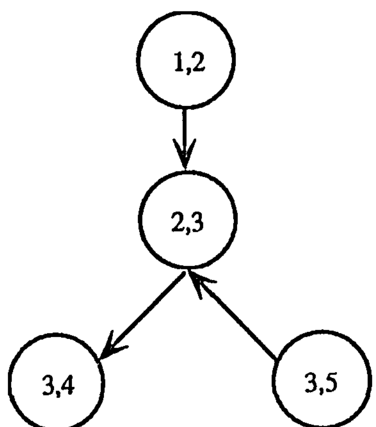

Figure VII

Let $M = (m_{ij})_{i,j=1}^{n}$ be a matrix. The directed graph $H = (V, \mathcal{F})$ is said to be a *directed graph for the nonzero-pattern* of M if $m_{ij} = 0$ whenever $(i,j) \notin \mathcal{F}$. Let $H = (V, \mathcal{F})$ be a directed graph and $\phi = [(x_1, y_1), ..., (x_n, y_n)]$ a perfect edge elimination scheme for H. If H is a directed graph of the nonzero-pattern of a matrix M, then M can be reduced by nonsymmetric perfect Gaussian elimination. This means that choosing the entries on the positions $(x_1, y_1), ..., (x_n, y_n)$ to act as pivots, M will be reduced to a matrix having only one nonzero entry on each row and column without ever changing (even temporarily) a zero entry to a nonzero.

1.3 Partial Matrices

A *partial matrix* is an $n - by - m$ array in which some entries are specified, while the remaining entries are "unspecified", i.e. independent free variables. For example,

$$\begin{pmatrix} 1 & 2 & ? \\ ? & 3 & 1+i \\ 1-i & ? & \sqrt{2} \end{pmatrix}$$

is a partial matrix in which the $(1,3)$, $(2,1)$ and $(3,2)$ entries are unspecified. The unspecified entries are denoted by ? or X, Y, Z, etc (perhaps also with indices).

Throughout this paper, we will consider the specified and unspecified entries of a partial matrix to be complex numbers, matrices or (bounded linear) operators acting between corresponding Hilbert spaces. The *operator partial matrices* will be the key objects of our investigation.

A *completion* of a partial matrix is simply a specification of each of the unspecified entries, resulting in a conventional matrix (or operator matrix). Of course, we will be interested in completions with certain properties such as: zero-blocks in the inverse on certain positions, positive definitness, contractivity and minimum number of negative eigenvalues.

A partial matrix R is called (combinatorially) symmetric if the following conditions are satisfied:

i) All the diagonal entries of R are specified.

ii) R_{ij} is specified if and only if R_{ji} is specified also.

With an $n-by-n$ symmetric partial matrix R an undirected graph $G = (V, E)$ is associated with vertex set $V = \{1, ..., n\}$ and edge set $E = \{(i,j) | i \neq j \textit{ and } R_{ij} \textit{ is specified}\}$.

With an $n - by - n$ nonsymmetric partial matrix R a directed graph $H = (V, \mathcal{F})$ is associated with vertex set $V = \{1, ..., n\}$ and edge set $\mathcal{F} = \{(i,j) | R_{ij} \textit{ is specified}\}$.

In Chapter IV we deal with $m - by - n$ partial matrices R with $m \neq n$. It is most convenient to associate with such a partial matrix a bipartite graph $G = (X, Y, E)$, in which $X = \{u_1, ..., u_m\}$, $Y = \{v_1, ..., v_n\}$ and $E = \{(u_i, v_j) | R_{ij} \textit{ is specified}\}$.

CHAPTER II
INVERTIBLE COMPLETIONS

The aim of this chapter is to find sufficient (and sometimes also necessary) conditions on a partial operator matrix in order that it admits a unique invertible completion with a certain property. This property is either a special type of UDL factorization of the inverse, or the property that the inverse has zero-blocks in the positions corresponding to the unspecified entries of the initial partial matrix. The first results of this type were obtained in [24] for banded partial matrices R. Necessary and sufficient conditions were established for the existence and uniqueness of an invertible completion F of R such that F^{-1} has a "band triangular" factorization and thus $(F^{-1})_{ij} = 0$ for $|i-j| > m$.

We start with a simple operator generalization of a scalar matrix factorization result of [46]. Then, Theorem 2.2 will be a generalization of the results of [24] in two directions. First, we shall allow the R_{ij} to be (bounded linear) operators acting between Hilbert spaces and second, we shall consider the graph of the partial matrix to be chordal.

In [43], the following was proven. *Let R be a partial matrix with a chordal support $G = (V, E)$ such that all of the principal submatrices of R corresponding to the maximal cliques and minimal vertex separators of G are invertible. Then there exists a unique invertible completion F of R such that $(F^{-1})_{ij} = 0$ for any $(i, j) \notin E$.* We present a different proof of this result that further allows the entries to be linear operators acting between Hilbert spaces. The proof is based on induction on the number of maximal cliques of G.

In all the above mentioned results the involved partial matrices R were symmetric. In Section 2.2 we deal with partial matrices with nonsymmetric support. It is most natural to consider directed graphs with a perfect edge elimination scheme in place of chordal graphs. First, a factorization result is proved for operator matrices whose sparsity pattern have a directed graph with the latter property.

Let $H = (V, \mathcal{F})$ be a directed graph and R a partial matrix with directed graph H. Assume that all the fully specified principal submatrices of R are invertible. The directed graph H is called completable whenever any such partial matrix R admits a unique invertible completion F with $(F^{-1})_{ij} = 0$ for any $(i,j) \notin \mathcal{F}$. As will be seen in Section 2.1, for undirected graphs this notion coincides with chordality. We show by means of an example that the property of having a perfect edge elimination scheme is not sufficient for a directed graph to be completable. Generalizing the methods of Section 2.1, we prove that any directed graph allowed by an oriented tree is completable. Several examples of completable and noncompletable directed graphs are presented, but a graph theoretical description of the set of all completable directed graphs is still open.

2.1 The Combinatorially Symmetric Case

Before starting some additional notation is necessary. Let Ω denote the algebra of matrices $F = (F_{ij})_{i,j=1}^{n}$ in which F_{ij} is a (bounded linear) operator acting between the Hilbert spaces $\mathcal{H}_j$ and $\mathcal{H}_i$. Also let

$$\Omega_G = \{F \in \Omega : F_{ij} = 0 \; for \; (i,j) \notin E\}$$

$$\Omega_- = \{F \in \Omega : F_{ij} = 0 \; for \; i < j\}$$

$$\Omega_+ = \{F \in \Omega : F_{ij} = 0 \; for \; i > j\}$$

$$\Omega_0 = \Omega_+ \cap \Omega_-$$

Ω_+, Ω_- respectively Ω_0 represent the set of all upper triangular, lower triangular respectively diagonal matrices in the class Ω.

When R is a partial matrix, $R \in \Omega_G$ will denote that R_{ij} is a linear operator acting between $\mathcal{H}_j$ and $\mathcal{H}_i$ and G is the graph of R (see Section 1.3). We have to make a clear distiction between the notation $F \in \Omega_G$ when F is a matrix with all of its entries specified and $R \in \Omega_G$ when R is a partial matrix. In the first case we refer to the nonzero-pattern of F, while the second notation refers to the structure of the pattern of the specified entries of R.

It is a classical result ([32]) that an operator matrix $H \in \Omega$ admits the factorization:

$$H = M_- J M_+ \tag{2.1}$$

with $M_\pm \in \Omega_\pm$, $(M_\pm)_{ii} = I$ and $J \in \Omega_0$ is invertible if and only if each $H(\{1,...,j\})$ is invertible for $j = 1,...,n$.

If a matrix is in the class Ω_G up to a permutation, each of its factors is in the same class. This is the content of the next proposition, which is an easy generalization of the scalar version in [59].

PROPOSITION 2.1. *Let G be a chordal graph and $[1,2,...,n]$ a perfect scheme for G and $H \in \Omega$ with all $H(\{1,...,j\})$ invertible. Then $H \in \Omega_G$ if and only if admits the factorization:*

$$H = X_- V X_+ \tag{2.2}$$

with $X_\pm \in \Omega_\pm \cap \Omega_G$, $(X_\pm)_{jj} = I$ and $V \in \Omega_0$ is invertible.

Proof. Let $H \in \Omega$ admit the factorization (2.2) and let $i, j \in V$ with $(i,j) \notin E$.

Thus

$$H_{ij} = \sum_{k=1}^{n} (X_- V)_{ik} (X_+)_{kj}. \tag{2.3}$$

If $(X_-V)_{ik}$ and $(X_+)_{kj}$ are nonzero we have $i \geq k$, $j \geq k$ and $(i,k), (j,k) \in E$. Since the vertex k is simplicial in the graph $G_{\{k,...,n\}}$ we obtain $(i,j) \in E$, a contradiction. Thus for every $k = 1,...,n$ we have $(X_-V)_{ik} = 0$ or $(X_+)_{kj} = 0$ and thus $H \in \Omega_G$.

Conversely, let $H \in \Omega_G$. Express H in the form (see (1.1)):

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} \tag{2.4}$$

with

$$A = H_{11}, B = [H_{12}, ..., H_{1n}]$$

$$C = [H_{21}, ..., H_{n1}]^t, D = H(\{2, ..., n\})$$

Consider $i, j \geq 2$ with $(i, j) \notin E$. Consequently:

$$(D - CA^{-1}B)_{ij} = H_{i1}H_{11}^{-1}H_{1j}.$$

Since the vertex 1 is simplicial and $(i, j) \notin E$ we get that $(1, i) \notin E$ or $(1, j) \notin E$ and so $D - CA^{-1}B \in \Omega_{G-\{1\}}$.

Take $\begin{pmatrix} I \\ CA^{-1} \end{pmatrix}$ to be the first column of X_- and $\begin{pmatrix} I & A^{-1}B \end{pmatrix}$ the first row of X_+. The factorization (2.4) of $D - CA^{-1}B$ gives us the second column of X_- and second row of X_+. Continuing in this way we eliminate all the vertices of G and obtain finally the factorization (2.2) of H. □

DEFINITION The factorization (2.2) of a matrix $H \in \Omega$ is called *triangular G-factorization.*

Note that the above definition requires $[1, ..., n]$ to be a perfect scheme for G.

For a given chordal graph G we establish next necessary and sufficient conditions on a partial operator matrix $R \in \Omega_G$ to admit a unique invertible completion F such that F^{-1} admits a triangular G-factorization.

THEOREM 2.2. *Let $R \in \Omega_G$ be a partial operator matrix and $[1, ..., n]$ a perfect scheme for G. Denote for $j = 1, ..., n$*

$$S_j = \{k \in Adj(j) | k > j\} = \{j_1, ..., j_s\} \tag{2.5}$$

with $j < j_1 < ... < j_s \leq n$ and let m be the least index for which the graph $G_{\{m+1,...,n\}}$ is complete. For $j = 1, ..., m$ express the operator $R(\{j\} \cup S_j)$ in the form:

$$R(\{j\} \cup S_j) = \begin{pmatrix} M_j & N_j \\ P_j & Q_j \end{pmatrix} \tag{2.6}$$

with

$$M_j = R_{jj}, N_j = [R_{jj_1}, ..., R_{jj_s}] \tag{2.7}$$

$$P_j = [R_{j_1 j}, ..., R_{j_s j}]^t, Q_j = R(S_j). \tag{2.8}$$

Then there exists a unique invertible completion F of R such that F^{-1} admits a triangular G-factorization:

$$F^{-1} = X_- V X_+ \tag{2.9}$$

if and only if the following conditions are satisfied:

i). The operators $\begin{pmatrix} M_j & N_j \\ P_j & Q_j \end{pmatrix}$ *are injective and have dense range for* $j = 1, ..., n$.

ii). The operators Q_j are injective and have dense range for $j = 1, ..., n$.

iii). $\mathcal{H}_j \oplus 0 \subseteq \mathcal{R}(\begin{pmatrix} M_j & N_j \\ P_j & Q_j \end{pmatrix})$ *and* $\mathcal{H}_j \oplus 0 \subseteq \mathcal{R}(\begin{pmatrix} M_j & N_j \\ P_j & Q_j \end{pmatrix}^*)$ *for* $j = 1, ..., m$.

iv). $\mathcal{R}(P_j) \subseteq \mathcal{R}(Q_j)$ *for* $j = 1, ..., m$.

v). The operators $R(\{k, ..., n\})$ are invertible for $k = m+1, ..., n$.

If the conditions i)-v) are satisfied we may constuct F as follows. Consider the unique solutions of the equations:

$$R(\{j\} \cup S_j) \begin{pmatrix} Z_{jj} \\ Z_{j_1 j} \\ \cdot \\ \cdot \\ \cdot \\ Z_{j_s j} \end{pmatrix} = \begin{pmatrix} I \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \tag{2.10}$$

and

$$\begin{pmatrix} W_{jj} \\ W_{jj_1} \\ \vdots \\ W_{jj_s} \end{pmatrix}^t R(\{j\} \cup S_j) = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}^t \tag{2.11}$$

Then put:

$$V_{jj} = Z_{jj} = W_{jj} \text{ for } j = 1, ..., n, \tag{2.12}$$

$$(2.13)\qquad \begin{pmatrix} X_{j_1 j} \\ \vdots \\ X_{j_s j} \end{pmatrix} = \begin{pmatrix} Z_{j_1 j} \\ \vdots \\ Z_{j_s j} \end{pmatrix} V_{jj}^{-1}$$

and

$$(2.14)\qquad [X_{jj_1}, ..., X_{jj_s}] = V_{jj}^{-1}[W_{jj_1}, ..., W_{jj_s}]$$

for $j = 1, ..., n-1$, *in which the* X_{ij} *are the block entries of* X_- *(resp.* X_+*) if* $i > j$ *(resp.* $i < j$*). Then we obtain* F *from (2.9).*

Proof. Suppose $R \in \Omega_G$ satisfies the conditions of the theorem. Consider the equations with the unknowns $Z \in \Omega_- \cap \Omega_G$ and $W \in \Omega_+ \cap \Omega_G$:

$$(2.15)\qquad (RZ)_{jj} = I \text{ for } j = 1, ..., n$$

and

$$(2.16)\qquad (RZ)_{ij} = 0 \text{ for } i > j \text{ and } (i,j) \in E,$$

respective,

$$(2.17)\qquad (WR)_{jj} = I \text{ for } j = 1, ..., n$$

and

$$(2.18)\qquad (WR)_{ij} = 0 \text{ for } i < j \text{ and } (i,j) \in E.$$

Consider $i \geq j$ and $(i,j) \in E$. Since

$$(2.19)\qquad (RZ)_{ij} = \sum_{k=j}^{n} R_{ik} Z_{kj}$$

and the vertex j is simplicial in the graph $G_{\{j,...,n\}}$, $(k,j) \in E$. Thus the equations (2.15) and (2.16) depend only on the specified entries of R and are equivalent to the equations (2.10). Expressing $R(\{j\} \cup S_j)$ in the form (2.6), the last mentioned equation is equivalent to:

$$(2.20)\qquad \begin{pmatrix} M_j & N_j \\ P_j & Q_j \end{pmatrix} \begin{pmatrix} Z_{jj} \\ Z^{(j)} \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}$$

in which $Z^{(j)} = [Z_{j_1 j}, ..., Z_{j_s j}]^t$.

In the same way one can show that the equations (2.17) and (2.18) are equivalent to:

$$(2.21) \qquad \begin{pmatrix} W_{jj} & W^{(j)} \end{pmatrix} \begin{pmatrix} M_j & N_j \\ P_j & Q_j \end{pmatrix} = \begin{pmatrix} I & 0 \end{pmatrix}$$

in which $W^{(j)} = [W_{jj_1}, ..., W_{jj_s}]$.

The conditions i) and iii) imply that the equations (2.20) and (2.21) have unique bounded linear solutions (see [22], Theorem 1).

Multiplying (2.21) from the right with $\begin{pmatrix} Z_{jj} \\ Z^{(j)} \end{pmatrix}$ one obtains that $W_{jj} = Z_{jj}$. We next prove $T_j = W_{jj}$ is invertible. Consider first $f \in \mathcal{H}_j$ with $T_j f = 0$. From (2.20) we obtain:

$$\begin{pmatrix} M_j & N_j \\ P_j & Q_j \end{pmatrix} \begin{pmatrix} 0 \\ Z^{(j)} f \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

Since $ker Q_j = \{0\}$, $Z^{(j)} = 0$ and thus $f = 0$ which means that T_j is injective. Take an arbitrary $f \in H_j$. The condition v) implies that there exists g in the domain of Q_j with $P_j f + Q_j g = 0$. Then (2.21) implies

$$\begin{pmatrix} T_j & W^{(j)} \end{pmatrix} \begin{pmatrix} M_j f + N_j g \\ 0 \end{pmatrix} = f$$

and T_j is also onto.

Note that for $j = m+1, ..., n$, (2.18) implies that

$$(2.22) \qquad T_j = (R(\{j\} \cup S_j)^{-1})_{11}$$

Let $T = diag(T_1, ..., T_n)$ and define the operator matrix:

$$(2.23) \qquad F = W^{-1} T Z^{-1}$$

We prove by induction that:

$$(2.24) \qquad F(\{k\} \cup S_k) = R(\{k\} \cup S_k)$$

Express $F(\{k\} \cup S_k)$ in the form

$$F(\{k\} \cup S_k) = \begin{pmatrix} M_k^1 & N_k^1 \\ P_k^1 & Q_k \end{pmatrix},$$

in which $Q_k = F(S_k) = R(S_k)$.

We obtain from the relations $FZ = W^{-1}T$ and $WF = TZ^{-1}$ the systems:

$$\begin{pmatrix} M_k^1 & N_k^1 \\ P_k^1 & Q_k \end{pmatrix} \begin{pmatrix} T_k \\ Z^{(k)} \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix} \tag{2.25}$$

and

$$\begin{pmatrix} T_k & W^{(k)} \end{pmatrix} \begin{pmatrix} M_k^1 & N_k^1 \\ P_k^1 & Q_k \end{pmatrix} = \begin{pmatrix} I & 0 \end{pmatrix}. \tag{2.26}$$

At this point M_k^1, N_k^1 and P_k^1 are considered unknowns. The relations (2.25) and (2.26) imply:

$$M_k^1 = T_k^{-1} + T_k^{-1} W^{(k)} Q_k Z^{(k)} T_k^{-1}$$

$$N_k^1 = -T_k^{-1} W^{(k)} Q_k$$

and

$$P_k^1 = -Q_k Z^{(k)} T_k^{-1}.$$

Thus (2.25) and (2.26) uniquely determine M_k^1, N_k^1 and P_k^1. The relations (2.20) and (2.21) imply that $M_k^1 = M_k$, $N_k^1 = N_k$ and $P_k^1 = P_k$ and so F is a completion of R.

Since

$$F^{-1} = ZT^{-1}W = (ZT^{-1})T(T^{-1}W),$$

denoting $X_- = ZT^{-1}$, $V = T$, $X_+ = T^{-1}W$, the factorization (2.9) follows.

We next prove the uniqueness of the completion F. We prove that in the hypothesis of the theorem, if R admits the completion F with (2.9) then the formulas (2.12)-(2.14) are satisfied.

From the $j-th$ column of the identity:

$$FX_-V = (X_+)^{-1}$$

we obtain:

$$(2.27) \qquad R(\{j\} \cup S_j) \begin{pmatrix} I \\ X_{j_1 j} \\ \cdot \\ \cdot \\ \cdot \\ X_{j_s j} \end{pmatrix} V_{jj} = \begin{pmatrix} I \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

which implies $V_{jj} = Z_{jj}$ and also the formula (2.13). The equality $V_{jj} = W_{jj}$ and the formula (2.14) are obtained from the $j-th$ row of the equality:

$$X_+VF = (X_-)^{-1}.$$

Finally we have to prove the necessity of the conditions. If $R \in \Omega_G$ has a unique invertible completion F with (2.9) then $F = X_+^{-1}V^{-1}X_-^{-1}$ and the condition v) is obviously satisfied. From the proof of the formulas (2.12)-(2.14) we deduce that the equations (2.20) and (2.21) must have unique bounded linear solutions with $Z_{jj} = W_{jj} = T_j$ invertible. This immediately implies that the conditions i) and iii) are satisfied.

Suppose that for some g in the domain of Q_j we have $Q_jg = 0$. Then:

$$T_jN_jg = \begin{pmatrix} T_j & W^{(j)} \end{pmatrix} \begin{pmatrix} M_j & N_j \\ P_j & Q_j \end{pmatrix} \begin{pmatrix} 0 \\ g \end{pmatrix}$$

Thus i) implies that $g = 0$ and Q_j is injective. The fact that Q_j^* is injective and thus Q_j has dense range can be proved in a similar way.

From (2.20) we obtain that:

$$P_jT_j + Q_jZ^{(j)} = 0$$

and so $P_j = -Q_jZ^{(j)}T_j^{-1}$ which implies iv) and finishes the proof. □

COROLLARY 2.3. *If the spaces $\mathcal{H}_1, ..., \mathcal{H}_n$ are finite dimensional, the conditions i)-v) are reduced to the invertibility of the following block matrices:*

a). $R(\{j\} \cup S_j)$ for $j = 1, ..., n$ and

b). $R(S_j)$ for $j = 1, ..., n-1$.

If the conditions a) and b) are satisfied, we have the more precise formulas in place of (2.12)-(2.14):

$$V_{jj} = (R\{j\} \cup S_j)^{-1})_{11} \tag{2.28}$$

$$\begin{pmatrix} X_{j_1 j} \\ \cdot \\ \cdot \\ \cdot \\ X_{j_s j} \end{pmatrix} = -R(S_j)^{-1} \begin{pmatrix} R_{j_1 j} \\ \cdot \\ \cdot \\ \cdot \\ R_{j_s j} \end{pmatrix} \tag{2.29}$$

and

$$[X_{jj_1}, ..., X_{jj_s}] = -[R_{jj_1}, ..., R_{jj_s}]R(S_j)^{-1} \tag{2.30}$$

Proof. If $R(\{j\} \cup S_j)$ is invertible, (2.28) is a consequence of the relation (2.27). Also (2.27) implies that:

$$\begin{pmatrix} R_{j_1 j} \\ \cdot \\ \cdot \\ \cdot \\ R_{j_s j} \end{pmatrix} + R(S_j) \begin{pmatrix} X_{j_1 j} \\ \cdot \\ \cdot \\ \cdot \\ X_{j_s j} \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

The invertibility of $R(S_j)$ implies (2.29). The formula (2.30) is obtained in a similar way. □

REMARK The next example shows that in the infinite dimensional case even for band partial matrices the invertibility conditions of Corollary 2.3 are not necessary. Consider the partial matrix:

$$R = \begin{pmatrix} A & 0 & ? \\ 0 & B & I \\ ? & I & I \end{pmatrix}$$

acting on $\mathcal{H}\oplus\mathcal{H}\oplus\mathcal{H}$ with $\mathcal{H}$ an infinite dimensional Hilbert space, A invertible, $\|B\| < 1$, $ker(B) = \{0\}$ and $\overline{\mathcal{R}(B)} = \mathcal{H}$. Since the equations:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} T_1 \\ Z^{(1)} \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} T_1 & W_{(1)} \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \end{pmatrix}$$

have unique solutions $T_1 = A^{-1}$, $Z^{(1)} = W^{(1)} = 0$ we obtain that the unique invertible completion F of R with $(F^{-1})_{13} = (F^{-1})_{31} = 0$ is:

$$F = \begin{pmatrix} A & 0 & 0 \\ 0 & B & I \\ 0 & I & I \end{pmatrix}$$

with

$$F^{-1} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -I & I \end{pmatrix}\begin{pmatrix} A^{-1} & 0 & 0 \\ 0 & (B-I)^{-1} & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} I & 0 & 0 \\ 0 & I & -I \\ 0 & 0 & I \end{pmatrix}$$

REMARK The condition v) of Theorem 2.2 is also necessary. Consider an infinite dimensional Hilbert space $\mathcal{H}$ and the operator:

$$\begin{pmatrix} A & I \\ I & A \end{pmatrix}$$

acting on $\mathcal{H}\oplus\mathcal{H}$ and assume $\overline{\mathcal{R}(A)} = \mathcal{H}$ but A is not invertible. It is easy to show that $\begin{pmatrix} A & I \\ I & A \end{pmatrix}$ is injective and has dense range. The equation:

$$\begin{pmatrix} A & I \\ I & A \end{pmatrix}\begin{pmatrix} T \\ Z \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix}$$

admits the solution $T = -A(I - A^2)^{-1}$ which is not invertible.

