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Strategy-Proofness on the Condorcet Domain

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Strategy-Proofness On the Condorcet Domain

A thesis submitted in partial fulfillment of the requirement for the degree of Bachelor of Science with Honors in Economics from the College of William and Mary in Virginia, by Lauren Nicole Merrill.

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A social choice rule \( g \) selects a member of a given set of alternative \( X \) as a function of individual preferences. The Gibbard–Satterthwaite theorem establishes that if preferences are unrestricted and the range of \( g \) has at least three members, only dictatorial rules are strategy–proof. However, if the domain of \( g \) is the set of profiles at which there exists a strong Condorcet winner, Campbell and Kelly [4] have shown that majority–rule is the only non–dictatorial strategy–proof rule for an odd number of individuals when the range of \( g \) contains at least three alternatives. Dasgupta and Maskin [6] consider the case of a continuum of voters as a means of circumventing the issue of parity. Although their analysis provides an approximation for a sufficiently large (but finite) set of individuals, no exact analysis exists for an arbitrary even number of individuals. We are therefore interested in characterizing the family of strategy–proof social choice rules over the Condorcet domain for an even number of individuals. We provide a full characterization when individual preferences are strict linear orderings, and prove several propositions concerning strategy-proof rules when individual preference orderings are permitted to be weak linear orders.
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Contents

1 Introduction 1
   1.1 Social Choice Theory . . . . . . . . . . . . . . . . . . . . . . . . . 2
   1.2 Notation and Definitions . . . . . . . . . . . . . . . . . . . . . . 4

2 The Strict Condorcet Domain 13
   2.1 An Odd Number of Individuals . . . . . . . . . . . . . . . . . . . . . 13
   2.2 An Even Number of Individuals . . . . . . . . . . . . . . . . . . . . . 18
      2.2.1 A Characterization for Two Individuals . . . . . . . . . . . . . . 18
      2.2.2 A Characterization for any Even Number of Individuals . 27

3 Weak Condorcet Domain 41

4 Conclusion 48
Chapter 1

Introduction

Social choice theory is often characterized as field of impossibility theorems. Indeed, its inception was largely tied to Arrow’s Theorem [1], which demonstrated the impossibility of constructing a voting mechanism that satisfies a number of seemingly innocuous properties. Since then numerous other impossibility theorems have been proved, and conditions have been identified under which possibility theorems may be obtained.

In this paper we focus on the Condorcet domain, the largest domain for which majority rule is well-defined. Campbell and Kelly [4] have shown that majority rule is the only non-dictatorial strategy-proof rule on the Condorcet domain for an odd number of individuals and at least three alternatives in the range. Dasgupta and Maskin [6] consider the case of a continuum of voters as a means of circumventing the issue of parity. Although their analysis provides an approximation for a sufficiently large (but finite) set of individuals, no exact analysis exists for an arbitrary even number of individuals. We are therefore interested in characterizing the family of strategy-proof social choice rules over the Condorcet domain for an even number of individuals.

Chapter 1 begins with a qualitative introduction to the field of social choice theory. Section 1.2 introduces the standard social choice notation and definitions in addition to new notation developed in
this project. Section 1.3 summarizes a number of well-known impossibility and possibility results that motivate our research.

In Chapter 2 we consider the strict Condorcet domain; that is, the Condorcet domain when individual preference orderings are strict linear orders. Section 2.1 summarizes the Campbell and Kelly [4] for an odd number of individuals, and concludes with a few examples. In Section 2.2 we present new results that characterize the set of strategy-proof social choice rules for an even number of individuals. We first prove a result for two individuals; the results of this case are used to structure the proof for an arbitrary even number of individuals. Section 2.3 consists of a number of examples of strategy-proof rules over the Condorcet domain, as well as examples and discussion of manipulable rules.

In Chapter 3 we alter our domain, and focus on the weak Condorcet domain; that is, the Condorcet domain when individual preference orderings are weak linear orders. Section 3.1 presents a number of propositions that restrict the behavior of strategy-proof social choice rules, while Section 3.2 contains examples that highlight features of the weak Condorcet domain.

Chapter 4 contains a summary of our results, as well as potential avenues for future research.

1.1 Social Choice Theory

There are many everyday situations in which a group of individuals must make a collective decision: an academic department electing a chair, a community deciding which public project to fund, a family choosing a restaurant at which to eat dinner. Similarly, there are situations in which a group must collectively order a set of alternatives, such as candidates for a job or college applicants. Social choice theory provides an axiomatic framework in which to analyze such decision processes.

The field of social choice theory is primarily concerned with the formal analysis of collective decision-making processes. The roots of the field date to 1785, when the French philosopher and mathe-
matician Marie Jean Antoine Nicolas de Caritat, marquis de Condorcet published his *Essays on the Application of Analysis to the Probability of Majority Decisions* [3]. Among other topics, the essay introduced the Condorcet method, a formulation of majority rule, and illustrated Condorcet’s Voting Paradox, which states that group preferences may be intransitive even when individual preferences are not. This paper will focus exclusively on the Condorcet domain, the largest domain for which Condorcet’s method is well-defined without the addition of a tie-breaking (or cycle-breaking) mechanism.

The field was founded in its modern form by the work of Kenneth Arrow in 1951, and was propelled forward by early landmark results, including Arrow’s Impossibility Theorem [1] and its refinements. Arrow’s theorem sets forth a number of natural properties we might desire a decision-making rule to possess, then shows that the properties are incompatible. We give a qualitative version of Arrow’s theorem here; a formal version follows in Section 1.2.

**Theorem 1.1.1** (Arrow’s Impossibility Theorem). Suppose there are at least two voters and at least three alternatives. Then no deterministic voting rule that produces a linear ordering of the alternatives as a function of voter preferences can simultaneously exhibit all of the following properties:

1. The rule is defined for all specifications of voter preferences.
2. There is no individual for whom the societal ranking always corresponds with his or her individual ranking.
3. Regardless of voter preferences, the introduction (or removal) of alternatives does not change the relative ranking of the existing (or remaining) alternatives.
4. If a voter promotes an alternative in his or her individual ranking, the alternative can never be demoted in the societal ranking.
5. Every possible ranking of the alternatives is attainable as the societal ranking of some set of voter preferences.
Another famous result was developed independently by Gibbard [9] and Satterthwaite [14]. The result is closely related to Arrow’s theorem; Reny [13] provides a single proof that yields both results. Again, we present the theorem informally.

**Theorem 1.1.2** (Gibbard–Satterthwaite Theorem). Suppose there are at least two voters and at least three alternatives, and that voters are allowed to have any preferences over the set of alternatives. Then one of the following must be true for any deterministic voting rule that selects one alternative as a function of the voter preferences:

1. One of the voters is a dictator, so the rule always chooses from that voter’s most–preferred set of alternatives.
2. There is some alternative that is never selected by the rule.
3. There exist situations in which an individual voter can benefit from misrepresenting his or her preferences over the set of alternatives.

Social choice theory therefore provides a framework in which collective decision–making processes can be analyzed axiomatically. A social choice rule is a mechanism for choosing an alternative as a function of individual preferences. Within the field, there are multitudes of identified properties a given rule might display; of particular interest in this paper are the properties of dictatorship and susceptibility to strategic manipulation. These properties, as well as the notation and definitions common in the field, are introduced in the next section.

### 1.2 Notation and Definitions

Let $N$ be the set of individuals indexed by the natural numbers, $1, \ldots, n$. Let $X = \{x_1, \ldots, x_m\}$ be the set of alternatives. In general, there can be any number of individuals and any number of alternatives.

Let $L_W(X)$ denote the set of weak linear orders on the set $X$ and let $L_S(X)$ denote the set of strict linear orders on $X$. Each individual
has a preference order over the alternatives in $X$ that is a member of $L_W(X)$ (we will sometimes further restrict individual preferences to $L_S(X)$). In the case of an infinite number of alternatives, we will restrict $L_W(X)$ and $L_S(X)$ to include only those strict linear orderings with a maximal (or most-preferred) element. Clearly, we have $L_S(X) \subset L_W(X)$. These axiomatic assumptions coincide with our intuitive understanding of individual preferences:

- **Complete**: For all pairs of alternatives $(x_i, x_j)$, individuals have some preference relation (potentially indifference) over the pair.
- **Transitive**: For all triples of alternatives $(x_i, x_j, x_k)$, no individual would simultaneously prefer $x_i$ to $x_j$, $x_j$ to $x_k$, and $x_k$ to $x_i$.
- **Asymmetric Part**: There exist pairs of alternatives $(x_i, x_j)$ for which individuals strictly prefer alternative $x_i$ to $x_j$.
- **Symmetric Part**: There exist pairs of alternatives $(x_i, x_j)$ for which individuals are indifferent between the alternatives $x_i$ and $x_j$.

A profile $p$ is a mapping that assigns a preference ordering to each individual. We let $p(i)$ indicate individual $i$’s preference ordering at profile $p$. When individual preference orderings are restricted to $L_S(X)$, let $p_k(i)$ indicate individual $i$’s $k$th most-preferred alternative at profile $p$. The notation $x_i \succ p(k) x_j$ is used to indicate that individual $k$ prefers alternative $x_i$ to alternative $x_j$ at profile $p$. Similarly, $x_i \sim p(k) x_j$ indicates that individual $k$ is indifferent over the pair $(x_i, x_j)$ at profile $p$.

**Example 1.2.1.** Suppose that $X = \{x_1, x_2, x_3\}$, and that at profiles $p$ and $q$ (respectively) individual $k$ has the preference orderings

$$p(k) = (x_1 \succ x_2 \succ x_3), \quad q(k) = (x_1 \sim x_2 \succ x_3).$$

These preference orderings indicate that at profile $p$ individual $k$ prefers alternative $x_1$ to $x_2$ and $x_3$, and alternative $x_2$ to $x_3$. At profile $q$
individual $k$ prefers alternative $x_1$ and $x_2$ to $x_3$, but is indifferent between $x_1$ and $x_2$.

Therefore, we have that $p \in L_S(X)^n \subset L_W(X)^n$ and $q \in L_W(X)^n \setminus L_S(X)^n$.  

We can imagine a profile in the form of a 2–dimensional array, in which each column represents a particular individual’s preference ordering. When individual preferences are in $L_S(X)$, the array has dimension $n \times m$; for convention we will retain these dimensions even when individual preferences contain indifference, and could be visualized in a smaller array. For clarity, we will generally append a row to the top of the profile matrix to indicate the individual to whom each column is assigned. The following example illustrates the use of this notation.

**Example 1.2.2.** Let $N = \{1, 2, 3\}$ and $X = \{x_1, x_2, x_3\}$. Then the profile

$$p = \begin{bmatrix}
1 & 2 & 3 \\
1 & x_2 & x_1 \\
x_2 & x_1 & x_3 \\
x_3 & x_3 & x_2 \\
\end{bmatrix}$$

indicates that $p(1) = (x_1 \succ x_2 \succ x_3)$ and $p(3) = (x_1 \succ x_3 \succ x_2)$. Furthermore, we have $p_1(2) = x_2$ and $p_3(3) = x_2$. Since all preferences are strict, we have $p \in L_S(X)$.

Next consider the profile

$$q = \begin{bmatrix}
1 & 2 & 3 \\
x_1 \sim x_2 & x_2 & x_1 \\
x_3 & x_1 \sim x_3 & x_3 \\
x_3 & x_2 & x_2 \\
\end{bmatrix},$$

where $q(1) = (x_1 \sim x_2 \succ x_3)$, $q(2) = (x_2 \succ x_2 \sim x_3)$, and $q(3) = (x_1 \succ x_3 \succ x_2)$. Since individuals 1 and 2 are indifferent over some pairs of alternatives, we have $q \in L_W(X) \setminus L_S(X)$.

We use the notation $N_p(x_i \succ x_j)$ to indicate the subset of individuals preferring alternative $x_i$ to $x_j$ at profile $p$. Similarly, we use
$N_p(x_i \succ x_j)$ to indicate the subset of individuals preferring alternative $x_i$ to $x_j$ at profile $p$. That is,

$$N_p(x_i \succ x_j) = \{k \in N : x_i \succ_p k \succ x_j\}$$

and

$$N_p(x_i \sim x_j) = \{k \in N : x_i \sim_p k \sim x_j\}.$$ 

We say that $N_p(x_i \succ x_j)$ is the coalition preferring $x_i$ to $x_j$ at profile $p$. In the above example, we had $N_p(x_1 \succ x_3) = \{1, 3\}$ and $N_p(x_1 \sim x_2) = \{1\}$. For all $x_i, x_j \in X$ and all profiles $p \in L_W(X)$, the sets $N_p(x_i \succ x_j), N_p(x_i \sim x_j), N_p(x_j \succ x_i)$ form a partition of the set $N$.

A social choice rule is a function $g : \varphi \longrightarrow X$, where the domain $\varphi$ is some profile space. In this paper we are interested in a specific profile space known as the Condorcet domain, denoted by $\varphi_C$. Moreover, we will consider two formulations of the Condorcet domain:

**Definition 1.2.3 (Weak Condorcet Domain).** The weak Condorcet domain is the profile space $\varphi_{CW} \subset L_W(X)^n$ for which at each profile $p \in \varphi_{CW}$ there exists some alternative $x_i \in X$ such that for all $x_j \in X \setminus \{x_i\}$, $|N_p(x_i \succ x_j)| > |N_p(x_j \succ x_i)|$.

