1994

A relativistic model of pion nucleon scattering and pion photoproduction on a single nucleon

Yohanes Surya

College of William & Mary - Arts & Sciences

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A RELATIVISTIC MODEL OF PION NUCLEON SCATTERING AND PION PHOTOPRODUCTION ON A SINGLE NUCLEON

A Dissertation

Presented to
The Faculty of the Department of Physics
The College of William and Mary in Virginia

In partial fulfillment of requirements for the Degree of
Doctor of Philosophy

by
Yohanes Surya

1994
APPROVAL SHEET

This dissertation is submitted in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

Yohanes Surya

Approved, April 1994

Prof. Franz L. Gross
Prof. John D. Walecka
Prof. Carl Carlson
Prof. Charles F. Perdrisat
Prof. J. Wallace Van Orden
Dedicated to my wife Christina and my daughter Chrisanthy

*I can do all things through Christ which strengtheneth me*

Phil. 4:13
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>viii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>ix</td>
</tr>
<tr>
<td>NOTATION AND CONVENTIONS</td>
<td>xii</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>xv</td>
</tr>
<tr>
<td>PART A: PION NUCLEON SCATTERING</td>
<td>1</td>
</tr>
<tr>
<td>CHAPTER I. OVERVIEW</td>
<td>2</td>
</tr>
<tr>
<td>I.1) Introduction</td>
<td>2</td>
</tr>
<tr>
<td>I.2) Restriction the Pion to Its Mass-shell</td>
<td>7</td>
</tr>
<tr>
<td>I.3) The $\pi NN$ Coupling</td>
<td>10</td>
</tr>
<tr>
<td>I.4) Nucleon Self Energy</td>
<td>15</td>
</tr>
<tr>
<td>I.5) Description of Spin $\frac{3}{2}$ Particles</td>
<td>18</td>
</tr>
<tr>
<td>I.6) The Roper</td>
<td>19</td>
</tr>
<tr>
<td>CHAPTER II. GENERAL THEORY</td>
<td>21</td>
</tr>
<tr>
<td>II.1) Why Should the Pion be On-shell?</td>
<td>21</td>
</tr>
<tr>
<td>II.2) Integral Equation</td>
<td>30</td>
</tr>
<tr>
<td>II.3) Unitarity</td>
<td>34</td>
</tr>
<tr>
<td>CHAPTER III. MODEL</td>
<td>36</td>
</tr>
<tr>
<td>III.1) Relativistic Contact Terms</td>
<td>36</td>
</tr>
<tr>
<td>III.2) Solving the Integral Equation for $M_c$</td>
<td>40</td>
</tr>
<tr>
<td>III.3) Form Factors</td>
<td>41</td>
</tr>
<tr>
<td>III.4) Treatment of the Coupled $N N^*$ System</td>
<td>42</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>III.5) Treatment of the $\Delta$</td>
<td>44</td>
</tr>
<tr>
<td>III.6) Treatment of the $D_{13}$</td>
<td>46</td>
</tr>
<tr>
<td>III.7) Inelastic Channels</td>
<td>47</td>
</tr>
<tr>
<td>CHAPTER IV. RESULTS AND DISCUSSIONS</td>
<td>49</td>
</tr>
<tr>
<td>CHAPTER V. CONCLUSIONS</td>
<td>65</td>
</tr>
<tr>
<td>PART B: PION PHOTO PRODUCTION</td>
<td>67</td>
</tr>
<tr>
<td>CHAPTER VI. OVERVIEW</td>
<td>68</td>
</tr>
<tr>
<td>VI.1) Introduction</td>
<td>68</td>
</tr>
<tr>
<td>VI.2) The $E2/M1$ Ratio</td>
<td>70</td>
</tr>
<tr>
<td>VI.3) Low Energy Theorem</td>
<td>73</td>
</tr>
<tr>
<td>VI.4) Unitarity</td>
<td>74</td>
</tr>
<tr>
<td>VI.5) Gauge Invariance</td>
<td>76</td>
</tr>
<tr>
<td>CHAPTER VII. GENERAL THEORY</td>
<td>78</td>
</tr>
<tr>
<td>VII.1) Integral Equation</td>
<td>78</td>
</tr>
<tr>
<td>VII.2) Unitarity</td>
<td>81</td>
</tr>
<tr>
<td>VII.3) Introduction to the Model</td>
<td>82</td>
</tr>
<tr>
<td>VII.4) Gauge Invariance</td>
<td>86</td>
</tr>
<tr>
<td>CHAPTER VIII. THE PHOTOPRODUCTION MODEL</td>
<td>95</td>
</tr>
<tr>
<td>VIII.1) Electromagnetic Couplings</td>
<td>95</td>
</tr>
<tr>
<td>VIII.1.a) Delta Current</td>
<td>96</td>
</tr>
<tr>
<td>VIII.1.b) Roper Current</td>
<td>98</td>
</tr>
<tr>
<td>VIII.1.c) $D_{13}$ Current</td>
<td>98</td>
</tr>
<tr>
<td>VIII.2) Off-shell Electromagnetic Currents</td>
<td>99</td>
</tr>
<tr>
<td>VIII.2.a) Nucleon Current</td>
<td>99</td>
</tr>
<tr>
<td>VIII.2.b) Pion Current</td>
<td>101</td>
</tr>
<tr>
<td>VIII.2.c) Resonances</td>
<td>101</td>
</tr>
</tbody>
</table>
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LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Parameters of the Pion-Nucleon Model</td>
<td>50</td>
</tr>
<tr>
<td>2.</td>
<td>Parameters of the Pion-Photoproduction Model</td>
<td>115</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Born (Driving) terms of pion-nucleon scattering.</td>
<td>4</td>
</tr>
<tr>
<td>2.</td>
<td>Diagrammatic representation of the integral equation</td>
<td>8</td>
</tr>
<tr>
<td>3.</td>
<td>S-wave $\pi - N$ scattering lengths.</td>
<td>13</td>
</tr>
<tr>
<td>4.</td>
<td>Nucleon self-energy.</td>
<td>15</td>
</tr>
<tr>
<td>5.</td>
<td>Infinite subclass of diagrams not included in our model.</td>
<td>16</td>
</tr>
<tr>
<td>6.</td>
<td>Self-energy of the nucleon (at the nucleon pole) as a function of $\lambda$.</td>
<td>17</td>
</tr>
<tr>
<td>7.</td>
<td>Feynman diagrams for $\pi N$ scattering</td>
<td>22</td>
</tr>
<tr>
<td>8.</td>
<td>Feynman diagrams for the scattering of two heavy particles.</td>
<td>24</td>
</tr>
<tr>
<td>9.</td>
<td>Pole structure of the diagrams in Fig.7.</td>
<td>25</td>
</tr>
<tr>
<td>10.</td>
<td>Pole structure of the diagrams in Fig.8.</td>
<td>28</td>
</tr>
<tr>
<td>11.</td>
<td>Diagrammatically representation of the equation.</td>
<td>32</td>
</tr>
<tr>
<td>12.</td>
<td>Dressed vertex, dressed propagator and self energy.</td>
<td>33</td>
</tr>
<tr>
<td>13.</td>
<td>The justification of our approximation.</td>
<td>38</td>
</tr>
</tbody>
</table>
14. Inelastic Channels.................................................. 48

15. Fits to the $S_{11}$ phase shift.................................. 52

16. Fits to the $S_{31}$ phase shift................................. 53

17. Fits to the isospin 1/2 $P-$ wave phase shifts............. 54

18. $P_{11}$ inelasticity parameter................................ 55

19. Fits to the isospin 3/2 $P-$wave phase shifts............. 56

20. Fits to the $D_{13}$ phase shift............................... 57

21. $D_{13}$ inelasticity parameter................................. 58

22. Fits to the $\pi^-p$ total cross section..................... 59

23. Fits to the $\pi^+p$ total cross section..................... 61

24. Form factors for the Nucleon................................. 63

25. Form factors for the Nucleon, Roper, $\Delta$ and $D_{13}$.. 63

26. Driving terms of pion-photoproduction..................... 71

27. Final state interactions....................................... 83

28. Contact and Vertex Interaction............................. 102
29. Inelastic Channels ..................................................... 113
30. Fits to the real part of $E_{0+}(I = 1/2)$ amplitudes. .............. 115
31. Fits to the real part of $E_{0+}(I = 3/2)$ amplitudes. .............. 116
32. Fits to the real part of $M_{1-}(I = 1/2)$ amplitudes. .............. 117
33. Fits to the real part of $M_{1-}(I = 3/2)$ amplitudes. .............. 118
34. Fits to the real part of $M_{1+}(I = 3/2)$ amplitudes. .............. 120
35. Fits to the real part of $E_{1+}(I = 3/2)$ amplitudes. .............. 121
36. Fits to the real part of $M_{2-}(I = 1/2)$ amplitudes. .............. 122
37. Fits to the real part of $E_{2-}(I = 1/2)$ amplitudes. .............. 123
38. Comparison of our $E_{0+}(I = 1/2$ and $I = 3/2)$ to SAID analysis. ... 125
39. Comparison of our $M_{1-}(I = 1/2$ and $I = 3/2)$ to SAID analysis. .. 126
40. Comparison of our $E_{1+}(I = 3/2)$ & $M_{1+}(I = 3/2)$ to SAID analysis. 127
41. Comparison of our $E_{2-}(I = 1/2)$ & $M_{2-}(I = 1/2)$ to SAID analysis. 128

xi
NOTATION AND CONVENTIONS

1. Vectors, Tensors and Related Definitions

In the natural system of units, the speed of light c is equal to unity so that the spacetime four vector can be denoted in terms of contravariant and covariant tensors. The contravariant components of a four-vector will be indicated, with an upper index,

\[ x^\mu = (x^0, x^1, x^2, x^3) \equiv (t, \mathbf{x}) . \]

And the covariant components of a four-vector will be indicated with a lower index,

\[ x_\mu = g_{\mu\nu} x^\nu \equiv (t, -\mathbf{x}) , \]

with the metric tensor \( g_{\mu\nu} \)

\[ g_{00} = 1 , \quad g_{ij} = -1 \quad (j = 1, 2, 3) ; \]
\[ g_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu ; \]
\[ g_{\mu\nu} = g^{\mu\nu} , \quad g^\nu = \delta^\nu_\mu . \]

The gradient operator is

\[ \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left( \frac{\partial}{\partial t}, -\nabla \right) , \]
\[ \partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial t}, \nabla \right) , \]

where \( \nabla \) is the gradient in the three-dimensional Euclidean space

\[ \nabla = (\partial_x, \partial_y, \partial_z) . \]

In manipulating three-vectors and tensors, we use the Kronecker \( \delta_{ij} \) function and the totally antisymmetric Ricci-Levi-Civita \( \epsilon_{ijk} \) tensor which is defined through

\[ \epsilon_{123} \equiv 1 , \]

xii
Useful identities are:

\[ \epsilon_{ijk} \epsilon_{ljk} = 2 \delta_{il} \]

\[ \epsilon_{ijk} \epsilon_{lmk} = \delta_{ij} \delta_{jm} - \delta_{im} \delta_{jl} \]

In four-dimensional Minkowski space, the totally antisymmetric \( \epsilon_{\mu\nu\rho} \) is defined by

\[ \epsilon_{\alpha\beta\gamma} \equiv \epsilon_{ijk} \cdot \]

Useful identities are

\[ \epsilon_{\mu\nu\lambda\delta} \epsilon^{\mu\nu\lambda\delta} = 24 \]

\[ \epsilon_{\mu\nu\lambda\delta} \epsilon_{\mu'\nu'\lambda'\delta'} = -\text{det}(g_{\alpha\alpha'}) \]

where

\[ \alpha = \mu, \nu, \lambda, \delta \]

and

\[ \alpha' = \mu', \nu', \lambda', \delta' \]

II. Dirac matrices

The 4 x 4 Dirac matrices representation, \( \gamma^\mu = (\gamma^0, \vec{\gamma}) \), used in this dissertation

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \]

where each element in the above expression is a 2x2 matrix and \( \vec{\sigma} \) are the Pauli matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

which satisfy the angular momentum algebra relations

\[ [\sigma_i, \sigma_j] = 2i\epsilon^{ijk}\sigma_k, \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}. \]
The Dirac matrices satisfy the following anti commutation relation:

\[ \{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \]

Antisymmetric combinations of the Dirac matrices define the \(\sigma_{\mu\nu}\) matrices:

\[ \sigma_{\mu\nu} \equiv \frac{1}{2}i[\gamma_\mu, \gamma_\nu]. \]

The chirality operator, \(\gamma^5\) matrix is defined through the relation

\[ \gamma^5 = \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \]

The "slash" of a 4-vector \(p^\mu\) is defined by

\[ p^' \equiv \gamma_\mu p^\mu. \]
A RELATIVISTIC MODEL OF PION NUCLEON SCATTERING AND
PION PHOTOPRODUCTION ON A SINGLE NUCLEON

ABSTRACT

Pion nucleon scattering is described by a manifestly covariant wave equation in which
the pion is restricted to its mass-shell. The kernel of the equation includes nucleon
($N$), Roper ($N^*$), delta ($\Delta$), and $D_{13}$ poles, with their corresponding crossed pole
terms approximated by contact interactions, and contact $\sigma$- and $\rho$-like exchange
terms. The $\pi NN$ vertex is treated as a mixture of $\gamma^5$ and $\gamma^\mu\gamma^5$ coupling, with
a mixing parameter $\lambda$ chosen so that the dressed nucleon pole will be unshifted
by the interaction. Chiral symmetry is maintained at threshold. The resonance
contributions are fully unitarized by the equation, with their widths determined by
the dynamics included in the model. The $\Delta$ and $D_{13}$ are treated as a pure spin 3/2
particles, with no spin 1/2 amplitude in the $S$-channel. Pion photoproduction is also
described by a manifestly covariant wave equation, which includes a treatment of the
final state $\pi N$ interactions consistent with the covariant, unitary, resonance model
of $\pi N$ scattering. The model is exactly gauge invariant to all orders in the strong
coupling, $g_{\pi NN}$, and satisfies the Low Energy Theorem. Unitarity is maintained up
to first order in the charge $e$ (Watson theorem). The complete development of the
model which gives a good fit to all the data up to 770 MeV photon energy lab, is
presented.

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DEPARTMENT OF PHYSICS
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xv
PART A:
PION NUCLEON SCATTERING
I OVERVIEW

1.1 Introduction

Pion-nucleon scattering has been studied thoroughly for many years. One of the best known early models, which treated the nucleon non-relativistically, was introduced by Chew and Low in 1956 [1]. This model described low energy $P$-wave scattering very well, but had to be modified in order to describe $S$-wave scattering [2]. Among later efforts is the work based in current algebra [3, 4], and Lagrangian models based on chiral symmetry [5, 6, 7]. More recently, Banerjee and Cammarata [8] have extended the Chew-Low model to include nucleon recoil and anti-nucleon contributions, and Pearce and Jennings [9], using a Lagrangian model and a relativistic wave equation, have extended the analysis up to pion laboratory kinetic energies of 400 MeV.

However, with the construction of powerful new facilities such as the Continuous Electron Beam Accelerator Facility (CEBAF), it is necessary to have a good description of $\pi N$ scattering which extends up to higher energies. Such a description must be covariant, and include not only the nucleon ($N$) and delta ($\Delta$) resonances, but also the Roper ($N^*$), which plays a prominent role in the $P_{11}$ channel at energies above 400 MeV, and the $D_{13}(1520)$, which makes a significant contribution to the total isospin $\frac{1}{2}$ cross section near 600 MeV.

In this dissertation (Part A) I present a simple, covariant, and unitary model
of $\pi N$ scattering which works well up to 600 MeV. These are essential features of a $\pi N$ model which is to be used as a reliable input to other model calculations, such as the calculation of $NN$ scattering up to nucleon laboratory kinetic energies of 1 GeV, where the excitations of pion degrees of freedom become important. It is with such applications in mind that this model has been developed.

In this work the $\pi N$ scattering amplitude is obtained as a solution of a relativistic wave equation in which the pion is restricted to its mass-shell in all intermediate states. The rationale for this approach is described in Chapter II.1. In order to describe the $\pi N$ resonances at $T_{\pi}^{\text{lab}} \sim 187$ MeV, $\sim 485$ MeV, and $\sim 611$ MeV, the kernel (sometimes referred to as Born or "driving" terms) of the relativistic integral equation includes undressed $\Delta$, $N^*$, and $D_{13}$ poles in addition to the undressed nucleon pole. We make no attempt to explain these bare, undressed states within the model; they are presumably explained by quark models in the same way that the nucleon state is explained. The kernel also includes contributions derived from crossed $N$, $\Delta$, $N^*$, and $D_{13}$ diagrams, and from $\sigma$- and $\rho$-like exchange terms. To simplify the equation, and obtain analytic solutions, the latter terms are approximated by a contact interaction, as described in Chapter III. The approximations used to obtain the contact terms give zero for the crossed $\Delta$ and $D_{13}$ poles. All of these driving terms are shown diagrammatically in Fig. 1. The solution which emerges from the integral equation automatically satisfies unitarity, and dresses the resonance poles by shifting their masses and giving them a width, and hence our model complements quark models by adding the pion interactions sometimes omitted from such models. For reasons which we will discuss in some detail below, we adjust the parameters of the model so that the dressed nucleon pole is not shifted by the interaction.
Figure 1: Born terms which make up the kernel of the integral equation used in this paper. The diagrams in box (h) are eventually approximated by a contact term, shown in (i).
For the $\pi NN$ coupling we use a superposition of both pseudoscalar ($\gamma^5$) and pseudovector ($\gamma^5 \gamma^\mu$) coupling, with a mixing parameter $\lambda$ defined so that the coupling is independent of $\lambda$ when both the incoming and outgoing nucleon are on-shell. This mixed coupling was used successfully in one boson exchange models of $NN$ scattering by Gross, Van Orden and Holinde [10, 11], who found that a small admixture (about 22%) of $\gamma^5$ coupling made it possible to fit the data with a minimum of exchanged mesons. One of the purposes of this study was to see if this mixed coupling had any justification within the framework of $\pi N$ scattering.

The $\Delta$ and $D_{13}$ are treated as pure spin 3/2 particles, using a spin 3/2 projection operator proposed by Behrends and Fronsdal [12], and recently discussed by Williams [13], and their widths emerge automatically as a dynamical consequence of unitarity. We also introduce a new form for the $\pi N\Delta$ and $\pi ND_{13}$ vertices. The combination of the spin 3/2 projection operator and this new vertex not only makes the calculation simpler but also eliminates all spin 1/2 amplitudes. Some authors [14] have argued that these virtual spin 1/2 amplitudes, which must be present if the spin 3/2 propagator is to have an inverse, are an important feature of the physics. We obtain a very successful fit without them. The $D_{13}$ is inelastic, and in this model we allow for this by coupling the $D_{13}$ to the $\pi\Delta$ channel, which gives an excellent description of the data.

The role of the Roper, especially at low energies, has been questioned for many years. Many authors [9, 15, 16] do not include the Roper, even in their description of the $P_{11}$ channel. They argue that a cancellation between the direct and crossed $N$ pole terms can explain the unique behavior of $P_{11}$ partial wave, which is negative at low energy and changes sign at $T_{\pi}$ lab \(\sim 100\) MeV. Oset et. al. [17] argue that
the change of sign is due to the cancellation of the \( N \) and Roper pole terms, but they treated the Roper only at the tree level. In this work we include \( N^* \leftrightarrow N^* \) and \( N^* \leftrightarrow N \) mixing to all orders, our result is unitary, and the Roper width emerges as a natural consequence of the dynamics. We also include the inelastic coupling of the Roper to the \( \pi\Delta \) channel, which is its dominant inelastic decay mode [18].

We conclude this brief introduction by summarizing the novel features of this model, which to our knowledge have not been studied before in the context of \( \pi N \) scattering:

1. the scattering amplitude is the solution of a relativistic wave equation in which the pion is restricted to its mass shell in all intermediate states;
2. the \( \pi N N \) coupling is taken to be a superposition of both pseudoscalar (\( \gamma^5 \)) and pseudovector (\( \gamma^5 \gamma^\mu \)) coupling;
3. the nucleon self energy is constrained to be zero at the nucleon pole, so that the nucleon mass remains unshifted by the interactions;
4. the \( \Delta \) and \( D_{13} \) are pure spin \( \frac{3}{2} \) particles, with widths which develop naturally from the unitarity of the solution; and
5. contributions from the Roper (\( N^* \)) and \( N^* \leftrightarrow N \) transition amplitudes are iterated to all orders, giving a consistent description of the Roper and its width.

These new features are discussed in the following sections of this introductory Chapter I. Chapter II, General Theory, presents the relativistic formalism including the
partial wave expansion and a discussion of unitarity. The construction and the development of the relativistic kernel are presented in Chapter III where the treatment of the Roper ($N^*$), $\Delta$, and $D_{13}$ is described in some detail. Chapter IV presents the numerical results, discussion and conclusions. The appendices discuss some technical points.

I.2 Restricting the Pion to Its Mass-shell

One way to insure that a scattering amplitude is both covariant and unitary is to obtain it as a solution of a covariant integral equation. Solving the equation automatically iterates its kernel to all orders, and gives a unitary amplitude.

The Bethe-Salpeter (BS) equation is one possible starting point for a relativistic description of $\pi N$ scattering. If the $\pi N$ scattering matrix is $M_{ji}^{\pi}$, then the BS equation is

$$M_{ji}^{\pi}(k',k,P) = v_{ji}^{\pi}(k',k,P) + i \int \frac{d^4k''}{(2\pi)^4} V_{ji}^{\pi}(k',k'',P) G_{BS}(k'',P) M_{li}^{\pi}(k'',k,P)$$

(I.1)

where $V_{ji}^{\pi}(k',k,P)$ is the relativistic kernel, $i$ and $j$ are the isospin of incoming and outgoing pion, $G_{BS}(k'',P)$ is the free relativistic two particle Green's function (two-body propagator) and $k, k', k''$ and $P$ are the four-momenta of the incoming pion, outgoing pion, intermediate pion and the total four-momentum of the system, as shown in Fig. 2. The integration is over all four components of the pion four-momentum, and for this reason the equation is described as a "four-dimensional" equation. The exact $\pi N$ scattering amplitude can be obtained from the BS equation only if its kernel includes the sum of all connected two particle irreducible diagrams.
There is an infinite number of these, and nobody knows how to sum them, so that the kernel must be approximated.

One approximation is to introduce a separable interaction. In Refs. [19, 20] this approach was used to parametrize the $S$ and $P$-wave phase shifts, with a different set of parameters for each phase shift. This worked well, but the parameters have no physical interpretation, and it is difficult to relate them to masses and coupling constants.

Since the kernel must be approximated (by using a few diagrams that we believe to be especially important physically) there is not necessarily any reason to retain the full four-dimensional BS equation. There are several covariant three-dimensional equations [21] which can be used, and the choice depends on the physics and on the approximations being made. Recently Pearce and Jennings [9] used what they refer to as a smooth propagator [22] to describe $\pi N$ scattering. They replace the two-body propagator of the BS equation,

$$G_{BS}(k, P) = \frac{m + P - k}{(\mu^2 - k^2 - i\epsilon)(m^2 - (P - k)^2 - i\epsilon)}$$  \hspace{1cm} (I.2)
by the propagator [22],

\[ G_{sm}(k, P) = \frac{2\pi \delta(\omega(W) - k_0)}{2W} \left( \frac{m + \gamma^0 E(W) + \gamma \cdot k}{m^2 + k^2 - E^2(W) - i\epsilon} \right) \]  

where \(\mu\) and \(m\) are the mass of the pion and the nucleon, respectively, \(W\) is the total energy in the cm system, and \(E(W)\) and \(\omega(W)\) are the energies of the nucleon and pion when both are on-shell. They derived this propagator by letting the mass of the nucleon become infinitely heavy and eliminating the short range structure from their relativistic kernel. They obtained a pretty good fit to the phase shifts up to 400 MeV by using 16 parameters.

Our approach follows from the examination of the singularities of a typical Feynman diagram which the equation will iterate, and study of these diagrams is carried out in detail in Chapter II.1. We are led to conclude that the most accurate method of summing the diagrams is to put the pion on its mass-shell. Since the pion is the light particle, and previous studies of scattering in which a light meson is exchanged between two heavy particles of masses \(m_1 \geq m_2\) led to the conclusion that the heaviest particle \((m_1)\) should be on-shell [23, 24], the new result seems surprising at first glance, and we also explain in Chapter II.1 why a different conclusion is reached for the \(\pi N\) system. The propagator we obtain can be written

\[ G_\pi(k, P) = \frac{2\pi \delta(\omega_k - k_0)}{2\omega_k} \left( \frac{m + \gamma^0 (W - \omega_k) + \gamma \cdot k}{E_k^2 - (W - \omega_k)^2 - i\epsilon} \right) \]  

where \(\omega_k = \sqrt{\mu^2 + k^2}\) is the on-shell energy of the pion. Not only is this propagator efficient in summing the relevant Feynman diagrams, but it also suggests some nice approximations for the relativistic kernel, as will be discussed in Chapter III.

To insure convergence of the integral equation, we multiply all of the driving terms by form factors. Since the pions are on-shell, the form factors will depend
only on the virtual mass (squared) of the off-shell incoming and outgoing baryons (the \( N, \Delta, D_{13}, \) and the \( N^* \)). For example, we attach a universal function \( f_N(p^2) \) to each off-shell nucleon entering or leaving any vertex, so the form factor for the \( \pi NN \) vertex automatically assumes a factorized form \( f_N(p^2) f_N(p'^2) \), where \( p \) and \( p' \) are the four-momenta of the incoming and outgoing nucleon, respectively. For clarity, we will defer all further discussion of the details of the definition of the form factors and the construction of the kernels to Chapter III.

### 1.3 The \( \pi NN \) Coupling

It is well known that in a model in which pions interact with nucleons which are on-shell, the pseudoscalar and pseudovector \( \pi NN \) coupling give the same results [25]. When the nucleon is off-shell this is no longer true, and the results may depend on which coupling is used. In some early pertubative calculations based on lowest order Feynman diagrams, \( \gamma^5 \) coupling was used because this coupling is renormalizable [26]. However, the use of \( \gamma^5 \) coupling for the nucleon Born terms gives an incorrect result for \( a_+ \), the \( \pi N \) isospin-even scattering length. This failure is associated with the fact that \( \gamma^5 \) coupling violates chiral symmetry unless it is accompanied by a \( \sigma \) exchange term with precisely the correct strength, as described (for example) by the linear \( \sigma \)-model introduced by Gell-Mann and Levy [27] in 1960. The Born terms in this model include the exchange of a \( \sigma \) particle with precisely the correct strength to give good predictions in the soft pion limit. This model was further improved by Weinberg [5] and others [28], who eliminated the \( \sigma \) and developed non-linear chiral Lagrangians. Models based on these Lagrangians give a good description of \( \pi N \) scattering in the soft pion limit without explicit reference to a \( \sigma \) particle. One
form of these effective Lagrangians replaces the pseudoscalar coupling and effective
σ term with a pseudovector coupling and a ρ term. Since then, some people have
preferred to use pseudovector coupling to describe πN scattering [6, 9].