The following is a simple and known fact, but it will be very useful in the rest of this chapter.

LEMMA 2.4. *Consider the operator matrix* $F = (A_{ij})_{i,j=1}^{3}$ *and assume that* $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, A_{22} *and* $\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}$ *are invertible. Then,* $(F^{-1})_{13} = 0$, *(respective* $(F^{-1})_{31} = 0$*) if and only if* $A_{13} = A_{12}A_{22}^{-1}A_{23}$, *(respective* $A_{31} = A_{32}A_{22}^{-1}A_{21}$*).*

Proof. By straightforward computation. It may also be obtained as a consequence of Theorem 2.2. □

We next present a new proof of a result in [43], in the case of operator matrices. The proof has the advantage that easily generalizes in Section 2.2 for nonsymmetric patterns allowed by oriented trees.

THEOREM 2.5. *Let $G = (V, E)$ be a chordal graph and $R \in \Omega_G$ a partial operator matrix such that all of the principal submatrices of R corresponding to the maximal cliques and minimal vertex separators of G are invertible. Then there exists a unique invertible completion F of R with $(F^{-1})_{ij} = 0$ whenever $(i,j) \notin E$.*

Proof. We prove the theorem by induction on m, the number of maximal cliques of G. For $m = 1$ it is obvious. For $m = 2$ the result follows from Lemma 2.4. Assume that the result is true for graphs with $m - 1$ maximal cliques and let prove it for m.

Let $G = (V, E)$ be a chordal graph with m maximal cliques and let $R \in \Omega_G$ be a partial matrix with the properties in the statement of the theorem. Let $T = (\mathcal{V}(T), \mathcal{E}(T))$ be a tree of G which must have m node sets (see Section 1.2). Select an arbitrary node set W of T and let W' be the unique neighbouring node set of W in T. Further, let $\beta = W \cap W'$, $\alpha = W - \beta$, $\gamma = W' - \beta$ and $\delta = V - (W \cup W')$. Consider the induced partial matrix $R(\beta \cup \gamma \cup \delta)$ with $G_{\beta\cup\gamma\cup\delta}$ as associated graph. Removing the node set W and the edge $\{W, W'\}$ from T, we obtain the tree $T' = (\mathcal{V}(T'), \mathcal{E}(T'))$ with the intersection property which is a tree of $G_{\beta\cup\gamma\cup\delta}$. The partial matrix $R(\beta \cup \gamma \cup \delta)$ will inherit the invertibility conditions in the stament of the theorem from R. Thus, by the assumption made for $m - 1$, $R(\beta \cup \gamma \cup \delta)$ has a unique invertible completion F' such that $G_{\beta\cup\gamma\cup\delta}$ is a graph of F'^{-1}. Consider now the graph G' having the tree:

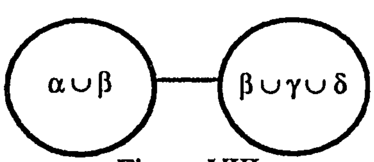

Figure VIII

and the partial matrix R' defined by $R'(\alpha \cup \beta) = R(\alpha \cup \beta)$ and $R'(\beta \cup \gamma \cup \delta) = F'$. Then R' is correctly defined and has G' as its graph. Since $R'(\beta) = F'(\beta) = R(\beta)$, R'

can be decomposed as:

$$R' = \begin{pmatrix} A_{11} & A_{12} & X_{13} \\ A_{21} & A_{22} & A_{23} \\ X_{31} & A_{32} & A_{33} \end{pmatrix} \tag{2.31}$$

in which $A_{11} = R(\alpha)$, $A_{22} = R(\beta)$ and $\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} = F'$. By Lemma 2.4, there exists an invertible completion F of R' (and also of R) with its inverse of the form:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 0 & \alpha_{32} & \alpha_{33} \end{pmatrix} \tag{2.32}$$

The relations $FF^{-1} = F^{-1}F = I$ imply that $\begin{pmatrix} \alpha_{23} \\ \alpha_{33} \end{pmatrix}$ is the second column, respectively $\begin{pmatrix} \alpha_{32} & \alpha_{33} \end{pmatrix}$ is the second row of $\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}^{-1}$ and thus $G_{\beta\cup\gamma\cup\delta}$ is a graph of F^{-1}.

It remains to prove the uniqueness part of the theorem. Assume that $\tilde{F}$ is an other completion with $(\tilde{F})_{ij} = 0$ whenever $(i,j) \notin E$. Let us decompose $\tilde{F} = (B_{ij})_{i,j=1}^{3}$ with respect to the partition $\alpha \cup \beta \cup (\gamma \cup \delta)$ of the index set. Then, in this decomposition $(\tilde{F}^{-1})_{13} = (\tilde{F}^{-1})_{31} = 0$. Thus, by the same argument as used for F, $G_{\beta\cup\gamma\cup\delta}$ is a graph of $\tilde{F}(\beta\cup\gamma\cup\delta)^{-1}$. By the uniqueness result for $m-1$, we have that $F(\beta\cup\gamma\cup\delta) = \tilde{F}(\beta\cup\gamma\cup\delta)$. It turns out that both F and $\tilde{F}$ are invertible completions of the partial matrix R' in (2.31) with the property that with respect to the partition $\alpha \cup \beta \cup (\gamma \cup \delta)$ of the index set their inverses have 0 on the (1,3) and (3,1) positions. Then Lemma 2.4 implies that $F = \tilde{F}$. This completes the proof. □

We next discuss the problem in Theorem 2.5 in the case in which the graph of the partial matrix fails to be chordal. In Chapter III we show that given any nonchordal graph $G = (V, E)$, there exists a partial matrix R with graph G and the property that all of the principal minors of R formed with specified entries are invertible, but there exist at least two invertible completions F of R with $(F^{-1})_{ij} = 0$ whenever $(i,j) \notin E$. Since the same examples will play a key role in Chapter III, we postpone their presentation.

Consider the partial matrix:

$$R = \begin{pmatrix} 1 & -1 & x & 1 \\ 1 & 1 & 0 & y \\ z & 0 & 1 & 1 \\ 0 & t & -1 & 1 \end{pmatrix}$$

with graph

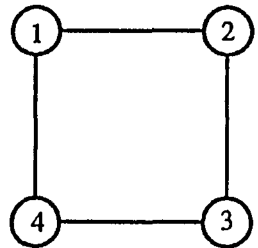

Figure IX

the simplest non-chordal graph. All the principal minors of R formed with specified entries are invertible. To find a completion F of R with $(F^{-1})_{ij} = 0$ whenever R_{ij} is unspecified we have to solve the equation system:

$$\left| \begin{pmatrix} 1 & 1 & y \\ z & 0 & 1 \\ 0 & t & 1 \end{pmatrix} \right| = yzt - z - t = 0$$

$$\left| \begin{pmatrix} 1 & -1 & x \\ z & 0 & 1 \\ 0 & t & -1 \end{pmatrix} \right| = xzt - z - t = 0$$

$$\left| \begin{pmatrix} 1 & x & 1 \\ 1 & 0 & y \\ z & 1 & 1 \end{pmatrix} \right| = xyz - x - y + 1 = 0$$

$$\left| \begin{pmatrix} -1 & x & 1 \\ 1 & 0 & y \\ t & -1 & 1 \end{pmatrix} \right| = xyt - x - y - 1 = 0$$

An elementary computation shows that this system has no solutions, which means that there are no invertible completions F with the desired zero-pattern of the inverse.

In conclusion, when the graph of the partial matrix is not chordal, there is no characterization of what may happen. We may have no completion, a unique completion or multiple completions.

The next example will show that even if the conditions of Theorem 2.5 are satisfied the unique completion given by the theorem may not be UDL factorable. Consider the partial matrix:

$$R = \begin{pmatrix} 0 & 1 & ? & ? \\ 1 & 1 & 1 & ? \\ ? & 0 & 1 & 1 \\ ? & ? & 1 & 0 \end{pmatrix}$$

having the graph:

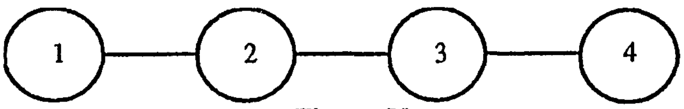

Figure X

The completion of R given by Theorem 2.5 is:

$$F = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

with

$$F^{-1} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Since the (1,1) and (4,4) entries of F are 0, neither in $[1,2,3,4]$ nor in the [4,3,2,1] orderings, the only perfect schemes of G, F admits UDL factorization. This shows that Theorem 2.5 does not imply Theorem 2.2, the extra conditions in Theorem 2.2 are necessary.

2.2 The Combinatorially Nonsymmetric Case

In Section 2.1 all the partial matrices involved R were combinatorially symmetric. In this section we consider $n-by-n$ nonsymmetric partial operator matrices R. When R is a partial matrix, $R \in \Omega_H$ will denote that $H = (V, \mathcal{F})$ is the directed graph of R.

We keep the notations Ω, Ω_+ and Ω_0 from Section 2.1. When F is a matrix with all of its entries specified, $F \in \Omega_H$ will denote that $F_{ij} = 0$ whenever $(i,j) \notin \mathcal{F}$, i.e. H is a directed graph of the nonzero-pattern of F.

It is natural to try to generalize the results of Section 2.1 for partial matrices with a "diagonal perfect edge elimination scheme", i.e. when there exists an ordering $[v_1, ..., v_n]$ of the set V such that $\phi = [(v_1, v_1), ..., (v_n, v_n)]$ is a perfect edge elimination scheme for H.

The following is a combinatorially nonsymmetric correspondence of Proposition 2.1.

PROPOSITION 2.6. *Let $H = (V, \mathcal{F})$ be a directed graph and $\phi = [(1,1), ..., (n,n)]$ a perfect edge elimination scheme for H. Let $M \in \Omega$ be such that all $M(\{1, ..., j\})$ are invertible for $j = 1, ..., n$. Then $M \in \Omega_H$ if and only if M admits the factorization:*

$$M = X_- V X_+$$

with $X_\pm \in \Omega_\pm \cap \Omega_H$, $(X_\pm)_{jj} = I$ and $V \in \Omega_0$ is invertible.

Proof. Similar to the proof of Proposition 2.1 taking into account that ϕ is a perfect edge elimination scheme for H. □

We expect a similar result to Theorem 2.5 for combinatorially nonsymmetric partial matrices, but this fails. Consider the following partial matrix:

$$\begin{pmatrix} 1 & 1 & 2 & x \\ y & 1 & z & 2 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 2 \end{pmatrix}$$

in which x, y and z denote unspecified entries. All the principal minors of R formed with specified entries are invertible. The directed graph of R has the perfect edge elimination scheme $\phi = [(1,1), (2,2), (3,3), (4,4)]$. We try to find x, y and z corresponding to an invertible completion F with $(F^{-1})_{14} = (F^{-1})_{21} = (F^{-1})_{23} = 0$. This latter equalities imply

$$\left| \begin{pmatrix} y & z & 2 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix} \right| = \left| \begin{pmatrix} 1 & 2 & x \\ y & z & 2 \\ 1 & 1 & 2 \end{pmatrix} \right| = \left| \begin{pmatrix} 1 & 2 & x \\ 1 & z & 2 \\ 1 & 1 & -1 \end{pmatrix} \right| = 0.$$

Then $y = z = 1$ and the third determinant in the above equality equals 3 regardless of x, a contradiction. The conclusion is that the existence of a perfect edge elimination scheme for a directed graph does not imply that the directed graph is completable.

We next prove that the directed graphs allowed by oriented trees (with the intersection property) are completable. The proof generalizes the proof of Theorem 2.5. We use the notation and results of Section 1.2.

THEOREM 2.7. *Let $T = (\mathcal{V}(T), \mathcal{E}(T))$, $\mathcal{V}(T) = \{V_1, ..., V_m\}$ be a tree with the intersection property and D an orientation on the edge set $\mathcal{E}(T)$. Let $H = (V, \mathcal{F})$ be the directed graph allowed by the pair (T, D). If $R \in \Omega_H$ is a partial operator matrix and the following matrices:*

(i) $R(V_k)$, for $k = 1, ..., m$

(ii) $R(V_i \cap V_j)$ for $\{V_i, V_j\} \in \mathcal{E}(T)$

are invertible, then there exists a unique invertible completion F of R with $(F^{-1})_{ij} = 0$ whenever $(i, j) \notin \mathcal{F}$.

Proof. We prove the theorem by induction on m, the number of node sets of T. For $m = 1$ the result is obvious, while for $m = 2$ the oriented tree (T, D) is of the form:

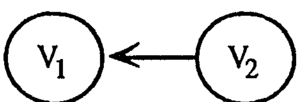

Figure XI

Let R let be a partial matrix having its directed graph allowed by the above tree and let: $\beta = V_1 \cap V_2$, $\alpha = V_1 - \beta$ and $\gamma = V_2 - \beta$. Then, R can be decomposed as:

$$\begin{pmatrix} A_{11} & A_{12} & X_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \tag{2.33}$$

with respect to the partition $\alpha \cup \beta \cup \gamma$ of the index set V and thus, $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = R(V_1)$, $\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} = R(V_2)$ and $A_{22} = R(V_1 \cap V_2) = R(\beta)$. Then, the invertibility conditions of the theorem imply via Lemma 2.4 that there exists an invertible completion F of R with the desired nonzero-pattern of the inverse.

Suppose that the result is true for $m-1$. We need to consider an extra assumption and prove it also by induction. Let $V_j \in \mathcal{V}(T)$ be a node set of T and define

$$\omega_j = \{k | \textit{there is an oriented path in } (T,D) \textit{ joining } V_k \textit{ with } V_j\}$$

$$\mu_j = \{k | \textit{there is an oriented path in } (T,D) \textit{ joining } V_j \textit{ with } V_k\}$$

Further denote $\mathcal{U}_{V_j} = \cup_{l \in \omega_j} V_l$ respective $\mathcal{V}_{V_j} = \cup_{l \in \mu_j} V_l$. Our assumption is: if F is the invertible completion of R with the desired nonzero-pattern of its inverse, then $F(\mathcal{U}_{V_j})$ and $F(\mathcal{V}_{V_j})$ are also invertible for any node set $V_j \in \mathcal{V}(T)$.

Let $T = (\mathcal{V}(T), \mathcal{E}(T))$, $\mathcal{V}(T) = \{V_1, ..., V_m\}$ be a tree with the intersection property and D an orientation on $\mathcal{E}(T)$. Let R be a partial operator matrix having its directed graph allowed by (T,D) and assume that R satisfies the conditions of the theorem. Select an extremal node set $W \in \{V_1, ..., V_m\}$ and let W' be its unique neighbour in T. Assume that the edge $\{W, W'\} \in \mathcal{E}(T)$ is oriented from W' to W. Let $\beta = W \cap W'$, $\alpha = W - \beta$, $\gamma = W' - \beta$, $\delta = \mathcal{V}_W - (W \cup W')$ and $\epsilon = V - (\alpha \cup \beta \cup \gamma \cup \delta)$. Let $T' = (\mathcal{V}(T'), \mathcal{E}(T'))$ be the tree obtained by removing the node set W and the edge $\{W, W'\}$ from T and D' be the orientation induced by D on T'. The partial matrix $R(V-\alpha)$ has its directed graph H' allowed by the oriented tree (T', D'). Since $R(V-\alpha)$ inherits from R the invertibility properties i) and ii) in the statement of the theorem, by the assumption made for $m-1$, $R(V-\alpha)$ has an invertible completion F' such that H' is a directed graph for F'^{-1}. Consider now the partial matrix R' obtained by replacing in R, $R(V-\alpha)$ with F'. With respect to the partition $\alpha \cup \beta \cup (\gamma \cup \delta) \cup \epsilon$ of the index set, R' can be decomposed as:

$$R' = \begin{pmatrix} A_{11} & A_{12} & X_{13} & X_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ X_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \tag{2.34}$$

in which $F' = (A_{ij})_{i,j=2}^4$. Since the union of the node sets W'' of T' with the property that there exists an oriented path in (T', D') joining W'' with W' is $\beta \cup \gamma \cup \delta$, by our second assuption $F'(\beta \cup \gamma \cup \delta) = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}$ is invertible. Let succesively define

$$(2.35) \qquad \begin{pmatrix} X_{13} & X_{14} \end{pmatrix} = A_{12}A_{22}^{-1} \begin{pmatrix} A_{23} & A_{24} \end{pmatrix}$$

and

$$(2.36) \qquad X_{41} = \begin{pmatrix} A_{42} & A_{43} \end{pmatrix} \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}^{-1} \begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix}.$$

Thus, by Lemma 2.4, the relations (2.35) and (2.36) will define an invertible completion F of R' in (2.34) (and of R also), with its inverse of the form:

$$(2.37) \qquad F^{-1} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ 0 & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}.$$

The relations $FF^{-1} = F^{-1}F = I$ imply that $\begin{pmatrix} \alpha_{23} & \alpha_{24} \\ \alpha_{33} & \alpha_{34} \\ \alpha_{43} & \alpha_{44} \end{pmatrix}$ are the last two columns and $\begin{pmatrix} \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}$ is the last row of the inverse of $F' = (A_{ij})_{i,j=2}^{4}$ and thus H' is a directed graph of $(\alpha_{ij})_{i,j=2}^{4}$. This together with (2.37) implies that H is a directed graph for F^{-1}.

To finish the existence part of the proof we must also prove our second assumption, namely that with the previous notation the matrices $F(\mathcal{U}_{V_j})$ and $F(\mathcal{V}_{V_j})$ are invertible for $j = 1, ..., m$. Taking into account the relation between the oriented trees (T, D) and (T', D') and the fact that by our assumption the result holds for T', it remains to prove that $F(\mathcal{V}_W)$ is invertible, when W is the selected extremal node set of T. Since $\mathcal{V}_W = \alpha \cup \beta \cup \gamma \cup \delta$ and $F(\alpha \cup \beta \cup \gamma \cup \delta)$ is the completion of $\begin{pmatrix} A_{11} & A_{12} & X_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$ with $X_{13} = A_{12}A_{22}^{-1}A_{23}$, Lemma 2.4 implies that $F(\mathcal{V}_W)$ is invertible.

Finally we prove the uniqueness of F. Assume that $\tilde{F}$ is an other completion of R with $(\tilde{F}^{-1})_{ij} = 0$ whenever $(i,j) \notin \mathcal{F}$. Let $F = (B_{ij})_{i,j=1}^{4}$ be the decomposition of F with respect to the partition $\alpha \cup \beta \cup (\gamma \cup \delta) \cup \epsilon$ of the index set. Then, in this decomposition, $(\tilde{F}^{-1})_{13}$, $(\tilde{F}^{-1})_{14}$ and $(\tilde{F}^{-1})_{41}$ are 0. Thus, by the same argument as used for F, H' is a directed graph for $\tilde{F}(V-\alpha)^{-1}$. By the uniqueness result for $m-1$, we have that $\tilde{F}(V-\alpha) = F(V-\alpha)$. It turns out that both F and $\tilde{F}$ are invertible completions of R' in (2.34), with the property that in the partition $\alpha \cup \beta \cup (\gamma \cup \delta) \cup \epsilon$ of the index set their inverses have 0 on the (1,3), (1,4) and (4,1) positions. Then by Lemma 2.4 we have that $\tilde{F} = F$. This finishes the proof. □

EXAMPLE Consider the following partial operator matrix:

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & X_{15} \\ R_{21} & R_{22} & R_{23} & R_{24} & X_{25} \\ X_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\ X_{41} & X_{42} & R_{43} & R_{44} & X_{45} \\ X_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{pmatrix}$$

The directed graph of R is the one presented in Fig. VI and allowed by the oriented tree in Fig. VII. Assume that all the submatrices $R(\{1,2\})$, $R(\{2,3\})$, $R(\{3,4\})$, $R(\{3,5\})$, R_{22} and R_{33} are invertible. Following Theorem 2.7, define succesively:

$$\begin{pmatrix} X_{25} \\ X_{45} \end{pmatrix} = \begin{pmatrix} R_{23} \\ R_{43} \end{pmatrix} R_{33}^{-1} R_{35}$$

$$X_{42} = \begin{pmatrix} R_{43} & R_{45} \end{pmatrix} R(\{3,5\})^{-1} \begin{pmatrix} R_{32} \\ R_{52} \end{pmatrix}$$

$$X_{15} = \begin{pmatrix} R_{12} & R_{13} & R_{14} \end{pmatrix} R(\{2,3,4\})^{-1} \begin{pmatrix} X_{25} & R_{35} & X_{45} \end{pmatrix}^t$$

$$\begin{pmatrix} X_{31} & X_{41} & X_{51} \end{pmatrix}^t = \begin{pmatrix} R_{32} & R_{42} & R_{52} \end{pmatrix}^t R_{22}^{-1} R_{21}$$

We obtain in this way an invertible completion F of R with the property that $(F^{-1})_{ij} = 0$ whenever R_{ij} is unspecified.

COROLLARY 2.8. *Let* $1 < \beta(1) ... < \beta(k) = n$ *and consider a partial operator matrix* R *with* R_{ij} *specified if and only if* $j \leq \beta(i)$. *If all the submatrices* $R(i, ..., \beta(i))$ *are invertible for* $i = 1, ..., k-1$, *then* R *admits a unique invertible completion* F *with* $(F^{-1})_{ij} = 0$ *whenever* $j > \beta(i), i = 1, ..., k-1$.

The partial matrices in Corollary 2.8 have all their unspecified entries situated above the main diagonal. They have a "triangular" form. Triangular partial matrices will be studied from an other point of view in Chapter IV.

REMARK Let us consider $1 \leq p, q \leq n-1$, $s = min\{p, q\}$ and a partial matrix R, in which R_{ij} is specified if and only if $i \leq j \leq i+p$ or $j \leq i \leq j+p$. If all the submatrices $R(\{k, ...k+s\})$, for $k = 1, ..., n-s$ and $R(\{k+1, ..., k+s\})$ for $k = 1, ..., n-s-1$ are invertible, then there exists a unique invertible completion F of R such that $(F^{-1})_{ij} = 0$ whenever $j > i+p$ or $i > j+p$.

The above result was proved in [10] for scalar matrices, but it is still true for partial operator matrices. The directed graph of these partial matrices is not allowed by an oriented tree.

Consider the following partial operator matrix:

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ X_{31} & R_{32} & R_{33} & R_{34} & R_{35} & X_{36} \\ X_{41} & R_{42} & R_{43} & R_{44} & R_{45} & X_{46} \\ X_{51} & R_{52} & X_{53} & R_{54} & R_{55} & R_{56} \\ X_{61} & R_{62} & X_{63} & R_{64} & R_{65} & R_{66} \end{pmatrix}$$

and assume that all the principal submatrices of R formed with specified entries are invertible. It will be proved in Section 5.2 that the directed graph of R is not allowed by any oriented tree. It is easy to see that the pattern of R is not permutation equivalent to any of the patterns discussed before. The directed graph of R is still completable, since succesively defining:

$$X_{46} = \begin{pmatrix} R_{42} & R_{45} \end{pmatrix} R(\{2,5\})^{-1} \begin{pmatrix} R_{26} \\ R_{56} \end{pmatrix}$$

$$\begin{pmatrix} X_{53} \\ X_{63} \end{pmatrix} = \begin{pmatrix} R_{52} & R_{54} \\ R_{62} & R_{64} \end{pmatrix} R(\{2,4\})^{-1} \begin{pmatrix} R_{23} \\ R_{43} \end{pmatrix}$$

$$X_{36} = \begin{pmatrix} R_{32} & R_{34} & R_{35} \end{pmatrix} R(\{2,4,5\})^{-1} \begin{pmatrix} R_{26} & X_{46} & R_{56} \end{pmatrix}^t$$

$$\begin{pmatrix} X_{31} & X_{41} & X_{51} & X_{61} \end{pmatrix} = \begin{pmatrix} R_{32} & R_{42} & R_{52} & R_{62} \end{pmatrix} R_{22}^{-1} R_{21}$$

we obtain an invertible completion F of R with $(F^{-1})_{ij} = 0$ whenever (i, j) corresponds to an unspecified entry of R.

In conclusion, the description of the set of all completable directed graphs still remains open.

