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Characterizations of Real Linear Algebras

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ABSTRACT

The purpose of this paper is to present characterizations and properties of linear algebras over the field of real numbers.

The thesis will consist of three parts. In chapter I, neither commutativity nor associativity of multiplication will be assumed. Algebras with identity, division algebras, normed algebras, and absolute valued algebras will be discussed, and theorems characterizing and relating these concepts will be presented.

In chapter II, algebras which are commutative or associative with respect to multiplication will be considered. In addition to characterizing these algebras, a proof of the classical result of Frobenius will be presented. That is, except for isomorphisms, the real numbers, the complex numbers, and the algebra of real quaternions form the only associative division algebras over the field of real numbers.

Chapter III will be primarily concerned with automorphisms on the algebra of real quaternions and their application to rotations on the real Euclidean vector space of dimension 3. Other characterizations and properties, which follow from the theorems of chapters I and II, will be presented.

CHARACTERIZATIONS OF REAL LINEAR ALGEBRAS

INTRODUCTION

The study of linear algebras (or hypercomplex systems) first began in 1841 with R. W. Hamilton's discovery of quaternions. Hamilton was then primarily interested in the solution of two problems:

1. Given an n dimensional vector space, is it possible to define multiplication in such a way that the resultant system is a field?

2. Can the product of two sums of n squares be expressed as a sum of n squares?

Hamilton defined a quaternion to be a quadruple of real numbers with the operations:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) + (\beta_1, \beta_2, \beta_3, \beta_4)$$

$$= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3, \alpha_4 + \beta_4)$$

and

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \cdot (\beta_1, \beta_2, \beta_3, \beta_4) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4),$$

where

$$\gamma_1 = \alpha_1 \beta_1 - \alpha_2 \beta_2 - \alpha_3 \beta_3 - \alpha_4 \beta_4$$

$$\gamma_2 = \alpha_1 \beta_2 + \alpha_2 \beta_1 + \alpha_3 \beta_4 - \alpha_4 \beta_3$$

$$\gamma_3 = \alpha_1 \beta_3 - \alpha_2 \beta_4 + \alpha_3 \beta_1 + \alpha_4 \beta_2$$

$$\gamma_4 = \alpha_1 \beta_4 + \alpha_2 \beta_3 - \alpha_3 \beta_2 + \alpha_4 \beta_1.$$

He showed that all the axioms for a field were satisfied with the exception of the commutative law of multiplication. He was also able to obtain the striking identity:

$$(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2) \cdot (\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2) = (\gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \gamma_4^2).$$

Since Hamilton's discovery, a great deal of interest and study has arisen in this area. Quaternions, for example, have proved to be a useful tool in some areas of both physics and mechanics.

The purpose of this paper is to present some of the characterizations and properties of both associative and nonassociative linear algebras over the field of real numbers. The first part will deal with essential definitions and theorems related to real linear algebras on which neither commutativity nor associativity is assumed. Throughout this paper, absolute valued algebras, as defined by A. A. Albert [1], will be of primary concern. It is interesting to note that without associativity of multiplication we are not assured of the existence of an identity element. However, it will be shown that given a real absolute valued algebra, we can always redefine multiplication such that the resultant algebra is an absolute valued algebra with identity. Theorems relating normed, absolute valued and division algebras will also be presented.

The next part deals with the characterizations of commutative and associative algebras over the field of real numbers. A construction of the algebra of quaternions is given together with a proof of Frobenius's theorem which shows the unique place of complex numbers and quaternions among the algebras.

The final objective will be to discuss, in more detail, the algebra of real quaternions. It will be shown here that all automorphisms on the algebra of real quaternions are of a certain type. These automorphisms form a group of linear orthogonal transformations which have a particularly interesting effect on the real Euclidean vector space of dimension three. Applications and properties of quaternions which follow from the theorems of this paper are also given.

Using the algebra of quaternions one can construct still another but less attractive algebra, the eight dimensional Caley numbers. Because of the length involved in defining this system, it will not be presented in this paper. For a discussion of the properties and a proof of the uniqueness of this algebra, the reader is referred to [1], [8], and [11].

CHAPTER I

GENERAL CHARACTERIZATIONS

As was mentioned in the Introduction, this chapter will be devoted to general characterization theorems of algebras over the field of real numbers. We begin with some essential definitions.

Definition: Let A be a vector space of finite dimension n over the field R. A is a linear algebra of order n (or simply an algebra) if there is defined on A a product xy which satisfies the conditions:

(1) $x(\alpha y) = (\alpha x)y = \alpha(xy)$ for α in R and x, y in A,

(2) $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$ for x, y, z in A.

If R is the field of real numbers, we call A a real algebra. Also, if A contains an element ϵ such that $\epsilon x = x\epsilon = x$ for all x in A, we say that A is an algebra with identity, and we denote this element by "1". Note that our definition does not assume commutativity or associativity of multiplication on the algebra A.

From this definition, we arrive at the following useful representation of a product in A. Let $e_1, e_2, \ldots, e_n$ be a basis for A, and let

$$x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$$

and

$$y = y_1 e_1 + y_2 e_2 + \ldots + y_n e_n$$

be any two elements in A. Then

$$xy = \sum_{i,j=1}^{n} x_i y_j e_i e_j$$

is an element in A. Therefore

$$e_i e_j = \sum_{k=1}^{n} \gamma_{ijk} e_k \qquad (i,j = 1, 2, \ldots, n; \gamma_{ijk} \text{ in } R)$$

so that

$$xy = \sum_{k=1}^{n} z_k e_k, \quad z_k = \sum_{i,j=1}^{n} x_i y_j \gamma_{ijk} \qquad (k = 1, 2, \ldots, n).$$

Thus, multiplication of elements in A is completely determined by n^3 constants γ_{ijk}. These constants are called the <u>structure</u> constants of the system. Throughout this paper, R will denote the field of real numbers and it will be understood that A is of finite dimension n.

<u>Definition</u>: Let A be a real algebra. A is said to be <u>absolute valued</u> if there is a function ϕ on A to R such that:

(1) $\phi(0) = 0$ and $\phi(x) > 0$ if $x \neq 0$,

(2) $\phi(xy) = \phi(x)\phi(y)$,

(3) $\phi(x + y) \leq \phi(x) + \phi(y)$, and

(4) $\phi(\alpha x) = |\alpha| \phi(x)$

for all x, y in A and α in R. If these properties hold we call ϕ an <u>absolute value</u> function on A. If $\phi(xy) \leq \phi(x)\,\phi(y)$, A is said to be a <u>normed</u> algebra and ϕ is called a <u>norm function</u> on A.

<u>Theorem 1.1 [1]</u>: Every real algebra is a normed algebra.

<u>Proof</u>: Let A be a real algebra having a basis $e_1, e_2, \ldots, e_n$. As we have seen multiplication on A is defined by

$$e_i\, e_j = \sum_k \gamma_{ijk}\, e_k,$$

where the γ_{ijk}'s are real. Now let $u_i = \alpha e_i$ $(i = 1, 2, \ldots, n)$ for any nonzero real number α. Then $u_1, u_2, \ldots, u_n$ forms a new basis for A such that

$$u_i\, u_j = \sum_k \delta_{ijk}\, u_k$$

and where $\delta_{ijk} = \alpha\gamma_{ijk}$. Let

$$x = \sum_i x_i\, u_i \quad \text{and} \quad y = \sum_j y_j\, u_j$$

be any two elements in A.

Then

$$xy = \sum_k z_k u_k, \quad z_k = \sum_{i,j} x_i y_j \delta_{ijk} .$$

Now we choose α such that

$$|\delta_{ijk}| \leq \frac{1}{n} \quad (i, j, k = 1, 2, \ldots, n)$$

so that

$$|z_k| \leq \frac{1}{n} \sum_{i,j} |x_i y_j| .$$

We define

$$\phi(w) = |w| = |w_1| + |w_2| + \cdots + |w_n|$$

for every

$$w = \sum_i w_i u_i$$

in A. Then ϕ is a norm function on A since

$$\phi(xy) = |xy| = \sum_k |z_k| \leq \sum_{i,j} |x_i| |y_j| = |x| \cdot |y|$$

and

$$\phi(x + y) = |x + y| = \sum_i |x_i + y_i| \leq |x| + |y| .$$

ϕ clearly satisfies the remaining properties which must hold for a norm function. Hence A is a normed algebra and the proof is complete.

Definition: A is a division algebra if the equations $ax = b$ and $ya = b$ always possess solutions if $a \neq 0$.

Theorem 1.2 [1, 6]: Every real absolute valued algebra is a division algebra.

Proof: Let A be a real algebra with absolute value function ϕ. For some a in A, define the mappings

$$R_a: \ x \to xa \quad \text{or} \quad xR_a = xa$$

and

$$L_a: \ x \to ax \quad \text{or} \quad xL_a = ax,$$

for all x in A. Then R_a and L_a are linear transformations on A. Now if $a \neq 0$ and $x \neq 0$, then $\phi(a) > 0$ and $\phi(x) > 0$ implies that $\phi(xa) > 0$ and hence $xa \neq 0$. Therefore, if $a \neq 0$, the null space of R_a consists of the zero vector alone. Similarly for L_a. Hence, if $a \neq 0$, R_a and L_a are nonsingular linear transformations on A. Thus, it follows that the equations $xa = b$ and $ax = b$ can be solved when $a \neq 0$.

Definition: Let A be an algebra and let P and Q be nonsingular linear transformations on A. The algebra A^* whose elements are those of A but whose product operation is defined by $x * y = xP \cdot yQ$ is called an <u>isotope</u> of A. A and A^* are said to be <u>isotopic</u>.

<u>Theorem 1.3 [1]</u>: If A is a real absolute valued algebra, then A has an absolute valued isotope with identity. Furthermore, the absolute value function of A is preserved in its isotope.

<u>Proof</u>: Let ϕ be an absolute value function defined on A. Since $\phi(\alpha x) = |\alpha| \phi(x)$ for every real α and x in A, there exists a nonzero element ϵ in A such that $\phi(\epsilon) = 1$. As before, since $\epsilon \neq 0$ and A is absolute valued,

$$xR_\epsilon = x\epsilon \quad \text{and} \quad xL_\epsilon = \epsilon x$$

define nonsingular linear transformations on A. Let x, z be any elements in A such that

$$xR_\epsilon^{-1} = z.$$

Then

$$x = zR_\epsilon = z\epsilon$$

and

$$\phi(x) = \phi(zR_\epsilon) = \phi(z) = \phi(xR_\epsilon^{-1}).$$

Similarly,

$$\phi(x) = \phi(xL_\epsilon^{-1}).$$

We now define an isotope A^* of A by the product

$$x * y = xR_\epsilon^{-1} \cdot yL_\epsilon^{-1}.$$

Since A and A^* are the same linear spaces over R, the properties of ϕ involving addition and scalar multiplication are preserved on A . Also

$$\phi(x * y) = \phi(xR_\epsilon^{-1}) \cdot \phi(yL_\epsilon^{-1}) = \phi(x)\ \phi(y)$$

for all x, y in A^*. Therefore A^* is absolute valued and preserves the absolute value function of A.