However, if one is careful to include the correct sigma term (which need not
be a real σ exchange, but could be a σ-like ππNN contact term), then it is still
possible to use γ⁵ coupling. Furthermore, one can show that a coupling consisting of
a mixture of pseudoscalar and pseudovector, with a corresponding mixture of σ-like
and ρ-like contact terms, is completely equivalent in Born approximation to either
γ⁵ or γ⁵γμ coupling alone. Specifically, consider a mixed πNN coupling of the form

\[ g \tau_i \left[ \lambda \gamma^5 - (1 - \lambda) \frac{(p - p') \cdot \gamma^5}{2m} \right], \tag{I.5} \]

where \( p \) and \( p' \) are the four-momenta of the initial and final nucleon, respectively, \( i \)
is the isospin index for the pion, and \( \lambda \) is the mixing parameter. (The vertex also
includes nucleon form factors, omitted here for simplicity; see Chapter III.) In this
form, we can easily see that when \( \lambda \) is zero the coupling is purely pseudovector and
when it is unity the coupling is pure pseudoscalar, and also that the coupling will
be independent of \( \lambda \) if both initial and final nucleons are on-shell. Next, consider a
ππNN contact term of the form

\[ -C \frac{g^2}{m} \left[ \lambda^2 \delta_{ij} + (1 - \lambda)^2 [\gamma_j, \tau_i] \frac{Q}{4m} \right], \tag{I.6} \]

where \( C \) is a strength parameter, \( Q = \frac{1}{2}(k + k') \), and \( k, i \) and \( k', j \) the four-momenta
and isospin indices of the incoming and outgoing pion, respectively. If the contact
term (I.6) is added to the nucleon Born terms (the \( N \) and crossed \( N \) pole terms
shown in Fig. 1(a) and (e)) computed from the coupling (I.5), the resulting πN
amplitude is independent of \( \lambda \) if the external nucleons are on-shell, and \( C = 1. \)
This comes about because the contact term in Eq. (I.6) also depends on the mixing
parameter \( \lambda \). It is pure \( \sigma \)-like if the \( \pi N \) coupling is pure \( \gamma^5 \) (corresponding to \( \lambda = 1 \)) and pure \( \rho \)-like if the \( \pi N \) coupling is pure \( \gamma^5 \gamma^\mu \) (corresponding to \( \lambda = 0 \)), and these contributions are just what is needed to cancel the \( \lambda \) dependence which arises from the nucleon Born terms, giving amplitudes independent of \( \lambda \). However, if these amplitudes are used as driving terms in an integral equation in which the nucleons are off-shell, they will no longer give identical results, and it is natural to ask whether a particular choice of \( \lambda \) is favored by the physics. It was found recently [10, 11] that relativistic \( NN \) scattering, in a formalism in which one nucleon is off-shell, is very sensitive to the mixing parameter \( \lambda \), and that a very good fit to \( NN \) data could be obtained using a one boson exchange (OBE) model with only the four mesons \( \pi, \sigma, \rho \) and \( \omega \), provided the \( \pi N \) coupling included an admixture of 22\% \( \gamma^5 \) coupling. Furthermore, this admixture of pseudoscalar coupling also gives a good description of the \( p^{40}Ca \) spin observables [29]. And recently Goudsmit et. al. [30] analyzed \( \pi N \) scattering at the tree level and found that an admixture of about 24\% \( \gamma^5 \) gives a good description of the scattering lengths. They obtained this value of the admixture from their analysis of data on pionic atoms with isoscalar nuclei using a relativistic mean field theory. To get a feeling for the dependence of our model on the parameter \( \lambda \), we plot the isospin-even scattering length, \( a_+ \), versus the isospin-odd scattering length, \( a_- \), in Fig. 3. For convenience, both scattering lengths have been made dimensionless by multiplying by the pion mass \( \mu \). Three analyses of the experimental results, labeled I [31], II [32], and III [33], are shown in the figure. The dashed line (almost overlaps to the solid line) shows how the scattering lengths, as calculated from the nucleon Born terms only (Figs. 1a and 1e) vary with the mixing parameter \( \lambda \). Note that the curve comes closest to the data if \( \lambda \approx 0.2 \). The solid line gives the dependence of the scattering lengths on \( \lambda \) when the nucleon Born terms are used for the kernel of our \( \pi N \) integral equation,
Figure 3: S-wave $\pi - N$ scattering lengths. The dots on the curves mark values of $\lambda$ incremented by 0.1, with $\lambda = 0$ at the top left end of the curves.
and we see that the iteration of the Born terms by the equation produces negligible effects. From this figure it is clear that if we use a kernel with nucleon Born terms only, pure pseudoscalar coupling ($\lambda = 1$) or pure pseudovector coupling ($\lambda = 0$) will not give as good a simultaneous description of the even and odd scattering lengths as a choice $\lambda \sim 0.22$. Since the OBE model of Ref. [10, 11] is most consistent with a model of $\pi N$ scattering based only on the nucleon Born terms, this result may partially explain why the result $\lambda \cong 0.22$ was obtained.

Next, consider a slightly more complete model in which the driving terms of the integral equation include the contact terms of Eq. (1.6), in addition to the nucleon Born terms. Now the Born term result for the scattering lengths is independent of $\lambda$, but it turns out that the scattering lengths obtained by solving the integral equation (the cross in Fig. 3) are also (nearly) independent of $\lambda$. Finally, the scattering lengths obtained from the solution of the integral equation with the full kernel, including all the terms shown in Fig. 1, also does not depend very much on $\lambda$. These results are represented by the star in Fig. 3. We therefore conclude that the scattering lengths (and most of the low energy observables) will be insensitive to $\lambda$ if the kernel is chirally symmetric.

Does it follow, therefore, that the mixing parameter $\lambda$ plays no role in the description of $\pi N$ scattering? To the level of sophistication to which we are working, this is not the case. To see where the dependence on $\lambda$ reappears, consider the nucleon self energy, $\Sigma(p)$ that will be discussed next.
Figure 4: Diagrams (excluding Roper contributions) which contribute to the nucleon self-energy in this model. (a) Bubble diagram, (b) contributions from connected pieces, with $M_{\pi ji}$ given by the iteration, or integral equation shown in (c).

### I.4 Nucleon Self Energy

If the Roper contributions are omitted for simplicity (they are discussed in detail in Chapter III.4) our integral equation produces a nucleon self energy which can be written diagrammatically as shown in Fig. 4. The elementary pion-nucleon bubble diagram is shown in Fig. 4a, and all other contributions which come from iterating the contact terms are shown in Fig 4b. [The integral equation for the unitarized contact amplitude, $M_{\pi ji}$, is shown in Fig. 4c.] Now the contributions shown in Fig. 4 include much physics, but also leave out many processes. An infinite class of diagrams excluded from our model are shown in Fig. 5. As the figure shows, this class could also be summed by the integral equation shown diagrammatically in Fig. 5d. We can allow for these contributions *approximately* if we demand that,
Figure 5: Infinite subclass of diagrams not included in our model. (a) is included, but its iterations (b), (c),..., are not included. All of these diagrams are summed by the integral equation shown in (d).

At the nucleon pole, the nucleon self energy not be shifted by the interactions. This requirement means that the infinite family of interactions is, in effect, included automatically, at least near the pole. It also means that the addition of the nucleon self energy to a model with bare nucleons will produce the minimum effect possible, meaning that the model is fairly stable under changes in the dynamics. We will impose this requirement on our model, and will refer to it as the stability condition.

At the nucleon pole the nucleon self energy is only a function of the parameters of the model, and its dependence on the parameter $\lambda$ (with the others held fixed) is shown in Fig. 6. Note that it is zero for $\lambda \approx 0.20$, and our stability condition is therefore realized practically as a constraint on the parameter $\lambda$. Note that this constraint yields a value for $\lambda$ which is in rough agreement with the value required to simultaneously minimize the error in the scattering lengths $a_+$ and $a_-$ obtained from the naive model which used only the nucleon poles as the driving term in the integral equation (recall the results shown in Fig. 3 and the subsequent discussion).

We do not believe that this is an accident; the value of $\lambda \approx 0.20$ which seems to
Figure 6: Self-energy of the nucleon (at the nucleon pole) as a function of $\lambda$. The two curves which are practically indistinguishable are the contribution of the bubble diagram only (dashed line) and the total result.
stabilize the model should also give the best physical approximation in situations where the model is incomplete.

Before leaving this section, we wish to emphasize that the stability condition can only be satisfied if a mixed coupling is used, and that it is almost completely determined by the bubble diagram shown in Fig. 4a. For pure pseudoscalar coupling, the self energy is positive definite, while for pure pseudovector coupling it is negative definite, so that only the parameter $\lambda$ can be determined from the stability condition.

I.5 Description of Spin $\frac{3}{2}$ Particles

It is well known that the $\Delta(1232)$-isobar plays an important role in describing interactions involving nucleons, and there are many works which include this resonance. However, in spite of the number of papers which have studied this particle, is still some disagreement about the best way to describe a spin 3/2 particle, and in this section we will discuss the choice we have adopted. The same choice is used for the $D_{13}$ resonance.

There are two spin 3/2 propagators used in the literature. One, which is known as the Rarita-Schwinger propagator, has the form

$$P_{\mu\nu} = \frac{i}{M - \not{P}} \left[ g_{\mu\nu} - \frac{1}{3} \gamma_{\mu}\gamma_{\nu} - \frac{2P_{\mu}P_{\nu}}{3M^2} - \frac{\gamma_{\mu}P_{\nu} - P_{\mu}\gamma_{\nu}}{3M} \right]$$

where $P_{\mu}$ is the four-momentum of the particle and $M$ its mass. This propagator, which was proposed by Fierz and Pauli [34] and simplified by Rarita and Schwinger [35] more than 50 years ago, can be obtained from the Lagrangian for a spin 3/2
particle [6, 14]. Another propagator
\[
P_{\mu\nu} = \left( \frac{i}{M - \not{P}} \right) \left[ g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu - \frac{1}{3P^2} (P\gamma_\mu P_\nu + P_\mu \gamma_\nu P) \right]
\] (I.8)
was recently popularized by Williams [13] and used by Jaus and Woolcock [36], who point out that the Rarita-Schwinger propagator \( P'_{\mu\nu} \) projects out a pure spin \( 3/2 \) state only when \( P^2 = M^2 \) (when the particle is on its mass shell). Moreover, Benmerrouche, Davidson and Mukhopadhyay [14] have recently pointed out that \( P_{\mu\nu} \) does not have an inverse, and claim that \( P'_{\mu\nu} \) is therefore the correct spin \( 3/2 \) propagator.

In this work we are not interested in developing a field theory of spin \( 3/2 \) particles. Instead, we need a propagator which gives a covariant, phenomenological description of a composite spin \( 3/2 \) state. Iteration of this term by our integral equation will then generate a dressed contribution which satisfies unitarity and has the correct width as determined by the dynamics. We will use the Williams propagator for this purpose, because it turns out to have a very nice property: when iterated by the integral equation, it retains its structure, giving a dressed propagator of the form
\[
P_{\mu\nu}^{\text{dressed}} = \left( \frac{i}{M - \not{P} + \Sigma_\Delta(P)} \right) \left[ g_{\mu\nu} - \frac{1}{3} \gamma_\mu \gamma_\nu - \frac{1}{3P^2} (P\gamma_\mu P_\nu + P_\mu \gamma_\nu P) \right]
\] (I.9)
where \( \Sigma_\Delta(p) \) is the self energy of the \( \Delta \). With our kernel, this self energy turns out to be a simple function. See the discussion in Chapter III.5 for more details.

### I.6 The Roper

The \( P_{11} \) phase shift is small and negative at low energy, then it changes sign at \( T_\pi \sim 100 \text{ MeV} \) and grows rapidly to pass through a resonance \([N^*(1440)] \) at \( T_\pi \sim 485 \)
MeV. There are two different explanations for this behavior in the literature. Oset, Toki, and Weise [17] argue that the Roper is needed to change the sign of the $P_{11}$, which is negative at low energies because of the nucleon pole term. However, Mizutani et. al. [15], Morioka and Afnan [16], and Pearce and Jennings [9] argue that this sign change is due to a cancellation between the repulsion from the nucleon pole and attractive non-pole contributions, and can be understood without the Roper. Our calculation supports this latter point of view, as we will show later.

In this work we study the role of the $N^*$ both at low energy and in the resonance region. To describe the $N^*$ consistently, we include a new “nucleon” pole term with a mass $m^*$ in the kernel of the integral equation, and iterate these contributions to all orders (in the same way other contributions are handled), being careful to include contributions from the amplitudes which describe the transition of a Roper to a nucleon, and vice versa. We describe the principal inelastic channel of the Roper by including the inelastic transitions $N^* \rightarrow \sigma^* + N$ and $\sigma^* + N \rightarrow N^*$. The final solution satisfies unitarity, and automatically dresses the Roper pole. This treatment is discussed in detail in Chapter III.4.
II General Theory

In this part the relativistic equation for the $\pi N$ scattering matrix $M_{ji}^{\pi}$ is presented, and we show that the theory is covariant and satisfies unitarity. We include a complete discussion of the justification for restricting the pion to its mass shell. Then we develop the technique used to solve the integral equations.

II.1 Why Should the pion be On-shell?

Before we construct the covariant integral equation used to describe $\pi N$ scattering, we discuss typical Feynman diagrams which contribute to the scattering amplitude. Following the historical route, we first consider a simplified problem where $\pi N$ scattering is dominated by diagrams like the direct and crossed nucleon poles (diagrams (a) and (e) in Fig. 1). A unitarized amplitude is obtained from these "driving terms" by iterating them to all orders. [The role of the integral equation is to carry out this iteration in a convenient, closed form.] The iteration of the direct pole diagrams (Fig. 1a) is straightforward; the most challenging case is the iteration of the crossed nucleon pole diagrams (Fig. 1e) and we are therefore led to look at the diagram in Fig. 7a, and the corresponding crossed diagram shown in Fig. 7b. The box and crossed box diagrams, which occur in the meson exchange theory, are shown in Figs. 8a and b for comparison, and will also be reviewed below.

For simplicity, we will carry out our analysis at threshold, with all of the external particles on-shell, so that the four-momenta of the external nucleons is $p_0 = (m, 0)$, and of the external pions is $q_0 = (\mu, 0)$. The four-momentum of the internal pion
Figure 7: Feynman diagrams used to construct the integral equation for $\pi N$ scattering.
is \( k = (k_0, k) \), and each of the diagrams (7) have four poles and two double poles in the complex \( k_0 \) plane. If the three momentum \( k \) is very small, the location of these poles is as shown in Fig. 9, and introducing the quantities \( \omega = \sqrt{\mu^2 + k^2} \) and \( E = \sqrt{m^2 + k^2} \), the poles for the box, Fig. 7a, are at

\[
\begin{align*}
  k_0^{1a} &= \omega - i\epsilon \\
  k_0^{2a} &= m + \mu - E + i\epsilon \\
  k_0^{3a} &= m - E + i\epsilon \\
  k_0^{4a} &= -\omega + i\epsilon \\
  k_0^{5a} &= E + m - i\epsilon \\
  k_0^{6a} &= E + m + \mu - i\epsilon .
\end{align*}
\]

Since \( m > \mu \), the poles (5a) and (6a) will give very small contributions, and we see that the box is very well approximated by closing the \( k_0 \) contour in the lower half plane, and keeping only the positive energy pion pole (1a). The same argument holds for the "crossed box", Fig. 7b, with singularities at

\[
\begin{align*}
  k_0^{1b} &= \omega - i\epsilon \\
  k_0^{2b} &= m - \mu - E + i\epsilon \\
  k_0^{3b} &= m - E + i\epsilon \\
  k_0^{4b} &= -\omega + i\epsilon \\
  k_0^{5b} &= E + m - i\epsilon \\
  k_0^{6b} &= E + m - \mu - i\epsilon .
\end{align*}
\]

Note that only the poles (2) and (6) have a location different for the corresponding
Figure 8: Feynman diagrams used to construct the integral equation for the scattering of two heavy particles exchanging a light meson.
Figure 9: Pole structure of the diagrams in Fig.7.
poles in (a), and that the pole in the lower half plane, (6) is still quite distant from the pole (1), which gives the dominant contribution.

We can use this analysis to make some very interesting estimates. If the pion three-momentum is small, so that $E \simeq m$, then the contribution from the dominant pion pole (1) to the box (a) and crossed box (b) is approximately

$$M_a^b = \frac{1}{(\omega - \mu)(2\omega^3)(8m^3)}$$

and the contributions from the poles (5) and (6) are approximately

$$M_a^m = M_b^m = \frac{15}{(2m)^7}.$$  

From these we conclude the following:

• If we neglect the crossed box, we make an "error" proportional to

$$\frac{M_b^b}{M_a^a} = \frac{\omega - \mu}{\omega + \mu}.$$  

Since momenta of the order of a few 100's of MeV are probably important, the crossed box contributions are not negligible when compared with the box contributions generated by the iteration of the crossed nucleon pole term. But these crossed boxes will not be included in our kernel, and hence our calculation of the effects of the crossed nucleon driving term is intrinsically approximate. Approximating the crossed nucleon driving term by a contact term is therefore not inconsistent with the precision of this method. This approximation will reduce the effects of the crossed boxes diagram and the error becomes,

$$\frac{M_b^b}{M_a^a} = \frac{(\omega - \mu)\mu}{\omega^2}.$$  

(II.4)
• The "error" which results from neglecting the two poles (5) and (6) is negligible:
\[
\frac{M_m^2}{M_p^2} \sim \frac{(\omega - \mu)\omega^3}{m^4}.
\]

The conclusion that the light particle (the pion) should be put on shell is very different from the result obtained in a theory in which a light meson is exchanged between two heavy particles with masses \(m_1\) and \(m_2\), where both \(m_1\) and \(m_2\) are much greater than \(\mu\). If we take \(m_2 > m_1\), then the best approximation leads to an equation in which the heavy particle \(m_2\) is on shell [24], and it is worthwhile to review the difference between these two cases here. The box and crossed box for the meson exchange case, Fig. 8, also have four poles and two double poles in the complex \(k_0\) plane, with the singularities as shown in Fig. 10. If we take \(m_2\) to be very large, so that \(E_2 = \sqrt{m_2^2 + k^2} \simeq m_2\), then the singularities of the box are at

\[
\begin{align*}
k_0^{1a} &= E_1 - i\epsilon \\
k_0^{2a} &= m_1 + i\epsilon \\
k_0^{3a} &= m_1 - \omega + i\epsilon \\
k_0^{4a} &= -E_1 + i\epsilon \\
k_0^{5a} &= m_1 + \omega - i\epsilon \\
k_0^{6a} &= 2m_2 + m_1 - i\epsilon,
\end{align*}
\]

where \(E_1 = \sqrt{m_1^2 + k^2}\) and \(\omega = \sqrt{\mu^2 + k^2}\). Because the exchanged particle is light, the situation is completely different; while the poles at (1) and (2) dominate, the singularities from the exchanged meson are very close, and are the most important source of "error". Because the singularity at (5) is now very close to (1), the "light" particle pole at (1) no longer clearly dominates. Here the crossed box plays an important role. As before, its singularities are at the same places as the box, except
Figure 10: Pole structure of the diagrams in Fig. 8.
for the poles at (2) and (6), which are at

\[ k_0^{2b} = m_1 - i\epsilon \]
\[ k_0^{6b} = m_1 - 2m_2 + i\epsilon \]

Now we see that the crossed box is well approximated by closing the contour in the upper half plane, and keeping only the double pole (3). However, this contribution is cancelled by a similar contribution from the same pole in the box. Neglecting the distant poles (6) and (4), the sum of the two diagrams is

\[ M^a + M^b \simeq \int dk_0 \frac{1}{(\omega - m_1 + k_0 - i\epsilon)^2(\omega + m_1 - k_0 - i\epsilon)^2(E_1 - k_0)} \times \left[ \frac{1}{(k_0 - m_1 - i\epsilon)} + \frac{1}{(m_1 - k_0 - i\epsilon)} \right], \]  

(II.6)

which displays the cancellation. However, (II.6) is not zero, because the two poles in the bracket “pinch”. The only way to evaluate (II.6) exactly, without considering the crossed box at all, is to close the contour in the upper half plane, in which case the total result comes only from the pole (2) in the box, corresponding to putting the heavy particle on shell.

Looking back over the arguments in the two cases, we see that the essential difference is the mass of the exchanged particle. If this mass is large (which is the case for \( \pi N \) scattering), then the singularities from the exchange are very distant, and the on-shell contributions of the light particle dominate. If the mass is light, the singularities from the exchange are important, and are best cancelled by putting the heavy particle on shell, and as the discussion shows [24], the exact answer is obtained in the limit when the mass of the heavy particle becomes very large.
II.2 Integral equation

To obtain the correct factors for our equation, it is convenient to start with the Bethe-Salpeter equation for $\pi N$ scattering,

$$M_{ji}^{\pi}(k', k, P) = V_{ji}^{\pi}(k', k, P)$$

$$+ \int \frac{d^4k''}{(2\pi)^4} V_{ji}^{\pi}(k', k'', P) G_{BS}(k'', P) M_{ji}^{\pi}(k'', k, P)$$

(II.7)

where $M_{ji}^{\pi}(k', k, P)$ and $V_{ji}^{\pi}(k', k, P)$ are the scattering matrix and the relativistic kernel (potential) of the scattering,(note that all the Dirac's indicies have been suppressed and the two-body propagator $G_{BS}(k, P)$ is

$$G_{BS}(k, P) = \frac{(m + P - k)_{\rho}\sigma^{\rho\mu}}{(m^2 - k^2 - i\varepsilon)(m^2 - (P - k)^2 - i\varepsilon)}.$$

(II.8)

The initial and final momentum of the nucleon are denoted by $p$ and $p'$, and the total momentum is $P$. In the center of mass system these momenta are written

$$P = (W, 0)$$

$$p = (W - k_0, -k); \quad k = (k_0, k)$$

$$p' = (W - k_0', -k'); \quad k' = (k_0', k')$$

(II.9)

where $W$ is the total energy of the system.

Eq. (II.7) can be reduced to the three-dimensional equation with the pion on shell by formally integrating over the internal pion energy $k_0$ and retaining only the contribution from the positive energy pion pole in the propagator (II.8), giving

$$M_{ji}^{\pi}(k', k, P) = V_{ji}^{\pi}(k', k, P)$$

$$- \int \frac{d^3k''}{(2\pi)^3 2\omega_{k''}} V_{ji}^{\pi}(k', k'', P) g(k'', P) M_{ji}^{\pi}(k'', k, P)$$

(II.10)
where the two-body propagator $G_{BS}(k, P)$ is now replaced by the off-shell nucleon propagator $g(k, P)$

$$g(k, P) = \frac{m + (P - k)}{(m^2 - (P - k)^2 - i\epsilon)}$$  \hspace{1cm} (II.11)

and $\omega_k = k_0 = \sqrt{\mu^2 + k^2}$ is the on shell energy of pion.

Consider a kernel which is a sum of a contact term $V_{\pi N}^{\pi N}(k', k, P)$ and baryon pole terms, collectively denoted by $B$ (the set $\{B\}$ includes the nucleon itself)

$$V_{ji}^{\pi N}(k', k, P) = V_{ji}^{\pi N}(k', k, P) + \sum_B \Gamma^0_B(k', P) G^0_B(P) \Gamma^0_B(P, k) ,$$  \hspace{1cm} (II.12)

where $\Gamma^0_B(P, k)$ are undressed vertex functions describing the coupling of baryon $B \rightarrow \pi N$, and $G^0_B(P)$ are the undressed propagators of the baryons. Then, if the baryons do not mix it can be shown that the solution to (II.10) can be written

$$M_{ji}^{\pi N}(k', k, P) = M_{ji}^{\pi N}(k', k, P) + \sum_B \Gamma^1_B(k', P) G_B(P) \Gamma_B(P, k)$$  \hspace{1cm} (II.13)

where $M_{ji}^{\pi N}(k', k, P)$ is the infinite sum of iterated contact diagrams,

$$M_{ji}^{\pi N}(k', k, P) = V_{ji}^{\pi N}(k', k, P) - \int \frac{d^3k''}{(2\pi)^3 2\omega_{k''}} V_{ji}^{\pi N}(k'', P) g(k'', P) M_{ji}^{\pi N}(k'', k, P) ,$$  \hspace{1cm} (II.14)

$\Gamma_B(P, k)$ is the dressed vertex for baryon $B$, which is computed from the bare vertex and $M_{ji}^{\pi N}$ using the following equation

$$\Gamma_B(P, k) = \Gamma^0_B(P, k) - \int \frac{d^3k''}{(2\pi)^3 2\omega_{k''}} \Gamma^0_B(P, k'') g(k'', P) M_{ji}^{\pi N}(k'', k, P) ,$$  \hspace{1cm} (II.15)

and $G_B(P)$ is the dressed baryon propagator, which is calculated from the equation

$$G_B(P) = G^0_B(P) \left( \frac{1}{1 + G^0_B(P) \Sigma_B(P)} \right) ,$$  \hspace{1cm} (II.16)
where $\Sigma_B(P)$ is the baryon self energy, given by

$$\Sigma_B(P) = \int \frac{d^3k''}{(2\pi)^3 2\omega_{k''}} \Gamma^0_{\mu B}(P, k'') g(k'', P) \Gamma^{0\mu}_{\nu B}(k'', P) \delta_{\mu\nu}$$

$$- \int \int \frac{d^3k''}{(2\pi)^6 4\omega_{k''}\omega_{k''}} \Gamma^0_{\mu B}(P, k'') g(k'', P) \Sigma^{\mu\nu}_{\epsilon B}(k'', k'''', P)$$

$$\times g(k''', P) \Gamma^{0\nu}_{\epsilon B}(k''', P) + \Sigma^{\text{inel}}_B(P), \quad (\text{II.17})$$

where $\Sigma^{\text{inel}}_B$ contains the effect of the coupling of baryon $B$ to inelastic channels (discussed in Chapter III.7). Equation (II.13) is illustrated diagrammatically in Fig. 11, and Eqs. (II.15), (II.16), and (II.17) in Figs. 12a, b, and c, respectively. The equivalence of Eqs. (II.13) – (II.17) with Eq. (II.10) is proved in Appendix A for the case of a single baryon, and the proof is trivially generalized to more than one if there is no mixing. If there is mixing, which is the case for the nucleon and the Roper, the self energies and propagators become matrices, and this case is discussed in detail in Chapter III.4.

It is more convenient to use Eq. (II.13) instead of Eq. (II.10) for several reasons:

(i) since we approximate the crossed terms by contact terms, all the factors which make up Eq. (II.13) can be expressed as geometric series, and summed to a
Figure 12: Diagrammatic representation of (a) Eq. (II.15) for the dressed vertex function, (b) Eq. (II.16) for the dressed propagator, and (c) Eq. (II.17) for the self-energy.
closed convenient form;

(ii) we want to keep the nucleon pole unshifted, and this requirement is conveniently implemented by requiring that Eq. (11.17), for $B = N$, be zero at $P^2 = m^2$;

(iii) The form of Eq. (11.13) enables us to separate the resonance contributions from the background, and the widths of resonances can be easily obtained from Eqs. (11.17).

All of the integral equations above are manifestly covariant. This is guaranteed by the covariance of the volume integration,

$$
\int \frac{d^3k}{2\omega_k} = \int d^4k \delta^4(m^2 - k^2)
$$

Furthermore, these equations automatically give a solution which satisfies unitarity exactly, as we will show in the next section.

### II.3 Unitarity

The derivation of the unitarity relation for pion-nucleon scattering is similar to the one given in Ref. [11] for NN scattering.

Let us start from Eq. (II.10), writing it in a compact form

$$
M = V - \int V G M
$$

where $\int = \int d^3k$. Taking the Dirac conjugate of this equation yields,

$$
\bar{M} = \bar{V} - \int \bar{M} \bar{G} \bar{V}
$$
Following [11] we obtain

\[ \overline{M} - M = -2i \int \overline{M} \Delta G M \]  

where in this case

\[ \Delta G = \pi \delta_+ \left( m^2 - (P - k)^2 \right) \left( \frac{m + P - \tilde{k}}{2\omega_k} \right) \]

Restoring the indices and integrating over the magnitude of \( k \) gives explicitly

\[ \overline{M}_{ji}^{\pi\pi}(k', k, P) - M_{ji}^{\pi\pi}(k', k, P) = \]
\[ -i \frac{|k''|}{16\pi^2W} \int d\Omega_{k''} \overline{M}_{ji}^{\pi\pi}(k', k'', P)(m + P - \tilde{k}'')M_{ji}^{\pi\pi}(k'', k, P) \]

where \( \tilde{k}'' = (\omega_{k''}, \tilde{k}'') \) is the pion momentum when both nucleon and pion are on shell.

