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Bounds for the Eigenvalues of a Matrix

Kenneth Ross Garren
College of William & Mary - Arts & Sciences

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A Thesis

Presented to

The Faculty of the Department of Mathematics

The College of William and Mary in Virginia

In Partial Fulfillment

Of the Requirements for the Degree of

Master of Arts

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APPROVAL SHEET

This thesis is submitted in partial fulfillment of

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Author

Approved, May 1965:

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Benjamin R. Cato, Chairman

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Fred W. Weiler, Ph.D.

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Michael H. Kutner

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TABLE OF CONTENTS

ABSTRACT

The purpose of this paper is to determine the eigenvalue bounds of a matrix defined over either the real or complex fields.

Well known theorems concerning the condition of eigenvalues as a function of the condition of the related matrix are stated. Theorems which determine the bounds are derived. Closed form solutions are expressed in terms of (1) the matrix elements, (2) matrix norms, and (3) vectors and the eigenvalues of related matrices.

A comparison is made in terms of the relative size of the areas of eigenvalue inclusion for the various solutions. Conditions for boundedness and unboundedness of these bounds are derived. Examples in terms of eigenvalue bounds for particular matrices are given.

Results of this paper are used in the problem of determining critical points of a function of n variables and in the problem of determining the convergence of iterative solutions for a system of linear equations.

BOUNDS FOR THE EIGENVALUES OF A MATRIX

INTRODUCTION

In various applications of matrix theory, the following question often arises: given a matrix A of order n for what scalars, λ, and corresponding nonzero vectors x, will

$$Ax = \lambda x \tag{1}$$

be satisfied. That is, for a given transformation, A, what vectors, x, will remain directionally invariant, and what is their change in magnitude. The answer to either of these questions almost immediately implies the answer to the other.

Equation (1), when written in the equivalent form

$$(A - \lambda I_n)x = 0,$$

where I_n is the identity matrix, yields n homogeneous linear equations in n unknown, these unknowns being the components of $x = (x_1, x_2, \ldots, x_n)$. This system of equation will have a nonzero solution if and only if the determinant of the coefficient matrix vanishes. Expansion of this determinant yields a nth degree polynomial in λ. The roots of this polynomial are called the eigenvalues of the matrix A.

For values of $n \leq 4$, the eigenvalues can always be found. However for $n > r$, this polynomial is not solvable by radicals.[1] Thus in general, for a matrix of order $n > 4$, its eigenvalues cannot be found by direct means (in closed form solutions).

Nevertheless, various techniques do exist for determining the bounds, both upper and lower, for the eigenvalues and quite often this information is sufficient to solve various types of problems.

This paper will be concerned with theorems which will determine the upper and lower bounds for the eigenvalues of a finite matrix defined over either the real or complex number fields.

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[1]B. L. van der Waerden, Modern Algebra, (New York: Frederick Ungar Publishing Company, 1953), p. 177.

CHAPTER I

WELL-KNOWN THEOREMS FOR EIGENVALUES

Some well-known results concerning the eigenvalues of particular types of matrices are given in a tabular form below.

Condition of A	Notation	Condition of λ	Footnotes
(1) Hermitian	$A = A^*$	All λ are real	2, 3, 4
(2) Real symmetric	$A = A^T$	All λ are real	2, 3
(3) Skew hermitian	$A = -A^*$	All λ are imaginary	2
(4) Real skew symmetric	$A = -A^T$	All λ are imaginary	2
(5) Unitary	$A^* A = I$	$\lvert\lambda_j\rvert = 1$	2, 4
(6) Orthogonal	$A^T A = I$	$\lvert\lambda_j\rvert = 1$	2
(7) Triangular, that is, $A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ or A^T		$\lambda_j = a_{jj}$	
(8) Permutation, that is, $A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$	$A = P$	$\lambda_j = e^{\frac{2\pi k}{j}}$	5

[2]E. T. Browne, "Limits to the Characteristic Roots of a Matrix," American Mathematical Monthly, vol. 46 (1939), p. 252.

[3]T. E. Hohn, Elementary Matrix Algebra, (New York: Macmillan Co., 1962), p. 225.

[4]W. V. Parker, "Characteristic Roots and Field and Value of a Matrix," Bulletin of The American Mathematical Society, vol. 57, no. 2 (March 1951), pp. 103-104.

[5]Marvin Marcus, Basic Theorems in Matrix Theory, (Washington, D.C.: U. S. Government Printing Office, 1960), p. 9.

Other results which are less well known than those above, but yet of some importance are listed below.

(1) If A is a positive real matrix, that is $a_{ij} > 0$, then there exists a real, positive eigenvalue which is simple and such that its absolute value is greater than that of any other eigenvalue.[6]

(2) If A is a non-negative real matrix, that is $a_{ij} \geq 0$, then there exists a real, positive eigenvalue.[7]

(3) If there exists a k such that A^k is a positive real matrix, then there exists an eigenvalue of A such that it is real, and its absolute value is greater than any other eigenvalue. If, in addition, k is an odd integer, then this eigenvalue is positive.[8]

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[6] Georg Frobenius, "Uber Matrizen aus Positiven Elementen," Sitzungsberichte der Berliner Akademie der Wissenchaften Mathem-Physikal (1908), pp. 471-476.

[7] Alfred Brauer, "On the Characteristic Roots of Power-Positive Matrices," Duke Mathematical Journal, vol. 28 (1961), p. 439.

[8] Ibid., pp. 439-443.

CHAPTER II

THEOREMS FOR EIGENVALUE BOUNDS

The bounds for eigenvalues may be determined in various ways. In general, these relations express the bounds in terms of (1) the elements of the matrix itself, (2) matrix norms, and (3) vectors and eigenvalues of related matrices. Although the eigenvalues may be approximated by considering the roots of the characteristic equations, the necessary procedures (Newton's method, Graffe's method, etc.) require a "first guess" of the roots combined with successive iterations. These relations do not lend themselves to closed form solutions of eigenvalue limits. Therefore, only those types of relations listed above will be investigated in this paper (in their listed order).

Section A - Bounds by Matrix Elements

An important relationship giving the eigenvalue bounds in terms of the matrix elements and matrix order is provided by the following theorem.[9]

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[9] M. A. Hirsch and M. I. Bendixson, "Sur Les Racines D'une Equation Fondamentale," Acta Mathematica, vol. 25 (1902), pp. 367-369.

Theorem II.A.1

Let A be a complex matrix of order n.

Define $G = 1/2\,(A + A^*)$ $T = 1/2\,(A - A^*)$

Let $a = \max |a_{ij}|$; $g = \max |g_{ij}|$; $t = \max |t_{ij}|$ $\lambda^A = \alpha + i\beta$

Then $|\lambda| \leq na$; $|\alpha| \leq ng$; $|\beta| \leq nt$

Proof: From $Ax = \lambda x$ follows $(x, Ax) = (x, \lambda x) = \lambda(x, x)$ and $(Ax, x) = (\lambda x, x)$ or $(x, A^* x) = \bar{\lambda}(x, x)$.

Then $(x, Ax) + (x, A^* x) = (\alpha + i\beta)(x, x) + (\alpha - i\beta)(x, x)$

or $(x, (A + A^*)x) = 2\alpha(x, x)$

$(x, Gx) = \alpha(x, x)$.

Likewise $(x, Ax) - (x, A^* x) = 2i\beta(x, x)$

or $(x, Tx) = i\beta(x, x)$

$-i(x, Tx) = \beta(x, x)$.

By the Cauchy-Schwarz inequality,

$$\left|\lambda(x, x)\right| = |\lambda|\;|(x, x)| = |(x, Ax)| \leq \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|\;|x_i|\;|x_j|$$

$$\leq a\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i|\;|x_j|\right) \leq \frac{a}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(|x_i|^2 + |x_j|^2\right)$$

$$= \frac{a}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i|^2 + \frac{a}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|x_j|^2 = \frac{na}{2} + \frac{na}{2} = na,$$

where the x are normalized such that $(x, x) = 1$. Thus $|\lambda| \leq na$.

Proceeding in a similar manner, since

$$\alpha(x, x) = (x, Gx)$$

then

$$|\alpha| \leq ng.$$

Likewise since $\beta(x, x) = -i(x, Tx)$

then

$$|\beta| \leq nt.$$

Bendixson essentially proved part of this result, and, in addition, found a bound for the imaginary part for a real matrix A.[10]

Theorem II.A.2

Let A be a real matrix of order n,

$$G = 1/2\ (A + A^T),\ T = 1/2\ (A - A^T),\ \text{and}\ \ \lambda^A = \alpha + i\beta.$$

Then

$$|\beta| \leq t\ \sqrt{n(n-1)/2}.$$

Proof: Since $Ax = \lambda x$ for $x = y + iz$, then

$$A(y + iz) = (\alpha + i\beta)(y + iz) = (\alpha y - \beta z) + i(\alpha z + \beta y).$$

Equating real and imaginary parts,

$$Ay = \alpha y - \beta z$$

$$Az = \alpha z + \beta y.$$

So

$$(y, Az) = (y, az) + (y, \beta y)$$

$$-(z, Ay) = -(z, \alpha y) + (z, \beta z)$$

---

[10] Ibid., p. 368-370.

and adding

$$(y, Az) - (z, Ay) = \beta((y, y) + (z, z)).$$

Now

$$(y, Az) - (z, Ay) = (y, Az) - (A^* y, z),$$

where A^* is the adjoint operator of A,

$$= (y, Az) - (y, A^T z) = (y, (A - A^T)z) = \beta((y, y) + (z, z))$$

or by definition of T, $\beta((y, y) + (z, z)) = 2(y, Tz)$.

Now, since $T = -T^T$, then $t_{ij} = -t_{ji}$ and $t_{ii} = 0$.

Thus

$$(y, Tz) = \sum_i \sum_j t_{ij} y_i z_j = \sum_{i <} \sum_j t_{ij} (y_i z_j - y_j z_i)$$

$$\leq \sum \sum |t_{ij}| \, |y_i z_j - z_i y_j|,$$

and upon squaring

$$\beta^2 \left(|y|^2 + |z|^2\right)^2 \leq 4t^2 \left(\sum \sum |y_i z_j - z_i y_j|\right)^2, \tag{2}$$

where $t = \max |t_{ij}|$

By the arithmetic-geometric mean inequality, for real numbers r_i,

$$(r_1 + \ldots + r_m)^2 \leq m(r_1^2 + \ldots + r_m^2).$$

Now there are n^2 elements in the matrix; the diagonals not appearing in the above sum since $t_{ij} = -t_{ji}$. For every two elements of the matrix, one combination is used in the summation. Thus there are

$$\frac{n^2 - n}{2} = \frac{n(n-1)}{2}$$

combinations.

Thus

$$\left(\sum \sum |y_i z_j - z_j y_i|\right)^2 \leq \frac{n(n-1)}{2} \sum \sum \left(y_i z_j - z_i y_j\right)^2. \tag{3}$$

Consider now

$$\left(|y|^2 + |z|^2\right)^2 - \left(|y|^2 - |z|^2\right)^2 = 4|y|^2 |z|^2.$$

By Lagrange's identity,

$$y^2 \; z^2 = (y, z)^2 + \sum_{i<j} \sum (y_i z_j - z_i y_j)^2.$$

Thus

$$\left(|y|^2 + |z|^2\right)^2 \geq 4 \sum \sum (y_i z_j - z_j y_i)^2.$$

Substituting this result in equation (2) yields

$$4\beta^2 \sum \sum (y_i z_j - z_i y_j)^2 \leq \beta^2 \left(|y|^2 + |z|^2\right)^2$$

$$\leq 4t^2 \left(\sum \sum |y_i z_j - z_i y_j|\right)^2$$

$$\leq 4t^2 \left[\frac{n(n-1)}{2}\right] \sum \sum (y_i z_j - y_j z_i)^2.$$

Thus $\beta^2 \leq t^2 \frac{n(n-1)}{2}$.

The major importance of these two theorems lies in their ability to determine an upper bound for the real and imaginary components separately. However, the following theorem proven by Levy-Hadamard-Gerschgorin gives an even more basic result and has since been used as a cornerstone for many more theorems of eigenvalue bounds.

Theorem II.A.3

The eigenvalues of a matrix are inside the closed domain consisting of all circles k_i $(i = 1, 2, \ldots, n)$ with centers a_{ii} and radius

$$r_i = \sum_{\substack{k=1 \\ k \neq i}}^{n} |a_{ik}| .$$

Proof: Let B be a matrix of order n. The system of equations $Bx = 0$ has a nontrival solution if and only if $\det B = 0$.

Let x_k be the dominant component of $x = (x_1, \ldots, x_n)$. Then the kth equation is

$$b_{kk}\, x_k = -\sum_{\substack{m=1 \\ m \neq k}}^{n} b_{km}\, x_m$$

or

$$|b_{kk}|\, |x_k| \leq \sum_{\substack{m=1 \\ m \neq k}}^{n} |b_{km}|\, |x_m|$$

and thus

$$|b_{kk}| \leq \sum_{\substack{m=1 \\ m=k}}^{n} |b_{km}| .$$

Now let $B = A - \lambda I$, where λ is such that $\det(A - \lambda I) = 0$, the "eigenvalue problem".

