ON THE PROOF OF CAUCHY'S THEOREM

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Raoul Louis Weinstein

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APPROVAL SHEET

This thesis is submitted in partial fulfillment of
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Raoul L. Weinstein

Author

Approved, July 1965

Fred W. Weiler

Fred W. Weiler, Ph.D.

Thomas L. Reynolds

Thomas L. Reynolds, Ph.D.

Hugh B. Easler

Hugh B. Easler, M.S.

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ON THE PROOF OF CAUCHY'S THEOREM

INTRODUCTION

This thesis is a presentation of two different approaches to the proof of Cauchy's Theorem for a Simply Connected Domain.

First, we prove the theorem by the widely-used method of approximating the integral of a function (analytic on a simply connected domain) over a closed rectifiable curve contained in the domain using the integral of this function over the boundaries of polygons (the polygons approximating the curve). As hinted above, this method is used in many texts but it is almost always simply sketched or outlined and the details are usually left to the reader. It is the principal effort of this thesis to insure that every detail and every argument needed in this proof has been thoroughly investigated and handled in a manner so as to leave nothing to the reader's imagination. As an example, the properties of polygons, including the tedious problem of triangulation, is approached with more rigor than is commonly used so as to solidify the foundation of the proof.

Second, we prove the theorem by showing that an analytic function defined on a bounded simply connected domain can be written as the limit of an almost uniformly convergent sequence of polynomials. Since polynomials have primitive functions, Cauchy's Theorem follows quite easily. This treatment was adapted from the method exhibited by S. Saks and A. Zygmund, [5], although they prove a stronger result than we desire, as they work in the extended plane, while we work in the unextended plane. Also, they work with rational functions while we prefer to work with polynomials. Consequently, an intricate examination was conducted to determine which parts of the work of Saks and Zygmund were not applicable and which parts could be used, so as to piece together a sound proof that would accomplish our goal, but include no extras.

Supplementing the proofs of the main theorems is an array of background material including many helpful definitions and key lemmas, with those not too obvious being proved, and those whose proofs are found in most texts just being stated for reference throughout the paper. The results in this paper are self-sufficient, carefully prepared, and rigorously detailed treatments of the proof of Cauchy's Theorem.

SYMBOLS AND NOTATION

$\underline{T}$	Theorem
$\underline{D}$	Definition
$\underline{L}$	Lemma
$\underline{R}$	Remark
$\underline{N}$	Notation
$\underline{C}$	Conclusion
$\underline{H}$	Hypothesis
$\underline{P}$	Proof
Z	Set of Complex Numbers
R	Set of Real Numbers
I	Set of Integers
I^+	Set of Positive Integers
$\mathcal{K}A$	Closure of Set A
$\mathcal{I}A$	Interior of Set A
$\mathcal{C}A$	Complement of Set A
$\mathcal{B}A$	Boundary of Set A
$\mathcal{J}(A,B)$	Juncture of the Sets A and B
$f:C \to Z$	Function f defined on C with values in Z
wrt	With respect to
iff	If and only if
$\Gamma(C)$	Boundary of curve C
$\ni$	Such that
$\exists$	There exists

$\forall$	For every; for all
$\Rightarrow$	Implies that; it follows that
$\emptyset$	Empty set
$A \subseteq B$	The set A is contained in the set B
$A \cup B$	The Union of sets A and B
$A \cap B$	The Intersection of sets A and B
$A \times B$	The Cartesian Product of the sets A and B
$a \in A$	a is an element of set A

CHAPTER I

CURVES

It will be assumed throughout this paper that the usual meaning of continuity, one-sided continuity, differentiability, and one-sided differentiability will be meant when these concepts are used.

D1.1 Consider $a,b \in R \ni a<b$ where $z=z(t) : I \to Z$ is a continuous function on the interval $I \equiv [a,b] = \{t \mid a \le t \le b\}$. Then, $z=z(t)$, $t \in I$ is defined to be a _continuous_ _curve_ (or _continuous_ _arc_).

N1.1 The notation for a continuous curve will be $C: z(t)$, $t \in I$.

R1.1 It is understood that the continuity of $z(t)$ at $t=a$ and $t=b$ is one-sided.

R1.2 The set $\{z \mid z=z(t), t\ I\}$ is a compact, connected set since I is compact and connected and $z=z(t)$ is continuous. Also, a curve C is not merely a set of points but very essentially also a succession of points, ordered by increasing values of t.

D1.2 Consider the interval $I \equiv [a,b]$ and the continuous curve $C: z=z(t), t \in I$. Then $z=z(t)$ is a _simple_ _curve_ (or _Jordan_ _arc_) iff $t_1, t_2 \in I \ni t_1 \neq t_2$ implies that $z(t_1) \neq z(t_2)$, i.e., z is 1-1, i.e., $z(t_1)=z(t_2)$ implies that $t_1=t_2$.

*D*1.3 We call $z(a)$ the *initial point* of C, and $z(b)$ the *terminal point* of C.

*D*1.4 $z=z(t)$ is said to be a *closed curve* iff $z(a)=z(b)$.

*D*1.5 $z=z(t)$ is a *simple closed curve* (or *Jordan curve*) iff

(1) $z=z(t)$ is a closed curve, and

(2) $t_1, t_2 \in I \ni t_1 < t_2$ and $z(t_1)=z(t_2)$ implies that $t_1=a$ and $t_2=b$.

*R*1.3 From the above it is noted that a simple closed curve is *not* simple.

In the following theorem, the usual meaning of *domain* will be assumed. However, a concise definition of domain is given in *D*2.2 as it is more easily defined then, rather than at this point.

*T*1.1 *H* $C: z=z(t), t\in I$ is a simple closed curve.

C There exist two domains D_1 and D_2 such that the following conditions hold: (1) $\mathcal{C}C=D_1 \cup D_2$; (2) $D_1 \cap D_2=\emptyset$; (3) D_1 is bounded while D_2 is unbounded; and (4) $\mathcal{B}D_1=\mathcal{B}D_2=C$. (*Jordan Curve Theorem*)

*R*1.4 The unbounded domain D_1 is termed the *interior* of the Jordan curve C and the unbounded domain D_2 is called the *exterior* of C. Thus *T*1.1 asserts that a Jordan curve divides the complex plane into an interior and an exterior.

*D*1.6 Consider the continuous curve $C: z=z(t), t\in I$ where I is the interval $I\equiv[a,b]$. We define Δ to be a *partition of I* $\equiv$ $\Delta: \{t_k\}_{k=0}^{n} \ni a=t_0<t_1<\ldots\ldots.<t_{n-1}<t_n=b$.

D1.7 We define $S(\Delta)\equiv \sum_{k=1}^{n}|z(t_k)-z(t_{k-1})|$.

D1.8 We say that $f(t)$ is of bounded variation on I iff the set $\{S(\Delta)\}$ is bounded for all Δ, i.e., lub $S(\Delta)<+\infty$ for all Δ.

D1.9 $z=z(t)$ is a rectifiable curve iff $\{S(\Delta)\}$ is bounded for all Δ, i.e., if there exists $M>0 \ni$ for all Δ implies that $|S(\Delta)|<M$.

D1.10 If $C:z(t),t\in I$ is a rectifiable curve, then $l(C)$ means the length of C, and $l(C)\equiv \text{lub}S(\Delta)$ for all Δ.

T1.2 H Given the continuous curve $C:z(t),t\in I$ where $I\equiv[a,b]$ and where $z(t)=x(t) + iy(t)$, $t\in I$.

C $C:z(t)$ is rectifiable iff $x(t)$ and $y(t)$ are both of bounded variation on I.

D1.11 Given the continuous curve $C:z(t),t\in I$ where $I\equiv[a,b]$. Then $z=z(t)$ is a differentiable curve iff

(1) $t\in I$ implies that $z'(t)$ exists, and

(2) $z'(t)$ is continuous on I.

(Again we assume the existence of one-sided differentiability and one-sided continuity at $t=a$ and at $t=b$).

T1.3 H Given the differentiable curve $C:z(t),t\in I$ where $I\equiv [a,b]$ and where $z(t)=x(t) + iy(t)$, $t\in I$.