Finally, consider the product $\epsilon^2 * y$ in A^*. The product of two linear transformations R and L is defined by

$$x(R \cdot L) = (xR) \cdot L \qquad \text{for all } x \text{ in } A.$$

Thus

$$\epsilon^2 * y = \epsilon^2 R_\epsilon^{-1} \cdot yL_\epsilon^{-1} = \epsilon(yL_\epsilon^{-1}) = (yL_\epsilon^{-1})L_\epsilon = y,$$

for all y in A. In a similar fashion we have

$$x * \epsilon^2 = x, \qquad \text{for all } x \text{ in } A.$$

Hence, ϵ^2 is the identity of A^*.

<u>Definition</u>: Let A be a real algebra with the basis $e_1, e_2, \dots, e_n$. Denote the vector scalar product of

$$x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

and

$$y = y_1 e_1 + y_2 e_2 + \dots + y_n e_n$$

by $\langle x, y \rangle$. The <u>norm</u> of the vector x is defined by

$$N(x) = \langle x, x \rangle = x_1^2 + x_2^2 + \dots + x_n^2.$$

In the theorems which follows, we will continuously use the fact that the scalar product $\langle x, y \rangle$ defines a nondegenerate and symmetric bilinear form on a real algebra A. Now consider the real algebras having the property that $N(xy) = N(x)\, N(y)$ for all x, y in A.

If A is not associative then x^n is not uniquely defined in A. Therefore, in order to give meaning to x^n, define

$$x^n = x^{n-1} \cdot x, \quad n = 2, 3, \dots .$$

<u>Lemma 1.4</u>: Let A be a real algebra such that $N(xy) = N(x)\, N(y)$ for all x, y in A. If $x^n = x^{n-1} \cdot x$, $n = 2, 3, \dots,$ then $N(x^k) = N(x)^k$ for all positive integers k.

Proof: By induction. Let x be any element in A. The lemma is obviously true for $k = 1$. Now let k be an arbitrary positive integer such that

$$N(x^k) = N(x)^k.$$

Then

$$N(x^{k+1}) = N(x^k \cdot x) = N(x^k)\, N(x) = N(x)^k\, N(x) = N(x)^{k+1}.$$

Hence

$$N(x^k) = N(x)^k$$

for all positive integers k.

Theorem 1.5: Let A be a real algebra with identity. If

$$N(xy) = N(x)\, N(y)$$

for all x, y in A, then A is absolute valued and the absolute value function defined on A is unique.

Proof: To show that A is absolute valued we define

$$\phi(x) = \|x\| = +\sqrt{N(x)}$$

for all x in A. Then

$$\phi(x + Y) = \| x + y \| = +\sqrt{|\langle x + y, x + y \rangle|}$$

so

$$\| x + y \|^2 \leq \| x \|^2 + 2 |\langle x, y \rangle| + \| y \|^2.$$

By the Cauchy-Schwarz inequality,

$$\| x + y \|^2 \leq \| x \|^2 + 2 \| x \| \cdot \| y \| + \| y \|^2 = (\| x \| + \| y \|)^2.$$

Hence

$$\phi(x + y) = \| x + y \| \leq \| x \| + \| y \| = \phi(x) + \phi(y).$$

The remaining properties which must hold for ϕ to be an absolute value function are clearly satisfied and thus A is absolute valued.

To avoid confusion between the identity in A and the identity in R we denote the first by ϵ. We first note that if ϕ is an absolute value function on A,

$$\phi(\epsilon) = \phi(\epsilon \cdot \epsilon) = \phi(\epsilon)\, \phi(\epsilon),$$

and hence

$$\phi(\epsilon) = 1.$$

Now suppose $\phi(x)$ is not unique in our algebra A. Then there exists an absolute value function $\theta(x)$ on A such that

$$\theta(a) \neq \phi(a)$$

for some $a \neq 0$ in A. Then either

$$\theta(a) > \phi(a)$$

or

$$\theta(a) < \phi(a).$$

We consider first the case when

$$\theta(a) > \phi(a).$$

If we let

$$y = \frac{a}{||a||},$$

then

$$||y|| = N(y) = 1.$$

We also have that

$$\theta(y) > 1 \quad \text{since} \quad \theta(y) = \frac{\theta(a)}{||a||} > \frac{\phi(a)}{||a||} = 1.$$

Furthermore, by lemma 1.4,

$$1 = N(y^k) = y_1^2 + y_2^2 + \dots + y_n^2,$$

which implies $|y_1| \leq 1$. Since θ is an absolute value function,

$$\theta(y)^k = \theta(y^k) \leq |y_1| \, \theta(\epsilon) + |y_2| \, \theta(e_2) + \dots + |y_n| \, \theta(e_n).$$

Hence

$$\theta(y)^k \leq 1 + \theta(e_2) + \dots + \theta(e_n).$$

But this is impossible since $\theta(y) > 1$ and k is arbitrary.

Now consider the case when $\theta(a) < \phi(a)$. Since A is absolute valued, it is a division algebra. Therefore we can solve the equation $az = \epsilon$ whenever $a \neq 0$. Let $y = ||a||z$.

Then

$$\theta(y) = ||a||\theta(z).$$

But since

$$\theta(a)\,\theta(z) = \theta(\epsilon) = 1,$$

we have that

$$\theta(y) = \frac{||a||}{\theta(a)} > \frac{||a||}{\phi(a)} = 1, \text{ and } \phi(y) = 1.$$

The remainder of this case follows exactly as the first. That is, we arrive at a contradiction to our assumption that $\theta(a) < \phi(a)$. Hence $\theta(x) = \phi(x)$ for all x in A, so $\phi(x)$ is unique.

The following lemma will aid us in the proof of our next characterization theorem.

<u>Lemma 1.5</u> [6, 10]: Let A be a real algebra such that $N(xy) = N(x)\,N(y)$ for all x, y in A. Then for all x, y, x', y' in A:

(i) $$\langle xy, x'y\rangle = \langle x, x'\rangle N(y),$$

and

$$\langle xy, xy'\rangle = N(x)\langle y, y'\rangle,$$

(ii) $$\langle xy, x'y'\rangle + \langle xy', x'y\rangle = 2\langle x, x'\rangle\langle y, y'\rangle.$$

<u>Proof</u>: We can easily establish that for all x, y in A,

$$\langle x, y\rangle = \frac{1}{2}\left[N(x + y) - N(x) - N(y)\right].$$

Then

$$\langle xy, x'y\rangle = \frac{1}{2}\left[N(xy + x'y) - N(xy) - N(x'y)\right] = \langle x, x'\rangle N(y).$$

Similarly

$$\langle xy, xy'\rangle = N(x)\langle y, y'\rangle.$$

Therefore

$$\langle x(y + y'), x'(y + y')\rangle = \langle x, x'\rangle N(y + y').$$

But

$$N(y + y') = 2\langle y, y'\rangle + N(y) + N(y').$$

Hence

$$\langle x(y + y'), x'(y + y')\rangle = 2\langle x, x'\rangle\langle y, y'\rangle + \langle x, x'\rangle N(y) + \langle x, x'\rangle N(y')$$

$$= 2\langle x, x'\rangle\langle y, y'\rangle + \langle xy, x'y\rangle + \langle xy', x'y'\rangle.$$

Now since,

$$\langle x(y + y'), x'(y + y')\rangle = \langle xy, x'y\rangle + \langle xy, x'y'\rangle$$

$$+ \langle xy', x'y\rangle + \langle xy', x'y'\rangle,$$

then

$$\langle xy, x'y'\rangle + \langle xy', x'y\rangle = 2\langle x, x'\rangle \langle y, y'\rangle.$$

Definition: Let A and B be algebras over a field F. A one-to-one mapping ψ of A onto B is called an isomorphism of A onto B if the operations of addition and multiplication are preserved under the mapping. That is,

(1) $$\psi(\alpha a + \beta b) = \alpha\psi(a) + \beta\psi(b),$$

(2) $$\psi(ab) = \psi(a)\ \psi(b),$$ for all a, b in A

and α, β in F. A and B are said to be isomorphic if there exists an isomorphism of A onto B. By an automorphism of an algebra A we shall mean an isomorphism of A onto itself. If a mapping ψ is an isomorphism (or automorphism) except that

$$\psi(ab) = \psi(b)\ \psi(a),$$

we say that ψ is an anti-isomorphism (or anti-automorphism).

Definition: An algebra A is termed an alternative algebra if for every x, y in A, $x^2 y = x(xy)$ and $xy^2 = (xy)y$.

Theorem 1.7 [6, 10]: Let A be a real algebra with identity 1. If $N(xy) = N(x)\,N(y)$ for all x, y in A, then A is an alternative algebra with involution (anti-automorphism) $\psi : x \to \bar{x}$ such that:

$$x\bar{x} = N(x)\cdot 1$$

and

$$x + \bar{x} = T(x)\cdot 1, \qquad T(x) \text{ real.}$$

Proof: Let I denote the subspace of A spanned by the identity and let $I^{\perp}$ denote its orthogonal complement. Then A is a direct sum of I and $I^{\perp}$ and we write $A = I \oplus I^{\perp}$. That is, every x in A can be written as $\alpha\cdot 1 + a$ for some real α and a in $I^{\perp}$. For a proof of this the reader is referred to [9, p. 157].