If we expand \((m + P - \tilde{k}'')\) in terms of Dirac spinors with helicity \( \lambda \) [38]

\[ (m + P - \tilde{k}'') = (m + p'') = 2m \sum_\lambda u(p'', \lambda) \bar{u}(p'', \lambda) \]

and introduce,

\[ M_{ji}^{\pi\pi}(k', k, P) = \bar{u}(p', \lambda') M_{ji}^{\pi\pi}(k', k, P)u(p, \lambda) \]

we obtain

\[ \overline{M}_{ij}^{\pi\pi}(k', k, P) - M_{ij}^{\pi\pi}(k', k, P) = -i \frac{m|k''|}{8\pi^2W} \sum_\lambda \int d\Omega_{k''} \overline{M}_{ij}^{\pi\pi}(k', k'', P)M_{ij}^{\pi\pi}(k'', k, P). \]

Eq. (II.26) is an exact statement of elastic unitarity.
III Model

The results obtained in the previous Part hold for any choice of the relativistic kernel (or potential). In this Part, details of the model of pion-nucleon scattering described in Chapter I are presented. The main goal is to calculate the scattering amplitudes shown diagrammatically in Fig. 11. The choice of form factors is discussed in Chapter III.3., followed by a discussion of the treatment of each baryon resonance ($N^*$, $\Delta$ and $D_{13}$) and the inelasticity.

III.1 Relativistic Contact Terms

The solution of the integral equation (II.14) is greatly simplified if the relativistic kernel $V_{cN}^{\pi}$ is approximated so that the two-pion production cut, which arises from the crossed pole driving terms, is eliminated. This approximation allows us to reduce the integral equation to a geometric series, which can be summed to give a closed form for the solution.

However, this approximation must be done very carefully, as these terms make important contributions to the $S$-waves. We require that the approximation preserves chiral symmetry at threshold (which will give the correct scattering lengths), that it not depart significantly from the tree level calculation (where all external particles are on-shell), and that it extrapolates smoothly to the nucleon pole. For the last requirement we extrapolate the amplitude off-shell in the manner suggested by the structure of our integral equation; we constrain the pions to their mass-shell and allow the nucleons to go off-shell.
The exact crossed nucleon Born term with mixed coupling is

\[ V^{\pi \pi}_{c(N) ji}(k', k, P) = g^2 \tau_i \tau_j \left( \gamma^5 + \frac{(1 - \lambda)}{2m} k' \gamma^5 \right) \frac{m + P - k - k'}{m^2 - (P - k' - k)^2} \]

\[ \left( \gamma^5 + \frac{(1 - \lambda)}{2m} k \gamma^5 \right) f_N^2 ((P - k' - k)^2) f_N ((P - k)^2) f_N ((P - k)^2) \]

(III.1)

where \( \tau_i \) and \( \tau_j \) denote the isospin of a nucleon coupled to the pion field \( i \) and \( j \) and \( g \) is the bare \( \pi NN \) coupling. The nucleon form factor \( f_N \) will be described later in Chapter III.3. This term can be written in the following form

\[ V^{\pi \pi}_{c(N) ji}(k', k, P) = g^2 \tau_i \tau_j f_N^2 ((P - k' - k)^2) f_N ((P - k)^2) f_N ((P - k)^2) \]

\[ \left[ \left( a_1 + b_1 \frac{Q}{\mu} \right) + \frac{m - \not{p}}{2m} \left( a_2 + b_2 \frac{Q}{\mu} \right) \right] + \left( a_3 + b_3 \frac{Q}{\mu} \right) \frac{m - \not{p}}{2m} + \frac{m - \not{p}'}{2m} \left( a_4 + b_4 \frac{Q}{\mu} \right) \frac{m - \not{p}}{2m} \]

(III.2)

which displays its coupling, through the factors \( m - \not{p} \) (or \( m - \not{p}' \)), to the negative energy sector. We will first assume that all the particles are on shell. This gives us

\[ V^{\pi \pi}_{c(N) ji}(k', k, P) = C g^2 \tau_i \tau_j f_N ((P - k')^2) f_N ((P - k)^2) f_N ((P - k)^2) \]

\[ \left( \lambda^2 - 1 \right) + \left[ \frac{1}{m^2 - u} - \frac{(1 - \lambda)^2}{4m^2} \right] Q \]

\[ V_{c(N) ji}^{\pi N}(P) f_N ((P - k')^2) f_N ((P - k)^2) \]

(III.3)

where \( C \) is a proportionality constant, \( u = (p' - k)^2 \), and

\[ Q = P - m. \]

(III.4)

Since we are interested in retaining the dominant \( S \)-wave terms only, we will also neglect the \( p' \cdot k \) term in \( u \) (this gives only a tiny contribution anyway). And in
order to obtain the correct limit at $W = m$, which is very important for a calculation of the stability condition, we have modified the second term of $V_{e(N)}^{\pi\pi}$ as follows

$$\frac{Q}{m^2 - u} = \frac{1}{2m - \mu \sqrt{P^2}}.$$  (III.5)

This approximate expression is very close to the result we would have obtained if we had averaged the exact crossed diagram (evaluated below threshold by putting the pions on-shell) over the pion three momentum (such as would occur when $V_{e(N)}^{\pi\pi}$ is used as a kernel); it gives only a 7% error when used to evaluate the fourth order diagram. It is also very close to the exact tree diagram above threshold, as shown in Fig. 13. Note that the “tree approximation” gives a very bad result below threshold. This approximation is simpler than the one originally used in Ref. [40]. It is covariant and has no pole at any energy, so that it can be easily imbedded into $NN$ scattering or other processes without producing unwanted singularities.

To restore the chiral symmetry which is broken by the pseudoscalar coupling, we introduce a $\sigma$-exchange term. A $\rho$-exchange term is also introduced in order
get a good description of the $S$ wave scattering lengths. The $\sigma$ and $\rho$-exchanges are approximated as contact terms, and the $\rho$-exchange is divided into two terms, one with a strength proportional to $(1 - \lambda)^2$, and one independent of $\lambda$. The first of these, when combined with the $\sigma$-like exchange term can be adjusted to give an interaction at the $\pi N$ threshold which is independent of $\lambda$ [recall Eq. (1.6)], while the second will have a strength which is independently adjustable [recall Eq. (IV.1)]. Specifically, with the approximation for $\mathcal{Q}$ made above and with the form factors added, these two contact terms are

\[ \tilde{V}^\pi_{c(\sigma, \rho)ji}(P) = -C \frac{g^2}{m} f_0^2 \left[ \delta_{ij} \lambda^2 + [\tau_j, \eta_i](1 - \lambda)^2 \frac{\mathcal{Q}}{4m} \right] \]

\[ \tilde{V}^\pi_{c(\rho)ji}(P) = -C \rho \frac{g^2}{4m^2} f_0^2 [\tau_j, \eta_i] \mathcal{Q}, \quad (\text{III.6}) \]

where $\tilde{V}^\pi_{cji}(P)$ was defined in Eq. (III.3). Since it is very difficult to preserve chiral symmetry to all energies, we maintained it at threshold, which required the same form factors in all of the contact terms, and the condition

\[ C f_0^2 = C f_N^2(u)_{\lambda=\text{th}} = C f_N^2((m - \mu)^2) = f_N^2((m + \mu)^2), \quad (\text{III.7}) \]

which determines the constant $C$.

The crossed diagrams for the baryon resonances ($N^*, \Delta, D_{13}$) also were approximated in the same way as we approximated the nucleon crossed diagram, and in this approximation the $\Delta$ and $D_{13}$ crossed diagrams are zero. The details of the treatment of the resonances will be discussed in Chapter III. 4–6.

Finally, the total relativistic contact interaction,

\[ V^\pi_{cji}(k', k, P) = V^\pi_{c(N)ji}(k', k, P) + V^\pi_{c(\sigma, \rho)ji}(k', k, P) + V^\pi_{c(\rho)ji}(k', k, P) \quad (\text{III.8}) \]
can be written in the following covariant form,

\[ V_{\pi ji}(k', k, P) = \left( A_0 \frac{P}{\sqrt{P^2}} + A + B P \right) f_N((P - k')^2)f_N((P - k)^2) \]  

(III.9)

### III.2 Solving the Integral Equation for \( M_{\pi ji}^{\pi\pi} \)

In this section we would like to solve the integral equation (II.14) for the background amplitude \( M_{\pi ji}^{\pi\pi} \). This equation is shown diagrammatically in Fig. 4c, and is the first term in the full solution, as represented in Fig. 11a. The driving terms for this equation are given in Eq. (III.9). The calculation of the dressed pole diagrams which complete the solution, as shown in Fig. 11b, will be postponed until we discuss the resonances.

To calculate both the background and the pole diagrams it is more convenient if we use the projection operators:

\[ \Lambda^\pm = \frac{1 \pm \gamma^0}{2} \]  

(III.10)

In terms of these projection operators Eq. (III.9) becomes

\[ V_{\pi ji}(k', k, P) = (V^+_c \Lambda^+ + V^-_c \Lambda^-)f_N((P - k')^2)f_N((P - k)^2) \]  

(III.11)

where

\[ V^\pm_c = A \pm BW \pm A_0. \]  

(III.12)

Since \( V^\pm_c \) only depends on the total momentum \( P = (W, 0) \), the integral equation (II.14) is a geometric series, which can be summed in closed form. The result is,

\[ M_{\pi ji}^{\pi\pi}(k', k, P) = (M^+_c \Lambda^+ + M^-_c \Lambda^-)f_N((P - k')^2)f_N((P - k)^2) \]  

(III.13)
where
\[ M_c^\pm = \frac{V_c^\pm}{1 + V_c^\pm (mI_0 \pm W I_0 \mp \mu I_1)} \]  

The integrals \( I_n \), which arise from the bubble integrations shown in Fig. 4c, are
\[ I_n = \int \frac{d^3k}{(2\pi)^3} \left( \frac{\omega_k}{\mu} \right)^n \frac{f_k^2 \left( (P - k)^2 \right)}{2\omega_k \left( m^2 - (P - k)^2 - i\epsilon \right)} \]  

### III.3 Form Factors

Form factors are needed to insure that the integrals in Eqs. (11.14), (11.15), and (11.17) converge. Ideally, the results should be insensitive to the details of the form factors.

The form factors for the nucleon and \( N^* \) have the following form
\[ f_B(p^2) = \left( \frac{(A_B^2 - m_B^2)^2}{(A_B^2 - m_B^2)^2 + (m_B^2 - p^2)^2} \right)^2 \left( \frac{p^4 \left( \mu^4 + (\mu^2 + m_B^2)^2 \right)}{m_B^4 \left( \mu^4 + (\mu^2 + p^2)^2 \right)} \right) \Theta(p^2) \]  

where \( A_B \) is the form factor cut-off mass, \( m_B \) is the bare baryon mass (recall, as we discussed above, that the dressed nucleon mass is equal to the bare nucleon mass) and \( \mu \) is the pion mass. The second term is introduced in order to avoid a pole which appears especially in the spin 3/2 projection operator. The theta function is introduced to insure that this formfactor is zero at \( p^2 < 0 \). This behaviour is useful for the extension of this model to NN calculation. Ideally, the same form should be used for the \( \Delta \) and \( D_{13} \) resonances, but we found that the same form (III.16) did not work unless we replaced \( m_B \) by \( m \), the nucleon mass. The behavior of these factors for various illustrative cases will be shown in Figs. 24 and 25.

The form factor (III.16) not only gives \( f_B(m_B^2) = 1 \), but also satisfies the criteria that were suggested for the form factor in Ref. [11]. The form factor should be only
a function of $p^2$, decrease at most like a power of $p^2$ as $p^2 \to \infty$ and have no pole on the real axis.

### III.4 Treatment of the Coupled $N N^*$ System

In this section we will first calculate the contribution of the $N^*$ to the background diagram, Fig. 11a, and then calculate the $N^*$ pole contributions. Since $N^*$ has the same properties as the nucleon, we treat it like a heavy nucleon. The Feynman rule for the $N^*N\pi$ vertex

$$g_{N^*} \left( \lambda^* + \frac{(1 - \lambda^*) \vec{k}}{m + m^*} \right) \gamma^5$$

where $g_{N^*}$ is the $\pi NN^*$ coupling constant, $m^*$ is the $N^*$ mass and $\lambda^*$ is the mixing parameter for the $\pi NN^*$ coupling. The reduced $N^*$ crossed diagram can be written (with the external nucleon form factors removed),

$$\tilde{\mathcal{V}}_{\pi N^*N\pi}(P) = g_{N^*}^2 \gamma^5 \left( \frac{m^* + p' - \vec{k}}{m^{*2} - (p' - \vec{k})^2} \right) \gamma^5 \tau_i \tau_j$$

$$\approx g_{N^*}^2 \left[ \frac{m^* - m}{m^{*2} - u} + \left( \frac{P - m}{m^{*2} - u} \right) \right] \tau_i \tau_j$$

(III.18)

where the poles are approximated as before and $m^{*2} - u = m^{*2} - m^2 + 2m\mu - \mu^2$.

We chose $\lambda^* = 1$.

To calculate the dressed pole terms (Fig. 11b) coming from the coupled $N N^*$ system we first calculate the dressed propagators for the $N$ and $N^*$, including the transition from $N^*$ to $N$ and vice versa. This requires that we diagonalize the inverse propagator matrix

$$G^{-1} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} m - P + \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & m^* - P + \Sigma_{22} \end{pmatrix}$$

(III.19)
where $\Sigma_{ij}$ is the self energy, and the indices $i, j$ can be either 1 (for nucleons) or 2 (for $N^*$). This matrix is symmetric, but not hermitian. It can be diagonalized by a complex symmetric matrix,

$$G_d^{-1} = A G^{-1} A \quad \text{(III.20)}$$

where, choosing the following simple form for $A$

$$A = \begin{pmatrix} 1 & Z \\ Z & 1 \end{pmatrix} \quad \text{(III.21)}$$

gives the following results for the diagonal elements of $G_d$

$$G_{11} = \frac{1}{(m - p + \Sigma_{11}) + (2\Sigma_{12})Z + (m^* - p + \Sigma_{22})Z^2}$$
$$G_{22} = \frac{1}{(m^* - p + \Sigma_{22}) + (2\Sigma_{12})Z + (m - p + \Sigma_{11})Z^2} \quad \text{(III.22)}$$

where

$$Z = \frac{-(g_{11} + g_{22}) \pm \sqrt{(g_{11} + g_{22})^2 - 4g_{12}^2}}{2g_{12}} \quad \text{(III.23)}$$

It turns out that the quantity $g_{11} + g_{22}$ is negative, and hence we must choose the minus sign in (III.23) in order that $Z \to 0$ as $g_{12} \to 0$.

The contributions of these terms to the scattering matrix $M$ is

$$M = \Gamma^T G \Gamma = \Gamma^T (G^{-1})^{-1} \Gamma$$
$$= \Gamma^T (A^{-1} A G^{-1} A A^{-1})^{-1} \Gamma = \Gamma^T (A^{-1} G_d^{-1} A^{-1})^{-1} \Gamma$$
$$= \Gamma^T A G_d A \Gamma \quad \text{(III.24)}$$

where the unmixed vertex column vector $\Gamma$ is

$$\Gamma = \begin{pmatrix} \Gamma_{\pi NN} \\ \Gamma_{\pi NN^*} \end{pmatrix} \quad \text{(III.25)}$$

The dressed vertices are therefore

$$\Gamma^{\text{dressed}}_{\pi NN} = \Gamma_{\pi NN} + Z \Gamma_{\pi NN^*}$$
$$\Gamma^{\text{dressed}}_{\pi NN^*} = \Gamma_{\pi NN^*} + Z \Gamma_{\pi NN} \quad \text{(III.26)}$$
The mixing therefore depends on $Z$, which is dependent on energy and is complex above the pion production threshold. The values of $Z$ at the nucleon mass and the dressed Roper mass are given in the Table; note that $Z$ is very small. Note that this treatment extends that of Pearce and Afnan [39], which can be applied only below the $\pi N$ threshold where all the matrix elements are real.

Each of these propagators and the corresponding dressed vertex functions can be written in terms of the projection operators $\Lambda_\pm$ of Eq. (III.10), and then the contributions from the dressed $N$ and $N^*$ pole terms to the scattering matrix, part of the sum shown in Fig. 11b, can be easily expressed in this form as well:

$$M = M_+ \Lambda_+ + M_- \Lambda_- , \quad (\text{III.27})$$

These two poles contribute to spin $\frac{1}{2}$ ($S$ and $P$) partial wave amplitudes.

### III.5 Treatment of the $\Delta$

In this Section we review the properties of the spin 3/2 propagator used in this calculation, and calculate the contribution of the dressed $\Delta$ pole to the scattering amplitude (Fig. 11b).

We start with the most general form of the spin 3/2 propagator:

$$P_{\mu\nu}(P) = \frac{-i(m_\Delta + P)}{(m_\Delta^2 - P^2 - i\epsilon)} \theta_{\mu\nu}(P) \quad (\text{III.28})$$

where

$$\theta_{\mu\nu}(P) = a g_{\mu\nu} + b \gamma_{\mu} \gamma_{\nu} + c \frac{P_\mu P_{\nu}}{P^2} + d \frac{P_\mu \gamma_{\nu} P}{P^2} \quad (\text{III.29})$$

where $P$ is the four-momenta of the $\Delta$ and $m_\Delta$ is its mass.
To get a pure spin $3/2$ propagator, we impose two conditions which eliminate the virtual spin $1/2$ and spin $1$ parts

$$\gamma^{\mu}\theta_{\mu\nu}(P) = 0 \quad \text{(III.30)}$$
$$P^{\mu}\theta_{\mu\nu}(P) = 0 \quad \text{(III.31)}$$

From these two conditions we get

$$b = c = d = \frac{1}{3}a \quad \text{(III.32)}$$

Choosing $a = -1$ gives

$$\theta_{\mu\nu}(P) = -g_{\mu\nu} + \frac{1}{3}\gamma_{\mu}\gamma_{\nu} + \frac{1}{3}\left(\frac{P\gamma_{\mu}P_{\nu} + P_{\mu}\gamma_{\nu}P}{P^2}\right) \quad \text{(III.33)}$$

We will exploit several properties of $\theta_{\mu\nu}(P)$. In addition to Eq. (III.31), we will use

$$P\theta_{\mu\nu}(P) = \theta_{\mu\nu}(P)P$$
$$\theta_{\mu}^{\alpha}(P)\theta_{\beta\nu}(P) = -\theta_{\mu\nu}(P) \quad \text{(III.34)}$$

To calculate the self energy of the $\Delta$, we need the Feynman rule for the coupling of the $\Delta$ with the pion–nucleon channel. For this coupling we take

$$\Gamma_{\Delta\lambda}(P, k) = -\left(\frac{g_{\Delta}}{\mu}\right) f_{\Delta}(P^2)\theta_{\mu\nu}(P)k^\nu f_N((P - k)^2)T_i \quad \text{(III.35)}$$

where $g_{\Delta}$ is the bare $\pi N \Delta$ coupling, $T_i$ is the isospin $3/2 \rightarrow 1/2$ transition operator for an incoming pion with isospin $i$, $f_N((P - k)^2)$ is the nucleon form factor, $f_{\Delta}(P^2)$ is the delta form factor, $k$ is the momentum of the incoming pion. (The Dirac index is suppressed in $\Gamma$.) Note that $T_i$ is related to $\tau_i$ by the relation [41]

$$T_{ji}T_i = (\delta_{ji} - \frac{1}{3}\tau_j\tau_i) \quad \text{(III.36)}$$
The A self energy (only the first term of Fig. 12c contributes because the second term does not couple to the $P_{33}$ channel) can be written:

$$\Sigma^{\mu\nu}(P) = \left(\frac{g_A}{\mu}\right)^2 \frac{1}{(2\pi)^3} \frac{f_\Delta^2(P^2)}{2\omega_k} \int \frac{d^3k}{m^2 - (P - k)^2 - i\epsilon} f_N^2((P - k)^2) \times \theta^{\mu\alpha}(P)k_{\alpha}(m + P - k)(-k_\beta)\theta^{\nu\beta}(P).$$  \hspace{1cm} (III.37)

The angular integration can be carried out using the formulae given in Appendix B. Using the properties of $\theta^{\mu\nu}(P)$, the dressed propagator becomes:

$$G_D^{\mu\nu}(P) = \frac{-i \theta^{\mu\nu}(P)}{m_\Delta - P + \Sigma_\Delta(P)}. \hspace{1cm} (III.38)$$

The dressed vertex is calculated from Fig. 12a. However only the first term (the bare vertex) will contribute; the second term is zero, using the properties of $\theta^{\mu\nu}(P)$ after doing the angular integration. Then the contribution of the dressed A pole to the scattering matrix is

$$M^{\pi \tau}_{\Delta j i} = -\left(\frac{g_A}{\mu}\right)^2 f_\Delta^2(P^2) T_j T_i \frac{k^{\alpha}\theta^{\alpha\mu}(P)\theta^{\mu\nu}(P)\theta^{\nu\rho}(P)k^\rho}{m_\Delta - P + \Sigma_\Delta(P)} \hspace{1cm} (III.39)$$

which can be reduced using the properties of $\theta_{\mu\nu}(P)$.

### III.6 Treatment of the $D_{13}$

In this section we calculate the scattering amplitude for the $D_{13}$ dressed pole term. The calculation of this pole term is similar to the $\Delta$ pole term just calculated, except that there is an extra $\gamma^5$ in the $\pi ND_{13}$ coupling. We write the interaction Lagrangian for the $\pi ND_{13}$ coupling as

$$L = i\left(\frac{g_D}{\mu}\right) \bar{\psi}_{D_{13}}^{\mu} \theta_{\mu\nu} \tau_i \gamma^5 \Psi_N \frac{\partial \phi^i}{\partial x^\nu} + h.c. \hspace{1cm} (III.40)$$
where $g_D$ is the coupling constant and $\theta_{\mu\nu}$ is the spin $3/2$ projection operator which is described in the previous section. From this Lagrangian one can derive a Feynman rule for the $\pi ND_{13}$ interaction vertex

$$\Gamma^D_{\mu}(P, k) = -i \left( \frac{g_D}{\mu} \right) \theta_{\mu\nu}(P) k^{\nu} \gamma^5 f_N((P - k)^2) f_D(P^2) \tau_i.$$  \hspace{0.5cm} (III.41)

Using this coupling, the $D_{13}$ propagator becomes

$$G^D_{\mu}(P) = \frac{-i \theta_{\mu\nu}(P)}{m_D - P + \Sigma_D(P)},$$  \hspace{0.5cm} (III.42)

where $\Sigma_D(P)$ is the self energy of the $D_{13}$ and $m_D$ is its bare mass. As in the case of the $\Delta$, the self energy of the $D_{13}$ is given completely by the first term in Fig. 12c

$$\Sigma^D_{\mu}(P) = - \left( \frac{g_D}{\mu} \right)^2 f_D(P^2) \frac{\tau_i \tau_i}{(2\pi)^3} \int \frac{d^3k''}{2\omega_{k''}} \frac{f_N^2((P - k'')^2)}{m_D^2 - (P - k'')^2 - i\epsilon} \times \theta^{\mu\alpha}(P) k'_{\alpha} \gamma^5(m_D + P - k'') \gamma^5(-k''_{\beta}) \theta^{\beta\nu}(P).$$ \hspace{0.5cm} (III.43)

The scattering amplitude is calculated in the same way as for the $\Delta$ channel. The $D_{13}$ resonance contributes only to the $D_{13}$ and $P_{13}$ partial waves.

### III.7 Inelastic Channels

It is well known that the inelastic channels become more and more important as we go to higher and higher energy. In this analysis we consider the inelasticity of the $P_{11}$ and $D_{13}$ channels are dominated by $\sigma^*N$ scattering. We assume that the two-pions are bound together as a scalar particle $\sigma^*$. The mass of this particle is taken to be the same as the mass of two-pions, 278 MeV. This is to insure that $N\pi\pi$ threshold should be in the right place. These inelastic channels are represented by Fig. 4a and 4b. They can be calculated by the following Feynman Rules:
Figure 14: Inelastic channel

$N^*\sigma^*N$ vertex:

$$\Gamma(P, k) = -i \left( g_1(\sigma^*N^*) + \frac{g_2(\sigma^*N^*)k}{m^* + m} \right) f_{N^*}(P)f_{N}(P - k)$$  \hspace{1cm} (III.44)

$D_{13}\sigma^*N$ vertex:

$$\Gamma_D(P, k) \alpha = -\Theta_{\alpha\pi}(P) k^\alpha f_D(P)f_{N}(P - k) \left( g_1(\sigma^*D) + \frac{g_2(\sigma^*D)k}{m_D + m} \right) \frac{1}{\mu}$$  \hspace{1cm} (III.45)

Explicitly Fig. 4a is given by,

$$\Sigma_{N^*}^{in\ell} = -\int \frac{d^3k''}{(2\pi)^3} \frac{f_N^3((p - k'')^2)}{2\omega_{k''}} g_1(\sigma^*N^*) \frac{1}{m_{N^*} - p + k'' - ic} f_{N^*}^2(p^2).$$  \hspace{1cm} (III.46)

We chose $g_2(\sigma^*N^*) = 0$.

And Fig. 4b is given by,

$$\Sigma_{D_{13}}^{in\ell,\mu\nu} = -\left( \frac{g_1(\sigma^*D)}{\mu} \right)^2 f_D^2(P) \int \frac{d^3k''}{(2\pi)^3} \frac{f_N^3((p - k'')^2)}{2\omega_{k''}}$$

$$\times \Theta^{\mu\nu}(P)k''^\nu \frac{1}{m_D - p + k'' - ic} k''^\alpha \Theta^{\alpha\nu}(P).$$  \hspace{1cm} (III.47)

We also chose $g_2(\sigma^*D) = 0$. 
IV Results and Discussions

Our principal results are shown in Figs. 15-23, and in Table 1. The $S$, $P$, and $D$ wave phase shifts and inelasticities are shown in Figs. 15-21, the total elastic $\pi^-\rho$ cross section in Fig. 22, and the total elastic $\pi^+\rho$ cross section in Fig. 23. In each of these figures, the solid line is the total result, including all of the driving terms shown in Fig. 1. Our fit to the phase shifts and inelasticities is very good, with an overall $\chi^2 \approx 1.3$ per phase point.

Table 1 gives the final values of all parameters. Those given in bold face (13 parameters) were adjusted to make the fit. The $\pi NN$ coupling constant was initially fixed at the value shown, but later we did try varying it and found that the fit could not be significantly improved and was not very sensitive to small variations in its value. Table 1 also gives values of parameters determined by the fit. These include effective resonance masses and widths (see below) and two other parameters fixed by consistency requirements. The $\pi NN$ mixing parameter, $\lambda$, was determined by the requirement that the nucleon self energy be unshifted by the interaction, as discussed above, and the overall strength $C$ of the combined $\sigma$- and $\rho$-like contact terms [recall Eq. (I.6)] was fixed so as to insure that they exactly cancel the nucleon pole terms at the $\pi N$ threshold, as required by chiral symmetry. (This adjustment is necessary because the nucleon form factors are not unity at the $\pi N$ threshold.)

Because of our choice of spin 3/2 propagator, and our approximation scheme which leads to the result that the crossed $\Delta$ and $D_{13}$ poles are zero when approximated as contact terms (for details see the discussion in Chapter III), the $\Delta$ contributes only to the $P_{33}$ channel (except for a tiny contribution to the $D_{33}$ chan-
Table 1: The parameters of the model. Those in bold face were varied during the fit; the others are determined by the fit.