CHAPTER III
POSITIVE SEMIDEFINITE AND HERMITIAN COMPLETIONS

Probably the area of positive definite matrices is the most fruitful from the point of view of matrix completion. We begin this chapter with the celebrated completion result of [38] stating that any partial positive matrix with a chordal graph admits a positive definite completion. Several proofs are indicated, one of them based on the results of Chapter II. In the same paper [38] it was proved that for an arbitrary partial positive matrix R which admits positive definite completions, there exist a unique determinant maximizing positive definite completion F_0 of R. Moreover, F_0 is the unique positive definite completion of R with the property that its inverse has 0 in all the positions corresponding to the unspecified entries of R. Their proof is based on the logconcavity of the determinant. Theorem 3.3 represents an operator correspondence for chordal graphs of the above mentioned maximum determinant principle. In Section 3.1, also based on the logcancavity of the determinant several optimization results are obtained.

In the chordal case, the unspecified entries of the maximum determinant positive definite completion can be obtained as rational functions of the given data. In [53] the problem was raised whether or not the above property characterizes only the chordal graphs. In the last part of Section 3.1 we give an affirmative answer to the above problem.

In Section 3.2 we study positive semidefinite completions of "generalized banded" operator matrices. The results follow the paper [7]. We first develop some distinguishing properties which uniquely characterize a so-called central completion, notion that appeared in different settings and names in [24], [25], [1] and [33]. Next, a linear fractional transform parametrization is presented for the set of all solutions. The coefficients of the transformations are obtained from the Cholesky factorizations of the central completion. This is a generalization of the results in [33].

In [49] it was proved that for any partial Hermitian matrix R with chordal support there exist an Hermitian completion F of R such that

$$i_0(F) + i_-(F) = max\{i_0(R(K)) + i_-(R(K)) | K \text{ is a clique of } G\}.$$

In Section 3.3 we prove that under certain circumstances the same result holds for operator partial matrices also. The problem of the number of negative eigenvalues of Hermitian extensions of partial matrices has been studied in [27], [18], [31] and [17].

Let F be an Hermitian matrix with a chordal nonzero-pattern of its inverse. In [44] a formula for the inertia of F was proved in terms of the inertias of certain key principal minors of F. This result was further generalized in [45] for a certain class of operator matrices. In Section 3.3 we prove this latter result in the most general operator setting.

3.1 Maximum Entropy Positive Definite Completions

A partial operator matrix R is called *partial positive* if all the principal submatrices of R formed with specified entries are positive definite. The following is a well-known result of [38]. The original proof is for scalar matrices, but as shown next, it works for operator matrices also. In [57], it was extended to matrices over certain C^*-algebras.

THEOREM 3.1. *Let G be a chordal graph. Then any partial positive operator matrix $R \in \Omega_G$ admits a positive definite completion.*

Proof. Consider $G = G_0, G_1, ..., G_t = K_n$ an arbitrary chordal sequence (see Section 1.2) of G. Let (u_j, v_j) be the unique edge added to G_{j-1} in order to obtain G_j and V_j the unique maximal clique of G_j which is not a clique of G_{j-1}. Consider the partial submatrix $R_0(V_1)$ of $R_0 = R$ corresponding to the index set V_1. After a reordering, if necessary, the partial matrix $R_0(V_1)$ has the following structure:

$$R_0(V_1) = \begin{pmatrix} A & B & X_{u_1,v_1} \\ B^* & C & D \\ X^*_{u_1,v_1} & D^* & E \end{pmatrix}$$

in which the matrices $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ and $\begin{pmatrix} C & D \\ D^* & E \end{pmatrix}$ are positive definite. Then, by (1.4), $R_0(V_1)$ admits a positive definite completion. Select any value for X_{u_1,v_1} which provides a positive definite completion of $R_0(V_1)$. R_0 will be transformed in this way into a partial positive matrix R_1 having G_1 as associated graph. Repeating the process for R_1 and G_1, we obtain a partial positive matrix R_2 and so on, until we obtain a positive definite completion M of R. □

The proof of Theorem 3.1 is also valid for positive semidefinite completions of partial positive semidefinite matrices.

In the scalar case, the proof of the above theorem together with Theorem 1.2 imply the following parametrization result ([5]).

THEOREM 3.2. *Let R be a partial positive matrix and let G be its associated graph which is supposed to be chordal. Fix a chordal sequence $G = G_0, G_1, ..., G_t = K_n$ of G. Then, any positive definite completion of R is uniquely determined by a set $\{g(u_j, v_j) | j = 1, ..., t\}$ of complex numbers with $|g(u_j, v_j)| < 1$, (u_j, v_j) being the unique edge added to G_{j-1} in order to obtain G_j.*

As the parameters $\{g(u_j, v_j) | j = 1, .., t\}$ depend only on the fixed chordal sequence of G, we call them the *parameters of M along the underlying chordal sequence.* These parameters will play a key role in Chapter V in computing the determinant of an arbitrary positive definite completion.

Theorem 3.1 can also be obtained as a corollary of Theorem 2.2. Indeed, consider a chordal graph G and without loss of generality assume that $\sigma = [1, ..., n]$ is a perfect scheme for G. For any partial positive matrix $R \in \Omega_G$ the conditions of Corollary 2.3 are satisfied. Thus there exists a unique completion F of R such that $F^{-1} = X_- V X_+$

with V and $X_\pm$ given by (2.28-2.30). It easily follows from the formulae that V is positive definite and $X_- = X_+^*$. This implies that F is a positive definite completion of R. For the same result, a proof based on the Arveson Extension Theorem was presented in [56].

It is known from [38] that for any nonchordal graph G, there exists a partial positive matrix $R \in \Omega_G$ without having a positive definite completion.

The following result is referred as the maximum entropy principle. The notion was first introduced in a particular case in [15]. In the scalar band case it was proved in [24], while in [35] appers in the operator band case as an example of a more general maximum entropy principle.

THEOREM 3.3. *Let G be chordal, $R \in \Omega_G$ be a partial positive operator matrix and assume that $[1, ..., n]$ is a perfect scheme for G. Let F be the unique completion of R with $F^{-1} \in \Omega_G$ and write $F^{-1} = X_+^* V X_+$ as in (2.9). Then for any positive definite completion H of R and factorization $H^{-1} = M_+^* J M_+$ with $M_+ \in \Omega_+$, $(M_+)_{jj} = I$ and $J \in \Omega_0$ we have that:*

$$V^{-1} \geq J^{-1} \tag{3.1}$$

Proof. Consider F and M as in the statement of the theorem. Then:

$$(X_+(H - F)X_+^*)_{jj} = 0. \tag{3.2}$$

In order to prove (3.2), note that

$$(X_+(H - F)X_+^*)_{jj} = \sum_{i,k=1}^{n} (X_+)_{ji}(H - F)_{ik}(X_+^*)_{kj}$$

If $(X_+)_{ji}$ and $(X_+^*)_{kj}$ are nonzero, $j \leq i$, $(i, j) \in E$, $k \geq j$, $(j, k) \in E$ then since the vertex j is simplicial in the graph $G_{\{j,...,n\}}$ we have that $(i, k) \in E$ and thus $(H - F)_{ik} = 0$. In the latter equality we used that both F and H are completions of R. This implies (3.2).

Following (3.2), since

$$X_+(H-F)X_+^* = X_+M_+^{-1}J^{-1}M_+^{*-1}X_+^{*-1} - V^{-1}$$

we have that

$$(V^{-1})_{jj} = (X_+M_+^{-1}J^{-1}M_+^{*-1}X_+^*)_{jj} \geq (J^{-1})_{jj}$$

for $j = 1, ..., n$. The last inequality holds since $(X_+M_+^{-1})_{jj} = I$ for $j = 1, ..., n$. Thus (3.1) is true and the equality holds if and only if $X_+M_+^{-1} = I$, consequently when $H = F$. □

In the scalar case, as consequence of Theorem 3.3, F is the unique maximum determinant positive definite completion of R. This result was proved in [38] for an arbitrary graph and partial matrix which admits a positive definite completion. Their result will be a particular case of a more general result ([67]) which will be considered next.

In what follows let $\mathcal{M}$ denote the set of all $n-by-n$ self-adjoint complex matrices, $\mathcal{P} = \{A \in \mathcal{M} | A \geq 0\}$ and $\mathcal{P}^{(0)} = \{A \in \mathcal{M} | A > 0\}$. Let $\mathcal{W} \subset \mathcal{M}$ be a linear subspace such that $\mathcal{W} \cap \mathcal{P} = \{0\}$. In the rest of this section we consider the scalar product $(C, D) = tr(CD^*)$ on $\mathcal{M}$.

We next present the approach of [67] to certain determinant optimization results.

THEOREM 3.4. *Let A, $B \in \mathcal{M}$ be such that $(A + \mathcal{W}) \cap \mathcal{P}^{(0)} \neq \emptyset$. Then there is a unique $F \in (A + \mathcal{W}) \cap \mathcal{P}^{(0)}$ such that $F^{-1} - B \perp \mathcal{W}$. Moreover, F maximizes the function*

$$f(X) = logdetX - tr(BX)$$

over $X \in (A + \mathcal{W}) \cap \mathcal{P}$.

Proof. Since $\mathcal{W} \cap \mathcal{P} = \{0\}$, $(A + \mathcal{W}) \cap \mathcal{P}$ is a bounded set (we are in a finite dimensional space). The set $(A + \mathcal{W}) \cap \mathcal{P}$ is convex. It is known that logdet is strictly concave on $\mathcal{P}$ (see e.g. [39]). Since $tr(BX)$ is linear in X, $f(X)$ is strictly concave and thus has a unique maximum on $(A+\mathcal{W})\cap\mathcal{P}$ denoted by F. Since near the boundary f tends to $-\infty$, F is an inner point of $(A+\mathcal{W})\cap\mathcal{P}^{(0)}$.

Fix an arbitrary $W\in\mathcal{W}$. Consider the function $f_W(x) = logdet(F+xW) - tr(B(F+xW))$ defined in a neighbourhood of 0 in $\mathbf{C}$. Then $f'_W(0)=0$ (since f has its maximum in F).

It is easy to see that:

$$f'_W(0) = \frac{(det(I+xF^{-1}W))'}{det(I+xF^{-1}W)}\Big|_{x=0} - (tr(B(F+xW)))'|_{x=0}$$

$$= tr(F^{-1}W) - tr(BW) = tr((F^{-1}-B)W) = 0$$

Since $W=W^*$ is an arbitrary element of $\mathcal{W}$ we have that $F^{-1}-B\perp\mathcal{W}$ which finishes the proof. □

Let $A\in\mathcal{P}^{(0)}$, $B\in\mathcal{M}$ and an arbitrary graph $G=(V,E)$ be given. We assume that for any $k\in V$, $(k,k)\in E$. Let

$$\mathcal{W} = \{W\in\mathcal{M}|W_{kj}=0 \ whenever \ (k,j)\in E\}$$

Then $\mathcal{W}\cap\mathcal{P}=\{0\}$ and $A\in(A+\mathcal{W})\cap\mathcal{P}^{(0)}$. By Theorem 3.4 there exists $F\in(A+\mathcal{W})\cap\mathcal{P}^{(0)}$ such that $F^{-1}-B\perp\mathcal{W}$. For any $(k,j)\notin E$, consider the matrix $W_R^{(k,j)}\in\mathcal{W}$ having all its entries 0 except those on the positions (k,j) and (j,k) which equal 1, respectively the matrix $W_I^{(k,j)}$ having i on the position (k,j), $-i$ on the position (j,k) and 0 in rest. The conditions $tr((F^{-1}-B)W_R^{(k,j)}) = tr((F^{-1}-B)W_I^{(k,j)}) = 0$ imply that $(F^{-1})_{kj}=B_{kj}$ for any $(k,j)\notin E$.

Thus Theorem 3.4 has the following consequence.

Corollary 3.5. *Let $A\in\mathcal{P}^{(0)}$, $B\in\mathcal{M}$ and the graph $G=(V,E)$ be given. Then there is a unique $F\in\mathcal{P}^{(0)}$ such that $F_{ij}=A_{ij}$ for any $(i,j)\in E$ and $F_{ij}=B_{ij}$ for any $(i,j)\notin E$. Moreover, F maximizes the function $f(X)=logdetX - tr(BX)$ over $(A+\mathcal{W})\cap\mathcal{P}$.*

The above result was first proved in [20] (see also [63]). The case $B = 0$ was independently proved in [38], which came in connection with the following completion result.

COROLLARY 3.6. *Let G be an arbitrary graph and R a partial positive matrix with graph G which admits positive definite completions. Then there exists a unique maximum determinant positive definite completion F of R. Moreover, F is the unique positive definite completion of R with $F^{-1} \in \Omega_G$. F is a real matrix whenever the partial matrix R is real.*

In the chordal case, Theorem 3.3 is more general, but unfortunately it can't be applied for nonchordal graphs.

In general under the hypothesis of Theorem 3.4 there is no precise formula for the optimal solution F. We next present an approximation of F.

Let $\{(i_k, j_k) | k = 0, 1, ..., s-1\}$ be an arbitrary ordering of the missing edges of G. For any $M \in (A + \mathcal{M}) \cap \mathcal{P}^{(0)}$ define the positive definite matrices $X_k^{(M)}$, $k = 0, 1, ..., s$ by $X_0^{(M)} = M$ and letting $X_{k+1}^{(M)}$ be obtained by modifying the (i_k, j_k) and (j_k, i_k) entries of $X_k^{(M)}$ such that $(X_{k+1}^{(M)})^{-1}_{i_k, j_k} = B_{i_k, j_k}$ for $k = 0, 1, ..., s-1$. (This is possible by Corollary 3.5). Define then the function

$$g : (A + \mathcal{W}) \cap \mathcal{P}^{(0)} \to (A + \mathcal{W}) \cap \mathcal{P}^{(0)}, g(M) = X_s^M.$$

Then F is the unique fixed point for g since for any other $M \in (A + \mathcal{W}) \cap \mathcal{P}^{(0)}$ we have $f(M) < f(g(M))$.

Define the following sequence: $Y_0 = A$, $Y_{m+1} = g(Y_m)$ for $m \geq 0$. Consider H to be a limit point of the sequence $\{Y_m\}_{m=0}^{\infty}$. Since H is a fixed point for g, it follows that $H = F$. Consequently, $Y_m \to F$. (This proof is based on the so-called coordinate descent, see e.g. the book [52] for more details on this method).

In the operatorial case, when $B = 0$ and $G = (V, E)$ is chordal, F can be precisely computed by the formulae (2.28-2.30).

Again let $\{(i_k, j_k) | k = 0, 1, ..., s-1\}$ be an ordering of the missing edges of an arbitrary graph $G = (V, E)$. For any $M \in \mathcal{P}^{(0)}$, order the $(M_{i_k, j_k})_{k=0}^{s-1}$, $i_k < j_k$ entries of

M in a vector $v(M) \in C^s$. Consider a matrix $Q \in C^{r\times s}$. Under these circumstances, Theorem 3.4 has the following consequence, originally proved in [54].

COROLLARY 3.7. *Let $A \in \mathcal{P}^{(0)}$ and $c \in C^r$ be such that $Qv(A) = c$. Then, among the matrices $M \in \mathcal{P}^{(0)}$ satisfying $M_{ij} = A_{ij}$ for $(i,j) \in E$ and $Qv(M) = c$ there is a unique one which maximazes the determinant. It is also the unique one with the property that $v(M) \in \mathcal{R}(Q^*)$.*

Proof. The result is a consequence of Theorem 3.4 for $B = 0$ and

$$\mathcal{W} = \{W \in \mathcal{M} | W_{ij} = 0 \textit{ whenever } (i,j) \in E \textit{ and } v(M) \in kerQ\}.$$

□

An important example is the Toeplitz case. One obtains here the following result, also proved in [54].

COROLLARY 3.8. *Let A be a partial Toeplitz matrix with a prescribed main diagonal which admits a positive definite Toeplitz completion. Then there exists a unique maximum determinant positive definite Toeplitz completion F. Moreover, F is the unique positive definite Toeplitz completion of R such that the sum of the entries of F^{-1} on each of the diagonals corresponding to unspecified diagonals of R equals 0.*

Proof. In the case of Theorem 3.4 consider $B = 0$ and $\mathcal{W}$ to be the span of Toeplitz matrices of the form

$$W_1^{(j)} = \begin{pmatrix} 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 & 0 & \dots & 1 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix} \textit{ and } W_2^{(j)} = \begin{pmatrix} 0 & 0 & \dots & i & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & i & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -i & 0 & \dots & 0 & 0 & \dots & i \\ 0 & -i & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -i & 0 & \dots & 0 \end{pmatrix}$$

supported on the j - th diagonal which is supposed to be unspecified in A. Then, $(M, W_k^{(j)}) = 0$, $k = 1, 2$ if and only if the sum of the elements on the j - th diagonal of M is zero. □

Given any partial positive matrix R with chordal associated graph then by Corollary 3.6 and Corollary 2.3, the unspecified entries of F, (the maximum determinant positive definite completion of R) can be obtained as a rational function of the given entries of R. In the thesis [53], it is stated (but not proved) that the above property characterizes the chordal graphs. Namely, given any nonchordal graph $G = (V, E)$, there exists a partial positive rational matrix with associated graph G, such that the maximum determinant positive definite completion F of R fails to be a rational matrix. It is easy to see that it is sufficient to prove the result in the case in which the graph G is a simple cycle of length $n \geq 4$.

Consider first the partial matrix

$$\begin{pmatrix} 1 & 1/2 & ? & 1/2 \\ 1/2 & 1 & 1/2 & ? \\ ? & 1/2 & 1 & 1/2 \\ 1/2 & ? & 1/2 & 1 \end{pmatrix}$$

having associated the simple cycle of length 4. Its maximum determinant positive definite completion is

$$\begin{pmatrix} 1 & 1/2 & (\sqrt{3}-1)/2 & 1/2 \\ 1/2 & 1 & 1/2 & (\sqrt{3}-1)/2 \\ (\sqrt{3}-1)/2 & 1/2 & 1 & 1/2 \\ 1/2 & (\sqrt{3}-1)/2 & 1/2 & 1 \end{pmatrix}$$

Consider the following partial positive matrix:

$$R = \begin{pmatrix} 1 & 1/2 & ? & ? & \dots & 1/2 \\ 1/2 & 1 & 1/2 & ? & \dots & ? \\ ? & 1/2 & 1 & 1/2 & \dots & ? \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1/2 & ? & ? & ? & \dots & 1 \end{pmatrix}$$

associated with the simple cycle of length n. Then R admits positive definite completions (for instance the completion with all $\frac{1}{2}$). Let F be the maximum determinant positive definite completion of R. Then by Kotelyanskii's inequality (see Section 1.1), we have that $detF < (det \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix})^{n-1} = (\frac{3}{4})^{n-1}$.

Taking into account Corollary 3.6 and the symmetry of R, F^{-1} has the following pattern:

$$F^{-1} = \begin{pmatrix} p & q & 0 & \dots & q \\ q & p & q & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ q & 0 & 0 & \dots & p \end{pmatrix}$$

in which obviously $p+q=1$. By (1.12) we have that:

$$detF < \frac{detF(\{1,2\})detF(\{2,...,n\})}{F_{22}}$$

consequently,

$$\frac{4}{3} < \frac{detF(\{2,...,n\})}{detF} = p. \tag{3.3}$$

The condition $R_{12} = \frac{1}{2}$ implies that $\frac{P(q)}{Q(q)} = \frac{1}{2}$, in which

$$P(q) = (-1)^{n+1} det \begin{pmatrix} q & 0 & 0 & ... & 0 & q \\ q & 1-q & q & ... & 0 & 0 \\ 0 & q & 1-q & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & q & 1-q \end{pmatrix}$$

and

$$Q(q) = det \begin{pmatrix} 1-q & q & 0 & ... & q \\ q & 1-q & q & ... & 0 \\ 0 & q & 1-q & ... & 0 \\ ... & .. & ... & ... & ... \\ q & 0 & 0 & ... & 1-q \end{pmatrix}$$

Since $P(0)=0$, $Q(0)=1$, P and Q have integer coefficients, the equation $Q(q) - 2P(q) = 0$ might have rational solutions only of the form $q = -\frac{1}{k}$, with k a nonzero integer. Thus $p = \frac{k+1}{k}$ and then (3.3) implies that the only possibilities might be $p = \frac{3}{2}$ or $p = 2$. An elementary computation shows that for any $n \geq 4$, the above choices of p do not provide a completion of R. Thus p and q are irrational.

By Corollary 3.6 the equation $Q(q) - 2P(q) = 1$ has at least one real solution. Since the equation has real coefficients and an irrational solution, it has at least two irrational solutions. Both solutions will provide a completion of R with the property that the simple cycle of legth n is a graph of their inverse. This result solves a question raised in Section 2.1. Thus, for any nonchordal graph G there exists a partial matrix $R \in \Omega_G$ such that all the fully specified principal matrices of R are invertible and still there are at least two invertible completions F_1 and F_2 of R such that $F_1^{-1}, F_2^{-1} \in \Omega_G$.

3.2 Generalized Banded Partial Matrices

In this section we prove for generalized banded partial operator matrices the existence of a positive semidefinite completion with some distinguishing properties. Based on this completion, linear fractional parametrization is obtained for the set of all solutions.

Consider first the following $3-by-3$ problem:

$$\begin{pmatrix} A_{11} & A_{12} & ? \\ A_{21} & A_{22} & A_{23} \\ ? & A_{32} & A_{33} \end{pmatrix} \geq 0, \tag{3.4}$$

in which

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \geq 0, \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} \geq 0. \tag{3.5}$$

Note that the positivity of the $2-by-2$ operator matrices in (3.5) implies via (1.3) that

$$A_{12} = A_{11}^{1/2} G_1 A_{22}^{1/2}, A_{23} = A_{22}^{1/2} G_2 A_{33}^{1/2}$$

in which $G_1 : \overline{\mathcal{R}(A_{22})} \to \overline{\mathcal{R}(A_{11})}$ and $G_2 : \overline{\mathcal{R}(A_{33})} \to \overline{\mathcal{R}(A_{22})}$ are contractions.

With the choice $G = 0$ in (1.4) we obtain the particular positive semidefinite completion

$$A_{13} = A_{11}^{1/2} G_1 G_2 A_{33}^{1/2}. \tag{3.6}$$

We shall call this the *central completion* of (3.4), referring to the fact that in the operator ball in which A_{13} lies (namely the one described by (1.4)) we choose the center.

Let F be a positive semidefinite operator matrix and let

$$F = V^*V = W^*W. \tag{3.7}$$

be the lower-upper and upper-lower Cholesky factorizations of F respectively. If $\hat{V}$ and $\hat{W}$ are upper (lower) triangulars with $F = \hat{V}^*\hat{V} = \hat{W}^*\hat{W}$, then there exists block diagonal unitaries $U : \overline{\mathcal{R}(V)} \to \overline{\mathcal{R}(\hat{V})}$ and $\hat{U} : \overline{\mathcal{R}(W)} \to \overline{\mathcal{R}(\hat{W})}$ with $UV = \hat{V}$ and

$\hat{U}W = \hat{W}$. This implies that if F is a positive semidefinite $n-by-n$ operator matrix, then the operators

$$\Delta_U(F) := diag(V_{ii}^* V_{ii})_{i=1}^n \tag{3.8}$$

and

$$\Delta_L(F) := diag(W_{ii}^* W_{ii})_{i=1}^n \tag{3.9}$$

do not depend upon the particular choice of V and W in (3.7).

Returning to our problem (3.4), if F is an arbitrary completion corresponding to the parameter G in (1.4) then F admits the factorization (3.7) with

$$V = \begin{pmatrix} A_{11}^{1/2} & G_1 A_{22}^{1/2} & (G_1G_2 + D_{G_1^*} G D_{G_2}) A_{33}^{1/2} \\ 0 & D_{G_1} A_{22}^{1/2} & (D_{G_1} G_2 - G_1^* G D_{G_2}) A_{33}^{1/2} \\ 0 & 0 & D_G D_{G_2} A_{33}^{1/2} \end{pmatrix} \tag{3.10}$$

and

$$W = \begin{pmatrix} D_{G^*} D_{G_1^*} A_{11}^{1/2} & 0 & 0 \\ (D_{G_2^*} G_1^* - G_2 G^* D_{G_1^*}) A_{11}^{1/2} & D_{G_2^*} A_{22}^{1/2} & 0 \\ (G_2^* G_1^* + D_{G_2} G^* D_{G_1^*}) A_{11}^{1/2} & G_2^* A_{22}^{1/2} & A_{33}^{1/2} \end{pmatrix}. \tag{3.11}$$

Further, using relations like $G_1^*(\mathcal{D}_{G_1^*}) \subseteq \mathcal{D}_{G_1}$, one easily obtains that $\overline{\mathcal{R}(V_{ij})} \subseteq \overline{\mathcal{R}(V_{ii})}$ and $\overline{\mathcal{R}(W_{ij})} \subseteq \overline{\mathcal{R}(W_{ii})}$, for all i and j. The triangularity of V and W now yields

$$\overline{\mathcal{R}(V)} = \overline{\mathcal{R}(A_{11}^{1/2})} \oplus \mathcal{D}_{G_1} \oplus \mathcal{D}_G, \overline{\mathcal{R}(W)} = \mathcal{D}_{G^*} \oplus \mathcal{D}_{G_2^*} \oplus \overline{\mathcal{R}(A_{33}^{1/2})}. \tag{3.12}$$

One immediately sees from these equalities that when $G = 0$ the closures of the ranges of the Cholesky factors of the completion are as large as possible.