Now for $x = \alpha\cdot 1 + a$ in A we define $\bar{x} = \alpha\cdot 1 - a$ and consider the mapping $\psi : x \to \bar{x}$ given by $\psi(x) = \bar{x}$. Clearly

$$\psi(\alpha x + \beta y) = \alpha\psi(x) + \beta\psi(y).$$

We now show that

$$\psi(xy) = \psi(y)\,\psi(x).$$

Consider (ii) of the previous lemma. If we put $x' = 1$ and take x in $I^{\perp}$ we have

$$\langle xy, y'\rangle + \langle xy', y\rangle = 2\langle x, 1\rangle\langle y, y'\rangle = 0$$

for all y, y' in A. Also, by the law of multiplication defined on A,

$$\langle(\alpha\cdot 1)y, y'\rangle - \langle y, (\alpha\cdot 1)y'\rangle = 0 \quad \text{for all real } \alpha.$$

Thus

$$\langle xy, y'\rangle + \langle(\alpha\cdot 1)y, y'\rangle + \langle xy', y\rangle - \langle(\alpha\cdot 1)y', y\rangle = 0$$

and

$$\langle(x + \alpha\cdot 1)y, y'\rangle = \langle(\alpha\cdot 1 - x)y', y\rangle$$

for all real α, x in $I^{\perp}$ and y, y' in A. Now if we put $w = \alpha\cdot 1 + x$, then $\langle wy, y'\rangle = \langle y, \bar{w}\, y'\rangle$, for all w, y, y' in A. Similarly, if we put $y' = 1$ and take y in $I^{\perp}$ in (ii) of our lemma, we find that $\langle xz, x'\rangle = \langle x, x'\bar{z}\rangle$ for all x, x', z in A. Combining these results we have that for all x, y, z in A,

$$\langle xy, z\rangle = \langle y, \bar{x}z\rangle = \langle y\bar{z}, \bar{x}\rangle = \langle \bar{z}, \bar{y}\bar{x}\rangle.$$

Now if we let $x = 1$, then $\langle y, z\rangle = \langle \bar{z}, \bar{y}\rangle$. Therefore

$$\langle xy, z\rangle = \langle \bar{z}, \overline{xy}\rangle.$$

Thus we have that

$$\langle \bar{z}, \bar{y}\bar{x}\rangle - \langle \bar{z}, \overline{xy}\rangle = 0$$

which implies that $\bar{y}\bar{x} = \overline{xy}$, and

$$\psi(xy) = \overline{xy} = \bar{y}\bar{x} = \psi(y)\,\psi(x).$$

Now suppose that $\psi(x) = \psi(y)$ for some x, y in A. Then since $\psi(\bar{x}) = x$ we have that

$$x = \psi(\psi(x)) = \psi(\psi(y)) = y.$$

ψ is an onto mapping and thus is an involution on A.

I contains all those elements of A left fixed by the involution while $I^{\perp}$ contains all x in A such that $\psi(x) = -x$. Now since $\psi(x\bar{x}) = x\bar{x}$, then $x\bar{x}$ is in I, so there exists a real number α such that $x\bar{x} = \alpha\cdot 1$. Now,

$$\alpha = \alpha\langle 1, 1\rangle = \langle \alpha\cdot 1, 1\rangle = \langle x\bar{x}, 1\rangle = \langle (1\cdot x), x\rangle = N(x).$$

Hence $x\bar{x} = N(x)\cdot 1$ for all x in A. Also,

$$\bar{x}x = N(\bar{x})\cdot 1 = N(x)\cdot 1.$$

Finally, since $x + \bar{x}$ is left fixed by ψ, there is a real number β such that $x + \bar{x} = \beta\cdot 1$. We write $x + \bar{x} = T(x)\cdot 1$, where $T(x)$ is a linear functional and is defined to be the <u>trace</u> of x.

To complete the proof, it remains to show that the alternative law is satisfied for all elements in A. Now from the first part of this proof we have that $\langle xy, xz\rangle = \langle y, \bar{x}(xz)\rangle$ for all x, y, z in A. But $\langle xy, xz\rangle = N(x)\langle y, z\rangle$ from (i) of lemma 1.6. Hence

$$\langle y, \bar{x}(xz)\rangle - \langle y, N(x)\cdot z\rangle = \langle y, \bar{x}(xz) - (x\bar{x})z\rangle = 0.$$

This implies that $\bar{x}(xz) = (x\bar{x})z$. Now since there exists some real number β such that $x + \bar{x} = \beta\cdot 1$, then

$$\bar{x}(xz) = (\beta\cdot 1 - x)(xz) = (\beta\cdot 1)(xz) - x(xz)$$

and

$$(x\bar{x})z = \left[x(\beta\cdot 1 - x)\right]z = (\beta\cdot 1)(xz) - x^2 z.$$

Hence

$$x(xz) = x^2 z \quad \text{for all } x, z \text{ in } A.$$

If we consider $\langle xz, yz\rangle$, we can similarly show that

$$xz^2 = (xz)z \quad \text{for all } x, z \text{ in } A.$$

Thus A is alternative and our proof is complete.

A direct consequence of this theorem is the following.

<u>Corollary</u>: Let A be a real algebra with identity 1 and let I denote the subspace spanned by the identity of A. If $N(xy) = N(x)\,N(y)$ for all x, y in A, then every element of A satisfies the quadratic equation $x^2 - T(x)\cdot x + N(x) = 0$ over I. Furthermore, the space I is the set of all elements left fixed by the involution $\psi(x) = \bar{x}$, while $I^{\perp}$ is the set of all a in A such that $\psi(a) = \bar{a} = -a$.

The proof of the converse to this theorem depends on the validity of the Moufang identity on an alternative algebra. That is,

$$(xy)(zx) = x\left[(yz)x\right] \quad \text{for all } x, y, z \text{ in } A.$$

In view of this, we present the following lemma which will be valuable in proving the converse. We begin by making the following definitions.

__Definition__: The __associator__ of an algebra A is a function S defined on A^3 to A by

$$S(x, y, z) = (xy)z - x(yz)$$

for all x, y, z in A.

__Definition__: Let A be an arbitrary algebra and let $f(x_1, x_2, \ldots, x_n)$ be a multilinear function defined on A^n to A. f is said to be __skew-symmetric__ provided:

1. f takes on the value 0 whenever at least two of its arguments are equal, and

2. f changes sign whenever two of its arguments are interchanged.

__Lemma 1.8 [4, 11]__: Let A be an alternative algebra over a field F. Define the function K from A^4 to A by

$$K(w, x, y, z) = S(wx, y, z) - xS(w, y, z) - S(x, y, z)\cdot w$$

for all w, x, y, z in A. Then S and K are linear skew symmetric functions.

__Proof__: The proof is contained in two parts:

I. That S is linear in x is readily verified by expanding $S(\alpha x_1 + \beta x_2, y, z)$ for any x_1, x_2, y, z in A and α, β in F. Similarly S is linear in y and z. It is also clear that $S(x, x, y) = 0 = S(x, y, y)$ when A is alternative.

Therefore,

$$S(x, y + z, y + z) = S(x, y, z) + S(x, z, y) = 0$$

and

$$S(y + z, y + z, x) = S(z, y, x) + S(y, z, x) = 0.$$

Thus

$$S(x, y, z) = -S(x, z, y) \quad \text{and} \quad S(z, y, x) = -S(y, z, x).$$

Finally,

$$S(x, y, z) = -S(x, z, y) = S(z, x, y) = -S(z, y, x).$$

Hence S is a linear skew-symmetric function from A^3 to A.

II. Now consider the function K. It is immediate that K is linear from the linearity of S. Also, from the definition of K, we note that $K(w, x, y, y) = 0$. Therefore

$$K(w, x, y, z) = -K(w, x, z, y),$$

since

$$K(w, x, y + z, y + z) = K(w, x, y, z) + K(w, x, z, y) = 0.$$

Now we define a function G on A^4 by

$$G(w, x, y, z) = S(wx, y, z) - S(w, xy, z) + S(w, x, yz)$$
$$- wS(x, y, z) - S(w, x, y)\cdot z.$$

By expanding all the associators we find that $G(w, x, y, z) = 0$. Therefore

$$-K(z, w, x, y) = G(w, x, y, z) - K(z, w, x, y).$$

Expanding G and K in terms of their associators and applying part I of this lemma we have

$$-K(z, w, x, y) = S(wx, y, z) - S(xy, z, w) + S(yz, w, x) - S(zw, x, y).$$

Using the fact that

$$S(wx, y, z) = K(w, x, y, z) + xS(w, y, z) + S(x, y, z).w$$

we find that a cyclic permutation of the elements z, w, x, y changes the sign on the right hand side of the expression for $-K(z, w, x, y)$. That is,

$$K(y, z, w, x) = -K(z, w, x, y).$$

Thus we have shown that for all w, x, y, z in A,

$$K(w, x, y, z) = -K(w, x, z, y)$$

and

$$K(w, x, y, z) = -K(z, w, x, y).$$

Since these two permutations of the elements w, x, y, z generate the entire symmetric group of permutations, we have proved the skew-symmetry of K.

We are now prepared to prove the converse to the last theorem.

Converse: [6, 10]: Let A be a real algebra with identity 1. If A is an alternative algebra with involution $\psi : x \to \bar{x}$, where $x\bar{x} = N(x)\cdot 1$ and $x + \bar{x} = T(x)\cdot 1$, $T(x)$ a real number, then $N(xy) = N(x)\, N(y)$ for all x, y in A.

Proof: We first prove the validity of the Moufang identity on A. We can easily verify that

$$(xy)(zx) = x[y(zx)] + S(x, y, zx).$$

From lemma 1.8,

$$S(x, y, zx) = -S(zx, y, x) = -S(x, y, x)\cdot z$$
$$- xS(z, y, x) - K(z, x, y, x).$$

Therefore

$$S(x, y, zx) = xS(y, z, x) = x[(yz)x] - x[y(zx)].$$

Hence

$$(xy)(zx) = x[(yz)x]$$

for all x, y, z in A. Now since $x + \bar{x} = T(x)\cdot 1$ for all x in A, then

$$x^2 y + (x\bar{x})y = [x(T(x)\cdot 1)]y$$

and

$$x(xy) + x(\bar{x}y) = x[(T(x)\cdot 1)y]$$

for all x, y in A. By the law of multiplication defined on A,

$$[x(T(x)\cdot 1)]\, y = T(x)\cdot(xy) = x[(T(x)\cdot 1)y].$$

Therefore, since A is alternative,

$$(x\bar{x})y = x(\bar{x}y).$$

Similarly,

$$x(y\bar{y}) = (xy)\bar{y}.$$

Now for every x, y in A,

$$N(xy)\cdot 1 = (xy)(\bar{y}\bar{x}) = (xy)[\bar{y}(T(x)\cdot 1 - x)]$$

$$= T(x)\cdot(xy)\bar{y} - (xy)(\bar{y}x).$$

By the Moufang identity

$$N(xy)\cdot 1 = T(x)\,[x(y\bar{y})] - x[(y\bar{y})x]$$

$$= T(x)\; N(y)\cdot x - N(y)\cdot x^2.$$

Hence

$$N(xy)\cdot 1 = N(y)\cdot x\,[T(x)\cdot 1 - x] = N(y)\cdot(x\bar{x}) = N(y)\; N(x)\cdot 1.$$

CHAPTER II

COMMUTATIVE AND ASSOCIATIVE ALGEBRAS

In chapter I we assumed neither commutativity nor associativity of multiplication. This chapter will be devoted to further characterizations and uniqueness of real linear algebras having these properties.