Qualitatively, the weak Condorcet domain consists of all profiles in $L_W(X)^n$ at which there exists some alternative $x_i$ that is preferred to every other alternative $x_j$ by a majority of the individuals who are not indifferent over the pair $(x_i, x_j)$. An alternative definition of the weak Condorcet domain exists in the literature; Chapter 3 we formally introduce the alternative definition and justify our preference for the definition above.

When studying the weak Condorcet domain, we will occasionally desire to know the number of individuals who have strict preferences over an alternative pair $(x_i, x_j)$. We will therefore define

$$\mu_p(x_i, x_j) = |N_p(x_i \succ x_j)| + |N_p(x_j \succ x_i)|$$

to be the number of individuals who have strict preferences over an
alternative pair \((x_i, x_j)\) at profile \(p\).

**Definition 1.2.4** (Strict Condorcet Domain). The strict Condorcet domain is the profile space \(\varphi_{CS} \subset L_S(X)^n\) for which at each profile \(p \in \varphi_{CS}\) there exists some alternative \(x_i \in X\) such that for all \(x_j \in X \setminus \{x_i\}\), \(|N_p(x_i, x_j)| > \frac{n}{2}\).

Qualitatively, the strict Condorcet domain consists of all profiles in \(L_S(X)^n\) at which there exists some alternative \(x_i\) that is preferred to every other alternative \(x_j\) by a majority of the individuals. In both definitions, we say that alternative \(x_i\) is the Condorcet winner at profile \(p\), and that a coalition is a majority coalition if \(|N_p(x_i \succ x_j)| > \frac{n}{2}\). In both the strong and weak definitions, it is clear that if a Condorcet winner exists, it must be unique.

We will next introduce a number of properties that social choice rules may or may not possess. The definitions of some of the properties depend on the profile space; when this is the case, we will present the definition of the property first on \(L_W(A)\) and then on \(L_S(A)\). Since the Condorcet domains are subsets of these more general domains, the definitions apply to the Condorcet domains in the natural way.

**Definition 1.2.5** (Weak Dictatorship). Let \(\varphi \subset L_W(X)\). Then a social choice rule \(g\) is dictatorial if there exists some \(i \in N\) such that for all \(p \in \varphi\), \(g(p) \in p_1(i)\). That is, the social choice rule selects some element from individual \(i\)'s set of most-preferred alternatives at every profile \(p \in \varphi\).

**Definition 1.2.6** (Strict Dictatorship). Let \(\varphi \subset L_S(X)\). Then a social choice rule \(g\) is dictatorial if there exists some \(i \in N\) such that for all \(p \in \varphi\), \(g(p) = p_1(i)\). That is, the social choice rule selects individual \(i\)'s most-preferred alternative at every profile \(p \in \varphi\).

Under both definitions, we say that individual \(i\) is the dictator.

**Definition 1.2.7** (Strategy-Proofness). A social choice rule is manipulable if there exists an individual \(i \in N\) and profiles \(p, q \in \varphi\) such that \(p(j) = q(j)\) for all \(j \neq i\) and alternative \(g(p) \succ_q g(q)\). If this is the case, we say that individual \(i\) can manipulate \(g\) at profile \(q\) via
the preference ordering $q(i)$. A social choice rule is strategy-proof if it is not manipulable.

**Definition 1.2.8** (Non–Reversal). A social choice rule satisfies non-reversal [5, 7] if for all $p \in \wp$, if $g(p) = x_i$, $x_j \in X$, and there is $k \in N$ for whom $x_j >_{p(k)} x_i$, then $g(p') \neq x_j$ for all $p' \in \mathcal{C}$ such that $p'(\ell) = p(\ell)$ for all $\ell \neq k$.

**Definition 1.2.9** (Weak Unanimity). Let $\wp \subset L_W(X)$. Then a social choice rule $g$ satisfies unanimity if $g(p) = x_j$ at any profile $p$ for which $x_j \in p_1(i)$ for all $i \in N$ and some $x_j \in X$.

**Definition 1.2.10** (Strict Unanimity). Let $\wp \subset L_S(X)$. Then a social choice rule $g$ satisfies unanimity if $g(p) = x_j$ at any profile $p$ such that $p_1(i) = x_j$ for all $i \in N$ and some $x_j \in X$. That is, the social choice rule selects the common most–preferred alternative at any profile where such an alternative exists.

In both definitions, we call such a profile $p$ a unanimous profile. When the profile space is a subset of $L_S(X)$, unanimity completely constrains social choice rules at unanimous profiles. When the profile space is a subset of $L_W(X)$ only, unanimity is much less binding, as the following example illustrates.

**Example 1.2.11**. Let $N = \{1, 2, 3\}$ be the set of individuals, $X = \{x_1, x_2, x_3, x_4\}$ be the set of alternatives, and $g$ be a unanimous social choice rule. Then at the unanimous profile in $L_S(X)$

$$
p = \begin{bmatrix}
1 & 2 & 3 \\
x_1 & x_1 & x_1 \\
x_4 & x_2 & x_3 \\
x_3 & x_4 & x_2 \\
x_2 & x_3 & x_4
\end{bmatrix}
$$

we have $g(p) = x_1$. However, at the unanimous profile in $L_W(X)$

$$
q = \begin{bmatrix}
1 & 2 & 3 \\
x_1 \sim x_2 & x_1 \sim x_2 \sim x_3 & x_1 \sim x_2 \sim x_4 \\
x_3 \sim x_4 & x_4 & x_3
\end{bmatrix},
$$

9
we could have either \( g(q) = x_1 \) or \( g(q) = x_2 \).

**Definition 1.2.12** (Monotonicity). Let \( g \) be a social choice rule, \( N = \{1, \ldots, n\} \) be the set of voters and \( X = \{x_1, \ldots, x_m\} \) be the set of alternatives. Let \( p, q \in L_W(X) \) be profiles such that there exists \( i \in N \) and \( x_k \in X \) such that \( p(j) = q(j) \) for all \( j \neq i \) and

\[
\{x_\ell \in X : x_k \succ_p(i) x_\ell\} \subset \{x_\ell \in A : x_k \succ_q(i) x_\ell\}.
\]

Then \( g \) is monotonic if and only if \( g(p) = x_k \Rightarrow g(q) = x_k \).

Qualitatively, monotonicity requires that if \( g \) selects an alternative \( x_k \) at some profile \( p \), it must also select \( x_k \) at every profile formed from \( p \) by promoting \( x_k \) in some individual’s preference ordering.

The previous definitions have referred to the behavior of a social choice rule over a fixed domain with a fixed number of individuals and alternatives. The following property concerns the behavior of a social choice rule with respect to varying numbers of individuals.

**Definition 1.2.13** (Consistency). Let \( g \) be a social choice rule, \( X \) be the set of alternatives, \( N_1 \) and \( N_2 \) be disjoint sets of individuals, \( p_1 \) a profile on \( N_1 \) and \( X \) and \( p_2 \) a profile on \( N_2 \) and \( X \), with \( g(p_1) = g(p_2) \). Let \( p \) be the profile on \( N = N_1 \cup N_2 \) such that \( p(i) = p_1(i) \) for all \( i \in N_1 \) and \( p(i) = p_2(i) \) for all \( i \in N_2 \). Then \( g \) is consistent if and only if \( g(p) = g(p_1) = g(p_2) \).

We have defined monotonicity and consistency so that we may formally introduce Arrow’s theorem in the next section; the reader may notice that the above definitions coincide with requirements (3) and (4) in the qualitative statement of Arrow’s theorem.

When studying the strategy-proofness of a particular rule, we often wish to create a “path” of profiles that lead from one specific profile to another. For arbitrary profiles \( p, q \in \varnothing \), we call \( \{q^t\} \) the standard sequence (“path”) from profile \( p \) to profile \( q \). This path starts at \( p \), so we set \( q^0 = p \). Next, we form \( q^{t+1} \) from profile \( q^t \) by replacing individual \( t+1 \)’s preference ordering with \( q(t+1) \). So for an arbitrary \( t \), \( q^t(i) = q(i) \) for all \( i = 1, \ldots, t \) and \( q^t(i) = p(i) \) for all \( i = t+1, \ldots, n \). Using this procedure, we have \( q^n = q \). When using a
standard sequence, we must ensure that \( q^t \in \varphi_C \) for all \( t = 0, \ldots, n \).

In general, \( p, q \in \varphi_C \) does not imply that \( \{q^t\} \in \varphi_C \), as the following example illustrates.

**Example 1.2.14.** Let \( N = \{1, 2, 3, 4\} \) be the set of individuals, \( X = \{x_1, x_2, x_3\} \) be the set of alternatives, and \( \varphi = \varphi_{CS} \). Consider the following profiles \( p, q \in \varphi_{CS} \).

\[
p = \begin{bmatrix} 1 & 2 & 3 & 4 \\ x_1 & x_3 & x_2 & x_1 \\ x_2 & x_1 & x_3 & x_3 \\ x_3 & x_2 & x_3 & x_2 \end{bmatrix}, \quad q = \begin{bmatrix} 1 & 2 & 3 & 4 \\ x_2 & x_2 & x_2 & x_1 \\ x_1 & x_3 & x_1 & x_3 \\ x_3 & x_1 & x_3 & x_2 \end{bmatrix}.
\]

At profile \( p \) alternative \( x_1 \) is the Condorcet winner, since

\[
|N_p(x_1 > x_2)| = |\{1, 2, 4\}| = 3 > \frac{4}{2}, \quad \text{and} \quad |N_p(x_1 > x_3)| = |\{1, 3, 4\}| = 3 > \frac{4}{2}.
\]

At profile \( q \) alternative \( x_2 \) is the Condorcet winner, since

\[
|N_q(x_2 > x_1)| = |\{1, 2, 3\}| = 3 > \frac{4}{2}, \quad \text{and} \quad |N_q(x_2 > x_3)| = |\{1, 2, 3\}| = 3 > \frac{4}{2}.
\]

If \( \{q^t\} \) is the standard sequence from \( p \) to \( q \), from above we have \( q^0, q^4 \in \varphi_{CS} \). However, the profile

\[
q^1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ x_2 & x_3 & x_2 & x_1 \\ x_1 & x_1 & x_3 & x_3 \\ x_3 & x_2 & x_3 & x_2 \end{bmatrix}
\]

is not in the domain \( \varphi_{CS} \), since in Condorcet winner exists at \( q^1 \).

Of the infinite social choice rules one could consider, we will focus primarily on majority rule, which is defined as follows.

**Definition 1.2.15.** Let \( N \) be the set of voters and \( X \) be the set of alternatives. Define majority rule to be the social choice rule \( g_C : \varphi \to X \), where for all \( p \in \varphi \) \( g \) selects the Condorcet winner at profile \( p \), if one exists.
We are motivated in our research first by the existence of impossibility theorems, which limit the results that can be obtained on arbitrary profile space domains. Having developed the necessary notation and definitions, we next formally introduce Arrow’s theorem, which largely motivated the modern study of social choice.

**Theorem 1.2.16** (Arrow’s Impossibility Theorem). Let $N$ be the set of voters and $X$ be the set of alternatives, with $|N| \geq 2$ and $|X| \geq 3$. Let $g : \wp^N \rightarrow X$ be a social choice rule. Then $g$ must fail to satisfy one of the following properties:

1. The rule is well defined over the domain $\wp = L_W(X)^n$.
2. $g$ is non–dictatorial.
3. $g$ is consistent.
4. $g$ is monotonic.
5. $g$ is onto.

We also formally introduce the Gibbard–Satterthwaite theorem, a corollary of Arrow’s theorem [13].

**Theorem 1.2.17** (Gibbard–Satterthwaite Theorem). Let $N$ be the set of voters and $X$ be the set of alternatives, with $|N| \geq 2$ and $|X| \geq 3$. Let $g : \wp \rightarrow X$ be a social choice rule. Then $g$ must satisfy one of the following properties:

1. $g$ is dictatorial.
2. $g$ does not have full range $X$ over the domain $\wp$.
3. $g$ is subject to strategic manipulation.

As we shall see, possibility results can be obtained by restricting the domain to be the Condorcet domain.
Chapter 2

The Strict Condorcet Domain

In this chapter we are concerned with characterizing non–dictatorial strategy–proof rules over the strict Condorcet domain. Our analysis will proceed in two general cases: when there are an odd number of individuals, and when there is an even number of individuals. The characterization for the odd case is provided by Campbell and Kelly [4]; this paper provides the characterization for the even case.

Throughout this chapter, we will use the notation $\varnothing$ to refer to $\varnothing_{C_S}$ since in every instance we refer to the strict Condorcet domain. Additional simplifying notation will be introduced as necessary.

2.1 An Odd Number of Individuals

The characterization of strategy–proof social choice rules over any domain is not trivial. In general, many of the standard rules are not strategy–proof on a given domain. The following example illustrates a common rule that is not strategy–proof on the strict Condorcet domain with an odd number of individuals.

