Relaxed Coloring of Sparse Graphs

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Relaxed Coloring of Sparse Graphs

by

Michael C. Kopreski

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Bachelor of Science in Mathematics

at the

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Abstract

A graph $G$ is $(d_1, d_2, \ldots, d_t)$-colorable if its vertices may be partitioned into subsets $V_1, V_2, \ldots, V_t$ such that for a given $d_i$, the maximum degree $\Delta(G[V_i]) \leq d_i$. We study this relaxed coloring of graphs with bounded maximum average degrees. Specifically, we use discharging and other methods to seek new upper and lower bounds for the maximum average degree of $(1, 1, 0)$-colorable graphs. We generalize this result to colorings of the type $(1_1, 1_2, \ldots, 1_a, 0_1, \ldots, 0_b)$, improving the results by Dorbec, Kaiser, Montassier, and Raspaud [7] for a large class of colorings.

Thesis Supervisor: Gexin Yu
Title: Associate Professor
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Chapter 1

Introduction

1.1 Background

1.1.1 Graphs and terminology

Definition 1.1.1. A graph $G$ is comprised of a vertex set $V(G)$ and an edge set $E(G)$, such that each edge $uv \in E(G)$ has exactly two endpoints $u, v \in V(G)$.

In this paper, we consider only finite, simple, undirected graphs:

Definition 1.1.2. A graph $G$ is simple if each pair $u, v \in V(G)$ has at most one edge $uv \in E(G)$. If $G$ is undirected then $uv = vu$.

![Figure 1-1: A simple undirected graph $G$.](image)

We define a subgraph $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ such that each edge $st \in E(H)$ has exactly two endpoints $s, t \in V(H)$. If $H$ is a subgraph such that $E(H)$ is maximal, then $H$ is the subgraph induced by $V(H)$; equivalently,
given a vertex set $M \subseteq V(G)$, the induced subgraph $G[M] \subseteq G$ includes all edges on $M$ present in $E(G)$. For an example, consider the simple, undirected graph $G$ in fig. 1-1. Here, $V(G) = \{t, w, x, y, z\}$ and $E(G) = \{tw, tx, ty, wx, wy, yz\}$; the triangle graph induced by $\{t, w, x\} \subset V(G)$, with edges $\{tw, tx, wx\} \subset E(G)$, is an induced subgraph of $G$.

Given two vertices $u, v \in V(G)$, $u$ and $v$ are **adjacent** if $uv \in E(G)$. If $v$ is an endpoint of an edge, e.g. $uv \in E(G)$, then $v$ and $uv$ are **incident**.

**Definition 1.1.3.** The **neighborhood** of a vertex $v \in G$, $N(v)$, is the set of all vertices adjacent to $v$ in $G$. The **degree** $d(v)$ of a vertex $v \in G$ is equal to the number of edges incident to $v$ in $G$.

$G$ is finite, hence we may express a minimum and maximum degree, respectively

$$\delta(G) = \min\{d(v) : v \in G\} \text{ and } \Delta(G) = \max\{d(v) : v \in G\}.$$ 

A **path** is a sequence of vertices and edges $v_1, e_1, v_2, e_2, \ldots, v_n$ such that $e_k = v_kv_{k+1}$ and each vertex appears only once. We say that a graph $G$ is **connected** if for every $u, v \in V(G)$, there exists a path with endpoints $u$ and $v$. A connected subgraph of $G$ is a **component** of $G$.

A **cycle** is a “closed” path, where the endpoints share an edge $e_n = v_nv_1$.

### 1.1.2 Graph coloring

We may **color** a graph $G$ by applying colors to each vertex of $G$, or equivalently by partitioning the vertex set into classes defined by the color(s) applied to each class. A **proper coloring** of a graph applies a single color to each vertex such that no adjacent vertices share a color, or equivalently, such that the subgraphs induced by each color class are edgeless.

**Definition 1.1.4.** $G$ is **$k$-colorable** if and only if it may be partitioned into $k$ or fewer spanning edgeless vertex sets, called color classes.

Proper coloring is well studied; perhaps the most famous of these results is the **four color conjecture**, which supposes that every planar graph is 4-colorable. The conjecture was proven in the affirmative by Appel and Haken in 1977 [1][2].
We examine a relaxation of proper coloring, wherein a vertex may have no more than a specified number of like-colored neighbors, or equivalently the subgraphs induced by each color class have a specified maximum degree:

**Definition 1.1.5.** $G$ is $(t_1, t_2, ..., t_n)$-**colorable** if and only if its vertices may be partitioned into sets $k_1, k_2, ..., k_n$, such that for a given set $k_i$,

$$\Delta(G[k_i]) \leq t_i.$$  

Consider again our graph $G$. $G$ is 3-colorable, as demonstrated in fig. 1-2. Moreover, $G$ requires at least 3 colors to be properly colored: $G$ contains 3-cycle subgraphs, which cannot be properly colored with only two colors.

![Figure 1-2: A proper 3-coloring of $G$.](image)

However, $G$ is $(1, 0)$-colorable. A $(1, 0)$ coloring of $G$ is shown in fig. 1-3, where 1 denotes the improper class and 0 the proper class. Note that the 1 class (necessarily, since $G$ is not 2-colorable) has maximum degree exactly 1.

![Figure 1-3: A $(1, 0)$-coloring of $G$.](image)

We say a vertex $u$ is **saturated** if it is colored $t_i$ and has at least $t_i$ neighbors.
colored with $t_i$. Note that all vertices except for $z$ are saturated in fig. 1-3.

1.1.3 Sparseness and maximum average degree

Qualitatively, we express that a graph is **sparse** if it has comparatively few edges over a given vertex set, or **dense** if it has comparatively many edges. The sparseness of a graph is strongly related to colorability, since a subgraph with high density is “difficult” to color: e.g., a complete graph $K_n$, where every vertex is adjacent to every other vertex in a graph of $n$ vertices, may be thought of having maximum density; it requires at least $n$ colors to be properly colorable. In fact, any graph with a “clique” $K_n$ as a subgraph requires at least $n$ colors in any proper coloring, since any coloring must also color the subgraph.

We note that each edge $u v \in E(G)$ contributes to the degree of both endpoints $u, v \in V(G)$; hence, we may express the average degree of $G$ in terms of the number of edges and vertices in $G$:

$$\bar{d}(G) = \frac{\sum_{v \in V} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|}.$$  \hspace{1cm} (1.1)

If the average degree of a graph $G$ may quantify density, then we are concerned with the average degree of the subgraph $H \subseteq G$ with maximum average degree.

**Definition 1.1.6.** The **maximum average degree** of a graph $G$ is

$$\text{mad}(G) = \max \{ \bar{d}(H) : H \subseteq G \}$$

1.2 Prior work

Appel and Haken demonstrated that every planar graph is (0, 0, 0, 0)-colorable (i.e. 4-proper-colorable) [1][2]. In 1986, Cowen, Cowen, and Woodall showed that every planar graph is (2, 2, 2)-colorable [3].

Relaxations of proper coloring, and in particular, establishing bounds for colorability in terms of graph sparseness, remain an active area of research. Recently, many
authors have published papers concerning improper 2-colorings, i.e. \((k, j)\)-colorings for natural \(k, j\), e.g. [4][5]. Borodin and Kostochka demonstrated \(\text{mad} \leq \frac{12}{5}\) implies (1,0)-colorability; moreover, the bound is sharp [6]. This result employed potential functions and discharging methods to establish the lower bound; sharpness (the “upper bound”) was demonstrated by construction.