Definition: A skew field is a ring in which the nonzero elements form a group under multiplication. A commutative skew field is called a field.

Theorem 2.1 [13, pp. 18-21]: Let A be a real division algebra. If multiplication on A is associative, then A is a skew field. (If, in addition, A is commutative with respect to multiplication, then A is a field.)

Proof: First, if A is associative with respect to multiplication, then A is a ring. Let a,b be nonzero elements of A. Then since A is a division algebra, there is an element x in A such that $ax = b$. Similarly, there is an element y in A such that $by = x$. Hence $a(by) = ax = b$ and since $b \neq 0$ and $a(by) = (ab)y$ then it follows that $ab \neq 0$. Thus A has no nonzero divisors of zero. We can now show that A has an identity.

Let a be any nonzero element in A. Then there exists an element ϵ in A such that $a\epsilon = a$. Then $\epsilon \neq 0$. Now $a\epsilon^2 = a\epsilon$, which implies that $\epsilon^2 = \epsilon$ since a is not a divisor of zero. Let x be any element in A. Then

$$(x - x\epsilon)\epsilon = 0 \quad \text{and} \quad \epsilon(x - \epsilon x) = 0.$$

Hence $x\epsilon = \epsilon x = x$, so ϵ is the identity element of A. As before, we denote this element by 1.

We next show that every nonzero element of A has a multiplicative inverse. Let a be any nonzero element of A. Then there exists an x in A such that $ax = 1$. Then $x \neq 0$. Furthermore,

$$(xa - 1)x = x(ax) - x = 0.$$

Therefore, $xa - 1 = 0$ or $xa = 1$ and hence, x is the inverse a^{-1} of a. We have shown that the nonzero elements of A form a multiplicative group. Hence A is a skew field. Furthermore, if A is commutative with respect to multiplication, then A is a commutative skew field or simply a field.

We shall now prove that except for isomorphisms, the real and complex numbers form the only commutative division algebras over the real numbers. As before, we denote the space spanned by the identity of A by I. Since A is real, I is clearly isomorphic to the field of real numbers. We begin with the following definition.

<u>Definition</u>: Let A be a division algebra with identity 1 over a field F. A is said to be <u>algebraic</u> over a field K if:

(1) K is contained in the center of A, and

(2) Every element a in A satisfies a nontrivial polynomial with coefficients in K.

In the theorems which follow we shall denote the center of A by C(A).

<u>Lemma 2.2 [7, p. 10]</u>: If A is a real associative division algebra, then A is algebraic over I. Furthermore, each element of A satisfies a nontrivial linear or quadratic equation over I.

Proof: Since A is an associative division algebra, A has an identity and since A is real, I is isomorphic to the field of real numbers. Also, if α is any real number, then by the rule of multiplication defined on A, $(\alpha\cdot 1)a = a(\alpha\cdot 1)$ for all a in A. Thus I is contained in C(A).

Now, since A is associative, we can express the product of k factors a by a^k. If A is of order n, the set of n + 1 elements $1, a, a^2, \ldots, a^n$ are linearly dependent with respect to R. Hence there exist real numbers $\alpha_0, \alpha_1, \ldots, \alpha_n$, not all zero, such that

$$\alpha_0\cdot 1 + \alpha_1 a + \alpha_2 a^2 + \ldots + \alpha_n a^n = 0.$$

Therefore a is a root of an equation of degree $\leq n$ with coefficients in I. Let

$$p(x) = \alpha_0\cdot 1 + \alpha_1 x + \ldots + \alpha_n x^n.$$

Since I is isomorphic to R, we have by the fundamental theorem of algebra that

$$p(x) = f_1(x)\cdot f_2(x) \ldots f_k(x),$$

$k \leq n$ and $f_i(x)$ is of degree 1 or 2. Now, since $p(a) = 0$, then some $f_i(a) = 0$ and thus a is a root of a linear or quadratic equation over I.

Lemma 2.3 [9, pp. 326-327]: Let A be an associative division algebra over the field C of complex numbers. If A is algebraic over $C^* = C\cdot 1$, then $A = C^*$.

Proof: Since A is algebraic over C^*, if a is any element of A, there exist complex numbers $c_0, c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_0\cdot 1 + c_1 a + \ldots + c_n a^n = 0.$$

Again making use of the fundamental theorem of algebra, the polynomial

$$p(x) = c_o \cdot 1 + c_1 x + \ldots + c_n x^n$$

can be factored into a product of linear factors. That is

$$p(x) = (x - \lambda_1 \cdot 1)(x - \lambda_2 \cdot 1)(x - \lambda_3 \cdot 1) \ldots (x - \lambda_n \cdot 1)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are in C. Now since $p(a) = 0$, some

$$a - \lambda_i \cdot 1 = 0.$$

Hence a is in C* and we have shown that $A \subseteq C^*$. Since A is algebraic over C^*, $C^* \subseteq A$. Therefore $A = C^*$.

<u>Theorem 2.4 [9, p. 327]</u>: Let A be a real associative division algebra. If A is commutative, then A is isomorphic to either the field of real numbers or the field of complex numbers.

<u>Proof</u>: By lemma 2.2, A is algebraic over I and hence I, which is isomorphic to R, is contained in C(A). Now suppose $I \neq A$. Then there exists an a in A which is not in I. Therefore, a satisfies some quadratic equation with real coefficients. Otherwise, a would be in I. Let

$$p(x) = x^2 + 2\alpha x + \alpha_o \cdot 1$$

such that $p(a) = 0$ and where α, α_o are real. Then

$$(a + \alpha \cdot 1)^2 = \alpha^2 \cdot 1 - \alpha_o \cdot 1.$$

We note here that for any x in A and γ' in R, if $x^2 = \gamma' \cdot 1$ and $\gamma' > 1$, then there is a real number γ such that

$$x^2 = \gamma^2 \cdot 1.$$

Then

$$x^2 - \gamma^2 \cdot 1 = (x + \gamma \cdot 1)(x - \gamma \cdot 1) = 0,$$

which implies that $x = \pm \gamma \cdot 1$. Hence we have that $\alpha^2 - \alpha_0 < 0$, for if this were positive, there would exist a γ in R such that

$$a + \alpha \cdot 1 = \pm \gamma \cdot 1.$$

But this implies that a is in I. Hence, there is a real number β such that $\alpha^2 - \alpha_0 = -\beta^2$. Therefore

$$(a + \alpha \cdot 1)^2 = -\beta^2 \cdot 1.$$

Thus, if a is in A but not in I, we can find real numbers α, β such that

$$\left(\frac{a + \alpha \cdot 1}{\beta}\right)^2 = -1.$$

We put

$$i = \left(\frac{a + \alpha \cdot 1}{\beta}\right)$$

so that $i^2 = -1$, and hence A contains $I + I \cdot i$ which is isomorphic to the field of complex numbers. We denote this field by C^*. It remains only to show that $A = C^*$.

Now since A is algebraic over I, then A is algebraic over C^*. For if a in A satisfies a polynomial with coefficients in I, then a clearly satisfies a polynomial with coefficients in C^*. Also, $C^* \subseteq C(A)$ since A is commutative. Hence, by lemma 2.3, $C^* = A$ and our proof is complete.

We now drop the property of commutativity on A and continue our characterization of real division algebras which are associative.

<u>Theorem 2.5 [2, pp. 240-241]</u>: Let A be an associative division algebra. For some a **contained in A,** let R_a and L_a be the linear transformations on A such that $xR_a = xa$ and $xL_a = ax$ for all x in A. Then A is isomorphic to

$$A_R = \{R_x \mid x \text{ in } A\}$$

and anti-isomorphic to

$$A_L = \{L_x \mid x \text{ in } A\}.$$

<u>Proof</u>: We define $\psi(x) = R_x$ for all x in A and we shall show ψ defines an isomorphism of A onto A_R. First consider

$$\psi(\alpha x + \beta y) = R_{\alpha x + \beta y}$$

for x, y in A and α, β in R. Note that for any a in A

$$aR_{\alpha x + \beta y} = a(\alpha x + \beta y) = \alpha\big(aR_x\big) + \beta\big(aR_y\big).$$

Hence

$$R_{\alpha x + \beta y} = \alpha R_x + \beta R_y,$$

so

$$\psi(\alpha x + \beta y) = \alpha\psi(x) + \beta\psi(y).$$

Now consider $\psi(xy) = R_{xy}$. For a in A,

$$aR_{xy} = a(xy) = (ax)y = \big(aR_x\big)R_y = a\big(R_x \cdot R_y\big).$$

Hence

$$R_{xy} = R_x R_y,$$

so

$$\psi(xy) = \psi(x)\,\psi(y).$$

Finally, suppose $\psi(x) = \psi(y)$ and let a be any nonzero element in A. Then $aR_x = aR_y$ so $a(x - y) = 0$ and since A is an associative division algebra, $x = y$. Hence, since ψ is an onto mapping, A is isomorphic to A_R.

Now consider the mapping $\psi'(x) = L_x$. Note that for a in A,

$$aL_{xy} = (xy)a = x(ya) = \big(aL_y\big)L_x = a\big(L_y \cdot L_x\big).$$

Hence

$$\psi'(xy) = \psi'(y)\ \psi'(x),$$

so ψ' defines an anti-isomorphism from A onto A_L.

Thus if A is an associative division algebra with basis $e_1, e_2, \ldots, e_n$ and if

$$x = x_1\ e_1 + x_2\ e_2 + \ldots + x_n\ e_n$$

is any element in A, then

$$x \leftrightarrow x_1\ R_{e_1} + x_2\ R_{e_2} + \ldots + x_n\ R_{e_n}$$

under the mapping ψ and

$$x \leftrightarrow x_1\ L_{e_1} + x_2\ L_{e_2} + \ldots + x_n\ L_{e_n}$$

under the mapping ψ'. Hence, $R_{e_1}, R_{e_2}, \ldots, R_{e_n}$ form a basis for A_R and $L_{e_1}, L_{e_2}, \ldots, L_{e_n}$ form a basis for A_L.

Theorem 2.6 [2, pp. 202-213]: Let A be a real associative division algebra. Then the algebras A_R and A_L of linear transformations on A are isomorphic to algebras of real $n \times n$ matrices.

