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Steepest Descent Techniques for Operator Equations

William T. Suit  
College of William & Mary - Arts & Sciences

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STEEPEST DESCENT TECHNIQUES FOR
OPERATOR EQUATIONS

A Thesis
Presented to
The Faculty of the Department of Mathematics
The College of William and Mary in Virginia

In Partial Fulfillment
Of the Requirements for the Degree of
Master of Arts

By
William T. Suit
1967
This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Arts.

Author

Approved, May 1967

Sidney M. Lawrence, Ph. D.

Benjamin R. Cato, M.A.

Luther T. Conner, Jr., M.A.
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ABSTRACT

The problem discussed in this paper is the solution of the equation $Ax = \phi$ where $A$ is an operator on a real Hilbert space, and $x$ and $\phi$ belong to this Hilbert space. The general method of solution is by successive approximations. The particular approach discussed is to apply steepest descent in Hilbert space in a manner analogous to the finite space application. In this case steepest descent is used to minimize the quadratic functionals,

$$H(x) = \langle Ax, x \rangle - 2 \langle x, \phi \rangle$$

and $||Ax - \phi||^2$, whose minimums are solutions of $Ax = \phi$.

A technique for bounding an unbounded operator by defining a new norm and scalar product, and then showing that the unbounded operator is bounded in the sense of the new norm is developed.

For the case where $A$ is a bounded operator, algorithms of the form $x_n = x_{n-1} + \epsilon_{n-1} z_{n-1}$, approximating the exact solution to $Ax = \phi$, were developed by minimizing the quadratic functionals. These algorithms gave convergent sequences and the speed of convergence was shown at least as fast as that of a geometric progression.

A different method of determining the direction of a modification to a particular approximate solution to give a better approximation to the actual solution was examined. By modifying a particular guess by a direction $w(x_n) = A^{-1} r(x_n)$, where $A^{-1}$ is determined through the use of a Neumann series, a new algorithm was obtained. The convergence of the new algorithm is shown and its overestimate indicated that the new algorithm might lead to faster convergence.

The techniques which were developed were applied to a second order, self adjoint, differential operator with zero end points. It was shown that this usually unbounded operator could be bounded using the techniques developed. If the coefficients of the differential operator, $A$ of $Ax = \phi$, are continuous and differentiable and $\phi$ is continuous, then an approximate solution may be generated for any second order differential equation meeting these conditions.
STEEPEST DESCENT TECHNIQUES FOR
OPERATOR EQUATIONS
INTRODUCTION

The problem discussed in this paper is the solution of the equation \( Ax = \emptyset \) where \( A \) is a linear operator on a real Hilbert space and \( x \) and \( \emptyset \) belong to this Hilbert space. The general method of solution will be by successive approximations. The particular approach discussed will be to apply steepest descent in Hilbert space in a manner analogous to the finite space application. In this case we will be using steepest descent to minimize a quadratic functional related to \( Ax = \emptyset \).

The first approximation will be denoted by \( x_0 \) where \( x_0 \) will be modified by \( \epsilon z \) in successive approximations. \( z \) is a unit vector called the direction whose actual spatial direction is suggested by steepest descent to give the maximum variation in the quadratic functional, in a restricted local sense, in the immediate vicinity of \( x_0 \). On the other hand for a given direction \( z \) the \( \epsilon \) determined by steepest descent is the best possible and leads to the minimization of \( H(x_0 + \epsilon z) \).

As will be shown in this paper directions which are better in the general sense of causing \( x_0 + \epsilon z \) to converge to a solution of \( Ax = \emptyset \) in a fewer number of steps may be found by a different method. In fact, the direction determined using steepest descent may also be determined by this different method. To show formally that one method of generating an approximate solution is better than another is
very difficult. However, by examining the features of the methods, some appear to give approximate solutions which give better approximations in a fewer number of steps.

The solution for $Ax = \emptyset$ will be constructed first for the case where $A$ is a bounded operator and then extended, by a method credited to Friedrichs, to the case where $A$ is an unbounded operator.

In the first chapter the theory for the extension of the results from the bounded to the unbounded operator will be developed. Also, the theorem that shows that the sequence $\{x_n\}$ which minimizes an appropriate quadratic functional is equivalent to the approximate solution of the equation $Ax = \emptyset$ is stated.

The next chapter is devoted to the development of approximate solutions to $Ax = \emptyset$ where $A$ is a bounded operator.

The final chapter extends the results of chapter two to the case of a unbounded operator and also gives an example of the application of the unbounded operator theory.
CHAPTER I

HILBERT SPACE PRELIMINARIES

When considering an equation of the form $Ax = \phi$, where $A$ is a bounded (with a positive lower bound), self adjoint operator, we first must establish the existence and uniqueness of solutions to this equation. The discussion [pp. 265-266, 6] shows that for a symmetric, positive definite operator an unique inverse exists. Therefore, $Ax = \phi$ where $A$ is a symmetric, positive definite operator will have an unique solution.

Now that we have verified that a unique solution to $Ax = \phi$ does exist we prove the following theorem which is also contained in [2], [4], and [6].

Theorem: The quadratic functional $H(x) = \langle Ax, x \rangle - 2\langle x, \phi \rangle$ attains its minimum only at the solution $x^*$ of $Ax = \phi$.

Proof: $\phi = Ax^*$ so that we write $H(x) = \langle Ax, x \rangle - 2\langle x, Ax^* \rangle$.

Adding and subtracting $Ax^*$ and $x^*$ in the first term and $x^*$ in the second term we obtain

$$H(x) = \langle Ax - Ax^* + Ax^*, x - x^* + x^* \rangle - 2\langle (x - x^*) + x^*, Ax^* \rangle$$

$$= \langle A(x - x^*), x - x^* \rangle - \langle Ax^*, x^* \rangle - \langle Ax^*, x^* \rangle.$$

Since $A$ is bounded $\langle Ax, x \rangle \geq m\|x\|^2$. Therefore,

$$H(x) \geq m\langle (x - x^*), (x - x^*) \rangle - \langle Ax^*, x^* \rangle \geq -\langle Ax^*, x^* \rangle = H(x^*).$$

so that $x^*$ minimizes $H(x)$. To prove $x^*$ is the only point where the minimum is attained let $H(\tilde{x}) = H(x^*)$. Then since

$$H(\tilde{x}) = \langle A(\tilde{x} - x^*), (\tilde{x} - x^*) \rangle + H(x^*),$$

$$0 = \langle A(x^* - \tilde{x}), (x^* - \tilde{x}) \rangle \geq m\|x^* - \tilde{x}\|^2 \geq 0$$

so $\|x^* - \tilde{x}\| = 0$ and $\tilde{x} = x^*$. 

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We would now like to develop the basis for the extension of bounded operator results to the case of unbounded operators.

**Definition:** If $A$ is a symmetric unbounded operator defined over a set $\Omega_0$, $B$ is a symmetric, positively semibounded operator, $\langle Bx, x \rangle \geq \langle x, x \rangle$, and if $|\langle Ax, x \rangle| \leq M\langle Bx, x \rangle$ for $x \in \Omega_0$ then $A$ is said to be $B$ bounded on $\Omega_0$.

We want to examine characteristics of $B$ such as the range, domain and existence of an inverse. First consider $B$ to have domain $\Omega_0$ where $\Omega_0$ is a linear set dense in a Hilbert space $\mathcal{H}$. Define a new scalar product $[u, v]$ in $\Omega_0$ as $[u, v] = \langle Bu, v \rangle$. This can be shown to be a true scalar product.

1. $[au, u] = \langle aBu, v \rangle = a\langle Bu, v \rangle = a[u, v]$
2. $[u + r, v] = \langle B(u + r), v \rangle = \langle Bu + Br, v \rangle = \langle Bu, v \rangle + \langle Br, v \rangle = [u, v] + [r, v]$
3. $[u, v] = \langle Bu, v \rangle = \langle u, Bv \rangle = \langle Bv, u \rangle = \langle v, u \rangle$

We don't have to worry about the complex conjugate when reversing the scalar products since we are dealing only with real numbers.

4. $[u, u] = \langle Bu, u \rangle > 0$ for $u \neq 0$ since $B$ is positively semibounded. We now define a norm in $\Omega_0$, which we will call the $B$ norm, as $u = [u, u]^{1/2}$.