C (1) $C:z(t),t\in I$ is rectifiable, and

(2) $l(C)=\int_a^b\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt =\int_a^b |z'(t)|dt.$

T1.4 H Given the differentiable curve $C:z(t),t\in I$ where $I\equiv [a,b]$ and where $z(t)=x(t) + iy(t)$, $t\in I$.

C (1) If $t\in I$ implies that there exists a bounded derivative

$z'(t)$, then $z(t)$ is of bounded variation.

(2) If $t \in I$ implies that there exists a continuous derivative $z'(t)$, then $z(t)$ is of bounded variation, since a continuous function on I is necessarily bounded on I.

D1.12 Consider the continuous curve $C: z(t), t \in I$ where $I \equiv [a,b]$. Then, $z=z(t)$ is a regular curve (or regular arc) iff $z=z(t)$, $t \in I$ is a differentiable curve and $z'(t) \neq 0$ for all $t \in I$.

D1.13 We say that $z=z(t)$ is a piecewise differentiable curve if there exists a Δ partition of I and $z=z(t), t \in I_k \equiv [t_{k-1}, t_k]$ $(k=1,\ldots,n)$ is a differentiable curve.

D1.14 We say that $z=z(t)$ is a piecewise regular curve if there exists a Δ partition of I and $z=z(t)$, $t \in I_k \equiv [t_{k-1}, t_k]$ $(k=1,\ldots,n)$ is a differentiable curve where $z'(t) \neq 0$ for all $t \in I_k$.

D1.15 Consider $a, b, \alpha, \beta \in R \ni$ there exist intervals $I \equiv [a,b]$ and $J \equiv [\alpha, \beta]$. Consider the two continuous curves $C_1: z=f(t), t \in I$ and $C_2: z=g(\tau), \tau \in J$. Then, C_1 and C_2 are said to be not essentially different if there exists a function $h: J \to I \ni h(\tau)$ is continuous on J and the following conditions hold:

(1) $h(\tau)$ is strictly monotonically increasing on J

(2) $h(\alpha)=a$ and $h(\beta)=b$

(3) $h(\tau), \tau \in J$ is piecewise regular and $\tau \in J$ implies that $g(\tau)=f(h(\tau))$.

D1.16 We write $C_1 = C_2$ iff C_1 and C_2 are not essentially different.

D1.17 Consider the continuous curve $C: z=f(t), t \in I$ where $I \equiv [a,b]$

and $t \in I$ implies that $g(t)=f(a+b-t)$. Then, the <u>inverse arc</u> (or <u>opposite arc</u>) of the arc $C: z=f(t), t\in I$ is defined to be $z=g(t)=f(a+b-t), t\in I$.

<u>N</u>1.2 $-C$, C^{-1}, $\overline{C} \equiv$ inverse arc of C.

<u>R</u>1.5 $-(-C) \equiv C$.

<u>D</u>1.18 Let the curves $C_1, C_2, \ldots\ldots, C_n$ be defined by the equations $z_1(t), t\in I \equiv [a_1, b_1]$, $z_2(t), t\in I_2 \equiv [a_2, b_2], \ldots\ldots$, $z_n(t), t\in I_n \equiv [a_n, b_n]$ respectively. Then, if $z_k(b_k)=z_{k+1}(a_{k+1})$ for $1\le k\le n-1$, we call the curve $C=\bigcup\limits_{k=1}^{n} C_k$ the <u>union</u> of the curves $C_1, C_2, \ldots$ $\ldots, C_n$ satisfying the following conditions:

(1) $C: z=z(t), t\in I\equiv [a_1, b_1+b_n-a_n]$, and

(2) $z=z(t)=z_k(t-b_{k-1}+a_k)$ if $t\in [b_{k-1}, b_{k-1}+b_k-a_k]$ since $t\in [b_{k-1}, b_{k-1}+b_k-a_k]$ implies that $a_k \le t-b_{k-1}+a_k \le b_k$.

<u>D</u>1.19 Consider the simple closed curve $C: z(t)$, $t\in I\equiv [a,b]$. We say that C is <u>oriented</u> since if two points are given on the curve, it is always clear which point precedes the other, and which part of the curve is to be regarded as "lying between" the two given points. ($z(t)$ describes the entire closed path precisely once when t runs over its interval).

<u>D</u>1.20 Keeping in mind <u>T</u>1.1, if the orientation of a simple closed path is such that the interior lies to the left, it is called <u>positive orientation</u>, i.e., the curve is described in the "positive sense." If the orientation is described in a manner that the interior lies to the right, it is called <u>negative orientation</u>, i.e., the curve is described in the "negative sense." (Note that this reasoning does not apply

and is therefore not meaningful at such points on a curve as the vertices of a rectangle, or a triangle, etc.).

CHAPTER II

POLYGONS

N2.1 The straight line segment joining the points a_1 and a_2 will be denoted by $L[a_1,a_2]$. Since a_1 and a_2 are not necessarily distinct, a line segment of the form $L[a_1,a_2]$ is degenerate, i.e., merely a point.

N2.2 An open disk with center z and radius r will be denoted by $N_r(z)$, is called an r-neighborhood (r-nhood) of z, and stands for the set of points: $N_r(z)\equiv\{z|\ |z|<r\}$.

R2.1 If $a_1,a_2\in N_r(z)$, then clearly $L[a_1,a_2] \subseteq N_r(z)$.

D2.1 Let $a_1,a_2,\ldots,a_n$ be complex numbers. The set $P\equiv P(a_1,a_2,\ldots,a_n)=\bigcup_{k=1}^{n-1} L[a_k,a_{k+1}]$ is called a <u>polygonal line</u>.

R2.2 Every polygonal line is a curve.

D2.2 If D is an open set such that $D\subseteq Z$, then D is called a <u>domain</u> if for every $z\neq z_0\in D \Longrightarrow \exists$ a polygonal line P with endpoints z and $z_0 \ni P \subseteq D$.

D2.3 A domain D is called a <u>simply-connected domain</u> if every simple closed curve contained entirely in D encompasses only points of D within its boundary.

D2.4 A polygonal line $P\equiv P(a_1,a_2,\ldots,a_n)$ is called a <u>closed polygonal line</u>, or <u>closed polygon</u>, if $a_1=a_n$.

D2.5 Consider the closed polygon $P: z=z(t), t \in I \equiv [a,b]$. P is a <u>simple closed polygon</u> iff $t_1, t_2 \in I \ni t_1 < t_2$ and $z(t_1)=z(t_2)$ implies that $t_1=a$ and $t_2=b$.

R2.3 A simple closed polygon divides the plane into an interior and an exterior (with respect to the polygon). We define the interior of the polygon P, i.e., all points contained within the boundary of P, to be $\mathcal{J}(P)$, and the exterior to be $\mathcal{E}(P)$.

N2.3 By P^*, we mean $P \cup \mathcal{J}(P)$.

D2.6 If $\{a_i\}_{i=1}^n$ are the vertices of a simple closed polygon, then a_j and a_k are said to be <u>adjacent vertices</u> if $|j-k|=1$. (Since $a_1=a_n$, we may use either to test the criterion.)

D2.7 By a <u>side</u> of a polygon, we will mean the line segment joining any two adjacent vertices.

D2.8 Let P be a simple polygon of n sides. A line segment joining two non-adjacent vertices of P is called a <u>diagonal</u> of P if it lies in $\mathcal{J}(P)$ except for the endpoints.

D2.9 A simple closed polygon P' is called a <u>subpolygon</u> of the simple closed polygon P if $\mathcal{J}(P') \subseteq \mathcal{J}(P)$ and every vertex of P' is also a vertex of P.

D2.10 A set of subpolygons [p] is said to constitute a decomposition of a polygon P if: (1) no two subpolygons in [p] have an interior point in common; (2) every interior point of P lies on or within a subpolygon in [p]; (3) every interior point of a subpolygon of [p] is an interior point of P.

R2.4 A diagonal of P decomposes P into two subpolygons each having a less number of sides than P has, and having the diagonal as a common side.

D2.11 We say that P can be triangulated if there exists a set of (n-3) diagonals of P which decompose P into (n-2) triangles (the subpolygons in this case are triangles).

Our basic definition of triangulation is mainly due to Einar Hille, [2].