Proof: Let $x = x_1\ e_1 + x_2\ e_2 + \ldots + x_n\ e_n$, where x_i is real and $e_1, e_2, \ldots, e_n$ form a basis for A. Now $xR_a = xa$ is in A so we put

$$xR_a = xa = y = y_1\ e_1 + y_2\ e_2 + \ldots + y_n\ e_n$$

$$y = \left(\sum_i x_i\ e_i\right) R_a = \sum_i x_i\ (e_i\ R_a)$$

and

$$e_i\ R_a = \sum_j \alpha_{ij}\ e_j.$$

Now

$$y = xR_a = \sum_i x_i \sum_j \alpha_{ij} \ e_j = \sum_j \left(\sum_i \alpha_{ij} \ x_i \right) e_j .$$

Hence

$$y_j = \sum_i \alpha_{ij} \ x_i$$

and we denote the matrix (α_{ij}) by $m(R_a)$. Thus the linear transformation R_a which sends the vector x having components $(x_1, x_2, \ldots, x_n)$ into the vector y having components $(y_1, y_2, \ldots, y_n)$ can be represented by the real $n \times n$ matrix (α_{ij}) where $e_i \ R_a = \sum_j \alpha_{ij} \ e_j$.

We now show that the mapping $\theta(R_x) = m(R_x)$ for all x in A, defines an isomorphism of A_R onto $M(A_R) = \{ m(R_x) \mid R_x \text{ in } A_R \}$.

First, we note that for real α, β and R_x, R_y in A_R,

$$\theta(\alpha R_x + \beta R_y) = m(\alpha R_x + \beta R_y) = \alpha \ m(R_x) + \beta \ m(R_y)$$

since

$$e_i(\alpha R_x + \beta R_y) = \alpha(e_i \ R_x) + \beta(e_i \ R_y).$$

Now suppose

$$e_i \ R_x = \sum_j \alpha_{ij} \ e_j \quad \text{and} \quad e_j \ R_y = \sum_k \beta_{jk} \ e_k .$$

Then

$$e_i(R_x \ R_y) = \sum_j \alpha_{ij}(e_j \ R_y) = \sum_j \alpha_{ij} \sum_k \beta_{jk} \ e_k .$$

Therefore

$$e_i(R_x \ R_y) = \sum_{j,k} \alpha_{ij} \ \beta_{jk} \ e_k ,$$

which implies

$$m(R_x\, R_y) = (\alpha_{ij})(\beta_{jk}) = m(R_x) \cdot m(R_y).$$

That is

$$m(R_x\, R_y) = (\tau_{ik}) = \sum_j \alpha_{ij}\, \beta_{jk}.$$

Hence

$$\theta(R_x\, R_y) = m(R_x)\, m(R_y).$$

Now suppose $\theta(R_x) = \theta(R_y)$. Then $m(R_x - R_y) = (0)$. If

$$m(R_x - R_y) = (\alpha_{ij}),$$

then each $\alpha_{ij} = 0$ so

$$e_i(R_x - R_y) = e_i\ (x - y) = 0.$$

This implies that $x = y$ so that $R_x = R_y$. Hence the mapping θ is an isomorphism of A_R onto $M(A_R)$.

Similarly, if we define

$$M(A_L) = \{m(L_x) \mid L_x \text{ in } A_L\}$$

we can show that the mapping θ' defined by

$$\theta'(L_x) = m(L_x)$$

for all x in A, is an isomorphism of A_L onto $M(A_L)$.

Corollary: Let A be a real associative division algebra. Then A is isomorphic to the algebra $M(A_R)$ and anti-isomorphic to the algebra $M(A_L)$.

Definition: The isomorphism $A \cong M(A_R)$ is known as the first regular representation of A and the anti-isomorphism $A \eqsim M(A_L)$ is called the second regular representation of A.

Hence, given an associative division algebra A with basis $e_1, e_2, \ldots, e_n$, we have for any

$$x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

in A the following correspondence:

$$x \leftrightarrow x_1 R_{e_1} + x_2 R_{e_2} + \dots + x_n R_{e_n}$$

and

$$x \leftrightarrow x_1 m(R_{e_1}) + x_2 m(R_{e_2}) + \dots + x_n m(R_{e_n}).$$

Similarly

$$x \leftrightarrow L_x \leftrightarrow m(L_x).$$

<u>Example</u>: <u>Regular representation of the algebra of complex numbers</u>: Let C denote the algebra of complex numbers. Then $1, i$ is a basis for C, so we have for any $\alpha + \beta i$ in C,

$$\alpha + \beta i \leftrightarrow \alpha R_1 + \beta R_i \leftrightarrow \alpha\, m(R_1) + \beta\, m(R_i),$$

where $m(R_1)$ is given by

$$1R_1 = 1 + 0i$$
$$iR_1 = 0 + i$$

and $m(R_i)$ is given by

$$1R_i = 0 + i$$
$$iR_i = 1 + 0i.$$

Hence

$$\alpha + \beta i \leftrightarrow \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Since C is a commutative algebra, the first and second regular representations of C are identical.

We shall now construct the algebra of real quaternions by imitating the construction of the complex numbers. Again, let C denote the algebra of complex numbers.

Consider the set Q of all ordered pairs (a, b) where a, b are complex numbers. Q is a vector space over the field of complex numbers. Each (a, b) in Q can be expressed as

$$(a, b) = a(1, 0) + b(0, 1).$$

We define multiplication on Q as follows:

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}),$$

where the bar indicates the complex conjugate. The multiplicative identity is clearly $1 = (1, 0)$. If we put $j = (0, 1)$ we find that $j^2 = -1$. Now, every (a, b) in Q is uniquely expressible in the form

$$(a, b) = a\cdot 1 + bj,$$

and the rule of multiplication on Q can be written as

$$(a\cdot 1 + bj)(c\cdot 1 + dj) = (ac - \bar{d}b)\cdot 1 + (da + b\bar{c})j.$$

Now let $a = \alpha_0 + \alpha_1\sqrt{-1}$ and $b = \alpha_2 + \alpha_3\sqrt{-1}$, where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are real. Then

$$(a, b) = a\cdot 1 + bj = \alpha_0\cdot 1 + \alpha_1\sqrt{-1}\cdot 1 + \alpha_2\, j + \alpha_3\sqrt{-1}\, j.$$

Let $(\sqrt{-1}, 0) = i$ and $(0, \sqrt{-1}) = k$. Thus, each element (a, b) in Q is uniquely represented in the form

$$(a, b) = \alpha_0\cdot 1 + \alpha_1\, i + \alpha_2\, j + \alpha_3\, k.$$

By our rule of multiplication we compute the following table which also defines multiplication on Q:

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k,$$

$$jk = -ki = i,$$

$$ki = -ik = j.$$

Under this rule of multiplication,

$$Q = \{ \alpha_0 \cdot 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_0, \alpha_1, \alpha_2, \alpha_3 \text{ real} \}$$

is a real associative algebra with identity. The elements of Q are known as <u>quaternions</u> and Q is called the <u>algebra of real quaternions</u>.

<u>Theorem 2.7</u>: The algebra of real quaternions is an absolute valued algebra.

<u>Proof</u>: We apply the converse to theorem 1.7.

First, for every q in Q we define the quaternion conjugate of q by

$$\bar{q} = \alpha_0 \cdot 1 - (\alpha_1 i + \alpha_2 j + \alpha_3 k).$$

By straight forward multiplication we can easily show that the mapping $\Psi(q) = \bar{q}$ defines an involution on Q. Similarly, we can show that $q\,\bar{q} = N(q) \cdot 1$ for all q in Q. Finally, we define $T(q) \cdot 1 = q + \bar{q}$ for all q. $T(q)$ is clearly real. Since Q is associative, we have that $N(pq) = N(p)\,N(q)$ and our proof is complete.

We conclude this chapter with the following proof of the uniqueness of the algebra of real quaternions.

<u>Theorem 2.8 [9, pp. 327-329; 6, pp. 10-12]</u>: Let A be a real associative division algebra. If A is not commutative, then A is isomorphic to the algebra of quaternions.

<u>Proof</u>: We first show that $I = C(A)$. By lemma 2.2, $I \subseteq C(A)$. Now suppose there exists an element a in $C(A)$ such that a is not in I. Then, as we have previously shown, there would exist real numbers α, β such that $\left(\frac{a + \alpha \cdot 1}{\beta}\right)^2 = -1$. Thus, $C(A)$ would contain a field C^* isomorphic to the field of complex numbers. Hence A would

be algebraic over C^* and so by lemma 2.3, $A = C^*$. This contradicts our assumption that A is not commutative. Therefore $I = C(A)$.

Now let a be any element of A such that a is not in I and take $i = \left(\frac{a + \alpha \cdot 1}{\beta}\right)$ such that $i^2 = -1$. Then i is not in I so there exists an element b in A such that

$$c = bi - ib \neq 0.$$

Note that

$$ic + ci = i(bi - ib) + (bi - ib)i = ibi - i^2 b + bi^2 - ibi = 0,$$

so $ic = -ci$.

Furthermore,

$$ic^2 = (ic)c = -(ci)c = c(ci) = c^2 i,$$

so c^2 commutes with i.

Now c satisfies some quadratic equation over I. Let

$$c^2 + \gamma c + \delta \cdot 1 = 0, \qquad \gamma, \delta \text{ real}.$$

Since

$$\gamma c = -c^2 - \delta \cdot 1,$$

then γc commutes with i. Hence

$$\gamma ci = i\gamma c = \gamma ic = -\gamma ci,$$

so

$$2\gamma ci = 0.$$

Since $2ci \neq 0$ and A is a division algebra, $\gamma = 0$. Therefore $c^2 = -\delta \cdot 1$. Also c can not be in I since $ic = -ci$. Hence $\delta > 0$ so we let $\delta = \xi^2$, ξ real.

Now let $j = \frac{c}{\xi}$. Then $j^2 = -1$. Also

$$ji + ij = \frac{ci + ic}{\xi} = 0.$$

Therefore $ij = -ji$. Let $k = ij$.

Hence A contains the algebra $C^* + C^*\cdot j$ which is isomorphic to the algebra of real quaternions. We denote this algebra by Q^*.

We finally show that $Q^* = A$. Suppose $Q^* \subsetneq A$. Then for some x in A but not in Q^*, we can determine an element l in A which is not in Q^* and such that $l^2 = -1$. Now $i \pm l$ are roots of a quadratic equations over I. For real $\alpha_1, \alpha_2, \beta_1, \beta_2$, we let

$$(i + l)^2 + \alpha_1(i + l) + \alpha_2\cdot 1 = 0$$

and

$$(i - l)^2 + \beta_1(i - l) + \beta_2\cdot 1 = 0.$$

Hence

$$(i + l)^2 = -2\cdot 1 + il + li = -\alpha_1(i + l) - \alpha_2\cdot 1$$

and

$$(i - l)^2 = -2\cdot 1 - il - li = -\beta_1(i - l) - \beta_2\cdot 1.$$

Adding, we get

$$(\alpha_1 + \beta_1)i + (\alpha_1 - \beta_1)l + (\alpha_2 + \beta_2 - 4)\cdot 1 = 0.$$

Since $1, i, l$ are linearly independent,

$$\alpha_1 = \beta_1 = 0.$$

Hence

$$il + li = \alpha\cdot 1, \qquad \alpha \text{ real.}$$

Similarly,

$$jl + lj = \beta\cdot 1$$

and

$$kl + lk = \gamma\cdot 1, \qquad \beta, \gamma \text{ real.}$$

Thus

$$lk = (li)j = (\alpha\cdot 1 - il)j = \alpha j - i(\beta\cdot 1 - jl) = \alpha j - \beta i + kl.$$

Then

$$2kl = \gamma\cdot 1 + \beta i - \alpha j.$$

Multiplying by k we get

$$-2l = \gamma k + \beta j + \alpha i.$$

This implies that l is in Q^* contradicting our assumption that $Q^* \subsetneq A$. Hence $A = Q^*$ completing the proof.