Consider a Cauchy sequence $\{x_n\}$ in the sense of the $B$ norm. That is if $\epsilon > 0$ there exists an $M$ such that for all $m, n \geq M$, $\|x_n - x_m\| < \epsilon; \lim_{n,m \to \infty} \|x_n - x_m\| = 0$. But, $\langle B(x_n - x_m), (x_n - x_m) \rangle \geq \|x_n - x_m\|^2$ so that $\lim_{n,m \to \infty} \|x_n - x_m\| = 0$ or $x_n$ converges to some $\hat{x}$ in $\mathcal{H}$. Let $K$ be the completion of $\Omega_0$. That is $K$ is a Hilbert space with $\Omega_0 \subset K$, $\Omega_0$ dense in $K$, and if $\langle \rangle$
is the scalar product in $K$, $(,) = [ , ]$ on $\Omega$. We will seek a one to one correspondence (a continuous linear transformation T) between $K$ and a subset of $\mathcal{H}$ such that for $f \in \Omega$, $T(f) = f$. For $h \in K - \Omega$, let $f_n \in \Omega$ with $f_n \rightarrow h \in K$. Define $T(h) = \lim_{n \rightarrow \infty} T(f_n) = \lim_{n \rightarrow \infty} f_n$. This limit exists since $f_n$ is Cauchy in $\Omega$ and hence in $\|\|$. If $g_n \rightarrow h$ in $K$ with $g_n \in \Omega$, $\|f_n - g_n\| \rightarrow 0$ so $\|f_n - g_n\| \rightarrow 0$. Therefore, $T(h)$ is well defined. Now we must show that $T$ is one to one. Suppose $T(f) = T(g)$, $f_n \rightarrow f \in K$ and $g_n \rightarrow g \in K$ with $f_n$ and $g_n \in \Omega$. Let $h \in \Omega$, then

$$(h, f - g) = \lim_{n \rightarrow \infty} [h, f_n - g_n] = \lim_{n \rightarrow \infty} < Bh, f_n - g_n >$$

$$= < Bh, \lim_{n \rightarrow \infty} f_n - \lim_{n \rightarrow \infty} g_n > = < Bh, T(f) - T(g) >$$

$$= < Bh, 0 > = 0.$$ 

Since $\Omega$ is dense in $K$, $f = g$. $K$ is, therefore, one to one with a subset of $\mathcal{H}$ where $T$ maps $K$ onto the subset of $\mathcal{H}$ which we will call $\Omega_B$.

The space $\Omega$ has been extended to the complete space $\Omega_B$. We would like to extend the operator $B$ on a subset of $\Omega_B$ such that the range of the extended operator will fill the whole Hilbert space $\mathcal{H}$.

To create a proper extension we consider an arbitrary element $h \in \mathcal{H}$ and define the functional $L_h(f) = < f, h >$ where $f \in \Omega_B$. Then $|L_h(f)| \leq \| f \| \| h \| \leq \| f \| \| h \|^{1/2}$ where $\| f \| = < Bf, f >^{1/2}$.

In the Hilbert space $\Omega_B$, $L_h(f)$ is a linear functional whose norm does not exceed $\| h \|$. To establish this result define a bounded linear functional $G_g$ such that $\| G_g \| \leq M \| f \|$ where $G$ is some
arbitrary operator. We now appeal to the following theorem [p. 120, 2]. "Whatever the linear functional \( f \) on a Hilbert space \( \mathcal{H} \), there exists an element \( y \in \mathcal{H} \), uniquely defined by the functional \( f \), such that \( f(x) = \langle x, y \rangle \) for any \( x \in \mathcal{H} \)." Moreover \( \| f \| = \| g \| \). Using the theorem, the notion of a linear functional on \( \Omega_B \) and that there exists an element \( g \) in \( \Omega_B \) such that \( L_h(f) = [f, g] \) and also \( \| g \| \) is the norm of the functional. Therefore, \( \| g \| \leq \| g \| \leq \| f \| \). The element \( g \) is uniquely determined by the functional \( L_h \) and, therefore, by \( h \) so we write \( g = Dh \) and define \( D \) as an unique transformation whose domain is \( \mathcal{H} \) and whose range is in \( \Omega_B \) and is denoted by \( \Omega^* \). \( \Omega_B \supset \Omega^* \). \( D \) is clearly linear. \( D \) may also be shown to be self adjoint.

\[
[f, Dh] = [Dh, f] = \langle h, f \rangle \quad \text{on} \quad \Omega_B.
\]

Also, \( [Df, h] = \langle f, h \rangle = \langle h, f \rangle \) on \( \Omega_B \) so that \( [f, Dh] = \langle h, f \rangle = [Df, h] \).

We know that \( L_h(f) = \langle f, h \rangle = [f, g] \) for \( f \in \Omega_B \). Assume that there are two \( h \)'s in \( \mathcal{H} \) such that \( Dh_1 = g = Dh_2 \). But \( [f, g] = \langle f, h_1 \rangle \) and \( [f, g] = \langle f, h_2 \rangle \). Since \( \Omega_B \) is dense then \( h_1 = h_2 \) so that there is a one to one correspondence between the elements in \( \Omega^* \) (the range of \( D \)) and the elements of \( \mathcal{H} \) (the domain of \( D \)). Therefore, \( D^{-1} \) may be defined as an operator with domain \( \Omega^*_B \) and range \( \mathcal{H} \).

We now wish to show that \( D^{-1} \) is an extension of \( B \) on \( \Omega^*_B \). For \( D^{-1} \) to be an extension of \( B \) we must show that for all \( f \in \Omega_B \), \( Bf = D^{-1}f \). Then since \( D^{-1} \) is defined on \( \Omega^*_B \), \( D^{-1} \) would be said to be an extension of \( B \) to \( \Omega^*_B \). Let \( L = Bg \), then
\[ L_{Bg}(f) = \langle f, Bg \rangle = [f, g] = \langle f, h \rangle = L_h(f) \] and \( D h = g \), therefore, \( h = D^{-1} g \) since \( D^{-1} \) exists and \( L = Bg \) \( D^{-1} g = Bg \) where \( g \in \Omega \). Therefore, \( D^{-1} \) is an extension of \( B \).

The possibility exists that there may be other symmetric extensions of \( B \) from \( \Omega \) to \( \Omega^* \). Let \( E \) be such an extension. Let \( f \) be an element of \( \Omega \) and \( f' \) an element of \( \Omega^* \). Then
\[ \langle f, Ef' \rangle = \left[ f, DEF' \right] \] since \( Ef' \) plays the part of \( h \) in the earlier definition of \( L_h(f) \). But
\[ \langle f, Ef \rangle = \langle Ef, f' \rangle = \langle D^{-1} f, f' \rangle = \left[ DD^{-1} f, f' \right] = \left[ f, f' \right] \]
so that \( DEF' = f' \) or \( Ef' = D^{-1} f' \) since \( \Omega^* \) is dense in \( \mathcal{H} \).
Therefore, \( E \subset D^{-1} \) so that any other extensions of \( B \) are equal to or contained in \( D^{-1} \).

We now have extended \( B \) on \( \Omega \) to \( D^{-1} \) on \( \Omega^* \). Also, we have created \( D \) which maps any element of \( \mathcal{H} \) into \( \Omega^* \). In considering the problem of solving \( Ax = \emptyset \), \( A \) is only defined for \( x \in \Omega \) while \( \emptyset \) can be any member of \( \mathcal{H} \). The range of \( A \) is in \( \mathcal{H} \). \( \Omega_B \) is a complete space which we have constructed by extending \( \Omega \). If we apply \( D \) to both sides of \( Ax = \emptyset \) we get \( DAx = D\emptyset \) an equation defined in \( \Omega^*_B \). Call \( D\emptyset = \emptyset_1 \). \( DA \) has range in \( \Omega^*_B \) but domain in \( \Omega \). We would like to show that \( DA \) is bounded on \( \Omega^*_B \) and extend the domain of \( DA \) to \( \Omega^*_B \).

We first must show that \( DA \) is symmetric at least on \( \Omega \).
Consider \( [DAf, g] \) where \( f \) and \( g \in \Omega \)
\[ [DAf, g] = \langle Af, g \rangle = \langle f, Ag \rangle = [Df, Ag] = [f, DAg] \]
so that \( DA \) is symmetric on \( \Omega \). \( B \) will be selected such that
\[ m \langle Bx, x \rangle \leq \langle Ax, x \rangle \leq [DAx, x] \leq M[x, x]. \]
Therefore, in $\Omega_0$ $DA$ is semibounded in the sense of the B metric.

However, to extend $DA$ to $A'$ in $\Omega_B$ we must show that
\[
[DAf,g]^2 \leq [DAf,f][DAg,g]
\]
where the metric is the B metric defined on $\Omega_B$ and $f$ and $g$ belong to $\Omega_0$.

For every real $\lambda$ an $h_\lambda \in \Omega_0$ may be defined so that
\[
h_\lambda = f + \lambda[DAf,g]g. \quad 0 \leq [DAh_\lambda,h_\lambda] = [DA(f + \lambda[DAf,g]g), f + \lambda[DAf,g]g],
\]
\[
f + \lambda[DAf,g]g = [(DA + \lambda[DAf,g][DAg,g], f + \lambda[DAf,g]g]
\]
\[
= [DAf,f] + 2\lambda[DAf,g]^2 + \lambda^2[DAf,g][DAg,g].
\]
The proceeding is a quadratic in $\lambda$ with real coefficients and always greater than zero. If we consider the general quadratic equation
\[
f(x) = ax^2 + bx + c > 0
\]
and form $f'(x) = 2ax + b$ and $f''(x) = 2a$
we see that $f'(x) = 0$ when $x = -\frac{b}{2a}$ and at that point $f''(x) > 0$ so that $x = -\frac{b}{2a}$ is a minimum point. Since $ax^2 + bx + c > 0$ has only one minimum point the graph of such an equation could touch the $\lambda$ axis only once. This would imply a single real root. If such occurs it would have to be a double root. If the graph of the quadratic does not cross the $\lambda$ axis there would be two imaginary roots. The roots, therefore, are equal or imaginary so that the discriminate
\[
b^2 - 4a \leq 0
\]
or
\[
h[D Af,g]^h - h[D Af,g]^2[D Ag,g][DAf,f] \leq 0
\]
or
\[
[DAf,g]^2 \leq [DAg,g][DAf,f] \leq M^2[g,g][f,f]
\]
so that $[DAf,g]$ is a bounded functional. Call $[DAf,g] = Lg(f)$.