<table>
<thead>
<tr>
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<th>dressed</th>
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nel, which we will not discuss), and the $D_{13}$ contributes only to the $D_{13}$ and $P_{13}$ channels. Furthermore, approximating the crossed nucleon and Roper poles by contact terms means that they only contribute to spin 1/2 channels. Hence the phase shifts decouple, with the $P_{33}$ channel driven only by the direct $\Delta$ pole, the $D_{13}$-$P_{13}$ channels driven only by the direct $D_{13}$ pole, and all the other (spin 1/2) channels driven only by the nucleon, $N^*$, and the effective $\sigma$- and $\rho$-like contact terms.

It is therefore convenient to describe the fits to each of the decoupled channels separately, and we will begin with the spin 1/2 channels, shown in Figs. 15-18. As discussed above, these channels are driven by the nucleon and $N^*$ Born terms, and the effective $\sigma$- and $\rho$-like contact terms. These driving terms depend on only 6 adjustable parameters: the undressed mass of the $N^*$ pole, $m^*$, the bare $\pi NN^*$ coupling, $g_{NN^*}$, the strength of an "additional" $\rho$-like $\pi\pi NN$ contact term $V_\rho$ not required by (but consistent with) chiral symmetry, parameterized by a constant $C_\rho$, where (omitting the form factors)

$$V_{\rho,ji} \simeq -C_\rho \frac{g^2}{4m^2} [r_j, r_i] \mathcal{Q},$$

(IV.1)

one parameters needed to describe the inelasticity of the $N^*$, and two form factor masses: the mass in the nucleon form factor, $\Lambda$ and the mass in the $N^*$ form factor, $\Lambda^*$.

The inelasticity of the $N^*$ is approximately described by coupling to the $\sigma^* N$ channel, with coupling constants $g_{1(\sigma^* N^*)}$. The mass of the scalar particle, $\sigma^*$, is chosen to be the mass of two-pions. This value is chosen to insure that the $N\pi\pi$ threshold should be in the right place.

Before we discuss the fits to the other channels, we wish to point out that the
Figure 15: Fits to the $S_{11}$ phase shift. As explained in the text, the dotted line is the nucleon Born terms only, the dashed line is the addition of the $\sigma$- and $\rho$-like contact terms required by chiral symmetry, the long-dashed line adds in the $N^*$ contributions, and the solid line is the total result, obtained by adding the additional $\rho$-like contact term.
Figure 16: Fit to the $S_{31}$ phase shift. The curves are the same as in Fig.7.
Figure 17: Fits to the isospin 1/2 $P-$ wave phase shifts.

$S$ waves, shown in Figs. 15 and 16, are particularly sensitive. To show how the total result is built up from individual contributions, the curves in the figures show the result when the kernel (i) includes only the direct nucleon pole term and the contact term derived from crossed nucleon exchange (the dotted line), (ii) the terms in (i) plus the combined $\sigma$- and $\rho$-like contact terms of Eq. (I.6) (the dashed line), (iii) the terms in (ii) plus $N^*$ driving terms (the long-dashed line), and finally (iv) the total result, which includes the terms in (iii) plus the additional $\rho$-like $\pi\pi NN$ contact term of Eq. (IV.1) (the solid line). Since the contributions add non-linearly, it is difficult to extract the separate contributions from the figures, but we can conclude
that the chiral model without the $N^*$ and $\rho$, (ii), gives a very good account of the scattering lengths, but the $N^*$ pushes the $S_{11}$ phase shift in the wrong direction, and only after the additional rho is added do we restore the correct low energy behavior. The bend in the $S_{11}$ is due to the $N^*$, and we have no need for the $S_{11}(1535)$ in our model. The isospin even and odd scattering lengths calculated from the $S_{11}$ and $S_{31}$ fits are: $\mu a^+ = 0.07$ and $\mu a^- = -0.05$. The values agree with the experimental results as shown in Fig. 3.

The same curves are shown for the spin 1/2 $P$ waves in Fig. 17. Note that the nucleon Born terms make a very small contribution to the $P_{11}$ channel above 200 MeV, but that below 200 MeV they already exhibit the change from repulsion to attraction. The Born terms alone give a zero in the $P_{11}$ phase shift at $\simeq 253$ MeV, but inclusion of the Roper moves this zero down to the correct region $\simeq 100$ MeV. At higher energies the $P_{11}$ phase shift is dominated by the Roper, with the contributions from the $\sigma$- and $\rho$-like contact terms being very small.

We now discuss the spin 3/2 channels, where the situation is much simpler.
Figure 19: Fit to the isospin $3/2$ $P$-wave phase shifts.
The results for the $P_{33}$ channel are shown in Fig. 19. This channel is fit by three parameters; the bare delta mass, $m_{\Delta}$, the bare $\pi N \Delta$ coupling constant, $g_{\Delta}$, and a mass in the delta form factor, $\Lambda_{\Delta}$. (The nucleon form factor has already been fixed by the fit to the spin 1/2 channels.) Note that the mass of the bare, unshifted delta pole is at 1302 MeV, considerably higher than the nominal delta mass of 1232 MeV, but that the dressed $\Delta$ mass is very close to the nominal value of 1232.

The $D_{13}-P_{13}$ channels are fit by 4 more parameters; the bare mass of the $D_{13}$
pole, $m_D$, the coupling of the $D_{13}$ to the $\pi N$ channel, $g_D$, the coupling of the $D_{13}$ to the inelastic $\sigma^* N$ channel, $g_{1(\sigma^*, D)}$, which describe the inelasticity of the $D_{13}$ approximately and the $D_{13}$ formfactor mass. Actually there is another parameter $g_{2(\sigma^*, D)}$, however it is very small and is neglected here. The fit to the $D_{13}$ channel, shown in Figs. 20 and 21, is good, and the bare $D_{13}$ mass, $m_D$, is about 1520 MeV, in very good agreement with the nominal value of 1520 MeV.

The total elastic $\pi^- p$ cross section is shown in Fig. 22. The data are from Ref. [31]. The first dotted, dashes and the second dotted curves are the result for a kernel with nucleon Born terms only (practically zero), the Born terms plus the chiral contact terms, and then these with the $\Delta$ contribution. The $\Delta$ clearly dominates the cross section below 300 MeV. The addition of the $N^*$ (the long–dashed line)
Figure 22: Fits to the $\pi^{-}p$ total cross section
followed by the additional rho (the dot-dashed line) suppresses the cross section up to 300 MeV, but gives needed strength above 400 MeV; adding the $D_{13}$ to get the final result (solid line) restores the cross section at very low energies and gives a very significant addition above 400 MeV.

The total elastic $\pi^+p$ cross section is shown in Fig. 23. The first dotted and the dashes curves which are practically zero are the result for a kernel with nucleon Born terms only and Born terms plus chiral contact terms. Then the results of adding the $\Delta$ and the crossed $N^*$ are the two overlapping dotted and long-dashed curves. Finally, the addition of the extra rho (solid line) makes small but important contributions at low and high energies.

A number of interesting parameters are determined by our fits, and these are also given in Table 1. We have already discussed how the $\pi N$ mixing parameter, $\lambda$, is fixed by the stability condition, and how the strength of the $\lambda$-dependent $\sigma$- and $\rho$-like contact terms, defined in Eq. (1.6), is fixed by chiral symmetry. In addition, we have looked at our solutions, and extracted an effective mass and width for each resonance by writing the solutions, near the resonance, in the following approximate form

$$M^{\pi} = \frac{A}{m_{\text{eff}} - W - i\frac{1}{2}\Gamma} + B,$$

where $m_{\text{eff}}$ and $\Gamma$ are constants obtained from the exact solutions (which depend on the total cm energy $W$) evaluated at $W = m_{\text{eff}}$. In particular, the value of $m_{\text{eff}}$ is the solution of the non-linear equation

$$m_{\text{R}} - m_{\text{eff}} + \text{Re}\Sigma_R(m_{\text{eff}}) = 0,$$

where $m_{\text{R}}$ is the bare mass and $\Sigma_R(W)$ is the self energy of the resonance $R$. The values of these effective masses and widths are given in the Table. The definition of
Figure 23: Fits to the $\pi^+ p$ total cross section
the effective coupling constants for the resonances, $g_{\text{eff}}$, is discussed in Appendix D. The renormalized $\pi NN$ coupling constant, $g^R$, is

$$\left(\frac{g^R}{g}\right)^2 = \left(1 - \frac{\partial}{\partial W} \Sigma_N(W)\right)^{-1} \bigg|_{W=m} .$$  \hspace{1cm} (IV.4)

Note that the renormalization of the $\pi NN$ coupling constant is insignificant.

The coupling of the nucleon to the $N^*$ implies that the dressed states are linear combinations of the bare nucleon and $N^*$ states. The admixture is given by a function $Z(W)$ (defined in Eq. (111.26), and the values of $Z$ at the nucleon mass and at the effective $N^*$ mass are given in Table 1. Note that the mixing is only a few percent. Finally, we close this review of our results by discussing the form factors used in this model (for a detailed discussion, see Sec. III.3). A form factor is needed to insure that the solutions of the integral equation exist, or alternatively, to cut off the integrals over the $\pi N$ loops which appear in the solution. This form factor cannot be associated with the pion mass, as is usually done in pion exchange models, because the pion is on-shell. Anticipating the extension of this model to the description of the electro-production of pions, where a gauge invariant treatment of electromagnetic interactions is possible following the procedure introduced in Ref. [37], we choose to make the form factor depend only on the off-shell nucleon mass, and to identify the form factor with the nucleon itself, so that the same universal form factor will be used for all off-shell nucleons, wherever they appear. When the nucleon form factor accompanies the intermediate nucleon in the direct nucleon pole term, the virtual nucleon mass (squared) is simply

$$W^2 = m^2 + \mu^2 + 2m(T_{\text{lab}} + \mu) ,$$  \hspace{1cm} (IV.5)

and the form factor is plotted versus $T_{\text{lab}}$ in Fig. 25. When the nucleon form factor
Figure 24: Form factors for the nucleon plotted as a function of the pion loop momentum.

Figure 25: Form factors for the nucleon, Roper, $D_{13}$ and $\Delta$. 
accompanies a virtual nucleon in a $\pi N$ loop, its mass (squared) is

$$p^2 = W^2 + \mu^2 - 2W\omega(k) ,$$

(IV.6)

where $k$ is the magnitude of the pion three-momentum in the loop, and $\omega = \sqrt{\mu^2 + k^2}$. The form factor is plotted versus $k$ for a fixed $W = m + \mu$ in Fig. 24. We emphasize that the same nucleon form factor is shown in both figures; only the variable on which it depends has been changed. The $N^*$, $\Delta$ and $D_{13}$ form factors are plotted versus $T_{lab}$ for the kinematics of Eq. (IV.5) in Fig. 25. In common with the nucleon form factor, the $\Delta$ and $D_{13}$ form factor was also chosen to peak at the nucleon mass $m$, but the $N^*$ form factor was chosen to peak at the $N^*$ mass. Unfortunately, our results are sensitive to the form factors, which are purely phenomenological.
V CONCLUSIONS

We draw the following conclusions from this work:

- A simple resonance pole model, with nucleon, delta, Roper, and $D_{13}$ poles and other couplings described by 13 adjustable parameters (including 4 resonance masses, 5 coupling constants, and 4 form factor masses) has been found to give a very good description of $\pi N$ scattering up to pion laboratory energies of 600 MeV. The model is simple, covariant, satisfies unitarity exactly, and is approximately chirally symmetric at threshold. A very good description of the data up to 400 MeV laboratory energy would require only 6 parameters.

- The requirement that the nucleon self energy be unshifted by the interaction (referred to as the stability condition) can be satisfied only if the $\pi N$ coupling is a mixture of pseudoscalar and pseudovector couplings, and the value we obtain (20% pseudoscalar and 80% pseudovector) is well constrained by our fit, and largely independent of the values of the other parameters. Furthermore, it is in good agreement with the value of this parameter obtained from a OBE model of $NN$ scattering [10].

- The spin 3/2 resonances in our model have no virtual spin 1/2 components, leading us to conclude that such components (which may very well be present in a less phenomenological treatment) are not necessary for a successful fit to the data.

- The position of the bare $\Delta$ pole ($m_\Delta \simeq 1302$ MeV) is surprizingly far from the effective mass of the $\Delta$ resonance ($m_{\Delta}^{\text{eff}} \simeq 1230$ MeV). This should be taken
into account in any quark model which neglects pion interactions. The same
is not true for the Roper and the $D_{13}$; in these cases the bare and effective
masses are quite close to each other.

- The existence of a zero in the $P_{11}$ phase shift does not depend on the Roper,
  but its precise location is sensitive to the presence of a Roper resonance.

- The value of the renormalized $\pi NN$ coupling constant, $(g^R)^2/4\pi$, is not well
determined by our model; a good fit is obtained for values in the range from
12 - 15.
PART B
PION PHOTO PRODUCTION
VI Overview

VI.1 Introduction

Pion Photoproduction has been studied for many years. One of the earliest models, developed by Chew, Goldberger, Low and Nambu, is based on dispersion theory[42]. It included nucleon Born terms and Δ-excitation and described the reaction up to 500 MeV photon laboratory energy. A further study (using pseudoscalar $\pi NN$ coupling) was undertaken by Donnachie [43]. Among later efforts is the work based on chiral Lagrangians carried out by Olsson and Osypowski [44]. They used pseudovector $\pi NN$ coupling and also introduced $\omega$ exchange. This work was further developed by Wittman, et al. [45]. More recently, Nozawa, Blankleider and Lee (NBL) [46] developed a dynamical model of pion-photoproduction in which they used a separable interaction to describe the final state $\pi N$ interactions. Lee and Pearce [47] improved on this description by using a reduction of Bethe-Salpeter equation to treat the meson exchange interaction in the final state. They calculated photoproduction observables up to 500 MeV photon lab energy. However, with the construction of new facilities such as the Continuous Electron Beam Accelerator Facility (CEBAF), it is necessary to have a good description of pion-photoproduction which extends up to higher energies. Such a description must be covariant, gauge invariant to all order of the strong coupling constants, and include not only the nucleon ($N$) and delta ($\Delta$) resonances, but also the Roper ($N^*$) which plays a prominent role in the isospin $\frac{1}{2}$ amplitudes and the $D_{13} (1520)$ which makes large contributions to $D$-waves.

In this work we present a simple, covariant, gauge invariant and unitary model for $\pi -$photoproduction which works well up to 770 MeV photon laboratory energy.
ergy. We have introduced a new form for the $\pi N\Delta$ and $\pi ND_{13}$ vertices which makes the calculations simpler. At all times we have tried to keep both the $\pi N$ and $\pi -$photoproduction models as simple as possible (without sacrificing essential physics) so that they may be consistently used as input to $NN$ scattering and deuteron photo-disintegration calculations.

In this work the pion-photoproduction multipole amplitudes are obtained from the solution of a relativistic wave equation, in which the pion is restricted to its mass shell in all intermediate states except in the pion pole diagram, which is needed to keep gauge invariance. The rationale for this approach has been described in part A of this dissertation. As in $\pi N$ scattering, in order to describe the resonances at photon laboratory energy $\sim 300$, $\sim 450$, and $\sim 760$ MeV, the kernel or driving terms of the relativistic integral equation include undressed $\Delta$, $N^*$, and $D_{13}$ poles in addition to the undressed nucleon pole. The kernel also includes contributions derived from crossed $N$, $\Delta$, $N^*$, and $D_{13}$ diagrams and from $\omega$ and $\rho$ exchange terms. The $\omega$ exchange is claimed to give a significant contribution to the $M_{1+}(1/2)$ amplitude and $M_{1-}(1/2)$ amplitudes (for an explanation of the multipole notation see subsection VI.2 below) [44]. Although the $\rho$ exchange contribution is small [48], it is still included in our model in order to get a picture of the $\gamma \pi \rho$ interaction. Our approximation scheme makes the crossed $\Delta$ and $D_{13}$ poles zero, as in the $\pi N$ model. This makes the model simpler and the numerical calculations easier, and is consistent with other approximations we have made. The crossed nucleon pole is treated exactly because of its importance in the proof of gauge invariance, and the crossed Roper is also treated exactly because it has the same properties as the nucleon. All of these driving terms are shown diagrammatically in Fig. 26. The Kroll-Ruderman term (contact diagram) and the additional interaction currents needed to make the
model gauge invariance are described in Chapter VII and VIII. The solution which emerges from the integral equation automatically satisfies unitarity up to the first order in $\epsilon$ (referred to as the Watson theorem)[49].

Features of our $\pi -$photoproduction model which are consistent with the $\pi N$ scattering model (Part A) include the following: (i) the $\pi NN$ coupling is taken to be a superposition of both pseudoscalar ($\gamma^5$) and pseudovector ($\gamma^\mu \gamma^5$) coupling; (ii) the nucleon self energy is constrained to be zero at the nucleon pole, so that the nucleon mass remains unshifted by the interaction; (iii) contributions from the Roper ($N^*$) and ($N^* \leftrightarrow N$) transition amplitudes are iterated to all orders, giving a consistent description of the Roper and its width; and (iv) the $\Delta$ and $D_{13}$ are treated as pure spin $3/2$ particles, which the same propagators used in the $\pi N$ model.

In the remainder of this section we will describe the history and background of some aspects of pion-photoproduction such as the $E2/M1$ ratio, low energy theorem, unitarity, and gauge invariance. The general theory is described in Chapter VII. The $\pi -$photoproduction model is described in Chapter VIII. The Appendices discuss some technical points.

**VI.2 The $E2/M1$ Ratio**

It has been known that the tensor interaction between quarks, such as the one which arises from the one-gluon-exchange interaction, gives a small $D-$state admixture to the predominantly $S-$state wave functions of the nucleon and the $\Delta$. This tensor interaction leads to a resonant electric quadrupole amplitude $E_{1+}(\frac{3}{2})$ (or $E2$) which is very small compared to the resonant magnetic dipole amplitude $M_{1+}(\frac{3}{2})$ (or $M1$).
Figure 26: Driving terms for pion-photoproduction
Here the amplitudes are denoted by $E_{l\pm}(I)$ and $M_{l\pm}(I)$, where $l$ is the orbital angular momentum of the photoproduced pion, the ± sign refers to the total $\pi N$ angular momentum $j = l \pm 1/2$, and $I$ is the isospin of the $\pi N$ system. The non-vanishing $E2$ amplitude will be one of the signals of the $D-$ state admixture. Therefore it is important to determine the $E2$ amplitude in order to test various quark model predictions.

There have been several attempts to measure the $E2$ amplitude, but it is difficult to get the accurate value because the $E2$ amplitude is very small compared to the dominant $M_{1+}(\frac{3}{2})$ amplitude, and the background is comparatively large [50]. The analyses of the data using several models show that although all of the calculations agree that $E2$ is small, there is considerable uncertainty as to its precise size. Results for the $E2/M1$ ratio which are listed in the Review of Particle Properties [18] are $E2/M1 = -1.1 \pm 0.4\%$, $-1.5 \pm 0.2\%$ [45], $3.7 \pm 0.4\%$ [51] and $-1.3 \pm 0.5\%$ [52]. Some other calculations give: $E2/M1 = -3.1\%$ [46], $-4\%$ [53], and $0\%$ [54]. These differences are a reflection of the fact that extraction of the $E2/M1$ ratio from the large experimental background requires a theoretical model for both the $\Delta$ resonance and the background, and the result one obtains is therefore sensitive to how the theoretical models are unitarized, and to how the background is described [46]. We expect that new, accurate data from CEBAF experiments, and new, more complete models of $\pi -$ photoproduction, will help to clarify the situation.

The value of the $E2/M1$ which we obtain from our fit (at the resonance pole $W_{tot} = M_{\Delta}$) is

$$E2/M1 = -2.73\%.$$ (VI.1)

This is small and negative, in agreement with some of the results given above. This
is calculated from the $\Delta$ contribution only.

VI.3 Low Energy Theorem

The low energy theorem (LET) was derived for the first time by Kroll and Ruderman [55] from an examination of the implications of gauge invariance in the framework of field theory. Later Fubini et al. [56], extended this theory by including the hypothesis of a partially conserved axial current (PCAC). In view of the LET, threshold pion production on the nucleon was considered to be well understood. According to this LET prediction the threshold value of the electric dipole of $\pi^0$ photoproduction is

$$E_{0^+}\big|_{\text{LET}} = -\frac{e g_{\pi} \mu}{8 \pi m^2} \left( 1 - \frac{\mu}{2m} (3 + \kappa_p) \right) + \mathcal{O} \left( \frac{\mu}{m} \right)^3$$

$$= \frac{2.3 \times 10^{-3}}{\mu} + \text{correction},$$

(VI.2)

where $\mu$ is the pion mass. However it was a big surprise when an analysis of the Saclay data [57] showed that the experimental threshold amplitude $E_{0^+}$ for this $\pi^0$ photoproduction was smaller than the prediction of LET by about a factor of five

$$E_{0^+}\big|_{\text{expt}} = \frac{(-0.5 \pm 0.3) \times 10^{-3}}{\mu}.$$  

(VI.3)

The Mainz analysis [58] confirmed this result, and renewed interest in the LET. Possible flaws in the derivation of the LET due to final state interactions [59], higher order corrections from the chiral perturbation expansion [60], or chiral symmetry breaking corrections [61, 62, 63], were proposed. Then, instead of extracting the low energy result from the differential cross section, Bernstein and Holstein [64] and Drechsel and Tiator [65] used the total cross section (which was not analyzed by
the Mainz group) and obtained results in good agreement with the LET:

\[ E_{0^+} = \frac{(-2.0 \pm 0.2) \times 10^{-3}}{\mu}. \]  
(VI.4)

and

\[ E_{0^+} = \frac{(-1.5 \pm 0.3) \times 10^{-3}}{\mu}. \]  
(VI.5)

There is no longer any evidence for a breakdown in the LET.

The result we obtain for the electric dipole amplitude at threshold,

\[ E_{0^+} = \frac{-1.23 \times 10^{-3}}{\mu}, \]  
(VI.6)

is in agreement with the result (VI.5).

**VI.4 Unitarity**

Symbolically, the unitarity statement can be written [see Eq. (VII.14) below]

\[ \text{Im} \, M_\alpha^{\pi\gamma} = -\rho^\pi M_\alpha^{\pi\pi} M_\alpha^{\pi\gamma} - \rho^\gamma M_\alpha^{\pi\gamma} M_\alpha^{\gamma\gamma}, \]  
(VI.7)

where \( M_\alpha^{\pi\pi}, M_\alpha^{\pi\gamma}, \) and \( M_\alpha^{\gamma\gamma} \) are the \( \pi N \), pion-photoproduction, and Compton scattering matrices for a state with quantum numbers \( \alpha \), and \( \rho^\pi \) and \( \rho^\gamma \) are phase space factors for the \( \pi N \) and \( \gamma N \) intermediate states. In 1954 Watson [49] pointed out that the second term in Eq. (VI.7) is very small because it contains no terms which are first order in \( e \) (the electric charge), and can therefore be neglected. Below the two-pion production threshold, the phase of the pion-photoproduction amplitude for a state \( \alpha \) will therefore be equal to the phase of \( \pi N \) scattering in the same channel. This statement can be explicitly written

\[ M_\alpha^{\pi\gamma} = |M_\alpha^{\pi\gamma}|e^{i\phi_\alpha^{\pi\gamma}}, \]  
(VI.8)
where $\delta^{\pi}_\pi$ is the partial wave phase shift for $\pi N$ scattering. The Watson statement (VI.8), sometimes called the Watson theorem, will start breaking down above the two-pion production threshold.

Unitarity was incorporated into models based on dispersion relations by Chew, Goldberger, Low, and Nambu (CGLN) [42] and by Fubini, Nambu, and Wataghin [66]. Early models based on effective Lagrangians were not unitary [44, 67] but were later unitarized [44, 68, 69]. As pointed out by Araki and Afnan [70] quark models based on effective Lagrangians are hard to interpret because it is difficult to establish the connection between the coupling constants in the Lagrangian and observed interaction strengths.

The importance of unitarity was recently pointed out by Nozawa, Blankleider and Lee [46], who claim that it is impossible to fit the $M_{1+}$ and $E_{1+}$ multipoles with a non-unitarity model. The observation was made by Wittman, Davidson, and Mukhopadhyay [45] who also showed that the result for these amplitudes can be improved by unitarizing the model. These models are unitary because they use a covariant integral equation with solutions which are automatically unitary. Ohta and Tanabe [51] and Yang [53] use an integral equation with a separable $\pi N$ potential. While their result is unitary, the value of $E_{0+}$ at threshold is sensitive to the particular separable expansion used, and they are not able to determine a unique value of $E_{0+}$.

Our model uses a relativistic wave equation in which the intermediate state pion is on shell and the intermediate state nucleon is off-shell. This is consistent with the $\pi N$ model previously discussed (see Part A). In our model, the same
equations are used to calculate both the scattering amplitude and the renormalized coupling constants, insuring that the renormalization of the propagators and vertices is carried out in a manner that is consistent with unitarity.

VI.5 Gauge Invariance

It has been known since 1954, when Kroll and Ruderman (KR) [55] wrote their well-known paper on pion-photoproduction, that the momentum dependence of the pseudovector $\pi NN$ coupling requires introduction of an interaction current (the famous Kroll-Ruderman term) in order to satisfy gauge invariance. More recently, using minimal substitution, Ohta [72] and Naus, Koch, and Friar [74] obtained a gauge invariant set of Born terms which included form factors. Antwerpen and Afnan [16] extended this theory to the treatment of pion-photoproduction with final state interactions, but have not obtained numerical results. In their approach they require the dressed $\pi NN$ vertex to be gauge invariant by itself. The NBL model [46, 71] also includes final state interactions, and satisfies gauge invariance by restricting both of the intermediate particles to their mass shell.

In this paper we apply the method originally introduced by Gross and Riska [37]. They show how the electromagnetic coupling to any two-body system described by a relativistic two-body equation (such as the Bethe-Salpeter equation or the Gross equation [11, 40]), will always conserve current provided the following three conditions are met: (i) the electromagnetic currents for the interacting off-shell nucleon and mesons satisfy the appropriate Ward-Takahashi (WT) identities; (ii) the interacting incoming and outgoing two-body system satisfy the same two-body relativistic equation (with the same interaction kernel); and (iii) the exchange (or
interaction) current is built up from the relativistic kernel by coupling the virtual photon to all possible places in the kernel. This method works even in the presence of strong form factors for the off shell nucleon; in this case it is only necessary to modify the structure of the off-shell $\gamma NN$ vertex so that it satisfies the a WT identity with dressed propagators (as discussed in Chapter. VIII).

Using this method, it is possible to construct a gauge invariant theory even when particles are off shell, but gauge invariance is achieved only through cancellations among all of the diagrams in the theory. To prove gauge invariance (as is done in Chapter VII), we use the WT identities, the relativistic wave equation satisfied by the $\pi N$ system, and must be careful to introduce interaction currents (in addition to the well-known KR interaction current) which arise from the momentum dependence of the interaction kernel.
VII General Theory

In this part the relativistic equation for the pion-photoproduction scattering matrix is presented, and we show that the theory is covariant, gauge invariant and satisfies unitarity.