Dorbec, Kaiser, Montassier, and Raspaud [7] use discharging (see section 1.3.1) to establish that \(\text{mad} < f(t, a, b)\) implies \((t_1, \ldots, t_a, 0_1, \ldots, 0_b)\)-colorability, where \(t_i\) is an improper class of degree \(t\), \(0_i\) is a proper class, and

\[
f(t, a, b) = a + b + \frac{ta(a + 1)}{(a + t + 1)(a + 1) + ab}. \tag{1.2}
\]

A construction recursing on \(a\) yields a non-\((t_1, \ldots, t_a, 0_1, \ldots, 0_b)\)-colorable graph with \(\text{mad}\)

\[
g(t, a, b) = 2a + b - \frac{2}{(t + 1)(b + 1) - 1} + \frac{2a + 2}{(t + a)^{a+1}(b + 1)^{a+1} - 1}. \tag{1.3}
\]

1.3 The problem

We consider a \((1,1,0)\)-coloring of some graph \(G\) and a number \(d\). We ask: **what is the maximum value \(d\) such that all \(G\) with \(\text{mad}(G) < d\) are necessarily \((1,1,0)\)-colorable?** That is, what is the minimum sparseness required to imply \((1,1,0)\)-colorability?

By (1.2) and (1.3), we find \(f(1,2,1) \leq d < g(1,2,1)\), hence

\[
3\frac{3}{7} \leq d < 4\frac{3}{7}. \tag{1.4}
\]

More generally, we ask **what is the maximum value \(d\) such that all \(G\) with \(\text{mad}(G) < d\) are necessarily \((1_1, \ldots, 1_a, 0_1, \ldots, 0_b)\)-colorable?** We seek a lower bound for \(d\) in the general case. Again, by (1.2),

\[
d \geq f(1,a,b) = a + b + \frac{a(a + 1)}{(a + 2)(a + 1) + ab}. \tag{1.5}
\]

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The upper bound in the special case is achieved by construction. The lower bound in both cases uses discharging, a method worthy of further introduction.

1.3.1 Discharging

Discharging methods associate vertex degree with a charge applied to each vertex of a graph. Since charge is related to the degree of a vertex, given the total charge it is possible to determine average degree of the graph. Typically, discharging is used to demonstrate a lower bound on the average degree of a graph. In this method, we propose a minimum counterexample and, using discharging, derive a contradiction. Hence, the set of counterexamples has no minimum element, i.e. it must be empty.

The general strategy follows:

(a) Determine the structure of the minimum counter example.

(b) Apply charge to each vertex as a function of degree.

(c) Redistribute charge according to specified rules (“discharging”).

(d) Demonstrate locally bounded final charge, and hence bound the total charge.

We provide here a brief example of a “typical” discharging proof, which demonstrates the lower bound in (1.4) in a novel way. In addition to offering an introduction to the method, the proof develops some structure we will employ later.

**Theorem 1.3.1.** If \( \text{mad}(G) < 3\frac{3}{7} \), then \( G \) is \((1,1,0)\)-colorable.

Let 1, 2 be improper classes and let 3 be the proper class. Let a vertex \( u \) be recolorable in some coloring \( c \) if there exists a coloring \( c' \) such that \( c(u) \neq c'(u) \) and for all \( t \neq u \), \( c(t) = c'(t) \). We will call a 3-vertex low if it is adjacent to another 3-vertex.

**Lemma 1.3.2.** Given \( v \in V(G) \) and a partial coloring \( c(G - v) \), if \( u \in N(v) \) is non-recolorable and 1 or 2-saturated, then \( d(u) \geq 4 \).
Let \( c(u) = 1 \). \( u \) is saturated, hence \( 1 \in c(N(u)) \). But \( u \) is non-recolorable, hence \( N(u) \) must saturate both 2, 3: \( v \) is uncolored, hence \( d(v) \geq 4 \).

Suppose \( G \) is a minimum counterexample, \( i.e. \ G \) is not \((1, 1, 0)\)-colorable and has \( \text{mad}(G) < 3 \frac{3}{7} \). Then every proper subgraph \( H \subset G \) has \( \text{mad}(H) \leq \text{mad}(G) < 3 \frac{3}{7} \): if \( H \) is not a (more) minimum counterexample, \( H \) must be colorable. Hence, for any \( v \in G \), there exists a partial coloring \( c(G - v) \).

**Characterizing the minimum counter example.**

**Lemma 1.3.3.** The following are properties of \( G \):

- \( \delta(G) \geq 3 \).
  
  Suppose there exists some \( v \in G \) with degree less than 3. \( c(G - v) \) exists; there are at most 2 colors represented in \( N(v) \). \( v \) may be colored with the remaining color, a contradiction.

- Every 3-vertex \( v \in G \) is adjacent to at least two \( 4^+ \)-vertices.
  
  Consider a partial coloring \( c(G - v) \). Every color appears exactly once in \( N(v) \), else \( c \) may be extended to \( v \), and hence must be saturated. Then \( v \) has neighbors \( x, y \) that are 1 and 2-saturated, respectively; by Lemma 1.3.2 each must have degree at least 4.

- Every 4-vertex \( v \in G \) is adjacent to at least one \( 4^+ \)-vertex.
  
  Consider a partial coloring \( c(G - v) \). Every color must be saturated in \( N(v) \), else \( c \) may be extended. Then at least one neighbor is 1, 2-saturated, hence degree at least 4.

- Every 4-vertex \( v \in G \) is adjacent to at most two low 3-vertices.
  
  Suppose \( v \) is adjacent to three 3-vertices (see fig. 1-4). Then only one (necessarily \( 4^+ \)) neighbor \( x \) may be saturated 1, 2; let \( c(x) = 1 \). The remaining vertices must be colored 2, 2, 3 respectively, to saturate the remaining classes; let \( u \in N(v) \) and \( c(u) = 3 \), \( d(u) = 3 \). Uncolor \( u \) and color \( v \) with 3 to produce the coloring \( c' \). Then if \( c' \) cannot be extended to \( u \), \( u \) must have two saturated
1, 2-neighbors $y, z$, both of which must have degree at least 4 by Lemma 1.3.2. Note that $y, z \neq v$, since $c'(v) = 3$. $u$ has three $4^+$-neighbors; $u$ is not low. $v$ has at most two low 3-neighbors.

![Figure 1-4: Neighborhood of a 4-vertex $v$ in $c(G - v)$.](image)

**Discharging.**

To each vertex $v \in G$, let

$$\mu(v) = 14d(v) - 48$$

and apply the following rule to determine $\mu^*(v)$:

**R1** For each $4^+$-vertex, donate a charge of 3 to every low 3-neighbor and a charge of 2 to every non-low 3-neighbor.

Consider the following cases:

- $d(v) \geq 5$. $v$ gives at most charge 3 to every neighbor: $\mu^*(v) \geq 14d(v) - 48 - 3d(v) = 11d(v) - 48 \geq 7$.

- $d(v) = 4$. $v$ has at most three 3-neighbors (requiring charge 2 or 3), at most two of which are low (requiring charge 3): $\mu^*(v) \geq 14 \times 4 - 48 - 2 \times 3 - 2 = 0$.

- $d(v) = 3$ and $v$ is low. $v$ has two $4^+$-neighbors and receives charge 3 from each: $\mu^*(v) = 14 \times 3 - 48 + 2 \times 3 = 0$. 


\( d(v) = 3 \) and \( v \) is not low. If \( v \) is not low, then \( v \) has three \( 4^+ \)-neighbors and receives charge 2 from each: \( \mu^*(v) = 14 \times 3 - 48 + 3 \times 2 = 0 \).

For every \( v \in G \), \( \mu^*(v) \geq 0 \).

We note that charge was redistributed, not created or destroyed:

\[
\sum_{v \in G} \mu(v) = \sum_{v \in G} \mu^*(v) \geq 0. \tag{1.7}
\]

Hence, recalling (1.6),

\[
\frac{1}{14n(G)} \sum_{v \in G} \mu(v) = \bar{\delta}(G) - \frac{24}{7} \geq 0 \tag{1.8}
\]

and \( G \) must have an average degree greater or equal to \( \frac{24}{7} \), a contradiction with our choice of \( G \). There cannot exist a minimum counter example. \( \square \)

We first bring attention to proof of the final claim of Lemma 1.3.3. Here, given a partial coloring \( c(G - v) \) and a uniquely colored neighbor \( u \), we generated a new partial coloring \( c'(G - u) \) by “recoloring” \( v \) with \( c(u) \). This recoloring process is central to the proofs in Chapter 3.