We now wish to extend this result to a more general $g$. Consider
\[ |[Df,g_n]| \leq M \forall g_n \forall f \forall \}, \text{ where } g_n \in \Omega_o, \text{ then} \]

\[ g_n \to g \in \Omega_B \text{ so that } |[Df,g]| \leq M \forall g \forall f \forall \].

Since \( Lg(f) \) is a bounded functional on \( \Omega_o \) there exists a \( k \in \Omega_B \) such that \( Lg(f) = [f,k] \). This element \( k \) is determined by the functional and, therefore, by \( g \). Let \( A'g = k \) where \( A' \) is an operator mapping \( \Omega_B \) into itself. If \( g \in \Omega_o \) then

\[ [Df,g] = [f,k], \text{ but } [Df,g] = \langle Af, g \rangle = \langle f,Ag \rangle = [f,DAg]. \text{ Then} \]

\[ [f,DAg] = [f,k]. \text{ Since } \Omega_o \text{ is dense } DAg = k \text{ or } DAg = A'g \text{ for } g \in \Omega_o. \text{ Therefore, } A' \text{ is an extension of } DA. \text{ DA is symmetric in } \Omega_o \text{ so } A' \text{ is symmetric in } \Omega_o. \text{ We now wish to show } A' \text{ is symmetric and semibounded on } \Omega_B. \]

Consider \( f \) and \( g \in \Omega_B \) where \( f_n \to f \) and \( g_n \to g \) with \( f_n \) and \( g_n \in \Omega_o \).

\[ |[f,A'g]| = \lim_{n \to \infty} |[f_n,A'g_n]| = \lim_{n \to \infty} |[A'f_n,g_n]| \leq \lim_{n \to \infty} M \forall f_n \forall g_n \forall \]

\[ = M \forall f \forall g \forall \].

Let

\[ f = A'g, \forall A'g \forall^2 = [A'g,A'g] \leq M \forall A'g \forall g \forall \]

so that \( \forall A'g \forall \leq M \forall g \forall \) or \( A' \) is a bounded operator and, therefore, continuous. Now

\[ [f,A'g] = \lim_{n \to \infty} [f_n,A'g_n] = \lim_{n \to \infty} [A'f_n,g_n] = [A'f,g] \]

so \( A' \) is symmetric on \( \Omega_B \). For \( h \in \Omega_o \),

\[ m[h,h] = m \langle Bh,h \rangle \leq \langle Ah,h \rangle = [DAh,h] = [A'h,h] = [DAh,h] \]

\[ = \langle Ah,h \rangle \leq M \langle Bh,h \rangle = M[h,h] \]

then \( m \forall h \forall^2 \leq [A'h,h] \leq M \forall h \forall^2 \) for \( h \in \Omega_o \). Since \( A' \) is
continuous by the above boundedness condition, $A'$ may be extended from $\Omega_0$ to $\Omega_B$ by continuity.
We would now like to consider the application of the method of steepest descent to the equation $Ax = \emptyset$ where $A$ is a bounded, symmetric, positive definite operator with domain and range in a Hilbert space $\mathcal{H}$. We proved in chapter I that the $x$ required to minimize the functionals $H(x)$ and $\|r(x)\|^2$ was also the solution to $Ax = \emptyset$.

The method of steepest descent will now be applied to the above mentioned functionals and approximations to the exact solution $x^*$ of $Ax = \emptyset$ will be generated. First we will consider the Kantorovich functional $H(x) = \langle Ax, x \rangle - 2\langle \emptyset, x \rangle$. If we designate some element of $\mathcal{H}$ by $z$ and a real parameter by $\epsilon$, then

$$H(x + \epsilon z) = \langle (Ax + \epsilon Az), x + \epsilon z \rangle - 2\langle (x + \epsilon z), \emptyset \rangle$$

$$= \langle Ax, x \rangle + \epsilon \left[ \langle Ax, z \rangle + \langle Az, x \rangle \right] + \epsilon^2 \langle Az, z \rangle - 2\langle x, \emptyset \rangle - 2\epsilon \langle z, \emptyset \rangle$$

$$= H(x) + 2\epsilon \langle Ax - \emptyset, z \rangle + \epsilon^2 \langle Ax, x \rangle.$$ 

In an attempt to minimize the functional $H(x)$ we will employ the method of steepest descent.

We begin with an initial guess at $x^*$ which we call $x_0$. We next want to find a unit vector $z$ which when used to modify $x_0$ will result in a vector as close as possible to the exact solution $x^*$. The steepest descent technique suggests a vector which gives the maximum variation of $H(x_0 + \epsilon z)$ in the immediate vicinity of $x_0$.

Assuming that we have found a suitable $z$ we will now determine the $\epsilon$ to be associated with the $z$ so that $H(x_0 + \epsilon z)$
will be a minimum. Setting \( \frac{d}{d\varepsilon} H(x_0 + \varepsilon z) = 0 \) we get
\[
2 \langle Ax_0 - \phi, z \rangle + 2\varepsilon \langle Az, z \rangle = 0 \quad \varepsilon = -\frac{\langle Ax_0 - \phi, z \rangle}{\langle Az, z \rangle}.
\]

The \( z \) indicated by steepest descent gives the maximum variation at \( x_0 \) and is determined by considering
\[
\frac{d}{d\varepsilon} H(x_0 + \varepsilon z) \bigg|_{\varepsilon=0} = 2 \langle Ax_0 - \phi, z \rangle
\]
where this will be a maximum when \( z \) is a unit vector parallel to \( Ax_0 - \phi \).

We call this first direction \( z_0 = Ax_0 - \phi \). Since \( z_0 \) need not be a unit vector when determining \( \varepsilon \) let \( \varepsilon_0 \) be the \( \varepsilon \) minimizing \( H(x_0 + \varepsilon z_0) \). The original guess is, therefore, modified by \( \varepsilon_0 z_0 \). The new \( x \) we call \( x_1 = x_0 + \varepsilon_0 z_0 \) and this \( x_1 \) will become the next guess. We then consider \( H(x_1 + \varepsilon_1 z_1) \) and repeat the process.

We next consider the problem; is the sequence \( \{x_n\} \) we are generating actually converging to the exact solution \( x^* \). In the case of the approximating sequence generated by considering the minimization of \( H(x) \) Kantorovich [2] has proved that: "The successive approximations \( x_0, x_1, \ldots \) converge in the norm to the solution \( x^* \) of \( Ax = \phi \) with the speed of a geometric progression."

The progression is given as
\[
|| x_n - x^* || \leq \frac{|| z_0 || || (M - m) ||^n}{m (M + m)}
\]
where \( M \) and \( m \) are the upper and lower bounds of \( A \) respectively.

Next we look at a different functional which may be used to generate an approximating sequence \( \{x_n\} \). In this case we will try to minimize \( || Ax_0 - \phi ||^2 \). Using a notation suggested by Petryshyn [5],
call \( r(x_n) = Ax_n - \phi = Aw(x_n) \) where
\[
w(x_n) = (x_n - x^*). \quad || r(x_n) ||^2 = || Ax_n - \phi ||^2.
\]
Consider
\[ \| r(x_0 + \epsilon_0 z_0) \|^2 = \| A(x_0 + \epsilon_0 z_0) - \phi \|^2 \]
\[ = \left\langle (Ax_0 + \epsilon_0 Az_0 - \phi), (Ax_0 + \epsilon_0 Az_0 - \phi) \right\rangle \]
\[ = \left\langle (r(x_0) + \epsilon_0 Az_0), (r(x_0) + \epsilon_0 Az_0) \right\rangle \]
\[ = \| r(x_0) \|^2 + \epsilon_0^2 \| Az_0 \|^2 + 2\epsilon_0 \langle r(x_0), Az_0 \rangle . \]
\[
\frac{d}{d\epsilon} \left[ \| r(x_0 + \epsilon_0 z_0) \|^2 \right] \\
= 2\epsilon_0 \| Az_0 \|^2 + 2 \langle r(x_0), Az_0 \rangle .
\]

For an arbitrary \( z_0 \) the \( \epsilon_0 \) which minimizes \( \| r(x_0 + \epsilon_0 z_0) \|^2 \) is given by setting \( \frac{d}{d\epsilon} \left[ \| r(x_0 + \epsilon_0 z_0) \|^2 \right] = 0 \). We get
\[ 2\epsilon_0 \| Az_0 \|^2 + 2 \langle r(x_0), Az_0 \rangle = 0 \] then \( \epsilon_0 = -\frac{\langle Ar(x_0), z_0 \rangle}{\| Az_0 \|^2} \) so that the next guess becomes \( x_1 = x_0 + \epsilon_0 z_0 \). The \( z_0 \) indicated by steepest descent is a unit vector determined by maximizing
\[ \frac{d}{d\epsilon} \left[ \| r(x_0 + \epsilon_0 z_0) \|^2 \right]_{\epsilon=0} = 2 \langle r(x_0), Az_0 \rangle = 2 \langle Ar(x_0), z_0 \rangle . \]

As before since \( z_0 \) does not have to be a unit vector when determining \( \epsilon_0, z_0 = Ar(x_0) \).

