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# Derivation of an Operator-Based Spatial Noncommutativity Parameter

A thesis submitted in partial fulfillment of the requirement  
for the degree of Bachelor of Science with Honors in  
Physics from the College of William and Mary in Virginia,

by

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Williamsburg, Virginia  
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## Abstract

The idea that the operators defining spacetime could be noncommuting has gained popularity in recent years. The formulation in which the commutators themselves are a set of commuting numbers has been applied to a number of quantum phenomena to determine what effects it might have. Most of the work overall has focused on this set of commuting numbers, but some earlier theories of noncommutative coordinates established the coordinate commutator as having an operator-based matrix value involving the angular momentum operator, and Döplicher et al. obtained a very similar algebra without mention of the angular momentum. Beginning from the relativistic geometry used to derive this result, we hope to explore an alternative foundation for a noncommutative geometry of space and determine its effects on some well-studied quantities in atomic physics.

## **Acknowledgements**

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# 1 Introduction

The commutativity of space-time has been challenged in at least two different contexts in the history of modern physics, and each of these instances resulted in a different interpretation of this noncommutativity. First, the concept of a quantized space-time was already developed and minimally laid out near the end of the 1940s [1], using a system of noncommuting coordinate operators which produced a commutator proportional to the angular momentum operator (this is analogous to the noncommuting directional components of the angular momentum operators themselves in standard quantum mechanics). This form of noncommutativity has only minimally developed beyond these roots [2], although it has largely entered the realm of algebraic topology [3]. The second development of noncommutative quantum mechanics has been one stemming from recent developments in string theory [4], especially contingent on a non-Abelian version of Yang-Mills theory. This theory leads to coordinate commutators given by a set of  $C$ -numbers, or commuting numbers, described by the definition

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \quad (1)$$

The quantity on the right-hand side is a set of ordinary numbers, and so this equation is not Lorentz covariant. Though this Lorentz-violating noncommutative geometry results, this set of numbers  $\theta^{\mu\nu}$  has been widely used to derive possible theories of noncommutativity, partly justified by its derivation from recent string theory results [5]. However, an alternative operator-based noncommutative theory has also developed; thus, one might also be able to study phenomenology using such an operator-based commutator as described by Snyder,

$$[\hat{x}^\mu, \hat{x}^\nu] = ia^2 L^{\mu\nu}. \quad (2)$$

This commutator will significantly increase the complexity of manipulations within any mechanics affected by it, but it will also produce a Lorentz-invariant space-time missing from

other theories. The  $C$ -number theory generates a space with some coordinate transformation definable by the matrix elements  $\theta^{\mu\nu}$ , as well as a momentum dependence [6] which exemplifies Lorentz violation. The operator construction, on the other hand, produces a geometric form known as the “fuzzy sphere” [3], which at large scales reduces to the typical continuous spatial representation, but at very small scales (out of necessity, smaller than that with which we can currently interact), the spatial coordinates become noncommutative and our definition of position becomes weaker. The reliance of this system on the angular momentum operator  $L$  is one possibility which does not reflect a Lorentz violation, as Eqn. 2 is symmetric under Lorentz transformation.

### 1.1 The Hamiltonian of a $C$ -number $\theta^{\mu\nu}$

A significant amount of work has been done to formulate a modified basic quantum mechanics from the  $C$ -number commutator mentioned previously,  $i\theta^{\mu\nu}$ , and the method of developing such a system has been well-documented, especially given a simple  $\frac{1}{r}$  potential [6]. We wish here to give an example of such a system. A typical set of commutators is given,

$$\begin{aligned} [\hat{x}^\mu, \hat{x}^\nu] &= i\theta^{\mu\nu} \\ [\hat{x}^\mu, \hat{p}^\nu] &= i\delta^{\mu\nu} \\ [\hat{p}^\mu, \hat{p}^\nu] &= 0, \end{aligned} \tag{3}$$

which are valid on the same Hilbert spaces as the usual commutative coordinates. To use this to perform meaningful calculations, we can reformulate the Hamiltonian based on these coordinates. Since these coordinates can be mapped onto a Hilbert space which contains their commutative counterparts, a new reference frame can be defined which allows the coordinate operators to commute in the usual way ( $[\hat{x}^\mu, \hat{x}^\nu] = 0$ ):

$$x^\mu = \hat{x}^\mu + \frac{1}{2}\theta^{\mu\nu}\hat{p}^\nu, \quad p^\mu = \hat{p}^\mu. \tag{4}$$