D2.12 Let $P \equiv P(a_1, a_2, \ldots, a_n, a_1)$ be a polygon. A vertex a_k of P is said to be a projecting vertex of P if there exists a line containing a point of each of the sides $a_{k-1}a_k$ and $a_k a_{k+1}$ and containing no other points of P.

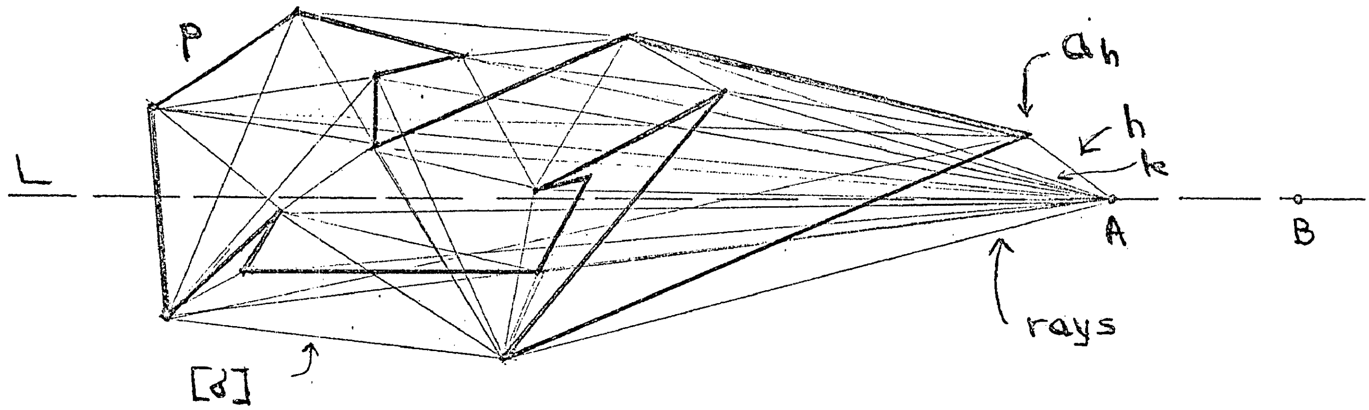
The following three lemmas are developed in the style used by N. J. Lennes, [4].

L2.1 For any set of n distinct points on a line, a notation $a_1, a_2, \ldots, a_n$ may be so arranged that the points a_i, a_j, a_k are in the order $a_i a_j a_k$ for all integral values of i, j, k such that $i<j<k$ and $1 \leq i$ and $k \leq n$.

L2.2 If two rays , h and k, eminating from the same vertex form an angle of less than π radians, this angle will be denoted by $<(h,k)$.

L2.3 Every polygon P has at least one projecting vertex.

P Consider the following general figure of the given polygon P for which we are proving this lemma:

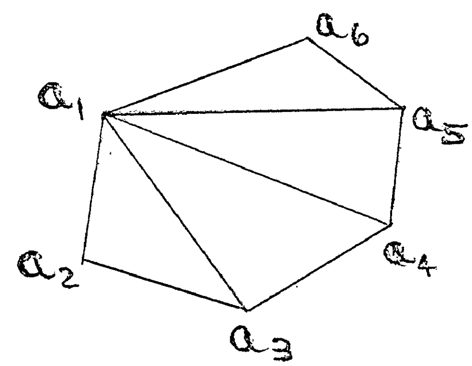


(figure 2.1)

Let [δ] be the set of all segments whose endpoints are vertices of P. Let L be a line meeting sides of the polygon but containing none of its vertices. By L2.1, there exists an ordering of the points of intersection of L with segments of [δ], starting with the first point of intersection to the last. In view of the ordered subscripts of these points of intersection, pick a point A on L satisfying the following conditions: (1) A is collinear with no two vertices of P, and (2) all intersection points of L with segments of [δ] are on the same side of A (i.e., A is situated on L so that A could be the highest subscripted point on L with regard to the above ordering). Then, choose point B further along L than A so as to subscript B even higher than A (see figure 2.1).

From A, construct rays through each of the vertices of P. Condition (1) above is needed to insure that none of these constructed rays will have more than one point in common (the point A). Now, as the ray from A through B is rotated around the point A in a counter-clockwise arc, let h be the first ray (connecting A with a vertex of P) that coincides with the rotating $\overrightarrow{AB}$. Let a_h be the vertex that h connects with A. Let k be the ray next to h. Then, any ray proceeding from A and lying within $<(h,k)$ (see L2.2) meets P in the sides $a_{h-1}a_h$ and $a_h a_{h+1}$ and in no other points, for otherwise, if it were to meet another side of P, there would have to be a vertex of P within $<(h,k)$, contradicting the manner in which k was selected. Hence, by definition, a_h is a projecting vertex.

D2.13 Let P be a simple closed polygon with n sides and vertices $a_1, a_2, \ldots, a_n$. If for any vertex a_i, $1 \leq i \leq n$, of P, the line segments drawn from a_i to <u>every</u> adjacent vertex are contained in the interior of P except for the endpoints, then P is called a <u>convex</u> <u>polygon</u>.



(figure 2.2)

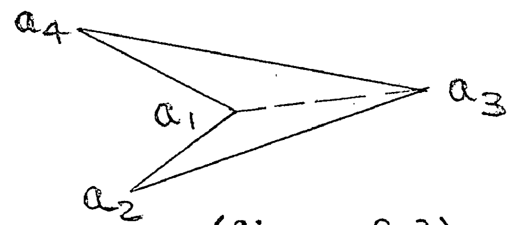
R2.5 Obviously, a convex polygon can be triangulated: each of the line segments $L[a_1,a_3]$, $L[a_1,a_4], \ldots, L[a_1,a_{n-1}]$ is a diagonal and the set $P_{j-1} \equiv (a_1, a_j, a_{j+1}, a_1)$, $2 \leq j \leq n-1$ are the corresponding triangles.

The following two theorems are outlined after, but more carefully presented than those used by Dr. K. Knopp, [3].

T2.1 Every simple closed polygon can be triangulated.

P (By induction) We show that $n \in I^+$ implies that a simple closed polygon P with $(n+3)$ sides can be triangulated.

If n=1, P is a quadrilateral, say $P=P(a_1,a_2,a_3,a_4,a_1)$. If P is convex, one of the vertices must have an interior angle of greater than π radians. (There can only be one, since the sum of the four interior angles of a quadrilateral is 2π radians). Assume it is a_1. Then $L[a_1, a_3]$ must be a diagonal decomposing P into two triangles.

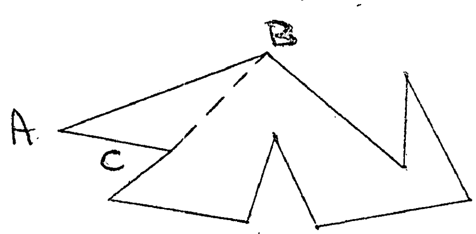


(figure 2.3)

Now let us assume the theorem is true for all simple closed polygons with not more than (n+3) sides, and let P be a simple closed polygon with (n+4) sides. We need only find an interior diagonal which decomposes P into two polygons, each having less than (n+4) sides.

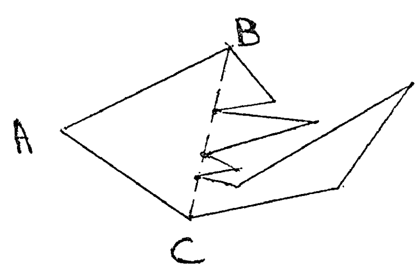
We proceed as follows: let a_i, which we will call A, be a projecting vertex of P (the existence of which is given by L2.3). Denote by B and C, the vertices a_{i-1} and a_{i+1} respectively. Then exactly one of the following is true:

(1) BC is a diagonal contained in $\mathcal{J}(P)$ except for the end-points, in which case we are finished.



(figure 2.4)

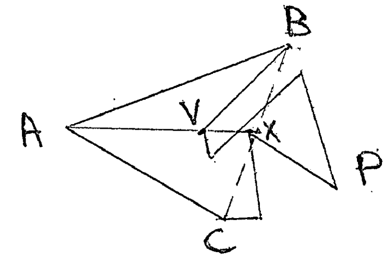
(2) There exists at least one vertex of P (other than B and C) on the line segment BC but no vertex in the interior of ΔABC. Since there may be more than one vertex of P on the line segment BC, we will denote by V, the vertex of P closest to B. Then clearly AV is the diagonal



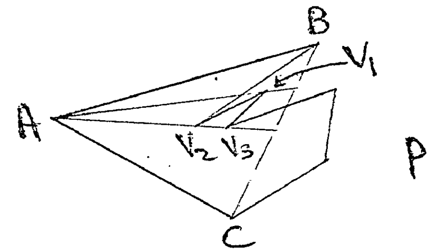
(figure 2.5)

we seek, and again we are finished.