As before we must show that \( x_n \to x^* \). We know that
\[ x_n = x_{n-1} + \epsilon_{n-1} z_{n-1} \] so that
\[ w(x_n) = x_n - x^* = x_{n-1} - x^* + \epsilon_{n-1} z_{n-1} = w_{n-1} + \epsilon_{n-1} z_{n-1} . \]
Consider \( Aw(x_n) = A(x_n - x^*) = Ax_n - \phi = r(x_n) \). Then
\[ \| Aw(x_n) \| = \| r(x_n) \| \] so that
\[ m \| w(x_n) \| \leq \| r(x_n) \| \leq M \| w(x_n) \| \] when \( M \) and \( m \) are the bounds of \( A \).
\[ w(x_n) \leq \frac{\| r(x_n) \|}{m} . \] Therefore, if
\[\|r(x_n)\| = \|Ax_n - \phi\| \text{ approaches zero as } n \text{ approaches infinity,}\]
then \[\|w(x_n)\| = \|x_n - x^*\| \text{ will also approach zero. From the}\]
above work \[w(x_n) = w(x_{n-1}) + \epsilon_{n-1}z_{n-1}\] and \[Aw(x_n) = r(x_n)\] so
\[r(x_n) = Aw(x_{n-1}) + \epsilon_{n-1}z_{n-1} = r(x_{n-1}) + \epsilon_{n-1}Az_{n-1}.\]
Substituting for \(\epsilon_{n-1}\) and for the \(z_{n-1}\) suggested by using steepest
descent we get
\[r(x_n) = r(x_{n-1}) - \frac{\langle Ar(x_{n-1}), Ar(x_{n-1}) \rangle A^2 r(x_{n-1})}{\|A^2 r(x_{n-1})\|^2}\]
\[= r(x_{n-1}) - \frac{\|Ar(x_{n-1})\|^2}{\|A^2 r(x_{n-1})\|^2} A^2 r(x_{n-1})\].
\[\|r(x_n)\|^2 = \langle r(x_n), r(x_n) \rangle\]
\[= \langle r(x_{n-1}) - \frac{\|Ar(x_{n-1})\|^2}{\|A^2 r(x_{n-1})\|^2} A^2 r(x_{n-1}), r(x_{n-1}) \rangle\]
\[- \frac{\|Ar(x_{n-1})\|^2}{\|A^2 r(x_{n-1})\|^2} A^2 r(x_{n-1})\rangle\]
so that
\[\|r(x_n)\|^2 = \|r(x_{n-1})\|^2 - 2 \frac{\|Ar(x_{n-1})\|^4}{\|A^2 r(x_{n-1})\|^2} + \frac{\|Ar(x_{n-1})\|^4}{\|A^2 r(x_{n-1})\|^2}\]
\[\|r(x_n)\|^2 = \|r(x_{n-1})\|^2 - \frac{\|Ar(x_{n-1})\|^4}{\|A^2 r(x_{n-1})\|^2}.\] (1)
We now wish to show that \( \| r(x_n) \|^2 \) approaches zero as \( n \) approaches infinity. An inequality will be created by overestimating the right hand side of equation (1). Since \( A \) is bounded
\[
\| A r(x_{n-1}) \| \geq m \| r(x_{n-1}) \| \quad \text{and} \quad \| A^2 r(x_{n-1}) \| \leq M^2 \| r(x_{n-1}) \|
\]
where \( M \) and \( m \) are the bounds of \( A \). Therefore, substituting into equation (1) we get
\[
\| r(x_n) \|^2 \leq \| r(x_{n-1}) \|^2 - \frac{m^2}{M^2} \| r(x_{n-1}) \|^2 \quad 0 < m \leq M
\]
so that \( 0 < \left( 1 - \frac{m^2}{M^2} \right) < 1 \). By induction
\[
\| r(x_n) \|^2 \leq \| r(x_0) \|^2 \left[ 1 - \left( \frac{m}{M} \right)^2 \right]^n.
\]
Therefore, as \( n \) approaches infinity \( \| r(x_n) \|^2 \) approaches zero.

We have developed sequences to minimize \( H(x) \) and \( \| r(x) \|^2 \). These sequences are also approximate solutions to \( Ax = 0 \). We would now like to see the form of the direction \( z \) and the \( \epsilon \) required to minimize \( \| x_n - x^* \|^2 \). If the \( n^{th} \) step \( x_n \) of an approximate solution is of the form \( x_n = x_{n-1} + \epsilon z \), where \( z \) is a given vector, the \( \epsilon \) which minimizes \( \| x_n - x^* \|^2 \) is determined as follows:
\[
\| x_n - x^* \|^2 = \| w(x_n) \|^2 = \| w(x_{n-1}) \|^2 + \epsilon^2 \| z \|^2 + 2 \epsilon \langle w(x_{n-1}), z \rangle.
\]
\[
\frac{d}{d\epsilon} \| w(x_n) \|^2 = 2\epsilon \| z \|^2 + 2 \langle w(x_{n-1}), z \rangle = 0 \quad \text{so that} \quad \epsilon = -\frac{\langle w(x_{n-1}), z \rangle}{\| z \|^2}.
\]
In general \( \epsilon \) can not be obtained since \( x^* \)
and hence \( w(x_{n-1}) \) are not known. However, if \( z \) is of the form \( Ay \)
where \( y \) is known the \( \epsilon \) becomes

\[
\epsilon = \frac{\langle Aw(x_{n-1}), y \rangle}{||Ay||^2} = -\frac{\langle r(x_{n-1}), y \rangle}{||Ay||^2}
\]

which may be calculated.

It is possible to obtain the "best" \( z \) of the form \( Ay \) in the sense of steepest descent applied to \( ||w(x_n)||^2 \). Let \( ||y|| = 1 \).

\[
\frac{d}{d\epsilon} ||w(x_n)||^2 \bigg|_{\epsilon=0} = 2\epsilon ||Ay||^2 + 2\langle w(x_{n-1}), Ay \rangle \bigg|_{\epsilon=0}
\]

\[
= 2\langle r(x_{n-1}), y \rangle.
\]

Thus \( ||w_n|| \) has steepest descent for \( y = \frac{r(x_{n-1})}{||r(x_{n-1})||} \)
and

\[
x_n = x_{n-1} - \frac{\langle r(x_{n-1}), r(x_{n-1}) \rangle}{||Ar(x_{n-1})||^2} \frac{Ar(x_{n-1})}{||r(x_{n-1})||} = x_{n-1} - \frac{||r(x_{n-1})||^2}{||Ar(x_{n-1})||^2} Ar(x_{n-1}).
\]

As can be seen \( \epsilon \) to minimize \( ||w_n||^2 \) has a form which is different from those for \( H(x) \) and \( ||r(x_n)||^2 \). Since every \( z \) has the form \( Ay \) for some \( y \), we see that the \( \epsilon \)'s previously obtained from consideration of \( H(x) \) and \( ||r(x_n)||^2 \) were not the best possible in the sense of minimizing the actual error \( ||x_n - x^*|| \). We still must show that sequence generated to minimize \( ||x_n - x^*||^2 \) actually converges.
\[
x_n - x^* = x_{n-1} - x^* - \frac{\|A_w(x_{n-1})\|^2}{\|A^2w(x_{n-1})\|_2} A^2w(x_{n-1})
\]

\[
v(x_n) = v(x_{n-1}) - \frac{\|A_w(x_{n-1})\|^2}{\|A^2w(x_{n-1})\|} A^2w(x_{n-1})
\]

\[
\|w(x_n)\|^2 = \langle w(x_n), w(x_n) \rangle
\]

\[
= \|w(x_{n-1})\|^2 - 2 \frac{\|A_w(x_{n-1})\|^2}{\|A^2w(x_{n-1})\|} \langle \frac{A^2w(x_{n-1})}{\|A^2w(x_{n-1})\|} w(x_{n-1}) \rangle
\]

\[
+ \frac{\|A_w(x_{n-1})\|^4}{4 \|A^2w(x_{n-1})\|^2}
\]

\[
\|w(x_n)\|^2 = \|w(x_{n-1})\|^2 - 2 \frac{\|A_w(x_{n-1})\|^2}{\|A^2w(x_{n-1})\|} \|A_w(x_{n-1})\|^2
\]

\[
+ \frac{\|A_w(x_{n-1})\|^4}{2 \|A^2w(x_{n-1})\|}
\]

\[
= \|w(x_{n-1})\|^2 - \frac{\|A_w(x_{n-1})\|^4}{\|A^2w(x_{n-1})\|} \quad (1)
\]

Since \( A \) is bounded,

\[
\|A_w(x_{n-1})\| \geq m \|w(x_{n-1})\| \quad \text{and} \quad \|A^2w(x_{n-1})\| \leq M^2 \|w(x_{n-1})\|
\]

where \( M \) and \( m \) are the bounds of \( A \). Therefore, substituting into
equation (1) we get \[ \| w(x_n) \| ^2 \leq \| w(x_{n-1}) \| ^2 \left[ 1 - \frac{m}{M} \right] \]. Since 0 < m < M 0 < \left( 1 - \frac{m}{M} \right) < 1. By induction

\[ \| w(x_n) \| ^2 \leq \| w(x_0) \| ^2 \left[ 1 - \frac{m}{M} \right]^n \]. As n approaches infinity \[ \| w(x_n) \| ^2 \] approaches zero.

