Supplementary Material

APPENDIX

Mathematical bounds of solutions

Here we show that solutions of the system of differential equations for juveniles J, adults A, dead oyster shell R and sediment S are bounded. The proof is based on two results of differential equations. First, if $u(t)$ satisfies $u' \leq u(D - Bu) + C$ with $u(0) = u_0 > 0$ and $B, C, D > 0$, then $\limsup\limits_{t\to\infty} u(t) \leq (D + \sqrt{D^2 + 4BC})/(2B)$, which is the unique positive equilibrium of the equation $u' = u(D - Bu) + C$. Similarly if $v(t)$ satisfies a linear inequality $v' \leq E - Fv$ with $v(0) = v_0$ and $E, F > 0$, then $\limsup\limits_{t\to\infty} v(t) \leq \limsup\limits_{t\to\infty}[E/F + (v_0 - E/F)e^{-Ft}] = E/F$.

For the juvenile oyster equation, from $L(A, R) \leq L_0$ and $f(d_J) \leq 1$, we get $J' \leq PL_0 - J$ and thus $\limsup\limits_{t\to\infty} J(t) \leq PL_0$. Secondly, for the live adult oyster equation, $\limsup\limits_{t\to\infty} \alpha J(t) \leq \alpha PL_0$, and the logistic growth term $\phi Af(d_A)(1-A/K) \leq A(\phi - \phi A/K)$, so asymptotically $A' \leq \alpha PL_0 + A(\phi - \phi A/K)$. Thus $\limsup\limits_{t\to\infty} A(t) \leq A_\infty = (K\phi + \sqrt{K^2\phi^2 + 4K\phi\alpha PL_0})/(2\phi)$. Next for the dead oyster shell equation, since $0 \leq f(d_A) \leq 1$, $R' \leq (\mu + \epsilon)A - \gamma R$, we have $\limsup\limits_{t\to\infty} R(t) \leq (\mu + \epsilon)A_\infty/\gamma$. Finally for the sediment equation, both g and $\exp[-F(Cg)A/Cg]$ are bounded above by 1, so $S' \leq C - \beta S$. Thus $\limsup\limits_{t\to\infty} S(t) \leq C/\beta$.

In summary, for any initial conditions $(J_0, A_0, R_0, S_0) \in \mathbb{R}^4_+$, none of the variables juvenile oysters, adult oysters, dead oyster shell, or sediment can grow indefinitely. One can also see that once the initial condition is non-negative, the solution stays non-negative. Hence the region $\Gamma = \{(J, A, R, S) : 0 \leq J \leq PL_0, 0 \leq A \leq A_\infty, 0 \leq R \leq (\mu + \epsilon)A_\infty/\gamma, 0 \leq S \leq C/\beta\}$ is positively invariant for the system of (J, A, R, S). In particular, all equilibria of the system lie in the region Γ.

Stability of extinction equilibrium

The equilibrium points of our model satisfy

$$0 = PL(A,R)f(d_J) - J, \tag{S1}$$

$$0 = \alpha J + \phi A f(d_A)\left(1 - \frac{A}{K}\right) - \mu f(d_A)A - \epsilon(1 - f(d_A))A, \tag{S2}$$

$$0 = \mu f(d_A)A + \epsilon(1 - f(d_A))A - \gamma R, \tag{S3}$$

$$0 = -\beta S + Cg \exp\left(-\frac{F(Cg)A}{Cg}\right). \tag{S4}$$

Linearizing (S1)-(S4) at the trivial equilibrium $E_0 = (0,0,0,C/\beta)$, we obtain the Jacobian matrix to be

$$\mathscr{J}(0,0,0,C/\beta) = \begin{pmatrix} -1 & Pf(-C/\beta)/(\psi\zeta) & 0 & 0 \\ \alpha & f(-C/\beta)(\phi - \mu + \epsilon) - \epsilon & 0 & 0 \\ 0 & f(-C/\beta)(\mu - \epsilon) + \epsilon & -\gamma & 0 \\ 0 & Cg'(0) - F(C) & Cg'(0) & -\beta \end{pmatrix}. \tag{S5}$$

We can see easily from the Jacobian that two of the four eigenvalues are $-\gamma$ and $-\beta$. To determine the stability, we need to further analyze the top left submatrix

$$L_{\mathscr{J}} = \begin{pmatrix} -1 & Pf(-C/\beta)/(\psi\zeta) \\ \alpha & f(-C/\beta)(\phi - \mu + \epsilon) - \epsilon \end{pmatrix}. \tag{S6}$$

The extinction equilibrium $(0,0,0,C/\beta)$ is locally asymptotically stable if the trace of $L_{\mathscr{J}}$ is negative ($Tr(L_{\mathscr{J}}) < 0$) and the determinant positive ($Det(L_{\mathscr{J}}) > 0$), where

$$Tr(L_{\mathscr{J}}) = -1 + f(-C/\beta)(\phi - \mu + \epsilon) - \epsilon, \tag{S7}$$

$$Det(L_{\mathscr{J}}) = -f(-C/\beta)[\phi - \mu + \epsilon + \alpha P/(\psi\zeta)] + \epsilon. \tag{S8}$$

We first argue that $Tr(L_{\mathscr{J}}) < 0$ parameters of interest. For equation (S7), as long as the maximum sediment deposition rate C remains positive, $f(-C/\beta) < f(0) = 1/2$, and $-1 + f(-C/\beta)(\phi - \mu + \epsilon) - \epsilon < -1 + (1/2)(\phi - \mu + \epsilon) - \epsilon$. Then one can estimate a bifurcation value $\phi_{bif} > \mu + \epsilon + 2$. Using the parameters in Table 2, we obtain $\phi_{bif} > 3.34$ which is unreasonably high (the one in Table 2 is $\phi = 0.649$). Because realistically

$f(-C/\beta)$ is a lot smaller than $f(0)$ and very close to 0, ϕ_{bif} for the actual trace would be larger and more unrealistic. Thus we can assume $Tr(L_{\mathscr{J}}) < 0$.

We next argue that $Det(L_{\mathscr{J}}) > 0$ parameters of interest.In order for the determinant to be positive, it is equivalent to write $f(-C/\beta)[\phi - \mu + \epsilon + \alpha P/(\psi\zeta)] - \epsilon < 0$. Similarly to above, $f(-C/\beta)[\phi - \mu + \epsilon + \alpha P/(\psi\zeta)] - \epsilon < (1/2)[\phi - \mu + \epsilon + \alpha P/(\psi\zeta)] - \epsilon$. Using the parameters in Table 2 and $P = 2000$, we obtain $\phi_{bif} \approx 1.06$ which is still relatively high comparing to 0.649. Again, we need to remember that $f(-C/\beta)$ is very close to 0, which makes ϕ_{bif} for the actual determinant a lot larger than the value 1.06. In fact, even when we take $C = 0.001$ (20 times smaller than our estimated $C = 0.02$), $\phi_{bif} \approx 7.07$ is still unreasonably high. So we can also safely assume $Det(L_{\mathscr{J}}) > 0$. In both cases, the scenario that the extinction equilibrium is unstable is not realistic, and we can always assume that the extinction equilibrium is locally asymptotically stable.