VII.1 Integral Equations

The Bethe-Salpeter equation for pion-photoproduction can be written in two equivalent ways. Keeping the lowest order terms in $e$ only, and suppressing all the Dirac indices gives

\[ M^\gamma\pi(k', q, P) = V^\gamma\pi(k', q, P) \]
\[ + i \int \frac{d^4k''}{(2\pi)^4} V^\pi\pi(k', k'', P) G(k'', P) M^\gamma\pi(k'', q, P) \]
\[ = V^\gamma\pi(k', q, P) \]
\[ + i \int \frac{d^4k''}{(2\pi)^4} M^\pi\pi(k', k'', P) G(k'', P) V^\gamma\pi(k'', q, P), \] (VII.1)

where $V^\gamma\pi(k', q, P)$ and $V^\pi\pi(k', k'', P)$ are the driving terms for the $\gamma\pi$ and $\pi\pi$ sectors, respectively, and $G(k'', P)$ is the two-body $\pi N$ propagator. The four-momenta of the incoming, outgoing, and intermediate nucleon are $p$, $p'$ and $p''$, of the outgoing, and intermediate pion are $k'$, and $k''$, and of the incoming photon is $q$, so that $P = p + q = p' + k' = p'' + k''$ is the total four-momentum. The equivalence of the two forms of Eq. (VII.1) follows from their Born series, which is identical. To see this, it is necessary to use the equations for the $\pi N$ scattering amplitude, which are

\[ M^\pi\pi(k', k, P) = V^\pi\pi(k', k, P) \]
\[ + i \int \frac{d^4k''}{(2\pi)^4} V^\pi\pi(k', k'', P) G(k'', P) M^\pi\pi(k'', k, P) \]
In Part A we have shown that pion-nucleon scattering is well described by a relativistic equation obtained from Eq. (VII.2) by putting the intermediate pion on mass-shell. To be consistent with this description of $\pi N$ scattering, we also put the intermediate pion on the mass-shell in the $\gamma N$ Eqs. (VII.1). Note that the only place where the pion will be off-shell is in one of the pion pole driving terms, which is needed to satisfy gauge invariance, as discussed below. If the pion is put on-shell, Eq. (VII.1) becomes

$$M^{\pi\gamma}_{\bar{j}}(k', q, P) = V^{\pi\gamma}_{\bar{j}}(k', q, P)$$

$$- \int \frac{d^3k''}{(2\pi)^3 2\omega_{k''}} V^{\pi\gamma}_{\bar{j}}(k', k'', P) S_N(P - k'') M^{\pi\gamma}_{\bar{i}}(k'', q, P)$$

$$= V^{\pi\gamma}_{\bar{j}}(k', q, P)$$

$$- \int \frac{d^3k''}{(2\pi)^3 2\omega_{k''}} M^{\pi\gamma}_{\bar{i}}(k', k'', P) S_N(P - k'') V^{\pi\gamma}_{\bar{j}}(k'', q, P)$$

where $\omega_{k''} = \sqrt{\mu^2 + k''^2}$ is the on-shell pion energy, and

$$S_N(P - k'') = \frac{1}{m - (p - \not{k}'') - i\epsilon}$$

is the nucleon propagator, and $\mu$ and $m$ are the pion and the nucleon masses.

The equations are regularized by adding a form factor, $f_N(p^2)$, to damp the high momentum behavior of the off-shell nucleon of momentum $p$. The Eqs. (VII.3) include these form factors in the interaction kernel $V^{\pi\gamma}_{\bar{j}}$. Alternatively, it is sometimes convenient (particularly in our discussion of gauge invariance below) to move these form factors from the kernel to the propagator. To this end we can introduce

$$= V^{\pi\gamma}_{\bar{j}}(k', k, P)$$

$$+ i \int \frac{d^4k''}{(2\pi)^4} M^{\pi\gamma}_{\bar{i}}(k', k'', P) G(k'', P) V^{\pi\gamma}_{\bar{j}}(k'', k, P) \cdot (VII.2)$$
reduced amplitudes and damped propagators as follows:

\[ V(k', k, P) = f_N[(P - k')^2] \tilde{V}(k', k, P) f_N[(P - k)^2] \]
\[ M(k', k, P) = f_N[(P - k')^2] \tilde{M}(k', k, P) f_N[(P - k)^2] \]
\[ \tilde{S}_N(P - k'') = f_N^2(p'^2) S_N(P - k''). \] (VII.5)

The symbol \( \tilde{M} \) will usually denote the reduced amplitude \( M \) that is the amplitude \( M \) with the form factors removed, and \( \tilde{S} \) the damped propagator with the (square of the) nucleon form factor added. It is easy to verify that the reduced amplitudes satisfy the same equations, but with damped propagators substituted for "bare" propagators.

We will have occasion to use the fact the \( \pi - N \) scattering matrix \( M_{\pi N}(k', k, P) \) can be written in the following form (see Part A):

\[ M_{\pi N}(k', k, P) = M_{\pi N}^\infty(k', k, P) + \sum_B \Gamma_B^I(k', P) G_B(P) \Gamma_B^I(B, k) \] (VII.6)

where the sum is over baryons B in the set \( \{N, N^*, \Delta, D_{13}\} \), \( M_{\pi N}^\infty(k', k, P) \) is the infinite sum of iterated contact diagrams, \( \Gamma_B^I(k, P) \) is the dressed vertex for baryon B, and \( G_B(P) \) is the dressed baryon propagator.

The integral equations (VII.3) are manifestly covariant. This is guaranteed by the covariance of the volume integration,

\[ \int \frac{d^3k}{2\omega_k} = \int d^4k \delta_+(\mu^2 - k^2). \] (VII.7)

Furthermore these equations automatically give a solution which satisfies unitarity to order \( e \) (the Watson theorem) as we will show in the next section.
VII.2 Unitarity

The proof of unitarity is very similar to the one given in Ref. [11] for $NN$ scattering. First, we write Eq. (VII.3), and a similar one for $\pi + N \rightarrow \gamma + N$ (pion-photoabsorption) in the following compact form (remove all the isospin indicies)

\[
M^{\pi\gamma} = V^{\pi\gamma} - \int V^{\pi\pi} S M^{\pi\gamma} \quad \text{(a)}
\]
\[
M^{\gamma\pi} = V^{\gamma\pi} - \int V^{\gamma\pi} S M^{\pi\pi} \quad \text{(b)}
\]

(VII.8)

where $M^{\pi\gamma}$, $M^{\gamma\pi}$, and $M^{\pi\pi}$ are the scattering matrices for photoproduction, photoabsorption, and pion-nucleon scattering, and $V^{\pi\gamma}$, $V^{\gamma\pi}$, and $V^{\pi\pi}$ are the driving terms (potentials) for photoproduction, photoabsorption, and $\pi N$ scattering. Taking the Dirac conjugate of Eq. (VII.8b), and using the fact that the driving terms are constructed from real invariant functions, so that

\[
V^{\pi\gamma} = \overline{V^{\gamma\pi}}
\]
\[
\overline{V^{\pi\pi}} = V^{\pi\pi},
\]

we obtain

\[
\overline{M}^{\pi\pi} = V^{\pi\gamma} - \int \overline{M}^{\pi\pi} \overline{S} V^{\pi\pi}.
\]

(VII.10)

Using Eq. (VII.8a) to replace the $V^{\pi\gamma}$ driving term under the integral in this equation gives the following non-linear equation for $\overline{M}^{\pi\pi}$

\[
\overline{M}^{\gamma\pi} = V^{\pi\gamma} - \int \overline{M}^{\pi\pi} \overline{S} M^{\pi\gamma} - \int \overline{M}^{\pi\pi} \overline{S} V^{\pi\pi} S M^{\pi\gamma}.
\]

(VII.11)

A second non-linear equation can be obtained from Eq. (VII.8a) by using the Dirac conjugate of the $\pi N$ equation

\[
\overline{M}^{\pi\pi} = V^{\pi\pi} - \int \overline{M}^{\pi\pi} \overline{S} V^{\pi\pi}
\]

(VII.12)
to replace the $V^{yy}$ driving term under the integral

$$M^{yy} = V^{yy} - \int \overline{M}^{xx} S M^{yy} - \int \overline{M}^{xx} \overline{S} V^{yy} S M^{yy}. \quad \text{(VII.13)}$$

Subtracting Eq. (VII.11) from Eq. (VII.13) gives the elastic unitarity condition

$$M^{yy} - \overline{M}^{yy} = - \int \overline{M}^{xx} (S - \overline{S}) M^{yy}. \quad \text{(VII.14)}$$

Using time reversal invariance, the Dirac conjugate $\overline{M}^{yy}$ can be related to the complex conjugate of $M^{yy}$.

In each eigen-channel, the elastic unitarity condition (VII.14) automatically implies that the pion-photoproduction amplitude has the same phase as the $\pi N$ scattering amplitude, which is a statement of the Watson theorem [49]. However, above the inelastic threshold, i.e. when the $\pi\pi N$ intermediate states become physical, the driving terms in our equation become complex, the elastic unitarity condition no longer holds, and the Watson theorem no longer applies.

### VII.3 Introduction to the Model

In this section we prepare the way for a demonstration of gauge invariance by giving a brief introduction to our model of pion-photoproduction. A detailed discussion of the structure of the couplings and the definition of parameters will be deferred to Chapter VIII. Here we will limit the discussion to those points essential to the proof of gauge invariance.

Our amplitude for pion-photoproduction is given by the sum of the Born diagrams shown in Fig. 26 and their final state interactions, shown in Fig. 27. The Born diagrams 26(a), (b), (e2) and (e3) include (in principle) contributions from all
Figure 27: Final state interactions
of the resonances \( B \), but the contributions of the \( \Delta \) and \( D_{13} \) to diagram 26(b) are zero in the approximation we employ. Furthermore, the \( \gamma NN^* \), \( \gamma N\Delta \), \( \gamma ND_{13} \), \( \rho \pi\gamma \), and \( \omega \pi\gamma \) couplings are all separately gauge invariant, and hence the contributions of the baryon resonances to the diagrams (a) and (b), and of the \( \rho \) and \( \omega \) to diagram (c), can be ignored in the proof of gauge invariance, and will not be discussed further here. Diagrams 26(e3) and (f) are interaction currents which arise because of the momentum dependence of the elementary \( \pi N \) contact interaction and the \( \pi NN \), \( \pi N\Delta \), and \( \pi ND_{13} \) couplings. In our model the \( \pi NN^* \) coupling does not depend on the momentum, and hence the Roper makes no contribution to diagram (e3).

Note that dressed vertices are needed in diagrams (b), (c), (e), and (f) because final state interactions cannot describe any \( \pi N \) interactions which take place before the photon is absorbed. Interactions which take place after the photon is absorbed are part of the final state interactions, and hence diagram (a) must contain only the bare vertex in order to avoid double counting.

The interaction kernel obtained from the Born diagrams in Fig. 26 has the form

\[
\tilde{V}_j^{\pi\gamma}(k', q, P) = \epsilon_\mu \tilde{J}_j^{\mu B}(k', q, P),
\]

(VII.15)

where \( \epsilon_\mu \) is the polarization vector of the incoming photon and \( j \) is isospin of outgoing pion. The reduced current \( \tilde{J}_j^{\mu B} \) for the diagrams (a)-(d), including nucleons only, is

\[
\left( \tilde{J}_j^{\mu B} \right)_{a-d}(k', q, P) = \tau_j \tilde{\Gamma}_{N0}(k', P)\tilde{S}_N(P)\tilde{j}_j^{\mu N}(P, p)
+ \tilde{j}_j^{\mu N}(p', p - k')\tilde{S}_N(p - k')\tilde{\Gamma}_j^{\dagger N}(k', p)
+ \tilde{j}_j^{\mu N}(k', k' - q)\tilde{\Delta}(k' - q)\tilde{\Gamma}_j^{\dagger N}(k' - q, p) + \tilde{j}_j^{\mu KR}(q)\tilde{\gamma}_{N16}\]

(VII.16)

where \( \tilde{\Gamma}_j^{\dagger N}(k', p) \) is the reduced dressed \( \pi NN \) vertex for an outgoing pion \( j \) with four-momentum \( k' \), \( \tilde{\Gamma}_{N0}(k', P) \) is the reduced bare \( \pi NN \) vertex, \( \tilde{\Delta} \) is the damped
pion propagator, \(j_N^{\gamma}(p', p)\) and \(j_{\pi\pi}^{\gamma}(k', k)\) are the reduced \(\gamma NN\) and \(\gamma \pi\pi\) current operators, and \(j_{\pi\pi}^{\gamma}(q)\) is the reduced Kroll-Ruderman term [Fig. 26(d)]. These quantities will be discussed below. Note that \(p\) and \(p'\) are the four-momenta of the incoming and outgoing (off-shell) nucleon, \(q = p' - p\).

The bare, reduced \(\pi NN\) vertex, \(\tilde{\Gamma}_{N0}^\dagger(k, P)\), is a superposition of pseudoscalar and pseudovector couplings

\[
\tilde{\Gamma}_{N0}^\dagger(k, P) = g \left( \lambda - \frac{1 - \lambda}{2m} k \right) \gamma^5.
\]  

(VII.17)

Note that it does not depend on \(P\). The dressed vertex, which includes all of the \(\pi N\) contact interactions, satisfies

\[
\tilde{\Gamma}_{jN}^\dagger(k', p) = \tau_j \tilde{\Gamma}_{N0}^\dagger(k', P) - \mathcal{T}_{i/2}^{ji} \int dk'' \tilde{M}_{c,1/2}^{\pi\pi}(k', k'', p) \tilde{S}_N(p - k'') \tilde{\Gamma}_{N0}^\dagger(k'', P)
\]

- \[
= \tau_j \tilde{\Gamma}_{N0}^\dagger(k', p) - \mathcal{T}_{i/2}^{ji} \int dk'' \tilde{V}_{c,1/2}^{\pi\pi}(k', k'', p) \tilde{S}_N(p - k'') \tilde{\Gamma}_{jN}^\dagger(k'', P)
\]

where \(\tilde{V}_{c,1/2}^{\pi\pi}\) is the reduced \(\pi N\) contact interaction (in the isospin \(1/2\) channel), and \(\tilde{M}_{c,1/2}^{\pi\pi}\) is the reduced iteration of these contact interactions to all orders (see Part A). The \(\mathcal{T}_{i/2}^{ji}\) is an isospin \(1/2\) projection operator. The isospin \(3/2\) will not contribute to the second term, because \(\mathcal{T}_{3/2}^{i}\) = 0. In the third term of Eq. (VII.16), the vertex \(\tilde{\Gamma}_{jN}^\dagger(k' - q, p)\) describes the coupling of a nucleon to an off-shell pion \(j\), which is, strictly speaking, an amplitude outside of the framework of our model. However, since the reduced contact interactions \(\tilde{V}_{c,1/2}^{\pi\pi}\) do not depend on the pion momenta (see the next section) and the reduced bare vertex depends on pion the momentum only through the \(\frac{(1-\lambda)(q)}{2m}\) term in Eq. (VII.17), the reduced off-shell vertex is easily obtained by simply using the (correct) off-shell pion four-momentum in the formula for the on-shell vertex.

The full result for pion-photoproduction, including final state interactions, will
be written

\[ M_j^{\pi}(k', q, P) = \epsilon_{\mu} \mathcal{J}_j^\mu(k', q, P) \]  

(VII.19)

where the current \( \mathcal{J}_j^\mu \) is a sum of the Born terms and integrals over the \( \pi N \) scattering amplitude [the diagrams shown in Fig. 27(a)–(f)]

\[ \mathcal{J}_j^\mu(k', q, P) = \tilde{J}_j^\mu_B(k', q, P) - \int dk'' \tilde{M}_j^\pi(k', k'', P) \tilde{S}_N(P - k'') \tilde{J}_j^\mu_B(k'', q, P). \]  

(VII.20)

Note that this equation is merely a statement of Eq. (VII.3), with

\[ \int dk'' = \int \frac{d^3k''}{2\omega_{k''}(2\pi)^3}. \]  

(VII.21)

We are now ready to prove that the expression (VII.20) is gauge invariant.

### VII.4 Gauge Invariance

Using the notation and the relativistic equations discussed above, we will now show that the photoproduction amplitude obtained from the driving terms shown in Fig. 26 is gauge invariant. As mentioned in the previous section, the \( \gamma NN^*, \gamma ND, \gamma ND_{13}, \rho \pi \gamma, \) and \( \omega \pi \gamma \) couplings are separately gauge invariant, so contributions to diagrams 26(a)–(c) from these resonances will be ignored here. The proof will follow the method introduced by Gross and Riska [37].

The reduced single nucleon current operator, denoted by \( \tilde{j}_N^\mu \) above, and the reduced single pion current operator, denoted by \( \tilde{j}_{\pi N}^\mu \) above, have the following structure:

\[ \tilde{j}_N^\mu = \tau_p \tilde{j}_N^\mu \]

\[ \tilde{j}_{\pi N}^\mu = -i\epsilon_{ij3} \tilde{j}_{\pi N}^\mu, \]  

(VII.22)
where \( \tau_p = \frac{1}{2}(1 + \tau_3) \) is the charge operator for the nucleon. In the pion current, \( l \) and \( j \) are the isospin of the incoming and outgoing pion, respectively.

The proof begins with the fact that the current operators \( \bar{J}^\mu_N \) and \( \bar{J}^\mu_\pi \), can be constructed so as to satisfy Ward-Takahashi (WT) identities involving the damped propagators. These WT identities are

\[
q_\mu \bar{J}^\mu_N (p', p) = (-ie) \left[ \bar{S}_N^{-1}(p) - \bar{S}_N^{-1}(p') \right]
\]

(VII.23)

and

\[
q_\mu \bar{J}^\mu_\pi (k', k) = (-ie) \left[ \bar{\Delta}_\pi^{-1}(k) - \bar{\Delta}_\pi^{-1}(k') \right].
\]

(VII.24)

The damped nucleon propagator \( \bar{S}_N(p) \) and the damped pion propagator \( \bar{\Delta}_\pi(k) \) are

\[
\bar{S}_N(p) = \frac{f_N^2(p^2)}{m - p - i\epsilon} = f_N^2(p^2) S_N(p^2)
\]

(VII.25)

and

\[
\bar{\Delta}_\pi(k) = \frac{f_\pi^2(k^2)}{\mu^2 - k^2 - i\epsilon} = f_\pi^2(k^2) \Delta_\pi(k^2),
\]

(VII.26)

where \( f_N(p^2) \) and \( f_\pi(k^2) \) are phenomenological form factors. The nucleon form factor, \( f_N(p^2) \), has already been discussed; the pion form factor, \( f_\pi(k^2) \), damps the off-shell contributions of the pion, which occur only in diagrams 26(c), (e1) and (e2), and their final state interaction contributions. Note the these form factors are unity when the particles are on their mass-shell: \( f_N(m^2) = 1 = f_\pi(\mu^2) \).

Now compute the four-divergence of the nucleon pole contributions to the Born terms 26(a)–(d), given in Eq. (VII.16) above. Allowing for the fact that the final nucleon will be off-shell when the Born terms are used to calculate the final state interactions, and that the form factors are unity when the particle is on-shell, the Ward-Takahashi identities give

\[
q_\mu \left( \bar{J}_B^\mu \right)_{a-d} = i e r_T \bar{\Gamma}^4_{N0}(k', P) \bar{S}_N(P) \left( \bar{S}_N^{-1}(P) - 0 \right)
\]
\[ + i e r_p \left( \check{S}_N^{-1}(p') - \check{S}_N^{-1}(p - k') \right) \check{S}_N(p - k') \tilde{\Gamma}_j N(k', p) \]
\[ \quad + i e(-i\epsilon_{j3}) \left( 0 - \Delta^{-1}(k' - q) \right) \Delta \pi(k' - q) \tilde{\Gamma}_j^t N(k' - q, p) + q_{\mu} \tilde{J}^\mu_{KR}(q) \]
\[ = i e r_p \tilde{\Gamma}_j^t N_0(k', P) - i e r_p \tilde{\Gamma}_j^t N(k', p) \]
\[ \quad + i e(i\epsilon_{j3}) \tilde{\Gamma}_j^t N(k' - q, p) + q_{\mu} \tilde{J}^\mu_{KR}(q) \]
\[ + i e r_p \check{S}_N^{-1}(p') \check{S}_N(p - k') \tilde{\Gamma}_j^t N(k', p) . \]  
(VII.27)

Using the relativistic wave equation (VII.18) for the dressed vertex permits us to write

\[ q_{\mu} \left( \tilde{J}_{jB}^\mu \right)_{a-d} = \left[ i e r_p \tilde{\Gamma}_j^t N_0(k', P) - i e r_p \tilde{\Gamma}_j^t N_0(k', p) \right] \]
\[ \quad + i e(i\epsilon_{j3} \tau_1) \tilde{\Gamma}_j^t N_0(k' - q, p) + q_{\mu} \tilde{J}^\mu_{KR}(q) \]
\[ + i e r_p \int dk'' \check{V}_c^{\pi\pi} (k', k'', p) \tilde{A}_{1/2}^{t} \check{S}_N(p - k'') \tilde{\Gamma}_j^{t} N(k'', p) \]
\[ \quad - i e(i\epsilon_{j3}) \int dk'' \check{V}_c^{\pi\pi} (k' - q, k'', p) \tilde{A}_{1/2}^{m} \check{S}_N(p - k'') \tilde{\Gamma}_j^{t} N(k'', p) \]
\[ + i e r_p \check{S}_N^{-1}(p') \check{S}_N(p - k') \tilde{\Gamma}_j^t N(k', p) . \]  
(VII.28)

Next, we recall that \( \tilde{\Gamma}_j^t N_0(k', P) \) does not depend on \( P \), and observe that

\[ \tilde{\Gamma}_j^t N_0(k' - q, p) = \tilde{\Gamma}_j^t N_0(k', p) + g \left( 1 - \frac{1}{\gamma^5} \right) \frac{1}{2m} . \]  
(VII.29)

Hence, since \( \tau_j \tau_p - \tau_p \tau_j = -i\epsilon_{j3} \tau_1 \), we see that the first four terms in square brackets in Eq. (VII.28) will be zero provided

\[ q_{\mu} \tilde{J}^\mu_{KR}(q) = i g e \left( 1 - \frac{1}{\gamma^5} \right) \frac{1}{2m} \gamma^5 . \]  
(VII.30)

This constraint will be satisfied by the Kroll-Ruderman term given in Chapter VIII. Using this constraint, and the fact that the reduced \( \pi N \) contact interaction, \( \check{V}_c^{\pi\pi}(k', k'', p) = \check{V}_c^{\pi\pi}(p) \), depends only on the total momentum \( p \), the divergence of the diagrams in Fig. 26(a)-(d) becomes finally

\[ q_{\mu} \left( \tilde{J}_{jB}^\mu \right)_{a-d} = \quad i e \int dk'' \check{V}_c^{\pi\pi} (p) \check{S}_N(p - k'') \tilde{A}_{1/2}^{t} \tilde{\Gamma}_j^{t} N(k'', p) \tau_p \]
\[ + i e r_p \check{S}_N^{-1}(p') \check{S}_N(p - k') \tilde{\Gamma}_j^t N(k', p) . \]  
(VII.31)
Now we add in the final state interactions from diagrams 27(a)-(d). It is convenient at this point to consider the final state interactions in the isospin $I = \frac{1}{2}$ and $\frac{3}{2}$ states separately. These states can be separated out by the isospin $\frac{1}{2}$ and $\frac{3}{2}$ projection operators, which are

\[
\mathcal{I}_{1/2}^{ji} = \frac{1}{3} \tau_j \tau_i
\]
\[
\mathcal{I}_{3/2}^{ji} = \delta_{ji} - \frac{1}{3} \tau_j \tau_i, \quad \text{(VII.32)}
\]

where $j$ and $i$ are the isospins of the outgoing and incoming pions, respectively. Hence the first term in Eq. (VII.31) is pure $I = \frac{1}{2}$

\[
\mathcal{I}_{3/2}^{ji} \tau_i = \left[ \delta_{ji} - \frac{1}{3} \tau_j \tau_i \right] \tau_i = 0 \quad \text{(VII.33)}
\]

and does not contribute to the discussion of $I = \frac{3}{2}$ gauge invariance. The second term in Eq. (VII.31) contributes to both isospin channels:

\[
\mathcal{I}_{1/2}^{ji} \tau_i = \frac{1}{2} \tau_j \left( 1 - \frac{1}{3} \tau_3 \right)
\]
\[
\mathcal{I}_{3/2}^{ji} \tau_i = \mathcal{I}_{3/2}^{ji}, \quad \text{(VII.34)}
\]

but is zero for the Born terms because the final nucleon is on shell. Hence the full contribution of the $I = \frac{3}{2}$ final states to the photoproduction amplitude, Eq. (VII.20), from the terms driven by the diagrams (a)-(d) is

\[
g_{\mu} \left( \mathcal{J}^{\Delta}_{3/2} \right)_{a-d} = -ie \int dk'' \bar{M}_{3/2}^{\tau_i}(k', k'', P) \tilde{S}_N(P - k'') \tilde{S}_N^{-1}(P - k'') \times \tilde{S}_N(p - k'') \tilde{\Gamma}_{N_0}^t(k'', p)
\]
\[
= -ie \int dk'' \bar{M}_{3/2}^{\tau_i}(k', k'', P) \tilde{S}_N(p - k'') \tilde{\Gamma}_{N_0}^t(k'', p), \quad \text{(VII.35)}
\]

where the isospin factors can be dropped after Eq.(VII.34). Note that the integral term in $\Gamma_{N}^t(k'', p)$ will not contribute to the isospin channel. If the amplitude, as presently constructed, were gauge invariant, Eq. (VII.35) would give zero. We must add several extra terms in order to get a gauge invariant result.
These extra terms are driven by the diagrams shown in Fig. 26(e1)–(e3). The diagrams (e1) and (e2) contain the overall isospin factor \(-i\epsilon_{j3}\), which can be decomposed into isospin \(\frac{1}{2}\) and \(\frac{3}{2}\) parts

\[-i\epsilon_{j3} = -\mathcal{I}_{3/2}^{j3} + 2\mathcal{I}_{1/2}^{j3}. \quad (VII.36)\]

Hence the \(I = \frac{3}{2}\) contribution from these diagrams is (dropping the overall factor of \(\mathcal{I}_{3/2}^{j3}\))

\[
\left(\tilde{J}^\mu_B\right)_{e,\frac{3}{2}} = \int dk'' \left[ \tilde{\mathcal{V}}_{\gamma\pi}^\tau(k', k'' + q, P) + \tilde{\Gamma}^\tau_\Delta_0(k', P)\tilde{G}_\Delta_0(P)\tilde{\Gamma}_\Delta_0(k'' + q, P) \right] 
\times \tilde{S}_N(p - k'', \bar{p}) 
+ \int dk'' \tilde{\Gamma}^\tau_\Delta_0(k', P)\tilde{G}_\Delta_0(P)\tilde{J}^\mu_\Delta_{\frac{3}{2}}(q, P)\tilde{S}_N(p - k'', \bar{p}) \quad (VII.37)
\]

where \(\tilde{J}^\mu_\Delta_{\frac{3}{2}}(q, P)\) is the isospin \(\frac{3}{2}\) interaction current for \(\gamma + \pi + N \rightarrow \Delta\), with \(q\) the momentum of the incoming photon and \(P\) the momentum of the outgoing \(\Delta\). This current is the \(\Delta\) contribution to the diagram shown in Fig. 26(e3). Using the WT identity to take the four-divergence of (VII.37) gives

\[
\left(\tilde{J}^\mu_B\right)_{e,\frac{3}{2}} = i \int dk'' \left[ e\tilde{\mathcal{V}}_{\gamma\pi}^\tau(P) + \tilde{\Gamma}^\tau_\Delta_0(k', P)\tilde{G}_\Delta_0(P) \left\{ e\tilde{\Gamma}_\Delta_0(k'' + q, P) + q_\mu\tilde{J}^\mu_\Delta_{\frac{3}{2}}(q, P) \right\} \right] 
\times \tilde{S}_N(p - k'', \bar{p}) \quad (VII.38)
\]

where we used the fact that \(\tilde{\mathcal{V}}_{\gamma\pi}^\tau\) depends on \(P\) only. In the next section we will show that the interaction current satisfies the following relation

\[ q_\mu\tilde{J}^\mu_\Delta_{\frac{3}{2}}(q, P) = -e \left( \tilde{\Gamma}_\Delta_0(k'' + q, P) - \tilde{\Gamma}_\Delta_0(k'', P) \right) \quad (VII.39) \]

Using this constraint, Eq. (VII.38) becomes

\[
\left(\tilde{J}^\mu_B\right)_{e,\frac{3}{2}} = ie \int dk'' \tilde{\mathcal{V}}_{\gamma\pi}^\tau(k', k'', P)\tilde{S}_N(p - k'', \bar{p})\tilde{\Gamma}_N_0(k'', \bar{p}) \quad (VII.40)
\]
where

\[ \tilde{V}_{3/2}^{\pi\pi}(k', k'', P) = \tilde{V}_{3/2}^{\pi\pi}(P) + \tilde{\Gamma}_{\Delta 0}^i(k', P) \tilde{G}_{\Delta 0}(P) \tilde{\Gamma}_{\Delta 0}(k'', P) \]  

is the full kernel for \( \pi N \) scattering in the \( I = \frac{3}{2} \) isospin channel.