In chapter I we introduced the idea of looking at a different method of choosing a direction which would lead to a more rapid convergence to the solution of \( Ax = \emptyset \). We now wish to expand these ideas. In the process of generating an approximate solution to \( Ax = \emptyset \) Kantorovich uses as his direction \( r(x_n) \) where

\[ r(x_n) = Ax_n - \emptyset = Aw(x_n) \] where \( w(x_n) = x_n - x^* \). If we knew the exact solution \( x^* \) the difference between the exact solution and any approximate solution would be the direction in which to modify the approximate solution to make it approach the exact solution. This direction would be \(-w(x_n)\). Since the exact solution cannot be calculated, \( w(x_n) \) is in general unknown. From the steepest descent approach to minimizing \( H(x) \) Kantorovich arrives at the direction

\[ Aw(x_n) = Ax_n - Ax^* = Ax_n - \emptyset = r(x_n) \] which can be calculated.

\[ Aw(x_n) = r(x_n) \] and \( w(x_n) = A^{-1} r(x_n) \), however \( A^{-1} \) in general cannot be determined directly. If A is symmetric and bounded \( A^{-1} \) may be represented as a Neumann series as given on [p. 266, 6] and is

\[ A^{-1} = I + (I - A) + (I - A)^2 + \ldots \] if \( \| I - A \| < 1 \). We now wish to determine what conditions this restriction places on the bounds of \( A \).
Consider \( \langle (I - A)x, x \rangle \) where \( m \langle x, x \rangle \leq \langle Ax, x \rangle \leq M \langle x, x \rangle \). Then
\[
\langle x, x \rangle - M \langle x, x \rangle \leq \langle (I - A)x, x \rangle \leq \langle x, x \rangle - m \langle x, x \rangle
\]

\[
(l - M) \langle x, x \rangle \leq \langle (I - A)x, x \rangle \leq (1 - m) \langle x, x \rangle.
\]

If in general \( m \langle x, x \rangle < \langle Gx, x \rangle < M \langle x, x \rangle \) the norm of \( G \) is \( \max \{M, m\} \). Then \( \| I - A \| = \max \{|M - 1|, |m - 1|\} \). In order for \( \| I - A \| \) to be less than one, we must have \( |M - 1| < 1 \) and \( |m - 1| < 1 \) so \( \| I - A \| < 1 \) implies \( |M - 1| \) and \( |m - 1| < 1 \) or \( M < 2 \) and \( m < 2 \). If \( A \) should have \( M < 2 \) then \( |M - 1| > 1 \).

But if we multiply \( A \) by \( c < \frac{2}{M} \) then \( \| I - cA \| < 1 \) since \( \| cA \| < 2 \). Call \( cA = A' \) then \( A^{-1} = c^{-1}A' \) = \( cI + c(I - cA) + \ldots \). Regardless of the norm of \( A, A^{-1} \) may be found. As previously mentioned the best direction \( z \) would be \( x^* - x_n \).

\( r(x_n) = A(x_n - x^*) \) so that \( -z = A^{-1} r(x_n) \) should minimize \( H(x) \) and \( \| x(x_n) \|^2 \) more rapidly than the direction obtained from steepest descent.

For convenience when calculating \( z \) we will use only the first two terms of the \( A^{-1} \) series \( A^{-1} = 2I - A \). However, we find that we can improve the speed of convergence of the estimate even more if the \( A^{-1} \) series is expanded about some arbitrary point \( a \) where \( 0 \leq m < a \leq M \). We now must show that such an expansion is possible.

We will use the \( A' = cA \) operator introduced above. Let

\[
A'^{-1} = \left[ aI - (aI - A') \right]^{-1} = \frac{1}{a} \left[ I - \left( I - \frac{A'}{a} \right) \right]^{-1} = \frac{1}{a} \sum_{n=0}^{\infty} 
(I - \frac{A'}{a})^n
\]

where \( a \geq \frac{cM}{2} \), when \( M \) is the upper bound of \( A \), for convergence to be guaranteed. Then
\[ A^{-1} = \frac{1}{a} \sum_{n=0}^{\infty} \left( \frac{aI - A'}{a} \right)^n = \frac{1}{a} I + \frac{aI - A'}{a^2} + \ldots = \frac{2}{a} I - \frac{1}{a^2} A' + \ldots \]

\[ \frac{1}{a^2} (2aI - A') + \ldots \] We may use \(2acI - c^2A\) as our representation for \(A^{-1}\) since the \(\frac{1}{a^2}\) is a constant and will not affect the direction obtained by using \(z_n = (A - 2aI) r(x_n)\).

In the following discussion we will assume \(M < 2\) unless otherwise noted. The recursive relation developed to minimize \(H(x)\)

is: \(x_n = x_{n-1} + \epsilon_{n-1}(A) r(x_{n-1})\) where \(\epsilon_{n-1}(A) = -\frac{\langle z_{n-1}', r(x_{n-1}) \rangle}{\langle A z_{n-1}', z_{n-1} \rangle}\).

The recursive relation for the minimization of

\[ ||Ax_n - Ax^*||^2 = \langle r(x_n) \rangle^2 \] is \(x_n = x_{n-1} + \epsilon_{n-1}(B) z_{n-1}'\)

\[ = x_{n-1} + \epsilon_{n-1} Ar(x_{n-1}) \]

where \(\epsilon_{n-1}(B) = -\frac{\langle Az_{n-1}', Ax_{n-1} - \phi \rangle}{||Az_{n-1}||^2}\). The form of

\(z_n = A^{-1} r(x_n)\) considering only the zero and first order terms is

\(z_n = - A^{-1} r(x_n) = -2r(x_n) + Ar(x_n)\).

The new recursive relations are:

1. To minimize \(H(x)\)

\(x_n = x_{n-1} - \frac{\langle (Ax_{n-1} - \phi), (-2ar(x_{n-1}) + Ar(x_{n-1})) \rangle (-2ar(x_{n-1}) + Ar(x_{n-1}))}{\langle A(-2ar(x_{n-1}) + Ar(x_{n-1})), (-2ar(x_{n-1}) + Ar(x_{n-1})) \rangle}\)
2. To minimize $||r_n||^2$

$$x_n = x_{n-1} - \frac{\langle A(-2ar(x_{n-1}) + Ar(x_{n-1}))\rangle r(x_{n-1})}{||A(-2ar(x_{n-1}) + Ar(x_{n-1}))||^2}$$

Another method of generating a solution to $Ax = \phi$ is to consider $x = A^{-1}\phi$ and expand $A^{-1}$ as a Neumann series

$$A^{-1} = I + (I - A) + (I - A)^2 + .......$$

Then

$$x = \phi + (I - A)\phi + (I - A)^2\phi + .......$$

For convenience we will terminate the series after the second order term $x = 3\phi - 3A\phi + A^2\phi$.

We would now like to compare these approximate solutions to $Ax = \phi$ with each other and with the original Kantorovich relation

$$x_n = x_{n-1} + \epsilon_n r(x_{n-1})$$

For this illustration let $x_o = \phi$. Then $r(x_o) = A\phi - \phi$. Using the Kantorovich algorithm for minimizing $H(x)$, where $z_o = r(x_o)$, we have $z_o = A\phi - \phi$. Then

$$x_1 = \phi + \epsilon_o (A\phi - \phi) = (1 - \epsilon_o)\phi + \epsilon_o A\phi.$$  Then

$$r(x_1) = (1 - \epsilon_o)A\phi + \epsilon_o A^2\phi - \phi$$

and $x_2 = x_1 + \epsilon_1 r(x_1)$ so

$$x_2 = [(1 - \epsilon_o) - \epsilon_1]\phi + [\epsilon_o + \epsilon_1(1 - \epsilon_o)] A\phi + \epsilon_o \epsilon_1 A^2\phi.$$  

After two steps using the Kantorovich algorithm, $x_2$ has the same form as the first three terms of the Neumann series. However, the $\epsilon$'s give an optimum distance to travel in the direction $z$ at each step so that the $x_2$ generated should be a better approximation of $x^*$ than the first three terms of the Neumann series which would result if $\epsilon_o = \epsilon_1 = -1$ in the Kantorovich algorithm.

We next consider $z_o = -2r + Ar$ so that $z_o = A^2\phi - 3A\phi + 2\phi$.

Then $x_1 = [1 + 2\epsilon_o]\phi - 3\epsilon_o A\phi + \epsilon_o A^2\phi$. After one step using the new algorithm, listed as relation (1) on page 21, we have the same form
as obtained using two steps of the original Kantorovich algorithm.

However, we do not have quite as much adjustment on the constants as before. If \( \epsilon_o = 1 \) then the \( x_1 \) given above would be the same as the first three terms of the Neumann series.

We have introduced the idea of using a Neumann series to determine a better direction in the sense of minimizing a particular functional. In this way new algorithms for the \( \{x_n\} \) to minimize \( H(x) \) and \( ||r(x_n)||^2 \) are obtained. We must now show that these actually converge to the exact solution \( x^* \).