Example 2.1.1. Let $N = \{1, 2, 3\}$ be the set of individuals and $X = \{x_1, x_2, x_3, x_4\}$ be the set of alternatives. At any profile $p \in \varnothing$, 

define Borda’s rule \( g_B(p) \) by the following procedure

1. For each \( i \in N \), individual \( i \) gives \( 4 - j \) points to the \( j \)th ranked in \( p(i) \).

2. Let \( g_B(p) \) select the alternative with the largest number of points.

Consider the profile

\[
p = \begin{bmatrix}
1 & 2 & 3 \\
x_1 & x_1 & x_2 \\
x_2 & x_2 & x_1 \\
x_3 & x_3 & x_3 \\
x_4 & x_4 & x_4 \\
\end{bmatrix},
\]

at which the Borda scores are

\[x_1 : 11, \quad x_2 : 10, \quad x_3 : 6, \quad x_4 : 3,\]

so that \( g_B(p) = x_1 \). Next consider the profile

\[
q = \begin{bmatrix}
1 & 2 & 3 \\
x_1 & x_1 & x_2 \\
x_2 & x_2 & x_3 \\
x_3 & x_3 & x_4 \\
x_4 & x_4 & x_1 \\
\end{bmatrix},
\]

which is formed from profile \( p \) by changing individual 3’s preference ordering. At \( q \) the Borda scores are

\[x_1 : 9, \quad x_2 : 10, \quad x_3 : 7, \quad x_4 : 4,\]

so that \( g_B(q) = x_2 \). Since \( x_2 = g_B(q) \succ_{p(3)} g_B(p) = x_1 \), individual 3 can manipulate at profile \( p \) via the preference ordering \( q(3) \).

Campbell and Kelly [4] have shown that majority rule is the unique non–dictatorial strategy–proof rule on the strict Condorcet domain when there are an odd number of voters.

**Theorem 2.1.2** (Campbell & Kelly). Assume that \( n > 1 \) is odd
and $X$ has at least three members. If $X$ is finite and $g : \emptyset \to X$ is a strategy–proof and non–dictatorial rule with range $X$, then $g$ is majority rule. If $X$ is infinite and $g : \emptyset \to X$ is a strategy–proof rule, with range $X$, then $g$ is majority rule.

The proof consists of three steps. The first step establishes that a strategy–proof rule on the strict Condorcet domain must be unanimous. The second step shows that if a social choice rule $g$ is non–dictatorial then $g$ must select alternative $x_i$ at every profile where more than half of the individuals have $x_i$ top–ranked. In the third step, the results of the first and second steps are used to show that in fact $g$ coincides with majority rule.

The hypothesis that $n$ is odd is crucial to their proof. Without this assumption it is impossible to complete the first step of their proof because in fact it is not true; strategy–proof social choice rules on the Condorcet domain with an even number of individuals need not satisfy unanimity. The following example demonstrates this result.

Example 2.1.3. Let $N = \{1, \ldots, n\}$ be the set of individuals and $X = \{x_1, x_2, x_3\}$ be the set of alternatives. Suppose that $n$ is even and define a social choice rule $g$ on $\emptyset$ as follows: if alternative $x_i$ is the Condorcet winner at $p$, then $g(p) = x_2$; if $x_2$ is the Condorcet winner, then $g(p) = x_3$; if $x_3$ is the Condorcet winner, then $g(p) = x_1$.

This rule is indeed strategy–proof over the domain (we will see that no individual can manipulate any social choice rule to change the Condorcet winner when there are an even number of individuals). However, it is not majority rule and furthermore does not satisfy unanimity: at any profile where all individuals have the same top–ranked alternative, that alternative is not selected. Thus in the case of an even number of individuals, there exist non–dictatorial strategy–proof social choice rules distinct from majority rule, including rules that do not satisfy unanimity.

We note that the rule in the previous example is defined over the Condorcet domain for an odd number of individuals, but is not
strategy–proof as the following example illustrates.

**Example 2.1.4.** Let $N = \{1, 2, 3\}$ be the set of individuals and $X = \{x_1, x_2, x_3\}$ be the set of alternatives. Define a social choice rule $g$ on $\varnothing$ as follows: if alternative $x_i$ is the Condorcet winner at $p$, then $g(p) = x_2$; if $x_2$ is the Condorcet winner, then $g(p) = x_3$; if $x_3$ is the Condorcet winner, then $g(p) = x_1$.

Consider the profile

$$p = \begin{bmatrix}
1 & 2 & 3 \\
 1 & x_3 & x_1 \\
 x_2 & 1 & x_3 \\
x_3 & x_2 & 2
\end{bmatrix}$$

where it can be verified that $x_1$ is the Condorcet winner, so that $g(p) = x_2$. Next, consider the profile

$$q = \begin{bmatrix}
1 & 2 & 3 \\
 3 & x_3 & x_1 \\
 x_2 & 1 & x_3 \\
x_1 & x_2 & 2
\end{bmatrix}$$

for which $q(i) = p(i)$ for $i = 2, 3$. At profile $q$, alternative $x_3$ is the Condorcet winner, so that $g(q) = x_1$. Since $x_1 = g(q) \succ_p 1 g(p) = x_2$, individual 1 can manipulate at $p$ by reporting the preference ordering $q(1)$. Therefore, $g$ is not strategy–proof.

By example there exist strategy–proof rules on the Condorcet domain for an even number of individuals that do not satisfy the unanimity property. To further investigate the Campbell–Kelly result as it pertains to an even number of voters, let us restrict our attention to rules that satisfy unanimity. The following example proposes such a rule.

**Example 2.1.5.** Let $N = \{1, \ldots, n\}$ be the set of individuals and $X = \{x_1, x_2, x_3\}$ be the set of alternatives. Suppose that $n > 2$ is even and define a social choice rule $g$ on $\varnothing$ as follows: if alternative $x_1$ is the Condorcet winner at $p$, $g(p) = p_1(1)$; if $x_2$ is the Condorcet
winner, \( g(p) = p_1(2) \); if \( x_3 \) is the Condorcet winner, \( g(p) = p_1(3) \). That is, individual \( i \) is the dictator over the subset of the domain for which alternative \( x_i \) is the Condorcet winner, for \( i = 1, 2, 3 \).

The social choice rule defined in the previous example obviously satisfies unanimity. Furthermore, it is non–dictatorial and strategy–proof over the entire domain but vastly different from majority rule. Thus the results of Campbell and Kelly [4] cannot be extended to an even number of individuals by simply requiring unanimity. Again we note that \( g \) is defined over the Condorcet domain for an odd number of individuals but is not strategy–proof, as the following example illustrates.

**Example 2.1.6.** Let \( N = \{ 1, \ldots, n \} \) be the set of individuals and \( X = \{ x_1, x_2, x_3 \} \) be the set of alternatives. Suppose that \( n > 2 \) is even and define a social choice rule \( g \) on \( \varnothing \) as follows: if alternative \( x_1 \) is the Condorcet winner at \( p \), \( g(p) = p_1(1) \); if \( x_2 \) is the Condorcet winner, \( g(p) = p_1(2) \); if \( x_3 \) is the Condorcet winner, \( g(p) = p_1(3) \).

Consider the profile

\[
p = \begin{bmatrix}
1 & 2 & 3 \\
3 & 1 & 2 \\
1 & x_2 & x_3 \\
x_2 & x_3 & 2 \\
\end{bmatrix}
\]

where it can be verified that \( x_1 \) is the Condorcet winner, so that \( g(p) = p_1(1) = x_3 \). Next, consider the profile

\[
q = \begin{bmatrix}
1 & 2 & 3 \\
3 & 3 & 1 \\
1 & x_2 & x_3 \\
x_2 & x_1 & 2 \\
\end{bmatrix}
\]

for which \( q(i) = p(i) \) for \( i = 1, 3 \). At profile \( q \), alternative \( x_3 \) is the Condorcet winner, so that \( g(q) = p_1(3) = x_1 \). Since \( x_1 = g(q) \succ_p(2) \), \( g(p) = x_3 \), individual 2 can manipulate at \( p \) by reporting the preference ordering \( q(2) \). Therefore, \( g \) is not strategy–proof.
We have seen several examples illustrating how the parity of the number of voters has a dramatic impact on the strategy–proofness of a given social choice rule. In the next section, we characterize strategy–proof rules over the Condorcet domain with an even number of individuals.

2.2 An Even Number of Individuals

When there are an odd number of individuals, Campbell and Kelly [4] have maj ority rule is the unique non–dictatorial strategy–proof social choice rule over the Condorcet domain. We will show that when there are are an even number of individuals, there is a large family of strategy–proof rules, including some that are rather exotic. This result hinges on our ability to partition the Condorcet domain into a number of subdomains between which individuals are unable to manipulate. Such a separation is not possible for an odd number of individuals. This restriction on individuals’ ability to manipulate allows flexibility in forming strategy–proof social choice rules, so that majority rule is but one of many.

We begin with a characterization of strategy–proof rules where there are two individuals. By considering such a limited case, our result and proof are relatively straightforward. We will therefore use this case as a means of developing intuition for the general case of an even number of individuals. When considering the general case, we will structure our proof to mimic the structure of the two individual cases.

2.2.1 A Characterization for Two Individuals

We next characterize the family of social choice rules that are non–dictatorial and strategy–proof over the strict Condorcet domain \( \mathcal{P} \) with two individuals and three or more alternatives. By limiting the number of individuals to \( n = 2 \) our domain can be characterized more precisely than in the general case of any even number \( n \) of individuals. In fact, the set of profiles that admit a Condorcet
winner with \( n = 2 \) individuals is the set of all profiles for which both individuals have the same most-preferred alternative.

This characterization leads to a useful partition of the domain. We use the notation \( \varphi_{x_i} \) to indicate the set of profiles in the domain for which alternative \( x_i \) is the Condorcet winner. Clearly, we have \( \varphi_{x_i} \subset \varphi \), and we will accordingly refer to \( \varphi_{x_i} \) as a subdomain. Furthermore, we will use the notation \( \mathcal{P}_{x_i} \) to indicate the set of profiles in the domain for which alternative \( x_i \) is the Condorcet winner. Clearly, we have \( \mathcal{P}_{x_i} \subset \mathcal{P} \), and we will accordingly refer to \( \mathcal{P}_{x_i} \) as a subdomain.

Furthermore, we will use the notation \( X_{x_i} \) to indicate the set of alternatives that are selected over the subset \( \varphi_{x_i} \); that is, the range of \( g \) restricted to \( \varphi_{x_i} \). This notation is necessary because while we require that \( g \) be onto with respect to the domain \( \varphi \), we cannot guarantee that every alternative will be selected over a given subset of the domain.

To motivate our discussion, let us first consider an example of a social choice rule over \( \varphi \) that is not strategy-proof.

**Example 2.2.1.** Let \( V = \{1, 2\} \) be the set of voters, \( X = \{x_1, x_2, x_3\} \) be the set of alternatives, and \( g \) be a social choice rule over \( \varphi \) defined by

\[
g(p) = \begin{cases} p_2(1) & \text{if } p(1) = p(2) \\ p_1(1) & \text{otherwise} \end{cases},
\]

for all profiles \( p \in \varphi \). For profiles

\[
p = \begin{bmatrix} 1 & 2 \\ x_1 & x_1 \\ x_2 & x_3 \\ x_3 & x_2 \end{bmatrix}, \quad q = \begin{bmatrix} 1 & 2 \\ x_1 & x_1 \\ x_2 & x_2 \\ x_3 & x_3 \end{bmatrix},
\]

we have \( g(p) = x_1 \) and \( g(q) = x_2 \). At profile \( q \), both individuals prefer alternative \( x_1 \) to the outcome, \( x_2 \). At \( q \), individual 2 can elicit the selection of alternative \( x_1 \) by reporting the preference ordering \( p(2) \), since \( g(p) = x_1 \succ_{g(2)} x_2 = g(q) \). Therefore, the social choice rule \( g \) is manipulable.

Having considered this example, we will state the following theorem and proceed with a formal proof.

**Theorem 2.2.2.** Let \( V = \{1, 2\} \) be the set of voters, \( X = \{x_1, x_2, x_3, \ldots\} \)
be the set of alternatives, and \( g \) be a social choice rule with domain \( \varphi \). Then \( g \) is strategy–proof and non–dictatorial if and only if:

(i) If \( g(p) = x_i \) for any profile \( p \in \varphi_{x_i} \), then \( g(p) = x_i \) for all profiles \( p \in \varphi_{x_i} \).

(ii) If there exists \( x_i \in X \) such that \( |X_{x_i}| \geq 2 \), then there exists \( k \in N \) such that individual \( k \) is dictatorial over \( \varphi_{x_i} \) with respect to the set of alternatives \( X_{x_i} \).

(iii) There exists some \( x_i \in X \) such that for all \( p \in \varphi_{x_i} \), \( g(p) \neq x_i \).

We prove the theorem by proving three lemmas concerning the strategy–proofness of three classes of social choice rules whose disjoint union is the set of all social choice rules: (1) rules that always select the common top–ranked alternative; (2) rules that never select the common top–ranked alternative; (3) rules that sometimes select the common top–ranked alternative and sometimes do not.

**Class 1 Rules**

We first consider rules that always select the common top–ranked alternative.