Relation (3.7) implies the existence of a unitary $U : \overline{\mathcal{R}(W)} \to \overline{\mathcal{R}(V)}$ with $UW = V$. A straightforward computation gives us the explicit expression of U, namely

$$U = \begin{pmatrix} D_{G_1^*} D_{G^*} & G_1 D_{G_2^*} - D_{G_1^*} G G_2^* & G_1 G_2 + D_{G_1^*} G D_{G_2} \\ -G_1^* D_{G^*} & D_{G_1} D_{G_2^*} - G_1^* G G_2^* & D_{G_1} G_2 - G_1^* G D_{G_2} \\ -G^* & -D_G G_2^* & D_G D_{G_2^*} \end{pmatrix}. \tag{3.13}$$

Note that the $(3,1)$ entry in U is zero if and only if $G = 0$. As it will turn out, this will be a characterization for the central completion, thus providing a generalization of the banded inverse characterization in the invertible case, discovered in [24]. We will state the result precise in the $n - by - n$ case.

Let us now consider the $n - by - n$ generalized banded positive semidefinite completion problem. Recall that $S \subseteq \underline{n} \times \underline{n}$ ($\underline{n} = \{1, ..., n\}$) is called a *generalized banded pattern* if

(1) $(i,i) \in S$, $i = 1, ..., n$;

(2) if $(i,j) \in S$ then $(j,i) \in S$; and

(3) $(i,j) \in S$ and $i \leq p, q \leq j$ imply $(p,q) \in S$.

Let us mention that the associated graph of banded pattern is a so-called "proper interval" graph, a particular case of chordal graphs.

The problem is the following. Given are $A_{ij} : \mathcal{H}_j \rightarrow \mathcal{H}_i$ for (i,j) in a prescribed generalized banded pattern S. We want to find all positive semidefinite completions of $\{A_{ij}, (i,j) \in S\}$. It is known (see Theorem 3.1) that a positive semidefinite completion of $\{A_{ij}, (i,j) \in S\}$ exists if and only if

$$(A_{ij})_{i,j \in J} \geq 0 \qquad (3.14)$$

for all $J \subseteq \underline{n}$ with $J \times J \subseteq S$. When $\{A_{ij}, (i,j) \in S\}$ verifies condition (3.14) we shall call this band *positive semidefinite.*

In [1] a parametrization was given for the set of all positive semidefinite completions of $\{A_{ij}, (i,j) \in S\}$ as follows. This parametrization is based on the result in [16] quoted above and the fact that making a completion of $\{A_{ij}, (i,j) \in S\}$ precisely corresponds to choosing the parameters $\{\Gamma_{ij}, 1 \leq i \leq j \leq n, (i,j) \notin S\}$. Thus there exists an one-to-one correspondence between the set of all positive semidefinite completions of $\{A_{ij}, (i,j) \in S\}$ and the completions of $\{\Gamma_{ij}, 1 \leq i \leq j \leq n, (i,j) \in S\}$ to a $(A_{ii})_{i=1}^{n}$ choice triangle. This parametrization is recursive in nature, because of the way the choice triangles are constructed.

The completion corresponding to the choice $\Gamma_{ij} = 0$ whenever $1 \leq i \leq j \leq n$ with $(i,j) \notin S$ is called the *central completion* of $\{A_{ij}, (i,j) \in S\}$. It shall be denoted by F_c, in which the subscript "c" stands for central.

An alternative way to obtain the central completion is described below. For a given $n-by-n$ positive generalized band $\{A_{ij}, (i,j) \in S\}$ one can proceed as follows: choose a position $(i_0, j_0) \notin S$, $i_0 \leq j_0$, such that $S \cup \{(i_0, j_0), (j_0, i_0)\}$ is also generalized banded. Choose A_{i_0,j_0} such that $(A_{ij})_{i,j=i_0}^{j_0}$ is the central completion of $\{A_{ij}, (i,j) \in S$ and $i_0 \leq i, j \leq j_0\}$. This is a $3-by-3$ problem and A_{i_0,j_0} can be found via a formula as in (1.4). Proceed in the same way with the partial matrix thereby obtained until all positions are filled. It turns out (see [1]) that the resulting positive semidefinite completion is the central completion F_c. Note that for $(i_0, j_0) \notin S$, $i_0 \leq j_0$, the entry A_{i_0,j_0} only depends upon $\{A_{ij}, (i,j) \in S$ and $i_0 \leq i, j \leq j_0\}$. This implies that the submatrix of F_c located in the rows and columns $\{k, k+1, ..., l\}$ is precisely the central completion of $\{A_{ij}, (i,j) \in S \cap \{k, k+1, ..., l\} \times \{k, k+1, ..., l\}\}$. This principle is referred to as the "inheritance principle" (or permanence principle in the positive definite case ([26]).

Our first result gives four equivalent conditions which characterize the central completion. This is a positive semidefinite operator analogue of Theorem 6.2 in [24].

THEOREM 3.9. *Let S be generalized banded pattern and F a positive semidefinite completion of $\{A_{ij}, (i,j) \in S\}$. Let $F = V^*V = W^*W$ be the lower-upper and upper-lower Cholesky factorizations of F. Then the following are equivalent:*

(i) *F is the central completion of $\{A_{ij}, (i,j) \in S\}$.*

(ii) *$\Delta_U(F) \geq \Delta_U(\tilde{F})$ for all positive semidefinite completions $\tilde{F}$ of $\{A_{ij}, (i,j) \in S\}$;*

(iii) *$\Delta_L(F) \geq \Delta_L(\tilde{F})$ for all positive semidefinite completions $\tilde{F}$ of $\{A_{ij}, (i,j) \in S\}$;*

(iv) *The unitary $U : \overline{\mathcal{R}(W)} \to \overline{\mathcal{R}(V)}$ with $UW = V$ verifies $U_{ij} = 0$ for $i \geq j, (i,j) \notin S$.*

Note that the uniqueness of the central completion implies that $\Delta_U(F) = \Delta_U(\tilde{F})$ (or $\Delta_L(F) = \Delta_L(\tilde{F})$) yields $F = \tilde{F}$. The maximality of $\Delta_U(F)(\Delta_L(F))$ can be viewed as a *maximum entropy principle* (see e.g., [15]).

Proof. The equivalence of (i) and (ii) can be read off immediately from (1.8), and similarly the equivalence of (i) and (iii) can be read off immediately from (1.9).

We prove the equivalence of (i) and (iv) by induction on the number of missing entries in the pattern S. For the $3 - by - 3$ problem (3.4), discussed at the beginning of this section, formula (3.13) proves immediately the equivalence.

Let $S \subseteq \underline{n} \times \underline{n}$ be a generalized banded pattern and $\{A_{ij}, (i,j) \in S\}$ positive semidefinite. Let F_c denote the central completion of $\{A_{ij}, (i,j) \in S\}$, and let V_c and W_c be upper and lower triangular operator matrices such that

$$F_c = V_c^* V_c = W_c^* W_c. \tag{3.15}$$

Consider the unitary operator matrix $U : \overline{\mathcal{R}(W_c)} \to \overline{\mathcal{R}(V_c)}$ so that $UW_c = V_c$. Let $\hat{S}$ denote the pattern $\hat{S} = S \cap (\underline{n-1} \times \underline{n-1})$, and $\hat{F} = \left(\hat{F}_{ij}\right)_{i,j=1}^{n-1}$ obtained from F_c by compressing its last two rows and columns. So, $\hat{F}_{ij} = (F_c)_{ij}$ for $i, j \le n-1$ and

$$\hat{F}_{i,n-1} = \hat{F}^*_{n-1,i} = ((F_c)_{i,n-1}\ (F_c)_{in})\,, i < n-1,$$

and

$$\hat{F}_{n-1,n-1} = \begin{pmatrix} (F_c)_{n-1,n-1} & (F_c)_{n-1,n} \\ (F_c)_{n,n-1} & (F_c)_{nn} \end{pmatrix}.$$

Consider the data $\{\hat{F}_{ij}, (i,j) \in \hat{S}\}$. From the way the central completion is defined one sees that $\hat{F}(= F_c)$ is the central completion of $\{\hat{F}_{ij}, (i,j) \in \hat{S}\}$. Now, in the same way, consider the operator matrices $\hat{U} = (\hat{U}_{ij})_{i,j=1}^{n}$, $\hat{V} = (\hat{V}_{ij})_{i,j=1}^{n}$ and $\hat{W} = (\hat{W}_{ij})_{i,j=1}^{n}$ obtained by the compression of the last two rows and columns of U, V_c and W_c, respectively. We obtain by the induction hypothesis that $\hat{U}_{ij} = 0$ for $(i,j) \notin \hat{S}$ with $i > j$. Thus it remains to show that $U_{nj} = 0$ for j with $(n,j) \notin S$ and $(n-1,j) \in S$. For this purpose let $\gamma = \min\{j, (n,j) \in S\}$ and consider the decomposition

$$U = \hat{U} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{pmatrix}$$

with $\Sigma_{11} = (U_{ij})_{i,j=1}^{\gamma-1}$, $\Sigma_{22} = (U_{ij})_{i,j=\gamma}^{n-1}$ and $\Sigma_{33} = U_{nn}$. Consider also the corresponding decomposition of $F_c = (\phi_{ij})_{i,j=1}^{3}$. Again we have that F_c is also the central completion of

$$\begin{pmatrix} \phi_{11} & \phi_{12} & ? \\ \phi_{21} & \phi_{22} & \phi_{23} \\ ?. & \phi_{32} & \phi_{33} \end{pmatrix}$$

Then, from the $3 - by - 3$ case we obtain that $\Sigma_{13} = 0$ and, consequently, $U_{nj} = 0$ for $j \leq \gamma - 1$, proving (iii).

Implication (iii) $\rightarrow$ (i) can be proved by the same type of induction process. One needs to use the observation that if S_1 and S_2 are two generalized banded patterns and F is the central completion of both $\{A_{ij}, (i,j) \in S_1\}$ and $\{A_{ij}, (i,j) \in S_2\}$, then F is the central completion of $\{A_{ij}, (i,j) \in S_1 \cap S_2\}$. We omit the details. □

THEOREM 3.10. *Let $S \subseteq \underline{n} \times \underline{n}$ be a generalized banded pattern and $\{A_{ij}, (i,j) \in S\}$ be positive semidefinite. Let F_c denote the central completion of $\{A_{ij}, (i,j) \in S\}$, and V_c and W_c be upper and lower triangular operator matrices such that*

$$F_c = V_c^* V_c = W_c^* W_c. \tag{3.16}$$

Further, let $U : \overline{\mathcal{R}(W_c)} \rightarrow \overline{\mathcal{R}(V_c)}$ be the unitary operator matrix so that

$$UW_c = V_c. \tag{3.17}$$

Then each positive semidefinite completion of $\{A_{ij}, (i,j) \in S\}$ is of the form

$$\mathcal{T}(G) = V_c^*(I + UG)^{*-1}(I - G^*G)(I + UG)^{-1}V_c \tag{3.18}$$

$$= W_c^*(I + GU)^{-1}(I - GG^*)(I + GU)^{*-1}W_c,$$

in which $G = (G_{ij})_{i,j=1}^{n} : \overline{\mathcal{R}(V_c)} \rightarrow \overline{\mathcal{R}(W_c)}$ is a contraction with $G_{ij} = 0$ whenever $i \geq j$ or $(i,j) \in S$. Moreover, the correspondence between the set of all positive semidefinite completions and all such contractions G is one-to-one.

The decompositions of $\overline{\mathcal{R}(V)}$ and $\overline{\mathcal{R}(W)}$ are given by

$$\overline{\mathcal{R}(V)} = \overline{\mathcal{R}(A_{11})} \oplus (\oplus_{k=2}^{n} \mathcal{D}_{\Gamma_{1k}}) \tag{3.19}$$

and

$$\overline{\mathcal{R}(W)} = \oplus_{k=1}^{n-1} D_{\Gamma_{kn}^*} \oplus \overline{\mathcal{R}(A_{nn})}. \tag{3.20}$$

Proof. Write $F_c = C + C^*$, in which C is upper triangular with $C_{ii} = 1/2F_{ii}$, $i = 1, ..., n$, and define for a contraction $G = (G_{ij})_{i,j=1}^{n} : \overline{\mathcal{R}(W_c)} \to \overline{\mathcal{R}(V_c)}$ with $G_{ij} = 0$ whenever $i > j$ or $(i,j) \in S$,

$$\mathcal{L}(G) = C - W_c^*(I + GU)^{-1}GV_c. \tag{3.21}$$

Since $U_{ij} = 0$ for $(i,j) \notin S$ with $i > j$, one easily sees that GU is strictly upper triangular and so $(I + GU)^{-1}$ exists and is upper triangular. Since W_c^* and V_c are both also upper triangular one readily obtains that

$$(\mathcal{L}(G))_{ij} = C_{ij}, (i,j) \in S. \tag{3.22}$$

Further, using (3.21) and the unitarity of U it is straightforward to check that $\mathcal{L}(G) + \mathcal{L}(G)^* = \mathcal{T}(G)$. This together with (3.22) yields that $\mathcal{T}(G)$ is a completion of $\{A_{ij}, (i,j) \in S\}$ and since $\|G\| \leq 1$ the operator matrix $\mathcal{T}(G)$ is positive semidefinite.

Assume that for two contractions G_1 and G_2 (of the required form) we have that $\mathcal{T}(G_1) = \mathcal{T}(G_2)$. Then also $\mathcal{L}(G_1) = \mathcal{L}(G_2)$ and since W_c^* and V_c^* are injective on $\overline{\mathcal{R}(W_c)}$ and $\overline{\mathcal{R}(V_c)}$, respectively, equation (3.21) implies that $(I + G_1U)^{-1}G_1 = (I + G_2U)^{-1}G_2$. Thus $G_1(I + UG_2) = (I + G_1U)G_2$ which yields $G_1 = G_2$.

Conversely, let F be an arbitrary positive semidefinite completion of $\{A_{ij}, (i,j) \in S\}$. Consider $\Omega = (\Omega_{ij})_{i,j=1}^{n}$ such that $\Omega_{ij} = 0$ whenever $i \leq j$ or $(i,j) \in S$, and $F_c - F = \Omega + \Omega^*$. Then there exists an operator $Q = (Q_{ij})_{ij}^{n} : \overline{\mathcal{R}(W_c)} \to \overline{\mathcal{R}(V_c)}$ with $Q_{ij} = 0$ whenever $i > j$ or $(i,j) \notin S$ and $\Omega = W_c^* Q V_c$. The proof of the existence of Q

is based on range inclusions. See [7] for details. Since UQ is strictly upper triangular, we can define

$$G = Q(I - UQ)^{-1},$$

which will give that $\Omega = W_c^*(I + GU)^{-1}GV_c$. Since $F = F_c - \Omega - \Omega^*$, and taking into account (3.21) we obtain that $F = \mathcal{T}(G)$. Since $F = \mathcal{T}(G)$ is positive semidefinite, the relation (3.18) implies that G is a contraction. This finishes our proof. □

3.3 Inertia Formulas for Hermitian Matrices

This section deals with Hermitian operator matrices. We obtain the operator version of two known results. The first one is related to inertia posibilities of Hermitian matrix completion and the second one represents an inertia formula for Hermitian matrices with sparse inverses.

LEMMA 3.11. *Let $M \in \mathcal{B}(\mathcal{H})$, $N \in \mathcal{B}(\mathcal{K})$ be invertible Hermitian operators with the property that $i_-(M) \leq i_-(N) < \infty$. Then there exists $Y \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $M > YNY^*$.*

Proof. Using the notation of Section 1.1, define $Y|\mathcal{H}_N^+$ to be 0. It remains to define Y on $\mathcal{H}_N^-$. Since $\mathcal{H}_N^-$ and $\mathcal{K}_M^-$ are finite dimensional, there exists ϵ, $\delta > 0$ such that $N^- \leq -\epsilon I$ and $M^- \geq -\delta I$. Let U be an isometry on $\mathcal{H}_N^-$ such that $\mathcal{K}_M^- \subseteq \mathcal{R}(U)$ and let $s > \sqrt{\delta/\epsilon}$. Then

$$(sU)N^-(sU)^* < M^-$$

and we can define $Y|\mathcal{H}_N^- = sU$. The operator Y defined in this way provides a solution to our problem. □

LEMMA 3.12. *Let $R = \begin{pmatrix} A_{11} & A_{12} & X_{13} \\ A_{12}^* & A_{22} & A_{23} \\ X_{13}^* & A_{23}^* & A_{33} \end{pmatrix}$ be a partial matrix and assume that $R(\{1,2\})$, $R(\{2,3\})$ and R_{22} are invertible and $i_-(R(\{1,2\})), i_-(R(\{2,3\})) < \infty$. Then* *there exists an invertible completion F of R such that*

$$(3.23)\qquad i_-(F) = max\{i_-(R(\{1,2\})), i_-(R(\{2,3\}))\}$$

Proof. Without loss of generality, assume that $i_-(R(\{1,2\})) \geq i_-(R(\{2,3\}))$. Let F be a completion of R corresponding to a certain choice X_{13}. Then F is invertible and satisfies (3.23) if and only if the Schur complement of $R(\{1,2\})$ in F is positive definite. A straightforward computation shows that this Schur complement equals

$$(3.24)\; A_{33} - A_{23}^*A_{22}^{-1}A_{23} - (X_{13} - A_{12}A_{22}^{-1}A_{23})^*(A_{11} - A_{12}A_{22}^{-1}A_{12}^*)(X_{13} - A_{12}A_{22}^{-1}A_{23})$$

Since $A_{33} - A_{23}^*A_{22}^{-1}A_{23}$ is the Schur complement of A_{22} in $R(\{2,3\})$, respective $A_{11} - A_{12}^*A_{22}^{-1}A_{12}$ is the Schur complement of A_{22} in $R(\{1,2\})$, the relation $i_-(R(\{1,2\})) \geq i_-(R(\{2,3\}))$ implies that

$$i_-(A_{11} - A_{12}^*A_{22}^{-1}A_{12}) \geq i_-(A_{33} - A_{23}^*A_{22}^{-1}A_{23})$$

The existence of an X_{13} which makes (3.24) positive definite is guaranteed by Lemma 3.11. Such an X_{13} produces a completion F with (3.23). □

PROPOSITION 3.13. *Consider the partial operator matrix* $R = \begin{pmatrix} A_{11} & A_{12} & X_{13} \\ A_{12}^* & A_{22} & A_{23} \\ X_{13}^* & A_{23}^* & A_{33} \end{pmatrix}$, *such that* $i_0(R(\{1,2\})) + i_-(R(\{1,2\})) < \infty$, $i_0(R(\{2,3\})) + i_-(R(\{2,3\})) < \infty$ *and 0 is an isolated point in the spectrum of* $R(\{1,2\})$ *and* $R(\{2,3\})$. *Then there exists a self-adjoint completion F of R, such that*

$$(3.25)\qquad i_-(F) + i_0(F) \leq$$

$$max\{i_-(R(\{1,2\})) + i_0(R(\{1,2\})), i_-(R(\{2,3\})) + i_0(R(\{2,3\}))\}$$

and if $0 \in \sigma(F)$ *then* 0 *is isolated in* $\sigma(F)$.

Proof. The relation $i_-(R(\{1,2\})) + i_0(R(\{1,2\})) < \infty$ implies that $i_-(A_{22}) + i_0(A_{22}) < \infty$ and since 0 is an isolated point in the spectrum of A_{22}, by Proposition

1.4, 0 will be an isolated point in the spectrum of A_{22} also. Thus, for sufficiently small $\lambda > 0$ $R(\{1,2\}) - \lambda I$ and $R(\{2,3\}) - \lambda I$ are invertible. Then

$$i_-(R(\{1,2\}) - \lambda I) = i_-(R(\{1,2\})) + i_0(R(\{1,2\}))$$

$$i_-(R(\{2,3\}) - \lambda I) = i_-(R(\{2,3\})) + i_0(R(\{2,3\}))$$

Thus, the partial matrix $R - \lambda I$ satisfies the conditions of Lemma 3.12 and this implies the existence of an invertible completion F_λ of $R - \lambda I$ such that

$$i_-(F_\lambda) = max\{i_-(R(\{1,2\}) - \lambda I), i_-(R(\{2,3\}) - \lambda I)\}.$$

Then $F = F_\lambda + \lambda I$ is a completion of R and

$$i_0(F) + i_-(F) \leq i_-(F_\lambda + \lambda I).$$

The interlacing inequalities imply now (3.25). Since $i_-(F_\lambda) < \infty$, 0 is isolated in $\sigma(F)$. □

The next result generalizes a result in [49].

THEOREM 3.14. *Let G be a chordal graph and $R \in \Omega_G$ be an Hermitian partial matrix. If $K_1, ..., K_r$ are the maximal cliques of G,*

$$max\{i_-(R(K_j)) + i_0(R(K_j)), j = 1, ..., r\} < \infty$$

and if $0 \in \sigma(R(K_j))$ then 0 is an isolated point in $\sigma(R(K_j))$, for $j = 1, ..., r$, there exists an Hermitian completion F of R such that

$$i_-(F) + i_0(F) \leq max\{i_-(R(K_j)) + i_0(R(K_j)), j = 1, ..., r\}$$

Proof. The proof is similar to the proof of Theorem 3.1. Consider a chordal sequence $G = G_0, ..., G_t = K_n$ of G and apply then repeatedly Proposition 3.13. □

In the rest of this section we prove inertia formulae for Hermitian operator matrices.

THEOREM 3.15. *Let $A = (A_{ij})_{i,j=1}^3$ be an invertible Hermitian operator matrix. Assume that $A^{-1} = B = (B_{ij})_{i,j=1}^3$ and $B_{13} = 0$. Then, if $i_-(A(\{1,2\})), i_-(A(\{2,3\})) < \infty$, we have:*

i). $i_-(B_{11})$ and $i_-(B_{33})$ are finite.

ii). $i_0(A(\{1,2\}))$ and $i_0(A(\{2,3\}))$ are finite and 0 is isolated in $\sigma(A(\{1,2\}))$ and $\sigma(A(\{2,3\}))$.

iii). $i_-(A)$ is finite and

$$i_-(A) = i_-(A(\{1,2\})) + i_-(A(\{2,3\})) - i_-(A_{22}) \tag{3.26}$$

Proof. As a consequence of $A^{-1} = B$ and $B_{13} = 0$ we have that

$$\begin{pmatrix} A_{22} & A_{23} \\ A_{23}^* & A_{33} \end{pmatrix} \begin{pmatrix} B_{23} \\ B_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{3.27}$$

Thus,

$$\begin{pmatrix} B_{23}^* & B_{33} \end{pmatrix} \begin{pmatrix} A_{22} & A_{23} \\ A_{23}^* & A_{33} \end{pmatrix} \begin{pmatrix} B_{23} \\ B_{33} \end{pmatrix} = B_{33}$$

Now $i_-(B_{33}) \leq i_-(A(\{2,3\}))$ is a consequence of Lemma 1.3. In a similar way one can prove $i_-(B_{11}) \leq i_-(A(\{1,2\}))$ and i) follows.