(3) There exists at least one vertex of P in the interior of ΔABC. Let a point X move from B to C along BC until AX encounters a vertex of P. AX may encounter more than one vertex (as in figure 2.6), in which case we denote by V, the vertex of P closest to A on AX. Clearly, Av is the desired diagonal in this case. But, (as in figure 2.7), the first vertex of P to be encountered by AX may be such that if this vertex is called V, then AV $\not\subset$ $\mathcal{D}$(P). Thus we



(figure 2.6)



(figure 2.7)

denote by V_1 the first vertex of P encountered by AX, V_2 the second, etc., up to V_m, the last one (there can only be a finite number <n). We denote by V, the closest of the V_i, $1 \leq i \leq m$, to A. If there is more than one with this property, pick V to be the one with the lowest subscript (for definiteness). Thus, AV will again be the desired diagonal, and the proof is completed.

T2.2 Every closed polygon P can be decomposed into a finite number of simple closed polygons and a finite number of double segments, i.e., "degenerate" closed polygons of the form $P=P(a_i,a_{i+1},a_i)$.

P Let $P=P(a_1,a_2,\ldots,a_n,a_{n+1})$ where $a_{n+1}=a_1$ and let us denote the sides $a_1a_2,a_2a_3,\ldots,a_{n-1}a_n,a_na_1$ by $S_1,S_2,\ldots,S_n$ respectively. Clearly we must have that $n \geq 2$, and since $a_{n+1} \equiv a_1$, a_1 is the initial and terminal point of P. We can add without loss of generality that the only way two adjacent sides may lie on the same straight line is to have more than one point in common. It is also understood that the line segment S_n is open at $a_{n+1} \equiv a_1$.

The proof is broken into two cases:

(1) Every side of P has only one point in common with its adjacent sides, and no point in common with any other side. Clearly, P is simple in this case, and we are finished.

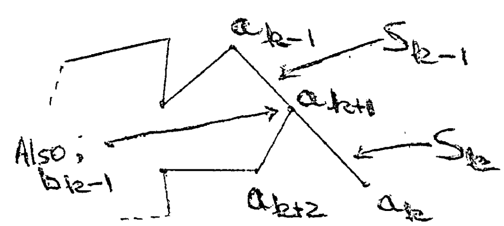
(2) There exists a side S_λ (where $\lambda=2,3,\ldots,n$) such that

a) S_λ has more than one point in common with $S_{\lambda-1}$, or

b) S_λ has at least one point in common with one or more of the segments S_μ ($\mu=1,2,\ldots,\lambda-2$).

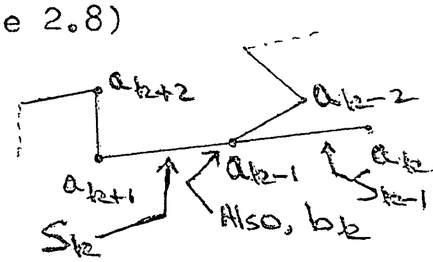
Let S_k be the side of P for which case (2) first occurs. If a) holds for S_k, then one of the following cases must exist:

(i) There exists a point b_{k-1} of S_{k-1} such that $b_{k-1}=a_{k+1}$, and then $b_{k-1}a_ka_{k+1}$ in that order is a straight line segment q' described twice, once in each direction, and thus $\overline{a_1a_2}\,\overline{a_2a_3}\ldots\overline{a_{k-1}b_{k-1}}\,\overline{a_{k+1}a_{k+2}}\ldots\ldots\overline{a_na_1}$ is a closed polygon P'. (See figure 2.8)



(figure 2.8)

(ii) There exists a point b_k of S_k such that $a_{k-1}=b_k$ (see figure 2.9) and then $a_{k-1}a_kb_k$ in that order is a straight line segment q' described

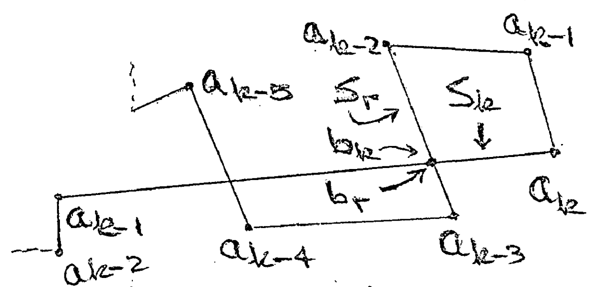


(figure 2.9)

twice, once in each direction, and $\overline{a_1a_2}\,\overline{a_2a_3}\cdots\overline{a_{k-2}a_{k-1}}$-$\overline{b_ka_{k+1}}\cdots\overline{a_na_1}$ is a closed polygon P'.

(The two straight line segments in (i) and (ii) above are considered degenerate forms of closed polygons.)

If b) holds and a) does not hold for S_k, let b_k on $a_ka_{k+1}\equiv S_k$ be the nearest point to a_k that S_k has in common with any of the segments S_μ $(\mu=1,2,\ldots,k-2)$ (see figure 2.10). Now, let $b_r\equiv b_k$, where b_r is a point on the segment S_r for some $r \leq k-2$. Thus b_r is uniquely determined, since if $\exists\, t \ni b_t=b_r$ and $\ni t \leq r-2$, then S_k would not have been chosen correctly, i.e., S_r should have been chosen instead of S_k. Hence, b_r is unique due to the way in which S_k was chosen. Then, $\overline{b_ra_{r+1}}\,\overline{a_{r+1}a_{r+2}}\cdots\overline{a_kb_k}$ is a closed polygon P'.



(figure 2.10)

R2.6 It is clear that with the aid of these last two theorems, any closed polygon can be triangulated.

D2.14 Given the curve $C:z(t)$, $t\in I\equiv[a,b]$. If there exists $\{t_k\}_{k=0}^{n} \ni a=t_0<t_1<\cdots<t_{n-1}<t_n=b$ and there exists $\{a_k\}_{k=0}^{n}$ $\ni 0 \leq k \leq n \implies a_k\equiv z(t_k)$, then the polygonal line $P(a_0,a_1,\ldots,a_n)$ is called an *inscribed polygon of C*.

CHAPTER III

INTEGRATION

D3.1 Consider $a,b \in R$, so that $I \equiv [a,b]$, where there exist two complex-valued functions h and g $\ni$ $g:I\to Z$ and $h:I\to Z$. Given also that Δ is a partition of I, $\Delta:\{t_k\}_{k=0}^{n} \ni a=t_0<t_1<\cdots<t_{n-1}<t_n=b$. We then define $\|\Delta\|$, called the <u>norm of Δ</u>, $\ni$ $\|\Delta\| \equiv \max\limits_{1\le k\le n} |t_k - t_{k-1}|$.

D3.2 $\delta_\Delta \equiv \{\xi_k\}_{k=1}^{n} \ni 1\le k\le n \implies t_{k-1}\le \xi_k \le t_k$.

D3.3 $S(\Delta,\delta_\Delta) \equiv \sum_{k=1}^{n} g(\xi_k)[h(t_k)-h(t_{k-1})]$.

D3.4 If $\alpha\in Z$, then $\exists \lim\limits_{\|\Delta\|\to 0} S(\Delta,\delta_\Delta)=\alpha$ means the following: given any $\epsilon>0$, $\exists\ \delta(\epsilon)>0 \ni$ for <u>every</u> δ_Δ of <u>any</u> partition Δ of I with $\|\Delta\|<\delta$, it will follow that $|S(\Delta,\delta_\Delta)-\alpha|<\epsilon$.

D3.5 We say in this case that g(t) is integrable on I with respect to h(t) (when the limit exists).

N3.1 $\lim\limits_{\|\Delta\|\to 0} S(\Delta,\delta_\Delta) = \int_a^b g(t)dh(t) \equiv$ the <u>Stieltjes Integral</u> of g(t) wrt h(t) over I.