**Lemma 2.2.3.** Let \( V = \{1,2\} \) be the set of voters, \( X = \{x_1, x_2, x_3, \ldots \} \) be the set of alternatives, and \( g \) be a social choice rule with domain \( \varphi \). If \( g(p) = p_1(1) \) for all \( p \in \varphi \), then \( g \) is strategy–proof.

**Proof.** Let \( V = \{1,2\} \) be the set of voters, \( X = \{x_1, x_2, x_3, \ldots \} \) be the set of alternatives, and \( g \) be a social choice rule such that \( g(p) = p_1(1) \) for all \( p \in \varphi \). Let \( p, q \in \varphi \) be such that \( p_1(1) = q_1(1) \), and let \( \{q^t\} \) be the standard sequence from \( p \) to \( q \). Since \( p_1(2) = q_1(1) = q_1(2) \), the have \( \{q^t\} \subset \varphi \).

At profile \( p \), \( g(p) = p_1(1) \) by construction. Furthermore, \( g(q) = q_1(1) \). Since \( q_1(1) = p_1(1) \), and all the profiles to which individual 1 can manipulate are of the form \( q \), then individual 1 cannot manipulate to elicit the selection of an alternative other than \( p_1(1) \). Since the social choice rule \( g \) treats both individuals symmetrically,
individual 2 cannot manipulate $g$ by the same reasoning. Therefore, $g$ is strategy–proof.

Unfortunately, it is clear that any rule that always selects the common top–ranked alternative is dictatorial on $\varphi$ by our previous definition. This can easily be shown by selecting either individual, say individual 1, and noting that at all profiles $p \in \varphi$, $g(p) = p_1(1)$ when $g$ always selects the common top–ranked alternative. Furthermore, we may note that any rule that does not always select the common top–ranked alternative cannot be dictatorial on $\varphi$: a rule that does not always select the common top–ranked alternative equivalently does not always select some individual’s top–ranked alternative, so that neither individual is a dictator.

Using our original definition of dictatorship, we must therefore require that there exists some subdomain $\varphi_{x_i}$ for which $x_i \notin X_{x_i}$. However, we note that when there only two individuals, all profiles in the domain $\varphi$ are unanimous, so that to reject a dictatorial rule is to reject a unanimous rule. We will therefore adopt an alternative definition of dictatorial rule.

**Definition 2.2.4 (Non–unanimous Dictatorship).** Let $\varphi \subset L_S(X)$. Then a social choice rule $g$ is dictatorial if there exists some $i \in N$ such that for all $p \in \varphi$, $g(p) = p_1(i)$ and there exists some $p \in \varphi$ that is not a unanimous profile.

If we adopt this new definition, we have that Class 1 rules are strategy–proof and non–dictatorial. We next consider a concrete example of a Class 1 social choice rule $g$ over $\varphi$ with two individuals.

**Example 2.2.5.** Let $V = \{1, 2\}$ be the set of voters, $X = \{x_1, x_2, x_3, \ldots\}$ be the set of alternatives, and $g$ be a Class 1 social choice rule. Then at profiles $p, q \in \varphi$,

$$p = \begin{bmatrix} 1 & 2 \\ x_i & x_i \\ \vdots & \vdots \end{bmatrix}, \quad q = \begin{bmatrix} 1 & 2 \\ x_j & x_j \\ \vdots & \vdots \end{bmatrix},$$

we have $g(p) = x_i$ and $g(q) = x_j$. Note that neither individual faces
the incentive to manipulate, because their most–preferred alternative is selected at both profiles \( p \) and \( q \). Furthermore, profile \( p \) if either individual reported a preference ordering with some alternative \( x_k \neq x_i \) top–ranked, the resulting profile would not admit a Condorcet winner, and would thus not be part of our domain.

Class 2 Rules

We next consider rules that never select the common top–ranked alternative.

**Lemma 2.2.6.** Let \( V = \{1, 2\} \) be the set of voters, \( X = \{x_1, x_2, x_3, \ldots \} \) be the set of alternatives, and \( g \) be a social choice rule with domain \( \varphi \). If \( g(p) \neq p_1(1) \) for all \( p \in \varphi \), then \( g \) is strategy–proof if and only if \( g \) is dictatorial over each subdomain \( \varphi_{x_i} \) with respect to the set of alternatives \( X_{x_i} \).

**Proof.** Let \( V = \{1, 2\} \) be the set of voters, \( X = \{x_1, x_2, x_3, \ldots \} \) be the set of alternatives, and \( g \) be a social choice rule such that \( g(p) \neq p_1(1) \) for all \( p \in \varphi \). Although we have assumed that \( g \) has full range \( X \) over the domain \( \varphi \), we have made no assumptions about the range of \( g \) over subsets of the domain, \( \varphi_{x_i} \). To this end, we will separately consider the cases when \(|X_{x_i}| = 1\), \(|X_{x_i}| = 2\), and \( X_{x_i} \geq 3\).

Suppose that \(|X_{x_i}| = 1\) for some sub–domain \( \varphi_{x_i} \), so that there exists \( x_j \in X \) such that \( g(p) = x_j \) for all \( p \in \varphi_{x_i} \) (since \( g \) never selects the common top–ranked alternative, \( i \neq j \)). Since \( g \) selects the same alternative at every profile \( p \in \varphi_{x_i} \), neither individual can manipulate \( g \) over \( \varphi_{x_i} \); no individual can elicit the selection of an alternative other than \( x_j \) on \( \varphi_{x_i} \), and no individual can report a preference ordering with an alternative other than \( x_i \) most–preferred. A social choice rule that never selects the common top–ranked alternative is therefore strategy–proof over all subsets \( \varphi_{x_i} \subset \varphi \) for which \(|X_{x_i}| = 1\).

Suppose that \(|X_{x_i}| = 2\), so that at all \( p \in \varphi_{x_i} \), \( g(p) = x_j \) or \( g(p) = x_k \) for some distinct \( x_j, x_k \in X \). By assumption, \( x_j, x_k \neq x_i \).
For some enumeration of the alternatives, either both individuals prefer alternative \( x_j \) to alternative \( x_k \), or one individual prefers \( x_j \) to \( x_k \) and the other prefers \( x_k \) to \( x_j \).

If both individuals prefer \( x_j \) to \( x_k \) and \( g(p) = x_j \), then neither individual can elicit the selection of a more-preferred alternative via manipulation, and our rule is strategy-proof. If both individuals prefer \( x_j \) to \( x_k \) and \( g(p) = x_j \), then \( g \) is strategy-proof if and only if \( g(p) = x_j \) for all \( p \in \wp_x \). To verify this, let \( p \) be a profile for which both individuals have \( x_j \) ranked above \( x_k \) and \( g(p) = x_j \), and let \( p' \) be any profile generated by permuting the ordering of individual 1’s non-top-ranked alternatives while holding individual 2’s preference ordering constant. If at any \( p' \), \( g(p') = x_j \), then individual 1 can manipulate at profile \( p \) by reporting a preference ordering of \( p'(1) \). Therefore, we must have that \( g(p') = x_k \) for all profiles \( p' \). Now, let \( q^{ij} \) be any profile generated from a profile \( p' \) by holding individual 1’s preference ordering constant and permuting the ordering of individual 2’s non-top-ranked alternatives. If at any \( q^{ij} \), \( g(q^{ij}) = x_j \), then individual 2 can manipulate at profile \( p' \) by reporting a preference ordering of \( q^{ij}(2) \). Therefore, we must have that \( g(q^{ij}) = x_k \) for all profiles \( q^{ij} \).

Since every profile in \( \wp_x \) can be generated by considering all permutations of non-top-ranked alternatives at profile \( p \), we therefore have that if \( g(p) = x_k \), then \( g \) is strategy-proof if and only if \( g \) selects alternative \( x_k \) at every profile \( p \in \wp_x \). Since this would contradict our assumption that \( |X_x| = 2 \), a strategy-proof social choice rule that never selects the common top-ranked alternative also cannot select the common bottom-ranked alternative from \( X_x \), when one exists at any \( p \in \wp_x \), for which \( |X_x| = 2 \).

Suppose now that for some arbitrary enumeration of individuals and alternatives, individual 1 prefers \( x_j \) to \( x_k \) and individual 2 prefers \( x_k \) to \( x_j \). If \( g(p) = x_j \), then individual 1 cannot precipitate the selection of a more preferred alternative by manipulation, since \( x_j \) is his or her most-preferred alternative in \( X_x \). Furthermore, individual 2 can only attempt to manipulate permuting the ordering of his non-top-ranked alternatives. If there exists a profile \( p' \) formed
by holding individual 1’s preference ordering constant at \( p \) and permuting the order of individual 2’s non–top–ranked alternatives such that \( g(p') = x_k \), then individual 2 can manipulate at profile \( p \) by reporting the preference ordering \( p'(2) \). Since the above enumeration of individuals was arbitrary, we have that if \( g \) is strategy–proof and \( |X_{x_i}| = 2 \), then \( g \) must select some fixed individuals’ most–preferred alternative from \( X_{x_i} \) at every \( p \in \varnothing_{x_i} \).

The result of the previous paragraph is consistent with our previous result that \( g \) cannot select a common–bottom ranked alternative from \( X_{x_i} \) at any profile where such an alternative exists. Since \( |X_{x_i}| = 2 \), we may suppose that \( X_{x_i} = \{x_j, x_k\} \) (by assumption, \( x_i \notin X_{x_i} \)). If there exists \( p \in \varnothing_{x_i} \), such that \( x_j \succ_{p(\ell)} x_k \) for \( \ell = 1, 2 \), then by the first result \( g(p) = x_j \), since there are only two alternatives in the range. Furthermore, \( x_j \) is some (in fact, both) individual’s most–preferred alternative from \( X_{x_i} \).

Suppose now that \( |X_{x_i}| \geq 3 \), and note that within the Condorcet domain preferences are unrestricted with respect to the set of alternatives in \( X \setminus \{x_i\} \). Moreover, note that by construction \( g \) has full range \( X_{x_i} \) over \( \varnothing_{x_i} \). These conditions satisfy the hypothesis of the Gibbard–Satterthwaite Theorem, so that we may apply the results to \( g \) over \( \varnothing_{x_i} \) with respect to the feasible alternatives \( X_{x_i} \). Therefore, \( g \) is strategy–proof over \( \varnothing_{x_i} \), if and only if \( g \) selects some individual i’s most–preferred alternative from the set \( X_{x_i} \) at every profile \( p \).

We note that the value of \( i \) must be constant over each subdomain \( p \in \varnothing_{x_i} \), but may vary between subsets domains.

To see that this characterization holds when we re–admit the alternatives that are never selected over the domain \( \varnothing_{x_i} \), first suppose that \( g \) is dictatorial over \( \varnothing_{x_i} \) with respect to the restricted set of alternatives \( X_{x_i} \). That is, there exists some individual \( i \) such that \( g \) selects individual \( i \)’s most–preferred alternative from the set \( X_{x_i} \) at every profile \( p \in \varnothing_{x_i} \). First, we note that individual \( i \) cannot precipitate the selection of any alternative he or she prefers to the one selected by \( g \), as \( g \) selects her most–preferred alternative from the feasible set \( X_{x_i} \). Furthermore, individual \( j \) cannot affect the outcome of \( g \) at any profile \( p \in \varnothing_{x_i} \), so that individual \( j \) cannot benefit from
reporting a false preference ordering. Moreover, individual $j$ cannot change the status of the Condorcet winner while remaining within the domain $\wp$.

Note further that any social choice rule $g$ that is not strategy-proof over $\wp_{x_i}$ with respect to the set of alternatives $X_{x_i}$ is also not strategy-proof when we readmit the remaining alternatives: if some individual $i$ can manipulate $g$ by permuting their ordering of alternatives in $X_{x_i}$, they can perform the same permutation when the irrelevant alternatives are reintroduced. (Imagine reintroducing the additional alternatives by appending them to the bottom of each individual’s preference ordering.) Therefore, when $|X_{x_i}| \geq 3$ a social choice rule $g$ is strategy-proof over domain $\wp_{x_i}$ if and only if $g$ selects individual $i$’s most-preferred alternative from $X_{x_i}$, for some value of $i$ that is fixed over each subdomain $\wp_{x_i}$ (but not necessarily across subdomains).

We have therefore completely characterized the set of strategy-proof social choice rules that never select the common top-ranked alternative. Next we consider a concrete example.

**Example 2.2.7.** Let $N = \{1, 2\}$ be the set of individuals, $X = \{x_1, x_2, x_3, x_4\}$ be the set of alternatives, and $g$ be a strategy-proof social choice rule with domain $\wp$ that never selects the common top-ranked alternative. Furthermore, suppose that $X_{x_1} = \{x_2, x_3\}$. Consider the profiles

$$
\begin{bmatrix}
1 & 2 \\
1 & 1 \\
2 & 2 \\
4 & 3 \\
3 & 4
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 2 \\
1 & 1 \\
4 & 3 \\
2 & 2 \\
3 & 4
\end{bmatrix},
$$

at which $g(p) = x_2$ and $g(q) = x_3$. Since by our previous lemma we know that $g$ must be dictatorial over $\wp_{x_1}$ with respect to the set of alternatives $X_{x_1}$, we can deduce that individual 2 is the dictator of $g$ over $\wp_{x_1}$. 