Thus, we can state the Hamiltonian in the noncommutative space and transform it using the above definition into a commuting coordinate system, converting the antisymmetric matrix  $\theta^{\mu\nu}$  into a vector  $\theta_i \equiv \epsilon_{ijk}\theta_{jk}$ :

$$\begin{aligned}
H^\mu &= \frac{\hat{p}^\mu \hat{p}_\mu}{2m} + V(\hat{x}^\mu); \\
V(\hat{x}^\mu) &= -\frac{Ze^2}{\sqrt{\hat{x}^\mu \hat{x}_\mu}}; \\
V(\hat{x}^\mu) &= -\frac{Ze^2}{\sqrt{(x^\mu - \frac{1}{2}\theta^{\mu\nu}p^\nu)(x_\mu - \frac{1}{2}\theta_{\mu\lambda}p^\lambda)}} \\
&= -\frac{Ze^2}{r} - Ze^2 \frac{\vec{L} \cdot \vec{\theta}}{4r^3} + \mathcal{O}(\theta^2),
\end{aligned} \tag{5}$$

and dropping the terms of order  $\theta^2$ , we produce the noncommutative Hamiltonian. The perturbative energy can be calculated from this Hamiltonian by assuming electron and proton spin applies here, and thus  $j = l \pm \frac{1}{2}$ , and simplified by choosing a coordinate system so the  $z$ -axis is parallel to  $\vec{\theta}$ :

$$\begin{aligned}
\Delta E_{NC}^H &= -\langle nl'jj'_z | \frac{Ze^2}{4\hbar} \frac{\vec{L} \cdot \vec{\theta}}{r^3} | nljj_z \rangle \\
&= \frac{-(Z\alpha^2)^4 m_e c^2 \theta_z}{4 \lambda_e^2 j_z} \left( 1 \mp \frac{1}{2l+1} \right) \cdot \frac{1}{n^3 l(l + \frac{1}{2}(l+1))} \delta_{ll'} \delta_{j_z j'_z},
\end{aligned} \tag{6}$$

which should then be able to provide perturbative energies to known transition values, dependent on the precision of the transition energy.

It should be noted, however, that building the Hamiltonian on the noncommutative space will alter its implementation, as typical vector and operator multiplication on this space will be replaced by the Moyal star product, defined by

$$f \star g(x) = f(\hat{x}) e^{\frac{i}{2}\theta^{\mu\nu} \overleftarrow{\partial}^\mu \overrightarrow{\partial}^\nu} g(\hat{x}). \tag{7}$$

This equation can be expanded to produce more complex terms for the operator-based

commutator we will define later, and we will have to explicitly interpret it using the Baker-Campbell-Hausdorff theorem [7] [See Appendix A].

## 2 Snyder Algebra [1]

We now proceed with a definition of the coordinate commutators, where the commutators themselves are operator valued rather than just  $C$ -number parameters. This particular derivation can be attributed to Snyder [1], who derived it as a theoretical framework for a noncommuting quantized space-time. Snyder's underlying motivation is deriving such a system which is Lorentz-invariant; he does so by defining coordinates  $\eta_0$  to  $\eta_4$  as the coordinates of a four-dimensional projected space, and defines invariant spatial operators (for  $\mu, \nu = [0, 1, 2, 3]$  and a quantization distance  $a$ )

$$\hat{x}^\mu = ia (\eta_4 \partial^\mu + \eta^\mu \partial^4), \quad (8)$$

angular momentum operators

$$L^{\mu\nu} = i (\eta^\mu \partial^\nu - \eta^\nu \partial^\mu), \quad (9)$$

and momentum operators

$$\hat{p}_\nu = -\frac{1}{a} \frac{\eta_\nu}{\eta_4}. \quad (10)$$

Now, with these definitions based on momentum-like coordinates, one can derive the applicable commutators of the  $x^\mu$ ,  $p_\nu$ , and  $L^{\mu\nu}$  operators, which can be used to replace the  $C$ -number parameter given in the previous derivation. Where  $g^{\mu\nu} \equiv \text{diag}(1, -1, -1, -1)$  is



the metric tensor:

$$[\hat{x}^\mu, \hat{x}^\nu] = ia^2 L^{\mu\nu} \quad (11)$$

$$[\hat{x}^\mu, \hat{p}_\nu] = -i (g_\nu^\mu - a^2 \hat{p}^\mu \hat{p}_\nu) \quad (12)$$

$$[\hat{x}^\lambda, L^{\mu\nu}] = i (g^{\lambda\mu} \hat{x}^\nu - g^{\lambda\nu} \hat{x}^\mu) \quad (13)$$

$$[\hat{p}_\mu, \hat{p}_\nu] = 0 \quad (14)$$

$$[\hat{p}_\lambda, L^{\mu\nu}] = i (g_\lambda^\mu \hat{p}^\nu - g_\lambda^\nu \hat{p}^\mu) \quad (15)$$

$$[L^{\mu\nu}, L^{\alpha\beta}] = i (g^{\mu\beta} L^{\nu\alpha} + g^{\nu\alpha} L^{\mu\beta} - g^{\mu\alpha} L^{\nu\beta} - g^{\nu\beta} L^{\mu\alpha}). \quad (16)$$

It is interesting to note that the commutators associated with the Snyder algebra form the symmetry group SO(4,1) with generators  $x^\lambda$  and  $L^{\mu\nu}$ .