Combining some of our previous results we have the following:

Corollary: Let A be a real associative absolute valued algebra. Then A is isomorphic to the real numbers, the complex numbers or the real quaternions.

CHAPTER III

THE ALGEBRA OF REAL QUATERNIONS

Let Q denote the algebra of real quaternions.

Theorem 3.1 [12, pp. 257-259]: Let p be a fixed nonzero quaternion. Then $\theta(q) = pqp^{-1}$ is an automorphism on Q. Furthermore, every automorphism on Q is of this type.

Proof: Since Q is an associative division algebra, every nonzero element of Q has a unique inverse. Now consider

$$\theta(q) = pqp^{-1},$$

for all q in Q and some fixed nonzero element p. If q_1, q_2 are arbitrary elements in Q, and α, β are real, then

$$\theta(\alpha q_1 + \beta q_2) = p(\alpha q_1 + \beta q_2)p^{-1} = \alpha p q_1 p^{-1} + \beta p q_2 p^{-1}$$

Hence

$$\theta(\alpha q_1 + \beta q_2) = \alpha\theta(q_1) + \beta\theta(q_2).$$

Also,

$$\theta(q_1 q_2) = pq_1\,(p^{-1}p)q_2\,p^{-1} = (pq_1p^{-1})(pq_2p^{-1})$$

so

$$\theta(q_1 q_2) = \theta(q_1)\cdot\theta(q_2).$$

Finally, if $\theta(q_1) = \theta(q_2)$, then $q_1 = q_2$ since Q has no non-zero divisors of zero. Hence θ is an inner-automorphism on Q.

Now suppose θ' is an automorphism of Q and let

$$\theta'(q) = q'.$$

Suppose that under the mapping θ'

$$1 \longleftrightarrow 1, \; i \longleftrightarrow e_1, \; j \longleftrightarrow e_2, \; k \longleftrightarrow e_3.$$

Then $1, e_1, e_2, e_3$ obey the same rule of multiplication as defined for $1, i, j, k$. Now there exist elements p_1, p_2, p_3 in Q such that

$$p_1 = e_3 j - e_2 k + e_1 + i$$

$$p_2 = e_1 k - e_3 i + e_2 + j$$

$$p_3 = e_2 i - e_1 j + e_3 + k.$$

We will show that for every q in Q

$$q'p_1 = p_1 q, \quad q'p_2 = p_2 q, \quad \text{and} \quad q'p_3 = p_3 q,$$

where q' is the image of q under the mapping θ'.

From our rule of multiplication on Q we have

$$e_1 p_1 = - e_2 j - e_3 k - 1 + e_1 i,$$

and

$$p_1 i = - e_3 k - e_2 j + e_1 i - 1,$$

so that

$$e_1 p_1 = p_1 i.$$

Similarly

$$e_2p_1 = p_1j \quad \text{and} \quad e_3p_1 = p_1k.$$

Hence for every q in Q,

$$q'p_1 = p_1q.$$

Similarly

$$q'p_2 = p_2q \quad \text{and} \quad q'p_3 = p_3q.$$

If one of the elements p_1, p_2, p_3 is not zero, the theorem is complete.

Now, suppose that

$$p_1 = p_2 = 0.$$

Then

$$e_1 + i = e_2k - e_3j.$$

Also

$$p_2 = e_1k - e_3i + e_3e_1 - ik = e_3(e_1 - i) + (e_1 - i)k = 0.$$

Since,

$$k^{-1} = \bar{k}/N(k) = -k,$$

$$(e_1 - i) = e_3(e_1 - i)k = e_2k + e_3j.$$

From above, we have

$$e_1 + i = e_2k - e_3j.$$

Hence

$$i = -e_3j,$$

which implies that $e_3 = k$. Similarly, if

$$p_2 = p_3 = 0, \quad i = e_1$$

and if

$$p_3 = p_1 = 0, \quad j = e_2.$$

Thus, if $p_1 = p_2 = p_3 = 0$, then θ' must be the identity mapping, in which case we take $p = 1$. This completes the proof of our theorem.

Theorem 3.2: The collection of all automorphisms on Q form a multiplicative group of linear orthogonal transformations on Q.

Proof: Let G be the collection of all automorphisms on Q. By the previous theorem, the elements of G are linear transformations of the form T_p where p is a fixed nonzero quaternion and

$$qT_p = pqp^{-1} \qquad \text{for all } q \text{ in } Q.$$

First we note that G is closed under multiplication. For suppose p_1, p_2 are fixed nonzero elements of Q. Then for any q in Q

$$q\left(T_{p_1} \cdot T_{p_2}\right) = \left(qT_{p_1}\right)T_{p_2} = \left(p_2\, p_1\right) q \left(p_1^{-1}\, p_2^{-1}\right).$$

But

$$p_1^{-1}\, p_2^{-1} = \bar{p}_1\, \bar{p}_2 \Big/ N(p_1\, p_2) = \overline{p_2\, p_1} \Big/ N(p_1\, p_2) = (p_2\, p_1)^{-1}.$$

Hence

$$T_{p_1}\, T_{p_2} = T_{p_2\, p_1}, \qquad \text{so } G \text{ is closed.}$$

Similarly, we can show

$$qT_{p_1}\left[\left(T_{p_2} \cdot T_{p_3}\right)\right] = q\left[\left(T_{p_1} \cdot T_{p_2}\right)T_{p_3}\right]$$

for fixed nonzero elements p_1, p_2, p_3 and q in Q. Thus G is associative.

Now, T_1 is clearly in G. If T_p is any element of G and q is an arbitrary element of Q, we have

$$q(T_1 \cdot T_p) = qT_p.$$

Hence T_1 is the identity in G. Finally, since each element of G is a nonsingular linear transformation on Q, T_p^{-1} exists for each T_p in G. Hence G is a multiplicative group of linear transformations on Q.

Note that for all q in Q and each T_p in G,

$$\langle qT_p, qT_p\rangle = N(pqp^{-1}) = N(q) = \langle q, q\rangle.$$

Thus G is a group of linear orthogonal transformations on Q.

As in theorem 1.7, we note that

$$Q = I \oplus I^{\perp},$$

where I is isomorphic to the field of real numbers. Furthermore, from our construction of Q, we have that $I^{\perp}$ is isomorphic to the real Euclidean vector space of dimension three. Denote $I^{\perp}$ by E_3. Then every element in Q is of the form

$$q = r + v,$$

where r is in I and v in E_3.

Theorem 3.3: Let G be the group of all automorphisms on Q. Then

(1) The elements of I are invarient under the transformations of G, and

(2) G defines the group of all rotations on E_3.

Proof: That the elements of G leave the elements of I fixed is clear since each r in I is of the form $\alpha \cdot 1$, where α is real. Thus for each r in I, and T_p in G,

$$rT_p = p(\alpha \cdot 1)p^{-1} = r.$$

Hence for any $q = r + v$ in Q, and T_p in G,

$$qT_p = r + pvp^{-1} .$$

Now consider the effect of an element in G on an element of E_3. Let

$$p = \alpha_0 \cdot 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k$$

be any fixed nonzero element of Q, and let

$$v = xi + yj + zk$$

be an arbitrary element of E_3. Then

$$vT_p = p\ v\ p^{-1} = \frac{1}{N(p)} \left[p\ v\ \bar{p} \right].$$

Expanding this expression we find that vT_p is in E_3. Since T_p is an orthogonal transformation, T_p defines either a rotation on E_3 or a rotation followed by a reflection on E_3.

Let $v' = x'i + y'j + z'k$ denote the vector in E_3 such that

$$vT_p = v'.$$

From the expansion of vT_p we have the following matrix representation of this transformation:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \frac{1}{N(p)} \begin{bmatrix} (\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2) & 2(\alpha_1 \alpha_2 - \alpha_0 \alpha_3) & 2(\alpha_1 \alpha_3 + \alpha_0 \alpha_2) \\ 2(\alpha_2 \alpha_1 + \alpha_3 \alpha_0) & (\alpha_0^2 + \alpha_2^2 - \alpha_3^2 - \alpha_1^2) & 2(\alpha_2 \alpha_3 - \alpha_0 \alpha_1) \\ 2(\alpha_3 \alpha_1 - \alpha_2 \alpha_0) & 2(\alpha_2 \alpha_3 + \alpha_0 \alpha_1) & (\alpha_0^2 + \alpha_3^2 - \alpha_1^2 - \alpha_2^2) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Let $m(T_p)$ denote the matrix of the transformation T_p and let

$$A(p) = N(p) \cdot m(T_p).$$

Since

$$\det\left[m(T_p)\right] = \pm 1,$$

then

$$\det A(p) = \pm N(p)^3.$$

Since $\det A(p)$ is a polynomial in $\alpha_o, \alpha_1, \alpha_2, \alpha_3$, it is a continuous function from

$$\left\{R^4 - (0, 0, 0, 0)\right\}$$

to R. Then $\det A(p)$ is either always positive or always negative. For suppose there exist fixed nonzero quaternions p_1 and p_2 such that

$$\det A(p_1) > 0 \quad \text{and} \quad \det A(p_2) < 0.$$

Now, since

$$\left\{R^4 - (0, 0, 0, 0)\right\}$$

is connected, we have by the intermediate value theorem [15, p.322] that there exists a p_3 in Q such that

$$\det A(p_3) = \pm N(p_3)^3 = 0.$$

But this implies that $p_3 = 0$ which is impossible since $p_3 = 0$ is not in the domain of $\det A(p)$. Hence,

$$\det\left[m(T_p)\right] = 1 \quad \text{for all} \quad T_p \quad \text{in} \quad G$$

or

$$\det\left[m(T_p)\right] = -1 \quad \text{for all } T_p \text{ in } G.$$

Now consider $T_{p'}$ in G defined by

$$p' = \alpha \cdot 1.$$

Then clearly $\det A(p) = \alpha^6 = +N(p')^3$, and so is positive for all transformations in G. Therefore

$$\det\left[m(T_p)\right] = +1 \quad \text{for all } T_p \text{ in } G.$$

Hence G is a group of rotations on E_3.