Therefore

$$\left|\lambda - a_{kk}\right| \leq \sum_{\substack{m=1 \\ m \neq k}}^{n} \left|a_{km}\right|.$$

An almost immediate consequence of the theorem is the well known "Theorem of Frobenius."

<u>Corollary II.A.3</u> (Frobenius)

$$\left|\lambda\right| \max \leq \max \sum_{m=1}^{n} \left|a_{km}\right|$$

$$\left|\lambda\right| \min \geq \min \left(\left|a_{kk}\right| - \sum_{\substack{m=1 \\ k \neq m}}^{n} \left|a_{km}\right| \right).$$

<u>Proof</u>: $\left|\lambda - a_{kk}\right| \geq \left|\lambda\right| - \left|a_{kk}\right|$, so that from the above

$$\left|\lambda\right| \leq \left|a_{kk}\right| + \sum_{\substack{m=1 \\ m \neq k}}^{n} \left|a_{km}\right| = \sum_{m=1}^{n} \left|a_{km}\right| .$$

also $\left|\lambda - a_{kk}\right| \geq \left|a_{kk}\right| - \left|\lambda\right|$, so that

$$\left|\lambda\right| \geq \left|a_{kk}\right| - \sum_{\substack{m=1 \\ m \neq k}}^{n} \left|a_{km}\right|.$$

Also, since $\det A = \det A^T$, then

$$\sum_{\substack{m=1\\ m\neq k}}^{n} |a_{km}| .$$

may be replaced in theorem II.A.3 and its corollary by

$$\sum_{\substack{m=1\\ m\neq k}}^{n} |a_{mk}|$$

Thus the centers of the circles containing the eignevalues will remain unchanged even though their radius will be changed.

As a further refinement of theorem 3, Alfred Brauer was able to restrict the regions containing the eigenvalues by means of the "ovals of Cassini" in the following.[11]

Theorem II.A.4

Each eigenvalue of A lies in at least one of the $\frac{n(n-1)}{2}$ ovals of Cassini

$$|\lambda - a_{kk}| \cdot |\lambda - a_{ll}| \leq \left(\sum_{\substack{j=1\\ j\neq k}}^{n} |a_{kj}|\right)\left(\sum_{\substack{j=1\\ j\neq l}}^{n} |a_{lj}|\right),$$

---

[11] Alfred Brauer, "Limits for the Characteristic Roots of a Matrix II," Duke Mathematical Journal, vol. 14 (1947), pp. 21-23.

and in at least one of the ovals

$$\left|\lambda - a_{kk}\right| \cdot \left|\lambda - a_{ll}\right| \leq \left(\sum_{\substack{i=1\\ i\neq k}}^{n} \left|a_{ik}\right|\right)\left(\sum_{\substack{j=1\\ j\neq l}}^{n} \left|a_{lj}\right|\right).$$

Proof: Let

$$\{x = x_1, x_2, \ldots, x_n\}, \ \xi_i = \left|x_i\right| .$$

as was shown in a previous theorem by Gerschgorin for the matrix $A = (a_{ij})$,

$$(\lambda - a_{ii})\, x_i = \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}\, x_j, \ i = 1, 2, \ldots, n .$$

Now let $\xi_k \geq \xi_l \geq \xi_j$, for $j \neq k$, $j \neq l$.

Then

$$\left|\lambda - a_{kk}\right| \xi_k \leq \left(\sum_{\substack{j=1\\ j\neq k}}^{n} \left|a_{kj}\right|\right) \xi_j \leq \left(\sum_{\substack{j=1\\ j\neq k}}^{n} \left|a_{kj}\right|\right) \xi_l$$

and

$$\left|\lambda - a_{ll}\right| \xi_l \leq \left(\sum_{\substack{j=1\\ j\neq l}}^{n} \left|a_{lj}\right|\right) \xi_j \leq \left(\sum_{\substack{j=1\\ j\neq l}}^{n} \left|a_{lj}\right|\right) \xi_k .$$

Therefore

$$|\lambda - a_{kk}| \cdot |\lambda - a_{ll}| \ \xi_k \ \xi_l \leq \left(\sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}| \right) \left(\sum_{\substack{j=1 \\ j \neq l}}^{n} |a_{lj}| \right) \xi_k \ \xi_l$$

so that

$$|\lambda - a_{kk}| \cdot |\lambda - a_{ll}| \leq \left(\sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}| \right) \left(\sum_{\substack{j=1 \\ j \neq l}}^{n} |a_{lj}| \right)$$

which proves the theorem.

Similarly it may be shown that all λ^A are contained in at least one of the ovals

$$|\lambda - a_{kk}| \cdot |\lambda - a_{ll}| \leq \left(\sum_{\substack{i=1 \\ i \neq k}}^{n} |a_{ik}| \right) \left(\sum_{\substack{i=1 \\ i \neq l}}^{n} |a_{il}| \right) .$$

There are n elements, a_{ii}, which, in part, form the ovals. The number of subsets with two elements that can be chosen from this set of n elements, is

$$\frac{n!}{2!(n-2)!} = \frac{n(n-1)(n-2)!}{2(n-2)!} = \frac{n(n-1)}{2} .$$

Thus there are $\frac{n(n-1)}{2}$ ovals.

Another inequality giving the regions in which the eigenvalues are contained is:[12]

Theorem II.A.5

For the matrix $A = (a_{ij})_n$,

$$|\lambda - a_{ii}| \leq \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right)^{\alpha} \left(\sum_{\substack{k=1 \\ k \neq i}}^{n} |a_{ki}| \right)^{1-\alpha}$$

for $0 \leq \alpha \leq 1$.

Proof: As was shown in theorem 3 and the following corollary, for the determinant of $A - \lambda I$ to vanish, the following inequalities must be satisfied:

$$|\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq n}}^{n} |a_{ij}| \quad \text{and} \quad |\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|.$$

Thus

$$|\lambda - a_{ii}| = \left(|\lambda - a_{ii}|^{\alpha} \right) \left(|\lambda - a_{ii}|^{1-\alpha} \right) \leq \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right)^{\alpha} \left(\sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \right)^{1-\alpha}$$

whenever $0 \leq \alpha \leq 1$.

---

[12] Alexander Ostrowski, "Ueber das Nichtverschwinden von Determinanten und die Lokalisierung der Characteristischen Wurzeln von Matrizen," Composito Mathematica, vol. 9 (1951), pp. 209-212.

The importance of this theorem lies in its ability to exclude certain regions in which the eigenvalues cannot exist. This is true since the region containing the eigenvalues is contained in

$$\bigcap_{\alpha}\left(\bigcup_{i,j}\left[\left(\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}\right)^{\alpha}\left(\sum_{\substack{i=1\\ i\neq j}}^{n} a_{ij}\right)^{1-\alpha}\right]\right).$$

As simple corollaries to this theorem, we have

<u>Corollary II.A.5a.</u>

$$\left|\lambda^{A}\right|_{max} \leq \left[\left|a_{ii}\right| + \left(\sum_{\substack{j=1\\ j\neq i}}^{n} \left|a_{ij}\right|\right)^{\alpha}\left(\sum_{\substack{i=1\\ i\neq j}}^{n} \left|a_{ij}\right|\right)^{1-\alpha}\right]$$

$$\left|\lambda^{A}\right|_{min} \geq \left[\left|a_{ii}\right| - \left(\sum_{\substack{j=1\\ j\neq i}}^{n} \left|a_{ij}\right|\right)^{\alpha}\left(\sum_{\substack{i=1\\ i\neq j}}^{n} \left|a_{ij}\right|\right)^{1-\alpha}\right]$$

$$\left|\lambda^{A}\right|_{max} \leq \left[\left|a_{ii}\right| + \sum_{\substack{j=1\\ j\neq i}}^{n} \left|a_{ij}\right|\right]^{\alpha}\left[\left|a_{ii}\right| + \sum_{\substack{i=1\\ i\neq j}}^{n} \left|a_{ij}\right|\right]^{1-\alpha}$$

$$\left|\lambda^{A}\right|_{min} \geq \left[\left|a_{ii}\right| - \sum_{\substack{j=1\\ j\neq i}}^{n} \left|a_{ij}\right|\right]^{\alpha}\left[\left|a_{ii}\right| + \sum_{\substack{i=1\\ i\neq j}}^{n} \left|a_{ij}\right|\right]^{1-\alpha}$$

all for $0 \leq \alpha \leq 1$. The corollaries and theorem hold likewise for $\alpha = 1 - \beta$, $1 - \alpha = \beta$ where $0 \leq \beta \leq 1$.

As a direct consequence of theorem 4, we have

Corollary II.A.5.b.

For each α, $0 \leq \alpha \leq 1$, every eigenvalue of A lies in at least one of the $\frac{n(n-1)}{2}$ ovals,

$$|z - a_{ii}| \cdot |z - a_{jj}| \leq \left[\left(\sum_{\substack{i=1\\i\neq j}}^{n} |a_{ij}|\right)\left(\sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}|\right)\right]^{1-\alpha}\left[\left(\sum_{\substack{i=1\\i\neq j}}^{n} |a_{ji}|\right)\left(\sum_{\substack{j=1\\j\neq k}}^{n} |a_{jk}|\right)\right]^{\alpha}$$

for $j \neq k$.

For $\alpha = 0$ or $\alpha = 1$, this reduces to theorem 4.

As a further extension of theorem 4, the largest eigenvalue may be bounded from the results of [13]

Theorem II.A.6.

Each eigenvalue, λ, satisfies

$$|\lambda| \leq \frac{1}{2} \max_{\substack{k,j=1,2,\ldots,n\\k\neq j}} \left\{ |a_{kk}| + |a_{jj}| + \left[\left(|a_{kk}| - |a_{jj}|\right)^2 + 4P_k P_j\right]^{1/2} \right\} = M$$

where
$$P_k = \sum_{\substack{j=1\\j\neq k}}^{n} |a_{kj}|.$$

---

[13] Brauer, "Limits ...", pp. 23-24.

Proof: By the previous theorem

$$|\lambda - a_{rr}||\lambda - a_{ss}| \leq P_r P_s .$$

assume that $|a_{rr}| \geq |a_{ss}|$.

(1) If $|\lambda| \leq |a_{rr}|$, then

$$\begin{aligned} |\lambda| &\leq \tfrac{1}{2}\left(|a_{rr}| + |a_{ss}|\right) + \tfrac{1}{2}\left(|a_{rr}| - |a_{ss}|\right) \\ &\leq \tfrac{1}{2}\left\{|a_{rr}| + |a_{ss}| + \left[\left(|a_{rr}| - |a_{ss}|\right)^2 \right.\right. \\ &\left.\left. + 4P_r P_s\right]^{1/2}\right\} \leq M \end{aligned}$$

since

$$P_r > 0,\ P_s > 0.$$

(2) If $|\lambda| > |a_{rr}| \geq |a_{ss}|$, then

$$0 < |\lambda| - |a_{rr}| \leq |\lambda - a_{rr}|$$

$$0 < |\lambda| - |a_{ss}| \leq |\lambda - a_{ss}| .$$

By the preceeding theorem,

$$\left\{|\lambda| - |a_{rr}|\right\}\left\{|\lambda| - a_{ss}|\right\} \leq |\lambda - a_{rr}|\ |\lambda - a_{ss}| \leq P_r P_s$$

or

$$|\lambda|^2 - \left(|a_{rr}| + |a_{ss}|\right) |\lambda| + |a_{rr}\, a_{ss}| - P_r\, P_s \leq 0$$

and

$$\left\{ |\lambda| - \frac{1}{2} \left(|a_{rr}| + |a_{ss}| + \sqrt{\left(|a_{rr}| - |a_{ss}|\right)^2 + 4P_r\, P_s} \right) \right\} \tag{7}$$

times

$$\left\{ |\lambda| - \frac{1}{2} \left(|a_{rr}| + |a_{ss}| - \sqrt{\left(|a_{rr}| - |a_{ss}|\right)^2 + 4P_r\, P_s} \right) \right\} \leq 0 . \tag{8}$$

Thus either equation (7) ≥ 0 and equation (8) ≤ 0

or equation (7) ≤ 0 and equation (8) ≥ 0 .

However since equation (7)

$$= |\lambda| - \frac{|a_{rr}| - |a_{ss}| - \sqrt{\left(|a_{rr}| - |a_{ss}|\right)^2 + 4P_r\, P_s}}{2}$$

$$\leq |\lambda| - \frac{|a_{rr}| - |a_{ss}| + \sqrt{\left(|a_{rr}| - |a_{ss}|\right)^2 + 4P_r\, P_s}}{2}$$

= equation (8),

then it must be true that equation (7) $\leq 0 \leq$ equation (8).