This algebra was created by Snyder in response to Heisenberg's attempt to use noncommutativity to eliminate divergences in field theories which existed prior to renormalization procedures now known to do so much more cleanly. Since its initial impact was weak and its raison d'être was eliminated shortly after its publication, the algebra was thought unnecessary for describing physical phenomena. However, particularly with the late 1990s observation that string theory could allow noncommuting coordinates, the question has arisen of whether noncommuting coordinates could be relevant at sufficiently small length scales. That is, given the accuracy of present data, how small a length scale might we have to probe in order to see noncommutativity?

Adapting this algebra to the task of redefining observable properties of the Hydrogen atom requires a re-expression in terms of a system of commuting coordinates. Two methods of doing this are documented here: reformulating the position operators to account for the  $\hat{\theta}^{\mu\nu}$  discrepancy [See Appendix B] and modifying the algebra to reflect the properties of an existing system (here, the system represented by Eqn. 3).

Now that we have a defined set of noncommutative coordinates  $\hat{x}^\mu$ , we need to find a

way to make them calculable in the commutative-coordinate model of the Hydrogen atom, modeling this on the  $C$ -number form derived above. To do so, we need a way to represent our position operators as a perturbation on the noncommutative form. Since the operator form of the literal Snyder algebra does not lend itself to creating such perturbations [See Appendix B], we shall entertain the idea of modifying the generators to represent a simpler algebra. One way to do this is to use a contraction of the Snyder algebra; this means to take the limit of the algebra as a set of parameters goes to zero, which can be set up to make our algebra generators reflect some properties of the  $C$ -number commutators. So we will contract the Snyder algebra to

$$\begin{aligned}
[\hat{x}^\mu, \hat{x}^\nu] &= i\hat{\theta}^{\mu\nu} \\
[\hat{\theta}^{\mu\nu}, \hat{x}^\lambda] &= 0 \\
[\hat{\theta}^{\mu\nu}, \hat{p}^\lambda] &= 0 \\
[\hat{\theta}^{\mu\nu}, \hat{\theta}^{\alpha\beta}] &= 0
\end{aligned}
\tag{17}$$

by taking the limit of a set of parameters multiplicatively relating these generators to those in the original Snyder algebra. Since  $\hat{\theta}^{\mu\nu}$  is the only operator not involved in the Snyder algebra, we need to relate it directly to one of that algebra's generators. Since it replaces  $L^{\mu\nu}$  in the first commutator (which is the most important for Lorentz conservation in our contraction), we will relate these two with some parameter  $b$ :

$$L^{\mu\nu} = \frac{\hat{\theta}^{\mu\nu}}{b}.$$
(18)

We can then derive our algebra from the condition  $b \rightarrow 0$  to square with the latter three commutators. Also, since in Snyder's commutators we want the  $\hat{x}^\mu$  commutators to yield a multiplicative parameter with the angular momentum operator  $ia^2L^{\mu\nu}$ , and since we want to retain the operator  $\hat{\theta}^{\mu\nu}$  in the first commutator, we will also consider the conditions  $a \rightarrow 0$

and  $\frac{a^2}{b} \rightarrow 1$ . Thus, we define our contraction in Eqn. 19:

$$L^{\mu\nu} = \frac{\hat{\theta}^{\mu\nu}}{b}; a \rightarrow 0; b \rightarrow 0; \frac{a^2}{b} \rightarrow 1, \quad (19)$$

which allows us to use the  $C$ -number position perturbation operators [6],

$$x^\mu = \hat{x}^\mu + \frac{1}{2}\hat{\theta}^{\mu\nu}\hat{p}_\nu, \quad (20)$$

and apply this perturbation to a  $\frac{1}{r}$  potential much more successfully, as we now have a physical meaning for  $\hat{\theta}^{\mu\nu}$  which can be evaluated in some situations with well-known quantities. We will use the error in the measurement of these quantities to predict the scale of this noncommutative perturbation, dependent on a parameter analogous to Snyder's  $a^2$ . An algebra with Lie commutators identical to this contraction (Eqn. 17) was proposed in 1994 by Döplcher, Fredenhagen, and Roberts [8] known as the DFR algebra. It was motivated by considerations emanating from general relativity, and was proposed as a complete algebra without mentioning its contraction from  $\text{SO}(4,1)$ .