T3.1 <u>H</u> Consider the interval $I\equiv[a,b]$ on which are defined the functions $g:I\to Z \ni g$ is continuous on I and $h:I\to Z \ni h$ is of bounded variation on I.

<u>C</u> $\exists \int_a^b g(t)dh(t)$.

<u>D</u>3.6 Given the continuous function $g:I \to Z$ where $I \equiv [a,b]$, and let $h(t) \equiv t$ for all $t \in I$. Then, the <u>Riemann Integral</u> is defined to be $\int_a^b g(t)dh(t) \equiv \int_a^b g(t)dt$.

<u>R</u>3.1 Since $\exists\ h'(t) \equiv 1$, by <u>T</u>1.4(2), $h(t)$ is of bounded variation on I for all $t \in I$, and hence we know that the integral defined above does exist.

<u>T</u>3.2 Given the three continuous functions $g:I \to Z$, $g_1:I \to Z$, and $g_2:I \to Z$, and the function $h:I \to Z \ni h(t)$ is of bounded variation on I, where $I \equiv [a,b]$.

<u>C</u> 1) $\exists \int_a^b \alpha g(t)dh(t) = \alpha \int_a^b g(t)dh(t)$, for $\alpha \in Z$.

2) $\exists \int_a^b [g_1(t)+g_2(t)]dh(t) = \int_a^b g_1(t)dh(t) + \int_a^b g_2(t)dh(t)$

3) $\int_a^b g(t)dh(t) = \int_a^c g(t)dh(t) + \int_c^b g(t)dh(t)$, where $a<c<b$.

<u>D</u>3.7 Consider the functions $g:I \to Z$ and $h:I \to Z$ where $I \equiv [a,b]$ and such that $\exists \int_a^b g(t)dh(t)$. Then we define :

$$\int_b^a g(t)dh(t) \equiv -\int_a^b g(t)dh(t).$$

The following material is on curvilinear integrals.

<u>D</u>3.8 Consider the continuous curve $C:z=z(t), t \in I$ where $I \equiv [a,b]$ and the function $f:C \to Z$, where we understand that $C \equiv \{\zeta \mid \zeta = z(t), t \in I\}$. Then, we say that $f(z)$ is integrable over C iff $f(z(t))$ is integrable wrt $z(t)$ over I.

<u>N</u>3.2 $\int_C f(z)dz \equiv \int_a^b f(z(t))dz(t)$.

<u>T</u>3.3 <u>H</u> Given $I \equiv [a,b]$. Let $C:z(t), t \in I$ be a continuous

rectifiable curve and let $f: C \to Z$ be a continuous function on C.

C $\exists \int_C f(z)dz$.

T3.4 H Let $C_1: z(t), t \in I \equiv [a,b]$ be a continuous rectifiable curve and let $C_2: z(\tau), \tau \in J \equiv [\alpha, \beta]$ be a continuous curve $\ni$ $C_1 = C_2$. Also, let $f: C_1 \to Z$ be a continuous function on C_1.

C $\exists \int_{C_2} f(z)dz = \int_{C_1} f(z)dz$.

T3.5 H Let $C: z(t), t \in I[a,b]$ be a continuous rectifiable curve and consider the three continuous functions $f: C \to Z$, $f_1: C \to Z$, and $f_2: C \to Z$. Note that $\exists \int_C f(z)dz$, $\exists \int_C f_1(z)dz$, and $\exists \int_C f_2(z)dz$.

C 1) $\exists \int_C \alpha f(z)dz = \alpha \int_C f(z)dz$, for $\alpha \in Z$.

2) $\exists \int_C [f_1(z)+f_2(z)]dz = \int_C f_1(z)dz + \int_C f_2(z)dz$

3) $\exists \int_{-C} f(z)dz = -\int_C f(z)dz$

T3.6 H Consider the two continuous rectifiable curves C_1: $z=z_1(t), t \in I_1 \equiv [a_1, b_1]$ and $C_2: z=z_2(t), t \in I_2 \equiv [a_2, b_2]$ such that $z_1(b_1) = z_2(a_2)$. Also, let $f: C_1+C_2 \to Z$ be a continuous function on C_1+C_2.

C $\exists \int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$.

R3.2 The above theorem can be extended to a finite number of cases.

T3.7 H Given $C: z(t), t \in I \equiv [a,b]$ to be a continuous rectifiable curve and $f: C \to Z$, a continuous function on C.

C 1) $\int_C |dz| \equiv \int_C |dz(t)| = l(c)$ (see D1.10)

2) $|\int_C f(z)dz| \leq \int_C |f(z)|\cdot|dz| \leq \max_{z\in C} |f(z)|\cdot l(C)$.

T3.8 H Consider the continuous rectifiable curve $C:z(t)$, $t\in I\equiv[a,b]$ and the continuous function $f:C\to Z$, so that $\exists \int_C f(z)dz$. Let there be given a sequence of partitions of I, $\{\Delta_k\}_{k=1}^{\infty} \ni$ for all $k\in I^+ \Longrightarrow \Delta_k\equiv\{t_k^k\}_{j=0}^{n_k}$ is a partition of I. Let there exist $\lim_{k\to\infty} \|\Delta_k\|=0$, and let S_k be such that $S_k\equiv \sum_{j=1}^{n_k} f(z(\xi_j^k))[z(t_j^k)-z(t_{j-1}^k)]$ where $1\leq j\leq n_k \Longrightarrow t_{j-1}^k\leq \xi_j^k\leq t_j^k$.

C $\exists \lim_{k\to\infty} S_k = \int_C f(z)dz$.

P Let $\epsilon>0$ be given. By definition, $\exists$ $\delta>0 \ni$ for all Δ where $\|\Delta\|<\delta$ and for any $S(\Delta, \delta_\Delta)$ it follows that $|S(\Delta, \delta_\Delta) - \int_C f(z)dz|<\epsilon$. Now, $\exists \lim_{k\to\infty} \|\Delta\|=0 \Longrightarrow [\exists\ K\ni k>K \Longrightarrow \|\Delta\|<\delta]$. Then, $k>K \Longrightarrow |S_k- \int_C f(z)dz|<\epsilon$. Hence, $\exists \lim_{k\to\infty} S_k= \int_C f(z)dz$.

T3.9 H Given the continuous curve $C:z=z(t), t\in I\equiv[a,b]$ and let $z(a)=z(b)$ so that C is a closed curve.

C 1) $\int_C dz \equiv \int_a^b dz(t) = 0$.

2) $\int_C zdz \equiv \int_a^b z(t)dz(t) = 0$.

P 1) Let $f(z)\equiv1$. Then $f(z)$ is continuous on C. We already know that $\int_a^b f(z(t))dz(t) = \lim_{\|\Delta\|\to 0} \sum_{k=1}^{n} f(z(\xi_k))[z(t_k)-z(t_{k-1})]$ where $\Delta:\{t_k\}_{k=0}^{n}$ is a partition on $I \ni a=t_o<t_1<\ldots<t_{n-1}<t_n=b$

and where $\exists\ \{\xi_k\}_{k=1}^{n}$ where $1 \le k \le n \implies t_{n-1} \le \xi_k \le t_k$. But, $1 \le k \le n \implies f(z(\xi_k)) \equiv 1$, so that $\int_a^b f(z(t))dz(t)$

$$= \lim_{\substack{n\to\infty \\ \|\Delta\|\to 0}} \sum_{k=1}^{n} 1\cdot[z(t_k)-z(t_{k-1})]$$

$$= \lim_{\substack{n\to\infty \\ \|\Delta\|\to 0}} \{[z(t_1)-z(t_o)]+[z(t_2)-z(t_1)]+\ldots+[z(t_n)-z(t_{n-1})]\}$$

$$= \lim_{\substack{n\to\infty \\ \|\Delta\|\to 0}} [z(t_n)-z(t_o)] = \lim_{\substack{n\to\infty \\ \|\Delta\|\to 0}} [z(b)-z(a)]$$

$$= z(b) - z(a) = 0 .$$

Therefore, $\int_a^b f(z(t))dz(t) = \int_a^b dz(t) \equiv \int_C dz = 0$.