Thus from equation (7), it follows that

$$|\lambda| \leq \left\{\frac{1}{2} \quad |a_{rr}| + |a_{ss}| + \left[\left(|a_{rr}| - |a_{ss}|\right)^2 + 4P_r\ P_s\right]^{1/2}\right\} \leq M .$$

In addition, if a third condition is satisfied, namely

$$|a_{kk}\ a_{jj}| > P_k\ P_j,$$

then a similar type of lower bound for the modulus of the eigenvalues of A can be formulated from the following [14]

Theorem II.A.6.

If

$$|a_{kk}\ a_{jj}| > P_k\ P_j, \quad k, j = 1, 2, \ldots, n$$

then

$$|\lambda| \geq \min_{k,j=1,2,\ldots n} \left\{|a_{kk}| + |a_{jj}| - \left[\left(|a_{kk}| - |a_{jj}|\right)^2 + 4P_k\ P_j\right]^{1/2}\right\}$$

$$= m > 0.$$

Proof: As was shown in the proof of theorem 2,

$$|\lambda|^2 - \left(|a_{rr}| + |a_{ss}|\right) \lambda + a_{rr} . a_{ss} - P_r . P_s \leq 0.$$

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[14] Ibid, pp. 24-25.

From equation (8) and the following result,

$$|\lambda| - \frac{1}{2}\left(|a_{rr}| + |a_{ss}| - \sqrt{\left(|a_{rr}| - |a_{ss}|\right) + 4P_r\, P_s}\right) \leq 0,$$

or

$$|\lambda| \geq \frac{1}{2}\left(|a_{rr}| + |a_{ss}| - \sqrt{\left(|a_{rr}| - |a_{ss}|\right) + 4P_r\, P_s}\right) \geq m,$$

regardless of the relation of λ to a_{rr} and a_{ss}.

Now assume that m is attained where $k = \gamma$, $j = \delta$, so that

$$\begin{aligned} m &= \frac{1}{2}\left\{|a_{\gamma\gamma}| + |a_{\delta\delta}| - \left[\left(|a_{\gamma\gamma}| - |a_{\delta\delta}|\right)^2 + 4P_\gamma\, P_\delta\right]\right\} \\ &= \frac{1}{2}\left\{|a_{\gamma\gamma}| + |a_{\delta\delta}| - \left[|a_{\gamma\gamma}|^2 - 2|a_{\gamma\gamma}\, a_{\delta\delta}| + |a_{\delta\delta}|^2 + 4P_\gamma\, P_\delta\right]^{1/2}\right\} \\ &> \frac{1}{2}\left\{|a_{\gamma\gamma}| + |a_{\delta\delta}| - \left[|a_{\gamma\gamma}|^2 - 2\,|a_{\gamma\gamma}\, a_{\delta\delta}| + |a_{\delta\delta}|^2 + 4\,|a_{\gamma\gamma}\, a_{\delta\delta}|\right]^{1/2}\right\} \\ &= \frac{1}{2}\left\{|a_{\gamma\gamma}| + |a_{\delta\delta}| - \left[\left(|a_{\gamma\gamma}| + |a_{\delta\delta}|\right)^2\right]^{1/2}\right\} = 0\,. \end{aligned}$$

Note that all of the previous theorems have given bounds for the <u>modulus</u> of the eigenvalues, only. However, for a particular case, more definite information may be implied from exact information regarding the values of the elementary symmetric functions of $\lambda_1, \lambda_2, \ldots, \lambda_n$. Several important results concerning these functions are given by the following:

Theorem II.A.8

For an arbitrary matrix $A = (a_{ij})_n$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} \tag{9}$$

$$\prod_{i=1}^{n} \lambda_i = \det A \tag{10}$$

and

$$\sum \lambda_i{}^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \tag{11}$$

if A is real.

Proof: Let

$$\psi(\lambda) = \det (A - \lambda I)$$

By a Maclaurin's series expansion of $\psi(x)$, the coefficient of λ^{k-1} is

$$\left.\frac{d^{k-1}\ \psi(\lambda)}{(d\lambda)^{k-1}}\right|_{\lambda=0} = (k - 1)! \left(a_{11} + \dots + a_{nn}\right) .$$

Also by the Fundamental Theorem of Algebra,

$$\psi(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda),$$

where the $\lambda_1, \dots \lambda_n$ are the eigenvalues of A.

Then

$$\left.\frac{d^{k-1}\ \psi(\lambda)}{(d\lambda)^{k-1}}\right|_{\lambda=0} = (k - 1)!\ (\lambda_1 + \ldots \lambda_n)\ .$$

Thus

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} \equiv \underline{\text{trace}} \text{ of } A\ .$$

Also for $\psi = \det\ (A - \lambda I)$

$$\psi(o) = \left.\frac{d^{o}\ \psi(\lambda)}{(d\lambda)^{o}}\right|_{\lambda=0} = \det A$$

and for $\psi(\lambda) = \prod_{i=1}^{n} (\lambda_i - \lambda)$, $\psi(0) = \prod_{i=1}^{n} \lambda_i$.

Thus $\prod \lambda_i = \det A$.

Likewise the other elementary symmetric functions are the corresponding coefficients of the characteristic equation.

Now since the multiplicity of λ_i in A is the same as the multiplicity of λ_i^k in A^k, then

$$\text{trace of } A^k = \sum_{i=1}^{n} \lambda_i^k\ .$$

(Note, if all $\lambda = 0$, then $\psi(\lambda) = \lambda^n = \det\ (A - \lambda I) = \det A = 0$)

Let $k = 2$. Trace $A^2 = \lambda_1^2 + \ldots + \lambda_n^2$.

Also

$$\text{Trace } A^2 = \sum_{i=1}^{n} a_{ii}^2 + \sum_{i=1}^{n} \sum_{\substack{k=1 \\ i \neq k}}^{n} a_{ik}\, a_{ki} \; .$$

$$\text{Trace } A^T A = \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ki}^2 = \sum_{i=1}^{n} a_{ii}^2 + \sum_{i \neq k} \sum a_{ik}^2 \; .$$

Since $a_{ik}^2 + a_{ki}^2 \geq 2\, a_{ik}\, a_{ki}$, then

$$\text{Trace } A^2 \leq \text{Trace } A^T A \; .$$

Thus

$$\sum_{k=1}^{n} |\lambda_k|^2 \leq \sum_{i=1}^{n} \sum_{k=1}^{n} |a_{ki}|^2 \; .$$

The possible accuracy (inaccuracy) of some of these theorems with several examples will be given in the section titled "Comparison and Computation."

Section B - Bounds by Matrix Norms

The purpose of this section is to determine the eigenvalue bounds in terms of matrix norms. A matrix norm $\|A\|$, of a square matrix, A, is any bounded, real valued function such that the following are true:

Matrix Norm Properties

(1) $\|A\| > 0$ whenever $A \neq 0$

(2) $\|\alpha A\| = |\alpha| \, \|A\|$ where α is a scalar

(3) $\|A + B\| \leq \|A\| + \|B\|$

(4) $\|A \cdot B\| \leq \|A\| \cdot \|B\|$

A relation between the eigenvalue bounds and the value of powers of the matrix are given in.[15]

<u>Theorem II. B.1</u>

All eigenvalues, λ^A, of the matrix A are contained within the unit circle, if and only if

$$\lim_{n\to\infty} A^n = 0.$$

<u>Proof (1)</u>: Assume that all eigenvalues of A are contained in the unit circle. Then choose an arbitrary $\epsilon > 0$ such that

$$\left|\lambda^A\right|_{max} + \epsilon < 1.$$

It is now desirable to find a matrix norm with

$$\|A\| \leq \left|\lambda^A\right|_{max} + \epsilon < 1$$

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[15] Alston S. Householder, <u>On the Convergence of Matrix Iterations</u>, (Oak Ridge, Tennessee: Oak Ridge National Laboratory, 1955), pp. 4-12.

for by property (4) of matrix norms it follows that

$$\|A^n\| < \|A\|^n.$$

So if $\|A\| < 1$, then

$$\lim_{n\to\infty} \|A\|^n = 0.$$

implies

$$\lim_{n\to\infty} \|A^n\| = 0$$

By the contrapositive of property (1) of matrix norms,

$$\lim_{n\to\infty} \|A^n\| = 0$$

implies that

$$\lim_{n\to\infty} A^n = 0$$

so that the sufficiency portion would be proven.

The desired matrix norm actually does exist. Define $\|A\|_g$ to be the maximal row sum of absolute values of $G^{-1}AG$; i.e.,

$$\|A\|_g = \|G^{-1}AG\|_e$$

where G is a diagonal matrix such that $Ge = g$, where

$$e = \begin{Bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{Bmatrix}.$$

This is a well defined norm, since $\|A\|_g$ properties (1) and (2) are obvious, while property (3) is immediate from Minkowski's inequality, and likewise is property (4) from the Cauchy-Schwarz inequality.

Let Λ be the Jordan canonical form of A, so that $A = T^{-1}\Lambda T$. Then Λ is an upper triangular matrix. Let $P = \text{diag}(\delta^n, \delta^{n-1}, \delta^{n-2}, \ldots)$ where $\delta > 0$.

Then

$$P^{-1}\Lambda P = \begin{bmatrix} \lambda_i & \delta & 0 & \cdots & 0 \\ 0 & \lambda_i & \delta & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix} .$$

Thus $\|P^{-1}\Lambda P\|_e \leq |\lambda^A|_{max} + \epsilon$, since the value of its maximal row sum may be made sufficiently close to $|\lambda|_{max}$ by choosing δ small enough. Then by transforming the e norm by P, i.e., $Pe = g$, we get

$$\|A\|_g = \|P^{-1}\Lambda P\|_e < |\lambda^A|_{max} + \epsilon \ .$$

Notice here that had the eigenvalues of A been distinct, δ could assume the value zero and thus so would $\epsilon = 0$.

Therefore, the norm $\|A\| \equiv \|A\|_g$ is sufficient to show the existence of the desired norms.

<u>Proof (2)</u>: Let $\lim_{n\to\infty} A^n = 0$. Now

$$A^n = (T^{-1}\Lambda T)^n = (T^{-1}\Lambda T)_1(T^{-1}\Lambda T)_2 \; . \; . \; . \; (T^{-1}\Lambda T)_n = T^{-1}\Lambda^n T$$

Therefore,

$$\lim_{n\to\infty} A^n = 0 \Rightarrow \lim_{n\to\infty} T^{-1}\Lambda^n T = 0$$

or

$$0 = T^{-1}(\lim_{n\to\infty} \Lambda^n)T \, .$$

Thus

$$\lim_{n\to\infty} \Lambda^n = 0 \, .$$

Then each element $(\lambda_{ij})^n$ of the Jordan matrix Λ^n must be such that

$$\lim_{n\to\infty} (\lambda_{ij})^n = 0$$

for all i,j.

Let Λ be partitioned into block diagonal form where each block corresponds to a distinct eigenvalue of A.

That is

$$\Lambda = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{bmatrix}$$

where

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}_{(P\times P)} .$$

Then

$$\Lambda^n = \begin{bmatrix} J_1{}^n & & 0 \\ & J_2{}^n & \\ 0 & & \ddots \; J_k{}^n \end{bmatrix} .$$

Notice, where $(\lambda_{ij})^n$ is the i,jth element of J^n, that

$$\lambda_{ij} = \begin{Bmatrix} 0 \quad \text{for} \quad i > j \\ \lambda_i \neq 0 \quad \text{for} \quad i = j \\ 1 \text{ or } 0 \quad \text{for } i < j \end{Bmatrix} .$$

Then the diagonal elements of J^2 may be described as

$$(\lambda_{ii})^2 = \sum_{k=1}^{p} \lambda_{ik}\lambda_{kj} = \sum_{k=1}^{p} \lambda_{ik}\lambda_{ki} = \sum_{k=1}^{i-1} \lambda_{ik}\lambda_{ki} + \sum_{k=i}^{i} \lambda_{ik}\lambda_{ki}$$

$$+ \sum_{k=i+1}^{p} \lambda_{ik}\lambda_{ki} = 0 \cdot \lambda_{ki} + \lambda_i{}^2 + \lambda_{ik} \cdot 0 = \lambda_i{}^2 .$$

Assume that for any m, $(\lambda_{ii})^m = \lambda_i{}^m$.