### 3 Noncommutative Lamb Shift

The Lamb shift is the name given to the common and significant electron energy level difference between the angular momentum states  ${}^2P_{\frac{1}{2}} \rightarrow {}^2S_{\frac{1}{2}}$ . This shift has a very well-known experimental value from a number of sources and methods, and provides a fertile testing ground for the scaling parameter  $b$  of our operator-based coordinate commutators. Since we now have a contracted Snyder algebra which improves the applicability of the commuting-matrix parameter by making  $\hat{\theta}^{\mu\nu}$  analogous to the angular momentum operator  $L^{\mu\nu}$ , we can use the basic Hydrogen-atom Hamiltonian, assuming a  $\frac{1}{r}$  potential, to determine the noncommutative perturbation in the energy  $\Delta E_{NC}^H$ . After doing so, we shall apply the

results to calculate the energy difference that coordinate noncommutativity might cause in the Lamb shift (see Fig. 1). A possible interpretation of the matrix elements of  $\hat{\theta}^{\mu\nu}$  is that they are related to matrix elements of an antisymmetric operator already known to be relevant to the Hydrogen atom. Based on Snyder's coordinate commutators, we set matrix elements of this operator proportional to those of the angular momentum operator,

$$\langle \Psi^* | \hat{\theta}^{\mu\nu} | \Psi \rangle \equiv \frac{1}{\Lambda_{NC}^2} \langle \Psi^* | L^{\mu\nu} | \Psi \rangle, \quad (21)$$

where  $\Lambda_{NC}$  is a parameter equivalent to  $b$  in the contracted algebra with dimensions of ( $\text{length}^{-1}$ ).

The Hamiltonian only depends on  $r$  in the potential, so we can reformulate the potential to appear as the  $C$ -number  $\theta^{\mu\nu}$  potential, replacing  $\hat{\theta}^{\mu\nu}$  with  $bL^{\mu\nu}$ :

$$\begin{aligned} V(x^\mu) &= -\frac{Ze^2}{r} - Ze^2 \frac{\vec{L} \cdot \vec{\theta}}{4\hbar r^3} \\ &= -\frac{Ze^2}{r} - \frac{Ze^2}{4\hbar r^3} \frac{L^2}{\Lambda_{NC}^2} \\ \Delta E_{NC}^H &= \frac{Ze^2}{4\hbar} \langle \Psi^* | \frac{L^2}{\Lambda_{NC}^2 r^3} | \Psi \rangle \\ &= \frac{Ze^2}{4\hbar} \langle R_{n,l} | \frac{1}{r^3} | R_{n,l} \rangle \langle l j j_z | \frac{L^2}{\Lambda_{NC}^2} | l j j_z \rangle \\ &= (Z\alpha)^4 \frac{m_e c^2}{\lambda_e^2 \Lambda_{NC}^2} \left( \frac{1}{n^3 (l + \frac{1}{2})} \right). \end{aligned} \quad (22)$$

Now that the  $\hat{\theta}^{\mu\nu}$  operator has been defined, we can write this perturbative energy entirely in terms of measurable quantities and the parameter  $\Lambda_{NC}$ . To actually calculate this value, then, we must find a well-defined, measured energy to which we can relate this perturbation. This measurable energy's uncertainty will be set equal to the perturbative noncommutative energy, thus allowing us to set an approximate maximum value for the parameter  $\Lambda_{NC}$ , representing  $b$  from the contraction relations.

We chose the Hydrogen atom Lamb shift, as this is a well-documented and very accurate

measurement [9], allowing us to set a relatively low value for this parameter. From the set of measured values of this energy difference, we choose the most accurate one (with the lowest error),  $\Delta E_{Lamb}^{(1)} = h \cdot [1,057,857.6(2.1)\text{kHz}]$ ; the quantity in parentheses represents the experimental error to the place of the respective final digits. This value is most important to us, as the uncertainty in the present measurement of the Lamb shift places a limit to the order at which we can reasonably expect noncommutative effects to appear; this uncertainty is represented by the experimental error in this measurement, 2.1kHz. Using the CODATA [10] values for the fine structure constant  $\alpha$ , the electron mass  $m_e$ , and Planck's constant  $h$ , we will derive an upper limit for the noncommutativity parameter  $\Lambda_{NC}$  from the Hydrogen atom Hamiltonian.

Since our Hamiltonian perturbation has both angular momentum and radius, we must deal with both radial wavefunctions and spherical harmonics. In the previous equation, the radial wavefunctions reduce to a set of parameters including  $\lambda_e$ , the Compton wavelength of the electron, which represents the uncertainty limit on the electron's position (thus, the minimum scattering length, to relate to Compton scattering). This is defined as

$$\lambda_e = \frac{\hbar}{m_e c}. \quad (23)$$

The noncommutative perturbation as given here does not affect the S-state of the orbital angular momentum due to the  $L^2$  in the numerator of Eqn. 22. Thus, the energy difference due to the Lamb shift is equivalent to the energy calculated for the  ${}^2P_{\frac{1}{2}}$  level. Considering both components of the Lamb shift in the  $n = 2$  radial state (referencing a known result for the expectation value of  $\frac{1}{r^3}$ ), this energy becomes

$$\begin{aligned} \Delta E_{NC}({}^2P_{\frac{1}{2}}) &= -(Z\alpha)^4 \frac{m_e c^2}{2\lambda_e^2} \left( \frac{1}{3n^3} \frac{1}{\Lambda_{NC}^2} \right); \\ &= -\alpha^4 \frac{m_e c^2}{48} \left( \frac{1}{\lambda_e^2 \Lambda_{NC}^2} \right). \end{aligned} \quad (24)$$