$\underline{P}$ 2) Consider $\{\Delta_k\}_{k=1}^{\infty} \ni k \in I^+ \implies \Delta_k \equiv \{t_j^k\}_{j=0}^{n_k}$ is a partition of I and let there exist $\lim_{k\to\infty} \|\Delta\| = 0$. We now define the following sums:

$$\overline{S}_k \equiv \sum_{j=1}^{n_k} z(t_j^k)[z(t_j^k)-z(t_{j-1}^k)] \text{ , and}$$

$$\underline{S}_k \equiv \sum_{j=1}^{n_k} z(t_{j-1}^k)[z(t_j^k)-z(t_{j-1}^k)] .$$

But here, for all $k \in I^+$ it follows that

$$\overline{S}_k+\underline{S}_k = \sum_{j=1}^{n_k} [z(t_j^k)^2 - z(t_j^k)z(t_{j-1}^k) + z(t_{j-1}^k)z(t_j^k) - z(t_{j-1}^k)^2]$$

$$= z(t_{n_k}^k)^2 - z(t_o^k)^2$$

$$= z(b)^2 - z(a)^2$$

$$= 0.$$

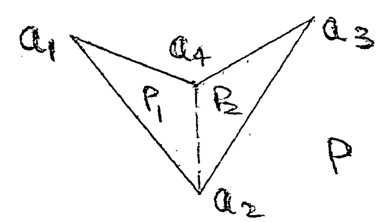
Therefore, there exists $\lim_{k\to\infty} \overline{S}_k = \int_C z dz$ and $\exists \lim_{k\to\infty} \underline{S}_k = \int_C z dz$

by $\underline{T}3.8 \implies \lim_{k\to\infty} (\overline{S}_k + \underline{S}_k) = 0 = \int_C z dz + \int_C z dz = 2\int_C z dz$

and thus $\int_C z\,dz = \int_a^b z(t)\,dz(t) = 0$.

T3.10 H Consider a simple closed quadrilateral P. By T2.1, P can be triangulated into two triangles P_1 and P_2 such that $P=P_1 \cup P_2$. Given also the function $f:\Gamma(P)\to Z \ni f$ is continuous on $\Gamma(P)$ and let $\int_{\Gamma(P)} f(z)\,dz$ exist.

C $\int_{\Gamma(P)} f(z)\,dz = \int_{\Gamma(P_1)} f(z)\,dz + \int_{\Gamma(P_2)} f(z)\,dz$



P Suppose for definiteness that Pis drawn as in figure 3.1. Then,

(figure 3.1)

$$\int_{\Gamma(P_1)} f(z)\,dz = \int_{[a_1,a_2]} f(z)\,dz + \int_{[a_2,a_4]} f(z)\,dz + \int_{[a_4,a_1]} f(z)\,dz$$

and $\int_{\Gamma(P_2)} f(z)\,dz = \int_{[a_2,a_3]} f(z)\,dz + \int_{[a_3,a_4]} f(z)\,dz + \int_{[a_4,a_2]} f(z)\,dz$

by T3.2. Therefore, $\int_{\Gamma(P_1)} + \int_{\Gamma(P_2)} = \int_{[a_1,a_2]} + \int_{[a_2,a_4]}$

$+ \int_{[a_4,a_1]} + \int_{[a_2,a_3]} + \int_{[a_3,a_4]} + \int_{[a_4,a_3]}$ and since

$\int_{[a_2,a_4]} = -\int_{[a_4,a_2]}$ by D3.7, we have that $\int_{\Gamma(P_1)} + \int_{\Gamma(P_2)}$

$= \int_{[a_1,a_2]} + \int_{[a_2,a_3]} + \int_{[a_3,a_4]} + \int_{[a_4,a_1]} = \int_{\Gamma(P)}$

and the proof is complete.

R3.3 The above theorem can be extended to an n-sided simple closed polygon, since the proof is based upon the fact that integrating over a line segment first in one direction, and then in the other direction, cancels each other.

CHAPTER IV

TOPOLOGICAL PRELIMINARIES

D4.1 Let X be a non-empty universal set and let $\mathcal{O}$ be a family of subsets of X. Then, $\{X,\mathcal{O}\}$ is called a topological space and $\mathcal{O}$ is called the topology on the space $\{X,\mathcal{O}\}$ if:

(1) $\mathcal{G} \subseteq \mathcal{O} \Longrightarrow \bigcup_{G\in\mathcal{G}} G \in \mathcal{O}$, and

(2) $\mathcal{G} \subseteq \mathcal{O} \ni \mathcal{G}$ is finite $\Longrightarrow \bigcap_{G\in\mathcal{G}} G \in \mathcal{O}$,

where $\mathcal{G}$ is a subset of $\mathcal{O}$.

R4.1 $\{Z,\mathcal{O}\}$ is a topological space ($Z\equiv$complex plane).

D4.2 Let $d{:}Z\text{x}Z\to R_o^+$ be a mapping from the product space ZxZ into the non-negative reals. For notation, we use $d(z_1,z_2)$ if $z_1,z_2\in Z$. Then, $\{Z,d\}$ is called a metric space iff for all $z_1,z_2,z_3\in Z$ the following holds:

(1) $d(z_1,z_2)=0$ iff $z_1=z_2$,

(2) $d(z_1,z_2)=d(z_1,z_2)$, and

(3) $d(z_1,z_2) \leq d(z_1,z_3) + d(z_3,z_2)$.

D4.3 We say that d is a metric on Z.

D4.4 Given the metric space $\{Z,d\}$ and let $\mathcal{B}_d\equiv\{N_r(z)\,|\,z\in Z, r\in R_o^+\}$ and $\mathcal{O}_d\equiv\{0\,|\,0 \subseteq Z$ and $\exists\ \mathcal{B}^*\subseteq \mathcal{B}_d \ni 0\equiv \bigcup_{B\in\mathcal{B}^*} B\ \}$. Then, $\mathcal{O}_d$ is called a topology induced by the metric d, and we call $\{Z,d,\mathcal{O}_d\}$ a metric topological space.

D4.5 Consider the topological space $\{Z, \mathcal{O}\}$ and a set $E \subseteq Z$. We say that E is <u>sequentially</u> <u>compact</u> iff $[\{x_j\}_{j=1}^{\infty} \ni j \Longrightarrow x_j \in E] \Longrightarrow \exists \{j_k\}_{k=1}^{\infty}$ and $\exists\, x_o \in E \ni \exists \lim_{k \to \infty} x_{j_k} = x_o$.

T4.1 H Given the metric topological space $\{Z, d, \mathcal{O}_d\}$ and the set $E \subseteq Z$.

C E is compact iff E is sequentially compact.

D4.6 Consider the topological space $\{Z, \mathcal{O}\}$ and the set $E \subseteq Z$. We say that E is a <u>Cantor</u> <u>set</u> iff [for all $\{C_n\}_{n=1}^{\infty} \ni n \Longrightarrow C_n$ is closed and (i) $C_{n+1} \subseteq C_n$ and (2) $E \cap C_n \neq \emptyset$] $\Longrightarrow E \cap \bigcap_{n=1}^{\infty} C_n \neq \emptyset$.

D4.7 Given sets A and $B \ni A, B \subseteq Z \ni A \cap B = \emptyset$. We define the <u>écart</u> of the sets A and B to be a non-negative real number representing the smallest "distance" (which could possibly be zero) between the sets A and B, i.e.,

$$\acute{e}(A,B) \equiv \operatorname{glb}_{a \in A,\, b \in B} [d(a,b)],$$

where d is the distance metric on Z.

T 4.2 H Given the metric topological space $\{Z, d, \mathcal{O}_d\}$ and the sets $A, B \subseteq Z$ where A is compact and B is closed and $A \cap B = \emptyset$.

C $\acute{e}(A,B) > 0$.