Secondly, like most discharging arguments, the above proof relies on local structure (i.e. in the neighborhood of a given vertex). In contrast, the arguments of Chapter 3 rely either explicitly or implicitly on global structures that are arbitrarily large.
Chapter 2

Proof of the upper bound

We seek to improve the upper bound in (1.4) for the special case of (1, 1, 0)-colorability. Specifically, we wish to show:

**Theorem 2.0.1.** Let $d$ be a number such that for all graphs $G$ with $\text{mad}(G) < d$, $G$ is $(1, 1, 0)$-colorable. Then $d < 4\frac{13}{40} < 4\frac{3}{7}$.

We provide a proof of this upper bound by construction. We demonstrate a non-$(1, 1, 0)$-colorable graph with $\text{mad} = 4\frac{13}{40}$; hence, for any value of $d \geq 4\frac{13}{40}$, there exists a (non-$(1, 1, 0)$-colorable) counterexample with mad less than $d$. Hence the following is sufficient to prove Theorem 2.0.1:

**Claim 2.0.2.** There exists a non-$(1, 1, 0)$-colorable graph with $\text{mad} = 4\frac{13}{40}$.

We achieve the construction using a collection of smaller graphs with a forced vertex color, which we use to build a large non-$(1, 1, 0)$-colorable graph; we then demonstrate a low maximum average degree for the counterexample.

2.1 Forcing a vertex color

Consider the following graph $H$:

We demonstrate that $H$ is not $(1,0)$-colorable (Borodin et al.\textsuperscript{3}). Suppose there exists a $(1,0)$ coloring of $H$. Consider any $(1,0)$ coloring of a 3-cycle $K$. For every
vertex $t \in K$, $t$ is adjacent to at least one 1-colored vertex in $K$. Hence, consider the vertex $v \in H$, which is a vertex in two 3-cycles. $v$ must be adjacent to a 1-colored vertex in each 3-cycle, hence $v$ must be colored 0. By symmetry, the same argument holds for $u$; however, not both $u$ and $v$ may be colored 0, a contradiction.

We use $H$ to construct a non-$(1,1,0)$-colorable graph. Let 1, 2 denote the 1-improper classes, and let 3 denote the proper class. If $H$ is $(1,1,0)$-colorable (in fact, it is $(1,1)$-colorable), in any $(1,1,0)$ coloring there must exist vertices colored both 1 and 2; else $H$ has been $(1,0)$ colored. We construct the fan $F_k$ by adding a vertex $x_k$ to two copies of $H$, and adding an edge between $x_k$ and every vertex of $2H$ (fig. 2-2).

We note that $x_k$ is adjacent to at least one 1 and 2 vertex in each copy of $H$, hence $x_k$ is saturated in 1 and 2: $x_k$ must be colored 3.

2.2 Constructing the counterexample

Let $A$ be a 5-vertex wheel graph, i.e. let $v_1, \ldots, v_4$ comprise a 4-cycle and let $v_0$ be adjacent to every other vertex (see fig. 2-3).

We demonstrate that $A$ is not $(1,1)$-colorable. Suppose there exists a $(1,1)$ coloring of $A$, with classes 1, 2. Then at most two vertices in the cycle $v_1, \ldots, v_4$ may be colored 1, else there exists some 1-colored $v_k$ with two 1-colored neighbors; equiv-
alently, there exists at most two vertices in the cycle colored 2. Hence exactly two vertices $v_1, \ldots, v_4$ are colored 1 and exactly two are colored 2. $v_0$ must be colored 1 or 2, but both colors are saturated on $v_1, \ldots, v_4$, a contradiction.

Hence, if $A$ is $(1,1,0)$-colorable, in every coloring it must have at least one vertex colored 3. Finally, we construct a graph $G$ formed with $A$ and five copies of the fan $F_0, F_1, \ldots, F_4$. From each vertex $v_k \in A$, an edge $v_k x_k$ is drawn (see fig. 2-4). Suppose $G$ is $(1,1,0)$-colorable. Then $A$ and each $F_k$ must be $(1,1,0)$ colored. Hence some $v_j$ and each $x_k$ must be colored 3; however $v_j$ is adjacent to $x_j$, a contradiction.
2.3 Determining the maximum average degree

We determine that \( n(G) := |V(G)| = 80 \) and \( e(G) := |E(G)| = 173 \). Hence, recalling (1.1), the average degree

\[
\bar{d}(G) = \frac{2e(G)}{n(G)} = \frac{173}{40}. \tag{2.1}
\]

For any subgraph \( H \subseteq G \), we define the potential function

\[
\rho(H) = 80e(H) - 173n(H). \tag{2.2}
\]

We wish to show that \( \text{mad}(G) = \bar{d}(G) \), i.e. that the maximum average degree is on \( G \) itself and that for any non-empty \( H \subseteq G \),

\[
\bar{d}(H) = \frac{2e(H)}{n(H)} \leq \frac{173}{40}, \tag{2.3}
\]

or equivalently (also noting that \( \rho(G) = 0 \)),

\[
\rho(H) \leq 0. \tag{2.4}
\]

For the graphs \( F_k \) and \( A \), it is easy to use a computer to determine the maximum average degree: for a small graph, we can enumerate all induced subgraphs and determine the maximum average degree. We find \( \text{mad}(F_k) = \bar{d}(F_k) = \frac{64}{15} \) (on the whole graph \( F_k \)) and \( \text{mad}(A) = \bar{d}(A) = \frac{16}{9} \) (on the whole graph \( A \)), or equivalently, for any non-empty \( F' \subseteq F \) and any non-empty \( A' \subseteq A \),

\[
\rho(F') \leq \rho(F) = -35, \quad \rho(A') \leq \rho(A) = -225. \tag{2.5}
\]

Consider any non-empty connected subgraph \( H \subseteq G \). If \( H \) is not connected, then \( \bar{d}(H) \leq \bar{d}(H') \), where \( H' \) is a maximum average degree component of \( H \); hence it suffices to consider only connected subgraphs of \( G \). Let \( H \) be a non-empty connected subgraph of \( G \). Let \( F'_k = H \cap F_k \) and \( A' = H \cap A \). Suppose that \( A' = \varnothing \). Then if \( H \) is connected, \( H = F'_k \neq \varnothing \). Then \( \rho(H) \leq -104 < 0 \). Let \( A' \) be non-empty, and
let $0 \leq q \leq 5$ subgraphs $F'_k$ be non-empty. $H$ is connected, hence for each non-empty $F'_k$, there exists exactly one edge connecting $F'_k$ to $A'$, namely $x_kv_k$. Then there are exactly $q$ edges in $H$ not in $(\bigcup_k F'_k) \cup A'$. Hence,

$$\rho(H) = \sum_k \rho(F'_k) + \rho(A') + 80q$$  \hspace{1cm} (2.6)

But if $F'_i$ is empty, then $\rho(F'_i) = 0$. Hence, noting (2.5) and that $q \leq 5$, we have

$$\rho(H) \leq q(\rho(F_k) + 80) + \rho(A) \leq 45q - 225 \leq 0.$$  \hspace{1cm} (2.7)

Hence, for any non-empty subgraph $H$ of $G$, $\rho(H) \leq 0$. \hfill \Box

We show that $\text{mad}(G) = 4\frac{13}{40}$: $G$ satisfies Claim 2.0.2 and hence Theorem 2.0.1 is demonstrated.
Chapter 3

Proofs of the lower bound

We present two proofs of improved lower bounds in the special \((1,1,0)\)-coloring case, and one proof of a generalized \((1_1,1_2,\ldots,1_a,0_1,\ldots,0_b)\)-coloring case. These proofs rely on the discharging method introduced in Section 1.3.1.