We shall now show that G is the group of all rotations on E_3. Let R denote any rotation on E_3. From analytic geometry we know that R can be defined by the direction cosines of the axis of rotation together with the angle of rotation about that axis. Let ξ, η, ζ denote the direction cosines of the axis of rotation with the x, y, z axis, respectively. Also let ω denote the angle of rotation. Whittaker [17, p.7] has shown that R has the following matrix representation:

$$\begin{bmatrix} 1 - 2(1 - \xi^2)\sin^2\frac{\omega}{2} & 2\sin\frac{\omega}{2}\left(\xi\eta\sin\frac{\omega}{2} + \zeta\cos\frac{\omega}{2}\right) & 2\sin\frac{\omega}{2}\left(\xi\zeta\sin\frac{\omega}{2} - \eta\cos\frac{\omega}{2}\right) \\ 2\sin\frac{\omega}{2}\left(\xi\eta\sin\frac{\omega}{2} - \zeta\cos\frac{\omega}{2}\right) & 1 - 2\left(1 - \eta^2\right)\sin^2\frac{\omega}{2} & 2\sin\frac{\omega}{2}\left(\eta\zeta\sin\frac{\omega}{2} + \xi\cos\frac{\omega}{2}\right) \\ 2\sin\frac{\omega}{2}\left(\xi\zeta\sin\frac{\omega}{2} + \eta\cos\frac{\omega}{2}\right) & 2\sin\frac{\omega}{2}\left(\eta\zeta\sin\frac{\omega}{2} - \xi\cos\frac{\omega}{2}\right) & 1 - \left(1 - \zeta^2\right)\sin^2\frac{\omega}{2} \end{bmatrix}.$$

But this is precisely the matrix we obtain when we make the substitution:

$$\alpha_o = \cos\frac{\omega}{2}$$

$$\alpha_1 = -\xi\sin\frac{\omega}{2}$$

$$\alpha_2 = -\eta\sin\frac{\omega}{2}$$

$$\alpha_3 = -\zeta\sin\frac{\omega}{2}$$

in the expansion of $vT_p = pvp^{-1}$ given earlier. Now $\alpha_o, \alpha_1, \alpha_2, \alpha_3$ can not all be zero since

$$\xi^2 + \eta^2 + \zeta^2 = 1.$$

Thus, we have found a fixed nonzero quaternion, namely

$$p = \cos \frac{\omega}{2} \cdot 1 - \xi \sin \frac{\omega}{2} i - \eta \sin \frac{\omega}{2} j - \zeta \sin \frac{\omega}{2} k,$$

such that

$$vT_p = vR.$$

Hence R is in G and the proof of the theorem is complete.

Corollary: The most general rotation of a vector v in E_3 can be defined by

$$vT_p = pvp^{-1},$$

where

$$p = \cos \frac{\omega}{2} \cdot 1 - \sin \frac{\omega}{2}(\xi i + \eta j + \zeta k).$$

ξ, η, ζ are the direction cosines of the axis of rotation with the x, y, z axis respectively, and ω is the angle of rotation about the axis.

We shall now list some of the properties and characterizations of Q which follow from the theorems of this paper.

1. Multiplication on Q in Gibbs notation [3, pp. 403-428]: Let $[v_1, v_2]$ denote the vector cross product of elements in E_3. Then by the rule of multiplication defined on Q, we can readily establish that

$$\text{(i)} \quad v_1 v_2 = -\langle v_1, v_2 \rangle \cdot 1 + [v_1, v_2].$$

Hence for all $q_1 = r_1 + v_1$ and $q_2 = r_2 + v_2$ we have

(ii) $$q_1 q_2 = \left(r_1 r_2 - \langle v_1, v_2\rangle \cdot 1\right) + \left(r_1 v_2 + r_2 v_1 + [v_1, v_2]\right).$$

The relationships (i) and (ii) yield the following interesting identities:

(iii) $$v_1 v_2 + v_2 v_1 = -2\langle v_1, v_2\rangle \cdot 1,$$

$$v_1 v_2 - v_2 v_1 = 2[v_1, v_2],$$

$$v_1 q_2 - q_2 v_1 = 2[v_1, v_2],$$

and

$$q_1 q_2 - q_2 q_1 = 2[v_1, v_2].$$

2. From theorem 1.5, we have that for all $q = r + v$ in Q, $\psi(q) = \bar{q} = r - v$ is an involution on Q. $\bar{q}$ is defined to be the conjugate of q. Thus

$$T(q)\cdot 1 = q + \bar{q} = 2r,$$

$$N(q) = q\bar{q} = \bar{q}q,$$

$$\overline{\alpha p + \beta q} = \alpha\bar{p} + \beta\bar{q},$$

and

$$\overline{pq} = \bar{q}\,\bar{p}$$

for all p, q in Q and real α, β.

3. <u>First regular representation of Q</u>:

Let

$$q = \alpha_0 \cdot 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k.$$

For some p in Q we define the linear transformation R_p by

$$qR_p = qp$$

for all q in Q. Then from theorems 2.5 and 2.6

$$q \longleftrightarrow \alpha_o R_1 + \alpha_1 R_i + \alpha_2 R_j + \alpha_3 R_k$$

and

$$q \longleftrightarrow \alpha_o m(R_1) + \alpha_1 m(R_i) + \alpha_2 m(R_j) + \alpha_3 m(R_k).$$

As an example, $m(R_i)$ is given by:

$$1R_i = i = 0\cdot 1 + i + 0\cdot j + 0\cdot k$$

$$iR_i = i^2 = -1 + 0\cdot i + 0\cdot j + 0\cdot k$$

$$jR_i = ji = 0\cdot 1 + 0\cdot i + 0\cdot j - k$$

$$kR_i = ki = 0\cdot 1 + 0\cdot 1 + j + 0\cdot k.$$

Thus we have

$$m(R_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad m(R_i) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$m(R_j) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}; \quad m(R_k) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$q \longleftrightarrow \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ -\alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ -\alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ -\alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{pmatrix}.$$

4. <u>Second regular representation of Q:</u>

In a similar manner we have

$$q \longleftrightarrow \alpha_0 L_1 + \alpha_1 L_i + \alpha_2 L_j + \alpha_3 L_k,$$

and

$$q \longleftrightarrow \alpha_0 m(L_1) + \alpha_1 m(L_i) + \alpha_2 m(L_j) + \alpha_3 m(L_k).$$

Here, L_p is defined by $qL_p = pq$ for all q in Q.

$$m(L_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \qquad m(L_i) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$m(L_j) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \qquad m(L_k) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ -\alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 \\ -\alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 \\ -\alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 \end{pmatrix}.$$

5. Rotations [14, 16]: The Euler angles ψ, θ, φ provide the most widely used technique for describing a rotation in E_3. Let O_{xyz} denote a right handed system of rectangular axes fixed in space and let v denote any vector in this system. In theorem 3.3 we have shown that any rotation of v can be defined by

$$vT_p = pvp^{-1},$$

where p is some fixed nonzero quaternion

$$p = \alpha_0 \cdot 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k.$$

This rotation can also be described by three successive Euler angle rotations. We shall now derive the relationship between the quaternion components $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and the Euler angles ψ, θ, φ.

Let $R_{\psi,z}$ denote a rotation of ψ about the z axis, rotating the 0_{xyz} system into $0_{x_1 y_1 z}$ and let

$$vR_{\psi,z} = v_1.$$

Then by theorem 3.3, there is a fixed nonzero quaternion p_1 such that,

$$vR_{\psi,z} = vT_{p_1} = p_1 \, v \, p_1^{-1} = v_1,$$

where

$$p_1 = \cos \frac{\psi}{2} \cdot 1 - \sin \frac{\psi}{2} k.$$

Similarly, let

$$R_{\theta,y_1} : 0_{x_1 y_1 z} \longrightarrow 0_{x' y_1 z_1},$$

such that

$$v_1 \, R_{\theta,y_1} = v_2.$$

Then there exists a p_2 in Q such that

$$v_1 R_{\theta, y_1} = v_1 T_{p_2} = p_2 v_1 p_2^{-1} = v_2,$$

where

$$p_2 = \cos \frac{\theta}{2} \cdot 1 - \sin \frac{\theta}{2} j.$$

Finally, let

$$R_{\varphi, x'} : O_{x' y_1 z_1} \to O_{x' y' z'},$$

such that

$$v_2 R_{\theta, x'} = v'.$$

Then there is a p_3 in Q such that

$$v_2 R_{\theta, x'} = v_2 T_{p_3} = p_3 v_2 p_3^{-1} = v',$$

where

$$p_3 = \cos \frac{\varphi}{2} \cdot 1 - \sin \frac{\varphi}{2} i.$$

The total rotation of a vector v in O_{xyz} into v' in $O_{x'y'z'}$ is given by

$$vT_p = pvp^{-1} = v',$$

where p is some fixed nonzero quaternion

$$p = \alpha_0 \cdot 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k.$$

Now since

$$v' = p_3\, p_2(p_1\, v\, p_1^{-1})p_2^{-1}\, p_3^{-1},$$

we have that

$$p = p_3\, p_2\, p_1.$$

Hence,

$$p = \left(\cos \frac{\varphi}{2}\cdot 1 - \sin \frac{\varphi}{2}\, i\right)\left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}\, j\right)\left(\cos \frac{\psi}{2} - \sin \frac{\psi}{2}\, k\right).$$

Expanding the right hand side of this expression we obtain the following relationships between the quaternion components and Euler angles:

$$\alpha_o = \cos \frac{\varphi}{2} \cos \frac{\theta}{2} \cos \frac{\psi}{2} + \sin \frac{\varphi}{2} \sin \frac{\theta}{2} \sin \frac{\psi}{2},$$

$$\alpha_1 = \cos \frac{\varphi}{2} \sin \frac{\theta}{2} \sin \frac{\psi}{2} - \sin \frac{\varphi}{2} \cos \frac{\theta}{2} \cos \frac{\psi}{2},$$

$$\alpha_2 = - \cos \frac{\varphi}{2} \sin \frac{\theta}{2} \cos \frac{\psi}{2} - \sin \frac{\varphi}{2} \cos \frac{\theta}{2} \sin \frac{\psi}{2},$$

$$\alpha_3 = - \cos \frac{\varphi}{2} \cos \frac{\theta}{2} \sin \frac{\psi}{2} + \sin \frac{\varphi}{2} \sin \frac{\theta}{2} \cos \frac{\psi}{2}.$$