P Using the method of contradiction, assume $\acute{e}(A,B) = 0$. Then $n \in I^+ \Longrightarrow a_n \in A, b_n \in B \ni d(a_n, b_n) < \frac{1}{n}$. Now, $\{Z, d, \mathcal{O}_d\}$ being a metric topological space along with A being compact implies that A is sequentially compact. Then, with $\{a_n\}_{n=1}^{\infty} \ni n \Longrightarrow a_n \in A$, we obtain by $\underline{T}4.1$ that $\exists \{n_j\}_{j=1}^{\infty}$ and $a_0 \in A$ such that $\exists \lim\limits_{j\to\infty} a_{n_j} = a_0$ (by $\underline{D}4.5$). Let $\epsilon > 0$ be given. Choose $j_0 \ni [\frac{1}{n_{j_0}} < \frac{\epsilon}{2}$ and $j > j_0 \Longrightarrow d(a_0, a_{n_j}) < \frac{\epsilon}{2}]$. Then, $j > j_0 \Longrightarrow$

$$\begin{aligned} d(a_0, b_{n_j}) &\leq d(a_0, a_{n_j}) + d(a_{n_j}, b_{n_j}) \quad , \quad (\underline{D}4.2) \\ &< \frac{\epsilon}{2} + \frac{1}{n_j} \\ &< \frac{\epsilon}{2} + \frac{1}{n_{j_0}} \qquad (j > j_0 \Longrightarrow \frac{1}{n_j} < \frac{1}{n_{j_0}}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon . \end{aligned}$$

Hence, $\exists \lim\limits_{j\to\infty} b_{n_j} = a_0$ and $j \Longrightarrow b_{n_j} \in B$ so that $a_0 \in {}^{\circ}\mathcal{K}B$ but since B is closed, $a_0 \in B$, but $a_0 \in A \Longrightarrow a_0 \in A \cap B \neq \emptyset$, a contradiction. Hence, $\delta(A,B) > 0$.

CHAPTER V

CLASSICAL PROOF OF CAUCHY'S THEOREM

In this chapter, the general treatment of the three theorems proved is influenced by that used by W. J. Thron, [6].

T5.1 (Cauchy-Goursat) Let $f(z)$ be differentiable in a simply connected domain D. Then, for every triangle $T \subseteq D$,

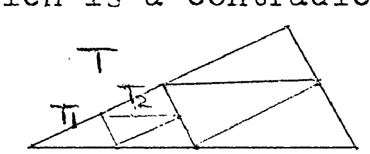
$$\int_{\Gamma(T)} f(z)dz = 0.$$

P $f(z)$ analytic on $D \Longrightarrow f(z)$ is analytic on $T \subseteq D \Longrightarrow f(z)$ is continuous on $\Gamma(T)$ which is a rectifiable curve $\Longrightarrow \exists \int_{\Gamma(T)} f(z)dz$. We divide T into four congruent oriented triangles by joining the midpoints of the three sides of T, and we call these $T^{(1)}, T^{(2)}, T^{(3)}, T^{(4)}$. For notation, we will use $I(T) \equiv \int_{\Gamma(T)} f(z)dz$ and $I(T^{(k)}) \equiv \int_{\Gamma(T^{(k)})} f(z)dz$, $k=1,2,3,4$.

Now, we denote by T_1 the triangle of these four smaller ones for which $|I(T^{(k)})|$ is the largest $(k=1,2,3,4)$. Hence we must have that $\frac{1}{4}|I(T)| \leq |I(T_1)|$, for otherwise

$$|I(T)| = |I(T^{(1)}) + I(T^{(2)}) + I(T^{(3)}) + I(T^{(4)})| \quad \text{(by T3.10)}$$
$$\leq \sum_{k=1}^{4} |I(T^{(k)})|$$
$$< 4(\tfrac{1}{4}|I(T)|) = |I(T)|, \text{ which is a contradiction.}$$

Now, let us divide T_1 into four congruent oriented triangles and let T_2 be the one of these four such that



(figure 5.1)

$\frac{1}{4}|I(T_1)| \leq |I(T_2)|$ for the same reasons argued above. Thus, we have that $\frac{1}{4^2}|I(T)| \leq \frac{1}{4}|I(T_1)| \leq |I(T_2)| \Rightarrow \frac{1}{4^2}|I(T)| \leq |I(T_2)|$. Proceeding in this manner, we arrive at $\frac{1}{4^2}|I(T)| \leq |I(T_n)|$, $n \geq 1$. Also, if $l(T)$ denotes the length of $\Gamma(T)$, then $l(T_1) = \frac{1}{2}l(T)$ (see figure 5.1) and $l(T_2) = \frac{1}{2}\cdot l(T_1) = \frac{1}{2^2}l(T)$ $l(T_n) = \frac{1}{2^n}l(T)$. Also, since $\ldots \subseteq T_{n+2} \subseteq T_{n+1} \subseteq T_n \subseteq T_{n-1} \subseteq \ldots \subseteq T_1 \subseteq T$, we have that there exists exactly one point $z_o \in T$ for $n \geq 1 \ni (z_o) = \bigcap_{n=1}^{\infty} T_n$, which is proved as follows: first, each triangle T_n is a non-empty compact set of complex numbers with a diameter d_n $(n=1,2,\ldots.)$. Next, $\exists \lim_{n\to\infty} d_n = 0$ and $T_n \supseteq T_{n+1} \supseteq T_{n+2} \supseteq \ldots$ for all n. If we assume that $\bigcap_{n=1}^{\infty} T_n$ contains two distinct points z_o and z_1, we would have that $0 < |z_o - z_1| \leq d_n$ for all $n \Longrightarrow \lim_{n\to\infty} d_n \neq 0$, a contradiction. Now, we must show that $z_o \in T_n$.

To show this, we use the Cantor Property (D4.6). We have that $n \Longrightarrow T_n \supseteq T_{n+1}$ and that each T_n is closed and that $T \cap T_n \neq \emptyset$ for all n. Hence, $T \cap \bigcap_{n=1}^{\infty} T_n = \bigcap_{n=1}^{\infty} T_n \neq \emptyset$, and since we've already shown that the intersection cannot be more than one point, it must be just the point z_o, proving that $\bigcap_{n=1}^{\infty} T_n = (z_o)$, and hence $z_o \in T$ also.

Now, let $\epsilon > 0$ be given, and let $\epsilon' = \frac{\epsilon}{[l(T)]^2}$.

Since $f(z)$ is analytic on T, $\exists\ f'(z_o)$ since $z_o \in T$. Hence,

$$\exists\ \delta>0 \ni 0<|z-z_o|<\delta \implies \left|\frac{f(z)-f(z_o)}{z-z_o} - f'(z_o)\right| < \varepsilon'.$$

Now, for all z in a δ-nhood of z_o, i.e., $z \in N_\delta(z_o)$, we

$$\text{define } \eta(z) \equiv \frac{f(z)-f(z_o)}{z-z_o} - f'(z_o), \quad \text{if } z \neq z_o$$

$$\equiv 0 \qquad\qquad\qquad\qquad , \quad \text{if } z = z_o .$$

$$\text{Now,} \quad \lim_{z\to z_o} \eta(z) = \lim_{z\to z_o} \left[\frac{f(z)-f(z_o)}{z-z_o} - f'(z_o)\right]$$

$$= f'(z_o) - f'(z_o) = 0 \equiv \eta(z_o).$$

Hence $\eta(z)$ is continuous at $z=z_o$, and is obviously continuous for $z \neq z_o$. It is clear that $z \in N_\delta(z_o)$ implies that $f(z) = (z-z_o) \cdot f'(z_o) + f(z_o) + \eta(z) \cdot (z-z_o)$, and also that $|\eta(z)| < \varepsilon'$.

Since $\exists \lim_{n\to\infty} d_n = 0$, $\exists\ N \ni l(T_N) < \delta$ (and thus $d_N < \delta$), so that $T_N \subseteq N_\delta(z_o)$ since $z, z_o \in T_N \implies |z-z_o| \leq d_N < \delta$.

$$\text{Hence, } I(T_N) = \int_{\Gamma(T_N)} (z-z_o) f'(z_o)\,dz + \int_{\Gamma(T_N)} f(z_o)\,dz$$

$$+ \int_{\Gamma(T_N)} \eta(z)\cdot(z-z_o)\,dz = \int_{\Gamma(T_N)} z f'(z_o)\,dz - \int_{\Gamma(T_N)} z_o \cdot f'(z_o)\,dz$$

$$+ \int_{\Gamma(T_N)} f(z_o)\,dz + \int_{\Gamma(T_N)} \eta(z)\cdot(z-z_o)\,dz = f'(z_o) \int_{\Gamma(T_N)} z\,dz$$

$$- z_o \cdot f'(z_o) \int_{\Gamma(T_N)} dz + f(z_o) \int_{\Gamma(T_N)} dz + \int_{\Gamma(T_N)} \eta(z)\cdot(z-z_o)\,dz$$

$= \int_{\Gamma(T_N)} \eta(z)\cdot(z-z_o)dz$, using T3.5 and T3.9.