The technique of proof follows the method used by Kantorovich to prove a similar convergence theorem \([2]\). Let \( \gamma = \max |x - a| \) on \([m, M]\). Assume \( a > \gamma \), that is \( a > \frac{M}{2} \).

Theorem: The sequences \( \{x_n\} \) resulting from the minimization of \( H(x) \) and \( ||r(x_n)||^2 \) are convergent to the element \( x^* \). The speed of convergence is indicated by the inequality

\[
||x_n - x^*|| \leq K^n \frac{||r(x_0)||}{m} \quad \text{for both } H(x) \text{ and } ||r(x_n)||^2
\]

where

\[
K = \frac{1}{2(a/\gamma)^2 - 1} \quad \text{and} \quad K \leq \left( \frac{M - m}{M + m} \right)^2 \text{ for } a = \frac{M + m}{2}.
\]

Proof: Consider \( Ax = \phi \), this equation may be transformed to

\[
x = x - kA\left[2aI - A\right]x + k\left[2aI - A\right]\phi.
\]

Let \( \left[2aI - A\right] = S \), then \( x = x - kASx + kS\phi \). Now call \( \left[I - kAS\right] = T \) so that \( x = Tx - kS\phi \) where \( k \) is to be determined. As previously discussed \( z_n = (A - 2aI)\ r(x_n) \) is the general form of the direction which will be used to minimize the functional \( H(x) \) and \( ||r(x_n)||^2 \).
\[ T \text{ is defined as } T = [I - kAS], \quad T = I - kA(2aI - A) \quad k > 0.\]

Define \( T = v(A) \) where \( V(x) = 1 - k(2a - x)x \) with \( x \in [m, M] \).

\[ V(x) = 1 - ka^2 + k(x - a)^2, \quad 1 - ka^2 \leq v(x) = 1 - ka^2 + k(x - a)^2. \]

Then \( 1 - ka^2 \leq v(x) \leq 1 - ka^2 + k\gamma^2 = 1 - k(a^2 - \gamma^2). \) To minimize \( \max |v(x)| \) choose \( k \) so that \( ka^2 - 1 = 1 - ka^2 + k\gamma^2. \) Then \( k = \frac{2}{2a^2 - \gamma^2} \), \( ||T|| \leq ka^2 - 1 \) by the discussion [p. 161, 3].

\[ ||T|| \leq \frac{2a^2}{2a^2 - \gamma^2} - 1 = \frac{\gamma^2}{2a^2 - \gamma^2} = \frac{1}{2\left(\frac{a}{\gamma}\right)^2 - 1}. \]

If \( a = \frac{M + m}{2} \) and \( \gamma = \frac{M - m}{2} \) then

\[ ||T|| \leq \frac{1}{2\left(\frac{M + m}{M - m}\right)^2 - 1} = \frac{(M - m)^2}{2(M + m)^2 - (M - m)^2} \]

\[ = \frac{(M - m)^2}{2M^2 + 4mM + 2m^2 - M^2 + 2Mm - m^2} \]

\[ = \frac{(M - m)^2}{M^2 + 6Mm + m^2} \]

\[ = \frac{(M - m)^2}{(M + m)^2 + 4mM} \leq \left[\frac{(M - m)^2}{(M + m)}\right]^2. \]

We must note here that our choice of "a" gives an improved inequality. However, since \( a > \gamma \) always, then \( ||T|| \leq \frac{1}{2\left(\frac{a}{\gamma}\right)^2 - 1} \) so that

\[ ||T|| < 1 \] regardless of the choice of \( a \) and \( \gamma \), \( ||T|| < 1 \) insuring convergence of the later inequalities.
Beginning with an initial guess $x_0$ we obtain $e_0 = (A - 2aI) r(x_0)$. Then $x_1 = x_0 + e_0 z_0$ to that

$$x_1 - x^* = x_0 - x^* + e_0 (A - 2aI) A(x_0 - x^*) = \left[ I + e_0 (A - 2aI) A \right] (x_0 - x^*)$$

Next define

$$\tilde{x}_1 = T x_0 + k S \phi ; \tilde{x}_1 = x_0 - k a S x_0 + k S \phi$$

$$\tilde{x}_1 = x_0 - k S [A x_0 - \phi] = x_0 - k S r(x_0)$$

Using a direction $S r(x_0)$ and our first guess $x_0$ we get $x_1 = x_0 + e S r(x_0)$, where the appropriate $e_0$ gives the best $x_1$ approximation for the $S r(x_0)$ direction in the sense of minimizing $H(x)$ or $\| r(x_n) \|^2$.

We will first examine the sequence $\{x_n\}$ which minimizes $H(x)$. We have developed approximations of the form $x_n = x_{n-1} + e_{n-1} z_{n-1}$ to minimize the functional $H(x) = \langle Ax, x \rangle - 2 \langle Ax^*, x \rangle$ which equals

$$H(x) = \langle A(x - x^*), (x - x^*) \rangle - \langle Ax^*, Ax^* \rangle.$$  

$A$ is positive definite, symmetric and bounded so, by \( [p. 265, 6] \) we can introduce the operator $V = A^{\frac{1}{2}}$. Then $H(x) = \langle V(x - x^*), V(x - x^*) \rangle - \langle Vx^*, Vx^* \rangle$

$$= \| V(x - x^2) \|^2 - \| Vx^* \|^2.$$  

Since $x_1 = x_0 + e_0 z_0$ gives the best $x_1$ for a given $x$, in the sense of minimizing $H(x)$, $H(\tilde{x}_1) \geq H(x_1)$ and $H(x_1) - H(x^*) \leq H(\tilde{x}_1) - H(x^*)$. Using the above equation for $H(x)$ in terms of $V$ we write

$$\| V(x_1 - x^*) \|^2 - \| Vx^* \|^2 \leq \| V(\tilde{x}_1 - x^*) \|^2 - \| Vx^* \|^2$$

so that

$$\| V(x_1 - x^*) \|^2 \leq \| V(\tilde{x}_1 - x^*) \|^2.$$  

Since $\tilde{x}_1 = T x_0 + k S \phi$ we write
\[ x^* = T x^* + kS \phi. \] Subtracting \( x_1 - x^* = T x_0 + kS \phi - T x^* - kS \phi = T(x_0 - x^*). \) Operating on both sides with \( V \) and commuting \( T \) and \( V; \)
\[ V(x_1 - x^*) = TV(x_0 - x^*). \] Then \[ ||V(x_1 - x^*)|| \leq ||T|| ||V(x_0 - x^*)|| \leq K ||V(x_0 - x^*)||. \]

In general we may define \( \tilde{x}_n = Tx_{n-1} + kS \phi = x_{n-1} - kS r(x_{n-1}), \) but \( \tilde{x}_n \) is at best only as good an estimate as \( x_n \) for the given direction \( Sr(x_{n-1}). \) Therefore, \( H(\tilde{x}_m) \geq H(x_n). \) Using the above arguments \[ ||V(x_n - x^*)|| \leq K ||V(x_{n-1} - x^*)||. \] Knowing that \[ ||V(x_{n-1} - x^*)|| \leq K ||V(x_{n-2} - x^*)||, \] \[ K ||V(x_{n-3} - x^*)|| \ldots \text{ etc. we write} \ ||V(x_n - x^*)|| \leq K^n ||(V x_0 - x^*)||. \]

Using \( [p. 266, 6] \) and since \( \sqrt{m} \leq ||V|| \leq \sqrt{M}, \)
\[ ||V^{-1}|| \leq \frac{1}{\sqrt{m}}. \] \[ ||x_n - x^*|| = ||V^{-1}V(x_n - x^*)|| \leq ||V^{-1}|| K^n ||V(x_0 - x^*)||. \]
\[ ||V^{-1}|| K^n ||V(x_0 - x^*)|| \leq \frac{1}{\sqrt{m}} K^n ||V(x_0 - x^*)||. \]
\[ ||V^{-1}A(x_0 - x^*)|| \leq ||V^{-1}|| ||A(x_0 - x^*)|| = \frac{1}{\sqrt{m}} ||r(x_0)||. \] Then \[ ||x_n - x^*|| \leq K^n ||r(x_0)||/m \] where \( n = 1, 2, \ldots. \)

Now for \( ||r(x_n)||^2 \) use \( ||r(x_n)||^2 = ||A(x_n - x^*)||^2. \) Since \( x_1 = x_0 + \epsilon_0 z_0 \) gives the best \( x_1 \) for a given \( z, \) in the sense of minimizing \( ||r(x_n)||^2, ||\tilde{r}(x_1)||^2 \geq ||r(x_1)||^2 \) so \( ||A(\tilde{x}_1 - x^*)||^2 \geq ||A(x_1 - x^*)||^2. \)
\( \tilde{x}_1 = T x_0 + kS \phi. \) so we write \( x^* = T x^* + kS \phi. \)

Subtracting \( \tilde{x}_1 - x^* = T x_0 + kS \phi - T x^* - kS \phi \) we write \( \tilde{x}_1 - x^* = T(x_0 - x^*). \)
Operating on both sides with $A$ and commuting $A$ and $T$ we get
\[ A(x_1 - x^*) = TA(x_0 - x^*). \]
\[ \|A(x_1 - x^*)\| \leq \|T\| \|A(x_0 - x^*)\| \leq K^2 \|A(x_0 - x^*)\|. \]
But \[ \|A(x_1 - x^*)\| \leq \|A(x_1 - x^*)\| \text{ then} \]
\[ \|A(x_1 - x^*)\| \leq K^2 \|A(x_0 - x^*)\|. \]