Using the following CODATA values for the constants represented<sup>1</sup>,

$$\begin{aligned}
\alpha &= \frac{1}{137.035999679(94)}, \\
m_e &= 9.10938215(45) \cdot 10^{-31} \text{kg}, \\
c &= 2.99792458 \cdot 10^8 \text{ m/s}, \text{ and} \\
h &= 6062606896(33) \cdot 10^{-34} \text{ J/Hz},
\end{aligned}
\tag{25}$$

we calculate the noncommutative perturbation to the Lamb shift energy to be

$$\begin{aligned}
\Delta E_{NC}^{(1)} &= - (32.44 \text{ J/m}^2) \frac{1}{\Lambda_{NC}^2}; \\
\Delta \nu_{NC}^{(1)} &= - (4.895 \cdot 10^{-31} \text{ kHz/m}^2) \frac{1}{\Lambda_{NC}^2}.
\end{aligned}
\tag{26}$$

When we compare this value to the error in the standard Lamb shift measurement [9], we can calculate an upper bound for the parameter  $\Lambda_{NC}$ :

$$\begin{aligned}
\Delta \nu_{NC}^{(1)} &< 2.1 \text{ kHz}; \\
\frac{1}{\Lambda_{NC}} &< 2.07 \cdot 10^{-16} \text{ m}.
\end{aligned}
\tag{27}$$

This upper bound is higher than is practical for reference, as it is only one order of magnitude smaller than the proton charge radius. As mentioned previously, the interpretations of such a spatial noncommutativity parameter would be either as a quantization of the 3 spatial dimensions or as some dependence of position measurement on a quantity like linear or angular momentum; either would not be likely to result in non-measurable effects at currently observable limits. Since the Lamb shift in Hydrogen is one of the most accurately measured energy-level discrepancies, such an atomic level-based upper bound on this parameter is unlikely to be effective in reasonably limiting noncommutativity.

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<sup>1</sup>Values are presented here at full reported accuracy for completion; this level of accuracy is not used in presenting further calculation.

## 4 Noncommutative Hyperfine Splitting

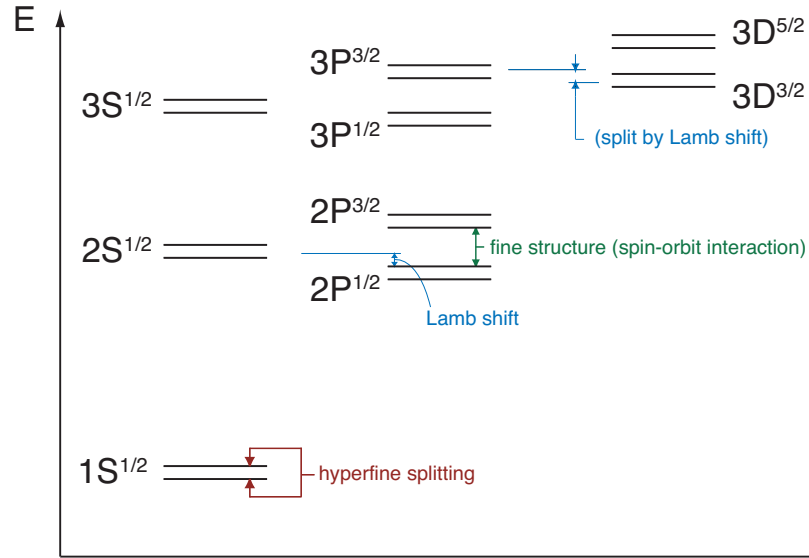


Figure 1: Hyperfine Structure of the Hydrogen Atom

A more precise quantum mechanical effect in which noncommutativity might be detectable is the spin-spin interaction, or hyperfine splitting. This is a small electron energy level shift which removes the degeneracy present between different spin states by the interaction of the nuclear spin with the magnetic field of the electrons (though the reverse gives the same splitting, this is how it will be analyzed here), which is simple to work out in the case of the Hydrogen atom's single proton in the nucleus (See Fig. 1. The structure is determined by the basic formula

$$\Delta E_{hfs} = -\vec{\mu}_p \cdot \vec{B}_e, \quad (28)$$

where we have values for the spin of the proton and must determine the magnetic field of the electron  $\vec{B}_e$  from the basic vector potential. This vector potential, in our commuting

coordinates, is

$$\begin{aligned}
\vec{A}_e &= \frac{\vec{\mu}_e \times \vec{r}_{NC}}{r_{NC}^3}; \\
\vec{r} &= \vec{r}_{NC} + \frac{1}{2} \overleftrightarrow{\theta} \cdot \vec{p}, \\
r^3 &\simeq r_{NC}^3 + \frac{3}{2} \left( \vec{r} \cdot \overleftrightarrow{\theta} \cdot \vec{p} \right) r_{NC}, \text{ so} \\
\vec{A}_e &\simeq \frac{\vec{\mu}_e \times \vec{r}}{r^3} + \frac{\vec{\mu}_e \times \left( \frac{3}{2} \vec{r} \cdot \overleftrightarrow{\theta} \cdot \vec{p} \right) \vec{r}}{r^5} - \frac{\vec{\mu}_e \times \left( \frac{1}{2} \overleftrightarrow{\theta} \cdot \vec{p} \right)}{r^3}.
\end{aligned} \tag{29}$$