Recall the “recoloring” argument used in the final claim of Lemma 1.3.3. Section 3.1 extends this idea to global structures of recolorable vertices called chains; using these structures, we succeed in improving the lower bound in (1.4) for the special \((1,1,0)\)-colorability case. Conceptually, these chains “connect” vertices deficient in charge to vertices with excess charge; when discharging, charge “flows” along chains from regions of high average degree to regions of low average degree. In practice, chains allow us to apply local arguments (which occur at the endpoints of the chain structure) to arbitrary vertices not local to the arguments in question.

Section 3.2 forces the arguments of Section 3.1 to be local by effectively excluding the chain structures from the counterexample. The proof is greatly simplified therein; hence we introduce more local structure, resulting in an improved lower bound. Finally, Section 3.3 generalizes Section 3.2 to \((1_1,\ldots,1_a,0_1,\ldots,0_b)\)-coloring, by simplifying and generalizing the notion of “special configurations”, which live at the endpoints of chains and contribute to the local discharging argument. The general lower bound improves that of (1.5) for a large class of colorings.
3.1 The special case

**Theorem 3.1.1.** If \( \text{mad}(G) < 3\frac{4}{7} \), then \( G \) is \((1, 1, 0)\)-colorable.

Let \( G \) be a minimum counterexample with at least one 4-vertex.

We iteratively construct the vertex subset \( F \subseteq V(G) \). Let \( F_0 \) be all \( 5^+ \)-vertices in \( G \). For a given \( F_k \), let a vertex in \( F_k \) be called flagged. We generate some \( F_{k+1} \) by adding to \( F_k \) all 4-vertices in one of the following *special configurations* in \( N[F_k] \), if and only if such a configuration exists:

(a) a 4-vertex adjacent to two flagged vertices.

(b) a 4-vertex adjacent to a flagged vertex and adjacent to a 3-vertex adjacent to two flagged vertices.

(c) two 4-vertices mutually adjacent and each respectively adjacent to at least one flagged vertex.

Note that this iteration must terminate, since \( G \) is finite. Let \( F := F_n \), where \( F_n \) is the final possible iteration of some sequence \( F_i \). Note that \( F_0 \subset F_1 \subset \ldots \subset F_n \).

Alternately, we may define \( F \) as follows. We consider a collection of vertex subsets \( Z \), such that \( f \in Z \) if and only if \( F_0 \to f \) through some sequence of the above iteration. Then \( F \) is some maximal element in \( Z \).

### 3.1.1 Characterizing the minimum counter example.

Let a vertex be *saturated* if it is colored 1/2 and has exactly one neighbor colored 1/2, else is colored 3. In some coloring \( c \), let a vertex \( v \) be *recolorable* if there exists some \( c^* \) such that \( c(u) = c^*(u) \) if and only if \( u \neq v \).

**Lemma 3.1.2.** If in some partial coloring \( c(G - v) \), if \( u \in N(v) \) is non-recolorable and 1/2-saturated, then \( u \) is a \( 4^+ \)-vertex.

If \( u \) is 1/2-saturated, then \( N(u) \) contains exactly one vertex colored 1/2. But \( u \) is non-recolorable, hence \( N(u) \) contains a vertex saturated 2/1 and 3 respectively. \( v \) is uncolored, hence \( d(u) \geq 4 \).
Definition 3.1.3. Let a **chain** be a path comprised of non-repeating, non-recolorable saturated vertices.

Let $J = u_1u_2\ldots u_r$ be a chain in some coloring $c(G - u_1)$. Then we recolor $J$ by uncoloring $u_r$ and coloring $c'(u_k) = c(u_{k+1})$ for all $k < r$. If $J$ is properly colored in $c'$, then $J$ is **properly recolorable** and $c'$ is a **proper recoloring** of $J$.

Lemma 3.1.4. For any 4-vertex $v$, in any partial coloring $c(G - v)$, there exists a neighbor $w$ such that $w$ is non-recolorable, 1,2-saturated, and $c(w)$ is unique on $N(v)$.

Consider any 4-vertex $v \in G$, and a partial coloring $c = c(G - v)$. If $c$ cannot be extended to $v$, then $N(v)$ saturates all colors. $N(v)$ contains at least one 3-colored neighbor, hence at most three neighbors are colored 1/2: both colors must be represented, hence there exists at least one neighbor $w$ uniquely colored 1/2, and hence saturated and non-recolorable.

We shall call a chain $J = u_1u_2\ldots$ good if (i) for all segments $J' \subseteq J$ such that $u_1 \in J'$, $J'$ is properly recolorable, (ii) $J$ does not terminate on a 3-colored vertex, and (iii) $J$ does not contain any flagged vertices.

Lemma 3.1.5. There does not exist any 4-vertex $v \in G - F$.

Let $v$ be a 4-vertex in $G - F$, and consider a partial coloring $c(G - v)$. By Lemma 3.2.3, there exists at least one neighbor of $v$ that is 1,2-saturated, non-recolorable, and unique on $c[N(v)]$; consider any such neighbor $x \in N(v)$. Then $x$ must be flagged, else $vx$ comprises a good chain.

If $c$ cannot be extended to $v$, then $N(v)$ must saturate all color classes. If $v$ has two 3-colored neighbors, then the remaining neighbors $x, y$ must be non-recolorably 1,2-saturated, respectively, and hence uniquely colored: both $x, y$ must be flagged. Then $v$ may be added to $F$, a contradiction since $F$ is maximal. Hence $v$ must have exactly one 3-colored neighbor $w$, two 1/2-colored neighbors $s, t$, and one flagged 2/1-colored neighbor $x$. Without loss of generality, let $s, t$ be colored 1 and $x$ be colored 2. If $w$ is flagged, then $v$ may be added to $F$ and $F$ is not maximal. Let $w$ not be flagged, and uncolor $w$ and color $u$ with 3, to produce the coloring $c'$. If
If $w$ is a 4-vertex, then by Lemma 3.2.3, there exists some $z \in N(w)$ such that $z$ is 1,2-saturated, non-recolorable, and unique on $c'[N(w)]$; note that $z \neq v$, since $v$ is properly colored. Then $z$ must be flagged, else $vwz$ comprises a good chain. But then $v, w$ may be added to $F$, and $F$ is not maximal. If $w$ is a 3-vertex, then let $k, l \neq v$ be neighbors of $w$. If $w$ cannot be colored, then $k, l$ are 4-vertices, saturated 1,2. If both $k, l$ are flagged, then $u$ may be added to $F$, and $F$ is not maximal. Let $k$ not be flagged and 1-saturated, noting that $c'(k)$ is unique on $N(w)$. Then $vwk$ comprises a good chain. There must exist a good chain extending from $v$.