Assume that $i_0(A(\{1,2\})) = i_0(B_{33}) = \infty$. For $h \in kerB_{33}$, (3.27) implies that:

$$\begin{pmatrix} A_{22} & A_{23} \\ A_{23}^* & A_{33} \end{pmatrix} \begin{pmatrix} B_{23}h \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ h \end{pmatrix}. \tag{3.28}$$

Thus the space $\mathcal{H}_0 = \overline{B_{23}(kerB_{33})}$ is infinite dimensional. Since $A_{22}B_{23}h = 0$, it follows that $rank[P_{kerA_{22}}\mathcal{H}_0] = \infty$, and then Proposition 3.1 of [17] implies that $i_-(A(\{2,3\})) = \infty$, a contradiction. Thus $i_0(B_{33}) = i_0(A(\{1,2\})) < \infty$. By a similar proof $i_0(B_{11}) = i_0(A(\{2,3\})) < \infty$.

We obtain that 0 is isolated in $\sigma(A(\{1,2\}))$ by a similar method used in the last part of the proof of Proposition 1.4. Since $i_0(B_{33}) + i_-(B_{33}) < \infty$, for suffiently small $\lambda > 0$, $B_{33} + \lambda I$ is invertible and $i_-(B_{33} + \lambda I) = i_-(B_{33})$. Denote

$$B_\lambda = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{12}^* & B_{22} & B_{23} \\ 0 & B_{23}^* & B_{33} + \lambda I \end{pmatrix}$$

and $A_\lambda = B_\lambda^{-1}$. By the Schur complement argument $A_\lambda(\{1,2\})$ is invertible. The equality $i_-(B_\lambda) = i_-(B) = i_-(A)$ holds for suffiently small $\lambda > 0$.

We have that $B_\lambda^- \searrow B^-$ for $\lambda \searrow 0$, and since $A_\lambda^- = (B_\lambda^-)^{-1}$, $A_\lambda^- \nearrow A^-$ (weakly) for $\lambda \searrow 0$. Thus $(A_\lambda(\{1,2\}))^- \nearrow (A(\{1,2\}))^-$ for $\lambda \searrow 0$. Then, $A_\lambda(\{1,2\})$ is invertible and $i_-(A(\{1,2\})) + i_0(A(\{1,2\})) < \infty$ implies that $i_-(A_\lambda(\{1,2\})) = i_0(A(\{1,2\})) + i_-(A(\{1,2\}))$. Applying Proposition 1.4 for A_λ, we get:

$$i_-(A) = i_-(A_\lambda(\{1,2\})) + i_-(B_{33} + \lambda I),$$

and finally

$$i_-(A) = i_0(A(\{1,2\})) + i_-(A(\{1,2\})) + i_-(B_{33}) < \infty. \tag{3.29}$$

which implies that A satisfies the conditions of Proposition 1.4. Consequently (1.4) implies

$$i_-(A) = i_-(A(\{2,3\})) + i_0(A(\{2,3\})) + i_-(B_{11}) \tag{3.30}$$

$$i_-(A) = i_-(A_{22}) + i_0(A_{22}) + i_-(B(\{1,3\})) \tag{3.31}$$

Since $B_{13} = 0$, we have that $i_0(B(\{1,3\})) = i_0(B_{11}) + i_0(B_{33})$ and $i_-(B(\{1,3\})) = i_-(B_{11}) + i_-(B_{33})$. Adding (3.29) to (3.30) and subtracting (3.31), we obtain that

$$i_-(A) = i_-(A(\{1,2\})) + i_-(A(\{2,3\})) - i_-(A_{22}), \tag{3.32}$$

which completes the proof. □

THEOREM 3.16. *Let $G = (V, E)$ be a chordal graph, $\mathcal{K}$ the set of all maximal cliques, respectively $\mathcal{S}$ the set of all minimal vertex separators of G. Let $A = (A_{ij})_{i,j=1}^n$ be an invertible Hermitian operator matrix such that $A^{-1} = B = (B_{ij})_{i,j=1}^n \in \Omega_G$. Assume that $i_-(A(K)) < \infty$ for any $K \in \mathcal{K}$. Then, we have:*

i). $i_0(A(K)) < \infty$ for any $K \in \mathcal{K}$.

ii). If $A(K)$ is not invertible, then 0 is isolated in $\sigma(A(K))$.

iii). $i_-(A) < \infty$ and $i_-(A) = \sum_{K\in\mathcal{K}} i_-(A(K)) - \sum_{S\in\mathcal{S}} i_-(A(S))$.

Proof. To make the proof easier, we first prove i) in a particular case.

Consider A to be a $7-by-7$ Hermitian operator matrix with inverse B. Assume that the chordal graph with the clique tree given in Fig. XII

Figure XII

is a graph for B and

$$i_-(A(K)) < \infty, \ for\ any\ maximal\ clique\ K\ of\ G. \tag{3.33}$$

Let assume that $i_0(A(K_0)) = \infty$ for a certain maximal clique K_0, for instance $\{4,5,6\}$. We shall prove that this assumption contradicts (3.33). Since by (1.13), $i_0(A(\{4,5,6\})) = i_0(B(\{1,2,3,7\})) = i_0(B(\{1,2,3\})) + i_0(B_{77})$, we must have either $i_0(B_{77}) = \infty$ or $i_0(B(\{1,2,3\})) = \infty$.

A. Assume that $i_0(B_{77}) = \infty$. Taking into account that G is a graph of B, the following relation holds:

$$A(\{5,6,7\})\begin{pmatrix} B_{57} \\ B_{67} \\ B_{77}\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I\end{pmatrix} \tag{3.34}$$

Then, as in the proof of Theorem 3.15, (relation (3.28)), (3.34) implies that $i_-(A(5,6,7)) = \infty$, a contradiction.

B. Assume that $i_0(B(\{1,2,3\})) = \infty$. The structure of B implies that:

$$A(\{4,5,1,2,3\})\begin{pmatrix} 0 & 0 & B_{43} \\ B_{51} & B_{52} & B_{53} \\ B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33}\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I\end{pmatrix} \tag{3.35}$$

Take $(h_1, h_2, h_3) \in kerB(\{1,2,3\})$. Then (3.35) implies that:

$$A(\{4,5,3\}) \begin{pmatrix} B_{43}h_3 \\ B_{51}h_1 + B_{52}h_2 + B_{53}h_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h_3 \end{pmatrix} \tag{3.36}$$

In the case in which $dim[P_{\mathcal{H}_3} kerB(\{1,2,3\})] = \infty$, as in the proof of Theorem 3.15, we deduce that $i_-(A(\{3,4,5\})) = \infty$, a contadiction. So, we only have to consider the case $dimP[\mathcal{H}_3 kerB(\{1,2,3\})] < \infty$. Under this assumption, since $i_0(B(\{1,2,3\})) = \infty$, there exists an infinite set of linearly independent vectors of the form $(h_1, h_2, 0)$ in $kerB(\{1,2,3\})$. For any such vector, (3.35) implies that:

$$A(\{5,2,3\}) \begin{pmatrix} 0 \\ B_{51}h_1 + B_{52}h_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ h_2 \\ 0 \end{pmatrix}$$

Then $dim[P_{\mathcal{H}_2} kerB(\{1,2,3\})] = \infty$ implies that $i_-(A(\{2,3,5\})) = \infty$, a contradiction. Thus, we also have to assume that $dim[P_{\mathcal{H}_2} kerB(\{1,2,3\})]$, in which case there exists an infinite family of linearly independent vectors $(h_1, 0, 0) \in kerB(\{1,2,3\})$. For any such vector, by (3.35) we obtain:

$$A(\{5,1,2\}) \begin{pmatrix} 0 \\ B_{51}h_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ h_2 \\ 0 \end{pmatrix},$$

which implies that $i_-(A(\{1,2,5\})) = \infty$, a contradiction. Finally, we conclude that our initial assumption $i_0(A(\{4,5,6\})) = \infty$ is false.

Let A be an $n - by - n$ Hermitian operator matrix. Assume that A verifies the conditions of the theorem and let $T = (\mathcal{V}(T), \mathcal{E}(T))$ be a tree of G. Assume that there is a maximal clique K_0 of G such that $i_0(A(K_0)) = \infty$. We shall prove that under this latter assumption there exists a maximal clique K' of T such that $i_-(A(K')) = \infty$, a contradiction.

If the node set corresponding to K_0 is extremal in T, then there exists a simplicial vertex $v_0 \in K_0$ of G. The structure of B and the simpliciality of v_0 imply that:

$$\begin{pmatrix} A(K_0 - \{v_0\}) & A(K_0 - \{v_0\}|v_0) \\ A(K_0 - \{v_0\}|v_0)^* & A_{v_0,v_0} \end{pmatrix} \begin{pmatrix} B(K_0 - \{v_0\}|v_0) \\ B_{v_0,v_0} \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix},$$

which in a similar way as (3.28) implies that $i_-(B_{v_0,v_0}) < \infty$ and $i_-(A(K_0)) < \infty$, a contradiction. Thus K_0 cannot be an extremal node set of T.

Denote by $T_i = (\mathcal{V}(T_i), \mathcal{E}(T_i))$, $i = 1, ..., r$ the distinct subtrees of T obtained by deleting K_0 and the edges joining K_0 in $\mathcal{E}(T)$. Since K_0 is not extremal, we have $r > 1$. Let $W_i = \cup_{V \in \mathcal{V}(T_i)} V$. Then $i_0(A(K_0)) = \sum_{i=1}^{r} i_0(B(W_i - K_0)) = \infty$, thus there exists an $i = 1, ..., r$ such that $i_0(B(W_i - K_0)) = \infty$. Without loss of generality we assume that $i = 1$ and $\mathcal{V}(T_1) = \{V_1, .., V_s\}$, this latter set being ordered as follows. V_1 is the unique node set of T_1 which neighbours K_0 in T. The rest of them are numbered in such a way that for any V_j, $j \geq 2$ the unique path in T_1 joining V_j to V_1 contains only node sets of index less than j. Taking into account the structure of B, we obtain that:

$$A(W_1)\begin{pmatrix} B(W_1 \cap K_0 | W_1 - K_0) \\ B(W_1 - K_0) \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{3.37}$$

Let $h = (h_j)_{j \in W_1 - K_0} \in kerB(W_1 - K_0)$. Then, using $W \cap K_0 = V_1 \cap K_0$, (3.37) implies that:

$$A(V_1)\begin{pmatrix} B(V_1 \cap K_0 | W_1 - K_0)h \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ P_{V_1 - K_0} h \end{pmatrix}.$$

If $dim[P_{V_1-K_0} kerB(W_1 - K_0)] = \infty$, by the same argument used in the proof of Theorem 3.15 (Proposition 3.1 of [17]), we obtain that $i_-(A(V_1)) = \infty$, a contradiction. So we have to assume that $dim[P_{V_1-K_0} kerB(V_1 - K_0)] < \infty$. Since $dim[kerB(W_1 - K_0)] = \infty$, we have that:

$$dim[kerB(W_1 - K_0) \cap (W_1 - (V_1 \cup K_0))] = \infty.$$

For any vector h in this latter space, (3.37) implies

$$A(V_2)\begin{pmatrix} B(W_2 \cap K_0 | W_1 - (V_1 \cup K_0))h \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ P_{V_2 - K_0} h \end{pmatrix}$$

thus $i_-(A(V_2)) = \infty$, unless $dim[P_{V_2-K_0} kerB(W_1 - K_0)] < \infty$. Assuming the latter, we continue our test on the node sets V_k, $k \geq 3$. Since $dim[kerB(W_1 - K_0)] = \infty$, we find $1 \leq l \leq s$ such that $i_-(A(V_l)) = \infty$, a contradiction.

In conclusion, our assumption $i_0(A(K_0)) = \infty$ is false and i) holds.

We prove ii) and iii) by induction on $n = |V|$. For $n = 3$, ii) and iii) are consequences of Theorem 3.15. Assume that the results hold for any chordal graph $G = (V, E)$ with $|V| \leq n-1$. Consider now the problem for an arbitrary chordal graph $G = (V, E)$ with $|V| = n$.

Select an arbitrary simplical vertex v_0 of G. Without loss of generality, let $v_0 = n$. Then, as in Theorem 3.15, we obtain that $i_0(B_{nn}) < \infty$ and $i_-(B_{nn}) < \infty$. For $\lambda > 0$ sufficiently small, $B_{nn} + \lambda I$ is invertible and $i_-(B_{nn} + \lambda I) = i_-(B_{nn})$. Let B_λ be the matrix obtained by replacing B_{nn} with $B_{nn} + \lambda I$ in B and let $A_\lambda = B_\lambda^{-1}$. Let K be an arbitrary clique of G such that $n \notin K$. Then, as in the proof of Theorem 3.15, we have that $(A_\lambda(K))^- \nearrow (A(K))^-$, thus, $i_-(A_\lambda(K)) < i_-(A(K)) + i_0(A(K)) < \infty$. When K' is the unique maximal clique which contains n, the latter inequality can be proved for K' in a similar way by selecting an other simplicial vertex of G. We conlude that

$$i_-(A_\lambda(K)) < \infty, \ \textit{for any maximal clique } K \textit{ of } G. \tag{3.38}$$

Since n is a simplicial vertex of G and $B_{nn} + \lambda I$ is invertible, $G_{\{1,\ldots,n-1\}}$ is a graph for $(A_\lambda(\{1, \ldots, n-1\}))^{-1}$. Assuming that iii) holds for $n-1$, (3.38) implies that $i_-(A_\lambda(\{1, \ldots, n\})) < \infty$. Then, by Proposition 1.13 we have that

$$i_-(A) = i_-(A_\lambda) = i_-(A_\lambda(\{1, \ldots, n-1\})) + i_-(B_{nn} + \lambda I)$$

$$= i_-(A_\lambda(\{1, \ldots, n-1\})) + i_-(B_{nn}) + i_0(B_{nn}) < \infty,$$

so the first part of iii) follows. Then, ii) is a consequence of Proposition 1.4.

The formula in iii) has been proved for the class of Hermitian matrices having the property that all of their principal submatrices have closed range. If the conditions of Theorem 3.16 are satisfied, we already know that $i_-(A) < \infty$ and then Proposition 1.4 implies that the range of any principal submatrix of A is closed. Finally, we conclude that the formula in iii) holds under the weaker conditions of Theorem 3.16. This completes our proof. □

COROLLARY 3.17. *Let $G = (V, E)$ be a chordal graph and $R \in \Omega_G$ a partial positive semidefinite matrix. If R has an Hermitian invertible completion F with $F^{-1} \in \Omega_G$ then F is positive definite. (And R is partial positive).*

Proof. For any clique K of G we have that $i_-(F(K)) = 0$, thus iii) of Theorem 3.16 implies $i_-(F) = 0$. Since F is invertible it follows that $F > 0$. □

CHAPTER IV
CONTRACTIVE AND LINEARLY CONSTRAINED CONTRACTIVE COMPLETIONS

The first contractive completion problem to be solved was of the following partial matrix:

$$(4.1) \qquad \begin{pmatrix} A & B \\ C & X \end{pmatrix}$$

in which A, B and C are given linear operators acting between corresponding Hilbert spaces such that $[A, B]$ and $\begin{pmatrix} A \\ C \end{pmatrix}$ are contractions. The case $A = A^*$, $B = C^*$ and $X = X^*$ was considered in [51] in connection with Hermitian extensions of unbounded operators. As proved in [55], the conditions $||[A, B]|| \leq 1$ and $||\begin{pmatrix} A \\ C \end{pmatrix}|| \leq 1$ are sufficient for the existence of a contractive solution to problem (4.1). Independently, in [2], [19] and [62] the set of all solutions has been described. The completion problem (4.1) for analytic operator valued functions was solved in [4].

Consider next the contractive completion problem of the following partial matrix:

$$(4.2) \qquad \begin{pmatrix} B_{11} & ? & ? & \cdots & ? \\ B_{21} & B_{22} & ? & \cdots & ? \\ \vdots & \vdots & \vdots & & \vdots \\ B_{n1} & B_{n2} & B_{n3} & \cdots & B_{nn} \end{pmatrix}$$

in which $B_{ij} : \mathcal{K}_j \to \mathcal{H}_i$, $1 \leq j \leq i \leq n$, are given linear operators acting between Hilbert spaces with the property that

$$(4.3) \qquad ||(B_{ij})_{i=p,j=1}^{n\ ,p}|| \leq 1, p = 1, \ldots, n.$$

We call a partial matrix (4.2) with the property (4.3) a *contractive triangle.* This problem came up in connection with the Arveson distance formula in nest algebras ([3] and [58]) respective engineering control problems ([8] and [9]).

A contractive completion problem concerning a partial matrix A can be transformed into a positive semidefinite completion problem of the partial matrix $\begin{pmatrix} I & A \\ A^* & I \end{pmatrix}$. Using this observation, we derive the results on contractive completions from our results in Chapter III on positive semidefinite completion.

DEFINITION A *partial contraction* is a partial operator matrix with the property that all of its fully specified submatrices are contractive.

It is obvious that any partial matrix which admits a contractive completion is a partial contraction.

DEFINITION A pattern S is called *(contractively) completable* if any partial contraction with pattern S can be completed to a contraction.

The first result in this chapter is the description of the structure of all (contractively) completable patterns S. This result was first proved in [47]. We present here a slightly modified proof of their result.

The results in Section 4.2 and 4.3 are based on Section 3.2 and are taken from [7]. A so-called "central completion" of the problem (4.2) is described. Thus, based on Theorem 3.10, a cascade transform description for the set of all contractive, isometric, co-isometric and unitary solutions of the problem (4.2) is constructed. Consequently we recover in a different way the results of [9] and [33] stating in the scalar case, respectively strict contractive case the existence of such a description. A parametrization for the set of all solutions of (4.2) in terms of the so-called "choice triangles" was obtained in [1].

In [29], the following linearly constrained contractive completion problem, named the *Strong-Parrott problem* was solved:

$$(4.4) \qquad \begin{pmatrix} B_{11} & B_{12} \\ ? & B_{22} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

in which the specified entries are linear operators acting between corresponding Hilbert spaces such that $S_1^*S_1 + S_2^*S_2 = T_1^*T_1 + T_2^*T_2^*$. The Strong-Parrott problem was a consequence of questions arising in connection with the Commutant Lifting Theorem (see, e.g. the recent book [29]).

Consider the following linearly constrained contractive completion problem:

$$(4.5)\qquad \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ ? & B_{22} & \dots & B_{2n} \\ \vdots & \vdots & & \vdots \\ ? & ? & \dots & B_{nn} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{pmatrix}$$

in which $B_{ij} : \mathcal{K}_j \to \mathcal{H}_i$, $1 \leq i \leq j \leq n$, are linear operators acting between Hilbert spaces and

$$(4.6)\qquad S = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{pmatrix} : \mathcal{H} \to \oplus_{i=1}^n \mathcal{K}_i, \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{pmatrix} : \mathcal{H} \to \oplus_{j=1}^n \mathcal{H}_j.$$

are also given.

Reducing the problem (4.5) to a positive semidefinite completion problem, necessary and sufficient conditions are obtained for the existence of a contractive solution to (4.5) generalizing in this way the results of [29]. In the case these conditions are met, we build a so-called "central completion", a solution with several distinguishing properties. From the central completion a cascade transform parametrization is constructed for the set of all contractive, isometric, co-isometric and unitary solutions.

4.1 The Structure of Contractively Completable Patterns

In Chapter II respective Chapter III we already proved that the properties of invertible respective positive semidefinite completability of an undirected graph coincide with the chordality of the graph. The aim of this section is to determine the structure of all contractively completable patters in the sense of the definition in the introduction of this chapter.

The following is an example of a non-completable pattern:

$$(4.7)\qquad \begin{pmatrix} ? & * & * \\ * & ? & * \end{pmatrix},$$

since the partial contraction:

$$\begin{pmatrix} ? & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & ? & 1/\sqrt{2} \end{pmatrix}$$

does not admit a contractive completion. Indeed, $\begin{pmatrix} ? & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ is a contraction if and only if ? equals 0 and $\begin{pmatrix} ? & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ is a contraction if and only if ? equals $-1/\sqrt{2}$. It is clear that the transposed of the above pattern is also non-completable.

With an $n-by-m$ pattern S a bipartite graph $G = (X, Y, E)$ is associated (see Section 1.3), with $X = \{u_1, ..., u_n\}$, $Y = \{v_1, ..., v_n\}$ and $(u_i, v_j) \in E$ if (i, j) corresponds to a specified entry of S. Let $G_k = (X_k, Y_k, E_k), k = 1, ..., s$ be the connected components of G.

If all the patterns S_k associated with the bipartite graphs G_k, $k = 1, .., s$ are completable then S is also completable. Indeed, consider in this case a partial contraction M_0 with pattern S. Complete all the partial submatrices associated with the connected components of G to contractions. Finally, put all the entries (i, j) in which u_i and v_j are in different connected components of G to be 0. We obtain in this way a contractive completion of M_0.

We next describe the structure of all completable patterns S. The proof is a modified version of the original one in [47]. By the above remark, without loss of generality the bipartite graph $G = (X, Y, E)$ associated with S is assumed to be connected. We also associate with S the graph $\hat{G} = (V, F)$ (not bipartite), with $V = X + Y$ and $F = E \cup (X \times X) \cup (Y \times Y)$.

THEOREM 4.1. *Let S be an $n-by-m$ pattern with G its bipartite graph and $\hat{G}$ the graph obtained from G as above. Then the following statements are equivalent:*

i). S is completable.

ii). S is permutation equivalent to the following "generalized triangular" pattern:

$$\begin{pmatrix} B_{11} & \dots & B_{1j_1} & ? & \dots & \dots & ? \\ B_{21} & \dots & \dots & \dots & B_{2j_2} & \dots & ? \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ B_{n1} & \dots & \dots & \dots & \dots & \dots & B_{nm} \end{pmatrix} \tag{4.8}$$

in which $1 \leq j_1 \leq j_2 \leq ... \leq j_n = m$.

iii). S has no subpattern of the form: $\begin{pmatrix} * & ? \\ ? & * \end{pmatrix}$ *or* $\begin{pmatrix} ? & * \\ * & ? \end{pmatrix}$.

iv). The graph $\hat{G}$ *is chordal.*

Proof. The implication ii) $\Rightarrow$ iii) is immediate. Assume that $\hat{G}$ is not chordal. Taking into account the structure of $\hat{G}$, the only possible chordless cycle of length greater than 3 in $\hat{G}$ might be of the form $[u, v, u', v']$ to which there corresponds a subpattern of the form iii). Thus we have ii) $\Rightarrow$ iv).

Consider the partial positive matrix $\begin{pmatrix} I & M_0 \\ M_0^* & I \end{pmatrix}$. It is easy to see that the undirected graph of this latter partial matrix coincides with $\hat{G}$. Since $\hat{G}$ is chordal, Theorem 3.1 implies the existence of a positive definite completion $\begin{pmatrix} I & M \\ M^* & I \end{pmatrix}$ of $\begin{pmatrix} I & M_0 \\ M_0^* & I \end{pmatrix}$. So, M is a contractive completion of M_0. Thus, iv) $\Rightarrow$ i).

Let assume that the pattern S is completable and $G = (X, Y, E)$ is the bipartite graph of S. Let $u, u' \in X$, $u \neq u'$. Since S is completable, S cannot have any subpattern of the form (4.7) or the transposed of (4.7). This implies that the sets $Adj(u)$ and $Adj(u')$ are either disjoint or one of them is included in the other. Since G is assumed to be connected, we deduce that there exists an ordering τ of the set $X = \{1, ..., n\}$ such that

$$\emptyset \neq Adj(u_{\tau(1)}) \subseteq Adj(u_{\tau(2)}) \subseteq ... \subseteq Adj(u_{\tau(n)}) = \{v_1, ..., v_m\}. \tag{4.9}$$

Permute the rows of S by the ordering of τ. Then (4.9) implies the existence of a column permutation which turns S into a pattern of type (4.8). Thus i) $\Rightarrow$ ii) and this completes the proof. □

Droping the assumption that the graph G is connected, we obtain that in general a pattern is completable if and only if it is permutation equivalent to a direct sum of patterns of the form (4.8).

4.2 Contractive Completions of a Contractive Triangle

Based on the existence of a central completion, we construct in this section a cascade transform parametrization for the set of all contractive completions of a contractive triangle. In a different decomposition, a partial contraction of type (4.8) is a particular case. We mention that we allow the spaces $\mathcal{H}_j$ and $\mathcal{K}_i$ to be the trivial space (0). So, for instance,

$$(4.10) \qquad \begin{pmatrix} ? & ? & ? \\ B_{22} & ? & ? \\ B_{32} & B_{33} & ? \end{pmatrix}$$

is a particular case. The problem (4.10) can be obtained by taking $\mathcal{H}_1 = (0)$ and $\mathcal{K}_4 = (0)$. Therefore all possible triangular patterns are covered, which by Theorem 4.1 are essentially the class of patterns for which the existence of a contractive completion is ensured as soon as the given submatrices are contractions.