Now that we have the vector potential  $\vec{A}_e$  of the electron, and taking into account the fact that the tensor  $\overleftrightarrow{\theta}$  is antisymmetric and hermitian, we can find the magnetic field<sup>2</sup>:

$$\begin{aligned}
\vec{B}_e &= \vec{\nabla} \times \vec{A}_e = \frac{3(\vec{\mu}_e \cdot \hat{r})\hat{r} - \vec{\mu}_e}{r^3} + \frac{9(\vec{\mu}_e \cdot \hat{r}) \left( (\vec{\theta} \times \vec{p}) \cdot \hat{r} \right) \cdot \hat{r}}{r^4} + \\
&\frac{9 \left( (\vec{\theta} \times \vec{p}) \cdot \hat{r} \right) \vec{\mu}_e}{2r^4} + \frac{3(\vec{\mu}_e \cdot \hat{r}) \left( \vec{\theta} \times \vec{p} \right)}{2r^4} + \frac{8\pi}{3} \vec{\mu}_e \delta^{(3)}(\vec{r}).
\end{aligned} \tag{30}$$

The first component of this magnetic field represents the typical dipole magnetic field, for the case of the electron as a perfect magnetic dipole, and thus would produce the commutative hyperfine splitting. To calculate the noncommutative hyperfine energy shift, then, we can ignore these terms and look specifically at the extra terms resulting from our noncommutative coordinates:

$$\begin{aligned}
\Delta E_{hfs} &= -\mu_p \cdot \vec{B}_e = \frac{-g_p e}{2m_p} \vec{s}_p \cdot \vec{B}_e \\
\Delta E_{hfs}^{NC} &= \frac{-g_e g_p e^2}{4m_e m_p} \left[ \frac{9(\vec{s}_e \cdot \hat{r})(\vec{s}_p \cdot \hat{r}) \left( (\vec{\theta} \times \vec{p}) \cdot \hat{r} \right)}{r^4} + \frac{9 \left( (\vec{\theta} \times \vec{p}) \cdot \hat{r} \right) (\vec{s}_e \cdot \vec{s}_p)}{2r^4} + \right. \\
&\left. \frac{3(\vec{s}_e \cdot \hat{r}) \left( (\vec{\theta} \times \vec{p}) \cdot \vec{s}_p \right)}{2r^4} \right].
\end{aligned} \tag{31}$$

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<sup>2</sup>N.b. Here, variables with carets are unit vectors, not operators



This equation then can be simplified further and written in terms of operators by vector rearrangement to simplify the terms involving momenta

$$\left(\vec{\theta} \times \vec{p}\right) \cdot \hat{r} = \frac{\vec{\theta}}{r} \cdot (\vec{p} \times \vec{r}) = -\frac{\vec{\theta} \cdot \vec{L}}{r} = -\frac{1}{\Lambda_{NC}^2} \frac{L^2}{r} \quad (32)$$

And the evaluation of standard spin eigenvalues for the possible spin states (that is,  $s = 1$  or  $0$ ):

$$\begin{aligned} (\vec{s}_e \cdot \hat{r})(\vec{s}_p \cdot \hat{r}) &\rightarrow s_e^i s_p^j, \text{ in operator form;} \\ \langle s = 1 | s_e^i s_p^j | s = 1 \rangle - \langle s = 0 | s_e^i s_p^j | s = 0 \rangle &= \frac{1}{12} \delta_{ij} - \left( -\frac{1}{4} \delta_{ij} \right) = \frac{1}{3} \delta_{ij}. \end{aligned} \quad (33)$$

By these operations, our energy (now as a Hamiltonian, since we are considering operators rather than valued vectors) can be written in the more compact form

$$\Delta H_{hfs}^{NC} = \frac{8\pi\alpha g_e g_p}{m_e m_p} \frac{1}{\Lambda_{NC}^2} \frac{L^2}{r^5}. \quad (34)$$

The  $\frac{1}{r^5}$  term here makes evaluation of this hyperfine splitting perturbation hamiltonian difficult for states below the  $3d$  orbital, due to divergences in the expectation-value integrals. Unfortunately, the hyperfine splitting of this state is not well-documented, and must be calculated. This energy expectation value will provide us with a standard for comparison with the noncommutative term in the form of a ratio, demonstrating the scale of the shift in the noncommutative case by applying the upper bound on the noncommutativity parameter calculated previously (See Eqn. 27).