In general we define $\tilde{x}_n = T\tilde{x}_{n-1} + kS\phi = x_{n-1} - kS\phi(x_{n-1})$. But $\tilde{x}_n$ is at best only as good as estimate as $x_n$ for the given direction $Sr(x_{n-1})$. Therefore, \[ \|r(\tilde{x}_n)\|^2 \geq \|r(x_n)\|^2. \] Using the above arguments \[ \|A(\tilde{x}_n - x^*)\| \leq K \|A(x_{n-1})x^*\|. \] Knowing that
\[ \|A(x_{n-1} - x^*)\| \leq K \|A(x_{n-2} - x^*)\|, \|A(x_{n-2} - x^*)\| \leq K \|A(x_{n-3} - x^*)\|, \text{ etc. we write} \]
\[ \|A(x_n - x^*)\| \leq K^n \|A(x_0 - x^*)\|. \]

Using [p. 266, 6] and if \( m \leq \|A\| \leq M \) then \[ \|A^{-1}\| \leq \frac{1}{m} \], then
\[ \|x_n - x^*\| = \|A^{-1}A(x_n - x^*)\| \leq \|A^{-1}\| K^n \|A(x_0 - x^*)\| = \frac{K^n \|r(x_0)\|}{m} \text{ where } n = 1, 2, \ldots. \]

We would now like to put together several ideas which have been introduced and write general expressions for the sequence \( \{x_n\} \), elements of which are in general given by $x_n = x_{n-1} + \varepsilon n^{-1}x_{n-1}$. Our direction $z = (A - 2I)r(x)$ in the most general case where the upper bound of $A$ is greater than 2 is given by $z = (c^2 A - 2caI)r(x)$ based on using the first two terms of the Neumann series and $c < \frac{2}{M}$. The $\varepsilon$ to minimize $H(x)$ is given by
\[ \epsilon = -\frac{\langle z, r \rangle}{\langle Az, z \rangle} . \]

Then to minimize \( H(x) \),

\[
x_n = x_{n-1} - \frac{\langle c^2\text{Ar}(x_{n-1}) - 2c\text{Ar}(x_{n-1}), r(x_{n-1}) \rangle}{\langle c^2 A^2 r(x_{n-1}) - 2c\text{Ar}(x_{n-1}), c^2\text{Ar}(x_{n-1}) - 2c\text{Ir}(x_{n-1}) \rangle} \]

The \( \epsilon \) to minimize \( \|r(x)\|^2 \) is given by \( \epsilon = -\frac{\langle \text{Ar}(x), z \rangle}{\|Az\|^2} \) so that

\[
x_n = x_{n-1} - \frac{\langle \text{Ar}(x_{n-1}), c^2\text{Ar}(x_{n-1}) - 2c\text{Ir}(x_{n-1}) \rangle}{\|c^2 A^2 r(x_{n-1}) - 2c\text{Ar}(x_{n-1})\|^2} \]

As a summary of chapter II the following review of some of the characteristics of the different methods of generating approximate solutions to \( Ax = \emptyset \), where \( A \) is a bounded operator, allows comparison of these characteristics.

**Characteristics**

**Direction** \( z \)

**\( \epsilon \)** Overestimate of \( \|x_n - x^*\| \)

indicating speed of convergence

For minimizing \( H(x) \)

\[
r(x) \quad x_n - x^* \leq \frac{\|z_0\|}{m} \left( \frac{M - m}{M + m} \right)^n
\]

\[
- \frac{\langle r(x), r(x) \rangle}{\langle \text{Ar}(x), r(x) \rangle}
\]
Direction \( z \) 

Overestimate of \( \| x_n - x^* \| \)

indicating speed of convergence

For minimizing \( \| r(x) \|^2 \)

\[
Ar(x) \\
x_n - x^* \leq \frac{\| r(x_0) \|}{m} \left[ 1 - \left( \frac{m}{M} \right)^{\frac{n}{2}} \right]
\]

\[
- \frac{\langle Ar(x), Ar(x) \rangle}{\langle A^2 r(x), A^2 r(x) \rangle}
\]

For minimizing \( \| v(x) \|^2 \)

\[
Ar(x) \\
\| x_n - x^* \| \leq \| v(x_0) \| \left[ 1 - \left( \frac{m}{M} \right)^{\frac{n}{2}} \right]
\]

\[
- \frac{\langle r(x), r(x) \rangle}{\langle A^2 r(x), A^2 r(x) \rangle}
\]

For minimizing \( H(x) \)

\[
(c^2 A - 2acI)r(x) \\
\| x_n - x^* \| \leq \frac{c \| r(x_0) \|}{m} \kappa^n
\]

\[
- \frac{\langle c^2 Ar(x_{n-1}) - 2caIr(x_{n-1}), r(x_{n-1}) \rangle}{\langle c^2 A^2 r(x_{n-1}) - 2caAr(x_{n-1}), c^2 Ar(x_{n-1}) - 2caIr(x_{n-1}) \rangle}
\]
Direction $z$ is not overestimated indicating speed of convergence

For minimizing $||r(x)||^2$

\[
(c^2A - 2acI)r(x)
\]

\[
||x_n - x^*|| \leq \frac{c ||r(x_0)||}{m} k^n
\]

\[
\frac{\langle Ar(x_{n-1}), c^2Ar(x_{n-1}) - 2caIr(x_{n-1}) \rangle}{||c^2A^2r(x_{n-1}) - 2caAr(x_{n-1})||^2}
\]

where $K = \frac{1}{2(\frac{a}{\gamma})^2 - 1}$

\[
\gamma = \max |x - a| \text{ on } [m,M]
\]

and $K \leq \left(\frac{M - m}{M + m}\right)^2$ for $a = \frac{M + m}{2}$.
CHAPTER III
UNBOUNDED OPERATOR THEORY

In chapter I we showed that beginning with an unbounded operator $A$, a related bounded operator $A'$ defined over $\Omega_B$, could be developed. The results of chapter II can be applied to such an operator.

In general we are not guaranteed that $Ax = \emptyset$ has a solution. If, however, $Ax = \emptyset$ does have a solution then the solution of $A'x = \emptyset$, will also be a solution of $Ax = \emptyset$. In this sense we will develop a generalized solution to $Ax = \emptyset$.

With this introduction we would now like to apply the theory that has been developed to the problem of solving a second order differential equation. In setting up the problem several conditions which we will impose on the problem must be discussed. First, we will be considering only self-adjoint differential operators. On [p. 37, 1] the proof of the fact that any second order differential equation may be made self adjoint, is given. Therefore, the self-adjoint requirement is actually no restriction. Secondly, we consider only problems with zero end conditions.

To determine the effect of the above condition consider an equation which does not have zero end points. If we have an equation of the form $Ax = \emptyset$ with $x(a) = a$ and $x(b) = b$ defined in $L_2$ we can easily find a twice continuously differentiable function $y$ defined in $L_2$ such that $y(a) = a$ and $y(b) = b$. Then $Ay = \xi$. Now the equation $A(x - y) = Ax - Ay = \emptyset - \xi$ with $x(a) - y(a) = 0$
and \( x(b - y(b)) = 0 \). Call \( x - y = z \) and \( \phi - \xi = \gamma \), so that we are solving \( Az = \gamma \) where \( z(a) = z(b) = 0 \). \( Az = \gamma \) is a zero end point equation and may be solved using the techniques herein. Since \( y \) is a known function the solution \( x \) of \( Ax = \phi \) is given by \( x = z + y \).

Therefore, any second order differential equation with constant nonzero end points may be transformed to a zero end point problem.

We will now examine a general second order self-adjoint differential equation with zero end conditions.

\[ Lx(t) = \frac{d}{dt}\left(p(t) \frac{dx}{dt}\right) - q(t)x(t) = \phi(t); \quad x(a) = x(b) = 0 \quad \text{and} \quad p(t) > 0 \]

is continuously differentiable, \( q(t) \geq \phi \) and \( \phi(t) \) are continuous. Call \( Ax(t) = -Lx(t) \). We now must choose a \( B \) which is bounded below and for which \( A \) is bounded in the sense of the \( B \) norm, or \( B \) bounded.