Consider first the following $2 - by - 2$ problem:

$$(4.11) \qquad \left\| \begin{pmatrix} B_{11} & ? \\ B_{21} & B_{22} \end{pmatrix} \right\| \leq 1$$

in which

$$\left\| \begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix} \right\| \leq 1, \left\| \begin{pmatrix} B_{21} & B_{22} \end{pmatrix} \right\| \leq 1.$$

Note that the contractivity of the latter operator matrices implies that

$$B_{11} = G_1 D_{B_{21}}, B_{22} = D_{B_{21}^*} G_2$$

with G_1 and G_2 uniquely determined contractions. It was proved in [2], [19] and [62] that there exists a one-to-one correspondence between the set of all contractive completions of (4.11) and the set of all contractions $G : \mathcal{D}_{G_2} \to \mathcal{D}_{G_1^*}$ given by

$$(4.12) \qquad B_{12} = -G_1 B_{21}^* G_2 + D_{G_1^*} G D_{G_2}.$$

With the choice $G = 0$ we obtain the particular completion $B_{12} = -G_1 B_{21}^* G_2$. We shall call this the *central completion* of (4.11).

Let $\{B_{ij}, 1 \leq j \leq i \leq n\}$ be a $n-by-n$ *contractive triangle*, i.e., let $B_{ij} : \mathcal{K}_j \rightarrow \mathcal{H}_i$, $1 \leq j \leq i \leq n$, be operators acting between Hilbert spaces with the property that

$$\|(B_{ij})_{i=p,j=1}^{n\ \ ,p}\| \leq 1, p = 1, ...n.$$

In order to make a contractive completion one can proceed as follows: choose a position (i_0, j_0) with $i_0 = j_0 - 1$, and choose B_{i_0,j_0} such that $(B_{ij})_{i=i_0,j=1}^{n\ \ \ ,j_0}$ is the central completion of $\{B_{ij}, i \geq i_0, j \leq j_0\}$ as in the $2-by-2$ case. Proceed in the same way with the partial matrix thereby obtained (some compressing of columns and rows is needed) until all positions are filled. We shall refer to F_c as the *central completion* of $\{B_{ij}, (i,j) \in T\}$.

THEOREM 4.2. *Let $\{B_{ij}, 1 \leq j \leq i \leq n\}$ be a contractive triangle. Let F_c denote the central completion of $\{B_{ij}, 1 \leq j \leq i \leq n\}$ and let Φ_c and Ψ_c be upper and lower triangular operator matrices such that*

$$\Phi_c^*\Phi_c = I - F_c^*F_c, \Psi_c^*\Psi_c = I - F_cF_c^*. \tag{4.13}$$

Further, let $\omega_1 : \mathcal{D}_{F_c} \rightarrow \overline{\mathcal{R}(\Phi_c)}$ and $\omega_2 : \mathcal{D}_{F_c^*} \rightarrow \overline{\mathcal{R}(\Psi_c)}$ be unitary operator matrices so that

$$\Phi_c = \omega_1 D_{F_c}, \Psi_c = \omega_2 D_{F_c^*}, \tag{4.14}$$

and put

$$\tau_c = -\omega_1 F_c^* \omega_2^*. \tag{4.15}$$

Then each contractive completion of $\{B_{ij}, 1 \leq j \leq i \leq n\}$ is of the form

$$\mathcal{S}(G) = F_c - \Psi_c^* G(I + \tau_c G)^{-1}\Phi_c. \tag{4.16}$$

in which $G = (G_{ij})_{i,j=1}^n : \overline{\mathcal{R}(\Phi_c)} \rightarrow \overline{\mathcal{R}(\Psi_c)}$ is a contraction with $G_{ij} = 0$ whenever $(i,j) \in T$. Moreover, the correspondence between the set of all positive semidefinite completions and all such contractions G is one-to-one.

Furthermore, $\mathcal{S}(G)$ is isometric (co-isometric, unitary) if and only if G is.

The decompositions of $\overline{\mathcal{R}(\Phi_c)}$ and $\overline{\mathcal{R}(\Psi_c)}$ are simply given by

$$\overline{\mathcal{R}(\Phi_c)} = \oplus_{i=1}^{n} \overline{\mathcal{R}((\Phi_c)_{ii})}, \overline{\mathcal{R}(\Psi_c)} = \oplus_{i=1}^{n} \overline{\mathcal{R}((\Psi_c)_{ii})}.$$

Proof. We apply Theorem 3.10 using the correspondence

$$\begin{pmatrix} I & B \\ B^* & I \end{pmatrix} \geq 0 \text{ if and only if } \|B\| \leq 1. \tag{4.17}$$

Consider the $(n+n)\times(n+n)$ positive semidefinite band which one obtains by embedding the contractive triangle $\{B_{ij}, 1 \leq j \leq i \leq n\}$ in a large matrix via (4.17). It is easy to check that when applying Theorem 3.10 on this $(n+n)\times(n+n)$ positive semidefinite band one obtains

$$V_c = \begin{pmatrix} I & F_c \\ 0 & \Phi_c \end{pmatrix}, W_c = \begin{pmatrix} \Psi_c & 0 \\ F_c^* & I \end{pmatrix}, U = \begin{pmatrix} \Psi_c^* & F_c \\ \tau_c & \Phi_c \end{pmatrix}$$

(use $F_c^* D_{F_c^*} = D_{F_c^*} F_c$). It follows now from Theorem 3.9 that $(\tau_c)_{ij} = 0$ for $i > j$. Further, it is easy to compute that

$$\mathcal{T}\left(\begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} I & \mathcal{S}(G) \\ \mathcal{S}(G)^* & Q(G) \end{pmatrix} = \begin{pmatrix} \hat{Q}(G) & \mathcal{S}(G) \\ \mathcal{S}(G)^* & I \end{pmatrix}, \tag{4.18}$$

and thus we have

$$I = Q(G) = \mathcal{S}(G)^*\mathcal{S}(G) + \Phi_c(I + \tau_c G)^{*-1}(I - G^*G)(I + \tau_c G)^{-1}\Phi_c, \tag{4.19}$$

and

$$I = \hat{Q}(G) = \mathcal{S}(G)\mathcal{S}(G)^* + \Psi_c(I + G\tau_c)^{-1}(I - GG^*)(I + G\tau_c)^{*-1}\Psi_c. \tag{4.20}$$

We obtain the first part of the theorem from (4.18) and Theorem 3.10. From relation (4.19) one immediately sees that G is an isometry if and only if $\mathcal{S}(G)$ is. Similarly, one obtains from (4.20) that G is a co-isometry if and only if $\mathcal{S}(G)$ is. This proves the last statement in the theorem. □

The existence of an isometric (co-isometric, unitary) completion is reduced to the existence of a strictly upper triangular isometry (co-isometry, unitary) acting between the closures of the ranges of Φ_c and Ψ_c. Taking into account the specific structures of Φ_c and Ψ_c one recovers the characterizations of existence of such completions given in [9] and [1].

REMARK We can apply Theorem 3.9 to characterize the central completion. We first mention that for an arbitrary completion F of $\{B_{ij}, 1 \leq j \leq i \leq n\}$ one can define Φ, Ψ and τ analogously as in (2.4), (4.14), and (4.15). The equivalence of i), ii) and iii) in Theorem 3.9 implies that the central completion is characterized by the maximality of $diag(\Phi_{ii}^*\Phi_{ii})_{i=1}^n$ or $diag(\Psi_{ii}^*\Psi_{ii})_{i=1}^n$. From the equivalence of i) and iv) in Theorem 3.9 one also easily obtains that the upper triangularity of τ characterizes the central completion.

4.3 Linearly Constrained Contractive Completions

We return to the problem (4.5). The next lemma will reduce this linearly constrained contractive completion problem to a positive semidefinite completion problem. The lemma is a slight variation of an observation in [65].

LEMMA 4.3. *Let $B : \mathcal{H} \to \mathcal{K}$, $S : \mathcal{G} \to \mathcal{H}$ and $T : \mathcal{G} \to \mathcal{K}$ be linear operators acting between Hilbert spaces. Then $\|B\| \leq 1$ and $BS = T$ if and only if*

$$\begin{pmatrix} I & S & B^* \\ S^* & S^*S & T^* \\ B & T & I \end{pmatrix} \geq 0. \tag{4.21}$$

Proof. The operator matrix (4.21) is positive semidefinite if and only if

$$\begin{pmatrix} S^*S & T^* \\ T & I \end{pmatrix} - \begin{pmatrix} S^* \\ B \end{pmatrix} \begin{pmatrix} S & B^* \end{pmatrix} = \begin{pmatrix} 0 & T^* - S^*B^* \\ T - BS & I - BB^* \end{pmatrix} \geq 0, \tag{4.22}$$

and this latter inequality is satisfied if and only if $\|B\| \leq 1$ and $BS = T$. □

THEOREM 4.4. *Let $B_{ij} : \mathcal{H}_j \to \mathcal{K}_i$, $1 \leq i \leq j \leq n$, $S_i : \mathcal{H} \to \mathcal{H}_i$, $i = 1, ..., n$ and $T_j : \mathcal{H} \to \mathcal{K}_j$ be given linear operators acting between Hilbert spaces and S and T be as* *in (4.5). Then there exist contractive completions B of $\{B_{ij}, 1 \leq i \leq j \leq n\}$ satisfying the linear constraint $BS = T$ if and only if*

$$(4.23) \qquad \begin{pmatrix} S^*S - S^{(i)*}S^{(i)} & T^{(i)*} - S^{(i)*}B^{(i)*} \\ T^{(i)} - B^{(i)}S^{(i)} & I - B^{(i)}B^{(i)*} \end{pmatrix} \geq 0$$

for $i = 1, .., n$, in which

$$(4.24) \qquad B^{(i)} = \begin{pmatrix} B_{1i} & . & . & . & B_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ B_{ii} & . & . & . & B_{in} \end{pmatrix}, S^{(i)} = \begin{pmatrix} S_i \\ . \\ . \\ S_n \end{pmatrix}, T^{(i)} = \begin{pmatrix} T_1 \\ . \\ . \\ T_i \end{pmatrix}$$

for $i = 1, ..., n$.

Proof. By Lemma 4.3 there exists a contractive completion B of $\{B_{ij}, 1 \leq i \leq j \leq n\}$ satisfying the linear constrain $BS = T$ if and only if there exists a positive semidefinite completion of the partial matrix

$$(4.25) \qquad \begin{pmatrix} I & 0 & \dots & 0 & S_1 & B_{11}^* & ? & \dots & ? \\ 0 & I & \dots & 0 & S_2 & B_{12}^* & B_{22}^* & \dots & ? \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & I & S_n & B_{1n}^* & B_{2n}^* & \dots & B_{nn}^* \\ S_1^* & S_2^* & \dots & S_n^* & S^*S & T_1^* & T_2^* & \dots & T_n^* \\ B_{11} & B_{12} & \dots & B_{1n} & T_1 & I & 0 & \dots & 0 \\ ? & B_{21} & \dots & B_{2n} & T_2 & 0 & I & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ ? & ? & \dots & B_{nn} & T_n & 0 & 0 & \dots & I \end{pmatrix}$$

As it is known, the existence of a positive semidefinite completion of (4.25) is equivalent to the positive semidefiniteness of the principal submatrices of (4.25) formed with specified entries. This latter condition is equivalent to (4.23). □

Let us examine the $2 - by - 2$ case a little further, i.e.,

$$(4.26) \qquad \begin{pmatrix} B_{11} & B_{12} \\ ? & B_{22} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

The necessary and sufficient conditions (4.23) for this case reduce to

$$(4.27) \qquad B_{11}S_1 + B_{12}S_2 = T_1, \| \begin{pmatrix} B_{11} & B_{12} \end{pmatrix} \| \leq 1$$

and

$$\begin{pmatrix} I - B_{12}^*B_{12} - B_{22}^*B_{22} & S_2 - B_{12}^*T_1 - B_{22}^*T_2 \\ S_2^* - T_1^*B_{12} - T_2^*B_{22} & S_1^*S_1 + S_2^*S_2 - T_1^*T_1 - T_2^*T_2 \end{pmatrix} \geq 0. \tag{4.28}$$

Assume that (4.27) and (4.28) are satisfied. Similar to Section 4.2, let $G_1 : \mathcal{H}_1 \to \mathcal{D}_{B_{12}^*}$ and $G_2 : \mathcal{D}_{B_{12}} \to \mathcal{K}_2$ be contractions such that

$$B_{11} = D_{B_{12}^*}G_1, B_{22} = G_2D_{B_{12}}. \tag{4.29}$$

Any solution of the constrained problem (4.26) is in particular a solution of the unconstrained problem (the lower triangular analogue (4.11)), and therefore we must have that (use the analogue of (4.12))

$$B_{21} = -G_2B_{12}^*G_1 + D_{G_2^*}\Gamma D_{G_1}, \tag{4.30}$$

in which $\Gamma : \mathcal{D}_{G_1} \to \mathcal{D}_{G_2^*}$ is some contraction. The equation $B_{21}S_1 + B_{22}S_2 = T_2$ implies that Γ is uniquely defined on $\overline{\mathcal{R}(D_{G_1}S_1)}$ by

$$D_{G_2^*}\Gamma D_{G_1}S_1 := T_2 - B_{22}S_2 + G_2B_{12}^*G_1S_1. \tag{4.31}$$

We define $\Gamma_0 : \mathcal{D}_{G_1} \to \mathcal{D}_{G_2^*}$ to be the contraction defined on $\overline{\mathcal{R}(D_{G_1}S_1)}$ as above, and 0 on the orthogonal complement, i.e.,

$$\Gamma_0 \mid \mathcal{D}_{G_1} \ominus \overline{\mathcal{R}(D_{G_1}S_1)} = 0 \tag{4.32}$$

We let $B_{21}^{(0)}$ denote the corresponding choice for B_{21}, that is,

$$B_{21}^{(0)} = -G_2B_{12}^*G_1 + D_{G_2^*}\Gamma_0D_{G_1}. \tag{4.33}$$

We shall refer to

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21}^{(0)} & B_{22} \end{pmatrix} \tag{4.34}$$

as the *central completion* of problem (4.26).

In the $n-by-n$ problem (4.5) (assuming conditions (4.23) are met) we construct step by step the central completion of (4.5) as follows. Start by making the central completion of the $2-by-2$ problem

$$\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ ? & B_{22} & \cdots & B_{2n} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \tag{4.35}$$

and obtain in this way $B_{21}^{(0)}$. Continue by induction and obtain at step p, $1 \leq p \leq n-1$, $B_{p1}^{(0)}, \cdots, B_{p,p-1}^{(0)}$ by taking the central completion of the $2-by-2$ problem

$$\begin{pmatrix} B_{11} & \cdots & B_{1,p-1} & B_{1p} & \cdots & B_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ B_{p-1,p}^{(0)} & \cdots & B_{p-1,p-1}^{(0)} & B_{p-1,p} & \cdots & B_{p-1,n} \\ ? & \cdots & ? & B_{pp} & \cdots & B_{pn} \end{pmatrix} \begin{pmatrix} S_1 \\ \vdots \\ S_{p-1} \\ S_p \\ \vdots \\ S_n \end{pmatrix} = \begin{pmatrix} T_1 \\ \vdots \\ T_{p-1} \\ T_p \end{pmatrix}. \tag{4.36}$$

The final result B_0 of this process is the *central completion* of the problem (4.5). The central completion is independent of procedure (see [1]).

LEMMA 4.5. *Let B_0 be a contractive completion of (4.5). Then B_0 is the central completion of (4.5) if and only if*

$$\begin{pmatrix} I & S & B_0^* \\ S^* & S^*S & T^* \\ B_0 & T & I \end{pmatrix} \tag{4.37}$$

is the central completion of the positive semidefinite completion problem (4.8).

Proof. By the inheritance principle and the way the central completion is defined it suffices to prove the lemma in the $2-by-2$ case. Take an arbitrary contractive completion B of (4.26), corresponding to the parameter Γ in (4.30), say. The lower-upper Cholesky factorization of the corresponding positive semidefinite completion problem is given by

$$V^*V = \begin{pmatrix} I & S & B^* \\ S^* & S^*S & T^* \\ B & T & I \end{pmatrix}, \tag{4.38}$$

in which

$$V = \begin{pmatrix} I & S & B^* \\ 0 & 0 & 0 \\ 0 & 0 & \Phi^* \end{pmatrix} \tag{4.39}$$

and Φ is lower triangular such that $I - BB^* = \Phi\Phi^*$. It is straightforward to check that

$$\Phi = \begin{pmatrix} D_{B_{12}^*} D_{G_1^*} & 0 \\ -G_2 B_{12} D_{G_1^*} - D_{G_2^*} \Gamma G_1^* & D_{G_2^*} D_{\Gamma^*} \end{pmatrix}. \tag{4.40}$$

Since for $\Gamma = \Gamma_0$ the operator $D^2_{\Gamma^*}$ is maximal among all Γ satisfying (4.31), the lemma follows from the equivalence of i) and ii) in Theorem 3.9. □

THEOREM 4.6. *Let B_0 be the central completion of the linearly constrained contractive completion problem (4.5) (for which the conditions (4.23) are satisfied). Let $\rho : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \overline{\mathcal{R}((S^*S - T^*T)^{1/2})}$ be such that*

$$(S^*S - T^*T)^{1/2}\rho = S^* D^2_{B_0}, \tag{4.41}$$

and Ψ and Φ lower triangulars such that

$$\Psi^*\Psi = I - \rho^*\rho - B_0^* B_0 \tag{4.42}$$

and

$$\Phi\Phi^* = I - B_0 B_0^*. \tag{4.43}$$

Consider the contraction $\omega_1 : \mathcal{D}_{B_0} \to \overline{\mathcal{R}(\Psi)}$ and the unitary $\omega_2 : \overline{\mathcal{R}(\Phi^*)} \to \mathcal{D}_{B_0^*}$ with the properties

$$\Psi = \omega_1 D_{B_0} \tag{4.44}$$

and

$$\Phi = D_{B_0^*}\omega_2 \tag{4.45}$$

Finally, define

$$\tau =: -\omega_1 B_0^* \omega_2. \tag{4.46}$$

Then there exists an one-to-one correspondence between the set of all contractive solutions of the problem (4.5) and the set of all strictly lower triangular contractions $G : \overline{\mathcal{R}(\Psi)} \to \overline{\mathcal{R}(\Phi^*)}$ *given by*

$$V(G) = B_0 - \Phi(I + G\tau)^{-1}G\Psi \tag{4.47}$$

Moreover, $V(G)$ *is a co-isometry if and only if* G *is a co-isometry and* $V(G)$ *is an isometry if and only if* $S^*S = T^*T$ *and* G *is an isometry.*

The decompositions of $\overline{\mathcal{R}(\Phi^*)}$ and $\overline{\mathcal{R}(\Psi)}$ are simply given by

$$\overline{\mathcal{R}(\Phi^*)} = \oplus_{i=1}^{n}\overline{\mathcal{R}(\Phi_{ii}^*)},\ \overline{\mathcal{R}(\Psi)} = \oplus_{i=1}^{n}\overline{\mathcal{R}(\Psi_{ii})}.$$

Proof. We shall obtain our results by applying Theorem 3.10 for the positive semidefinite completion problem (4.25). Straightforward computation yield that

$$V_c = \begin{pmatrix} I & S & B_0^* \\ 0 & 0 & 0 \\ 0 & 0 & \Phi^* \end{pmatrix} \tag{4.48}$$

and

$$W_c = \begin{pmatrix} \Psi & 0 & 0 \\ \rho & (S^*S - T^*T)^{1/2} & 0 \\ B_0 & T & I \end{pmatrix} \tag{4.49}$$

We remark here that the relation

$$S^*S - T^*T = S^*D_{B_0}^2 S \geq S^*D_{B_0}^4 S \tag{4.50}$$

gives the existence of the contraction ρ with (4.41).

Now we have to determine the unitary $U = (U_{ij})_{i,j=1}^3$ so that $UW_c = V_c$. Note that the existence of ω_1 and ω_2 is assured by the relations (4.43) and (4.43). An immediate computation shows that

$$U = \begin{pmatrix} \Psi^* & \rho^* & B_0^* \\ 0 & 0 & 0 \\ -\omega_2^* B_0 \omega_1^* & -\omega_2^* B_0 \hat{\omega}_1^* & \Phi^* \end{pmatrix}$$

in which $\begin{pmatrix} \omega_1 \\ \hat{\omega}_1 \end{pmatrix}$ is unitary with

$$\begin{pmatrix} \Psi \\ \rho \end{pmatrix} = D_{B_0} \begin{pmatrix} \omega_1 \\ \hat{\omega}_1 \end{pmatrix}.$$

Substituting these data in the first equality of (3.18) gives

$$\mathcal{T}(\begin{pmatrix} 0 & 0 & G^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} I & S & V(G)^* \\ S^* & S^*S & T^* \\ V(G) & T & Q(G) \end{pmatrix}, \tag{4.51}$$

with $V(G)$ given by (4.47) and

$$I = Q(G) = V(G)V(G)^* + \Phi(I + G_{\mathcal{T}})^{-1}(I - GG^*)(I + G_{\mathcal{T}})^{*-1}\Phi^* \tag{4.52}$$

The first part of the theorem now follows from (4.51) and Lemma 4.3. Further, (4.52) implies that $V(G)$ is a co-isometry if and only if G is.

If the contractive solution $V(G)$ to the constrained problem (4.5) is isometric, then clearly we must have that $S^*S = T^*T$ and thus $\rho = 0$. In this case,

$$W_c = \begin{pmatrix} \Psi & 0 & 0 \\ 0 & 0 & 0 \\ B_0 & T & I \end{pmatrix}.$$

Using the second inequality in (3.18) in this special case, we obtain that

$$\mathcal{T}(\begin{pmatrix} 0 & 0 & G^* \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}) = \begin{pmatrix} \hat{Q}(G) & S & V(G)^* \\ S^* & S^*S & T^* \\ V(G) & T & I \end{pmatrix} \tag{4.53}$$

in which

$$I = \hat{Q}(G) = V(G)^*V(G) + \Psi^*(I + G_{\mathcal{T}})^{-1}(I - G^*G)(I + G_{\mathcal{T}})^{*-1}\Psi. \tag{4.54}$$

Relation (4.54) implies that when $S^*S = T^*T$, the spaces $\mathcal{D}_{V(G)}$ and $\mathcal{D}_G$ have the same dimensions and thus $V(G)$ is isometric if and only if G is. This finishes the proof. □

In the $2 - by - 2$ case another parametrization was derived in [6].

REMARK 4.7. By Theorem 4.6 we can reduce the existence of a co-isometric completion of the problem (4.5) to the existence of a strictly lower triangular co-isometry acting between $\overline{\mathcal{R}(\Psi)}$ and $\overline{\mathcal{R}(\Phi^*)}$. Also, when $S^*S = T^*T$, the existence of a isometric completion of the problem (4.5) reduces to the existence of a strictly lower triangular isometry acting between $\overline{\mathcal{R}(\Psi)}$ and $\overline{\mathcal{R}(\Phi^*)}$.

REMARK 4.8. There exists a unique solution to (4.5) if and only if 0 is the only strictly lower triangular contraction acting $\overline{\mathcal{R}(\Psi)} \to \overline{\mathcal{R}(\Phi^*)}$. This can be translated in the following. If i_0 denotes the minimal index for which $\Psi_{i_0 i_0} \neq 0$, then there exists a unique solution if and only if $\Phi_{kk} = 0$ for $k = i_0 + 1, ..., n$.

REMARK 4.9. The upper triangularity of τ characterizes the central completion. For this one can simply use Theorem 3.9 and Lemma 4.5. Also the maximality of $diag(\Phi_{ii}\Phi_{ii}^*)_{i=1}^n$ or $diag(\Psi_{ii}^*\Psi_{ii})_{i=1}^n$ characterizes the central completion.

For a further analysis in the $2 - by - 2$ case we refer to [6].

CHAPTER V
DETERMINANT FORMULAE

The results of this chapter are applications and extensions of the results in the previous chapters. The aim of this chapter is to prove determinant formulae for matrices with sparse inverses. We also obtain a formula in terms of some "free" parameters for the determinant of an arbitrary positive definite completion of a partial positive matrix with a chordal associated graph. Since we deal with determinant formulae, all the involved matrices are assumed to be scalar.