25
Finally, we will restrict our attention to those rules that sometimes select the common top–ranked alternative and sometimes do not. We shall see that the strategy–proofness of these rules can be decomposed into the previous two lemmas.

Lemma 2.2.8. Let \( V = \{1, 2\} \) be the set of voters, \( X = \{x_1, \ldots, x_m\} \) be the set of alternatives, and \( g \) be a social choice rule with domain \( \wp \). If there exist \( p, q \in \wp \) such that \( g(p) = p_1(1) \) and \( g(q) = q_k(1) \) for some \( k = 2, \ldots, m \), then \( g \) is strategy–proof if and only if

(i) If there exists \( p \in \wp_{x_i} \) such that \( g(p) = x_i \), then \( g(p) = x_i \) for all \( p \in \wp_{x_i} \).

(ii) If there exists a subdomain \( \wp_{x_i} \), over which \( |X_{x_i}| \geq 2 \), then \( x_i \notin X_{x_i} \) and \( g \) is dictatorial over \( \wp_{x_i} \) with respect to the set of alternatives \( X_{x_i} \).

Proof. Let \( V = \{1, 2\} \) be the set of voters, \( X = \{x_1, x_2, x_3, \ldots\} \) be the set of alternatives, and \( g \) be a social choice rule with domain \( \wp \). Moreover, suppose that there exist \( p, q \in \wp \) such that \( g(p) = p_1(j) \) and \( g(q) = q_k(j) \) for some \( k \neq 1 \) and \( j = 1, 2 \).

For an arbitrary subdomain \( \wp_{x_i} \), suppose that at some profile \( p \in \wp_{x_i} \), \( g(p) = p_1(1) \), so that our social choice rule selects the common top–ranked alternative at profile \( p \). Consider any profile \( q \in \wp_{x_i} \) formed from profile \( p \), in which we hold individual 1’s preference ordering constant and permute the non–top–ranked alternatives in individual 2’s preference ordering. If \( g(q) \neq p_1(1) \), then individual 2 can manipulate at \( q \) via \( p \) by reporting a preference ordering \( p(2) \), which will ensure the selection of \( p_1(1) = p_1(2) = q_1(2) \), individual 2’s top–ranked alternative at profile \( q \). At profile \( p \) (respectively, at every profile \( q \) created by permuting individual 1’s non–top–ranked alternatives at \( p \)) we may next consider any profile \( q' \in \wp_{x_k} \) for which we hold individual 1’s preference ordering constant and permute the non–top–ranked alternatives in individual 2’s preference ordering. If \( g(q') \neq p_1(1) \) then individual 2 can manipulate at \( q' \) via \( p \) (respectively, via \( q \)) to ensure the selection of \( p_1(1) = q_1(2) \),
which individual 2 prefers to every other alternative at all profiles \( q' \). Therefore, if \( g \) selects the common top-ranked alternative at some profile in \( \varphi_{x_i} \), it is strategy-proof if and only if it selects the common top-ranked alternative at every profile in \( \varphi_{x_i} \).

Let us return to our arbitrary subset \( \varphi_{x_i} \) and suppose that \( g \) never selects the common top-ranked alternative at any \( p \in \varphi_{x_i} \). Then we can consider our social choice rule \( g \) a Class 2 rule over domain \( \varphi_{x_i} \) and apply the previous lemma.

We have therefore completely characterized the set of strategy-proof rules on domain \( \varphi \) with \( n = 2 \). Using our alternative definition of non-unanimous dictatorial rules, we may state the following corollary:

**Corollary 2.2.9.** Let \( V = \{1, 2\} \) be the set of voters, \( X = \{x_1, x_2, x_3, \ldots\} \) be the set of alternatives, and \( g \) be a social choice rule with domain \( \varphi \). Then \( g \) is strategy-proof and non-dictatorial if and only if:

1. If there exists \( p \in \varphi_{x_i} \) such that \( g(p) = x_i \), then for all \( p \in \varphi_{x_i} \), \( g(p) = x_i \).
2. If there exists a subdomain \( \varphi_{x_i} \) over which \( |X_{x_i}| \geq 2 \), then \( x_i \notin X_{x_i} \) and \( g \) must be dictatorial over \( \varphi_{x_i} \) with respect to the set of alternatives \( X_{x_i} \).

Having characterized the family of non-dictatorial and strategy-proof social choice rules over the Condorcet domain with two individuals, we next consider the case of an arbitrary even number of individuals. Our proof in the general case will follow the structure of the case of two individuals.

### 2.2.2 A Characterization for any Even Number of Individuals

Let \( g \) be a strategy-proof social choice rule for an even number of individuals \( n \) with domain \( \varphi \) and range \( X \). We will first establish some useful lemmas and definitions.
Definition 2.2.10 (Restriction of a Social Choice Rule). Let $g$ be a social choice rule with domain $\varphi$. Let $P \subseteq \varphi$. Define $g|_P$ to be the restriction of $g$ to $P$ defined by mapping $p \in P$ into $g(p)$.

Throughout our proof, we will consider the restriction of our social choice rule $g$ to certain subsets of the domain.

Lemma 2.2.11 (Decomposition Lemma). Let $g$ be a social choice rule with domain $\varphi$. Suppose there exists a partition $\Pi$ of $\varphi$ such that for any $P \in \Pi$ and any $i \in N$, if $p \in P$, $q \in \varphi$, and $q(j) = p(j)$ for all $j \neq i$, then $q \in P$. Then $g$ is strategy-proof if and only if $g|_P$ is strategy-proof for all $P \in \Pi$.

The proof of this lemma is straightforward: if from any profile $p \in P$ no individual can manipulate to a profile $q \in \varphi \setminus P$, then strategy-proofness within each $P$ is equivalent to strategy-proofness over the entire domain.

Definition 2.2.12. Denote by $\varphi_{x_i}$ the set of all profiles at which alternative $x_i$ is the Condorcet winner. We will call the set $\varphi_{x_i}$ a Condorcet section.

It can be easily verified that the set of Condorcet sections produces a partition for even $n$. Furthermore this partition satisfies the Decomposition Lemma, so that we can reduce the question of strategy-proofness over the entire Condorcet domain $\varphi$ to a question of strategy-proofness over an arbitrary Condorcet section, $\varphi_{x_i}$. It is therefore sufficient to characterize $g|_{\varphi_{x_i}}$ for an arbitrary $x_i$. As before, let $X_{x_i}$ denote the range of $g|_{\varphi_{x_i}}$. With this partition and decomposition established, we state our main result.

Theorem 2.2.13. Suppose $g$ is a social choice rule over the strict Condorcet domain for an even number of individuals, with range $X$. Then $g$ is strategy-proof if and only if:

(i) If there exists $x_i \in X$ such that $|X_{x_i}| = 2$, then $g|_{\varphi_{x_i}}$ satisfies non-reversal;

(ii) If there exists $x_i \in X$ such that $|X_{x_i}| \geq 3$, then $x_i \notin X$ and $g|_{\varphi_{x_i}}$ is dictatorial with respect to the set of alternatives $X_{x_i}$.
To structure our proof of the theorem, we consider three cases: $|X_x| = 1$, $|X_x| = 2$, and $|X_x| \geq 3$. In each case we will state and prove a proposition characterizing strategy-proof rules within the specified framework.

For notational convenience, we will drop the subscripts from our standard notation for the remainder of this section, as we are only interested in a single arbitrary subdomain $\wp_x$. To this end, let our Condorcet section be simply $\wp$, over which alternative $x$ is the Condorcet winner. Furthermore, let us label the elements of the range of $g|_\wp$, so that $X_x = \{y, z, \ldots\}$; in general we will make no assumptions as to whether $x \in X_x$ or $x \notin X_x$. Lastly, we will simply refer to our social choice rule over this subdomain as $g$ instead of $g|_\wp$.

Case I: $|X_x| = 1$

Proposition 2.2.14. Any social choice rule $g$ over $\wp$ with $|X_x| = 1$ is strategy-proof.

Of course, any social choice rule on any domain is strategy-proof if it has a singleton range. We note that the proof in this case holds whether $X_x = \{x\}$ or $X_x \neq \{x\}$. We further note that majority rule is comprised of this case, with the range over each Condorcet section being a singleton set containing the Condorcet winner.

Case II: $|X_x| = 2$

Proposition 2.2.15. A social choice rule $g$ over $\wp$ with $|X_x| = 2$ is strategy-proof if and only if $g$ satisfies non-reversal.

Proof. To show necessity, suppose that $g$ does not satisfy the non-reversal condition. So there are profiles $p, p' \in \wp$ such that $g(p) = z$ and $g(p') \neq z$. Furthermore, $g(p') = y$ and there exists some individual $i$ for whom $y \succ_{p(i)} z$, and $p'(j) = p(j)$ for all $j \neq i$. Therefore, individual $i$ could manipulate at profile $p$ via $p'(i)$ to precipitate the selection of alternative $y$. 

29
To show sufficiency, suppose that $g$ satisfies the non-reversal condition. If $g(p) = z$ and $z \succ_{p(i)} y$, then individual $i$ cannot precipitate the selection of a more-preferred alternative by reporting a preference ordering other than $p(i)$. If $g(p) = z$ and $y \succ_{p(i)} z$, then by the non-reversal condition individual $i$ cannot unilaterally precipitate the selection of any other alternative. Thus, $g$ is strategy-proof at all profiles $p$.

Here again, we note that the result holds whether $x \in X_x$ or $x \notin X_x$. Furthermore, this result applies more generally: any social choice rule over any domain with a two-element range is strategy-proof if and only if it satisfies non-reversal.

Case III: $|X_x| \geq 3$

**Proposition 2.2.16.** A social choice rule $g$ over $\wp$ with $|X_x| \geq 3$ is strategy-proof if and only if $g$ is dictatorial with respect to the set of alternatives $X_x$.

Sufficiency is clear in this case, so our proof will focus on necessity and is based on two subcases: when $x \in X_x$ and when $x \notin X_x$. When $x \in X_x$, the structure and content of the proof is similar to Campbell and Kelly [4], but differs at several crucial steps. In the first step, we show that if $x$ is selected anywhere over $\wp$ then $x$ must be selected at all unanimous profiles. The second step establishes that if $g$ is non-dictatorial, then $x$ must be selected when over half of the individuals have $x$ top-ranked. The third step shows that either $g$ is majority rule or $g$ is dictatorial, and that majority rule is inconsistent with $|X_x| \geq 3$.

When $x \notin X_x$, the proof consists of two steps. In the first step we invoke the Gibbard–Satterthwaite theorem over a certain subset of the Condorcet section to show that $g$ restricted to this subset must be dictatorial. In the second step, we show that strategy-proofness implies that the dictator over this subset must indeed be the dictator over the entire Condorcet section with respect to the set of alternatives $X_x$. 
Lemma 2.2.17. Let $g$ be a strategy–proof rule with domain $\wp$ and range $X_x$, where $|X_x| \geq 3$ and $x \in X_x$. Then $g$ is dictatorial with respect to the set of alternatives $X_x$.

Proof. Let $g$ be a strategy–proof rule with domain $\wp$ and range $X_x$, where $|X_x| \geq 3$.

Step 1. Let $p \in \wp$ be such that $g(p) = x$, and let $u \in \wp$ be a profile at which $u_1(i) = x$ for all $i \in N$. Let $\{u^t\}$ be the standard sequence from profile $p$ to $u$. First, we show that $u^t \in \wp$ for all $t$, so that $\{u^t\} \in \wp$. By assumption $p \in \wp$, so that $|N_p(x, y)| > \frac{n}{2}$ for all $y \neq x$. At each step $u^t$ in the standard sequence, we promote alternative $x$ to the top of individual $t$’s preference ordering. So we have $|N_{u^t}(x, y)| \geq |N_u(x, y)| > \frac{n}{2}$ for every $y \neq x$ and all $t$.

Alternative $x$ will therefore remain the Condorcet winner at each step in the sequence, so that $u^t \in \wp$ for $t = 0, \ldots, n$.

We know that $g(p) = g(u^0) = x$. Suppose that $g(u^t) = x$, and suppose further that $g(u^{t+1}) = y$, where $y \neq x$. At profile $u^{t+1}$ individual $t + 1$ has $x$ top–ranked, so that $x \succ_{u^{t+1}(t+1)} y$. Thus, individual $t + 1$ could manipulate at profile $u^{t+1}$ by reporting a preference ordering of $u^t(t + 1)$, precipitating the selection of the preferred alternative $x$. Since we have assumed that $g$ is strategy–proof, we must therefore have that $g(u^{t+1}) = x$. Thus by induction we have that $g(u^n) = g(u) = x$. So, if $g$ selects alternative $x$ at any profile in the subdomain, then $g$ must select $x$ at every unanimous profile in $\wp$.

Step 2. Next, we show that if $g$ is non–dictatorial, $x$ need only be top–ranked for more than $\frac{n}{2}$ individuals at an arbitrary profile $p \in \wp$ to ensure that $g(p) = x$.

Define $K(p) = \{k \in N : x \text{ is top–ranked by individual } k \text{ at profile } p\}$. Our proof for $\frac{n}{2} < |K(p)| \leq n$ will proceed by induction. From Step 1 we have already established that $g(p) = x$ when $|K(p)| = n$. Next we suppose that $g(p) = x$ whenever $|K(p)| = k$, for $\frac{n}{2} + 2 \leq k \leq n$. We will then show that if $g(p) \neq x$ when $|K(p)| = k – 1$ (that is, when $\frac{n}{2} < |K(p)| \leq n$), then $g$ is dictatorial.