Including the final state interactions, the full contributions generated by diagram 26(e) are

\[ q_\mu \left( \mathcal{J}_{1/2}^- \right) = \text{ie} \int dk'' \left[ \tilde{V}_{3/2}^{\pi\pi}(k', k'', P) - \int dk \tilde{M}_{3/2}^{\pi\pi}(k', k, P) \tilde{S}_N(P - k) \tilde{V}_{3/2}^{\pi\pi}(k, k'', P) \right] \times \tilde{S}_N(p - k'') \tilde{\Gamma}_{\Delta 0}^i(k'', p) \]

\[ = \text{ie} \int dk'' \tilde{M}_{3/2}^{\pi\pi}(k', k'', P) \tilde{S}_N(p - k'') \tilde{\Gamma}_{\Delta 0}^i(k'', p), \]  

where, in the second step, we used the wave equation for \( \tilde{M}_{3/2}^{\pi\pi} \) to reduce the expression. Note that the contributions from diagrams (e), Eq. (VII.42), cancel the contributions from diagrams (a)-(d), Eq. (VII.35), proving that the \( I = \frac{3}{2} \) amplitude is gauge invariant.

We now turn to a discussion of the \( I = \frac{1}{2} \) amplitude. The proof for this channel is similar to the one given above, but we must add the additional contributions from Eq. (VII.31), and also be careful to consider the different isospin operators which can contribute to this channel. Using the results from Eqs. (VII.31) and (VII.42) we get

\[ q_\mu \left( \mathcal{J}_{1/2}^- \right)_{a-e} = \text{ie} \int dk'' \tilde{V}_{1/2}^{\pi\pi}(p) \tilde{S}_N(p - k'') \mathcal{I}_{1/2}^i \tilde{\Gamma}_{\Delta 0}^i N(k'', p) \tau_p \]

\[- \text{ie} \int dk' \tilde{M}_{1/2}^{\pi\pi}(k', k, P) \tilde{S}_N(P - k) \times \int dk'' \tilde{V}_{1/2}^{\pi\pi}(p) \tilde{S}_N(p - k'') \mathcal{I}_{1/2}^i \tilde{\Gamma}_{\Delta 0}^i N(k'', p) \tau_p \]

\[- \text{ie} \int dk'' \tilde{M}_{1/2}^{\pi\pi}(k', k'', P) \tilde{S}_N(p - k'') \mathcal{I}_{1/2}^i \tau_p \tilde{\Gamma}_{\Delta 0}^i N(k'', p) \]

\[- \text{ie} \int dk'' \tilde{M}_{1/2}^{\pi\pi}(k', k'', P) \tilde{S}_N(p - k'') \mathcal{I}_{1/2}^i (-\text{ie}^{im3}) \tilde{\Gamma}_{\Delta 0}^i N(k'', p) (\text{VII.43}) \]
where the first term is the contribution of the Born terms from diagrams (a)-(d), the next two terms are the final state interactions generated by these Born terms, and the last term is the contribution from diagrams 26(e) and 27(e). To obtain the last term in the form given above, we followed steps similar to those leading to Eq. (VII.42), eliminating the isospin $\frac{1}{2}$ interaction currents, associated with the diagrams 26(e3) and 27(e2), using a generalization of the constraint (VII.39)

$$q_{\mu}\tilde{J}^{\mu}_{B_{1/2}}(q, P) = -e\left(\tilde{\Gamma}_{B_{0}}(k'' + q, P) - \tilde{\Gamma}_{B_{0}}(k'', P)\right),$$

(VII.44)

where $B = \{N, D_{13}\}$. Note that the Roper has no interaction current because, by construction, its coupling is independent of the pion momentum. In the next section we will show that these constraints are satisfied.

Adding the last two terms in Eq. (VII.43), and replacing $\tilde{M}$ by its integral equation, $\tilde{M} \rightarrow \tilde{V} - \int \tilde{M} \tilde{S} \tilde{V}$, allows us to rewrite Eq. (VII.43) in the following form:

$$q_{\mu}\left(\mathcal{J}_{i_{1/2}}^{\mu}\right)_{a-e} = ie\int dk'' \left[\tilde{V}_{c_{1/2}}^{\tau}(p) - \tilde{V}_{c_{1/2}}^{\tau}(k', k'', P)\right] \tilde{S}_{N}(p - k'')\mathcal{I}_{1/2}^{\tau} \tilde{N}(k'', p)\tau_{p}$$

$$- ie\int dk\tilde{M}_{1/2}^{\tau}(k', k, P)\tilde{S}_{N}(P - k)\int dk'' \left[\tilde{V}_{c_{1/2}}^{\tau}(p) - \tilde{V}_{c_{1/2}}^{\tau}(k, k'', P)\right]$$

$$\times \tilde{S}_{N}(p - k'')\mathcal{I}_{1/2}^{\tau} \tilde{N}(k'', p)\tau_{p}.$$  

(VII.45)

Next, we recall from Eq. (VII.41) that $\tilde{V}^{\pi\pi}$ is the sum of a connected part and a resonance part. The contributions from the resonance part to Eq. (VII.45) involves the following integrals

$$I_{l\Gamma} = \int dk'' \tilde{\Gamma}_{B_{0}}(k'', P)\tilde{S}_{N}(P - k'')\mathcal{I}_{1/2}^{\tau} \tilde{N}(k'', p)$$

$$= \int dk'' \tilde{\Gamma}_{B_{0}}(k'', P)\tilde{S}_{N}(P - k'')\mathcal{I}_{1/2}^{\tau} \tilde{N}(k'', p),$$

(VII.46)

where the second expression follows from the fact that $\tilde{\Gamma}_{B_{0}}$ is not a function of the nucleon momentum. However, for different reasons, these integrals (VII.46)
are all zero. The integral describing the $N \rightarrow D_{13}$ transition is zero because the nucleon and $D_{13}$ are orthogonal in our model, and the transition to the Roper is zero because the physical nucleon is defined by the condition that it be orthogonal to the Roper resonance at the Roper pole (see the discussion in Part A). Finally the $N \rightarrow N$ contribution is the value of the nucleon self energy at the nucleon pole, and, as discussed in Part A, we adjust the parameters of the $\pi N$ driving terms so as to insure that this quantity is zero. This constraint, which we call the stability condition, is an approximate way to include higher order interactions and ensures that the model is stable under small changes in the physical input (see Part A). Because of these conditions, Eq. (VII.45) reduces to

$$q_{\mu} \left( \mathcal{J}_{\frac{3}{2}}^\mu \right)_{a-e} = i e \int d k'' \left[ \tilde{V}^{\pi\pi}_{c,\frac{3}{2}}(p) - \tilde{V}^{\pi\pi}_{c,\frac{3}{2}}(P) \right] \tilde{S}_N(p - k'') \tilde{\mathcal{T}}_{1/2}^{\pi\pi}_{i} \tilde{N}(k'', p) \tau_p$$

$$- i e \int d k' \tilde{M}^{\pi\pi}_{c,\frac{3}{2}}(k', k, P) \tilde{S}_N(p - k) \int d k'' \left[ \tilde{V}^{\pi\pi}_{c,\frac{3}{2}}(p) - \tilde{V}^{\pi\pi}_{c,\frac{3}{2}}(P) \right]$$

$$\times \tilde{S}_N(p - k'') \tilde{\mathcal{T}}_{1/2}^{\pi\pi}_{i} \tilde{N}(k'', p) \tau_p .$$

(VII.47)

This term is canceled by the second type of interaction current, illustrated in Figs. 26(f) and 27(f). This interaction current contributes the following terms to the amplitude

$$\left( \mathcal{J}_{\frac{3}{2}}^\mu \right)_{a} = \int d k'' \tilde{\mathcal{J}}_{c,\frac{3}{2}}^\mu (q, P) \tilde{S}_N(p - k'') \tilde{\mathcal{T}}_{1/2}^{\pi\pi}_{i} \tilde{N}(k'', p) \tau_p$$

$$- \int d k' \tilde{\mathcal{M}}^{\pi\pi}_{c,\frac{3}{2}}(k', k, P) \tilde{S}_N(p - k) \int d k'' \tilde{\mathcal{J}}_{c,\frac{3}{2}}^\mu (q, P) \tilde{S}_N(p - k'')$$

$$\times \tilde{\mathcal{T}}_{1/2}^{\pi\pi}_{i} \tilde{N}(k'', p) \tau_p ,$$

(VII.48)

where the first term is the Born term shown in Fig. 26(f) and the second is the final state interactions shown in Fig. 27(f). Later we will show that this term satisfies the following constraint

$$q_{\mu} \tilde{\mathcal{J}}_{c,\frac{3}{2}}^\mu (q, P) = i e \left( \tilde{V}^{\pi\pi}_{c,\frac{3}{2}}(P) - \tilde{V}^{\pi\pi}_{c,\frac{3}{2}}(P - q) \right) ,$$

(VII.49)
which is precisely what is needed to cancel the contribution from Eq. (VII.47).
Hence, the gauge invariance of the $I = \frac{1}{2}$ channels has been proven.

We have proved that our theory involving the driving terms shown in Fig. 26
and the final state $\pi N$ interactions shown in Fig. 27 is gauge invariant provided

(i) the interaction currents satisfy the constraints (VII.39), (VII.44), and (VII.49),

(ii) the $\gamma NN^*$, $\gamma N\Delta$, $\gamma ND_{13}$, $\rho \pi \gamma$, and $\omega \pi \gamma$ couplings are all explicitly gauge in-

variant, and

(iii) the reduced one body currents satisfy the WT identities (VII.23) and (VII.24).

These results will be demonstrated in the following sections.

We turn now to a detailed description of the couplings, parameters and the
driving terms.
VIII The Photoproduction Model

This Chapter is divided into four sections. In the first we write down all of the couplings which describe the direct electromagnetic production of the Roper, $\Delta$, and $D_{13}$ from the nucleon. These expressions contain the precise definitions of the resonance photoproduction parameters given in Table 2, and are individually gauge invariant, which justifies neglecting them in the discussion given in Chapter VII. Next, we construct off-shell current operators for the single nucleon and single pion which are consistent with the WT identities Eqs. (VII.23) and (VII.24). These current operators are modified by the presence of the nucleon and pion form factors. In the next section we construct the interaction currents implied by the momentum dependence of the electromagnetic couplings and the contact interaction $\tilde{V}_e^{\pi\pi}$. To obtain these interaction currents, we use minimal substitution, and then demonstrate that they satisfy the necessary constraints obtained in Chapter VII. Finally, we assemble the pieces and construct the actual pion-photoproduction driving terms which fully define the model.

VIII.1 Electromagnetic Couplings

In this section we define the electromagnetic transition currents for the baryon resonances, $\gamma NB$. 
VIII.1.a Delta Current

According to Jones and Scadron [77] the $\gamma N\Delta$ transition current can be written in terms of a standard "normal parity" set of invariants $\mathcal{O}_i^{\mu\nu}\gamma_5$. For real photons this gives

$$j_\Delta^{\mu\nu}(P,p) = (-ie)T_3[G_1\mathcal{O}_1^{\mu\nu} + G_2\mathcal{O}_2^{\mu\nu}]\gamma_5,$$  \hspace{1cm} (VIII.1)

where $T_3$ is the third component of the isospin $1/2 \rightarrow 3/2$ transition operator, and the current conserving spin invariants are

$$\mathcal{O}_1^{\mu\nu} = (q g^{\mu
u} - q^{\mu} r^{\nu})$$

$$\mathcal{O}_2^{\mu\nu} = (q^{\mu} P^{\nu} - q P^r g^{\mu\nu}).$$  \hspace{1cm} (VIII.2)

Here $q$ is the photon momentum, and $P' = \frac{1}{2}(p + P)$, where $p$ and $P$ are the four-momentum of nucleon and $\Delta$, respectively. The $G_1$ and $G_2$ couplings are often written in terms of the magnetic coupling $G_M$ and the electric coupling $G_E$:

$$G_M = \left[ \frac{(3M + m)(M + m)G_1}{M} + (M^2 - m^2)G_2 \right] \frac{m}{3(M + m)}$$

$$G_E = \left[ \frac{(M^2 - m^2)G_1}{M} + (M^2 - m^2)G_2 \right] \frac{m}{3(M + m)},$$  \hspace{1cm} (VIII.3)

where $M$ is the $\Delta$ mass.

Benmerrouche et al. [14] obtain $N\Delta$ transition currents from the following two contributions to the Lagrangian

$$L_{\gamma N\Delta}^1 = \frac{ieq_1}{2m}T_3 \bar{\psi}_\mu \Sigma^{\mu\lambda}(Y)\gamma_\nu \gamma_5 \psi F^{\nu\lambda} + h.c.$$  \hspace{1cm} (VIII.4)

$$L_{\gamma N\Delta}^2 = -\frac{eg_2}{4m^2}T_3 \bar{\psi}_\mu \Sigma^{\mu\lambda}(X)\gamma_5 \partial_\lambda \psi F^{\nu\lambda} + h.c.,$$

where $\psi$ and $\Psi_\mu$ are the nucleon and delta fields, respectively, and $\Sigma_{\mu\nu}(X)$ is

$$\Sigma_{\mu\nu}(X) = g_{\mu\nu} + \left[ \frac{1}{2}(1 + 4X)A + X \right] \gamma_\mu \gamma_\nu,$$  \hspace{1cm} (VIII.5)
where $A$ and $X$ are parameters. The interaction derived from Eq. (VIII.4) using the $g_{\mu\nu}$ term in $\Sigma_{\mu\nu}(X)$ (and removing the factor of $e$) gives Eq. (VIII.1) with

$$G_1 = \frac{g_1}{2m}$$
$$G_2 = \frac{g_2}{4m^2}. \quad \text{(VIII.6)}$$

The couplings of Refs. [14] and [77] therefore differ by an extra term which depends on $X$, and which can be shown to vanish at the $\Delta$ pole.

In order to be consistent with our pion-nucleon model, we introduce a new $\gamma N\Delta$ current which has almost the same form as the current derived from the Lagrangian (VIII.4). Our current is

$$j^{\mu\nu}_\Delta(P, p) = (-ie)T_3 \Theta_\Delta^{\mu}(P) \left[ \frac{g_1}{2m} \Theta^{\lambda\nu}_1 + \frac{g_2}{4m^2} \Theta^{\lambda\nu}_2 \right] \gamma_5 \frac{1}{f_\Delta(P^2)} \frac{P^2}{m^2}. \quad \text{(VIII.7)}$$

where $f_\Delta$ is the delta form factor. We found in difficult to fit the $E_1^+$ and $M_1^+$ amplitudes without removing the strong form factor from the $\Delta$ pole diagram, and to accomplish this we have divided the transition current (VIII.7) by this form factor (which then cancels the form factor connected to the $\pi N\Delta$ vertex). This detail will be discussed ini Chapter VIII.2.C. The last factor in the above equation is introduced to kill the pole in the $\Theta_\Delta^{\mu}(P)$ This can be done without spoiling gauge invariance because the $\gamma N\Delta$ transition current is separately gauge invariant. Because of the properties of the spin 3/2 projection operator, our coupling (VIII.7), the coupling derived from Eq. (VIII.4), and the coupling (VIII.1) give the same scattering amplitude.
VIII.1.b  Roper Current

The $\gamma N N^*$ transition current is

$$j_{N^*}^{\mu}(p', p) = -ie\gamma \left( g_{N^*} + \frac{(p' + p)^{\mu}}{p'^2 - p^2} + g_{2N^*} \frac{i\sigma^{\mu\nu}q_{\nu}}{2m} \right) \frac{1}{f_{N^*}} , \quad (\text{VIII.8})$$

where $q, p$ and $p'$ are the momenta of the photon, the nucleon and the Roper, respectively, and $g_{N^*}$ and $g_{2N^*}$ are the strength of the two independent couplings. We multiply the Roper current with $\frac{1}{f_{N^*}}$ in order to be consistent with the Delta (see section VIII.2.c.). Note that

$$q_{\mu}j_{N^*}^{\mu}(p', p) = 0 , \quad (\text{VIII.9})$$

showing that all diagrams containing the Roper transition current are individually gauge invariant.

VIII.1.c  $D_{13}$ Current

Like the $\Delta$, the $D_{13}$ also has two independent couplings. The $D_{13}$ current is similar to the $\Delta$ current except it has an opposite parity and isospin 1/2. The current is

$$j_{D}^{\mu}(P, p) = -e\tau_3 \Theta_{A}^{\mu}(P) \left[ \frac{g_{1D}}{2m} \sigma_{1}^{\lambda\nu} + \frac{g_{2D}}{4m^2} \sigma_{2}^{\lambda\nu} \right] \frac{1}{f_{D}(P^2)} \frac{P^2}{m^2} . \quad (\text{VIII.10})$$

In order to maintain consistency with the treatment of the $\Delta$ described above, we have also divided this current by the form factor of the $D_{13}$. We multiply the current by factor $\frac{P^2}{m^2}$ to eliminate the pole in the spin 3/2 projection operator $\Theta_{A}^{\mu}(P)$. We now turn to a discussion of the construction of the off-shell current operators for the nucleon and the pion.
VIII.2 Off-shell Electromagnetic Currents

As discussed in Chapter VII, the current operators must satisfy the WT identities Eqs. (VII.23) and (VII.24). These involve damped propagators, instead of bare propagators, and as a result the current operators will have a different structure from those usually encountered.

VIII.2.a Nucleon Current

A complete description of the general off-shell nucleon current requires 12 invariant functions:

\[ \tilde{j}_N(p',p) = -ie \left( F_1 \gamma^\mu + F_2 \frac{i \sigma^{\mu\nu} q^\nu}{2m} + F_3 q^\mu \right) \]
\[ + \Lambda_-(p') \left[ F_4 \gamma^\mu + F_5 \frac{i \sigma^{\mu\nu} q^\nu}{2m} + F_6 q^\mu \right] \]
\[ + \left[ F_7 \gamma^\mu + F_8 \frac{i \sigma^{\mu\nu} q^\nu}{2m} + F_9 q^\mu \right] \Lambda_-(p) \]
\[ + \Lambda_-(p') \left[ F_{10} \gamma^\mu + F_{11} \frac{i \sigma^{\mu\nu} q^\nu}{2m} + F_{12} q^\mu \right] \Lambda_-(p) \]  

(VIII.11)

where the negative energy projection operator is

\[ \Lambda_-(p) = \frac{m - \not{p}}{2m}. \]  

(VIII.12)

This current operator must satisfy the Ward-Takahashi identity (VII.23)

\[ q\mu \tilde{j}_N(p',p) = -ie \left( \tilde{S}_N^{-1}(p) - \tilde{S}_N^{-1}(p') \right) = -ie \left( \frac{m - \not{p}}{f_N(p^2)} - \frac{m - \not{p'}}{f_N(p'^2)} \right) \]  

(VIII.13)

where \( f_N(p^2) \) is the nucleon form factor. Writing out both sides of this equation gives

\[ F_1 \not{q} + F_3 q^2 + \Lambda_-(p') [F_{10} \not{q} + F_{12} q^2] \Lambda_-(p) \]
Equating the coefficients of the four independent Dirac matrices on each side of this equation gives four relations between the invariant functions which permits us to eliminate $F_3$, $F_6$, $F_9$, and $F_{12}$ from this equation; we can get:

$$F_3 = F_7 \left( \frac{m^2 - p^2}{2mq^2} \right) - F_4 \left( \frac{m^2 - p'^2}{2mq^2} \right)$$

$$F_{12} = \frac{2m}{q^2} (F_7 - F_4)$$

$$F_6 = \left( \frac{-2m}{q^2 f^2} \right) + F_{10} \left( \frac{m^2 - p^2}{2mq^2} \right) + \frac{2m}{q^2} (F_1 + F_4)$$

$$F_9 = \left( \frac{-2m}{q^2 f^2} \right) + F_{10} \left( \frac{m^2 - p'^2}{2mq^2} \right) + \frac{2m}{q^2} (F_1 + F_7),$$

where $f = f_N(p^2)$ and $f' = f_N(p'^2)$. Substituting these constraints into Eq. (VIII.11), and taking the limit as $q^2 \to 0$, gives the following most general form for the current operator of a real photon:

$$\tilde{J}_N^\mu(p', p) = -ie \left( F_0 \gamma^\mu + F_2 \frac{i \sigma^{\mu\nu}q_\nu}{2m} \right.$$

$$+ \Lambda_-(p') F_5 \frac{i \sigma^{\mu\nu}q_\nu}{2m} + F_8 \frac{i \sigma^{\mu\nu}q_\nu}{2m} \Lambda_-(p)$$

$$- ie \left( \frac{1}{f^2} - \frac{1}{f'^2} \right) \frac{4m^2}{p'^2 - p^2} \Lambda_-(p') \gamma^\mu \Lambda_-(p)$$

$$+ F_{11} \Lambda_-(p') \frac{i \sigma^{\mu\nu}q_\nu}{2m} \Lambda_-(p) \left. \right),$$

where

$$F_0 = \left( \frac{1}{f^2} \frac{m^2 - p^2}{p^2} + \frac{1}{f'^2} \frac{m^2 - p'^2}{p'^2} \right).$$

For simplicity, in this calculation we take $F_5 = F_8 = 0$ and $F_2 = F_0 \kappa_N$, where $\kappa_N$ is the magnetic moment of the nucleon and $F_{11} = F_{10}$. 
VIII.2.b Pion Current

Following Gross and Riska [37], a simple off-shell current operator which satisfies the WT identity (VII.24) is

$$\bar{j}^{\mu}_{\pi\nu}(k', k) = -ie(k + k')^\mu \left[ 1 + \frac{\Pi(k'^2) - \Pi(k^2)}{k'^2 - k^2} \right], \quad (VIII.18)$$

where $k$ and $k'$ are the momenta of the incoming and outgoing pion, and

$$\Pi(k^2) = \left[ \frac{1}{f^2_{\pi}(k^2)} - 1 \right] (k^2 - \mu^2) \quad (VIII.19)$$

and $f_{\pi}(k^2)$ is the pion form factor. If one of the pion is on-shell the formula will reduce to

$$\bar{j}^{\mu}_{\pi\nu}(k', k) = -ie(k + k')^\mu \frac{1}{f^2_{\pi}(k^2)}, \quad (VIII.20)$$

the form factor will be cancelled by the pion form factor connected to the $\pi NN$ vertex.

VIII.2.c Resonances

The strong form factor in the resonances channel ($\Delta, D_{13}, N^*, \rho$ and $\omega$) will not spoil the gauge invariance because the $\gamma NN^*, \gamma N\Delta, \gamma ND_{13}, \rho \pi \gamma$ and $\omega \pi \gamma$ are all separately gauge invariant. However it is found that it is difficult to fit the $E_{1+}$ and $M_{1+}$ amplitudes without modifying the $\gamma N\Delta$ vertex. The modification is done by multiplying the vertex by $1/f_\Delta(p^2)$, where $f_\Delta(p^2)$ is the $\Delta$ form factor. This factor will cancel the $\Delta$ form factor connected to the $\pi NN$ vertex.

To be consistent with the $\gamma N\Delta$ vertex, the other electromagnetic couplings ($N, N^*, D_{13}, \rho$ and $\omega$) are also multiplied by $1/f_R(p^2)$, where $R$ stands for ($N, N^*, D_{13}, \rho$ and $\omega$).
VIII.3 Interaction Currents

In this section we derive the exact forms of the interaction currents introduced in Sec. III and shown in Figs. 26(e3) and (f), and 27(e2) and (f).

VIII.3.a The Five-Point Current

We begin with a discussion of the five-point current, \( J_{c/2}^{\mu}(q, P) \), shown in Fig. 26(f). This current appears because of the momentum dependence in the crossed \( \pi N \) diagram that was approximated as a contact interaction.

The structure of this current can be obtained by minimal substitution. To obtain this current, we start from Fig. 28(a) which describes a contact interaction of pions and nucleons in pion-nucleon scattering. In part \( A \) we have shown that this contact interaction only depends on the total momentum \( P \). The reduced contact interaction as shown in Fig. 28(a) can be written as the following,

\[
\bar{V}_{c/2}^{\mu, jli}(k', k, P) = (A_{1/2} + B_{1/2}P + A_{0,1/2}\gamma^0)\mathcal{I}_{1/2}^{jl} + (A_{3/2} + B_{3/2}P + A_{0,3/2}\gamma^0)\mathcal{I}_{3/2}^{jl} 
\]

\[
= (A_{1/2} + B_{1/2}(p + \kappa) + A_{0,1/2}\gamma^0)\mathcal{I}_{1/2}^{jl} + (A_{3/2} + B_{3/2}(p + \kappa) + A_{0,3/2}\gamma^0)\mathcal{I}_{3/2}^{jl}. \quad \text{(VIII.21)}
\]

where \( A, A_0 \) and \( B \) are constant. In the coordinate space the isospin \( 1/2 \) part is
given by,

\[ \tilde{V}_{\pi, j}^{\pi} (x, y) = \int \frac{d^4k d^4p}{(2\pi)^8} \tilde{V}_{\pi, j}^{\pi}(k', k, P)e^{ik.x}e^{ip.y} \]

\[ = \int \frac{d^4k d^4p}{(2\pi)^8} T_{1/2}^{lm} \delta_{ml} \]

\[ \times \left( A_{1/2} + A_{0,1/2} \gamma^0 - iB_{1/2} \gamma^\mu \left( \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu} \right) \right) e^{ik.x}e^{ip.y}. \quad (VIII.22) \]

The minimal substitution is done by applying the following substitution to Eq. (VIII.22).

\[ \partial_i^\mu \delta_{ml} = \partial_i^\mu \delta_{ml} + ie_i^{ml} A^\mu(x_i) \quad (VIII.23) \]

where \( x_1 = x, x_2 = y, e_2^{ml} = e_{\tau \delta_{ml}} = e \left( \frac{1+\Delta}{2} \right) \delta_{ml} = \) proton charge, and \( e_1^{ml} = e \left( -i e^{ml} \right) = \) pion charge. After the minimal substitution we have,

\[ \tilde{V}_{\pi, j}^{\pi} (x, y) = \int \frac{d^4k d^4p}{(2\pi)^8} e^{ik.x}e^{ip.y} T_{1/2}^{lm} \left( A_{1/2} \delta_{ml} + A_{0,1/2} \delta_{ml} \right) \]

\[ - iB_{1/2} \gamma^\mu \left( \partial_\mu \delta_{ml} + ie_1^{ml} A_\mu(x) + \partial_\mu \delta_{ml} + ie_2^{ml} A_\mu(y) \right) \quad (VIII.24) \]

Following Ohta [72], the interaction current generated by this contact interaction can be obtained by differentiating Eq. (VIII.24) with respect to \( A_\mu \),

\[ \Delta \tilde{M}_{1/2}^{\mu j} (x, y) = - \left[ \frac{\delta \tilde{V}_{\pi, j}^{\pi} (x, y)}{\delta A_\mu} \right] \left. \right|_{A \rightarrow 0} \]

\[ = - \int \frac{d^4k d^4p}{(2\pi)^8} B_{1/2} \gamma^\mu T_{1/2}^{lm} (e_1 + e_2)^{ml} e^{ik.x}e^{ip.y} \]

\[ = \int \frac{d^4k d^4p}{(2\pi)^8} e^{ik.x}e^{ip.y} \tilde{J}_{\pi, j}^{\mu l} (q, P) \quad (VIII.25) \]

where,

\[ \tilde{J}_{\pi, j}^{\mu l} (q, P) = - B_{1/2} \gamma^\mu T_{1/2}^{lm} (e_1 + e_2)^{ml} \]

\[ = - e B_{1/2} \gamma^\mu T_{1/2}^{lm} (-ie^{ml3} + \tau_p \delta_{ml}) \quad (VIII.26) \]
Using this current, finally the interaction vertex in Fig. 26 (f) can be written as the following (for isospin 1/2 in the final state),

\[
\left( \tilde{J}_{jB}^\mu \right)_{I, \frac{1}{2}} = -i \int \frac{d^3k''}{(2\pi)^3 2\omega_{k''}} \tilde{J}_{jB}^{\mu \iota \jmath}(q, P) \tilde{S}_N(p - k'') \tilde{I}_N^{\dagger}(k'', p)
\]

\[= i e \int \frac{d^3k''}{(2\pi)^3 2\omega_{k''}} B_{\frac{1}{2}} \gamma^\mu \tilde{T}_{1/2}^{\iota \jmath m} \left( -i \epsilon^{m \ell \alpha} + \tau_\nu \delta_{m \ell} \right) \tilde{S}_N(p - k'') \tilde{I}_N^{\dagger}(k'', p)
\]

\[= i e \int \frac{d^3k''}{(2\pi)^3 2\omega_{k''}} B_{\frac{1}{2}} \gamma^\mu \tilde{S}_N(p - k'') \tilde{T}_{1/2}^{\iota \jmath \ell} \tilde{I}_N^{\dagger} \tau_\nu(k'', p) \tag{VIII.27}
\]

This current will give the first part of Eq. (VII.48), if it is contracted with \( q^\alpha \).