Consider $(\lambda_{ii})^{m+1}$. Since the product of a finite number of upper triangular matrices (as are the J_i's) are upper triangular matrices, it must be true that

$$(\lambda_{ij}) = 0 \quad \text{for} \quad i > j .$$

Thus

$$(\lambda_i)^{m+1} = \sum_{k=1}^{i-1} (\lambda_{ik})^m(\lambda_{ki}) + \sum_{k=i}^{i} (\lambda_{ik})^m(\lambda_{ki})$$

$$+ \sum_{k=i+1}^{p} (\lambda_{ik})^m(\lambda_{ki}) =$$

$$0(\lambda_{ki}) + \lambda_i^m \cdot \lambda_i + (\lambda_{ik})^m \cdot 0 = \lambda_i^{m+1} .$$

Then for any J_i^n, $(\lambda_{ii})^n = \lambda_i^n$, where $(\lambda_{ii})^1 = \lambda_i$ as was stated before,

$$\lim_{n\to\infty} A^n = 0$$

implies

$$\lim_{n\to\infty} (\lambda_{ij})^n = 0 .$$

Therefore in particular for $i = j$,

$$\lim_{n\to\infty} (\lambda_{ii})^n = 0$$

or

$$\lim_{n\to\infty} (\lambda_{ii})^n = \lim_{n\to\infty} \lambda_i^n = 0 ;$$

that is $|\lambda_i| < 1$, for all i.

The importance of this relation is obvious when iterative (numerical) techniques, defining the matrix A as the error in the approximate solution, are considered.

One of the most significant and generalized results is given by the following theorem.[16]

[16] *Ibid.*, p. 12.

Theorem II. B.2

For an arbitrary matrix A, the largest possible eigenvalue modulus $|\lambda^A|_{max} \leq \|A\|$, the matrix norm of A.

Proof: Let $\|A\| = \alpha$, real scalar. Also define $B_\epsilon = \dfrac{A}{(\alpha + \epsilon)}$ where ϵ is a positive real.

Consider

$$\|B_\epsilon\| = \left\|\frac{A}{(\alpha + \epsilon)}\right\| = \frac{1}{(\alpha + \epsilon)} \|A\| = \frac{\alpha}{\alpha + \epsilon} < 1$$

for all $\epsilon > 0$, that is, $\|B_\epsilon\| < 1$. By the proof of the previous theorem (first part of Sufficiency), $\|B_\epsilon\| < 1$ implies that $\lim_{n \to \infty} B^n = 0$, which, by the result of the previous theorem, implies that for all eigenvalues of B_ϵ, $\lambda_i^{B_\epsilon}$, it is true that $|\lambda_i^{B_\epsilon}| < 1$. Now if λ_i^A is any eigenvalue of A, then there will exist a corresponding eigenvalue of B such that

$$\lambda_i^{B_\epsilon} = \frac{\lambda_i^A}{(\alpha + \epsilon)} .$$

From the above, since $|\lambda_i^{B_\epsilon}| < 1$, for all eigenvalues of B_ϵ, then

$$\left|\frac{\lambda_i^A}{(\alpha + \epsilon)}\right| < 1$$

or

$$|\lambda_i^A| < \alpha + \epsilon = \|A\| + \epsilon .$$

Therefore $|\lambda_i^A|_{max} \leq \|A\|$, since the relation is true for all $\epsilon > 0$.

The following is a listing of several possible norms for an arbitrary matrix, $A = (a_{ij})_n$.

$$\|A\|_E = \sqrt{\text{trace } (A^*A)} = \text{square root of sum of squares of } A \quad \text{(Euclidean)} \tag{9}$$

$$\|A\|_e = \text{maximal row (column) sum of } A \tag{10}$$

$$\|A\|_{e^T} = \text{maximal row (column) sum of } A^* \tag{11}$$

$$\|A\|_g = \|G^{-1}AG\|_e \tag{12}$$

where G is any nonsingular matrix and $\|A\|$ is any matrix norm, where $Ge = g$.

$$\|A\|_s = \sum_{i,j=1}^{n} |a_{ij}| \tag{13}$$

$$\|A\|_I = \max_{\|x\| = 1} \|Ax\|, \quad \text{induced norm} \tag{14}$$

It is interesting to note at this point that for an arbitrary matrix, there does not exist a smallest norm. To show this, let $\|A\|_{min}$ equal minimal matrix norm of A. Then by definition $\|A\|_{min} \leq \|A\|$ for an arbitrary choice of norm, and $\|A\|_{min}$ satisfies all other matrix norm properties. Let $\|A\|' = \min\{\|A\|_{min}, \|A^T\|_{min}\}$ and let $\lambda^{AA^T}_{max}$ be the largest eigenvalue of AA^T. (Note that $\lambda^{AA^T}_{max}$ is real by Theorem II. C.3.) Then $\lambda^{AA^T}_{max} \leq \|AA^T\|_{min}$ by Theorem II. B.2. $\|AA^T\|_{min} \leq (\|A\|')^2$ or $(\lambda^{AA^T}_{max})^{1/2} \leq \|A\|'$.

It has been shown that $\left(\lambda^{AA^T}{}_{max}\right)^{1/2}$ satisfies all requirements for a matrix norm.[17]

Thus, $\left(\lambda^{AA^T}{}_{max}\right)^{1/2} = \|A\|_{min}$ is such a minimal norm exists.

To show that it does not exist, consider

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

so that

$$AA^T = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$$

and take

$$\|A\| = \max_j \sum_{i=1}^{n} |a_{ij}| .$$

Then, $\|A\| = 1$, $\lambda^{AA^T}{}_{max} = 2$, so

$$\|A\| = \max \sum_{i=1}^{n} |a_{ij}| = 1 < \sqrt{2} = \left(\lambda^{AA^T}{}_{max}\right)^{1/2} \quad ;$$

a contradiction since no norm can be less than the minimal norm.

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[17] Albert H. Bowker, "On the Norms of a Matrix," Annals of Mathematical Statistics, vol. XVIII, no. 2 (June 1947), pp. 287-288.

As has been shown in Theorem II. B.2, the largest modulus of any eigenvalue of a matrix A, $|\lambda^A|_{max}$, is less than the matrix norm of A, regardless of how the norm is defined. Thus, $|\lambda^A|_{max}$ is a lower bound for the matrix norms of A.

A reasonable question becomes, is $|\lambda^A|_{max}$ the greatest lower bound of these matrix norms?

A. H. Bowker, in requiring a fourth property of matrix norms, namely for $\| e_i > < e_j \| = 1$, shows that[18]

$$\max_{i,j} |a_{ij}| \leq \|A\|$$

where ($e_j = (0, 0, \ldots, \underset{\text{jth position}}{1}, 0, \ldots, 0)$.

Thus, for a simple triangular matrix, B,

$$A = \begin{bmatrix} 10 & 0 & 0 \\ 1 & 3 & 0 \\ 20 & 4 & 5 \end{bmatrix}, \quad \max_{i,j} |a_{ij}| = 20$$

while $|\lambda^A|_{max} = 10$. Therefore, in general $|\lambda^A|_{max}$ is not the greatest lower bound for the norms of A. However, $|\lambda^A|_{max}$ is the greatest lower bound of a set of constant multiples of related matrices. This result is given as:

[18] *Ibid.*, pp. 285-286.

Theorem II. B.3

For the matrix B, the eigenvalue of largest modulus $|\lambda^B|_{max}$, is given as the greatest lower bound of the $k_i > 0$ where

$$K = \left\{ k_i \;\middle|\; A = \frac{B}{k_i} \text{ and } \lim_{n\to\infty} A^n = 0 \right\} .$$

Proof: From Theorm II. B.1, it follows that

$$\lim_{n\to\infty} \frac{B^n}{k_i{}^n} = \lim_{n\to\infty} A^n = 0$$

if and only if $|\lambda^A|_{max} < 1$, so that $1 > |\lambda^A|_{max} = \frac{|\lambda^B|_{max}}{k_i}$, or

$$|\lambda^B|_{max} < k_i$$

for

$$k_i \epsilon K$$

Thus $|\lambda^B|_{max}$ is a lower bound.

Also

$$k_o \equiv |\lambda^B|_{max}$$

is not an element of K, since

$$A = \frac{B}{k_o}$$

and

$$|\lambda^A|_{max} = 1 \;,$$

so that

$$\lim_{n\to\infty} A^n \neq 0.$$

Now since

$$1 = |\lambda^A|_{max}$$

is the least upper bound of the

$$|\lambda^A|_{max}$$

such that

$$\lim_{n\to\infty} A^n = 0$$

then it follows that $|\lambda^B|_{max}$ is the greatest lower bound of the $k_i \in K$.

If the matrix is partitioned such that each diagonal submatrix is square, then eigenvalue bounds may be determined by procedures similar to those used in Section A, Chapter II.

Let A be any matrix order n, which is partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ A_{N1} & \cdots & \cdots & A_{NN} \end{bmatrix}$$

where the diagonal submatrices A_{ii} are square of order n_i.

Define the matrix norm by:

$$\|A\| \equiv \sup_{\substack{x \epsilon \Omega_j \\ x \neq 0}} \frac{\|A_{ij}x\|_{\Omega_i}}{\|x\|_{\Omega_j}}$$

for an arbitrary vector norm over the subspace Ω_k.

If the diagonal submatrices, A_{ii}, are nonsingular, and if

$$(\|A_{jj}^{-1}\|)^{-1} > \sum_{\substack{k=1 \\ k \neq j}}^{N} \|A_{j,k}\|$$

for all

$$1 \leq j \leq N$$

then the matrix A is to be <u>block</u> <u>strictly</u> <u>diagonally</u> <u>dominant</u>.[19]

[19] David G. Feingold, and Richard S. Varga, "Block Diagonally Dominant Matrices and Generalizations of the Gerschgorin Circle Theorem," <u>Pacific Journal of Mathematics</u>, vol. 12, (1962), pp 1241-1244.

Theorem II. B.4

For the partitioned matrix, A, each eigenvalue, λ^A, satisfies

$$(\| (A_{j,j} - \lambda I_j)^{-1} \|)^{-1} \leq \sum_{\substack{k=1 \\ k \neq j}}^{n} \| A_{j,k} \|$$

for at least one j, $1 \leq j \leq N$.

Proof: Assume that $A - \lambda I$ is singular. Then there exists a nonzero vector $X = \begin{bmatrix} X_1 \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{bmatrix}$ such that

$$(A - \lambda I)X = 0 \ .$$

Consider $A - \lambda I$ in its partitioned form; this is equivalent to

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} A_{ij}X_j = -(A_{ii} - \lambda I_i)X_i \tag{15}$$

Let X_r be the largest component of X, i.e., $\|X_r\| \geq \|X_j\|$; $1 \leq j \leq N$.

Divide X by $\|X_r\|$. Then from equation (15)

$$\left\| \sum_{\substack{j=1 \\ j \neq r}}^{n} A_{rj}X_j \right\| = \|(A_{rr} - \lambda I_r)X_r\| \ . \tag{16}$$

From the Cauchy-Schwarz inequality, the left side of equation (16) is such that

$$\leq \sum_{\substack{j=1\\j\neq r}}^{n} \left\|A_{rj}\right\| \cdot \left\|X_j\right\| \leq \sum_{\substack{j=1\\j\neq r}}^{n} \left\|A_{rj}\right\| \qquad (17)$$

since

$$1 = \left\|X_r\right\| \geq \left\|X_j\right\|$$

by the division of X.

Now assume that the matrix, $A - \lambda I$, is block strictly diagonally dominant.

Let $Z_{rr} = (A_{rr} - \lambda I_r)X_r$.

Then

$$\left\|(A_{rr} - \lambda I_r)X_r\right\| = \frac{\left\|(A_{rr} - \lambda I_r)X_r\right\|}{\left\|X_r\right\|} = \frac{\left\|Z_r\right\|}{\left\|(A_{rr} - \lambda I_r)^{-1} Z_{rr}\right\|} \geq (\left\|(a_{rr} - \lambda I_r)^{-1}\right\|)^{-1}$$

since

$$\frac{\left\|(A_{rr} - \lambda I_r)^{-1}X_r\right\|}{\left\|X_r\right\|} \leq \sup \frac{\left\|(A_{rr} - \lambda I_r)^{-1}X_r\right\|}{\left\|X\right\|} = \left\|(A_{rr} - \lambda I_r)^{-1}\right\|.$$

Then from equations (16) and (17)

$$(\left\|(A_{rr} - \lambda I_r)^{-1}\right\|)^{-1} \leq \sum_{\substack{j=1\\j\neq r}}^{n} \left\|A_{rj}\right\|$$

Thus $A - \lambda I$ being singular implies that it cannot be block strictly diagonally dominant.

However, for λ to be an eigenvalue of A, then in order for there to exist a nonzero eigenvector corresponding to λ, then $A - \lambda I$ must be singular, and the conclusion follows.