Suppose that $x$ is top–ranked by exactly $k – 1$ individuals at profile $p \in \wp$, so that $|K(p)| = k – 1$. Furthermore, suppose that
$g(p) = y$ where $y \neq x$. We will show that this second assumption implies that for every $z \in X_x$ there exists a profile in $\varnothing$ where $z$ is selected even though $k - 1$ individuals have $x$ top-ranked. This will allow us to show that that $g$ is dictatorial, so that we must have $g(p) = x$ when $g$ is non–dictatorial.

Assuming that $g(p) = y \neq x$, if $x \succ_{p(i)} y$ for some $i \in N \setminus K(p)$, then the induction hypothesis implies that this individual $i$ could manipulate $g$ at profile $p$ by reporting a preference ordering that has alternative $x$ top–ranked. Since $g$ is strategy–proof, we conclude that $y \succ_{p(i)} x$ for all $i \in N \setminus K(p)$.

Let $q \in \varnothing$ be such that $q(i) = p(i)$ for all $i \in K(p)$ and $q(i) = (y, x, \ldots)$ for all $i \in N \setminus K(p)$. Let $\{q^t\} \subset \varnothing$ be the standard sequence from $p$ to $q$. Then $g(q^0) = y$. Furthermore, $g(q^t) = y$ and the strategy–proofness of $g$ imply that $g(q^{t+1}) = y$, since $q^{t+1} \neq q^t$ implies that $t+1 \in N \setminus K(p)$. Hence, $y$ is the top–ranked alternative for individual $t+1$ at profile $q^{t+1}$, so that $g(q) = g(q^n) = y$ by induction.

Next, let $r \in \varnothing$ be a profile such that $r(i) = (x, \ldots, y)$ for all $i \in K(p)$ and $r(i) = q(i)$ for $i \in N \setminus K(p)$. Once again, let $\{r^t\} \subset \varnothing$ be the standard sequence from profile $q$ to $r$. Then $g(r^0) = y$. Furthermore, for any $t$ we have that $g(r^t) \in \{x, y\}$ by the induction hypothesis, because any $i \in N \setminus K(p)$ can precipitate the selection of $x$ by reporting a preference ordering with $x$ top–ranked and increasing the cardinality of $K(p)$ to $k$.

Now, suppose that $g(r^t) = y$. If $g(r^{t+1}) = x$, then $t+1 \in K(p)$ and individual $t+1$ can manipulate $g$ at $r^t$ by reporting the preference ordering $r(t+1)$. Therefore $g(r^{t+1}) = y$, so we have $g(r) = g(r^n) = y$.

Let $\varnothing^*$ denote the set of profiles in $\varnothing$ such that $s(i) = r(i)$ for all $i \in K(p)$, where $s(i)$ is some linear ordering on the set $X$ for all $i \in N \setminus K(p)$. Define a new social choice rule $g^*$ on $\varnothing^*$ by setting $g^*(s) = g(s)$ for all $s \in \varnothing^*$. We will show that the range of $g^*$ is $X_x$.

We know that $g(r) = y = g^*(r)$, since in particular $r \in \varnothing^*$. Also, if $h \in N \setminus K(p)$, if $r'(h) = (x, \ldots)$, and if $r'(i) = r(i)$ for all $i \neq h$, we have $g(r') = x = g^*(r')$ by the induction hypothesis.

32
Hence, \( x \) and \( y \) belong to the range of \( g^* \). Now choose an arbitrary \( z \in X_x \setminus \{x, y\} \). Next let \( s \in \wp^* \) be such that \( s(i) = r(i) \) for all \( i \in K(p) \) and \( s(i) = (z, y, \ldots) \) for all \( i \in N \setminus K(p) \). Let \( \{s^i\} \subset \wp^* \) be the standard sequence from \( r \) to \( s \). We have \( g(s^0) = y \). Suppose that \( g(s^t) = y \) and \( g(s^{t+1}) \neq z \) for some \( t \in \{1, \ldots, n\} \). Then \( g(s^{t+1}) = y \), otherwise individual \( t + 1 \) could manipulate \( g \) at \( s^{t+1} \) via \( s^t(t+1) \).

Therefore, either \( g(s^t) = y \) for all \( t \), or else \( g(s^t) = z \) for some \( t \). In the latter case, we know that \( z \) belongs to the range of \( g^* \). Suppose however that \( g(s^t) = y \) for all \( t \). In particular, then \( g(s) = y \). Recall that for \( i \in K(p) \) we have \( s(i) = r(i) = (x, \ldots, y) \). We will next show that \( g(s) = y \) leads to a contradiction in this case.

Let \( w \in \wp \) be some profile such that \( g(w) = z \), where \( z \in X_x \setminus \{x, y\} \). Such a profile exists since \( |X_x| \geq 3 \). Let us reorder our individuals, so that the first \( k - 1 \) individuals correspond to those in \( K(p) \), and let \( \{w^t\} \) be the standard sequence from \( s \) to \( w \). First, we will show that each member of \( \{w^t\} \) is in \( \wp \).

We have \( s \in \wp \), so that \( |N_s(x, x')| > \frac{n}{2} \) for all \( x' \neq x \). So, \( w^0 \in \wp \), since \( w^0 = s \). Next, we have that \( w^1 \in \wp \), since by replacing individual 1’s preference ordering at \( w^0 \) with \( (x, \ldots, y) \) we have that \( |N_{w^1}(x, x')| \geq |N_{w^0}(x, x')| > \frac{n}{2} \) for all \( x' \neq x \). Similarly, for each of the first \( k - 1 \) steps in the sequence, the size of the coalitions preferring alternative \( x \) to any other alternative will not decrease. Therefore, \( \{w^0, \ldots, w^{k-1}\} \subset \wp \).

Next we note that at profile \( w^{k-1} \), there are \( k - 1 > \frac{n}{2} \) individuals who have alternative \( x \) top–ranked; namely, those individuals in \( K(p) \). Therefore, a majority of individuals prefer alternative \( x \) to every other alternative regardless of the profiles of the last \( n - k + 1 \) individuals, or those in \( N \setminus K(p) \). Therefore, alternative \( x \) will remain the Condorcet winner at each of \( \{w^k, \ldots, w^n\} \) as we replace each of the remaining individual’s preference ordering with their ordering at \( s \). Therefore, we have that \( \{w^t\} \subseteq \wp \).

Next we will show that \( g(w) = z \) and the strategy–proofness of \( g \) are not consistent with \( g(s) = y \). Starting at first step of the standard sequence, we must have \( g(w^1) = y \), otherwise individual 1 would manipulate at \( s \) via \( w^1(1) \), since individual 1 prefers any other
alternative to \( y \) at profile \( s \). Similarly, \( q \) must select alternative \( y \) at each of \( \{w^1, \ldots, w^{k-1}\} \). At the next step in the standard sequence we will change \( w^{k-1}(k) = (z, y, \ldots) \) to \( w^k(k) = w(k) \). Therefore, \( g(w^k) \neq z \) or else individual \( k \) would manipulate at \( w^{k-1} \) via \( w^k(k) \) to precipitate the selection of her top-ranked alternative \( z \). Similarly, we must have \( g(w^t) \neq z \) for \( t = k, \ldots, n \). However, this contradicts the fact that \( g(w^n) = g(w) = z \). Therefore, we must drop our previous assumption that \( g(s^t) = y \) for all values of \( t \). In fact, the standard sequence argument shows that strategy-proofness requires that \( g(s) = z \), so that \( g(s^t) = z \) for some value of \( t \). Since \( s^t \in \varphi^* \) for all \( t \), we have therefore shown that \( X_x \) is the range of \( g^* \).

We can use \( g^* \) to induce a social choice rule for society \( N \setminus K(p) \) with a domain of all linear orderings on \( X_x \) for the individuals in \( N \setminus K(p) \) in the natural way, and we may also use \( g^* \) to denote that induced rule. The main result of Aswal, Chatterji, and Sen [2] establishes the Gibbard–Satterthwaite Theorem for the domain \( L(X_x)^H \) and arbitrary finite \( H \). Therefore, some individual \( h \in N \setminus K(p) \) is a dictator for \( g^* \). We will next show that \( h \) is a dictator for \( g \) itself. This will establish that either \( g \) is dictatorial, or for every \( k > \frac{n}{2} \) and arbitrary \( p \in \varphi \), if \( k \) individuals have a common top-ranked alternative then \( g \) must select that alternative at profile \( p \).

Choose any \( z \in X_x \setminus \{x, y\} \) and \( v \in \varphi \) such that \( v(h) = (z, x, \ldots) \) and \( v(i) = (x, \ldots, z) \) for all \( i \neq h \). Let \( v' \in \varphi \) denote the profile at which \( v'(i) = r(i) \) for all \( i \in K(p) \) and \( v'(i) = v(i) \) for all \( i \in N \setminus K(p) \). We have \( g(v') = z \) because \( v' \in \varphi^* \) and \( h \in N \setminus K(p) \) is a dictator for \( g^* \). If \( \{v'\} \) is the standard sequence from \( v' \) to \( v \) we have \( g(v') \in \{x, z\} \) for all \( t \) because individual \( h \) can precipitate the selection of \( x \) by reporting an ordering with \( x \) as the top-ranked element; such a change would result in \( k \) individuals declaring \( x \) to be their top-ranked alternative, since \( h \in N \setminus K(p) \). We have \( g(v^0) = z \), and if \( g(v^t) = z \) then \( g(v^{t+1}) = z \), because if \( g(v^{t+1}) = x \) then \( t + 1 \in K(p) \) and individual \( t + 1 \) could manipulate \( g \) at \( v^t \) via \( v^{t+1}(t + 1) \). Therefore, \( g(v^t) = z \) for all \( t \); in particular, \( g(v) = g(v^n) = z \). Because \( z \) is bottom-ranked for all \( i \in K(p) \) and \( v(i) \)
was arbitrarily chosen for all \( i \neq h \in N \setminus K(p) \), a standard sequence argument will establish that \( g(v'') = z \) for all \( v'' \in \varnothing \) such that \( v''(h) = z \). We can use the same argument with \( z \) replaced by \( y \) to show that \( g(v'') = y \) for any \( v'' \in \varnothing \) such that \( v''(h) = y \).

It therefore remains to show that \( g(v'') = x \) for any \( v'' \in \varnothing \) such that \( v''(h) = x \). We note that the arguments of the previous paragraph do not suffice in this case, as we cannot freely change the placement of the Condorcet winner within individual preference orderings while remaining in the domain. Let \( v \in \varnothing \) be such that \( v_1(h) = x \) and \( z \in X_x \) is the second–most preferred member of \( X_x \) by individual \( h \) at \( v \). Furthermore, let \( v_m(i) = z \) for all \( i \neq h \). From the previous paragraph, we know that \( g(v) \in \{ x, z \} \) because individual \( h \) could precipitate the selection of \( z \) by reporting a preference ordering with \( z \) top–ranked.

Let \( v' \in \varnothing \) be such that \( v'(h) = v(h) \) and for all \( i \neq h \), \( v'(i) \) is formed from \( v(i) \) by promoting alternative \( x \) to the top of the preference ordering. We know that \( g(v') = x \) since \( g \) satisfies unanimity. Let \( \{v'\} \subset \varnothing \) be the standard sequence from \( v \) to \( v' \) and suppose \( g(v) = g(v^0) = z \). Next suppose \( g(v') = z \), and note that strategy–proofness requires that \( g(v^{t+1}) = z \) since individual \( i \) prefers any other alternative to \( z \). By induction we therefore have that \( g(v^t) = z \) for all \( t \); however, this contradicts \( g(v^n) = g(v') = x \). Therefore, we must have that \( g(v) = x \).

Next, let \( v'' \in \varnothing \) be any profile for which \( v''(h) = x \). At this profile, there exists some alternative \( y \in X_x \) such that \( y \) is the second–most preferred member of \( X_x \) by individual \( h \) at profile \( v'' \). Next we form profile \( v' \) from \( v'' \) as follows: let \( v''(h) = v'(h) \) and for all \( i \neq h \) form \( v'(i) \) from \( v''(i) \) by moving alternative \( y \) to the bottom of individual \( i \)'s preference ordering at \( v' \). From the previous paragraph, we know that \( g(v') = x \). Let us now consider the coalition \( N_{v''}(x, y) \), and let \( |N_{v''}(x, y)| = a \). Let us reorder the individuals so that the members of \( N_{v''}(x, y) \) are the first \( a \) individuals, with individual \( h \) placed first. Let \( \{v^t\} \in \varnothing \) be the standard sequence from \( v' \) to \( v'' \). From above, we know that \( g(v^t) \in \{ x, y \} \) for all \( t \) and \( g(v') = g(v^0) = x \). For \( t = 1, \ldots, a \) we must have \( g(v^t) = x \);
otherwise \( g(v^t) = y \) and individual \( t \) would manipulate at \( v^t \) via \( v'(t) \) to precipitate the selection of \( x \), which individual \( i \) prefers to \( y \) at \( v^n \). Next, we must have that \( g(v^t) = x \) for \( t = a + 1, \ldots, n \); otherwise \( g(v^t) = y \) and individual \( t \) would manipulate at \( v^{(t-1)} \) via \( v''(t) \) to precipitate the selection of \( y \), which individual \( i \) prefers to \( x \) at \( v' \). Therefore, we have that \( g(v^n) = g(v^n) = x \), so that \( g(v^n) = x \) for any \( v^n \in \varphi \) such that \( v^n(h) = x \).