Let $J$ be a maximal good chain extending from $v$ with a terminating vertex $u$. Note that $c(u) = 1, 2$; without loss of generality, let $c(u) = 1$. Properly recolor $J$ such that $u$ is uncolored in $c'$. By Lemma 3.2.3, there exists at least one neighbor of $u$ that is 1,2-saturated, non-recolorable, and unique on $c'[N(u)]$; consider any such neighbor $x \in N(u)$ and note that $x \notin J$, since $J$ is properly colored. Then $x$ must be flagged, else $J + x$ is good and $J$ is not maximal. If $c'$ cannot be extended to $u$, then $N(u)$ must saturate all color classes. Let $t$ precede $u$ on $J$. Then $c'(t) = 1$ and $t$ is properly colored. Hence $u$ must have exactly one 3-colored neighbor $w$, two 1-colored neighbors $s, t$, and one flagged 2-colored neighbor $x$. $s \notin J$, else consider a recoloring $c^*$ of $J^* = v \ldots t \subset J$. Note that $c^*(s) = c'(s) = 1$ and that $u \notin J'$, hence $c^*(u) = c(u) = 1$. Since $u \in N(s)$, $s$ is not properly colored in $c^*$, a contradiction since $J$ is good, hence $J^*$ must be properly recolorable. We may also demonstrate that $w \notin J$. $s \notin J$, hence $c(s) = c'(s) = 1$. Noting that $c'(w) = 3$ and that $c(u) = c(s) = 1$, we observe that either $c(w) = 2$ or $w$ is uncolored in $c$, i.e. $w = v$. Let $w \neq v$. Let $p, q$ precede/follow $w$ on $J$ and $r$ be the remaining neighbor. Then $c'(p) = 2$ and we note $c(r) = 2$. Uncolor $w$ and recolor $c^*(u) = 3$. If $r \in J$, then $w$ has four properly colored neighbors, and $c^*$ may be extended. Else, $c^*(r) = 2$ and $w$ may be colored 1.

If $w = v$, then note that we may always choose a partial coloring $c^*(G - v)$ and a maximal good chain $J^*$ extending from $v$ such that $v$ is not colored 3 with a recoloring of $J^*$. Consider the coloring $c'$ above, and let $w = v$. Uncolor $v$ and recolor $u$ with 3 to form $c^*$. Note that for any maximal good chain $J^*$, $u \notin J^*$, since $c^*(u) = 3$ and $N(u)$ is either unsaturated, uncolored, or flagged. Hence in any proper recoloring on
$J^4$, $u$ is colored 3 and $v$ cannot be recolored 3. Let $w \notin J$.

If $w$ is flagged, then $u$ may be added to $F$, and $F$ is not maximal. Let $w$ not be flagged, and uncolor $w$ and color $u$ with 3, to produce the coloring $c''$. If $w$ is a 4-vertex, then by Lemma 3.2.3, there exists some $z \in N(w)$ such that $z$ is 1, 2-saturated, non-recolorable, and unique on $c''[N(w)]$; note that $z \notin J$, since $J$ is properly colored. Then $z$ must be flagged, else $J + w + z$ is good and $J$ is not maximal. But then $u, w$ may be added to $F$, and $F$ is not maximal. If $w$ is a 4-vertex, then let $k, l \neq u$ be neighbors of $w$. If $w$ cannot be colored, then $k, l$ are 4-vertices, saturated 1, 2. If both $k, l$ are flagged, then $u$ may be added to $F$, and $F$ is not maximal. Let $k$ not be flagged and 1-saturated, noting that $c''(k)$ is unique on $N(w)$. Then $J + w + k$ comprises a good chain: $J$ is not maximal, a contradiction. 

### 3.1.2 Discharging

For a vertex $v \in G$, define charge $\mu(v)$ as follows:

$$\mu(v) = 7 \times d(v) - 25$$

We discharge iteratively by the following rules: for every $v \in V(G)$,

R1 if $v \in F_0$, give charge 2 to each $u \in N(v) - F_0$.

R2 if $v \in F_k - F_{k-1}$ for some $k > 0$, then give charge 2 to each vertex $u \in N(v) - F_k$, unless $u$ is a 3-vertex adjacent to two vertices in $F_{k-1}$.

For every $v \in V(G)$ such that $d(v) = 5^+$, $v \in F_0$. Hence,

$$\mu^*(v) \geq 7 \times d(v) - 25 - 2 \times d(v) = 5 \times d(v) - 25 \geq 0$$

Every 4-vertex $v, v \in F$ and $v \notin F_0$, hence $v \in F_k - F_{k-1}$ for some $k > 0$. Then $v \in H = F_k - F_{k-1}$, where $H$ is a some special configuration. We note that $\mu(v) = 7 \times 4 - 25 = 3$ and consider the following cases:

- $H$ is type (a): $\mu^*(v) \geq 3 + 2 \times 2 - 2 \times 2 = 3$. 

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\* \* H is type (b) or (c): \( \mu^*(v) \geq 3 + 2 - 2 \times 2 = 1. \*

Finally, we note that every 3-vertex \( u \in V(G) \) must be adjacent to at least two \( 4^+ \)-vertices; moreover, if \( u \) is adjacent to only two \( 4^+ \)-vertices, then it receives charge from both. If \( u \) does not receive charge from some 4-neighbor, then it presupposes that \( u \) has two additional \( 4^+ \)-neighbors, from which it has received charge from each. Hence, noting that \( \mu(u) = 7 \times 3 - 25 = -4 \), then

\[ \mu^*(u) \geq -4 + 2 \times 2 = 0. \]

For every \( v \in V(G) \), \( \mu^*(v) \geq 0. \) We note that \( \sum_{v \in V(G)} \mu(v) = \sum_{v \in V(G)} \mu^*(v) \geq 0. \) Hence,

\[ \sum_{v \in V(G)} \mu(v) = \sum_{v \in V(G)} 7 \times d(v) - 25 \geq 0, \]

and hence

\[ \frac{1}{7n(G)} \sum_{v \in V(G)} \mu(v) = \bar{d}(G) - \frac{25}{7} \geq 0. \]

We find that \( \text{mad}(G) \geq \bar{d}(G) \geq \frac{25}{7} \), a contradiction with our choice of \( G. \) \( \square \)

### 3.2 Excluding the chain structure

**Theorem 3.2.1.** If \( \text{mad}(G) < 3 \frac{2}{3} \), then \( G \) is \((1,1,0)\)-colorable.

Let \( G \) be a minimum counterexample with at least one 4-vertex.

We iteratively construct the vertex subset \( F \subseteq V(G) \). Let \( F_0 \) be all \( 6^+ \)-vertices in \( G \) and every 5-vertex adjacent to another 5-vertex. For a given \( F_k \), let a vertex in \( F_k \) be called \textit{flagged}. We generate some \( F_{k+1} \) by adding to \( F_k \) all vertices in one of the following \textit{special configurations} in \( N[F_k] \), if and only if such a configuration exists:

(a) a 5-vertex adjacent to a flagged vertex.

(b) a 5-vertex adjacent to a 4-vertex adjacent to a flagged vertex.

(c) a 5-vertex adjacent to a 3-vertex adjacent to two flagged vertices.
(d) a 4-vertex adjacent to two flagged vertices.

(e) two 4-vertices mutually adjacent and each respectively adjacent to at least one flagged vertex.

(f) a 4-vertex adjacent to a flagged vertex and adjacent to a 3-vertex adjacent to two flagged vertices.

Note that this iteration must terminate, since $G$ is finite. Let $F := F_n$, where $F_n$ is the final possible iteration of some sequence $F_i$. Note that $F_0 \subset F_1 \subset \ldots \subset F_n$.

Alternately, we may define $F$ as follows. We consider a collection of vertex subsets $Z$, such that $f \in Z$ if and only if $F_0 \rightarrow f$ through some sequence of the above iteration. Then $F$ is some maximal element in $Z$.

### 3.2.1 Characterizing the minimum counter example.

Let a vertex be saturated if it is colored 1/2 and has exactly one neighbor colored 1/2, else is colored 3. In some coloring $c$, let a vertex $v$ be recolorable if there exists some $c^*$ such that $c(u) = c^*(u)$ if and only if $u \neq v$.

**Lemma 3.2.2.** If in some partial coloring $c(G - v)$, if $u \in N(v)$ is non-recolorable and 1/2-saturated, then $u$ is a 4$^+$-vertex.

If $u$ is 1/2-saturated, then $N(u)$ contains exactly one vertex colored 1/2. But $u$ is non-recolorable, hence $N(u)$ contains a vertex saturated 2/1 and 3 respectively. $v$ is uncolored, hence $d(u) \geq 4$.