In Section 5.1, a determinant formula in terms of the determinants of some key principal minors is obtained for matrices with the property that their inverse has a chordal associated graph. We prove that after a cancellation process our formula leads to a determinant formula proved earlier in [11]. Also, an algorithmic method of constructing minimal vertex separators for chordal graphs is presented.

In Section 5.2 the results of Section 5.1 are generalized for matrices with non-symmetric nonzero-pattern of their inverse. Thus, based on the connection between Gaussian elimination and graph theory pointed out in [37], a determinant formula is obtained for matrices with the property that their inverse has a directed graph with a perfect edge elimination scheme. As consequence, we obtain a formula proved in [12]. Next, a counterexample to a conjecture of [12] is presented which was found by the method developed in this section.

In Section 5.3 we follow [5] to further study positive definite completions of partial positive matrices with chordal associated graphs. A formula for the determinant of an arbitrary completion in terms of the parameters along a chordal sequence introduced in Section 1.3 is derived. As application we obtain a proof of a conjecture stated in [48] concerning an inheritance principle for chordal graphs, generalizing a result for band matrices in [25]. The conjecture was independently proved by different methods in [13]. Finally, another variant of the inheritance principle is presented by the means of a graph theoretical result of [60].

5.1 Determinant Formulae for Matrices with Chordal Inverses

If $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a block matrix with A invertible, then as a consequence of (1.2) we have that:

$$det(D - CA^{-1}B) = \frac{detH}{detA} \tag{5.1}$$

We present our first determinant formula.

THEOREM 5.1. *Let $G = (V, E)$ be a chordal graph and $\sigma = [v_1, ..., v_n]$ a perfect scheme for G. Denote*

$$S_j = \{v_k \in Adj(v_j) | k > j\} \tag{5.2}$$

for $j = 1, ..., n$. If R is an invertible matrix with $R^{-1} \in \Omega_G$ and each of the submatrices $R(S_j)$, $j = 1, ..., n$ is invertible, then (with the convention $detR(\emptyset) = 1$)

$$detR = \prod_{k=1}^{n} \frac{detR(\{v_k\} \cup S_k)}{detR(S_k)} \tag{5.3}$$

Proof. We can apply the results of Corollary 2.3 for the perfect scheme σ also, since if we reorder the rows and colums of R by the ordering of σ, $[1, ..., n]$ becomes a perfect scheme. Thus, in the hypothesis of the theorem, since R is invertible, it follows that all the submatrices $R(\{v_k\} \cup S_k)$ are invertible and

$$detR = \prod_{j=1}^{n} detV_{v_j}^{-1} \tag{5.4}$$

in which $V_{v_j}^{-1}$ is the Schur complement of $R(S_j)$ in $R(\{v_j\} \cup S_j)$. Thus (5.4) implies (5.3) via (5.1). □

In the paper [11] it is proved that if G is chordal, $T = (V(T), E(T))$ is a tree of G, R is invertible with $R^{-1} \in \Omega_G$, then

$$(5.5) \qquad detR = \frac{\prod_{V \in V(T)} detR(V)}{\prod_{\{V_1,V_2\} \in E(T)} detR(V_1 \cap V_2)}$$

provided that the terms of the denominator are nonzero.

We next present how formula (5.5) can be obtained from (5.3).

PROPOSITION 5.2. *For any perfect scheme* $\sigma = [v_1, ..., v_n]$ *and tree* $T = (V(T), E(T))$ *of* G *the formula (5.5) can be obtained from (5.3) by cancellation.*

Proof. The proposition is proven by induction on n, the number of vertices of G. For $n = 1$ it is obvious. Suppose now that $T' = (V(T'), E(T'))$ is a tree of the graph $G_{\{v_2,...,v_n\}}$. Assume that

$$(5.6) \qquad \prod_{k=2}^{n} \frac{detM(\{v_k\} \cup S_k)}{detM(S_k)} = \frac{\prod_{W \in V(T')} detM(W)}{\prod_{\{W,W'\} \in E(T)} detM(W \cap W')}$$

There are two possibilities:

A. The clique S_1 is not maximal in $G_{\{v_2,...,v_n\}}$. Then a tree $T = (V(T), E(T))$ can be obtained by adding to $V(T')$ a new vertex corresponding to $\{v_1\} \cup S_1$ and a new edge joining this vertex with the vertex in $V(T')$ corresponding to a maximal clique of $G_{\{v_2,...,v_n\}}$ containing S_1.

Thus

$$\frac{\prod_{W \in V(T)} detM(W)}{\prod_{\{W,W'\} \in E(T)} detM(W \cap W')} = \frac{detM(\{v_1\} \cup S_1)}{detM(S_1)} \frac{\prod_{W \in V(T')} detM(W)}{\prod_{\{W,W'\} \in E(T')} detM(W \cap W')}$$

and the equality is proved for G without any new cancellation.

B. The clique S_1 is maximal in $G_{\{v_2,...,v_n\}}$. A tree $T = (V(T), E(T))$ of G can be obtained from T' by renaming the vertex corresponding to S_1 by $\{v_1\} \cup S_1$.

Thus, in the product

$$\frac{detM(\{v_1\} \cup S_1)}{detM(S_1)} \prod_{k=2}^{n} \frac{detM(\{v_k\} \cup S_k)}{detM(S_k)}$$

the term $detM(S_1)$ will be cancelled. The right member of (5.6) after multiplication with $\frac{detM(\{v_1\}\cup S_1)}{detM(S_1)}$ and cancellation of $detM(S_1)$ becomes $\frac{\prod_{W\in V(T)} detM(W)}{\prod_{\{W,W'\}\in E(T)} detM(W\cap W')}$. The denominator of this latter expression coincides with the denominator of (5.5) since v_1 is contained in a unique maximal clique of V. This completes the proof. □

We next illustrate the result of the above proposition with a simple example. Consider G to be the graph in Fig. XIII

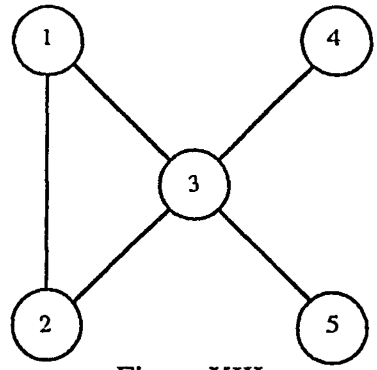

Figure XIII

with maximal cliques $C_1 = \{1,2,3\}$, $C_2 = \{3,4\}$, $C_3 = \{3,5\}$ and tree $C_1 - C_2 - C_3$. Consider the perfect scheme $\sigma = [1,2,3,4,5]$ for G. Then for any matrix $R \in \Omega_G$ we have (by assuming that the denominators are nonzero):

$$\prod_{k=1}^{5} \frac{detR(\{v_k\}\cup S_k)}{detR(S_k)} =$$

$$= \frac{detR(\{1,2,3\})}{detR(\{2,3\})} \frac{detR(\{2,3\})}{detR(\{3\})} \frac{detR(\{3,4\})}{detR(\{3\})} \frac{detR(\{3,5\})}{detR(\{5\})} detR(\{5\}) =$$

$$= \frac{detR(C_1)detR(C_2)detR(C_3)}{detR(C_1\cap C_2)detR(C_2\cap C_3)}.$$

In [13], Theorem 3.5 it is proved that for any tree T of G, the set of cliques appearing in the denominator of (5.5) is the set of minimal vertex separators of the graph G. From Proposition 5.2, it follows that for any perfect scheme $\sigma = [v_1, ..., v_n]$ of G, in the denominator of (5.5) appear the cliques of the form S_i which are not maximal in $G_{\{v_{i+1},...,v_n\}}$. The following result can be viewed as a consequence of Proposition 5.2 and Theorem 3.5 of [13] but it can also be proved directly. It represents an algorithmic method of constructing the minimal vertex separators of a chordal graph.

Proposition 5.3. *Let $G = (V, E)$ be a chordal graph and $\sigma = [v_1, ..., v_n]$ a perfect scheme for G. A subset $S \subseteq V$ is a minimal vertex separator of G if and only if S equals some $S_i (i < n)$ that is not a maximal clique in $G_{\{v_{i+1},...,v_n\}}$.*

Proof. The proposition is proven by induction on the cardinality of V. For $n \leq 3$ it is immediate. Assume that it holds for $G' = G_{\{v_2,...,v_n\}}$.

Since v_1 is simplicial any minimal $v_k - v_m$ separator is the same in G' and G for any $k, m \geq 2$. If S_1 is not maximal in G' then there exists a vertex $v_m, m \geq 2$, with $S_1 \subseteq Adj(v_m)$, so S_1 is a minimal $v_1 - v_m$ separator. Conversely, if S_1 is a minimal vertex separator in G, by Ex.12 page 102 from [37], S_1 is not maximal in G'. After removing any $v_1 - v_k$ separator S from G, $S \neq S_1$, the connected component of v_1 must contain a vertex $v_r, r \geq 2$. Since v_1 is simplicial, S must also be a minimal $v_r - v_k$ separator and by the assumption made for G', S is of the desired form. So the statement is completely proved. □

5.2 Determinant Formulae and Nonsymmetric Gaussian Elimination

As in Chapter II, Section 5.2 will be a generalization of Section 5.1 for nonsymmetric nonzero-patterns. The directed graph model for the nonsymmetric Gaussian perfect elimination is the basic tool.

Let $H = (V, \mathcal{F})$ be a directed graph and $\phi = [(x_1, y_1), ..., (x_n, y_n)]$ a perfect edge elimination scheme for H. Consider an invertible matrix R such that $R^{-1} \in \Omega_H$. Recall that under these conditions R^{-1} can be reduced by perfect Gaussian elimination. This means that choosing the entries on the positions $(x_1, y_1), ..., (x_n, y_n)$ to act as pivots, R^{-1} will be reduced to a matrix having only one nonzero entry on each row and column without ever changing during this process a zero entry to a nonzero.

Let denote:

$$X_k = \{y_j \in Adj(x_k) | j > k\}$$

$$(5.7) \qquad Y_k = \{x_j \in Adj^{-1}(y_k) | j > k\}$$

$$Z_k = \{y_1, ..., y_{k-1}\} \cup \{y_j \notin Adj(x_k) | j > k\}$$

$$U_k = \{x_1, ..., x_{k-1}\} \cup \{x_j \notin Adj^{-1}(y_k) | j > k\}$$

for $k = 1, ..., n$. Thus $X = \{x_k\} \cup Y_k \cup U_k$ and $Y = \{y_k\} \cup X_k \cup Z_k$ for $k = 1, ..., n$.

Then we have:

LEMMA 5.4. *In the above conditions, after reducing the matrix R^{-1} by Gaussian elimination by succesively choosing the entries on the positions $(x_1, y_1), ..., (x_n, y_n)$ to act as pivots, we obtain a matrix $D = (d_{ij})_{1 \le i,j \le n}$ with the only nonzero entries $d_{x_k y_k}$, $k = 1, ..., n$ given by the formulae*

$$(5.8) \qquad d_{x_k y_k} = (-1)^{s_k + t_k} \frac{det R^{-1}(\{x_k\} \cup \alpha_k | \{y_k\} \cup Z_k)}{det R^{-1}(\alpha_k | Z_k)}$$

$$= (-1)^{s'_k + t'_k} \frac{det R^{-1}(\{x_k\} \cup U_k | \{y_k\} \cup \beta_k)}{det R^{-1}(U_k | \beta_k)}$$

provided that the terms of the denominators are nonzero, in which $\alpha_k = \{x_1, ..., x_{k-1}\} \cup \alpha'_k$, $\beta_k = \{y_1, ..., y_{k-1}\} \cup \beta'_k$ with $\alpha'_k \subseteq \{x_{k+1}, ..., x_n\}$, $\beta'_k \subseteq \{y_{k+1}, ..., y_n\}$ arbitrary sets with the property $card\alpha_k = cardZ_k$ and $card\beta_k = cardU_k$, s_k and t_k (respective s'_k and t'_k) are the indices of the rows and columns of the entry (x_k, y_k) in the matrix $R^{-1}(\{x_k\} \cup \alpha_k | \{y_k\} \cup Z_k)$ (respective $R^{-1}(\{x_k\} \cup U_k | \{y_k\} \cup \beta_k)$).

Proof. Since $Z_k = \{y_1, ..., y_{k-1}\} \cup \{y_j \notin Adj(x_k) | j > k\}$, after performing partial Gaussian elimination on the matrix $R^{-1}(\{x_k\} \cup \alpha_k | \{y_k\} \cup Z_k)$, (in which we keep the same indices as in R^{-1}) by choosing the entries on the positions $(x_1, y_1), ..., (x_{k-1}, y_{k-1})$ to act as pivots, we obtain a matrix having on the rows $x_1, ..., x_{k-1}$ and on the columns $y_1, ..., y_{k-1}$ exactly one nonzero entry and since no zero entry is changed into a nonzero, all the entries on the positions (x_k, y_s) with $y_s \in \{y_j \notin Adj(x_k) | j > k\}$ are zero.

Performing the same operations on the matrix $R^{-1}(\alpha_k | Z_k)$ we obtain the same matrix as before but without its x_k row and y_k column. Dividing the determinants of these two matrices we obtain the first equality in (5.8). The second one can be obtained in a similar way. □

THEOREM 5.5. *The elements* $d_{x_k y_k}$ *are given by the formulae*

$$d_{x_k y_k} = (-1)^{s_k+t_k+x_k+y_k} \frac{detR(X_k|\gamma_k)}{detR(\{y_k\} \cup X_k|\{x_k\} \cup \gamma_k)} \tag{5.9}$$

$$= (-1)^{s'_k+t'_k+x_k+y_k} \frac{detR(\delta_k|Y_k)}{detR(\{y_k\} \cup \delta_k|\{x_k\} \cup Y_k)}$$

in which $\gamma_k \subseteq \{x_{k+1}, ..., x_n\}$ *and* $\delta_k \subseteq \{y_{k+1}, ..., y_n\}$ *are arbitrary sets with* $card\gamma_k = cardX_k$ *and* $card\delta_k = cardY_k$.

Proof. By the Jacobi identity (see e.g. [39] p.21) for any α, $\beta \subseteq \{1, ..., n\}$ with $card\alpha = card\beta$,

$$detR^{-1}(\alpha|\beta) = (-1)^u \frac{detR(C_\alpha|C_\beta)}{detR}$$

in which C_α and C_β are the complementary sets of α and β in $\{1, ..., n\}$ and $u = \sum_{i\in\alpha} i + \sum_{j\in\beta} j$. Thus the formulae (5.9) follow directly from (5.8). □

COROLLARY 5.6. *The determinant of R is given by the formula*

$$detR = \frac{sgn\theta}{\prod_{k=1}^{n} d_{x_k y_k}} \tag{5.10}$$

in which $d_{x_k y_k}$ *are given by (5.9) and* θ *is the permutation in which* y_k *corresponds to* x_k.

We illustrate the above results with an example. Let

$$R = \begin{pmatrix} 3/5 & -4/5 & -4/5 & 2/5 \\ 1/5 & 2/5 & 2/5 & -1/5 \\ 2/5 & -1/5 & -4/5 & -2/5 \\ -1/5 & -2/5 & -7/5 & 6/5 \end{pmatrix}$$

with

$$R^{-1} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & 0 & 2 & 1 \\ -1 & 1 & 2 & 2 \end{pmatrix}$$

Since only the complete graph is an undirected graph of R^{-1}, applying the results of Section 5.1, we are only able to get the formula $detR = detR$. The directed graph $H = (V, \mathcal{F})$ in Fig. V is a directed graph of R^{-1} and $\phi = [(3,4),(1,1),(2,2),(4,3)]$ is a perfect edge elimination scheme for H.

Consider the corresponding reduction of R^{-1};

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & 0 & 2 & 1 \\ -1 & 1 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -5/2 & 0 \end{pmatrix}$$

In this case $X_1 = \{1,3\}$. Since $x_1 = 3$ for γ_1 we have the possibilities $\{1,2\}$, $\{1,4\}$ and $\{2,4\}$, thus

$$d_{34} = \frac{detR(\{1,3\}|\{1,2\})}{detR(\{1,3,4\}|\{1,2,3\})} = \frac{detR(\{1,3\}|\{1,4\})}{detR(\{1,3,4\})} = \frac{detR(\{1,3\}|\{2,4\})}{detR(\{1,3,4\}|\{2,3,4\})} = 1.$$

Since $Y_1 = \{4\}$ and $y_1 = 4$ for δ_1 we have the possibilities $\{1\}$, $\{2\}$ and $\{3\}$ and so

$$d_{34} = -\frac{r_{14}}{detR(\{1,4\}|\{3,4\})} = -\frac{r_{24}}{detR(\{2,4\}|\{3,4\})} = -\frac{r_{34}}{detR(\{3,4\})} = 1.$$

Since $X_2 = \{2\}$, $Y_3 = \{4\}$, for d_{11} we have the formulae

$$d_{11} = -\frac{r_{24}}{detR(\{1,2\}|\{1,4\})} = -\frac{r_{22}}{detR(\{1,2\})} = -\frac{r_{34}}{detR(\{1,3\}|\{1,4\}} = 1.$$

Since $X_3 = \{3\}$, $Y_3 = \{4\}$, we have

$$d_{22} = \frac{r_{34}}{detR(\{2,3\}|\{2,4\})} = 2,$$

and finally

$$d_{43} = \frac{1}{r_{34}} = -\frac{5}{2}.$$

In this way $detR$ can be obtained by computing only $2 - by - 2$ determinants.

We apply the previous results to obtain the main result of [12]. Consider the directed graph $H = (V, \mathcal{F})$ allowed by the oriented tree (T, D) in which $T = (V(T), E(T))$ and $V(T) = \{V_1, ..., V_m\}$. When $R \in \Omega_H$ it is said that R has *a nonzero-pattern allowed* *by the pair* (T, D). Consider also the chordal graph $G = (V, E)$, the intersection graph of T.

We construct by the aid of T a perfect scheme $\sigma = [v_1, ..., v_n]$ for G as follows. Choose first an extremal node set $V_s \in V(T)$. Then there exists a simplicial vertex $v_1 \in V_s$. If $V_s - \{v_1\}$ is a maximal clique of $G_{\{v_2,...,v_n\}}$ then a tree T' for $G_{\{v_2,...,v_n\}}$ can be obtained by replacing V_s by $V_s - \{v_1\}$ in T. If $V_s - \{v_1\}$ is not maximal in $G_{\{v_2,...,v_n\}}$ then T' can be obtained by deleting V_s and the unique edge joining V_s in T. Let D' be the orientation induced by D on T'. Continue now by choosing v_2 from an extremal node set of T', and so on, until we obtain the perfect scheme $\sigma = [v_1, ..., v_n]$ of G.

LEMMA 5.7. *If* $\sigma = [v_1, ..., v_n]$ *is constructed as above, then* $\phi = [(v_1, v_1), ..., (v_n, v_n)]$ *is a perfect edge elimination scheme for* H.

Proof. We first prove that (v_1, v_1) is a bisimplicial edge. Consider $(v_1, v), (u, v_1) \in \mathcal{F}$. Since $v \in V_s$ and V_s is an extremal node set of T, we have that $u \in V_s$ or $v \in V_s$. Assume that $u \in V_s$. If $v \in V_s$ it is clear that $(u, v) \in \mathcal{F}$. If $v \in V_t$, $t \neq s$ since $(v_1, v) \in \mathcal{F}$ there exists a path in D from V_s to V_t and since $u \in V_s$ we have that $(u, v) \in \mathcal{F}$ and (v_1, v_1) is bisimplicial. The same holds in the case when $u \in V_t$, $t \neq s$. Using the oriented tree (T', D') we obtain that (v_2, v_2) is a bisimplicial edge in the induced directed graph $H_{\{v_2,...,v_n\}}$. We continue this operation until all the vertices are eliminated. □

In the particular case of the directed graph H and perfect edge elimination scheme ϕ constructed above, we have that $X_k = S_k$ or $Y_k = S_k$, for $k = 1, ..., n$, in which S_k is given by (5.2) for G and σ, X_k and Y_k are given by (5.7) for H, $x_k = v_k$ and $y_k = v_k$.

By choosing $\gamma_k = S_k$ respective $\delta_k = S_k$ in the formula (5.9) and replacing this in (5.10), Theorem 5.5 has the the following corollary:

COROLLARY 5.8. *Let* R *be an invertible matrix such that its inverse has a nonzero-pattern allowed by the pair* (T, D). *Then by the previous notation*

$$detR = \prod_{k=1}^{n} \frac{detM(\{v_k\} \cup S_k)}{detM(S_k)} \tag{5.11}$$

provided that the terms of the denominator are nonzero.

After a cancellation as in Proposition 5.2, we obtain

COROLLARY 5.9. *In the above conditions the following formula holds:*

$$detR = \frac{\prod_{V \in V(T)} detR(V)}{\prod_{\{V_1,V_2\} \in E(T)} detR(V_1 \cap V_2)} \tag{5.12}$$

The above result was first obtained in [12].

Let us next consider the index sets $V_1, ..., V_m \subseteq \{1, ..., n\} = V$ having the property

$$\cup_{k=1}^{m} V_k = V \tag{5.13}$$

We introduce some notation and definitions.

If $V_1, ..., V_m \subseteq V$ are index sets satisfying (5.13) and $Z \subseteq V \times V$, we say that Z *lies outside the profile of* $V_1, ..., V_m$ if $Z \cap [\cup_{k=1}^{m}(V_k \times V_k)] = \emptyset$.

If $Z \subseteq V \times V$, the $n - by - n$ matrix M is said to have a *nonzero-pattern allowed by* Z if $m_{rs} = 0$ for all $(r, s) \in Z$. Let $\mathcal{A}_Z$ be the set of all $n - by - n$ matrices with nonzero-pattern allowed by Z. Given an oriented tree (T, D) let $Z(T, D)$ be the set of all $(r, s) \in V \times V$ satisfying neither i) nor ii) of the definition in Section 1.2. Let $V_1, ..., V_m \subseteq V$ be index sets satisfying (5.13) and let T_1 and T_2 be distinct trees with node sets $V_1, ..., V_m$. Then T_1 and T_2 are said to be *equivalent* if the two collections $\{V_i \cap V_j : \{V_i, V_j\} \in E(T_1)\}$ and $\{V_i \cap V_j : \{V_i, V_j\} \in E(T_2)\}$ are identical.

The following was conjectured in [12]:

CONJECTURE Let $V_1, ..., V_m \subseteq V$ be index sets satisfying (5.13) and T a tree with node set $V_1, ..., V_m$. Let $Z \subseteq V \times V$ lie outside the profile of $V_1, ..., V_m$ and assume that

$$\prod_{k=1}^{m} detR(V_k) = detR \prod_{\{V_i,V_j\} \in E(T)} det(V_i \cap V_j) \tag{5.14}$$

for all nonsingular matrices R for which $R^{-1} \in \mathcal{A}_Z$. Then T satisfies the intersection property (1.14). Furthermore there is a tree T' equivalent to T and an orientation D on T' such that $Z \subseteq Z(T', D)$.

It was proved in [12] that in this case T satisfies the intersection property (see Section 1.2). We next give a counterexample to the conjecture.

For $n = 6$, consider $V_1 = \{1,2\}$, $V_2 = \{2,3,4\}$, $V_3 = \{2,4,5\}$, $V_4 = \{2,5,6\}$ and the following tree denoted by T:

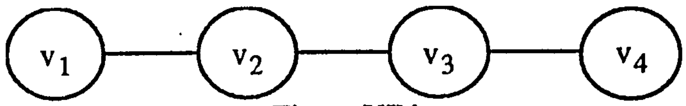

Figure XIV

Let $Z = \{(3,1),(3,6),(4,1),(4,6),(5,3),(6,1),(6,3)\}$. Let consider an invertible matrix R with $R^{-1} \in \mathcal{A}_Z$. Then R^{-1} has the following nonzero-pattern:

$$\begin{pmatrix} X & X & X & X & X & X \\ X & X & X & X & X & X \\ 0 & X & X & X & X & 0 \\ 0 & X & X & X & X & 0 \\ 0 & X & 0 & X & X & X \\ 0 & X & 0 & X & X & X \end{pmatrix}$$

The relation

$$(5.15) \qquad \prod_{k=1}^{4} detR(V_k) = (detR) \prod_{\{V_i,V_j\} \in E(T)} detR(V_i \cap V_j)$$

is equivalent by the Jacobi identity to:

$$detB(\{1,3,4,5,6\})detB(\{1,3,5,6\})detB(\{1,3,4,6\}) =$$

$$= detB(\{3,4,5,6\}detB(\{1,5,6\})detB(\{1,3,6\})detB(\{1,3,4\})$$

for every $B \in \mathcal{A}_Z$. Since $b_{31} = b_{41} = b_{51} = b_{61} = 0$, this relation is equivalent to

$$detB(\{3,5,6\})detB(\{3,4,6\}) = detB(\{5,6\})detB(\{3,6\})detB(\{3,4\})$$

Since $b_{53} = b_{63} = 0$, we have to prove that

$$b_{33}detB(\{3,4,6\}) = detB(\{3,6\})detB(\{3,4\}).$$

The last relation is true since by $b_{36} = b_{46} = 0$, $detB(\{3,4,6\}) = b_{66}detB(\{3,4\})$ and by $b_{36} = b_{63} = 0$, $detB(\{3,6\}) = b_{33}b_{66}$.