If, in theorem II. A.4, $|\lambda - a_{ii}|$ is replaced by the general form $(\|(A_{ii} - \lambda I_i)^{-1}\|)^{-1}$ and

$$\sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{kj}|$$

is replaced by

$$\sum_{\substack{l=1 \\ l\neq i}}^{n} \|A_{il}\|$$

then an identical proof will give the following.[20]

Corollary I. B.4a

All eigenvalues of A, λ^A, lie in the union of the $[N(N-1)]/2$ point sets C_{ij}, defined by

$$(\|(A_{ii} - \lambda I_i)^{-1}\|^{-1} \cdot \|(A_{jj} - \lambda I_j)^{-1}\|)^{-1} \leq \left(\sum_{\substack{l=1 \\ l\neq i}}^{n} \|A_{i,l}\|\right)\left(\sum_{\substack{l=1 \\ l\neq j}}^{n} \|A_{j,l}\|\right)$$

where

$$1 \leq i,\ j \leq N \quad \text{and} \quad i \neq j\ .$$

In a similar manner, if the above substitutions are made in theorem II. A.5, and an identical proof is used, the result will be the following corollary.[21]

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[20] Ibid., p. 1247.

[21] Ibid., p. 1248.

<u>Corollary II. B.4b</u>

For any α with $0 \leq \alpha \leq 1$, each eigenvalue of A satisfies

$$\left(\left\| (A_{jj} - \lambda I_j)^{-1} \right\| \right)^{-1} \leq \left(\sum_{\substack{k=1 \\ k \neq j}}^{n} \left\| A_{jk} \right\| \right)^{\alpha} \cdot \left(\sum_{\substack{k=1 \\ k \neq j}}^{n} \left\| A_{kj} \right\| \right)^{\alpha-1}$$

for at least one j, $1 \leq j \leq N$.

Section C— Bounds by Vectors and Related Matrices

The purpose of this section is to determine eigenvalue bounds in terms of vectors or in terms of the eigenvalues of related matrices. The majority of the following proofs will depend upon the quadratic form of a matrix combined with simple geometric inequalities.

Bendixson proved the following result for a real matrix $A = (a_{ij})_n$; extended by Hirsch to complex case.[22]

<u>Theorem II.C.1</u>

If $\lambda^A = \alpha + i\beta$, and $\lambda^{\frac{1}{2}(A+A^T)}_{max}$, $\lambda^{\frac{1}{2}(A+A^T)}_{min}$ are the largest and smallest eigenvalues of $\frac{1}{2}(A+A^T)$,

then $\lambda^{\frac{1}{2}(A+A^T)}_{max} \geq \lambda^A \geq \lambda^{\frac{1}{2}(A+A^T)}_{min}$.

Proof: Let H be an arbitrary Hermitian matrix and U be the unitary transformation such that U^*HU is a diagonal matrix. If the equality

$$\sum h_{ij}\ \bar{x}_i\ x_j = \sigma \sum x_i\ \bar{x}_j$$

is satisfied by a non-trivial x, then

$$\lambda^H_{max} \geq \sigma \geq \lambda^H_{min} .$$

For $\left(x, Hx\right) = \left(Uy,\ HUy\right) = \left(y,\ U^*HUy\right) = \sum \lambda^H_i\ y^2_i$.

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[22] Hirsch, 368-370.

Now if $(x, Hx) = \sigma(x, x)$, then $\sigma(x, x) = \sigma(Uy, Uy) = \sigma(y, U^*Uy)$
$= \sigma \sum y_i^2$.

Thus $\lambda_{max}^H y_1 \bar{y}_1 + \ldots + \lambda_{min}^H y_n \bar{y}_n = \sigma \sum_{i=1}^{n} y_i \bar{y}_i$, or

$$\lambda_{max}^H \left(\sum_{i=1}^{n} y_i \bar{y}_i\right) \geq \sigma\left(\sum_{i=1}^{n} y_i \bar{y}_i\right) \geq \lambda_{min}^H\left(\sum_{i=1}^{n} y_i \bar{y}_i\right).$$

Therefore $\lambda_{max}^H \geq \sigma \geq \lambda_{min}^H$.

Now as was shown in theorem II.A.1

$$\alpha = \frac{1}{2}\sum \left(a_{ij} + \overline{a_{ji}}\right) \bar{x}_i x_j,$$

so that

$$\lambda_{max}^{\frac{1}{2}(A+A^T)} \geq \alpha \geq \lambda_{min}^{\frac{1}{2}(A+A^T)}.$$

Similarly, since

$$\left(\frac{A - A^*}{2i}\right)$$

is also Hermitian and since

$$\beta = \frac{1}{2i}\sum \left(a_{ij} - \overline{a_{ji}}\right) \bar{x}_i x_j, \text{ then}$$

<u>Corollary II.C.1</u> (to proof)

$$\lambda_{max}^{\frac{1}{2i}(A-A^*)} \geq \beta \geq \lambda_{min}^{\frac{1}{2i}(A-A^*)}.$$

Just as in the proof of theorem II.C.1 related vectors may be used to define eigenvalues and their bounds.

Assume that an eigenvalue λ_i with a corresponding eigenvector x_i exist for the complex matrix A. Then $Ax_i = \lambda x_i$, so that

$$(x_i, Ax_i) = (x_i, \lambda x_i) = \lambda(x_i, x_i), \text{ or}$$

$$\lambda_i = \frac{(x_i, Ax_i)}{(x_i, x_i)} .$$

In the more general form, the above quotient

$$q(x) \equiv \frac{(x, Ax)}{(x, x)},$$

for arbitrary vector x, is called the Rayleigh's quotient.

Now if either A has distinct eigenvalues, or if A is Hermitian, then there exists a unitary matrix U such that U^*AU = Diagonal and $U^*U = I$. In either case

$$(x, Ax) = (Uy, AUy) = (y, U^*AUy) = (y, \left[\text{diag } \lambda\right] y) .$$

Also if $(x, x) = 1$, then

$$1 = (x, x) = (x, U^*Ux) = (Ux, Ux) = (y, y) .$$

Thus the values assumed by (x, Ax) on $(x, x) = 1$ are equivalent to the values assumed by $(y, \left[\text{diag } \lambda_i\right] y)$ on $(y, y) = 1$.

However, $(y, \left[\text{diag } \lambda_i\right] y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2$,

so that $(y, \left[\text{diag } \lambda_i\right] y) \geq \lambda_{min} (y, y) = \lambda_{min}$

and $(y, \left[\text{diag } \lambda_i\right] y) \leq \lambda_{max} (y, y) = \lambda_{max}$.

Thus $\lambda^{A}_{min} = \min_{y} \dfrac{(y, [\mathrm{diag}\ \lambda_i]\ y)}{(y, y)} = \min_{x} \dfrac{(x, Ax)}{(x, x)}$

and

$$\lambda^{A}_{max} = \max_{y} \frac{(y\ [\mathrm{diag}\ \lambda_i]\ y)}{(y, y)} = \max_{x} \frac{(x, Ax)}{(x, x)}\ .$$

Although the preceding was predicated on the assumption that A had distinct eigenvalues or was Hermitian, this concept can be extended to include a larger class of matrices, as will now be shown.

Let $A = (a_{ij})$ be an arbitrary complex matrix and $B = (b_{ij})$ a Hermitian matrix whose order is equal to that of A, and such that $(x, Ax) = (x, Bx)$.

Let $a_{ij} = \alpha_{ij} + i\beta_{ij}$

$$x_j = r_j + is_j$$

$b_{ij} = \gamma_{ij} + i\ \delta_{ij}$

Note that $a_{ii} = b_{ii}$, $i = 1, \ldots, n$, and we require that

$$a_{ij}\ \overline{x}_i\ x_j + a_{ji}\ x_i\ \overline{x}_j = b_{ij}\ \overline{x}_i\ x_j + b_{ji}\ x_i\ \overline{x}_j$$

where $b_{ij} = \overline{b_{ji}}$. Thus,

$(\alpha_{ij} + i\beta_{ij})(r_i - is_i)(r_j + is_j) + (\alpha_{ji} + i\beta_{ji})(r_i + is_i)(r_j - is_j) =$

$\left[\alpha_{ij}\ (r_i\ r_j + s_i\ s_j) - \beta_{ij}\ (r_i\ s_j - r_j\ s_i)\right] + i\left[\beta_{ij}\ (r_i\ r_j + s_i\ s_j)\right.$

$\left. + \alpha_{ij}\ (r_i\ s_j - r_j\ s_i)\right] + \left[\alpha_{ji}(r_i\ r_j + s_i\ s_j) + \beta_{ji}\ (r_i\ s_j - r_j\ s_i)\right]$

$+ i\left[-\alpha_{ji}(r_i\ s_j - r_j\ s_i) + \beta_{ji}(r_i\ r_j + s_i\ s_j)\right]$

$= \left[(\alpha_{ij} + \alpha_{ji})(r_i\ r_j + s_i\ s_j) + (\beta_{ji} - \beta_{ij})(r_i\ s_j - r_j\ s_i)\right]$

$$+ i\left[(\alpha_{ij} - \alpha_{ji})(r_i\ s_j - r_j\ s_i) + (\beta_{ij} + \beta_{ji})(r_i\ r_j + s_i\ s_j)\right] =$$

$$2\gamma_{ij}\ (r_i\ r_j + s_i\ s_j) - 2\delta_{ij}\ (r_i\ s_j - r_j\ s_i) + 0.i\ .$$

Thus equating real and imaginary parts

$$(\alpha_{ij} + \alpha_{ji})(r_i\ r_j + s_i\ s_j) - (\beta_{ij} - \beta_{ji})(r_i\ s_j - r_j\ s_i)$$

$$= 2\gamma_{ij}\ (r_i\ r_j + s_i\ s_j) - 2\delta_{ij}\ (r_i\ s_j - r_j\ s_i)$$

and

$$\left(\frac{\alpha_{ij} - \alpha_{ji}}{\beta_{ij} + \beta_{ji}}\right) = \left(\frac{r_i\ r_j + s_i\ s_j}{r_j\ s_i - r_i\ s_j}\right) .$$

Notice that for equality in the real part, it is sufficient that

$$\gamma_{ij} = \frac{1}{2}\,(\alpha_{ij} + \alpha_{ji})$$

and

$$\delta_{ij} = \frac{1}{2}\ (\beta_{ij} - \beta_{ji}),$$

conditions independent of the vector components. However for equality in the imaginary, it is necessary to restrict the vector components.

Thus we have proven :

<u>Theorem II.C.2</u>

An arbitrary complex matrix $A = (a_{ij})_n$ is equivalent to a Hermitian matrix, B, under the quadratic form if: for

$$b_{ij} = \gamma_{ij} + i\delta_{ij} \qquad x_j = r_j + is_j$$

$$a_{ij} = \alpha_{ij} + i\beta_{ij}$$

$$\gamma_{ij} = \frac{1}{2} (\alpha_{ij} + \alpha_{ji})$$
$$\delta_{ij} = \frac{1}{2} (\beta_{ij} - \beta_{ji}) \tag{18}$$

$$\left(\frac{\alpha_{ij} - \alpha_{ji}}{\beta_{ij} + \beta_{ji}}\right) = \left(\frac{r_i\, r_j + s_i\, s_j}{r_j\, s_i - r_i\, s_j}\right) \tag{19}$$

whenever the domain is restricted to values of equation (19) for all $1 \leq i,\ j \leq n$.

Notice that if the matrix A and vectors x are all real, so that the real scalar product is used, then necessary conditions reduce to equation (18).

A relation which gives eigenvalue bounds of the matrix A in terms of eigenvalues of the related matrix A^*A is from:

<u>Theorem II.C.3</u>

$$\lambda_{\min}^{A^*A} \leq \left|\lambda_i^A\right|^2 \leq \lambda_{\max}^{A^*A}$$

Proof: Let x_i be an eigenvector corresponding to the eigenvalue, λ_i, of A, so that $Ax_i = \lambda_i\, x_i$ and

$$(Ax_i,\ Ax_i) = (\lambda_i\, x_i,\ \lambda_i\, x_i) \text{ or}$$

$$(x_i,\ A^*Ax_i) = \lambda_i\, (\lambda_i\, x_i,\ x_i) = \lambda_i\, \bar{\lambda}_i\, (x_i,\ x_i)\,.$$

Since

$$(A^*Ax,\ x) = (Ax,\ A^{**}x) = (Ax,\ Ax) = (x,\ A^*Ax)$$
$$= ((A^*A)^*x,\ A),\ \text{then}\ \ (A^*A)^* = A^*A$$

so that A^*A is Hermitian. Thus $|\lambda_i|^2 = \frac{(x_i, A^*Ax_i)}{(x_i, x_i)}$, and by the same reasoning as that of theorem II.C.1

$$\lambda_{min}^{A^*A} \leq \left|\lambda_i^A\right|^2 \leq \lambda_{max}^{A^*A} .$$

Corollary I.C.3a

$$\lambda_{min}^{A^*A} \leq \left|\lambda_{min}^A\right|^2 \leq \left|\lambda_{max}^A\right|^2 \leq \lambda_{max}^{A^*A} .$$

Corollary II.C.3b

If the matrix A is real, then:

$$\lambda_{min}^{A^TA} \leq \left|\lambda_{min}^A\right| \leq \left|\lambda_{max}^A\right|^2 \leq \lambda_{max}^{A^TA} .$$

The largest eigenvalue cannot only be bounded by considering related vectors, but in fact, can be approximated as closely as desired. This result is due to Collatz.