The second part of Eq. (VII.48) can be derived from Fig. 28(a) using the minimal substitution as we did before. In this case we include the final state interaction (see Fig. 27(f)).

For the isospin 3/2 we have,

\[
\left( \tilde{J}_{jB}^\mu \right)_{I, \frac{3}{2}} = i e \int \frac{d^3k''}{(2\pi)^3 2\omega_{k''}} B_{\frac{3}{2}} \gamma^\mu \tilde{S}_N(p - k'') \tilde{T}_{3/2}^{\iota \jmath \ell} \tilde{I}_N^{\dagger} \tau_\nu(k'', p)
\]

\[= 0 \tag{VIII.28}
\]

This result is consistent with our discussion in section VII.4.

**VIII.3.b The Four-Point Current**

The four-point current, \( \tilde{J}_{jB}^\mu \) shown in Fig. 26(c3) appears because of the momentum dependence in the \( \pi NN, \pi N\Delta \) and \( \pi ND_{13} \) vertices.

The structure of this current can also be obtained by minimal substitution. To obtain this current, we start from Fig. 28(b) which describes \( \pi NN, \pi N\Delta, \pi ND_{13} \) and \( \pi NN^* \) interactions. The \( \pi NN^* \) vertex does not depend on the momentum therefore it will not contribute to this four-point current.
First let's consider the $\pi N\Delta$. The reduced $\pi N\Delta$ vertex is given by, is,

$$\tilde{\Gamma}_{i\Delta}^{\mu}(P, k) = -\left(\frac{g_{\Delta}}{\mu}\right) \Theta^{\nu\mu}(P)k_{\nu}T_{i}$$ (VIII.29)

where $g_{\Delta}$ is the coupling strength of baryon $\Delta$ and $T_{i}$ is the isospin $3/2 \rightarrow 1/2$ transition operator.

In the coordinate space this interaction is given by,

$$\tilde{\Gamma}_{i\Delta}^{\mu}(x, y) = -\int \frac{d^{4}kd^{4}p}{(2\pi)^{8}} \left(\frac{g_{\Delta}}{\mu}\right) \Theta^{\nu\mu}(P)T_{m}\delta_{l\mu}e^{ikx}e^{ipy}$$

$$= i\int \frac{d^{4}kd^{4}p}{(2\pi)^{8}} \left(\frac{g_{\Delta}}{\mu}\right) \Theta^{\nu\mu}(P)T_{m}\delta_{l\mu}e^{ikx}e^{ipy}$$ (VIII.30)

After the minimal substitution, we have

$$\tilde{\Gamma}_{i\Delta}^{\mu}(x, y) = i\int \frac{d^{4}kd^{4}p}{(2\pi)^{8}} \left(\frac{g_{\Delta}}{\mu}\right) \Theta^{\nu\mu}(P)T_{m}(\partial_{\mu}\delta_{m} + ie^{m}_{l}\partial_{l\mu}(x))e^{ikx+ipy}$$ (VIII.31)

Differentiating this formula with respect to $A_{\mu}$ and multiply by $-1$ (consistent with the previous section) yield,

$$\tilde{M}_{i}^{\nu\mu}(x, y) = \int \frac{d^{4}kd^{4}p}{(2\pi)^{8}} e^{ikx+ipy} \tilde{\Gamma}_{l\Delta}^{\nu\mu}$$ (VIII.32)

where,

$$\tilde{\Gamma}_{l\Delta}^{\nu\mu} = e\left(\frac{g_{\Delta}}{\mu}\right) \Theta^{\nu\mu}(P)T_{m}(-ie^{m}_{l})$$ (VIII.33)

This current is exactly the same as the current in Eq.(VII.44). Using this current Fig. 26(e3) can be written as following,

$$\tilde{J}_{i\Delta}^{\mu}(k', P) = -i\tilde{\Gamma}_{\Delta}^{\mu}(k', P)\tilde{G}_{\Delta\alpha\alpha\nu}(P)T_{j}^{\rho\nu}$$

$$\times \int \frac{d^{3}k''}{(2\pi)^{3}2\omega_{k''}} \tilde{S}_{N}(p - k'')\tilde{T}_{l}N(p - k'', p)$$

$$= -\tilde{\Gamma}_{\Delta}^{\mu}(k', P)\tilde{G}_{\Delta\alpha\alpha\nu}(P)\left(ie\frac{g_{\Delta}}{\mu}\right) \Theta^{\nu\mu}(P)$$

$$\times \int \frac{d^{3}k''}{(2\pi)^{3}2\omega_{k''}} \tilde{S}_{N}(p - k'')T_{j}T_{m}(-ie^{m}_{l})\tilde{T}_{l}N(p - k'', p)$$ (VIII.34)
The following four-point current generated from $\pi ND_{13}$ vertex can be obtained by the same manner. For the $D_{13}$ current we have,

$$\left( J_{jD}^\mu \right)_{e_3} = \tilde{\Gamma}_{D0}(k', P) \tilde{G}_{D0,\alpha\nu}(P) \left( e^{-}\frac{g_D}{\mu} \right) \Theta^{\mu\nu}(P) \gamma^5$$

$$\times \int \frac{d^3k''}{(2\pi)^3 2\omega_{k''}} \tilde{S}_{N}(p - k'') \tau_j \tau_m (-ie^{\text{em}}) \tilde{I}_{1N}(p - k'', P)$$

(VIII.35)

The four-point current generated from the $\pi NN$ vertex can also be derived by the same way, we have

$$\left( J_{jN}^\mu \right)_{e_3} = \tilde{\Gamma}_{N0}(k', P) \tilde{G}_{N0}(P) \left( i ge^{-}\frac{1}{2m} \gamma^\mu \gamma^5 \right)$$

$$\times \int \frac{d^3k''}{(2\pi)^3 2\omega_{k''}} \tilde{S}_{N}(p - k'') \tau_j \tau_m (-ie^{\text{em}}) \tilde{I}_{1N}(p - k'', P)$$

(VIII.36)

VIII.4 Driving terms

The main goal of this part is to calculate explicitly all the driving terms in Fig. 26 using the electromagnetic currents as described in the previous sections.

VIII.4.a Nucleon

The direct nucleon pole diagram [Fig. 26(a)] is:

$$\left( J_{jN}^\mu \right)_a (k', q, P) = -ige \left( \lambda - \frac{(1 - \lambda)k'}{2m} \right) \gamma^5 \left( \frac{1}{m - P} \right)$$

$$\times (\gamma^\mu \tau_j \tau_p - \frac{1}{2} [\gamma^\mu \hat{a} - \hat{a}^\mu] \tau_j \mu N)$$

(VIII.37)

where $\mu$ is the photon polarization vector index, $q$ and $k'$ are the photon and pion momenta respectively, $\tau_j$ is the isospin of nucleon coupled to the pion field with isospin index $i$, and $\mu N = \frac{1}{2} [\mu_p + \mu_n + (\mu_p - \mu_n) \tau_3]$ is the nucleon anomalous magnetic
moment. Note that the formfactor doesn’t appear in this formula. When one of the nucleons in the $\gamma NN$ vertex is on-shell, Eq. (VIII.16) will reduce to $-ie\gamma_\mu/F_N^2(p+k)$. This term will exactly cancel the formfactor of this direct nucleon pole diagram. The $p+k$ is momentum of the off-shell nucleon. The overall factor of $i$ in this term is needed to insure that the driving term satisfies Eq. (VII.9).

$$
\left(\tilde{J}_{jN}^\mu\right)_a(k', q, P) = \gamma_0 i ge (\gamma^\mu \tau_p \tau_j - \frac{1}{2}[\gamma^\mu - \gamma^\nu \gamma^\nu] \mu_N \tau_j) \left(\frac{1}{m - P^*}\right) \gamma^5 \\
\times \left(\lambda - \frac{(1 - \lambda)k^*}{2m}\right) \gamma_0
$$

This term will exactly cancel the formfactor of this direct nucleon pole diagram. The $p+k$ is momentum of the off-shell nucleon. The overall factor of $i$ in this term is needed to insure that the driving term satisfies Eq. (VII.9).

$$
\left(\tilde{J}_{jN}^\mu\right)_a(k', q, P) = \gamma_0 i ge (\gamma^\mu \tau_p \tau_j - \frac{1}{2}[\gamma^\mu - \gamma^\nu \gamma^\nu] \mu_N \tau_j) \left(\frac{1}{m - P^*}\right) \gamma^5 \\
\times \left(\lambda - \frac{(1 - \lambda)k^*}{2m}\right) \gamma_0
$$

This condition is needed to give the correct unitarity relation, as we saw in Sec. VII.

The crossed nucleon pole diagram [Fig. 26(b)] is

$$
\left(\tilde{J}_{jN}^\mu\right)_b(k', q, P) = -ige \left(F_0 \left[\gamma^\mu \tau_p \tau_j - \frac{1}{2}(\gamma^\nu \gamma^\nu - \gamma^\nu \gamma^\nu) \mu_N \tau_j\right]ight) \\
+ G_0 \Lambda_-(p') \left[\gamma^\mu \tau_p \tau_j - \frac{1}{2}(\gamma^\nu \gamma^\nu - \gamma^\nu \gamma^\nu) \mu_N \tau_j\right] \Lambda_-(Q) \\
\times \left(\frac{f_2(Q^2)}{m - Q}\right) \tilde{\Gamma}_N(k', p)
$$

where $Q = p - k'$, and $\tilde{\Gamma}_N(k', p)$ is the dressed, but reduced, $\pi NN$ vertex, which satisfies Eq. (VII.18). The $F_0$ and $G_0$ are defined in Eq. (VIII.16) and Eq. (VIII.17). In this term the formfactor is not cancelled, because both nucleons in the $\gamma NN$ vertex are off-shell.

The Kroll-Ruderman term, Fig. 26(d), has two terms. The first term can be obtained from the momentum dependence of the $\pi NN$ coupling using minimal sub-
stitution (Eq. VII.30), and the second is needed in order to insure that contributions from pseudoscalar and pseudovector coupling give identical results at threshold, as required by the low energy theorem [64]. The complete Kroll-Ruderman is

\[ J^\mu_{KR} = -ig \left[ \epsilon_{ji3} \tau_i \frac{(1 - \lambda)\gamma^\mu}{2m} + \frac{\lambda}{4m} [\gamma^\mu \not{d} - \not{d} \gamma^\mu] \left( \mu_N \tau_j + \tau_j \mu_N \right) \right] \gamma^5. \] (VIII.40)

Note that the second term is separately gauge invariant, and therefore did not enter into the proof of gauge invariance presented in Sec. VII. Gauge invariance requires no formfactor in this diagram. The additional interaction current driving terms are obtained from the interaction currents worked out above. The nucleon contribution from the diagram shown in Fig. 26(e3) is (Eq. (VIII.35))

\[ \left( \tilde{J}^\mu_{jN} \right)_{e3} (k', q, P) = i eg \tilde{\Gamma}_{N0}(P) \tilde{G}_{N0}(P) \frac{(1 - \lambda)\gamma^\mu}{2m} \gamma^5 \]
\[ \times \int \frac{d^3k''}{(2\pi)^32\omega_{k''}} \frac{F^2_N[(p - k'')^2]}{(m - \not{p} + \not{k}'')\tau_j\tau_i \left( -ie^{im3} \right) \tilde{\Gamma}_m N(k'', p)}. \] (VIII.41)

The contribution from the five-point contact current shown in Fig. 26(f) is (Eq. (VIII.27)),

\[ \left( \tilde{J}^\mu_{jN} \right)_{f, \frac{1}{2}} (q, P) = ie B_{\frac{1}{2}} \gamma^\mu \int \frac{d^3k''}{(2\pi)^32\omega_{k''}} \frac{F^2_N[(p - k'')^2]}{(m - \not{p} + \not{k}'')\tau_j^1 \tilde{\Gamma}_1^1 N(k'', p) \tau_j^1} \] (VIII.42)

where \( B_{\frac{1}{2}} \) is defined in Eq. (VIII-21). This current only contributes to isospin \( \frac{1}{2} \) channel.

VIII.4.b Meson Exchange

The pion pole diagram, Fig. 26(c), is

\[ \left( \tilde{J}^\mu_{j\pi} \right)_c (k', q, P) = -ige \frac{(k' + k)\mu}{\mu^2 - k^2} \left( -ie^{im3} \right) \tilde{\Gamma}_m N(k, p). \] (VIII.43)
where \( k = p - p' = k' - q \) is the four-momentum of the off shell pion. Note that the square of the pion form factor associated with the damped propagator of the pion is cancelled by the factor of \( 1/f_\pi(k^2) \) in the off shell current. The vertex function \( \tilde{\Gamma}_{IN}^+(k, p) \) describes the coupling to an off-shell pion \( l \), which, because pions are on-shell in our propagators, does not appear as an elementary amplitude in our model. However, as discussed in Sec. VII.3, the simple structure of our model permits us to obtain the reduced off-shell vertex function from the reduced on-shell one by simply using the correct off-shell pion four-momentum.

Pion pole terms also enter into the diagrams shown in Figs. 26(e1), and (e2). These are

\[
\left( \tilde{J}_{j,m}^\mu \right)_{e1+e2} (k', q, P) = -ie \int \frac{d^3k''}{(2\pi)^3 2\omega_{k''}} \tilde{V}_{ji}^{\pi\pi}(k', k'', P) \epsilon^{im3} \frac{(2k'' + q)^\mu}{\mu^2 - (k'' + q)^2} \times \frac{F_k^2((p - k'')^2)}{m - \not{p} + \not{k''}} \tilde{\Gamma}_{mN}^+(k'', p),
\]

(VIII.44)

where \( \tilde{V}_{ji}^{\pi\pi}(k', k, P) \) is the driving term for \( \pi N \) scattering for incoming and outgoing pions with isospin \( l \) and \( j \), respectively. Note that the pion form factor has again disappeared, showing that it does not enter into the final result. (This is because the only place where pions are off-shell is when they are connected to the off-shell current (VIII.20), which has a factor which cancels the form factor.)

The driving terms also include additional contributions to Fig. 26(c) coming from \( \omega \) and \( \rho \) exchange. The \( \omega \) exchange diagram is

\[
\left( \tilde{J}_{j,\omega}^\mu \right) (k', q, P) = \delta_{j,3} \frac{e \int \omega_{NNN} \omega_{\pi\pi}}{\mu m_\omega^2 - (k - q)^2} \epsilon^{\mu\nu\lambda\rho} \epsilon_{\nu\lambda\rho} (k - q) \gamma_5 \left( \gamma_\rho + \frac{\kappa_\omega}{2m} i \sigma_{\rho \eta} (k - q)^\eta \right)
\]

(VIII.45)

where \( \epsilon_{0123} = 1 \). Using the following identity,

\[
\gamma^5 = -\frac{i}{24} \epsilon^{\mu\nu\lambda\rho} \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho
\]

(VIII.46)
or
\[ 1 = -\frac{i}{24} \varepsilon^{\mu\nu\lambda\rho} \gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho \gamma^5 \tag{VIII.47} \]

and using the identity of the multiplication of the totally antisymmetric tensor as written in the beginning of this dissertation, the \( \omega \) exchange diagram becomes,
\[
(J_{j, \omega})_{c} (k', q, P) = -i \delta_{j,3} \frac{e f_{\omega NN G_{\mu \nu \gamma}}}{\mu} \left( 1 - \frac{\kappa_{\omega}}{2m} 2(k - \bar{q}) \right) \gamma^5 \frac{-k.q_{\ell} + k.c_{\ell} + k_{\ell} q_{\ell}}{m_{\omega}^2 - (k - q)^2} \tag{VIII.48} \]

The \( \rho \) exchange diagram is the same as the \( \omega \) exchange diagram, except for the isospin. The \( \rho \) is pure isovector.
\[
(J_{j, \rho})_{c} (k', q, P) = -i \tau_{j} \delta_{j,3} \frac{e f_{\rho NN G_{\mu \nu \gamma}}}{\mu} \left( 1 - \frac{\kappa_{\rho}}{2m} 2(k - \bar{q}) \right) \gamma^5 \frac{-k.q_{\ell} + k.c_{\ell} + k_{\ell} q_{\ell}}{m_{\rho}^2 - (k - q)^2} \tag{VIII.49} \]

VIII.4.c Roper

A roper has the same spin structure as a nucleon, therefore the direct and crossed roper diagram have the same structure as the nucleon diagrams.

The direct roper pole diagram Fig. 26(a) is:
\[
(J_{j, N^*})_{a} (k', q, P) = -i g_{N^*} \gamma^5 \left( \frac{1}{m - \bar{P}} \right) (g_{1N^*} \gamma^\mu \tau_j - g_{2N^*} \frac{1}{2} [\gamma^\mu \bar{q} - \bar{q} \gamma^\mu] \tau_j) \gamma^5 \tag{VIII.50} \]

and for the crossed roper pole diagram Fig. 26(b) is:
\[
(J_{j, N^*})_{b} (k', q, P) = -i g_{N^*} \gamma^5 \tau_j (g_{1N^*} \gamma^\mu - \frac{(P + p - 2k')^\mu}{(P - k')^2 - (p - k')^2})
- g_{2N^*} \frac{1}{2} [\gamma^\mu \bar{q} - \bar{q} \gamma^\mu] \gamma^5 \tag{VIII.51} \]
There is no formfactor in this diagram. It is cancelled by the electromagnetic formfactor (see section VIII.2.c).

VIII.4.d Delta

The crossed $\Delta$ in Fig. 26(b) is approximated to be zero. This approximation is consistent with the one that we did to the crossed $\Delta$ pole in our $\pi N$ model. This approximation will make the numerical calculation simpler because the spin 3/2 channel will almost decouple to spin 1/2 channel.

The direct $\Delta$ channel in Fig. 26(a) is

$$\left(\tilde{J}_j^{\mu}\right)_a(k',q,P) = -i \left(\frac{g_\Delta}{\mu}\right) k''\Theta_{\eta\nu}(P) \left(\frac{1}{m_\Delta - P}\right) [G_1 \mathcal{O}_1^{\nu} + G_2 \mathcal{O}_2^{\nu}] \gamma^5 \tau_j T_3$$  

(VIII.52)

Where $\Theta_{\eta\nu}$ is the spin 3/2 projection operator.

The four-point function in Fig 26(e3) is :

$$\left(\tilde{J}_j^{\mu}\right)_{E3}(k',q,P) = -ie \left(\frac{g_\Delta}{\mu}\right)^2 \frac{F_{\Delta}^2 F_{\pi}^2}{m_\Delta - P} k''\Theta_{\eta\nu}(P) \int \frac{d^3k'' F_{\pi}^2(p - k'')}{(2\pi)^3 2\omega_{k''}} \left(\frac{1}{(m - \not{q} + \not{k}')}\right)$$

(VIII.53)

VIII.4.e $D_{13}$

This resonance is similar to the previous one, except that it has the opposite parity and different isospin. The $D_{13}$ has an isospin 1/2.
The direct $D_{13}$ channel in Fig. 26(a) is

$$\langle \bar{J}_{j_{D_{13}}}^{\mu} (k', q, P) = -i \left( \frac{G_{D_{13}}}{\mu} \right) k'^{\nu} \Theta_{\eta\nu}(P) \left( \frac{1}{m_{D_{13}} + P} \right) \gamma^5 [G_{1D} \sigma_{\mu}^{\nu} + G_{2D} \sigma_{\mu}^{\nu}] \tau_j \tau_3 \rangle$$

(VIII.54)

The four-point function in Fig 26(e3) is:

$$\langle \bar{J}_{j_{D_{13}}}^{\mu} (k', q, P) = i \left( \frac{g_{D_{13}}}{\mu} \right)^2 \frac{F_{D}^2(P^2)}{m_{D_{13}} + P} k'^{\nu} \Theta_{\eta\nu}(P) \int \frac{d^3k''}{(2\pi)^3} \frac{F_{N}^2(p - k'')}{2\omega_{k''}} \frac{1}{(m - \not{p} + \not{k}'') \Sigma_{N}} \rangle$$

(VIII.55)

**VIII.4.f Inelasticity**

In our model we do not couple the photon to the inelastic channel, instead we add some diagrams to the driving terms as shown in fig. 29. With this new extra terms the Roper propagator can be written as following:

$$G(P) = \frac{-i}{m_{N} - P + \Sigma_{N}}$$

(VIII.56)

And for the $D_{13}$ propagator,

$$G_{D_{13}}(P) = \frac{-i \Theta_{\mu\nu}(P)}{m_{D} - P + \Sigma_{D}}$$

(VIII.57)

where $\Sigma_{N}$ and $\Sigma_{D}$ the self energy of Roper and $D_{13}$ which were discussed in Chapter III.7.
Figure 29: Inelastic Channel
IX Results and Discussions

Our principal results are shown in Figs. 30-41 and Table 2. The various contributions to the real part of the multipole amplitudes for pion-photoproduction from a proton are shown in Figs. 30-37 and the comparisons of the real and imaginary part of these amplitudes to the VPI interactive SAID program of Arndt and Roper [31] are shown in Figs. 38-41. The experimental phases shown in the figures come from the data base in the SAID program. The definitions of the parameters shown in Table 2 are given in Chapter VIII. In Table 2, the parameters $g_{1B}$ and $g_{2B}$ (where $B = \{N^*, \Delta, D\}$) describe the $\gamma NB$ couplings (there are two independent forms for each coupling, see Chapter VIII), the products $g_{\pi\gamma}g_{\nu NN}$ (where $\nu = \{\rho, \omega\}$) are the strengths of the $\rho\pi\gamma$ and $\omega\pi\gamma$ couplings (the fit can determine the product of these factors only), and the $f_{\nu NN}/g_{\nu NN}$ are the ratio of the tensor ($f_{\rho NN}$) to vector ($g_{\rho NN}$) strengths of the $\rho NN$ and $\omega NN$ couplings. The $f_{\rho NN}/g_{\rho NN}$ value given in Table II was taken from the $NN$ Model IA of Ref. [11], while the $f_{\omega NN}/g_{\omega NN}$ was adjusted to improve the fit. Without varying this parameter, it is very difficult to fit $E_{2+}(3/2)$.

Because of our choice of the spin 3/2 propagator and our approximation scheme which sets the crossed $\Delta$ and $D_{13}$ pole terms to be zero, the $\Delta$ and the $D_{13}$ only contribute to the $j = 3/2$ channels. It is therefore convenient to describe our fits to the $j = 1/2$ and $j = 3/2$ channels separately.

We begin with the $j = 1/2$ channels, shown in Figs. 30-37. These channels are driven by the nucleon and $N^*$ poles and crossed poles, and the $\pi$, $\omega$ and $\rho$ exchange terms (see Chapter VIII for details). These driving terms depend on only
Table 2: The parameters of the model. Those in bold face were varied during the fit; the others are determined by the fit.

<table>
<thead>
<tr>
<th>parameter</th>
<th>bare</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{1N}^*$</td>
<td>-0.129</td>
</tr>
<tr>
<td>$g_{2N}^*$</td>
<td>0.744</td>
</tr>
<tr>
<td>$g_{1\Delta}$</td>
<td>1.187</td>
</tr>
<tr>
<td>$g_{2\Delta}$</td>
<td>1.053</td>
</tr>
<tr>
<td>$g_{1D}$</td>
<td>-2.287</td>
</tr>
<tr>
<td>$g_{2D}$</td>
<td>-2.352</td>
</tr>
<tr>
<td>$g_{\rho\pi\gamma}g_{pNN}$</td>
<td>0.319</td>
</tr>
<tr>
<td>$g_{\omega\pi\gamma}g_{\omega NN}$</td>
<td>7.390</td>
</tr>
<tr>
<td>$f_{pNN}/g_{pNN}(=\kappa_\rho)$</td>
<td>7.52525</td>
</tr>
<tr>
<td>$f_{\omega NN}/g_{\omega NN}(=\kappa_\omega)$</td>
<td>-0.727</td>
</tr>
</tbody>
</table>

Figure 30: Fits to the real part of $E_{0^+}(I=1/2)$ amplitudes
Figure 31: Fits to the real part of $E_{0^+}(I = 3/2)$ amplitudes. The dashed line and widely spaced dotted line overlap the solid line. The $\rho$ doesn't contribute to this channel and the $N^*$ gives a very small contribution.
Figure 32: Fits to the real part of $M_{1-}(I = 1/2)$ amplitudes
Figure 33: Fits to the real part of $M_1^-(I=3/2)$ amplitudes. The dashed line and widely spaced dotted line overlap the solid line. The $\rho$ doesn't contribute to this channel and the $N^*$ gives a very small contribution.
five adjustable parameters: two \( \gamma NN^* \) couplings, denoted by \( g_{1N^*} \) and \( g_{2N^*} \), the \( \rho \pi \gamma \) and \( \omega \pi \gamma \) couplings, denoted by \( g_{\rho \pi \gamma} \times g_{\rho NN} \) and \( g_{\omega \pi \gamma} \times g_{\omega NN} \) and the magnetic moment coupling of the omega \( f_{\omega NN}/g_{\omega NN} \). Our fits to both the real and imaginary parts of the \( j = 1/2 \) multipole amplitudes is very good. In the \( S_{11} \pi N \) channel (Fig. 30) there is a small peak near 700 MeV that we can not describe. This peak is associated with \( \eta \) production, not included in our model. It is known that \( \eta \) production also contributes to the \( S_{31} \) channel (Fig. 31) at high energy.

In Figure 31 and 33, the dashed line is overlapping the wider space dotted line and it almost overlaps the solid line. This shows that the Roper gives a very small contribution to these channels. Because of the isospin structure, the \( \rho \) doesn’t contribute to these channels. Note that in the \( E_{0+}(I = 3/2) \) amplitude, the omega is very important, especially the magnetic moment term. This channel could not be fitted, unless we vary \( (f_{\omega NN}/g_{\omega NN}) \) coupling. The small value of \( (f_{\omega NN}/g_{\omega NN}) \) from one boson exchange model [11] doesn’t work.