Thus (5.15) is verified for any R with $R^{-1} \in \mathcal{A}_Z$.

There are the following two equivalent trees to T, denoted T' and T'':

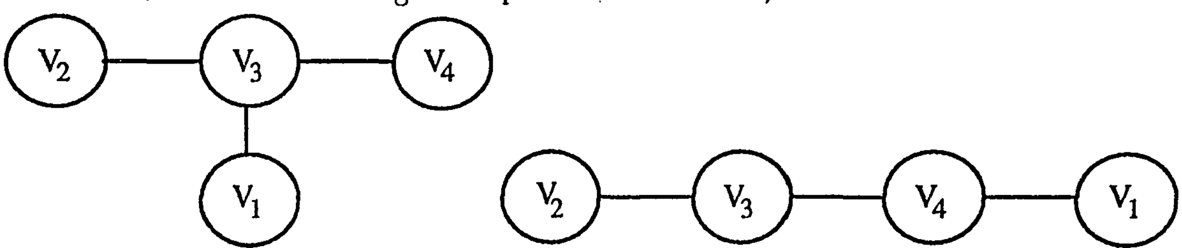

Figure XV

We prove that there is no orientation on any of the trees T, T' or T'' such that its set of mandatory zeros is included in Z. Let assume that there is an orientation on one of these trees such that the corresponding set of mandatory zeros is included in Z. Since $(3,5) \notin Z$ and $(6,4) \notin Z$, on the subtree corresponding to the node set V_2, V_3 and V_4 we must have the following orientation:

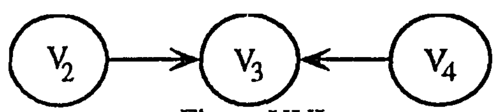

Figure XVI

Since $(k,1) \in Z$ for $k \geq 3$, the unique edge involving V_1 must have the orientation $V_1 \rightarrow V_i$. Thus we may have the following orientations D, D' and D'' on T, T' respective T'':

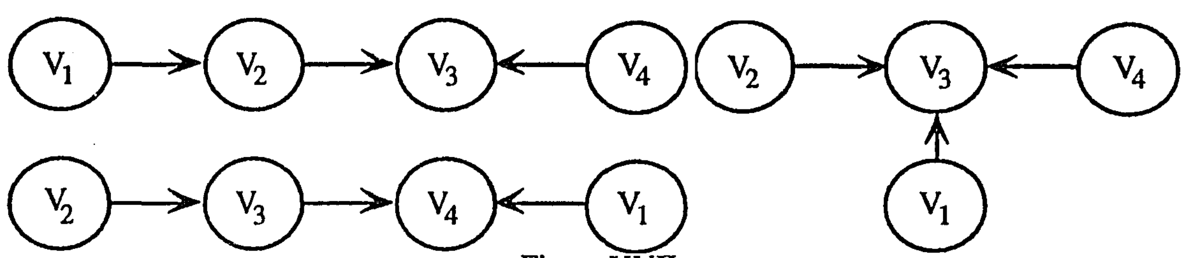

Figure XVII

But $(1,6) \in Z(T,D)$, $(1,3) \in Z(T,D')$, and so none of them is included in Z and the conjecture is not true.

5.3 Inheritance Principles for Chordal Graphs

In this section we obtain a formula for the determinant of each positive definite completion of a partial positive matrix with a chordal graph in terms of the parameters along a fixed chordal sequence and answer a conjecture of [48] concerning an inheritance principle for chordal graphs. Another variant of the inheritance principle is presented.

Let R be a partial positive matrix with a chordal graph $G = (V, E)$. It was proved in [38] that under this circumstances, the unique determinant maximazing positive definite completion F_0 verifies the condition $F_0^{-1} \in \Omega_G$, i.e. $(F_0^{-1})_{ij} = 0$ whenever $(i, j) \notin E$. Thus, given any tree $T = (V(T), E(T))$ and perfect scheme of G, as consequence of the results in Section 5.1, we have

COROLLARY 5.10. *The maximum of the determinants of all positive definite completions of R denoted $D(R)$ is given by the formulae*

$$D(R) = \frac{\prod_{V \in V(T)} detR(V)}{\prod_{\{V_1,V_2\}\in E(T)} detR(V_1 \cap V_2)} \tag{5.16}$$

and

$$D(R) = \prod_{k=1}^{n} \frac{detM(\{v_k\} \cup S_k)}{detM(S_k)} \tag{5.17}$$

Formulae (5.16) and (5.17) depend only on the given data. The formula (5.16) was first proved in [42]. Another formula for $D(R)$ was given in [13].

Before passing to the main result of this section, we discuss a particular example.

EXAMPLE Take a chordal graph $G = (V, E)$ with four vertices, and a partial positive matrix R having G as associated graph. Take an arbitrary chordal sequence $G_0, G_1, ..., G_t$ of G and a positive definite completion F of R. Let $\{g(u_j, v_j)\} | t = 1, ...t\}$ be the parameters of F along the fixed chordal sequence of G. Then

$$detF = \prod_{j=1}^{t} [1 - |g(u_j, v_j|)^2] D(R).$$

This formula can be directly verified for all possible cases. Let us illustrate with the graph G in Fig. XVIII

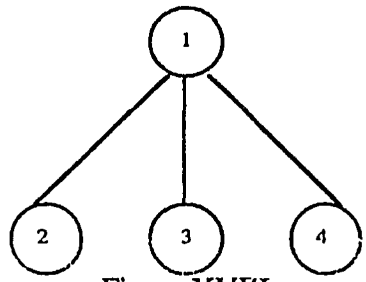

Figure XVIII

which is the only graph with four vertices that is not a proper interval graph.

For the considered graph there exists exactly one chordal sequence (up to a reordering of the vertices) given by $E_1 = E \cup \{(1,2)\}$, $E_2 = E_1 \cup \{(2,4)\}$, $E_3 = E_2 \cup \{(3,4)\}$. Let $g(2,3)$, $g(2,4)$, $g(3,4)$ be the parameters of F along this chordal sequence. Using (1.11) we get

$$detF = [1 - |g(3,4)|^2]\frac{detF(\{1,2,3\})detF(\{1,2,4\})}{detF(\{1,2\})}$$

$$= [1 - |g(3,4)|^2][1 - |g(2,4)|^2]detF(\{1,2,3\})detF(\{1,4\})$$

$$= [1 - |g(3,4)|^2][1 - |g(2,4)|^2][1 - |g(2,3)|^2]detF(\{1,2\})detF(\{1,3\})detF(\{1,4\}),$$

which is exactly the required formula.

Now we can state the main result of this section. The point of the proof is a splitting process based on the Fischer-Hadamard type formula (1.11). In what follows, in the case R is a partial matrix, $G(R)$ will denote the associated graph of R.

THEOREM 5.11. *([5]). Let M_0 be a partial positive matrix with chordal associate graph $G = G(M_0)$. Let $G_0, G_1, ..., G_t$ be a fixed chordal sequence of G.*

Consider a positive definite completion M of M_0, and let $\{g(u_j, v_j) | j = 1, ..., t\}$ be the parameters of M along the fixed chordal sequence. Then the following formula holds:

$$detM = \prod_{j=1}^{t}[1 - |g(u_j, v_j)|^2]D(M_0). \tag{5.18}$$

Proof. We prove (5.18) by induction on the number n of vertices of G. For $n \leq 4$ the formula can be verified directly (see Example above). Assume the statement of Theorem 5.11 is true for any partial positive matrix for which the associated graph has at most $n-1$ vertices.

Fix a partial positive matrix M_0 such that $G = G(M_0)$ has n vertices. Moreover, fix a chordal sequence $G = G_0, G_1, ..., G_t = K_n$ of G and a positive definite completion $M = \{s_{ij} | 1 \leq i, j \leq n\}$ of M_0. Let $\{g(u_j, v_j) | j = 1, ..., t\}$ be the parameters of M along the fixed chordal sequence. Define for $0 \leq m \leq t$ the partial positive matrices $M_0(G_m) = \{s_{ij}(M_0(G_m)) | 1 \leq i, j \leq n\}$ by

$$s_{ij}(M_0(G_m)) = \begin{cases} s_{ij} \; if \; (i,j) \in E_m, \\ unspecified \; otherwise \end{cases} \tag{5.19}$$

Of course, $M_0(G_0) = M_0$, $M_0(G_t) = M$, and M is a positive definite completion of any $M_0(G_m)$, $0 \leq m < t$. Moreover, G_m is the associated graph of $M_0(G_m)$, and $G_m, G_{m+1}, ..., G_t$ is a chordal sequence of G_m. The parameters of M, viewed as a positive definite completion of M_0, along this chordal sequence are obviously $\{g(u_j, v_j) | j = m+1, ..., t\}$.

Now, we get by (1.11),

$$detM = (1 - |g(u_t, v_t)|^2) \frac{detM(V - \{u_t\}) detM(V - \{v_t\})}{detM(V - \{u_t, v_t\})}. \tag{5.20}$$

But we can show that

$$\frac{detM(V - \{u_t\}) detM(V - \{v_t\})}{detM(V - \{u_t, v_t\})} = D(M_0(G_{t-1})). \tag{5.21}$$

Indeed, $[u_t, v_t, (V - \{u_t, v_t\})]$ is a perfect scheme of G_{t-1}, in which the order in $V - \{u_t, v_t\}$ is arbitrary, and (5.21) is a consequence of Corollary 5.10. From (5.20) and (5.21) it results that

$$detM = [1 - |g(u_t, v_t)|^2] D(M_0(G_{t-1})). \tag{5.22}$$

Suppose now that we have proved the formula

$$(5.23)\qquad detM = [1 - |g(u_t, v_t)|^2]...[1 - |g(u_{k+1}, v_{k+1}|^2]D(M_0(G_k)).$$

for every $m \leq k \leq t-1$, in which $0 < m < t$. We will show that the same formula holds for $m-1$.

Let $[v_1, ..., v_n]$ be a perfect scheme of G_{m-1}. There are two possibilities.

A. $v_1 \neq u_m$, $v_1 \neq v_m$.

In this case, the vertices u_m and v_m are not simultaneously adjacent to v_1 in G_{m-1}, because v_1 is simplicial in G_{m-1}. As a first consequence, v_1 remains simplicial also in G_{m-1} and by Theorem 1.6 we find a perfect scheme $[v_1, w_2, ..., w_n]$ of G_m.

Now, denote by $\hat{G}_m$, $0 \leq m \leq t$, the induced graph $(G_m)_{\{v_2,...,v_n\}}$. In particular, $[v_2, ..., v_n]$ remains a perfect scheme of $\hat{G}_{m-1}$ and $[w_2, ..., w_n]$ remains a perfect scheme of $\hat{G}_m$. Moreover, some of these graph may coincide. Taking into account those consecutive graphs only once, we obtain a chordal sequence $\hat{G} = \hat{G}_0, ..., \hat{G}_{t'} = K_{n-1}$ of $\hat{G}$, in which $\hat{G}_{m-1}$ and $\hat{G}_m$ remain consecutive, but possibly at other positions in the sequence. Despite this fact, we keep the same notation for them.

Further, denote by $\hat{M}$ the principal submatrix of M subordinate to $\{v_2, ..., v_n\}$. By (5.19), $\hat{M}_0 = \hat{M}_0(\hat{G}_0)$ is a partial positive matrix with associated graph $\hat{G}$. $\hat{M}$ can be viewed as a positive definite completion of $\hat{M}_0$; let $\{\hat{g}(\hat{u}_j, \hat{v}_j) | j = 1, ..., t'\}$ be the parameters of $\hat{M}$ along the chordal sequence $\hat{G}_0, \hat{G}_1, ..., \hat{G}_{t'}$.

By the previous remark, $\{g(u_j, v_j) | j = m, ..., t'\}$ are the parameters of $\hat{M}$, as a positive definite completion of $\hat{M}_0(\hat{G}_m)$, along the chordal sequence $\hat{G}_m, \hat{G}_{m-1}, ..., \hat{G}_{t'}$ of $\hat{G}_m$. By the induction hypothesis,

$$det\hat{M} = \prod_{j=m}^{t} [1 - |\hat{g}(u_j, v_j)|^2]D(\hat{M}_0(\hat{G}_{m-1})).$$

and

$$det\hat{M} = \prod_{j=m+1}^{t} [1 - |\hat{g}(u_j, v_j)|^2]D(\hat{M}_0(\hat{G}_m)).$$

as $|\hat{g}(u_j, v_j)| < 1$ for $j = 1, ..., t'$, we deduce

$$D(\hat{M}_0(\hat{G}_m)) = [1 - |\hat{g}(u_m, v_m)|^2] D(\hat{M}_0(\hat{G}_{m-1})).$$

But now, the main point is the following: as another consequence of the fact that u_m and v_m are not simultaneously adjacent to v_1 in G_{m-1}, the unique maximal clique in G which is not a clique in $\hat{G}_{m-1}$ is also the unique maximal clique in $\hat{G}_m$ which is not a clique in $\hat{G}_{m-1}$. In view of the dependence on parameters in Theorem 1.2, this means that

$$\hat{g}(u_m, v_m) = g(u_m, v_m),$$

and we obtain the formula

$$D(\hat{M}_0(\hat{G}_m)) = [1 - |g(u_m, v_m)|^2] D(\hat{M}_0(\hat{G}_{m-1})). \tag{5.24}$$

By Corollary 5.10, the equation (5.24) can be written in the form

$$\prod_{s=2}^{n} \frac{det M(\{w_s\} \cup S_s)}{det M(S_s)} = [1 - |g(u_m, v_m)|^2] \prod_{s=2}^{n} \frac{det M(\{v_s\} \cup S_s)}{det M(S_s)}. \tag{5.25}$$

As $[v_1, v_2, ..., v_n]$ and $[v_1, w_2, ..., w_n]$ are perfect schemes of G_{m-1} and G_m, respectively, we multiply both sides of (5.25) by

$$\frac{det M(\{v_1\} \cup Adj(v_1))}{det M(Adj(v_1))},$$

and by Corollary 5.10, we obtain

$$D(M_0(G_m)) = [1 - |g(u_m, v_m)|^2] D(M_0(G_{m-1})). \tag{5.26}$$

Equation (5.23) was supposed to be true for $k = m$. Using (5.26), we obtain the same formula (5.23) for the required case $k = m - 1$.

B. By Dirac's Lemma 1.5, the only possibility is that u_m and v_m are the only simplicial vertices of G_{m-1}. We show that in this case $G_m = K_n$, so that this situation can occur only for $m = t$, a case already covered in (5.22).

By Lemma 3 in [38], G_m has one more maximal clique which is not a clique in G_{m-1}, containing both u_m and v_m. As u_m and v_m are simplicial vertices, this clique is exactly $V_m = \{u_m, v_m, Adj_{G_{m-1}}(u_m) \cap Adj_{G_{m-1}}(v_m)\}$.

Excepting the cliques $W_1 = \{u_m, Adj_{G_{m-1}}(u_m)\}$ and $W_2 = \{v_m, Adj_{G_{m-1}}(v_m)\}$, the other maximal cliques in G_{m-1} remain maximal cliques in G_m.

Now, we prove that it is not possible for both W_1 and W_2 to remain maximal cliques in G_m. Indeed, assuming the contrary, we use the simple remark that in a chordal graph a vertex is simplicial if and only if it is contained into exactly one maximal clique, in order to obtain that G_m has no simplicial vertex, thus contradicting the chordality of G_m. Consequently, we can suppose that W_2 is not a maximal clique in G_m (and so, v_m is also a simplicial vertex in G_m). As the only clique in G_m which can contain W_2 is V_m, it follows that $Adj_{G_{m-1}}(v_m) \subset Adj_{G_{m-1}}(u_m)$.

Now, we prove that W_1 is also not a maximal clique in G_m. Suppose it is, so u_m is not a simplicial vertex of G_m. By Dirac's Lemma 1.5, we search for a second simplicial vertex v of G_m containing v. As $W \neq V_m$ because v is not adjacent to v_m, it follows that v is also simplicial in G_{m-1}, a contradiction, which shows us that either $G_m = K_n$ or W_1 is not a maximal clique in G_m.

Supposing the later case holds, we get that $Adj_{G_{m-1}}(u_m) = Adj_{G_{m-1}}(v_m)$. Finally, supposing that G_m is different from K_n, we search, again by Dirac's Lemma 1.5 for a simplicial vertex v of G_m, $v \notin V_m$. Then the unique maximal clique in G_m containing v was the unique maximal clique in G_{m-1} containing v, a contradiction, which shows that, in any case, $G_m = K_n$.

From the analysis of the two cases A and B it follows that the formula (5.23) holds for any $k \in \{0, 1, ..., t\}$. In particular, for $k = 0$, this is exactly the required formula (5.18). □

The first inheritance (or permanence) principle was proved in [25] for the band matrices studied in [24]. In [48] the relevance of this principle was pointed out, and the following result was conjectured, which is now a consequence of (5.11).

THEOREM 5.12. *For a chordal graph G, every chordal sequence $G = G_0, G_1, ..., G_t = K_n$ of G has the following inheritance property. For every partial positive matrix M_0 with G as associate graph, construct a (unique) sequence of partial positive matrices as follows: M_j is obtained from M_{j-1} by completing the (u_j, v_j) entry in such a way that its principal submatrix subordinate to V_j is the maximum determinant positive definite completion of $M_0(V_{j-1})$. Then the last matrix M_t in the sequence is the maximum determinant positive definite completion of M_0.*

Proof. Completing the (u_j, v_j) entry in the partial positive matrix $M_0(V_{j-1})$ for $j = 1, ..., t$ is exactly the completion process considered in Theorem 3.1 for the fixed chordal sequence of G and applied to the matrix M_t. Let $\{g(u_j, v_j) | j = 1, ..., n\}$ be the parameters of M_t along the fixed chordal sequence of G. By (1.10) we have to choose $g(u_j, v_j) = 0$ at every step in order to obtain the principal submatrix of M_j subordinate to V_j as the maximum determinant positive definite completion of $M_0(V_{j-1})$. So M_t has the parameters $g(u_j, v_j) = 0$, $j = 1, ..., t$ along the fixed chordal sequence of G. On the other hand, by (5.18) in Theorem 5.11, the maximum determinant positive definite completion M^0 of M_0 has the parametrs $g^0(u_j, v_j) = 0$, $j = 1, ..., t$, along the same fixed chordal sequence of G. In other words, $M_t = M^0$. □

Another variant of the inheritance principle can be obtained using the following result ([60], Lemma 2).

LEMMA 5.13. *Let $G = (V, E)$ and $G' = (V, E')$ be two chordal graphs with $E \subset E'$ and $|E'| \geq |E| + 2$, in which $|E|$ denotes the cardinality of E. Then there exists a chordal graph $G" = (V, E")$ with $E \subset E' \subset E"$ and $E \neq E' \neq E"$.*

Proof. By induction on $n = |V|$. For $n = 4$ we can simply verify the statement of the lemma. Suppose it to be true for any graph with at most $n - 1$ vertices, and let now $G = (V, E)$ and $G' = (V, E')$ be with $E \subset E'$ and $|E'| \geq |E| + 2$.

Let $[v_1, v_2, ..., v_n]$ be a perfect scheme of G. There are two possibilities:

A. $G_{\{v_2,...,v_n\}} \neq G'_{\{v_2,...,v_n\}}$. If $G_{\{v_2,...,v_n\}} = (V - \{v_1\}, \hat{E})$ and $G'_{\{v_2,...,v_n\}} = (V - \{v_1\}, \hat{E}')$ then, by induction there exists a chordal graph $\hat{G}" = (V - \{v_1\}, \hat{E}")$ with

$\hat{E} \subset \hat{E}" \subset \hat{E}'$. In the case $G'_{\{v_2,...,v_n\}}$ has only one more edge than $G_{\{v_2,...,v_n\}}$ we can take $\hat{G}" = G'_{\{v_2,...,v_n\}}$. We now construct $G"$ by adding to $V - \{v_1\}$ the vertex v_1 and to $\hat{E}"$ all the edges in G having v_1 as an endpoint. Taking a perfect scheme $[w_2, ..., w_n]$ of $\hat{G}"$, remark that $[v_1, w_2, ..., w_n]$ is a perfect scheme of $G"$, and so $G"$ is a chordal graph satisfying the required properties.

B. $G_{\{v_2,...,v_n\}} = G'_{\{v_2,...,v_n\}}$. In this case, by Dirac's Lemma 1.5 there exists one more simplicial vertex in G starting another perfect scheme of G satisfying condition A. □

First, we obtain an extension of Lemma 4 in [38].

PROPOSITION 5.14. *Let two chordal graphs $G = (V, E)$ and $G' = (V, E')$ with $E \subset E', E \neq E'$ be given. Then there exists a sequence of chordal graphs $G = G_0, G_1, ..., G_s = G'$ such that G_j is obtained by adding exactly one edge to G_{j-1}, for all $j = 1, ..., s$.*

Proof. This a consequence of Lemma 5.13. □

In analogy with the case $G' = K_n$, a sequence of chordal graphs satisfying the requirement of Proposition 5.13 is called a *chordal sequence connecting G to G'*.

Finally, let $R' = (r'_{ij})_{i,j=1}^n$ be a partial positive matrix, and $R = (r_{ij})_{i,j=1}^n$ a partial submatrix of R', i.e. R is a partial matrix in its turn, but having more unspecified entries than R', and $r_{ij} = r'_{ij}$ for the specified entries. Let $G' = (V, E')$ and $G = (V, E)$ be the associated graphs of R' and R respectively, which are supposed to be chordal. Then $E \subset E'$ and $E \neq E'$.

Take a chordal sequence $G = G_0, G_1, ..., G_s = G'$ connecting G to G', and let $G' = G'_0, G'_1, ..., G'_t = K_n$ be a chordal sequence of G'. Then $G = G_0, G_1, ..., G_s = G" = G' = G'_0, ..., G'_t = K_n$ is a chordal sequence of G.

Fix a positive definite completion F of R'. Of course, F can also be viewed as a positive definite completion of R. So, let $\{g'(u_j, v_j) | j = 1, ..., t\}$ be the parameters of F along the chordal sequence $G'_0, ..., G'_t$ of G', and $\{g(u_j, v_j) | j = 1, ..., s + t\}$ be the parameters of F along the chordal sequence $G_0, G_1, ..., G_s, G'_1, ..., G'_t$ of G. The next result establishes the connection between $D(R)$ and $D(R')$.

PROPOSITION 5.15. *With the above notation, the product*

$$\prod_{j=1}^{s} [1 - |g(u_j, v_j)|^2]$$

is the same for any chordal sequence connecting G to G' and any positive definite completion F of R, and

$$D(R') = \prod_{j=1}^{s}[1 - g(u_j, v_j)|^2]D(R). \tag{5.27}$$

Proof. In view of the dependence on the parameters in Proposition 3.2, the numbers $g(u_j, v_j)$, $j = 1, ..., s$, do not depend on the chosen positive definite completion F of R. By the same remark, we have that $g'(u_j, v_j) = g(u_j, v_j)$ for $j > s$. Then, by Theorem 5.11,

$$detF = D(F) = \prod_{j=1}^{t}[1 - g(u_j, v_j)|^2]D(R').$$

and

$$D(F) = \prod_{j=1}^{s+t}[1 - g(u_j, v_j)|^2]D(R).$$

Consequently, the formula (5.27) holds and the product

$$\prod_{j=1}^{s}[1 - g(u_j, v_j)|^2].$$

does not depend on the chosen chordal sequence connecting G and G'. □

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