We have therefore established that individual \( h \) is the dictator with respect to \( g \) itself.

**Step 3.** We have established by induction that if \( g \) is non–dictatorial then \( g(p) = x \) for any profile \( p \in \varphi \) at which \( x \) is the top–ranked alternative for over half the members of \( N \). Assuming that \( g \) is non–dictatorial, it remains to prove that \( g(p) = x \) if \( x \) defeats every other member of \( X_x \) by a strong majority at \( p \); that is, at all profiles \( p \in \varphi \).

Suppose that \( g \) is non–dictatorial and there exists a profile \( p \in \varphi \) such that \( g(p) = y \), where \( y \) is an arbitrary element from \( X_x \setminus \{x\} \). At this profile there is some majority of individuals who prefer alternative \( x \) to alternative \( y \), namely those in \( N_p(x, y) \). Let \( q \in \varphi \) be such that \( q_1(i) = x \) for all \( i \in N_p(x, y) \) and \( q(i) = p(i) \) for all \( i \in N_p(y, x) \). Next, let us reorder our set of individuals so that \( N_p(x, y) = \{1, 2, \ldots, k\} \), where \( k > \frac{n}{2} \) and consider the standard sequence \( \{q^t\} \) from \( p \) to \( q \).

By assumption, we have that \( g(p) = g(q^0) = y \). If \( g \) is strategy–proof, we must also have \( g(p^t) \neq x \), or else individual 1 would manipulate at profile \( p \) via \( p^t(1) \) to precipitate the selection of her top–ranked alternative at \( p \). Similarly, we must have that \( g(p^t) \neq x \) for all \( t = 0, \ldots, k \). By Step 2 we must have that \( g(p^{k+1}) = x \), since at this profile more than half of the individuals will have alternative \( x \) top–ranked. However, we know that \( k \geq \frac{n}{2} + 1 \), and strategy–proofness requires that \( g(p^k) \neq x \). Therefore, we must have that \( g(p) = x \) for all profiles \( p \) at which alternative \( x \) is the Condorcet winner. However, this result is inconsistent with the assumption that \( |X_x| \geq 3 \); therefore, strategy–proof \( g \) must be dictatorial rule.
Lemma 2.2.18. Let \( g \) be a strategy-proof rule with domain \( \wp \) and range \( X_x \), where \( |X_x| \geq 3 \) and \( x \notin X_x \). Then \( g \) must be dictatorial with respect to the set of alternatives \( X_x \).

Proof. Let \( g \) be a strategy-proof rule with domain \( \wp \) and range \( X_x \), where \( |X_x| \geq 3 \) and \( x \notin X_x \).

Step 1. First we define a subset of \( \wp \) over which we may invoke the Gibbard–Satterthwaite theorem. Define \( U \subset \wp \) as follows: for all \( p \in U \) and all \( i \in N \), \( p_1(i) = x \); \( p_k(i) \in X_x \) for \( k = 2, \ldots, |X_x|+1 \), and \( \{p_{|X_x|+2}(i), \ldots, p_m(i)\} \) is some fixed ordering of the alternatives in \( X \setminus \{X_x \cup x\} \) for all \( i \in N \).

It is obvious that every member of \( U \) is indeed a member of \( \wp \): \( x \) is unanimously top-ranked. Furthermore, preferences with respect to the elements of the range \( X_x \) are unrestricted over \( U \) and \( |X_x| \geq 3 \). We may therefore invoke the Gibbard–Satterthwaite theorem to show that \( g \) must be dictatorial over \( U \). Without loss of generality suppose individual 1 is the dictator.

Step 2. Let \( p \in U \), and \( p^{(1)} \in \wp \) be any profile such that \( p^{(1)}(i) = p(i) \) for all \( i \neq 1 \). Let \( y \) denote individual 1’s most-preferred alternative from \( X_x \) at \( p^{(1)} \). Strategy-proofness therefore requires that \( g(p^{(1)}) = y \); otherwise, individual 1 could manipulate at \( p^{(1)} \) to a profile in \( U \) where \( y \) would be selected. Therefore, we must have that individual 1 is the dictator over all such profiles \( p^{(1)} \) with respect to the set \( X_x \).

Next consider any profile \( p^{(2)} \in \wp \) such that \( p^{(2)}(i) = p^{(1)}(i) \) for all \( i \neq 2 \). Strategy-proofness requires that \( g(p^{(2)}) \not\succ_{p^{(2)}(2)} y \), otherwise individual 2 would manipulate at \( p^{(1)} \) via \( p^{(2)}(2) \). Strategy-proofness further requires that \( g(p^{(2)}) \not\prec_{p^{(2)}(2)} y \), otherwise individual 2 would manipulate at \( p^{(2)} \) via \( p^{(1)}(2) \). Therefore, we have \( g(p^{(2)}) = y \), so that individual 1 is the dictator over all such profiles \( p^{(2)} \). Proceeding by induction we therefore have that individual 1 is the dictator over all such profiles \( p^{(k)} \), where \( k = 1, \ldots, n \).

We may reconstruct all of \( \wp \) from such a sequence of profiles \( p^{(k)} \). Note that from any \( q \in \wp \) we can produce a profile \( r \in U \) at which each individual’s relative ordering of the alternatives in \( X_x \) is identical to their relative ordering at \( q \). Let \( \{q^i\} \) be the standard sequence
from \( r \) to \( q \). First we show that \( q^t \in \wp \) for all \( t = 0, \ldots, n \). By assumption \( q^n = q \in \wp \), so \( |N_{q^n}(x, y)| > \frac{\theta}{2} \) for all \( y \neq x \). Furthermore we have \( |N_{q^t}(x, y)| \geq |N_{q^{t-1}}(x, y)| \) for all \( t \), because at each step backwards in the sequence we promote alternative \( x \) to the top of individual \( t-1 \)'s preference ordering. Therefore, \( \{q^t\} \in \wp \). Furthermore, \( q^t \) corresponds to a profile of the type \( p^{(t)} \) from above, so that individual 1 is the dictator at all \( q^t \). In particular, individual 1 is the dictator at an arbitrary profile \( q \in \wp \).

Combining the propositions from the previous section, we achieve a proof of Theorem 2.2.13 which characterizes all strategy-proof social choice rules over the Condorcet domain for an even number of individuals. From the Gibbard–Satterthwaite theorem, we know that a strategy-proof \( g \) must therefore satisfy unanimity over any Condorcet section \( \wp_{x_i} \) for which \( |X_{x_i}| \geq 3 \). If we require that \( g \) satisfy unanimity over every Condorcet section, we arrive at the following corollary:

**Corollary 2.2.19.** Suppose \( g \) is a unanimous social choice rule over the Condorcet domain for an even number of individuals, with range \( X \). Then \( g \) is strategy-proof and satisfies unanimity if and only if:

(i) If there exists \( x_i \in X \) such that \( |X_{x_i}| = 1 \), then \( g|_{\wp_{x_i}}(p) = x_i \) for all \( p \in \wp_{x_i} \);

(ii) If there exists \( x_i \in X \) such that \( |X_{x_i}| = 2 \), then \( g|_{\wp_{x_i}} \) satisfies non-reversal and \( g|_{\wp_{x_i}}(p) = x_i \) when there exists \( p \in \wp_{x_i} \) and \( x_j \in X_{x_i} \) such that \( x_i \succ_p k x_j \) for all \( k \in N \);

(iii) If there exists \( x_i \in X \) such that \( |X_{x_i}| \geq 3 \), then \( x \in X_{x_i} \) and \( g|_{\wp_{x_i}} \) is dictatorial.

**Proof.** The proof of Corollary 2.2.19 follows directly from Theorem 2.2.13. Unanimity implies that \( x_i \in X_{x_i} \), so that \( g|_{\wp_{x_i}}(p) = x_i \) whenever \( |X_{x_i}| = 1 \), as in (i). Case (iii) remains unchanged save for this additional requirement.

In case (ii), let \( v \in \wp_{x_i} \) be any profile such that for \( x_j \in X_{x_i} \), \( x_i \succ_v k x_j \) for all \( k \in N \); that is, everyone prefers \( x_i \) to all other
alternatives in $X_{x_i}$ at profile $v$. Furthermore, let $v'$ be a unanimous profile in $\varphi_{x_i}$ formed from $v$ by promoting alternative $x_i$ to the top of each individual’s preference ordering. Let \{ $v^t$ \} be the standard sequence from $v$ to $v'$, and suppose that $g(v) = g(v^0) = x_j$. Next suppose $g(v^t) = x_j$, then we must have $g(v^{t+1}) = x_j$, otherwise individual $t + 1$ could manipulate at $v^t$ via $v^{t+1}(t + 1)$ to precipitate the selection of the more–preferred alternative $x_i$. By induction, we therefore have that $g(v^t) = x_j$ for all $t$; however, $g(v^n) = g(v') = x_i$ by unanimity. Therefore, we must have that $g(v) = x_i$, so that alternative $x_i$ must be selected at any profile at which every individual prefers it to the other alternative in $X_{x_i}$.

Thus requiring $g$ to satisfy unanimity does little to alter the family of strategy–proof social choice rules, and is not sufficient to extend the results of Campbell and Kelly [4] as demonstrated by previous example. However, we may construct an analogous theorem by introducing two new definitions.

**Definition 2.2.20.** (Dictatorial Section) $P \in \Pi$ is a dictatorial section of $g$ if $P$ is a Condorcet section and $g|_P$ is dictatorial.

**Definition 2.2.21.** (Quasi–Majority Rule) A social choice rule $g$ is quasi–majority rule if for any profile $p \in P \subset \mathcal{C}$, if $g(p)$ is not the Condorcet winner at $p$ then the range of $g|_P$ contains exactly two members, one of which is the Condorcet winner at $p$, and $g|_P$ satisfies non–reversal.

With these definitions, we state an additional result.

**Theorem 2.2.22.** Suppose that $g$ is a strategy–proof social choice function on the Condorcet domain with an even number of individuals. If $g$ has no dictatorial sections and satisfies unanimity then it is quasi–majority rule.

The proof of this theorem follows from Theorem 2.2.13 and the above definitions. If $g$ has no dictatorial sections, the range of $g$ over any Condorcet section $P$ must be either a singleton or contain exactly two members. We note that by requiring unanimity we have
insured that the Condorcet winner over each Condorcet section is indeed an element of the range of \( g \) over that section.
Chapter 3

Weak Condorcet Domain

On the strict Condorcet domain, we found a large disparity between the characterizations of strategy–proof social choice rules when there were an odd or even number of individuals. In this chapter, we attempt to close the gap between the two results by admitting profiles at which individuals are indifferent between alternatives. In this chapter we are concerned with characterizing non–dictatorial strategy–proof rules over the weak Condorcet domain. We present a number of propositions concerning strategy–proof social choice rules on the domain.

Throughout this chapter, we will use the notation \( \mathcal{P} \) to refer to \( \mathcal{P}_{\mathcal{C}_{\mathcal{W}}} \) since in every instance we refer to the weak Condorcet domain.

The weak Condorcet domain differs in many vital respects from the strict Condorcet domain. In particular, our analysis does not depend on the parity of the number of individuals. As the proof of the next lemma establishes, we cannot partition the space as in the case of the strict Condorcet domain and an even number of individuals.

**Proposition 3.0.23.** If \( g \) is a strategy–proof social choice rule on \( \mathcal{P} \), then \( g \) has the unanimity property.

**Proof.** Let \( p \in \mathcal{P} \) be any profile such that \( p_1(k) = \{ x_i \} \) for all \( k \in N \), so that alternative \( x_i \) is strictly preferred to every other alternative by every individual. We will show that \( g(p) = x_i \). Since \( g \) is onto,
there exists some $r \in \wp$ such that $g(r) = x_i$.

Let $g_C$ be majority rule. Suppose that $g_C(r) = x_i$ and consider the standard sequence $\{p^t\}$ from $r$ to $p$. By construction $x_i$ is the Condorcet winner at $p^0 = r$. Suppose that $x_i$ is the Condorcet winner at $p^t$ for some $t \in \{0, \ldots, n\}$. At profile $p^{t+1}$, alternative $x_i$ must remain the Condorcet winner since we have promoted $x_i$ to the top of individual $t + 1$’s preference ordering. That is, at each step the size of the coalition preferring $x_i$ to $x_j$ increases or is unchanged for every alternative $x_j \in X \setminus \{x_i\}$, so that

$$|N_{p^t+1}(x_i \succ x_j)| = |N_{p^t}(x_i \succ x_j)| + 1$$

or

$$|N_{p^t+1}(x_i \succ x_j)| = |N_{p^t}(x_i \succ x_j)|$$

and in either case,

$$|N_{p^t}(x_i \succ x_j)| > \frac{\mu_{p^t}(x_i, x_j)}{2} \implies |N_{p^{t+1}}(x_i \succ x_j)| > \frac{\mu_{p^{t+1}}(x_i, x_j)}{2}.$$  

Therefore, $x_i$ is the Condorcet winner at every step of the standard sequence, so that $\{p^t\} \subset \wp$. Note that $g(p^0) = g(r) = x_i$. Suppose that $g(p^t) = x_i$ and $g(p^{t+1}) = x_j \neq x_i$. Then individual $t + 1$ can manipulate at profile $p^{t+1}$ via the preference ordering $p^t(x_i \succ x_j)$ since $g(p^{t+1}) = x_i \succ p^{t+1}(x_i \succ x_j) = g(p^{t+1})$. Strategy-proofness therefore implies that $g(p^{t+1}) = x_i$, and by induction we have that $g(p^n) = g(p) = x_i$.