Theorem II.C.4.

For a matrix A of order k, with k distinct eigenvalues and for an arbitrary $\epsilon>0$, there exists an $N > 0$ such that for all $n > N$,

$$\left|\frac{A^n \nu}{A^{n-1} \nu} - \lambda_{max}^A\right| < \epsilon.$$

Proof: Let $\nu = \beta_1 y_1 + \beta_2 y_2 + \dots + \beta_n y_n$, and let $x_i = \beta_i y_i$, so that $\nu = x_1 + x_2 + \dots + x_n$. Assume that the eigenvalues of A

are ordered such that

$$|\lambda_1| > |\lambda_2| > \ldots. > |\lambda_n| .$$

Then

$$A^n \, \nu = \lambda_1^n \left(x_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^n x_2 + \ldots + \left(\frac{\lambda_n}{\lambda_1}\right)^n x_n \right).$$

As $n \to \infty$, then $\left(\frac{\lambda_i}{\lambda_1}\right) \to 0$ for $i = 2$, thus $\lim_{n \to \infty} A^n \, \nu = \lambda_1^n \, x_1$;

likewise $\lim_{n \to \infty} A^{n-1} \, \nu = \lambda_1^{n-1} \, x_1$. Therefore

$$\lim_{n \to \infty} \frac{A^n \, \nu}{A^{n-1} \, \nu} = \frac{\lambda_1^n \, x_1}{\lambda_1^{n-1} \, x_1} = \lambda_1 .$$

Several theorems which give eigenvalue bounds in terms of eigenvalues of related matrices were shown by Wittmeyer.[23] Several of these are given below.

Theorem II.C.5

$$\lambda_{max}^{(AB)^*(AB)} \leq \lambda_{max}^{A^*A} \cdot \lambda_{max}^{B^*B}$$

Proof: Let $\begin{matrix} h = Br \\ z = Ah \end{matrix}$ so that $z = ABr$.

As was shown in the proof of theorem II.C.1 and 2

$$|h| \leq \left(\lambda_{max}^{B^*B}\right)^{\frac{1}{2}} |r|$$

---

[23]Helmut Wittmeyer, "Einfluss der Anderung einer Matrix," Zeitschrift für Angewandte Mathematik und Mechanik, band 16 (October, 1936), pp. 287-300.

$$\left|z\right| \le \left(\lambda_{max}^{A^*A}\right)^{\frac{1}{2}} \left|h\right|$$

so that

$$\left|z\right| \le \left(\lambda_{max}^{A^*A}\right)^{\frac{1}{2}} \left(\lambda_{max}^{B^*B}\right)^{\frac{1}{2}} \left|r\right|.$$

Let r, be the eigenvector of $(AB)^*(AB)$ corresponding to $\lambda_{max}^{(AB)^*(AB)}$, that is $(AB)^*(AB)\ r_1 = \lambda_{max}^{(AB)^*(AB)}\ r_1$.

Let $z_1 = ABr_1$. Then $\left|z_1\right|^2 = (z_1, z_1) = (ABr_1, ABr_1)$

$$= (r_1, (AB)^*(AB)r_1) = \lambda_{max}^{(AB)^*(AB)} \left|r_1\right|^2.$$

Thus, let $z = z_1$, so that

$$\left|r_1\right| \left(\lambda_{max}^{(AB)^*(AB)}\right)^{\frac{1}{2}} = \left|z_1\right| \le \left|r_1\right| \left(\lambda_{max}^{A^*A}\right)^{\frac{1}{2}} \left(\lambda_{max}^{B^*B}\right)^{\frac{1}{2}},$$

and conclusion.

It follows from theorem II.C.3 and its corollary that

Corollary II.C.5

$$\left|\lambda_{max}^{AB}\right| \le \left(\lambda_{max}^{(AB)^*(AB)}\right)^{\frac{1}{2}},$$

and also

$$\left|\lambda_{max}^{AB}\right| \le \left(\lambda_{max}^{A^*A}\right)^{\frac{1}{2}} \cdot \left(\lambda_{max}^{B^*B}\right)^{\frac{1}{2}}.$$

Now in a manner similar to that of theorem II.C.5, we have

$$\left|h\right| \ge \left(\lambda_{min}^{B^*B}\right)^{\frac{1}{2}} \left|r\right|$$

$$\left|z\right| \ge \left(\lambda_{min}^{A^*A}\right)^{\frac{1}{2}} \left|r\right|$$

so that

$$z \geq \left(\lambda_{min}^{A^*A}\right)^{\frac{1}{2}} \left(\lambda_{min}^{B^*B}\right)^{\frac{1}{2}} |r| .$$

Let r_1 be such that $(AB)^*(AB)\ r_1 = \lambda_{min}^{(AB)^*(AB)}\ r_1$, so that for $z_1 = ABr_1$ results in:

$$|z_1| = \lambda_{min}^{(AB)^*(AB)}\ |r_1| .$$

Letting $z = z_1$,

$$|r_1| \left(\lambda_{min}^{(AB)^*(AB)}\right)^{\frac{1}{2}} = |z_1| \geq |r_1| \left(\lambda_{min}^{A^*A}\right)^{\frac{1}{2}} \cdot \left(\lambda_{min}^{B^*B}\right)^{\frac{1}{2}},$$

thus proving

<u>Theorem II.C.6</u>

$$\lambda_{min}^{(AB)^*(AB)} \geq \lambda_{min}^{A^*A} \cdot \lambda_{min}^{B^*B} .$$

Again from this and the corollary to theorem II.C.5, we have

<u>Corollary II.C.6</u>

$$\left|\lambda_{min}^{AB}\right| \geq \left(\lambda_{min}^{(AB)^*(AB)}\right)^{\frac{1}{2}}$$

and

$$\left|\lambda_{min}^{AB}\right| \geq \left(\lambda_{min}^{A^*A}\right)^{\frac{1}{2}} \cdot \left(\lambda_{min}^{B^*B}\right)^{\frac{1}{2}} .$$

Another theorem which follows from a somewhat different geometric consideration if A and B are normal matrices is:

<u>Theorem II.C.7</u>

$$\left(\lambda_{max}^{(A+B)^*(A+B)}\right)^{\frac{1}{2}} \geq \left(\lambda_{max}^{A^*A}\right)^{\frac{1}{2}} + \left(\lambda_{max}^{B^*B}\right)^{\frac{1}{2}} .$$

<u>Proof</u>: Consider

$$\lambda_{max}^{(A+B)^*(A+B)} = \max_{x} \frac{(x_1\ (A + B)^*(A + B)\ x)}{(x,\ x)}$$

$$= \max_x \left[\frac{(x,\ A^*Ax)}{(x,\ x)} + \frac{(x,\ B^*Bx)}{(x,\ x)} + \frac{(x,\ B^*Ax)}{(x,\ x)} + \frac{(x,\ A^*Bx)}{(x,\ x)}\right]$$

$$\leq \max_x \frac{(x,\ A^*Ax)}{(x,\ x)} + \max_x \frac{(x,\ B^*Bx)}{(x,\ x)} + \max_x \frac{(x,\ B^*Ax)}{(x,\ x)} + \max_x \frac{(x,\ A^*Bx)}{(x,\ x)}$$

$= \lambda^{A^*A}_{\max} + \lambda^{B^*B}_{\max} + \lambda^{B^*A}_{\max} + \lambda^{A^*B}_{\max}$, which by the corollary to Theorem II.C.5 is

$$\leq \lambda^{A^*A}_{\max} + \lambda^{B^*B}_{\max} + \left(\lambda^{BB^*}_{\max} \cdot \lambda^{A^*A}_{\max}\right)^{\frac{1}{2}} + \left(\lambda^{AA^*}_{\max} \cdot \lambda^{B^*B}_{\max}\right)^{\frac{1}{2}} = \lambda^{A^*A}_{\max} + \lambda^{B^*B}_{\max}$$

$$+ 2\left(\lambda^{A^*A}_{\max} \cdot \lambda^{B^*B}_{\max}\right)^{\frac{1}{2}} = \left[\left(\lambda^{A^*A}_{\max}\right)^{\frac{1}{2}} + \left(\lambda^{B^*B}_{\max}\right)^{\frac{1}{2}}\right]^2 .$$

Thus

$$\left[\lambda^{(A+B)^*(A+B)}_{\max}\right]^{\frac{1}{2}} \leq \left(\lambda^{A^*A}_{\max}\right)^{\frac{1}{2}} + \left(\lambda^{B^*B}_{\max}\right)^{\frac{1}{2}},$$

and likewise we have

<u>Corollary II.C.7</u>

$$\left|\lambda^{A + B}_{\max}\right| \leq \left(\lambda^{A^*A}_{\max}\right)^{\frac{1}{2}} + \left(\lambda^{B^*B}_{\max}\right)^{\frac{1}{2}} .$$

Theorems II.C.5 through 7 with their respective corollaries may be repeated as corollaries for the special case where A and B are real matrices and tranjugate is replaced by transpose.

CHAPTER III

COMPARISON AND COMPUTATION

From the results of the preceeding sections, it is seen that there are many ways to compute the bounds for eigenvalues. Two obvious questions at this point are:

(1) Do any theorems give more precise eigenvalue bounds in all cases than other theorems?

(2) Are the eigenvalue bounds, themselves, bounded?

The answer to the first question will be given as the major portion of this chapter.

With respect to the second question, it can be stated that:

Theorem III.1 (a) If the eigenvalue bound is expressed in terms of the off-diagonal elements in powers not less than 1, then the eigenvalue bound is unbounded.

(b) If the bound is expressed in terms of the diagonal elements only, the eigenvalue bound is bounded.

Proof: (a) Thus

$$\left|\lambda_{max}\right| \leq f_n\,(a_{12}, a_{21}, \ldots, a_{ij}) \qquad \begin{array}{l} i \neq j \\ i, j = 1, 2, \ldots, n\,. \end{array}$$

For a triangular matrix, $\lambda_{max} = \max a_{ii}$.
For, say a_{21} unbounded, λ_{max} is as given above for all possible values of a_{21}.

Then

$$\lim_{a_{21} \to \infty} f_n (a_{12}, a_{21}, \ldots, a_{ij}) = \infty \qquad i \neq j$$

$$i, j = 1, 2, \ldots n$$

since the off-diagonal elements are in powers not less than 1.

Thus the bound, f_n, is unbounded, even though the λ_{max} remains constant foı all possible values of b.

The bounds given in: chapter 1, section A, theorems 1 and 2 (whenever a, g, or t correspond to off-diagonal elements), 3 with its corollary, 4, 5 with its corollaries, 6, 8 (part 3); section B, matrix norms 1-5, theorem 4 and its corollaries, are all covered by theorem III.1.

<u>Proof</u>: (b) by theorem II.A.8, the values of the diagonal elements are explicitly connected to the eigenvalues of the matrix A, by

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$$

and

$$\prod_{i=1}^{n} \lambda_i = \det A .$$

Then if any diagonal element is increased without bound, that is $a_{ii} \to \infty$ implies

$$\lim_{a_{ii} \to \infty} \sum_{i=1}^{n} \lambda_i \to \infty$$

so that at least one of the eigenvalues increases without bound. This part applies to theorems 1 and 2 of section A whenever a, g or t are diagonal elements.

Comparison will now be made between the inclusion regions of eigenvalues for several theorems of section A.

Since

$$\max_i \sum_{j=1}^{n} |a_{ij}| \leq n \; |\max a_{ij}|,$$

then the corollary to theorem II.A.3 gives a smaller region than theorem II.A.1 in all cases.

Theorem II.A.4 (ovals of Cassini) give a smaller region than theorem II.A.3 since every point of the oval lies in at least one of the two circles which form it. This is readily seen in figure 1.