Considering one of the two possible spin states,  $3d^{5/2}$ , the expectation value for  $L^2$  is 6 and the expectation value  $\langle \frac{1}{r^5} \rangle$  is given by

$$\langle 3d | \frac{1}{r^5} | 3d \rangle = \frac{1}{720} \frac{32}{243} \frac{1}{a_0^5}, \quad (35)$$

where  $a_0$  is the bohr radius. The commutative energy eigenvalue for the  $3d^{5/2}$  state is given by

$$\Delta E_{hfs}(3d^{5/2}) = \mu_e \mu_p \frac{144}{35} \langle \frac{1}{r^3} \rangle = \mu_e \mu_p \frac{144}{35} \left[ \frac{1}{120} \frac{8}{27a_0^3} \right], \quad (36)$$

collecting the constants into the numerical values of the magnetic moments  $\mu_e$  and  $\mu_p$ . Thus, we have everything we need to calculate the ratio of the noncommutative shift in the hyperfine energy splitting to the hyperfine shift itself:

$$\frac{\Delta E_{hfs}^{NC}(3d^{5/2})}{\Delta E_{hfs}(3d^{5/2})} = \frac{1}{\Lambda_{NC}^2} [1.23 \cdot 10^{-21} \text{m}^{-2}]. \quad (37)$$

with the noncommutativity parameter given by  $\frac{1}{\Lambda_{NC}} < 2.07 \cdot 10^{-16} \text{m}$ , we find that at this maximum we would expect a proportional shift in the hyperfine energy splitting as

$$\frac{\Delta E_{hfs}^{NC}(3d^{5/2})}{\Delta E_{hfs}(3d^{5/2})} < 5.3 \cdot 10^{-11}. \quad (38)$$

The scaling of this shift puts it beyond the current range of both experimental observation and theoretical calculation of the hyperfine splitting at any level, much less the  $3d$  states; it is one order of magnitude lower than the accuracy of any observation, and lower still than any accurate theoretical calculations. This means that we are unlikely to observe noncommutativity in investigation of the spin-spin interaction, especially considering the fact that our current upper limit is larger than we realistically expect, as discussed in the previous section.

## 5 Conclusions and Future Goals

The projected derivation of a number of values for the noncommutativity parameter  $\Lambda_{NC}$  produced a single result, which does set a limit on the parameter but tells us more in the

negative for future approaches to this subject. The limit

$$\frac{1}{\Lambda_{NC}} < 2.07 \cdot 10^{-16} \text{m} \quad (39)$$

tells us that most atomic physics will not be affected even if noncommutativity of coordinates is realized in nature at some small level. Additionally, we calculated a small shift in the hyperfine structure of the  $3d^{5/2}$  state of Hydrogen, given in Eqn. 38, which indicates that it is unlikely the hyperfine structure will yield observable noncommutative effects.

We have defined one modification of the original Snyder algebra; further attempts to produce a full Lie algebra which retains the desired Lorentz symmetry may suggest modifying the Snyder algebra itself to make the production of commuting coordinates from the noncommutative simpler. A number of other atomic and some subatomic interactions might provide better limits than that given in this paper. Other theoretical issues which may prove insightful are learning to work with the full algebra describing our operator-based noncommutativity parameter,  $ia^2\hat{\theta}^{\mu\nu}$ , and an investigation into the angular momentum aspect of this parameter and how an innate  $L^{\mu\nu}$  component would affect the geometry of space on its own terms. Spatial noncommutativity is a potentially very fruitful concept which has yet to be explored in real depth, and may hold insight into the debate over the validity of string theory and other theoretical physics models. It may also provide a mode of analysis and interpretation of deviations potentially to be found through the LHC and higher-energy explorations of the Standard Model.

# A Appendices

## A.1 Development of BCH Theorem

In this appendix, we wish to record a term in the expansion of the BCH theorem which, while we did not find a direct use for it, is beyond what we have found in the texts on the subject and may prove useful in the future.

It was mentioned that the Moyal star product for our particular situation must be altered for the application of our new Hamiltonian to its eigenstates in the new operator space. The original star product was developed for the system of  $C$ -numbers  $\theta^{\mu\nu}$ , which allowed for a simplification of the Fourier transforms used to convert between operators. This simplification comes from the application of the Baker-Campbell-Hausdorff theorem within the transform:

$$H(\hat{x}) = f(\hat{x}) \cdot g(\hat{x}) \quad (40)$$

$$= \int dp_\mu dp_\nu e^{ip_\mu \hat{x}} e^{ip_\nu \hat{x}} \tilde{f}(p_\mu) \tilde{g}(p_\nu). \quad (41)$$