Choose \( B = -\frac{d^2}{dt^2} \) which has domain in \( L^2([a,b]) \) and in particular in \( \Omega_0 \), the space of twice continuously differentiable functions zero at the end points. The scalar product of \( x(t) \) and \( y(t) \) is given by \( \int_a^b x(t)y(t)dt \). The following inequality [p. 129, 4]

\[
\int_a^b x^2(t)dt \leq \frac{(b - a)^2}{2} \int_a^b x'^2(t)dt
\]

is for functions which are zero at \( a \) and \( b \). \( Bx = -x'' \) so that

\[
\langle Bx, x \rangle = -\int_a^b x''x dt = -x \bigg|_a^b - \int_a^b x'^2 dt + \int_a^b x'^2 dt = N \langle x, x \rangle
\]

so that \( B \) is bounded below.
\[
\langle Ax, x \rangle = \int_a^b q(t)(x(t))^2 dt - \int_a^b \frac{d}{dt} (p(t)x'(t))x dt
\]

\[
= \int_a^b q(t)x^2(t) dt - p(t)x'(t)x(t) \bigg|_a^b
\]

\[
+ \int_a^b p(t)(x'(t))^2 dt
\]

\[
\int_a^b p(t)(x'(t))^2 dt \leq \int_a^b q(t)(x(t))^2 dt + \int_a^b p(t)(x'(t))^2 dt
\]

\[
\leq E \frac{b-a}{2} \int_a^b (x'(t))^2 dt + \int_a^b p(t)(x'(t))^2 dt
\]

\[
= \int_a^b \left[ \frac{(b-a)^2}{2} q(t) + p(t) \right] (x'(t))^2 dt,
\]

where \( E \) is \( \max q(t) \), so that \( \alpha \langle Bx, x \rangle \leq \langle Ax, x \rangle \leq \beta \langle Bx, x \rangle \)

where \( \alpha \leq \min_p \frac{p(t)}{t} \) and \( \beta \leq \max_p \frac{p(t)}{t} + \frac{E}{N} \).

We will now discuss briefly the extensions introduced in chapter I as they apply to this particular problem.

First the completion of \( \Omega_o \) will be summarized. The completion process indicates that an \( f_n \in \Omega_o \) converges to some element in the completed space, which we call \( \Omega_B \), in the sense of \( L^2 \) mean convergence. This result leads to the conclusion that Cauchy sequences \( f_n \in \Omega_o \) converges uniformly to elements \( f \in \Omega_B \). The functions in \( \Omega_B \) are still zero at the end points but are not necessarily differentiable.

Next we consider \( B = -\frac{d^2}{dt^2} \). \( B \) maps \( \Omega_o \) into \( L^2 \). The \( B^{-1} \)

necessary to map members of \( L^2 \) into \( \Omega_o \) is given
\[ B^{-1} = - \int_a^t \int_a^x z \, d\xi \, dx + \frac{t-a}{b-a} \int_a^b \int_a^x z \, d\xi \, dx. \]

\[ B^{-1} \] may be extended to \( D \), which maps \( L_2 \) into a subset of \( \Omega_B \) by observing that the double integrals exist for members of \( L_2 \) and that the members of \( \Omega_B \) mapped onto have at least one derivative and zero end points. Therefore, \( D \), the extension of \( B^{-1} \), maps \( L_2 \) onto \( \Omega^*_B \), a space of at least once differentiable functions with zero end points.

We next look at the operator \( D A \).

\[ D Az = + \int_a^t p(x) \frac{dz}{dx} \, dx - \int_a^b p(x) \frac{dz}{dx} \, dx \]

\[ - \int_a^t \int_a^x q(\xi)z(\xi) \, d\xi \, dx + \frac{t-a}{b-a} \int_a^b \int_a^x q(\xi)z(\xi) \, d\xi \, dx. (\ast) \]

Examination of \( D A \) shows that \( D A \) maps \( \Omega_o \) back onto \( \Omega^*_B \). We now show that \( D A \) can be extended to \( A' \) on \( \Omega_B \). Consider

\[ D Az = + p(t)z(t) - \int_a^t z(x) \frac{dp(x)}{dx} \, dx + \frac{t-a}{b-a} \int_a^b z(x) \frac{dp(x)}{dx} \, dx \]

\[ - \int_a^t \int_a^x q(\xi)z(\xi) \, d\xi \, dx + \frac{t-a}{b-a} \int_a^t \int_a^x q(\xi)x(\xi) \, d\xi \, dx. \]

There are now no \( \frac{dz}{dx} \) terms involved so that \( D Az \) can be extended to \( A' \) on \( \Omega_B \).

A final observation relative to the extensions is that when operating in an extended space the extended \( B \) or \( D^{-1} \) will be used to define the so called \( B \) scalar product or \( [ , ] \). In \( \Omega_o [ , ] = \langle B , \rangle \) but on \( \Omega^*_B [ , ] = \langle D^{-1} , \rangle \).
We must now find the expression for the extension of \( \langle Bx, z \rangle \).

\[
\langle Bx, z \rangle = -\int_{t_0}^{t_1} x''z \, dx = \int_{t_0}^{t_1} x'z' \, dt,
\]
since the functions have zero end points, so that \( \langle Bx, z \rangle = \int_{t_0}^{t_1} x'z' \, dt \) may be extended to

\[
\langle D^{-1}x, z \rangle = \int_{t_0}^{t_1} x'z' \, dt \text{ since } D^{-1} \text{ is defined on } \Omega_B^x \text{ the space of at least once differentiable functions which have zero end points.}
\]

In many of the problems of the form \( Ax = \phi \), the problems of most interest have \( \phi \) continuous and also the choices of \( x \) are continuous. When generating approximate solutions to

\[
-\frac{d}{dt}\left( p(t) \frac{dx}{dt} \right) + q(t)x(t) = -\phi(t) \text{ if } \phi(t), q(t), p(t) \text{ and the first guess } x_0(t) \text{ are all continuous, with } p(t) \text{ being continuously differentiable and } x_0(t) \in \Omega_o \text{, then referring to equation } \star \text{ on page } 34 \text{ we see by inspection that } DAX_0 \in \Omega_o. \text{ Since } \phi \text{ is continuous}
\]

\[
D\phi = \phi_1 \in \Omega_o. \text{ Therefore, the residual } r(x_0) = DAX_0 + D\phi \in \Omega_o. \text{ In all the algorithms developed the direction is equal to } r(x_n) \text{ or}
\]

\[
A'r(x_n) \in \Omega_o \text{ for continuous } x_0 \in \Omega_o \text{ and } \phi. \text{ Then}
\]

\[
x_1 = x_0 + \epsilon_o r(x_0) \text{ will belong to } \Omega_o. \text{ Repeated application of the steepest descent process shows that all } x_n \in \Omega_o.
\]

The form of \( r(x_0) \) given above is obtained by considering

\[
hx = \phi, \quad -hx = -\phi \quad \text{so } Ax = -\phi. \text{ To obtain a bounded form of } Ax = -\phi \text{ consider } DAX = -D\phi \text{ where } DA \text{ is bounded in the sense of a } D \text{ metric. Then for an } x_0 \text{ guess } DAX_0 + D\phi = r(x_0). \text{ We will now look at the algorithm for the minimization } H(x) \text{ using the direction obtained through steepest descent. The } \epsilon = -\frac{\left[ r(x_{n-1}), r(x_{n-1}) \right]}{\left[ B^{-1}A r(x_{n-1}), r(x_{n-1}) \right]}
Since we are operating in $\Omega_c$, $B^{-1} = D$ so that $B$ and $B^{-1}$ will be used instead of $D^{-1}$ in this example. The numerator

$$\begin{aligned}
[r(r_{n-1}), r(x_{n-1})] & = \langle Br(x_{n-1}), r(x_{n-1}) \rangle \\
& = \int_a^b \frac{d^2}{dt^2} r(x_{n-1}) \frac{d}{dt} r(x_{n-1}) dt = \int_a^b \frac{dr(t)}{dt} (x_{n-1})^2 
\end{aligned}$$

and the denominator

$$\begin{aligned}
\left[ B^{-1}Ar(x_{n-1}), r_{n-1} \right] & = \langle BB^{-1}Ar(x_{n-1}), r(x_{n-1}) \rangle \\
& = \langle Ar(x_{n-1}), r(x_{n-1}) \rangle 
\end{aligned}$$

so that

$$\epsilon = -\frac{\int_a^b \frac{d}{dt} r(x_{n-1}) dt}{\int_a^b Ar(x_{n-1}) r(x_{n-1}) dt}$$

$$\begin{aligned}
&= -\frac{\int_a^b \frac{d}{dt} r(x_{n-1}) dt}{\int_a^b p(t) \left[ \frac{dr(x_{n-1})}{dt} \right]^2 dt + \int_a^b q(t) \left[ r(x_{n-1}) \right]^2 dt} 
\end{aligned}$$

then

$$\begin{aligned}
x_n(t) = x_{n-1}(t) - \frac{\int_a^b \frac{d}{dt} r(x_{n-1}) dt}{\int_a^b p(t) \left[ r(x_{n-1}) \right]^2 dt + \int_a^b q(t) \left[ r(x_{n-1}) \right]^2 dt} \frac{r(x_{n-1})}{r(x_{n-1})}
\end{aligned}$$
This algorithm is identical to that derived by Kantorovich \([7]\). Any other algorithm developed in chapter II may similarly be applied to the equation \(Ax = -\emptyset\) and the extended operators shown to be defined on \(\Omega_0\) for this particular equation. Therefore, for all the algorithm considered each estimate \(x_n \in \Omega_0\) and expressions similar to those above can be developed.


VITA

William Tull Suit

The author was born on July 17, 1937, in Charlottesville, Virginia. He obtained his elementary and secondary schooling in the Augusta County, Virginia public schools. He was graduated from Virginia Polytechnic Institute at Blacksburg, Virginia, in June of 1959, with a Bachelor of Science degree in Physics. Since graduation he has been employed by the Langley Research Center of the National Aeronautics and Space Administration, located at Hampton, Virginia.