Now suppose that $g_C(r) = x_j \neq x_i$. We will consider two cases: when $\mu_r(x_i, x_j)$ is odd and when $\mu_r(x_i, x_j)$ is even. In either case, the standard sequence $\{p^t\}$ from $r$ to $p$ may no longer fall within our domain $\wp$; we will therefore employ an auxiliary profile $q \in \wp$, at which $x_i \succ q(i) x_j \succ q(i) x_t$ for all $i \in N$ and all $x_t \in X \setminus \{x_i, x_j\}$.

Suppose that $\mu_r(x_i, x_j)$ is odd. Let $\{q^t\}$ denote the modified standard sequence from $r$ to $q$ in which we change the preferences of the individuals in $N_r(x_i \succ x_j)$ first, $N_r(x_j \succ x_i)$ second, and $N_r(x_i \sim x_j)$ last. For $0 \leq t \leq |N_r(x_i \succ x_j)|$, alternative $x_j$ will
remain the Condorcet winner; at each step we have

\[ |N_{q^t}(x_j \succ x_i)| = |N_r(x_j \succ x_i)| \forall x_\ell \in X \setminus \{x_i, x_j\} \]

and

\[ |N_{q^t}(x_j \succ x_\ell)| \geq |N_r(x_j \succ x_\ell)| \forall x_\ell \in X \setminus \{x_i, x_j\}. \]

Furthermore, \( x_j \) will remain the Condorcet winner at \( q^t \) for all

\[ |N_r(x_i \succ x_j)| < t < \frac{\mu_r(x_i, x_j) + 1}{2}, \]

since at each step

\[ |N_{q^t}(x_j \succ x_i)| > |N_r(x_j \succ x_i)|, \forall x_\ell \in X \setminus \{x_i, x_j\} \]

and

\[ |N_{q^t}(x_j \succ x_\ell)| \geq |N_r(x_j \succ x_\ell)|, \forall x_\ell \in X \setminus \{x_i, x_j\}. \]

For all \( \frac{\mu_r(x_i, x_j) + 1}{2} \leq t < \mu_r(x_i, x_j) + 1 \), alternative \( x_i \) will be the Condorcet winner at profile, since \( |N_{q^t}(x_i \succ x_\ell)| > |N_{q^t}(x_j \succ x_\ell)| \) and for all \( x_\ell \in X \setminus \{x_i\} \) since \( x_i \) is strictly top-ranked by a majority of the individuals with preferences over the pair \((x_i, x_j)\).

For at each profile \( q^t \) for \( \mu_r(x_i, x_j) < t \leq n \), we change the preferences of individuals who were previously indifferent between the pair \((x_i, x_j)\) so that they now prefer \( x_i \) to \( x_j \). Therefore for \( \mu_r(x_i, x_j) < t \leq n \),

\[ |N_{q^{t+1}}(x_i \succ x_\ell)| = |N_{q^t}(x_i \succ x_\ell)| + 1, \forall x_\ell \in X \setminus \{x_i\} \]

which ensures that

\[ |N_{q^{t+1}}(x_i \succ x_\ell)| > |N_{q^{t+1}}(x_\ell \succ x_i)|, \forall x_\ell \in X \setminus \{x_i\}. \]

Therefore, we have that alternative \( x_j \) is the Condorcet winner at all \( q^t \) with \( t = 0, \ldots, \frac{\mu_r(x_i, x_j) + 1}{2} \) and \( x \) is the Condorcet winner for all \( q^t \) with \( t = \frac{\mu_r(x_i, x_j) + 1}{2}, \ldots, n \); so \( \{q^t\} \subseteq \varnothing \).

By assumption \( g(q^0) = g(r) = x_i \). If \( g(q^t) = x_i \) and \( g(q^{t+1}) \neq x_i \),
then individual $t+1$ can manipulate at $q^{t+1}$ via the preference ordering $q^i(t+1)$ to elicit the selection of alternative $x_i$, since $x_i \succ q_i^{t+1}(t+1) z$ for all $z \in X \setminus \{x_i\}$. By induction, we have $g(q^i) = x_i$ for $t = 0, \ldots, n$, so that $g(q) = g(q^n) = x_i$. Since $p$ is an arbitrary profile at which each individual has alternative $x_i$ strictly top–ranked, and each individual has $x_i$ strictly top–ranked at profile $q$, we may again use a standard sequence argument to show that $g(p) = x_i$.

Suppose now that $\mu(x_i, x_j)$ is even. In this case, we will construct another profile $r'$ by choose exactly one individual $j$ from $N_r(x_i \sim x_j)$ and promoting alternative $x_j$ to the top of their preference ordering, so that $x_j \succ r'(j) x_\ell$ for all $x_\ell \neq x_j$; for all $i \neq j$ we will set $r'(i) = r(i)$. At profile $r$, we have

$$|N_r(x_j \succ x_\ell)| > |N_r(x_\ell \succ x_j)|, \forall x_\ell \in X \setminus \{x_j\}$$

and by construction we have

$$|N_r(x_j \succ x_\ell)| \geq |N_r(x_\ell \succ x_j)|$$

which together imply that

$$|N_{r'}(x_j \succ x_\ell)| > |N_{r'}(x_\ell \succ x_j)|, \forall x_\ell \in X \setminus \{x_j\}$$

so that indeed $r' \in \varnothing$. Furthermore, suppose that $g(r') = x_\ell \neq x_j$; then individual $j$ can manipulate at profile $r'$ via the preference ordering $r(j)$, since $g(r) = x_j \succ r(j) x_\ell = g(r')$ for any $x_\ell \in X \setminus \{x_j\}$. Since $\mu_r(x_i, x_j)$ was even, we have that $\mu_r(x_i, x_j)$ is odd since we have increased the number of individuals preferring $x_j$ to $x_i$ by one. Therefore, we may use the arguments of the previous paragraphs with $r'$ in place of $r$ to show that $g(p) = x_i$. \hfill \Box

The next two propositions utilize our results on the strict Condorcet domain. We first introduce two new definitions.

**Definition 3.0.24 (Fixed Odd Subdomain).** Let $N = \{1, \ldots, n\}$ be the set of voters and $X = \{x_1, \ldots, x_m\}$. Let $N_{FO} \subset N$ denote a fixed subset of individuals such that $|N_{FO}| = n_{FO}$ is odd. Denote the
fixed odd subdomain with respect to the set $N_{FO}$ as $\varphi_{FO} \subset \varphi_{CW}$ and define it as follows: for all $p \in \varphi_{FO}$, for all $i \in N_{FO}$, $p(i) \in L_S(X)$ and for all $i \in N \setminus N_{FO}$,

$$p(i) = (x_1 \sim x_2 \sim \cdots \sim x_m).$$

That is, all individuals in $N_{FO}$ have preference orderings that are strict linear orders, and all other individuals are completely indifferent between all of the alternatives.

The next definition is analogous to the previous, except that the parity of the set of individuals with strict preferences is even.

**Definition 3.0.25** (Fixed Even Subdomain). Let $N = \{1, \ldots, n\}$ be the set of voters and $X = \{x_1, \ldots, x_m\}$. Let $N_{FE} \subset N$ denote a fixed subset of individuals such that $|N_{FE}| = n_{FE}$ is even. Denote the fixed odd subdomain with respect to the set $N_{FE}$ as $\varphi_{FE} \subset \varphi_{CW}$ and define it as follows: for all $p \in \varphi_{FE}$, for all $i \in N_{FE}$, $p(i) \in L_S(X)$ and for all $i \in N \setminus N_{FE}$,

$$p(i) = (x_1 \sim x_2 \sim \cdots \sim x_m).$$

That is, all individuals in $N_{FE}$ have preference orderings that are strict linear orders, and all other individuals are completely indifferent between all of the alternatives.

With these definitions, we state the following two propositions

**Proposition 3.0.26** (Restricted to a Fixed Odd Subdomain). If $g$ is a strategy–proof social choice rule over $\varphi_{CW}$, then or any fixed odd subdomain $\varphi_{FO}$, $g|_{\varphi_{FO}}$ must be dictatorial or majority rule.

**Proposition 3.0.27** (Restricted to a Fixed Even Subdomain). If $g$ is a strategy–proof social choice rule over $\varphi_{CW}$, then for any fixed even subdomain $\varphi_{FE}$, $g|_{\varphi_{FE}}$ must be dictatorial or satisfy the following properties

(i) If there exists $x_i \in X$ such that $|X_{x_i}| = 2$, then $g|_{\varphi_{x_i}}$ satisfies non–reversal;
(ii) If there exists \( x_i \in X \) such that \( |X_{x_i}| \geq 3 \), then \( x_i \not\in X \) and \( g|_{\varphi_{x_i}} \) is dictatorial with respect to the set of alternatives \( X_{x_i} \), where \( \varphi_{x_i} \) is the set of profiles in \( \varphi_{FE} \) for which alternative \( x_i \) is the Condorcet winner, and \( X_{x_i} \) is the range of \( g \) over \( \varphi_{x_i} \).

In each of these propositions, we have applied the results of the strict Condorcet domain to subdomains for which some fixed number of individuals have strict preferences and the remaining individuals are completely indifferent over all the alternatives. The proof of these propositions follows almost immediately from our previous result. Since any strategy-proof rule \( g \) must be unanimous, \( g \) must have full range over any fixed odd or even subdomain \( \varphi_{FO} \) or \( \varphi_{FE} \) because in particular each of these subdomains include the profiles at which all the individuals in the fixed set \( N_{FO} \) or \( N_{FE} \) have alternative \( x_i \) top-ranked for every alternative \( x_i \).

To prove Propositions 3.0.26 (respectively, Proposition 3.0.27) using the previous results, each standard sequence argument should be augmented so that the individuals in \( N_{FO} \) (respectively, \( N_{FE} \)) come first, followed by the individuals who are completely indifferent. Because preferences are strict on the first set of individuals, for any standard sequence \( \{q^t\} \) the arguments of the previous chapter go through for all \( t = 0, \ldots, n_{FO} \) (respectively, \( t = 0, \ldots, n_{FE} \)). Moreover, since the preferences of the remaining individuals are fixed over the entire domain, \( g(q^t) = g(q^{t+1}) \) for \( t = n_{FO} + 1, \ldots, n \) (respectively, \( t = n_{FO} + 1, \ldots, n \)). Therefore, we can essentially drag the individuals in \( N \setminus N_{FO} \) (respectively, \( N \setminus N_{FO} \)) through the proofs of Chapter 2 with respect to the subdomain \( \varphi_{FO} \) (respectively, \( \varphi_{FE} \)) without substantial changes.

We have therefore shown that any strategy-proof rule \( g \) over the weak Condorcet domain must

- Satisfy unanimity;
- Coincide with either dictatorial or majority rule over all fixed odd subdomains, and;
- Satisfy two conditions (see Proposition 3.0.27) over all fixed even subdomains.
Although we have not shown that these results are sufficient to characterize strategy-proof rules over the weak Condorcet domain, we remain hopeful that further work will provide a complete characterization.
Chapter 4

Conclusion

Combining the results of this paper with those of Campbell and Kelly [4], we have established a complete theory of non–dictatorial strategy–proof social choice rules on the strict Condorcet domain for an arbitrary finite number of individuals. In this paper we have characterized the family of strategy–proof social choice rules over the strict Condorcet domain for an even number of individuals. As made evident through examples, this class of rules is much broader than in the case of odd number of individuals. We note that majority rule remains strategy–proof in the even case, and that under this rule the range associated with each Condorcet section is a singleton, namely the Condorcet winner.

The disparity between the results of this paper and the results of Campbell and Kelly [4] are entirely due to the Condorcet section partition in the case of even number of individuals. It is this partition that allows for more exotic strategy–proof rules to exist, and it is the barriers between Condorcet sections that cause the method of proof used in the odd case to fail when applied generally to the even case. We believe that it may be possible to achieve results more similar to the odd case by expanding the domain to the weak Condorcet domain.

Although we do not yet have a complete characterization of strategy–proof rules over the weak Condorcet domain, we have es-
established the unanimity lemma for strategy–proof rules as well as two restriction conditions. In the proof of this lemma, we showed that the weak Condorcet domain cannot be partitioned like the strict Condorcet domain in the case of even number of individuals. In the proof of the propositions, we utilized the results of Chapter 2 on well–specified subdomains. With these results, we maintain hope that a characterization can be discovered and that majority rule may be the unique non–dictatorial and strategy–proof social choice rule on the weak Condorcet domain.
Bibliography