Before we discuss the fits to the other channels, we wish to point out that the \( E_{0+}(I = 1/2) \) and \( M_{1-}(I = 1/2) \) amplitudes shown in Figs. 32 and 33, are particularly sensitive. To show how the total result is built up from individual contributions, the curves in the figures show the result when the kernel (i) includes only the direct nucleon pole term, the crossed nucleon exchange, the pion pole, and all the interaction currents associated with the nucleon (the dotted line), (ii) the terms in (i) plus the \( \omega \) exchanged (the dashed line), (iii) the terms in (ii) plus \( \rho \) exchange term (the dotted line, with wider space between dots), and finally (iv) the total result, which includes the terms in (iii) plus the \( N^* \) (the solid line). Since the contributions add non-linearly, it is difficult to extract the separate contributions
Figure 34: Fits to the real part of $M_{1+}(I = 3/2)$ amplitudes. The $\rho$ doesn’t contribute to this channel and the $N^*$ gives only a very small contribution from the figures. but we can conclude that $\omega$ and $\rho$ exchange contribution are very important to describe these two amplitudes. The roper is very significant especially in the $M_{1-}(I = 1/2)$ amplitudes. Without it one can not fit this amplitude.

The $j = 3/2$ channels are driven by the direct spin 3/2 resonance poles (from the $\Delta$ and $D_{13}$), the crossed $N$ and $N^*$ pole diagrams, and the $\pi$, $\rho$, and $\omega$ exchange diagrams. All of the parameters for the crossed and exchange diagrams were already determined by the $j = 1/2$ fit. The direct $\Delta$ pole, which contributes only to the $P_{33}$ final state, requires two new parameters (the couplings $g_{1\Delta}$ and $g_{2\Delta}$), and the
Figure 35: Fits to the real part of $E_{1+}(I = 3/2)$ amplitudes. The $\rho$ doesn't contribute to this channel and the $N^*$ gives only a very small contribution.
Figure 36: Fits to the real part of $M_2 - (I = 1/2)$ amplitudes. The long-dashed line almost overlaps the widely spaced dotted line (the $N^*$ gives only a very small contribution).
Figure 37: Fits to the real part of $E_{2^{-}}(I = 1/2)$ amplitudes. The long-dashed line almost overlaps the widely spaced dotted line (the $N^*$ gives only a very small contribution)
direct $D_{13}$ pole, which contributes only to the $D_{13}$ final state, requires two more (the couplings $g_{1D}$ and $g_{2D}$) The values of the $\gamma N\Delta$ couplings which we obtain are within range of other calculations [14] which used the Rarita Schwinger propagator to describe the spin 3/2 resonances. The $j = 3/2$ amplitudes are also fit reasonably well by our model. In Figure 34 and 35 rho does not contribute. The dashed line and the wider space dotted line almost overlap ($N^*$ gives only a very small contribution).
Figure 38: Comparison of our $E_{0^+}(I = 1/2$ and $I = 3/2)$ to SAID analysis.
Figure 39: Comparison of our $M_1^{-}(I = 1/2$ and $I = 3/2$) to SAID analysis.
Figure 40: Comparison of our $M_{1+}(I = 3/2)$ and $E_{1+}(I = 3/2)$ to SAID analysis.
Figure 41: Comparison of our $M_2-(I = 1/2)$ and $E_2-(I = 1/2)$ to SAID analysis.
The following conclusions can be drawn from the present work. A relativistic resonance model of pion-photoproduction, fully consistent with the $\pi N$ scattering model, has been found to give a good description of the process up to 750 photon laboratory energy. The model is covariant, satisfies unitarity up to first order in the electric charge $e$, and is gauge invariant to all orders. The simplicity and consistency of the two models means that they can be used as a basis for a treatment of the coupled $NN \leftrightarrow \pi NN$ system, and its electromagnetic extension to $\gamma NN$ and $\gamma \pi NN$.

The ratio $E_2/M_1 = -2.73\%$, implies that the $D$ state admixture in the predominantly $S$ state is not zero. This ratio is calculated by considering the $\Delta$ contribution only. This result show that the tensor interaction between quarks should not be neglected at all.

Finally as the last words I present the worst and the best features of our pion-nucleon scattering and pion photoproduction model. Hopefully in the next future someone will think how to improve the worst features but still keep the best features:
A. Worst Features:

1. The crossed $\pi N$ diagram was approximated by a contact interaction. This approximation destroyed the crossing symmetry that could be obtained by interchanging the particles momenta. The reason for this approximation was because I thought that the symmetry was only important at the lowest order Feynman diagram (tree approximation) and it was very difficult to maintain the symmetry at the higher order.

2. The inelasticity was described by a fictitious scalar, $\sigma^*$ particle instead of two-pion production which was believed to dominate the Roper and $D_{13}$ inelastic channels. We assumed that the two pions are bound together to be a scalar particle with its mass was equal to the mass of the two pions ($278 MeV$). This mass was chosen to insure that the $N\pi\pi$ threshold should be in the right place. This value was different from the $\sigma$ mass (obtained by other models) which is about $560 MeV$.

3. The model was very sensitive to the kind of formfactor, especially the $\Delta$ and $N^*$ formfactors. The $\Delta$ required the same form as the nucleon form factor which had a peak at the nucleon pole. However the Roper required a formfactor that had a peak at the $N^*$ resonance.

4. We kept the chiral symmetry only at threshold. The reason for that was because it was very difficult to maintain the symmetry in our model. The formfactor that was needed to ensure the convergence destroyed this symmetry.
B. Best Features:

1. The model worked well up to 1525 MeV total energy ($D_{13}$ resonance). This was the first time pion-nucleon scattering and pion-photoproduction were described very well up to that high energy by a dynamical model.

2. The unitarity was maintained up to the first order of the charge $e$. This feature preserved the conservation probability.

3. The model was covariant. This meant that the model was invariant under a Lorentz transformation. Therefore it was applicable to any frame.

4. The model was gauge invariant to all orders in the strong coupling. This was possible by adding several extra diagrams that were needed to recover the invariance that was spoiled by the strong formfactor. This was the first successful gauge invariant model of pion-photoproduction which included the final state interaction.

5. The model was simple, no pole associated with the crossed diagram. This simplicity made the calculation much simpler. Because of this simplicity the model can be easily extended to other processes such as: electroproduction, NN scattering etc.

6. The resonance parameters (such as resonance mass, coupling constant and width) were extracted automatically within the model. In the future this model may be used to analyze new data of $\pi - N$ scattering and $\pi - \gamma$ production giving the new value of the resonance parameters.
7. The model was able to justify the need of the mixed coupling in the $\pi NN$ vertex. The value of the mixing parameter was calculated from stability condition, a situation in which the nucleon mass was unshifted.

8. Nucleon and Roper were treated as coupled channels. The Nucleon and the Roper had the same spin structure, therefore it was necessary to treat them carefully and consistently. In my knowledge not many people treated the Nucleon and the Roper like this.

9. The $\pi N$ and $\pi \gamma$ scattering matrices were treated consistently. We used the same propagator for both processes. This consistency was very important. It gave a better physics because it considered all the multiple scattering (pion dynamics). Nobody had considered this consistency before.
Appendix A.

Alternative Forms for the Integral Equation

First, we show that

\[ \Gamma(p', P) = \Gamma_0(p', P) + \int d^3k \, V_{c}^{\pi\pi}(p', k, P)G_0(k, P)\Gamma(k, P) \]  \hspace{1cm} (A.1)

is equivalent to

\[ \Gamma(p', P) = \Gamma_0(p', P) + \int d^3k \, M_c^{\pi\pi}(p', k, P)G_0(k, P)\Gamma_0(k, P) \]  \hspace{1cm} (A.2)

where \( \Gamma_0 \) is the bare vertex, \( \Gamma \) is the dressed vertex, \( M_c^{\pi\pi} \) is the scattering matrix with the crossed diagrams as driving terms and \( V_{c}^{\pi\pi} \) stands for the potential (crossed diagrams). In this Appendix we absorb the minus sign in front of the integral in Eq. (11.10) into \( G_0 \), giving a plus sign in Eq. (A.1)

To carry out the proof, simplify the notation, and use the scattering equation

\[ M_c^{\pi\pi} = V_c^{\pi\pi} + V_c^{\pi\pi}G_0M_c^{\pi\pi} \]  \hspace{1cm} (A.3)

to write the second term in Eq. (A.1) in the form

\[ V_c^{\pi\pi}G_0\Gamma = M_c^{\pi\pi}G_0\Gamma - V_c^{\pi\pi}G_0M_c^{\pi\pi}G_0\Gamma \]  \hspace{1cm} (A.4)

Now since

\[ V_c^{\pi\pi}G_0M_c^{\pi\pi} = M_c^{\pi\pi}G_0V_c^{\pi\pi} \]  \hspace{1cm} (A.5)

Eq. (A.4) becomes

\[ V_c^{\pi\pi}G_0\Gamma = M_c^{\pi\pi}G_0\Gamma - M_c^{\pi\pi}G_0V_c^{\pi\pi}G_0\Gamma \]  \hspace{1cm} (A.6)
Substituting Eq. (A-1) into (A-6) gives,

\[ V_c^{\pi\pi}G_0\Gamma = M_c^{\pi\pi}G_0\Gamma - M_c^{\pi\pi}G_0(\Gamma - \Gamma_0) = M_c^{\pi\pi}G_0\Gamma_0 \]  

(A.7)

Finally substituting Eq. (A.7) into Eq (A.1) gives Eq. (A.2), in the short hand notation:

\[ \Gamma = \Gamma_0 + M_c^{\pi\pi}G_0\Gamma_0 \]  

(A.8)

Next, prove that

\[ M^{\pi\pi} = M_c^{\pi\pi} + \Gamma G \Gamma^\dagger \]  

(A.9)

is equal to the infinite sum of all the possible diagrams generated from driving terms which are the sum direct and contact diagrams

\[ M^{\pi\pi} = (V_c^{\pi\pi} + V_d^{\pi\pi}) + (V_c^{\pi\pi} + V_d^{\pi\pi})G_0M^{\pi\pi} \]  

(A.10)

Here \( M^{\pi\pi} \) is the scattering matrix, \( V_d^{\pi\pi} \) is the direct potential, \( G_0 \) is the two-body propagator, and \( G_0 \) and \( G \) are the bare and dressed propagator of the baryon, where

\[ G = \tilde{G}_0 + \tilde{G}_0 \Gamma_0 G_0 G \]  

(A.11)

Proof:

\[ V_c^{\pi\pi}G_0M^{\pi\pi} = V_c^{\pi\pi}G_0M_c^{\pi\pi} + V_c^{\pi\pi}G_0\Gamma G \Gamma^\dagger \]

\[ = M_c^{\pi\pi} - V_c^{\pi\pi} + V_c^{\pi\pi}G_0\Gamma G \Gamma^\dagger \]

\[ = M_c^{\pi\pi} - V_c^{\pi\pi} + \Gamma G \Gamma^\dagger - \Gamma_0 G \Gamma^\dagger \]

\[ = M^{\pi\pi} - V_c^{\pi\pi} - \Gamma_0 G \Gamma^\dagger \]  

(A.12)

where we used Eq. (A.9), (A.3) and (A.1).
Now consider the direct term, which in this notation is \( V_d^{\pi\pi} = \Gamma_0 \tilde{G}_0 \Gamma_0^\dagger \). Hence

\[
V_d^{\pi\pi} G_0 M^{\pi\pi} = \Gamma_0 \tilde{G}_0 \Gamma_0^\dagger G_0 (M_\pi^{\pi\pi} + \Gamma G \Gamma^\dagger) \\
= \Gamma_0 \tilde{G}_0 (\Gamma^\dagger - \Gamma_0^\dagger) + \Gamma_0 (G - \tilde{G}_0) \Gamma^\dagger \\
= \Gamma_0 \Gamma^\dagger - \Gamma_0 \Gamma_0^\dagger \\
\tag{A.13}
\]

where we used the complex conjugate of Eq. (A.8) and (A.11).

Adding Eq. (A.12) to Eq. (A.13), yields

\[
(V_c^{\pi\pi} + V_d^{\pi\pi}) G_0 M^{\pi\pi} = M^{\pi\pi} - V_c^{\pi\pi} - \Gamma_0 G_0 \Gamma_0^\dagger \\
= M^{\pi\pi} - V_c^{\pi\pi} - V_d^{\pi\pi} \\
\tag{A.14}
\]

which is the same as Eq. (A.10)
Appendix B

Angular Integrals

\[ \int k d\Omega_k = \frac{4\pi P}{W} \omega_k \quad \text{(B.1)} \]

\[ \int k^\mu d\Omega_k = \frac{4\pi P^\mu}{W} \omega_k \quad \text{(B.2)} \]

\[ \int k^\mu k d\Omega_k = 4\pi \frac{P^\mu P^\nu}{W^2} \left( \omega_k^2 + \frac{1}{3} k^2 \right) - 4\pi g^{\mu\nu} \frac{1}{3} k^2 \quad \text{(B.3)} \]

\[ \int k^\mu k^\nu d\Omega_k = 4\pi \frac{P^\mu P^\nu}{W^2} \left( \omega_k^2 + \frac{1}{3} k^2 \right) - 4\pi g^{\mu\nu} \frac{1}{3} k^2 \quad \text{(B.4)} \]

\[ \int k^\mu k^\nu k d\Omega_k = 4\pi \frac{P^\mu P^\nu P^\rho}{W^3} \omega_k \left( \omega_k^2 + k^2 \right) - 4\pi g^{\mu\nu} \frac{1}{3} k^2 \frac{P^\rho}{W} \omega_k - \left( \frac{4\pi}{3} k^2 \omega_k \right) \frac{\gamma^\mu P^\nu + P^\mu \gamma^\nu}{W} \quad \text{(B.5)} \]
Appendix C

Phase Shifts and Cross Section

The phase shift is calculated using,

\[ -\frac{M_{l+}^{\pi\pi}}{a} = \frac{\eta_{l+} e^{2i\delta_{l+}^{\pi\pi}} - 1}{2i} \]  \hspace{1cm} (C.1)

where \( \eta_{l\pm} \) and \( \delta_{l\pm}^{\pi\pi} \) are the inelasticity parameter and phase shift. The \( \pm \) refers to \( j = l \pm \frac{1}{2} \) and \( a \) is defined by

\[ a = \frac{8\pi^2 W}{m|\tilde{k}|} \]  \hspace{1cm} (C.2)

where \( \tilde{k} \) is the on-shell momentum in the cm system. The inelasticity parameter \( \eta_{l\pm} \) can be calculated from

\[ \frac{\eta_{l\pm}^2}{4} = \Re \left( \frac{M_{l\pm}^{\pi\pi}}{a} \right)^2 + \left( \frac{1}{2} + \Im \left( \frac{M_{l\pm}^{\pi\pi}}{a} \right) \right)^2 \]  \hspace{1cm} (C.3)

This formula can be obtained easily from Eq. (C.1).

The total crossed section formula is,

\[ \sigma_{tot}^t = \frac{4\pi}{k} \sum_{i=0}^{\infty}((1 + 1)\Re(f_{i+}) + i\Im(f_{i-})) \]  \hspace{1cm} (C.4)

where,

\[ f_{i\pm} = -\frac{M_{i\pm}^{\pi\pi}}{a} \]  \hspace{1cm} (C.5)

For \( \pi^+P \) system,

\[ \sigma_{tot} = \sigma_{tot}^{I=3/2} \]  \hspace{1cm} (C.6)

and for \( \pi^-P \) system we have,

\[ \frac{1}{3}(\sigma_{tot}^{I=3/2} + 2\sigma_{tot}^{I=1/2}) \]  \hspace{1cm} (C.7)
Appendix D

Width, Effective Mass and Effective Coupling

To calculate the width, the effective mass of resonances and effective coupling constant of resonance particles, we start from the pole diagrams in the scattering amplitudes (for the $D_{13}$ and $\Delta$ we have only 1 diagram, but for $P_{11}$ channel, after diagonalization, we have two pole diagrams as discussed in Chapter III).

These scattering amplitudes can be written in form:

$$M^{\pi\pi}(W) = \frac{g_B^2 f_B(W)}{m_B - W - i\Gamma/2} \tag{D.1}$$

where $g_B$ is the pion-baryon coupling constant. The imaginary part of this $M(W)$ (at $W = m_B$) is

$$\Im[M(m_B)] = \frac{g_B^2 f_B(m_B)}{\Gamma/2} \tag{D.2}$$

Taking the real part of the derivative of Eq. (D.1) gives

$$\Re\left(\frac{\partial M(m_B)}{\partial W}\right) = -\frac{g_B^2 f_B(m_B)}{\Gamma^2/4} \tag{D.3}$$

and from Eq. (D.2) and Eq.(D.3)

$$\Gamma = -\frac{2\Im[M(m_B)]}{\Re\frac{\partial M(m_B)}{\partial W}} \tag{D.4}$$

The effective mass $m_B$ is calculated using Eq. (IV.3).

The effective pion-baryon coupling constant can also be derived from Eq. (D.2)

$$g_B^2 = \frac{\Gamma \Im[M^*]}{2 f(m^*)} \tag{D.5}$$
Appendix E.

Isospin Decomposition

The scattering $S-$ matrix for $\pi - \gamma$ production is written in the following form:

$$S_{ji}^{\pi\gamma} = 1 - (2\pi)^4i\delta^4(k + p' - q - p)\frac{m}{\sqrt{4\omega_k E_p E_{p'}}}M_{ji}^{\pi\gamma}$$ (E.1)

Where: $k = (\omega_k, k), q(q, q), p(E_p, p), p'(E_{p'}, p')$ are the four momenta of the pion, photon, incoming and outgoing nucleon, $\omega_k = \sqrt{\mu^2 + k^2}, E_p = \sqrt{m^2 + p^2}, E_{p'} = \sqrt{m^2 + p'^2}, \mu$, and $m$ are the masses of pion and nucleon.

The isospin structure of the matrix elements of the $M^{\pi\gamma}$ matrix is determined by the transformation properties of the electromagnetic flux of the hadron with respect to the isotopic rotations. We know that the current is transformed as the sum of the isoscalar and the third component of the isovector. Therefore the isotopic structure of the photoproduction matrix elements can be written as:

$$M_{ji}^{\pi\gamma} = M_{i}^{\pi\gamma}\delta_{i3} + M_{-}\gamma\frac{1}{2}[\tau_i, \tau_3] + M_{0}^{\pi\gamma}\tau_i$$ (E.2)

where $\tau_i$ and $\tau_3$ are the Pauli spin matrices and $i$ is the isospin index of the pion. The isovector transition amplitudes $M_{i,(+,-)}^{\pi\gamma}$ may be expressed in terms of the amplitudes $M_{i,(\frac{1}{2},\frac{3}{2})}^{\pi\gamma}$ with isospin $\frac{1}{2}, \frac{3}{2}$ in the final state:

$$M_{i,\frac{1}{2}}^{\pi\gamma} = M_{i,\frac{1}{2}}^{\pi\gamma} + 2M_{i,\frac{3}{2}}^{\pi\gamma} \quad M_{i,\frac{3}{2}}^{\pi\gamma} = M_{i,\frac{1}{2}}^{\pi\gamma} - M_{i,\frac{3}{2}}^{\pi\gamma}$$ (E.3)

The isoscalar amplitudes $M_{i}^{\pi\gamma}$ always lead to a final state with isospin $\frac{1}{2}$. 
Appendix F.

Multipole Amplitudes

First let me express the scattering matrix of the pion-photoproduction in terms of operators $O^i_{\pm}$

$$ M = \sum_{i=1,2} O^i_{+} M^i_{+} + O^i_{-} M^i_{-} $$  \hspace{1cm} (F.1)

where:

$$ O^i_{\pm} = \frac{1 \pm \gamma^0}{2} \epsilon^\pm $$

$$ O^2_{\pm} = \frac{1 \pm \gamma^0}{2} - 2 k \cdot \epsilon^\pm $$  \hspace{1cm} (F.2)

where the photon polarization vector is:

$$ \epsilon_{\pm} = \mp (e_1 \pm i e_2) \frac{1}{\sqrt{2}} \text{ for } \lambda_\gamma = \pm 1 $$  \hspace{1cm} (F.3)

Now denote the photon helicity, the incoming nucleon helicity and the outgoing nucleon helicity by $\lambda_\gamma$, $\lambda_N$ and $\lambda_{N'}$. Following Jacob and Wick [38], the angular momentum decomposition of the helicity amplitudes $M_{\lambda',\lambda}(\theta, \phi)$ is given by:

$$ M_{\lambda',\lambda}(\theta, \phi) = \sum_j (j + \frac{1}{2}) <\lambda_N | M^i | \lambda_N > e^{-i(\lambda_\gamma - \lambda_{N'} + \lambda_{N'})\phi} d^{j}_{\lambda',\lambda}(\theta, \phi) $$  \hspace{1cm} (F.4)

where: $\lambda' = $ final state helicity = $\lambda_\pi - \lambda_{N'}$, $\lambda = $ initial state helicity = $\lambda_\gamma - \lambda_N$.

Since $\lambda_\gamma = \pm 1$ for real, transverse photons, then we have eight helicity amplitudes, however the parity reduces these amplitudes to four. Choose $\phi = 0$, then we can write explicitly the operator $O^i_{\pm}$ for each helicity:

**Helicity ++**:

...
\[
\begin{align*}
< +|O_+^1|+> &= 0 \\
< +|O_+^1|+> &= 0 \\
< +|O_+^2|+> &= \frac{1}{\sqrt{2}} \sin \theta \cos \frac{\theta}{2 m} \frac{z_2}{z_1} |q||k| \\
< +|O_-^2|+> &= \frac{1}{\sqrt{2}} \sin \theta \cos \frac{\theta}{2 m} \frac{z_1}{z_2} |k|^2 \\
\end{align*}
\]

Helicity ++:

\[
\begin{align*}
< +|O_+^1|-> &= \frac{1}{\sqrt{2}} \cos \frac{\theta}{2 m} \frac{z_1 z_2}{z_1 z_2 m} \\
< +|O_+^1|-> &= \frac{1}{\sqrt{2}} \cos \frac{\theta}{2 m} \frac{z_1 z_2}{z_1 z_2 m} |q||k| \\
< +|O_+^2|-> &= \frac{1}{\sqrt{2}} \sin \theta \sin \frac{\theta}{2 m} \frac{z_2}{z_1} |q||k| \\
< +|O_-^2|-> &= \frac{1}{\sqrt{2}} \sin \theta \sin \frac{\theta}{2 m} \frac{z_1}{z_2} |k|^2 \\
\end{align*}
\]

Helicity --:

\[
\begin{align*}
< -|O_+^1|+> &= 0 \\
< -|O_+^1|+> &= 0 \\
< -|O_+^2|+> &= \frac{1}{\sqrt{2}} \sin \theta \sin \frac{\theta}{2 m} \frac{z_2}{z_1} |q||k| \\
< -|O_-^2|+> &= \frac{1}{\sqrt{2}} \sin \theta \sin \frac{\theta}{2 m} \frac{z_1}{z_2} |k|^2 \\
\end{align*}
\]

Helicity --:
\(<-|O^1_+|-> = \frac{-1}{\sqrt{2}} \sin \frac{\theta}{2} \frac{z_1 z_2}{m}

\(<-|O^1_-|-> = \frac{-1}{\sqrt{2}} \sin \frac{\theta}{2} \frac{q||k|}{z_1 z_2 m}

\(<-|O^2_+|-> = \frac{1}{\sqrt{2}} \sin \frac{\theta}{2} \frac{z_2 q||k|}{z_1}

\(<-|O^2_-|-> = \frac{-1}{\sqrt{2}} \sin \frac{\theta}{2} \frac{z_1 |k|^2}{z_2} \) (F.8)

where \(z_1 = \sqrt{E_p + m}\) and \(z_2 = \sqrt{E_{p'} + m}\). Now the next step is to write the helicity amplitudes in terms of states of definite angular momentum and parity.

\[ M^{T\gamma}_{\lambda',\lambda}(k', k, P) = \sum M^{T\gamma}_{\lambda',\lambda}(k', k, P) \frac{2j + 1}{4\pi} d_{\lambda',\lambda}(\theta, \phi) e^{i(\lambda - \mu)\phi} \] (F.9)

where:

\[ M^{T\gamma}_{\lambda',\lambda}(k', k, P) = 2\pi \int M^{T\gamma}_{\lambda',\lambda}(\theta, \phi) d\cos \theta \] (F.10)

where \(d_{\lambda',\lambda}\) for \(j = \frac{1}{2}\) and \(\frac{3}{2}\) is written explicitly in appendix G. The orthogonality of these functions makes it easy to express the integrated cross section \(\sigma_{\text{total}}\) in terms of \(M^j_{\lambda',\lambda}\).

Now to calculate the multipole amplitudes, let me define parity conserving amplitudes by:

\[ A_{l+} = \frac{-1}{\sqrt{2}} \frac{1}{4\pi} (M^j_{\frac{1}{2}, \frac{1}{2}} + M^j_{-\frac{1}{2}, -\frac{1}{2}}) \]

\[ A_{l+1} = \frac{1}{\sqrt{2}} \frac{1}{4\pi} (M^j_{\frac{1}{2}, \frac{3}{2}} - M^j_{-\frac{1}{2}, -\frac{3}{2}}) \]

\[ B_{l+} = \frac{\sqrt{2}}{\sqrt{l(l + 2)}} \frac{1}{4\pi} (M^j_{\frac{1}{2}, \frac{1}{2}} + M^j_{-\frac{1}{2}, -\frac{1}{2}}) \quad l > 0 \]

\[ B_{l+1} = -\frac{\sqrt{2}}{\sqrt{l(l + 2)}} \frac{1}{4\pi} (M^j_{\frac{1}{2}, \frac{3}{2}} - M^j_{-\frac{1}{2}, -\frac{3}{2}}) \quad l > 0 \] (F.11)
where $l = j - 1/2$

Finally the multipoles amplitudes are following:

\[ E_{l+} = \frac{1}{l+1} \left( A_{l+} + \frac{1}{2} l B_{l+} \right) \]
\[ M_{l+} = \frac{1}{l+1} \left( A_{l+} - \frac{1}{2} (l + 2) B_{l+} \right) \]
\[ E_{(l+1)-} = \frac{-1}{l+1} \left( A_{(l+1)-} - \frac{1}{2} (l + 2) B_{(l+1)-} \right) \]
\[ M_{(l+1)-} = \frac{1}{l+1} \left( A_{(l+1)-} + \frac{1}{2} l B_{(l+1)-} \right) \]  
(F.12)
Appendix G.

The $d^j_{\nu \lambda}$ Matrices

For $j = \frac{1}{2}$

\[
\begin{align*}
    d^{\frac{1}{2}}_{\frac{1}{2} \frac{1}{2}} &= \cos \frac{\theta}{2} \\
    d^{\frac{1}{2}}_{\frac{1}{2} \frac{-1}{2}} &= -\sin \frac{\theta}{2} \\
    d^{\frac{1}{2}}_{\frac{-1}{2} \frac{1}{2}} &= \sin \frac{\theta}{2} \\
    d^{\frac{1}{2}}_{\frac{-1}{2} \frac{-1}{2}} &= \cos \frac{\theta}{2}
\end{align*}
\]

For $j = \frac{3}{2}$

\[
\begin{align*}
    d^{\frac{3}{2}}_{\frac{3}{2} \frac{3}{2}} &= -\sqrt{3} \cos^2 \frac{\theta}{2} \sin \frac{\theta}{2} \\
    d^{\frac{3}{2}}_{\frac{3}{2} \frac{1}{2}} &= \cos \frac{\theta}{2} \left( 1 - 3 \sin^2 \frac{\theta}{2} \right) \\
    d^{\frac{3}{2}}_{\frac{3}{2} \frac{-1}{2}} &= \sqrt{3} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \\
    d^{\frac{3}{2}}_{\frac{3}{2} \frac{-3}{2}} &= \sin \frac{\theta}{2} \left( 1 - 3 \cos^2 \frac{\theta}{2} \right)
\end{align*}
\]
References


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