**Lemma 3.2.3.** For any 4-vertex $v$, in any partial coloring $c(G - v)$, there exists a neighbor $w$ such that $w$ is non-recolorable, 1,2-saturated, and $c(w)$ is unique on $N(v)$.

Consider any 4-vertex $v \in G$, and a partial coloring $c = c(G - v)$. If $c$ cannot be extended to $v$, then $N(v)$ saturates all colors. $N(v)$ contains at least one 3-colored neighbor, hence at most three neighbors are colored 1/2: both colors must be represented, hence there exists at least one neighbor $w$ uniquely colored 1/2, and hence saturated and non-recolorable.
Lemma 3.2.4. For any 5-vertex \( v \), in any partial coloring \( c(G - v) \), there exists a neighbor \( w \) such that \( w \) is non-recolorable, saturated, and \( c(w) \) is unique on \( N(v) \).

Consider any 5-vertex \( v \in G \), and a partial coloring \( c = c(G - v) \). If \( c \) cannot be extended to \( v \), then \( N(v) \) saturates all colors. \( N(v) \) contains at least one 3-colored neighbor. If \( N(v) \) contains exactly one 3-colored neighbor \( w \), then it is non-recolorable, saturated, and unique. Suppose \( N(v) \) contains at least two 3-colored neighbors. Then at most three neighbors are colored 1/2: both colors must be represented, hence there exists at least one neighbor \( w \) uniquely colored 1/2, and hence saturated and non-recolorable.

Lemma 3.2.5. There does not exist any 4\(^+\)-vertex \( v \in G - F \).

Let \((v, c(G - v))\) be a 5\(^+\)-vertex in \( G - F \) and a partial coloring \( c(G - v) \) with a minimum number of 1, 2-saturated 4\(^+\)-vertices. All 6\(^+\)-vertices are flagged in \( F_0 \), hence \( d(v) = 4, 5 \). Let \( v \) be a 5-vertex. By Lemma 3.2.4, there exists at least one neighbor that is saturated, non-recolorable, and unique on \( c[N(v)] \). Consider any such neighbor \( x \), and note that \( d(x) \leq 4 \), else \( v \) is adjacent to a 5\(^+\)-vertex and flagged in \( F_0 \). If \( x \) is 1, 2-saturated, then by Lemma 3.2.2, \( x \) is a 1, 2-saturated 4-vertex. \( x \) is not flagged, else \( v \) may be added to \( F \) by configuration (a), and \( F \) is not maximal. Color \( v \) with \( c(x) \) and uncolor \( x \) to produce \( c'(G - x) \): since \( c(x) = c'(v) \) is unique on \( N(v) \), no additional vertex is saturated and \((x, c'(G - x))\) has fewer 1, 2-saturated 4\(^+\)-vertices than \((v, c(G - v))\), a contradiction.

Let \( c(x) = 3 \), and color \( v \) with 3 and uncolor \( x \) to produce another coloring \( c'(G - x) \); note that no additional vertices are saturated. We note that \( x \) must have at least one unflagged unique 1, 2-saturated neighbor \( u \). If \( x \) is a 3-vertex, then it must have two 1, 2-saturated neighbors \( u, w \), where \( u, w \neq v \) since \( c'(v) = 3 \). Not both \( u, w \) may be flagged, else by configuration (c) \( v \) may be added to \( F \), a contradiction; let \( u \) be unflagged. Similarly, if \( x \) is a 4-vertex, then by Lemma 3.2.3, it must at least one unique, 1, 2-saturated neighbor \( u \neq v \). \( u \) is unflagged, else by configuration (b) \( v, x \) may be added to \( F \), a contradiction. In both cases, recall that \( u \) must be a 4\(^+\) vertex. Recolor \( x \) with \( c'(u) \) and uncolor \( u \) to produce the coloring \( c''(G - u) \): \( c'(u) \)
is unique on $N(x)$, hence no additional vertices are saturated and $(u, c''(G - u))$ has fewer 1,2-saturated $4^+$-vertices than $(v, c(G - v))$, a contradiction. There exists no minimum $5^+$-vertex in $G - F$: all $5^+$-vertices are flagged.

Let $(v, c(G - v))$ be a 4-vertex in $G - F$ and a partial coloring $c(G - v)$ with a minimum number of 1,2-saturated $4^+$-vertices. By Lemma 3.2.3, there exists at least one neighbor of $v$ that is 1,2-saturated, non-recolorable, and unique on $c[N(v)]$; consider any such neighbor $x \in N(v)$. Then $x$ must be flagged, else color $v$ with $c(x)$ and uncolor $x$ to produce $c'(G - x)$: since $c(x) = c'(v)$ is unique on $N(v)$, no additional vertex is saturated and $(x, c'(G - x))$ has fewer 1,2-saturated 4-vertices than $(v, c(G - v))$, a contradiction.

If $c$ cannot be extended to $v$, then $N(v)$ must saturate all color classes. If $v$ has two 3-colored neighbors, then the remaining neighbors $x, y$ must be non-recolorably 1,2-saturated, respectively, and hence uniquely colored: both $x, y$ must be flagged. Then $v$ may be added to $F$, a contradiction since $F$ is maximal. Hence $v$ must have exactly one 3-colored neighbor $w$. If $w$ is flagged, then $v$ may be added to $F$ and $F$ is not maximal. Let $w$ not be flagged, and hence $d(w) = 3, 4$. Uncolor $w$ and color $u$ with 3, to produce the coloring $c'$; note that $c'$ has an equal number of saturated 1,2 vertices as $c$. If $w$ is a 4-vertex, then by Lemma 3.2.3, there exists some $z \in N(w)$ such that $z$ is 1,2-saturated, non-recolorable, and unique on $c'[N(w)]$; note that $z \neq v$, since $v$ is properly colored. Then $z$ must be flagged, else uncolor $z$ and color $w$ with $c'(z)$ to produce the coloring $c''(G - z)$: since $c'(z) = c''(w)$ is unique on $N(w)$, no additional vertex is saturated and $(z, c''(G - z))$ has fewer 1,2-saturated 4-vertices than $(v, c(G - v))$, a contradiction. But then $v, w$ may be added to $F$, and $F$ is not maximal. If $w$ is a 3-vertex, then let $k, l \neq v$ be neighbors of $w$. If $w$ cannot be colored, then $k, l$ are 4-vertices, saturated 1,2. If both $k, l$ are flagged, then $v$ may be added to $F$, and $F$ is not maximal. Let $k$ not be flagged and 1-saturated, noting that $c'(k)$ is unique on $N(w)$. Then uncolor $k$ and color $w$ with $c'(k)$: $(v, c(G - v))$ is not minimum, a contradiction. All $4^+$-vertices must be flagged. □
3.2.2 Discharging

For a vertex \( v \in G \), define charge \( \mu(v) \) as follows:

\[
\mu(v) = 3 \times d(v) - 11
\]

We discharge iteratively by the following rules: for every \( v \in V(G) \),

- **R1** if \( v \in F_0 \), give charge 1 to each \( u \in N(v) - F_0 \).

- **R2** if \( v \in F_k - F_{k-1} \) for some \( k > 0 \), then give charge 1 to each vertex \( u \in N(v) - F_k \).

Let \( H_k = F_k - F_{k-1} \) for \( k > 0 \) and \( H_0 = F_0 \), and let \( D = \{ v \in V(G) : d(v) = 3 \} - F \). By Lemma 3.2.5, \( D \) and \( F \) partition \( V(G) \). Moreover, since \( F_{k-1} \subseteq F_k \), \( F \) is partitioned by \( \{ H_k \} \). We consider the vertices in each partition separately.