We can see that these exponents contain operators; the way such units behave is defined in the Baker-Campbell Hausdorff (BCH) theorem,

$$e^{\hat{A}} e^{\hat{B}} = e^{\sum^n c_n(A:B)},$$

$$c_0 = 0$$

$$c_1 = A + B \quad (42)$$

$$c_n = \frac{1}{2(n+1)} [A - B, c_n] + \sum_{p=1}^{n/2} K_{2p} \sum_{k_1 + \dots + k_{2p} = n} [c_{k_1}, [c_{k_2}, [\dots [c_{k_{2p}}, A + B] \dots ]]] ,$$

where the  $K_{2p}$  are defined by

$$\begin{aligned}
1 + \sum_{p=1}^{\infty} K_{2p} z^{2p} &= \frac{z}{1 - e^{-z}} - \frac{1}{2}z \\
&= 1 + \frac{1}{12}z^2 - \frac{1}{720}z^4 + \mathcal{O}(z^6).
\end{aligned} \tag{43}$$

The usual  $C$ -number commutator produces only one term from this expression, expressed as

$$e^A + e^B = e^{A+B+\frac{1}{2}[A,B]}, \tag{44}$$

which produces a Fourier transform that results in the Moyal star product shown above in Eqn. 7. However, with the commutator of the coordinate operators resulting in another commutator related to the angular momentum, we must expand this further from the definition given above. Doing so will then allow us to put our Hamiltonian into the noncommutative operator space and apply it to wavefunctions which will allow us to measure the quantized distance  $a$ . This expansion of the Baker-Campbell-Hausdorff theorem should accommodate as many terms as possible which we might potentially need. This resulted in the following terms, from  $c_2$  to  $c_5$ :

$$\begin{aligned}
c_2 &= \frac{1}{2} [A, B] \\
c_3 &= \frac{1}{12} ([A, [A, B]] - [B, [A, B]]) \\
c_4 &= -\frac{1}{48} ([B, [A, [A, B]]] + [A, [B, [A, B]]]) \\
c_5 &= \frac{1}{480} ([B, [B, [A, [A, B]]]] + [B, [A, [B, [A, B]]]] \\
&\quad - [A, [B, [A, [A, B]]]] - [A, [A, [B, [A, B]]]]).
\end{aligned} \tag{45}$$

This concept was initiated with the goal of developing a commutative coordinate system based on the Snyder algebra defining the  $\hat{\theta}^{\mu\nu}$  operator. Were we able to develop such a system, we would be able to redefine our Hamiltonian in terms of this new position operator

and thus redefine the Moyal product, Eqn. 7. However, we were unable to find such a coordinate system with an additive factor related to Eqn. 4 [See Appendix B], and thus for this project this redefinition was not used. It would be very useful were a different algebra or commuting coordinate applied.

## A.2 A Commuting Coordinate System from the Snyder Commutators

Snyder's algebra describes a noncommutative system of spatial coordinates, which describe a space that has yet to have physical application observed. Thus, to make predictions about how this noncommutativity will manifest itself, we must reformulate the noncommutative operators generating this space to a set of commuting operators, with which we are accustomed to working. These commuting operators, in their simplest and most straightforward form, would involve redefining the position operators as the basic position operators  $x^\mu$  with an added "perturbation", similar to the form of Chaichian's system (see Eqn. 4). Since the  $C$ -number system's perturbation involves only commuting units in the perturbation, its derivation is relatively simple. However, using the Snyder algebra, one must take the operator nature of  $\theta^{\mu\nu} = L^{\mu\nu}$  into account in devising such a perturbation. So starting with the basic generating algebra, we can devise the condition for a perturbative operator to make the coordinate system commutative:

$$\begin{aligned}
 [\hat{x}^\mu, \hat{x}^\nu] &= i\theta^{\mu\nu} \\
 [\hat{x}^\mu + B^\mu, \hat{x}^\nu + B^\nu] &= 0 \\
 [B^\nu, B^\mu] + [\hat{x}^\nu, B^\mu] + [B^\nu, \hat{x}^\mu] &= ia^2 L^{\mu\nu}.
 \end{aligned}
 \tag{46}$$

Thus, the operator  $B$  must satisfy the above condition to create a commuting coordinate system from the noncommutative operators  $\hat{x}^\mu$ . We tried the simplest and more straightforward

possible values for  $B$ , which must all be operator-based to match:

$$B^\mu \stackrel{?}{=} A\hat{p}^\mu \quad (47)$$

$$B^\mu \stackrel{?}{=} A\hat{x}^\mu \quad (48)$$

$$B^\mu \stackrel{?}{=} AL^{\mu\nu}\hat{p}_\nu \quad (49)$$

$$B^\mu \stackrel{?}{=} AL^{\mu\nu}\hat{x}_\nu. \quad (50)$$

Unfortunately, the Snyder commutators yield too few commuting relations between any of these three operators, thereby making it difficult or perhaps impossible to create such a  $B^\mu$  operator. The makeup of the Snyder algebra itself suggests that such a solution is unlikely, and that altering the construction of the algebra itself may be necessary to force the commutation relation to exist. This led to the development of the alternative theory discussed in the main body of this study, which allowed us to develop an applicable operator system to the Hydrogen-atom Hamiltonian.

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