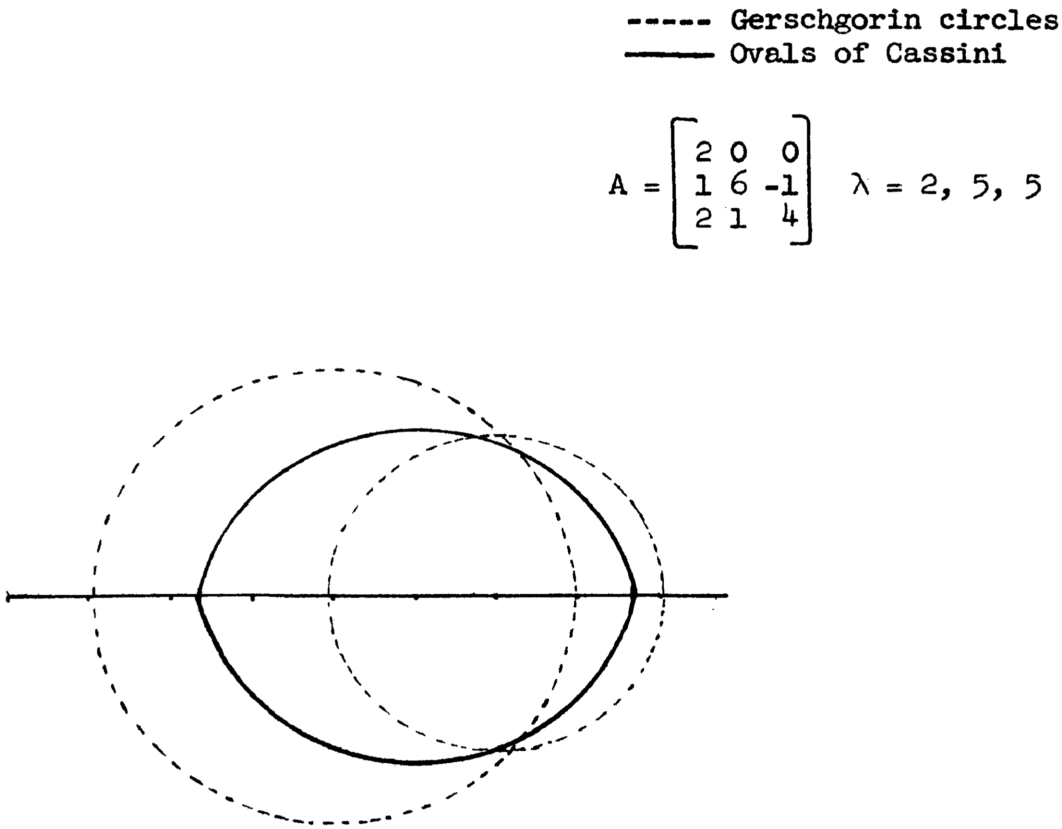

Figure 1

Theorem II.A.5 gives a smaller inclusion region (when considered over all $0 \leq \alpha \leq 1$) than theorem II.A.3. An example of this is seen in figure 2 for the matrix A, whose approximate eigenvalues are 30.55, 10.07, and 0.38

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 4 & 10 & 2 \\ 3 & 2 & 30 \end{bmatrix}$$

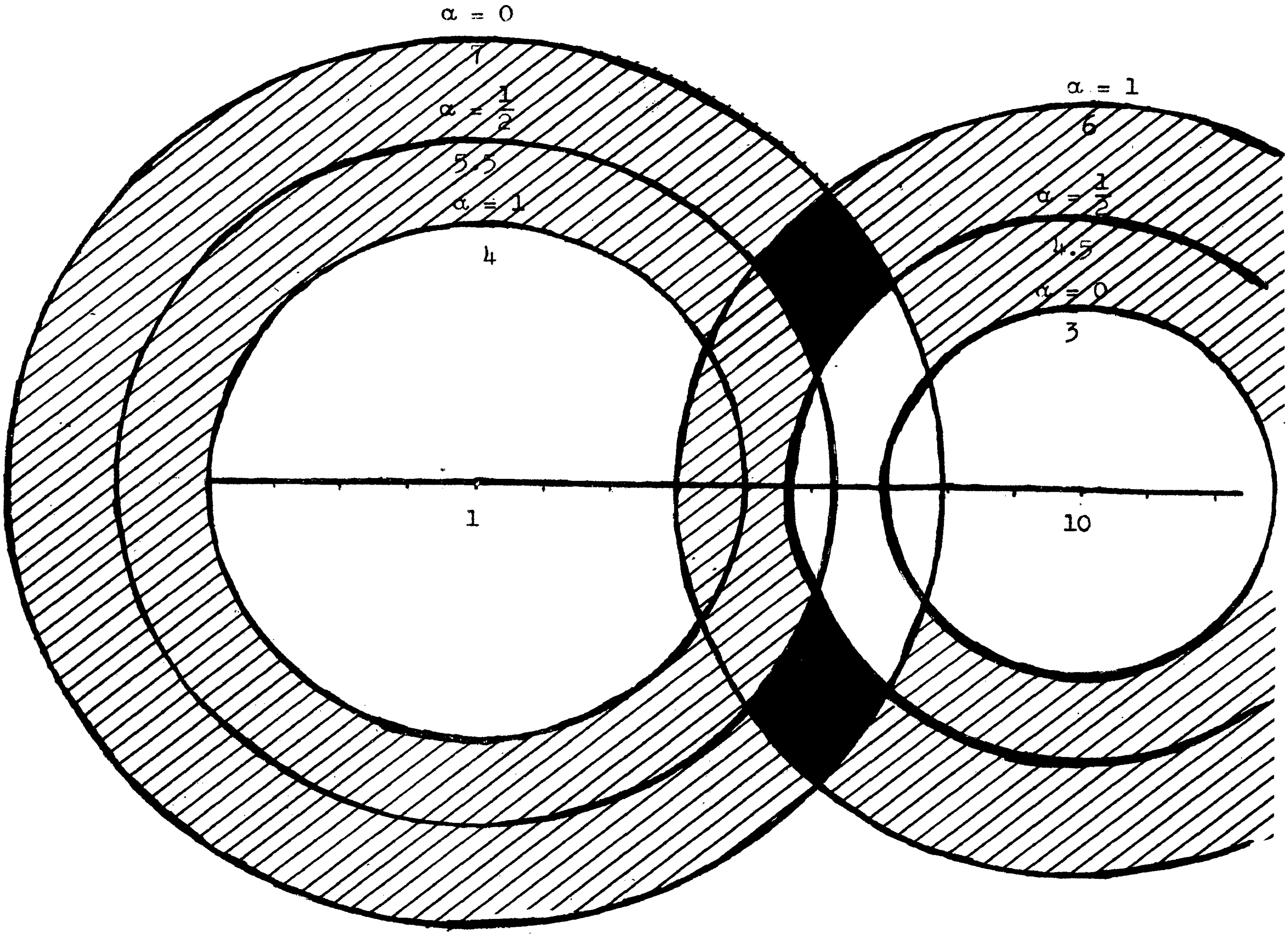

Figure 2

The figure contains the region of the first two eigenvalues only.[24] The curves for theorem II.A.3 correspond to $\alpha = 1$. Note that theorem 5 thus excludes region (shaded portion of figure) for eigenvalues. From this it is clear that the second corollary to theorem II.A.5 is superior to theorem II.A.4.

With respect now to the theorems of section 2, it is clear that the matrix norms (1), (2), (3) and (5) are no better than the results of theorem II.A.3. However theorem II.B.4 gives a fundamental result superior to that of theorem II.A.3. As an example, consider the partitioned matrix,[25]

$$A = \left[\begin{array}{cc|cc} 4 & -2 & -1 & 0 \\ -2 & 4 & 0 & -1 \\ \hline -1 & 0 & 4 & -2 \\ 0 & -1 & -2 & 4 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

with eigenvalues, $\lambda = 1, 3, 5, 7$ and where the vector norm is taken as

$$\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} .$$

[24]Ostrowski, pp. 219-220.

[25]Feingold and Varga, pp. 1246-1247.

Clearly $\|A_{12}\| = \|A_{21}\| = 1$.

For $\|A_{11}\| = \|A_{22}\| = \sup \dfrac{\|A_{11}\, \mathbf{x}\|}{\|\mathbf{x}\|}$, consider

$$\left\| \left(A_{ii} - \lambda I\right)^{-1} \right\|^{-1} = \inf \left(\frac{\left\| \left(A_{ii} - \lambda I\right) x \right\|}{\|x\|} \right) =$$

$$\inf \frac{\|A_{ii}\, x - \lambda x\|}{\|x\|} \geq \left| \inf \left(\frac{\|A_{ii}\, x\|}{\|x\|} - \frac{\|\lambda\, x\|}{\|x\|} \right) \right| = \inf \left| \frac{\|A_{ii}\, x\|}{\|x\|} - |\lambda| \right| .$$

For $i = 1, 2$

$$\frac{\|A_{ii}\, x\|}{\|x\|} = \frac{\left\| \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|}{\|x\|} = \frac{2\sqrt{5x_1^2 + 5x_2^2 - 8x_1\, x_2}}{\sqrt{x_1^2 + x_2^2}} = 2\sqrt{5 - \frac{8x_1\, x_2}{x_1^2 + x_2^2}} .$$

Let $k\left(x_1, x_2\right) = - \dfrac{8x_1\, x_2}{x_1^2 + x_2^2}$; for a minimum value of k,

$$\frac{\partial k}{\partial x_1} = \frac{\partial k}{\partial x_2} = 0,$$

so that

$$\frac{\partial k_2}{\partial x_1} = \frac{-8x_2 \left[x_2^2 - x_1^2 \right]}{x_1^2 + x_2^2} = 0$$

$$\frac{\partial k}{\partial x_2} = \frac{-8x_1 \left[x_1^2 - x_2^2 \right]}{x_1^2 + x_2^2} = 0,$$

so that

$$x_1 = x_2$$

$$x_1 = -x_2.$$

Thus

$$\inf \frac{\|A_{ii}\ x\|}{\|x\|} = 2 \text{ for } x_1 = x_2$$

$$= 6 \text{ for } x_1 = -x_2 .$$

By theorem II.B.4 all eigenvalues are contained in the region defined by: $|2 - \lambda| \leq 1$ and $|6 - \lambda| \leq 1$.

A comparison between theorem II.B.4 (whose inclusion region is shaded) and theorem II.A.3 is given in figure 3.

From this result is clear that corollary II.B.4 a is superior to theorem II.A.4. Likewise corollary II.B.4b is superior to theorem II.A.5.

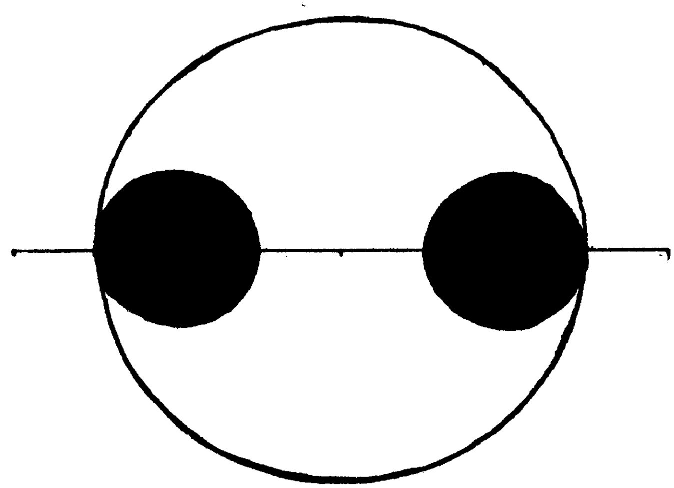

Figure 3

CHAPTER IV

APPLICATIONS

Questions relating to the convergence of series and sequences of matrices arise in many situations. The eigenvalue bounds of the related matrices can give sufficiency conditions for convergence. For example, consider the system of linear equations, $Ax = y$, where A is a $n \times n$ nonsingular matrix of coefficients and x and y are n dimensional vectors.[26]

Let G be an approximate inverse of A, so that the approximate solution is $t = Gy$.

It can be shown by induction that

$$x = \sum_{h=0}^{p} (I - GA)^h t + (I - GA)^{p+1} t .$$

Denote the error in t by $\epsilon = x - t$ and let $D = I - GA$. Thus

$$\epsilon = \sum_{h=1}^{p} D^h t + D^{p+1} x .$$

Now if $\left| \lambda_{max}^{D} \right| < 1$, the $\lim_{p \to \infty} D^p = 0$. Thus

$$\epsilon = \sum_{h=1}^{\infty} D^h t$$

places a bound on the error ϵ.

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[26] A. de la Garzo, "Error Bounds on Approximate Solutions to Systems of Linear Algebraic Equations." Aids to Computation, vol. 7, no. 43 (July 1953), pp 81-84.

As another example, consider again the system of linear equations $Ax = y$ where A is nonsingular.[27] Let x_p be the pth approximation of x, so that $S_p \equiv x - x_p$.

Define $r_p \equiv A(x - x_p) = y - Ax_p$. Assume that $A = A_1 + A_2$ where A_1 is nonsingular. To form the iterative solution of x, let each successive x_{p+1} be defined by the equation

$$A_1 x_{p+1} = y - A_2 x_p \; .$$

Now if the iteration converges, the sequence of x_p will do so in the Cauchy sense. Thus

$$\lim_{p \to \infty} (A_1 x_{p+1} + A_2 x_p) = Ax \; .$$

Then

$$A_1(x - x_{p+1}) = - A_2(x - x_p)$$

or

$$A_1 S_{p+1} = - A_2 S_p$$

so that

$$S_{p+1} = - (A_1{}^{-1} A_2) S_p \; .$$

[27] Alston S. Householder, On Norms of Vectors and Matrices, (Oak Ridge, Tennessee, Oak Ridge National Laboratory, 1954, pp 6-8.