For every \( v \in H_0 = F_0 \), \( d(v) = 5^+ \). If \( v \) is a 6\(^+\)-vertex, then

\[
\mu^*(v) \geq 3 \times 5 - 11 - 1 \times 4 = 0.
\]

Let \( d(v) = 5 \). Then if \( v \in F_0 \), \( v \) has a 5-vertex neighbor also in \( F_0 \), i.e., at most 4 neighbors not in \( F_0 \):

\[
\mu^*(v) \geq 3 \times 5 - 11 - 1 \times 4 = 0.
\]

Hence for every \( v \in H_0 \), \( \mu^*(v) \geq 0 \), and

\[
\mu^*[H_0] := \sum_{v \in H_0} \mu^*(v) \geq 0.
\]

We consider the total charge of each partition \( H_k \), \( k > 0 \). By **R2**, we note that every neighbor in \( F_{k-1} \) gives charge 1 to \( H_k \), and that \( H_k \) gives charge 1 to every neighbor not in \( F_k \). \( H_k \) comprises a special configuration, hence (noting that \( \mu = -2, 1, 4 \) for 3, 4, 5-vertices respectively):

- **H_k** is type (a): \( \mu^*[H_k] \geq 4 + 1 - 4 \times 1 = 1 \).

- **H_k** is type (b): \( \mu^*[H_k] \geq 4 + 1 + 1 - 6 \times 1 = 0 \).
\begin{itemize}
  \item $H_k$ is type (c): $\mu^*[H_k] \geq 4 - 1 + 2 \times 1 - 4 \times 1 = 1$.
  \item $H_k$ is type (d): $\mu^*[H_k] \geq 1 + 2 \times 1 - 2 \times 1 = 1$.
  \item $H_k$ is type (e): $\mu^*[H_k] \geq 2 \times 1 + 2 - 4 \times 1 = 0$.
  \item $H_k$ is type (f): $\mu^*[H_k] \geq 1 - 2 + 3 \times 1 - 2 \times 1 = 0$.
\end{itemize}

Finally, for every $v \in D$, $v$ is a 3-vertex adjacent to at least two flagged vertices (add lemma above). $v$ is unflagged, hence recieves charge 1 from all flagged neighbors:

$$\mu^*(v) \geq -2 + 2 \times 1 = 0,$$

and we note that $\mu^*[D] \geq 0$.

Since $\{H_k\}$, $D$ partition $V(G)$, $\mu^*[V(G)] = \sum_k \mu^*[H_k] + \mu^*[D] \geq 0$. But charge is conserved, hence $\mu[V(G)] = \mu^*[V(G)] \geq 0$, and

$$\sum_{v \in V(G)} \mu(v) = \sum_{v \in V(G)} (3 \times d(v) - 11) \geq 0,$$

and hence

$$\frac{1}{3n(G)} \sum_{v \in V(G)} \mu(v) = \bar{d}(G) - \frac{11}{3} \geq 0.$$

We find that $\text{mad}(G) \geq \bar{d}(G) \geq \frac{11}{3}$, a contradiction with our choice of $G$. \hfill $\square$

### 3.3 Generalization to $(1, a, b)$ colorings

We say a graph is $(t, a, b)$-colorable if $V(G)$ may be partitioned into $b$ independent sets $0_1, 0_2, \ldots, 0_b$ and $a$ sets $t_1, t_2, \ldots, t_a$ whose induced graphs have maximum degree less than or equal to $t$. We consider the case $t = 1$ with $a$ independent sets $1_1, 1_2, \ldots, 1_a$, given $a > 1$ and $b > 0$.

If a vertex is colored “1”, then it belongs to some color class $1_k$; similarly if a vertex is colored “0”, then it belongs to some color class $0_k$. A 1-class refers to some color class $1_k$, and a 0-class refers to some color class $0_k$. 

Theorem 3.3.1. Given $a > 1$ and $b > 0$, if $\text{mad}(G) < \frac{4}{3}a + b$, then $G$ is $(1, a, b)$-colorable.

For a given relaxation $(1, a, b)$, for a vertex $v \in G$ define $h(v) = d(v) - (a + b)$. Then $v$ is

- **small** if $d(v) = a + b$, i.e. if $h(v) = 0$.
- **medium** if $a + b < d(v) < 2a + b$, i.e. if $0 < h(v) < a$. In particular, we say $v$ is an $h$-medium vertex.
- **large** if $2a + b \leq d(v) < 2a + 2b$, i.e. if $a \leq h(v) < a + b$. In particular, we say $v$ is an $h$-large vertex.
- **huge** if $d(v) \geq 2a + 2b$, i.e. if $h(v) \geq a + b$.

Let $G$ be a minimum counterexample.

We iteratively construct the vertex subset $F \subseteq V(G)$. Let $F_0$ be all huge vertices in $G$. For a given $F_k$, let a vertex in $F_k$ be called flagged. We define “special configurations” of $F_k$.

Definition 3.3.2. Given a non-empty vertex set $H_k \subseteq V(G)$ disjoint from a flagged vertex set $F_k$, $H_k$ is a special configuration of $F_k$ if for every $v \in H_k$

(i) $v$ has at least $\max\{a - h(v), 0\}$ flagged neighbors, and

(ii) $v$ has degree at least $a + b - h(v)$ in the induced subgraph $G[H_k \cup F_k]$.

If there exists some special configuration $H_k$ of $F_k$, let $F_{k+1} = F_k \cup H_k$. $H_k$ is non-empty and disjoint from $F_k$, hence $H_k - F_k \neq \emptyset$: since $G$ is finite, the above iteration must terminate. Hence let $F := F_n$, where $F_n$ is the final possible iteration of some sequence $F_k$. Note that $F_1 \subset F_2 \subset \ldots \subset F_n$.

Alternately, we may define $F$ as follows. We consider a collection of vertex subsets $Z$, such that $f \in Z$ if and only if $F_0 \rightarrow f$ through some sequence of the above iteration. Then $F$ is some maximal element in $Z$. 
3.3.1 Characterizing the minimum counter example.

Let a vertex be *saturated* if it is colored $0_i$ or if it is colored $1_j$ and has exactly one neighbor colored $1_j$. In some coloring $c$, let a vertex $v$ be *recolorable* if there exists some $c^*$ such that $c(u) = c^*(u)$ if and only if $u \neq v$. A *partial coloring* is a coloring excluding exactly one vertex in $G$.

**Lemma 3.3.3.** Every vertex $v \in G$ is at least small, i.e. $\delta(G) \geq a + b$.

Consider a partial coloring $c(G - v)$. If $c$ cannot be extended to $v$, then $N(v)$ must saturate $a + b$ colors: $v$ has at least $a + b$ neighbors. 

Given some neighbor $u$ of a vertex $v$, we say $u$ is *uniquely colored* on $N(v)$ if $c(u)$ is unique on $N(v)$.

**Lemma 3.3.4.** Consider any non-huge vertex $v \in G$ and a partial coloring $c(G - v)$. Then $v$ has at least $a + b - h(v)$ uniquely saturated neighbors, at least $\max\{a - h(v), 0\}$ of which are uniquely 1-saturated.

We note that $d(v) = a + b + h(v)$. Suppose that $v$ has $q < a + b - h(v)$ uniquely colored neighbors. If $c$ cannot be extended to $v$, then every color class must be represented on $N(v)$, hence $a + b - q$ classes are not unique on $N(v)$. Hence, $v$ has at least two neighbors of each of $a + b - q$ classes: $d(v) \geq 2(a + b - q) + q = 2a + 2b - q > a + b + h(v)$, a contradiction. $v$ has at least $a + b - h(v)$ uniquely colored neighbors, each of which must be saturated if $c$ cannot be extended to $v$.

Similarly, let $v$ be medium or small, hence $a - h(v) > 0$. Suppose that $v$ has $p < a - h(v)$ uniquely colored 1-colored neighbors. Every 1-class must be represented on $N(v)$, hence $a - p$ 1-classes are not unique on $N(v)$. $v$ has at least two neighbors of each of $a - p$ 1-classes and at least $b$ 0-colored neighbors, hence $d(v) \geq 2(a - p) + b + p = 2a + b - p > a + b + h(v)$, a contradiction. $v$ has at least $a - h(v)$ uniquely 1-colored neighbors, each of which must also be saturated.