The rotation of v into v' has the following matrix representation in terms of the Euler angles [14]:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\varphi - \cos\varphi\sin\psi & \sin\varphi\sin\theta\sin\psi + \cos\psi\cos\varphi & \sin\varphi\cos\theta \\ \cos\varphi\sin\theta\cos\psi + \sin\varphi\sin\psi & \cos\varphi\sin\theta\sin\psi - \sin\varphi\cos\psi & \cos\varphi\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Since the Euler transformation is identical to the quaternion transformation we easily determine the additional relationships:

$$\psi = \tan^{-1}\left[\frac{2(\alpha_1\alpha_2 - \alpha_o\alpha_3)}{(\alpha_o^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2)}\right],$$

$$\theta = \sin^{-1}\left[\frac{-2(\alpha_1\alpha_3 + \alpha_o\alpha_2)}{N(q)}\right],$$

$$\varphi = \tan^{-1}\left[\frac{2(\alpha_2\alpha_3 - \alpha_o\alpha_1)}{(\alpha_o^2 + \alpha_3^2 - \alpha_1^2 - \alpha_2^2)}\right].$$

6. <u>Rate equations</u> [5, 14, 16]: Suppose that the system $O_{x'y'z'}$ is rotating with an angular velocity ω. Let

$$\omega = pi + qj + rk$$

where p, q, r are the angular velocities about the x', y', z' axes respectively. The Euler angle rates are expressed as follows [14]:

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & \sin\varphi/\cos\theta & \cos\varphi/\cos\theta \\ 0 & \cos\varphi & -\sin\varphi \\ 1 & \sin\varphi\sin\theta/\cos\theta & \cos\varphi\sin\theta/\cos\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}.$$

The obvious disadvantage of this system of equations is the singularity existing at

$$\theta = (2n - 1)\frac{\pi}{2},\ n = 1, 2, \cdots .$$

We shall now show that no such problem exists in the corresponding rate equations for the quaternion components. Suppose that the quaternion q is a function of the scalar quantity t. That is,

$$q = q(t) = \omega(t)\cdot 1 + x(t)i + y(t)j + z(t)k.$$

Then analogous to our definition of a derivative in the Euclidean vector space of dimension 3, we define

$$\frac{dq(t)}{dt} = \lim_{\Delta t \to 0} \frac{q(t + \Delta t) - q(t)}{\Delta t}$$

or

$$\frac{d}{dt} q(t) = \frac{d}{dt} \omega(t) \cdot 1 + \frac{d}{dt} x(t) i + \frac{d}{dt} y(t) j + \frac{d}{dt} z(t) k .$$

From this it follows that

$$\frac{d}{dt} (q_1 q_2) = \left(\frac{d}{dt} q_1\right) q_2 + q_1 \left(\frac{d}{dt} q_2\right) .$$

Now let T_p be the quaternion transformation rotating the vector v in O_{xyz} into v' in $O_{x'y'z'}$. That is,

$$v' = vT_p = pvp^{-1}$$

where p, v are functions of t. Furthermore, let $O_{x'y'z'}$ be rotating with an angular velocity ω as defined above. Finally, define the quaternion $\lambda = \lambda_0 \cdot 1 + \lambda_1 i + \lambda_2 j + \lambda_3 k$ by

$$\lambda = \frac{p}{||p||} = \frac{p}{+\sqrt{N(p)}},$$

where

$$\lambda_i = \frac{\alpha_i}{||p||}, \quad i = 0, 1, 2, 3.$$

Then

$$v' = vT_p = pvp^{-1} = \lambda v\bar{\lambda}.$$

Also

$$v = v'T_p^{-1} = v'T_p^{\,T} = p^{-1}\,vp = \bar{\lambda}v\lambda.$$

From this, we have the following matrix representation for T_p:

$$\begin{bmatrix} \lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2 & 2(\lambda_1\lambda_2 - \lambda_0\lambda_3) & 2(\lambda_1\lambda_3 + \lambda_0\lambda_2) \\ 2(\lambda_2\lambda_1 + \lambda_3\lambda_0) & \lambda_0^2 + \lambda_2^2 - \lambda_3^2 - \lambda_1^2 & 2(\lambda_2\lambda_3 - \lambda_0\lambda_1) \\ 2(\lambda_3\lambda_1 - \lambda_2\lambda_0) & 2(\lambda_2\lambda_3 + \lambda_0\lambda_1) & \lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2 \end{bmatrix}.$$

From theoretical mechanics [5, pp.141-145] , we have the following relationship:

$$\left(\frac{dv}{dt}\right)' = \frac{dv'}{dt} + [\omega, v'].$$

The prime denoting the $O_{x', y', z'}$ or moving system. Hence, in terms of the quaternion transformation T_p, we can write

$$\left(\frac{dv}{dt}\right)T_p = \frac{dv'}{dt} + [\omega, v'] .$$

Now,

$$\left(\frac{dv}{dt}\right)T_p = \lambda\left[\frac{d}{dt}(\bar{\lambda}v'\lambda)\right]\bar{\lambda} = \frac{dv'}{dt} + \lambda\left(\frac{d}{dt}\,\bar{\lambda}\right)v' + v'\left(\frac{d}{dt}\,\lambda\right)\bar{\lambda}.$$

Since $\lambda\bar{\lambda} = 1$,

$$\left(\frac{dv}{dt}\right)_{T_p} = \frac{dv'}{dt} - \left(\frac{d}{dt}\lambda\right)\bar{\lambda}v' + v'\left(\frac{d}{dt}\lambda\right)\bar{\lambda}.$$

Thus

$$[\omega, v'] = v'\left(\frac{d}{dt}\lambda\right)\bar{\lambda} - \left(\frac{d}{dt}\lambda\right)\bar{\lambda}v'.$$

Note that for any λ in Q we can write

$$\left(\frac{d}{dt}\lambda\right)\bar{\lambda} = r^* + v^*$$

where r^* is in I and v^* is in E_3. Then from (iii) of (1) we have

$$[\omega, v'] = -2[v^*, v'],$$

or

$$[\omega + 2v^*, v'] = 0.$$

Since this expression is valid for all v' in E_3, and since the vector cross product is nondegenerate,

$$\omega = -2v^*.$$

Finally,

$$v^* = \frac{1}{2}\left[\left(\frac{d}{dt}\lambda\right)\bar{\lambda} - \overline{\left(\frac{d}{dt}\lambda\right)\bar{\lambda}}\right] = \frac{1}{2}\left[\left(\frac{d}{dt}\lambda\right)\bar{\lambda} - \lambda\left(\frac{d}{dt}\bar{\lambda}\right)\right].$$

Again, since $\lambda\bar{\lambda} = 1$

$$v^* = \left(\frac{d}{dt}\lambda\right)\bar{\lambda}.$$

Thus

$$\omega = -2\left(\frac{d}{dt}\lambda\right)\bar{\lambda},$$

or

$$\frac{d}{dt}\lambda = -\frac{1}{2}\omega\lambda.$$

Expanding, we have the following matrix representation:

$$\begin{bmatrix}\dot{\lambda}_0\\ \dot{\lambda}_1\\ \dot{\lambda}_2\\ \dot{\lambda}_3\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}0 & -p & -q & -r\\ p & 0 & -r & q\\ q & r & 0 & -p\\ r & -q & p & 0\end{bmatrix}\begin{bmatrix}\lambda_0\\ \lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix}.$$

Equivalently,

$$\begin{bmatrix}\dot{\lambda}_0\\ \dot{\lambda}_1\\ \dot{\lambda}_2\\ \dot{\lambda}_3\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}-\lambda_1 & -\lambda_2 & -\lambda_3\\ \lambda_0 & \lambda_3 & -\lambda_2\\ -\lambda_3 & \lambda_0 & \lambda_1\\ \lambda_2 & -\lambda_1 & \lambda_0\end{bmatrix}\begin{bmatrix}p\\ q\\ r\end{bmatrix},$$

where

$$\dot{\lambda}_i = \frac{d}{dt}\lambda_i, \qquad i = 0, 1, 2, 3.$$

Note: The substitution

$$\lambda = \frac{p}{+\sqrt{N(p)}}$$

is equivalent to using $vT_p = pvp^{-1}$ and applying the constraint

$$p\bar{p} = N(p) = 1.$$

BIBLIOGRAPHY

1. Albert, A. A.: "Absolute-Valued Algebraic Algebras", Bulletin of The American Mathematical Society, Vol. 55, 1949, pp 763-768.
2. Birkhoff, G.; and MacLane, S.: A Survey of Modern Algebra, The Macmillan Company, New York, 1946.
3. Brand, Louis: Vector and Tensor Analysis, John Wiley and Sons, Inc., New York 1948.
4. Bruck, R. H.; and Kleinfeld, E.: "The Structure of Alternative Division Rings", Proceedings of the American Mathematical Society, Vol. 2, 1951, pp 878-890.
5. Corben, H. C.; and Stehle, Philip: Classical Mechanics, John Wiley and Sons, Inc., New York 1960.
6. Curtis, C. W.: "The Four and Eight Square Problem and Division Algebras", MAA Studies in Mathematics, Vol. 2, 1963, pp 100-125.
7. Dickson, L. E.: Linear Algebras, Hafner Publishing Co., New York, 1914.
8. Dickson, L. E.: "On Quaternions and Their Generalization and the History of the Eight Square Theorem", Annals of Mathematics, Vol. 20, 1919, pp 155-171.
9. Herstein, I. N.: Topics in Algebra, Blaisdell Publishing Co., New York, 1964.
10. Jacobson, N.: "Composition Algebras and Their Automorphisms", Rendiconti del Circolo Mathematico di Palermo, Vol. 7, 1958, Ser. 11, pp 55-80.

11. Kleinfeld, E.: "A Characterization of the Caley Numbers," MAA Studies in Mathematics, Vol. 2 (1963), pp. 126-143.

12. MacDuffee, C. C.: An Introduction to Abstract Algebra, John Wiley and Sons, Inc., New York, 1940.

13. McCoy, N. H.: "Rings and Ideals," The Carus Mathematical Monographs No. 8, 1948.

14. Niemz, W.: "Anwendung der Quaternionen auf die allgemeinen Bewegungsgleichungen der Flugmechanik," Zeitschrift für Flugwissenschaften, Vol. 11, No. 9 (1963), pp. 368-372.

15. Olmsted, J. M. H.: Real Variables, Appleton-Century-Crafts, Inc., 1959.

16. Surber, T. E.: "On the Use of Quaternions to Describe the Angular Orientation of Space Vehicles," Journal of Aerospace Science, Vol. 28, pp. 79-80, 1961.

17. Whittaker, E. T.: Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Cambridge University Press, 1960.

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