Now if x_o is the initial guess for x, then $S_o = x - x_o$ so that $S_p = (-1)^p (A_1{}^{-1}A_2)^p S_o$. Thus a sufficient condition for the iteration to converge is that $\left|\lambda^{(A_1{}^{-1}A_2)}_{max}\right|_{max} < 1$.

Along the same lines, the ratio

$$\frac{\lambda^A_{max}}{\lambda^A_{min}} = P$$

gives a rough measure of the probable accuracy of the computation of the inverse of A. This ratio, P, is called the "P-condition number" of the matrix.[28] In general, the accuracy of the results is in proportion to the reciprocal of P.

As a further extension of the results given in chapter I, iterative schemes may be established to give close approximations to all of the eigenvalues of a matrix. A survey of these techniques together with comparative accuracy and computation time is given by White.[29]

As a final application of the results of the preceding chapters, consider the problem of determining for a function of n variables, the values for which it attains its maximum and minimum.

Let $f(x_1, x_2, \ldots, x_n)$ be a real function of the set of real variables $x_1, x_2, \ldots, x_n$ over a closed interval, I, in E_n. Assume that it possesses a convergent Taylor series about each point in the interior of I, with continuous partial derivations.

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[28]Philip Davis et al., "Bound for the P-Condition Number of Matrices With Positive Roots," Journal of Research of the National Bureau of Standards, vol. 65(B), no. 1, (January-March, 1961), pp. 13-14.

[29]Paul A. White, "The Computation of Eigenvalues and Eigenvectors of a Matrix, "Journal of the Society for Industrial and Applied Mathematics, vol. 6, no. 4 (December, 1958).

Since by Rolle's theorem, if $f(a_1, \ldots, a_n) = f(b_1, \ldots, b_n)$, where a and $b \epsilon I$, then there exists $c \epsilon$ interior of I such that $f'(c_1, c_2, \ldots, c_n) = 0$. Assume that such a "c" exists.

If

$$f''(c_1, \ldots, c_n) > 0$$

f has a relative minimum at $x = c$.

If

$$f''(c_1, \ldots, c_n) < 0$$

f has a relative maximum at $x = c$.

If

$$f''(c_1, \ldots, c_n) = 0$$

higher order derivative of f must be considered.

Let

$$\begin{aligned} x_1 &= c_1 + \alpha_1 t \\ x_2 &= c_2 + \alpha_2 t \\ &\vdots \\ x_n &= c_n + \alpha_n t \ . \end{aligned} \tag{20}$$

Then

$$F(t) \equiv f(x_1, x_2, \ldots, x_n) = f(c_1 + \alpha_1 t, \ldots, c_n \alpha_n t).$$

Expanding $F(t)$ in a Maclaurin's series results in

$$F(t) = F(0) + F'(0)t + \frac{F''(0)}{2!}t^2 + \ldots \tag{21}$$

Then

$$F'(t) = \frac{\partial f}{dx_1}\frac{dx_1}{dt} + \ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt} = \alpha_1 \frac{\partial f}{\partial x_1} + \ldots + \alpha_n \frac{\partial f}{\partial x_n} = \sum_{k=1}^{n} \alpha_k \frac{\partial f}{\partial x_k} .$$

$$F''(t) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{n} \alpha_k \frac{\partial f}{\partial x_k}\right)\frac{dx_i}{dt} = \sum_{i=1}^{n} \alpha_i{}^2 f_{x_i x_i} \tag{22}$$

$$+ 2 \sum_{\substack{i=1 \\ i<j}}^{n} \alpha_i \alpha_j f_{x_i x_j};\ j = 2, \ldots, n . \tag{23}$$

since $t = 0$ implies that

$$\begin{aligned} x_1 &= c_1 \\ x_2 &= c_2 \\ &\vdots \\ x_n &= c_n . \end{aligned}$$

Thus

$$F''(0) = \sum_{i=1}^{n} \alpha_i{}^2 f_{x_i x_i}(c_1, \ldots, c_n)$$

$$+ 2 \sum_{\substack{i=1 \\ i<j}}^{n} \alpha_i \alpha_j f_{x_i x_j}(c_1, c_2, \ldots, c_n);\ j = 2, \ldots, n . \tag{24}$$

Then, assuming the $c = (c_1, \ldots, c_n)$ is a critical point, so that

$$f_{x_1}(c_1, \ldots, c_n) = \ldots = f_{x_n}(c_1, \ldots, c_n) = 0$$

the nature of $f(x_1, \ldots, x_n)$ at $c_1, c_2, \ldots, c_n$ will depend upon the values of $\frac{F''(0)t^2}{2!}$ in equation (21). Notice that from equation (1), $\alpha_i t = (x_i - c_i)$. Thus

$$\frac{F''(0)t^2}{2!} = \frac{1}{2}\sum_{i=1}^{n} \alpha_i{}^2 t^2 f_{x_i x_i}{}^{(c)} + \sum_{\substack{i=1 \\ i<j}}^{n} \alpha_i \alpha_j t^2 f_{x_i x_j}{}^{(c)};\ j = 2, \ldots, n$$

$$= \frac{1}{2}\sum_{i=j}^{n} (x_i - c_i)^2 f_{x_i x_i}{}^{(c)} + \sum_{\substack{i=1 \\ i<j}}^{n} (x_i - c_i)(x_j - c_j) f_{x_i x_i}{}^{(c)};$$

$$j = 2, \ldots, n \tag{25}$$

Let $y_i = x_i - c_i$ substitute in equation (25) and define the result as $Q(y_1, \ldots, y_n)$.

Define

$$a_i = \frac{1}{2} f_{x_i x_i}{}^{(c)},\ b_{ij} = \frac{1}{2} f_{x_i x_j}{}^{(c)}. \tag{26}$$

From this substitution into equation (25) results in a quadratic form in n variables, so that equation (25) becomes

$$Q(y_1, \ldots, y_n) = \sum_{i=1}^{n} a_i y_i^2 + 2 \sum_{\substack{i=1 \\ i\neq j}}^{n} b_{ij} y_i y_j \tag{27}$$

Thus consider the critical points of $Q(y_1, \ldots, y_n)$ under the constraining condition that $y_1^2 + \ldots + y_n^2 = 1$.

Using the general method of Lagrange multipliers, form the function

$$F = Q(y_1, \ldots, y_n) - \lambda(y_1^2 + \ldots + y_n^2) \tag{28}$$

where λ is the parameter to be determined.

Using the n equations

$$\frac{\partial F}{\partial y_i} = 0, \; i = 1, \ldots, n \tag{29}$$

and equation (28), λ can be determined.

From equation (29) it follows that

$$\begin{array}{l} 0 = \dfrac{\partial F}{\partial y_1} = 2a_1 y_1 + 2 \displaystyle\sum_{j=2}^{n} b_{1j} y_j - 2\lambda y_1 \\ \vdots \\ 0 = \dfrac{\partial F}{\partial y_n} = 2a_n y_n + 2 \displaystyle\sum_{j=1}^{n-1} b_{nj} y_j - 2\lambda y_n \end{array} \tag{30}$$

or

$$(A - \lambda I)y = \begin{pmatrix} (a_1 - \lambda)y_1 + b_{12}y_2 + b_{13}y_3 + \ldots + b_{1n}y_n \\ b_{21}y_1 + (a_2 - \lambda)y_2 + b_{23}y_3 + \ldots + b_{2n}y_n \\ \vdots \\ b_{n1}y_1 + \ldots\ldots\ldots + (a_n - \lambda)y_n \end{pmatrix} = 0 \tag{31}$$

In order for a nontrivial solution of equation (31) to exist, the determinant of the coefficient matrix of equation (31) must vanish.

To determine the nature of $f(x_1, \ldots, x_n)$ in the neighborhood of $(c_1, \ldots, c_n)$, the nature of $Q(y_1, \ldots, y_n)$ must be determined for those $y_1, \ldots, y_n$ in the ϵ neighborhood of zero, that is, all $y' = (y_1', y_2', \ldots, y_n')$ such that y contained in $N_\epsilon(0)$, $\epsilon > 0$. Notice that just by determining the values of $Q(y_1, \ldots, y_n)$ considering only those values of y which are on the radius of the ϵ neighborhood, is sufficient for determining the nature of $Q(y_1, \ldots, y_n)$ over the entire space. This is because Q is homogeneous of degree 2, that is, $Q(\lambda y_1, \ldots, \lambda y_n) = \lambda^2 Q(y_1, \ldots, y_n)$. Thus, the values assumed on $y_1^2 + y_2^2 + \ldots + y_2^2 = \epsilon^2$ are related to the values on $y_1^2 + \ldots + y_n^2 = 1$.

Now if $Q(y_1, \ldots, y_n)$ is to be positive for all values of $y_1, y_2, \ldots, y_n$, then,

$$\underset{y_1^2 + \ldots + y_n^2 = 1}{\text{minimum}} \quad Q(y_1, \ldots, y_n) > 0 \quad \text{(positive definite)}$$

and likewise if $Q(y_1, \ldots, y_n)$ is to be negative for every $y_1, \ldots y_n$, then,

$$\underset{y_1^2 + \ldots + y_n^2 = 1}{\text{maximum}} \quad Q(y_1, \ldots, y_n) > 0 \quad \text{(positive indefinite)}$$

If either of the above conditions for $Q(y_1, \ldots, y_n)$ hold, then it is necessary and sufficient for $Q(y_1, \ldots, y_n)$ to satisfy either of the above inequalities at both its maximum and minimum. Since it was assumed that the function had continuous partial derivatives, and was a real valued function, then the coefficient matrix is Hermitian so that values for λ are real. Thus the values of $y(\lambda)$ corresponding to distinct values of λ are orthogonal.

These values of y are determined by roots of the characteristic equation corresponding to equation (31). When the condition that $(y(\lambda),y(\lambda)) = 1$ is added, the resulting $y'(\lambda)$ will be the points at which Q attains its maximum and minimum. From these values of Q, it may be determined at what points in the space that $f(x_1, \ldots, x_n)$ is a maximum or minimum.

Since the matrix in equation (31) is real and symmetric, there exists an orthogonal transformation matrix, T, whose columns consist of the associated eigenvectors of A, such that

$$T^T AT = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

where the λ are the eigenvalues of A.

Now

$$Q(y) = (y,Ay) .$$

Let

$$Tz = y .$$

Then

$$(y,Ay) = (Tz,ATz) = (z,T^T ATz) .$$

Thus

$$Q(y) = (z,z\mathrm{diag}(\lambda)) = \sum_{i=1}^{n} \lambda_i z_i^2 .$$

Therefore, a necessary and sufficient condition for

$$\underset{y_1^2+\ldots+y_n^2=1}{\text{minimum}} \quad Q(y) > 0$$

is that all λ's be positive.

Likewise a necessary and sufficient condition for

$$\underset{y_1^2+\ldots+y_n^2=1}{\text{maximum}} \quad Q(y) < 0$$

is that all λ's be negative.

Thus in this example, it is not necessary to know the exact values of the λ to determine if one or the other of the above inequalities exist. To be more specific, if, for the matrix under consideration,

$$a_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} |b_{ij}| > 0$$

for all $i=1, \ldots, n$ and $a_i > 0$ then it is positive definite. Likewise for $a_i < 0$ and

$$|a_i| - \sum_{\substack{j=1 \\ j=i}}^{n} |b_{ij}| > 0$$

then the matrix is positive indefinite.

APPENDIX

Notation

$A = (a_{ij})_n$	a matrix of order n
$A^T = (a_{ji})_n$	matrix transpose of A
$\overline{A} = (\overline{a}_{ij})_n$	complex conjugate of A
$A^* = (\overline{a}_{ji})_n$	conjugate transpose of A
$\mathrm{diag}(a_1, \ldots, a_n)$	diagonal matrix with $a_1, \ldots, a_n$ down the main diagonal
$\|\|A\|\|$	matrix norm of A
$\det A$	determinant of A
λ_i^A	the ith eigenvalue of A
Λ_A	Jordan canonical form of A
J_i	the ith Jordan block corresponding to λ_i
I_n	n-square identity matrix
$(x$	row vector, $(x_1, x_2, \ldots, x_n)$
$x)$	column vector $(x_1, x_2, \ldots, x_n)^T$
e_i	n-tuple with 1 as ith coordinate, 0 otherwise
(x,y)	complex scalar product

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VITA

Kenneth Ross Garren

Born in Augusta, Georgia, August 30, 1940, the author graduated from William Fleming High School, Roanoke, Virginia, in June of 1958. He received a Bachelor of Science degree from Roanoke College in June of 1962, with a concentration in physics and mathematics. He was an instructor at that school during the summer session of 1962. In August 1962 he became an Aerospace Engineer with the National Aeronautics and Space Administration, Langley Research Center. The course requirements for this degree and the thesis, "Bounds for the Eigenvalues of a Matrix," will be completed in June 1965.

In September 1963 the author entered the College of William and Mary as a graduate student in the Department of Mathematics.