Given a flagged vertex set $F$, we call a partial coloring $c$ *minimum* if the number of unflagged 1-saturated vertices is minimum on the set of all partial colorings $\{c(G - v) : v \in V(G) - F\}$.
Lemma 3.3.5. Every vertex is flagged in $F$.

Let $H$ be the set of all unflagged vertices with minimum partial colorings. The following must be true for every $v \in H$:

(i) $v$ has at least $\max\{a - h(v), 0\}$ flagged neighbors.

If $v$ is large, then $a - h(v) < 0$. Suppose $v$ is medium and has fewer than $a - h(v)$ flagged neighbors. Consider any minimum partial coloring $c(G - v)$. By Lemma 3.3.4, $v$ has at least $a - h(v)$ uniquely 1-saturated neighbors. Hence, $v$ must have at least one uniquely 1-saturated neighbor $u$ that is unflagged. Uncolor $u$ and color $v$ with $c(u)$ to produce the coloring $c'(G - u)$. $v$ is unsaturated and no other vertices have been saturated, since $c(u)$ is unique on $N(v)$: $c'$ has fewer 1-saturated vertices than $c$, hence $c$ is not minimum, a contradiction.

(ii) $v$ has at least $a + b - h(v)$ neighbors in the induced graph $G[H \cup F]$.

If $v$ is unflagged, then $v$ is non-huge. By Lemma 3.3.4, $v$ has at least $a + b - h$ uniquely saturated neighbors. Consider any such neighbor $w$, and assume that $w$ is unflagged. Uncolor $w$ and color $v$ with $c(w)$ to produce the coloring $c'(G - w)$. $v$ is not 1-saturated and no other vertices have been 1-saturated, since $c(w)$ is unique on $N(v)$. $c'$ has no more 1-saturated vertices than $c$, hence $c'$ is minimum as well: $w \in H$.

Hence $H$ is a special configuration of $F$. If $F \cup H \neq F$, then $F$ is not maximal, a contradiction. $H$ and $F$ are disjoint, hence $H = \emptyset$. There are no unflagged vertices.

3.3.2 Discharging

For a vertex $v \in G$, define charge $\mu(v)$ as follows:

$$\mu(v) = 3d(v) - 4a - 3b$$

We discharge iteratively by the following rules: for every $v \in V(G)$,
**R1** if $v \in F_0$, give charge 1 to each $u \in N(v) - F_0$.

**R2** if $v \in F_k - F_{k-1}$ for some $k > 0$, then give charge 1 to each vertex $u \in N(v) - F_k$.

Let $H_k = F_{k+1} - F_k$ for $k \geq 0$. By Lemma 3.3.5, $F = V(G)$. Since $F_{k-1} \subset F_k$, $F$ is partitioned by $\{H_k\}_k \cup \{F_0\}$. We consider the vertices in each partition separately.

For every $v \in F_0$, $v$ is huge. By **R1** $v$ gives charge 1 to every neighbor:

$$\mu^*(v) = 3d(v) - 4a - 3b - d(v) = 2d(v) - 4a - 3b \geq 2(2a + 2b) - 4a - 3b > 0$$

given $d(v) \geq 2(a + b)$ and $b > 0$. $\mu^*[F_0] > 0$.

We consider the total charge of each partition $H_k$. By **R2**, we note that every neighbor in $F_k$ gives charge 1 to $H_k$, and that $H_k$ gives charge 1 to every neighbor not in $F_{k+1}$. $H_k$ comprises a special configuration, hence for any $v \in H_k$,

- $v$ is non-huge,
- $v$ has at least $\max\{a - h(v), 0\}$ neighbors in $F_k$, and
- $v$ has at least $a + b - h(v)$ neighbors in $H_k \cup F_k = F_{k+1}$, i.e. $v$ has at most $2h$ neighbors not in $F_{k+1}$.

Hence for any $v \in H_k$, we consider the following cases:

- $v$ is **small** or **medium**: $v$ has at least $a - h(v)$ flagged neighbors in $F_k$. Then

  $$\mu^*(v) \geq 3(a + b + h(v)) - 4a - 3b + (a - h(v)) - 2h = 0.$$  

- $v$ is **large**: noting that $h(v) \geq a$,

  $$\mu^*(v) \geq 3(a + b + h(v)) - 4a - 3b - 2h(v) = h(v) - a \geq 0.$$  

Hence we find that $\mu^*(v) \geq 0$ for all $v \in H_k$: $\mu^*[H_k] \geq 0$. 

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Since \( \{H_k\}_k \cup \{F_0\} \) partition \( V(G) \), \( \mu^*[V(G)] = \sum_k \mu^*[H_k] + \mu^*[F_0] \geq 0 \). But charge is conserved, hence \( \mu[V(G)] = \mu^*[V(G)] \geq 0 \), and

\[
\sum_{v \in V(G)} \mu(v) = \sum_{v \in V(G)} (3 \times d(v) - 4a - 3b) \geq 0,
\]

and hence

\[
\frac{1}{3n(G)} \sum_{v \in V(G)} \mu(v) = \bar{d}(G) - \left( \frac{4}{3}a + b \right) \geq 0.
\]

We find that \( \text{mad}(G) \geq \bar{d}(G) \geq \frac{4}{3}a + b \), a contradiction with our choice of \( G \). \( \square \)

Finally, we note that if a graph is \( (1, a, b) \)-colorable, then it is certainly \( (t, a, b) \)-colorable, for \( t \geq 1 \). Hence Theorem 3.3.1 holds for the fully general \( (t, a, b) \)-coloring case, although it is not always an improvement of the previous lower bound in (1.2).
Chapter 4

Conclusion

For the special \((1, 1, 0)\)-coloring case, we demonstrate the bounds

\[
\frac{2}{3} \leq d < \frac{13}{40},
\]

where \(d\) is the maximum number such that for all \(G\) with \(\text{mad}(G) \leq d\), \(G\) is \((1, 1, 0)\)-colorable. Moreover, for a coloring with \(a \geq 2\) improper classes and \(b \geq 1\) proper classes, we demonstrate a lower bound

\[
\frac{4}{3}a + b \leq d,
\]

where \(d\) is again the maximum number such that for all \(G\) with \(\text{mad}(G) \leq d\), \(G\) is colorable with \(a\) improper classes and \(b\) proper classes.

We note that for the case of \((1, a, b)\)-colorings (here denoting \(a \geq 2\) 1-improper classes and \(b \geq 1\) proper classes), \((4.2)\) is an improvement over the existing bound in
for $a \geq 2$,

$$a + 2 + \frac{ab}{a + 1} > 3$$

$$a((a + 2)(a + 1) + ab) > 3a(a + 1)$$

$$\frac{1}{3}a > \frac{a(a + 1)}{(a + 2)(a + 1) + ab}$$

$$\frac{4}{3}a + b > a + b + \frac{a(a + 1)}{(a + 2)(a + 1) + ab},$$

hence for colorings with mixed 1-improper and proper classes, our result improves the previously existing result in [7].

### 4.1 Future work

Often work supporting the lower bound develops structure that can inform efforts toward improving the upper bound. Our upper bound for the special $(1, 1, 0)$-coloring case suggest significant room for improvement: ideally, rather than employing structures developed for simpler colorings, we might produce color-forcing graphs “designed” for $(1, 1, 0)$-colorings. Our work with the lower bound presents novel structure that could inform a sparse non-colorable graph, or family of graphs.

Moreover, we have yet to seriously explore constructions for a generalized upper bound. The structure employed in the proof of the generalized lower bound suggests improvements for an upper bound in the general case, which would complement our completed efforts.
Bibliography