Now, using T3.7, we have that

$$|I(T_N)| = \left| \int_{\Gamma(T_N)} \eta(z)\cdot(z-z_o)dz \right| \leq \max_{z\in\Gamma(T_N)} |\eta(z)|\cdot|z-z_o|\cdot l(T_N)$$

$$< \varepsilon'\cdot[l(T_N)]\cdot[l(T_N)] = \frac{\varepsilon}{[l(T)]^2}\cdot[l(T_N)]^2$$

$$= \frac{\varepsilon}{[l(T)]^2}\cdot\frac{[l(T)]^2}{2^{2N}} = \frac{\varepsilon}{4^N}\ .$$

Hence, $\frac{1}{4^N}|I(T)| \leq |I(T_N)| \Longrightarrow |I(T)| \leq 4^N|I(T_N)|$

$< 4^N\cdot\frac{\varepsilon}{4^N} = \varepsilon\ , \Longrightarrow |I(T)| < \varepsilon$, and the theorem is proved.

T5.2 Let $f(z)$ be differentiable in a simply connected domain D, and let P be a closed polygon in D. Then,

$$\int_{\Gamma(P)} f(z)dz = 0.$$

P By T2.2, we can decompose P into a finite number of simple closed polygons (say $P_1, P_2, P_3, \ldots, P_n$) and a finite number of segments (say $S_1, S_2, \ldots, S_m$) described twice, once in each direction. Now, by T2.1, $P_1, P_2, \ldots, P_n$ can be triangulated. Hence, we have that P can be decomposed (or divided) into a finite number of triangles (say $T_1, T_2, \ldots, T_r$) and a finite number of segments $S_1, S_2, \ldots, S_m$.

By T3.10 and R3.3, we have that $\int_{\Gamma(P)} f(z)dz = \int_{\Gamma(T_1)} f(z)dz$

$$+ \int_{\Gamma(T_2)} f(z)dz + \ldots\ldots + \int_{\Gamma(T_r)} f(z)dz + \int_{\Gamma(S_1)} f(z)dz + \ldots + \int_{\Gamma(S_m)} f(z)dz.$$

But, by T5.1, $\int_{\Gamma(T_i)} f(z)dz=0$, $(i=1,2,\ldots,r)$ and obviously, $\int_{\Gamma(S_i)} f(z)dz=0$, $(i=1,2,\ldots,m)$ by D3.7. Hence $\int_{\Gamma(P)} f(z)dz=0+0+\ldots$ $\ldots+0+0+0\ldots+0=0$, and the theorem is proved.

T5.3 Let $f(z)$ be differentiable in a simply connected domain D, and let C be a closed and rectifiable curve contained in D. Then, $\int_C f(z)dz=0$.

P We will prove that using the above hypothesis for any given $\epsilon>0$, $\exists$ a polygon $P(\epsilon)$ inscribed (D2.13) on C $\left|\int_C f(z)dz - \int_{\Gamma(P(\epsilon))} f(z)dz\right| < \epsilon$. Then, applying T5.2, we complete our proof.

Let $\{\Delta_k\}_{k=1}^{\infty}$ be a sequence of partitions of I, where $C:z=z(t), t\in I\equiv[a,b]$, $\ni$ for all $k\in I^+ \Rightarrow \Delta_k:\{t_j^k\}_{j=0}^{n_k}$ is a partition of I as in the hypothesis of T3.8, and also $\ni$ $\exists \lim_{k\to\infty}\|\Delta_k\|=0$. This last condition $\Rightarrow \exists k_o \in I^+ \ni k>k_o \Longrightarrow$ $\left|\int_C f(z)dz - \bar{S}_k\right| < \frac{\epsilon}{2}$ where $\bar{S}_k$ is defined as in the proof to T3.9(2).

We point out first that if $D\neq Z$, then $\Gamma(D)$ is closed and non-empty. However, if $D=Z$, then clearly $\exists$ another simply connected domain D' containing C, since C is compact, $\ni \Gamma(D')$ is closed and non-empty. Now, D being open implies that $C \cap \Gamma(D)=\emptyset$ and since C is compact and $\Gamma(D)$ is closed, then $\acute{e}(C,\Gamma(D))>0$ (by $\underline{T}4.2$), say $\acute{e}(C,\Gamma(D))=r$. Then, for all $z\in C \Rightarrow N_r(z) \subseteq D$; otherwise, $N_r(z) \cap D\neq\emptyset$ for some $z\in C$ and $\left.\begin{matrix} N_r(z) \text{ connected} \\ N_r(z) \cap D\neq\emptyset \end{matrix}\right\} \Rightarrow N_r(z) \cap \Gamma(D)\neq\emptyset$. Let

$\zeta \in N_r(z) \cap \Gamma(D) \subseteq \Gamma(D)$. Then $r=\acute{e}(C,\Gamma(D))\leq d(z,\zeta)<r$, a contradiction. Hence, $N_r(z) \subseteq D$ for all $z\in C$.

Now, let $K=\mathcal{K} \bigcup\limits_{z\in C} \bar{N}_{\frac{r}{2}}(z)$. Then K is compact since K is closed and bounded, and $K \subseteq D$, a result of the following argument: suppose $w\in K=\mathcal{K} \bigcup\limits_{z\in C} N_{\frac{r}{2}}(z)$. Consider $N_{\frac{r}{2}}(w)$. Here, $w\in K \Rightarrow N_{\frac{r}{2}}(w) \cap \bigcup\limits_{z\in C} N_{\frac{r}{2}}(z)\neq\emptyset \Rightarrow \exists\, w^*,w_o\in C$ such that $w^*\in N_{\frac{r}{2}}(w) \cap N_{\frac{r}{2}}(w_o)$. Therefore, $d(w,w_o)\leq d(w,w^*)+d(w^*,w_o) < \frac{r}{2} + \frac{r}{2} = r \Rightarrow w\in N_r(w_o)$ and $w_o\in C \Rightarrow w\in D$ since we showed earlier that every r-nhood of a point on C is contained in D. Hence $K \subseteq D$.

Hence, since $f(z)$ is differentiable on D, $f(z)$ is continuous on D and since K is compact and $K \subseteq D$, $f(z)$ must be uniformly continuous on K. Thus, $\exists\ \delta>0 \ni |f(z')-f(z'')| < \dfrac{\varepsilon}{2l(C)}$ for $|z'-z''|<\delta$, $z',z''\in K$. We point out at this time that since $z=z(t)$ is continuous, $t\in I$, a compact set, then $z(t)$ is uniformly continuous on I. Thus, if we are given as our "epsilon" the value $\varepsilon'>0$, where $\varepsilon'=\min(\frac{r}{2},\delta)$ (δ being the δ used above in the definition of uniform continuity of $f(z)$ on K), then $z(t)$ being uniformly continuous on I implies that $\exists\, \eta>0 \ni |z(t')-z(t'')|<\varepsilon'$ for $|t'-t''|<\eta$, $t',t''\in I$.

Now, choose $k_1 \ni k>k_1 \Rightarrow \|\Delta_k\|<\eta$. This means that if $t_j^k, t_{j-1}^k \in \Delta_k$ (and hence $t_j^k, t_{j-1}^k \in I$), then $|t_j^k - t_{j-1}^k|<\eta$ $\Longrightarrow |z(t_j^k) - z(t_{j-1}^k)|<\varepsilon'\leq\delta$, a fact we will need later, but also, we can write $|z(t_j^k) - z(t_{j-1}^k)|<\varepsilon'\leq\frac{r}{2}$ implying that $z(t_{j-1}^k)\in N_{\frac{r}{2}}(z(t_j^k))$. But since $z(t_j^k)\in C$, we have that $L[z(t_j^k),z(t_{j-1}^k)] \subseteq N_{\frac{r}{2}}(z(t_j^k)) \subseteq \bigcup_{z\in C} N_{\frac{r}{2}}(z) \subseteq K$.

For brevity, we will write $L_j^k \equiv L[z(t_j^k),z(t_{j-1}^k)]$ and $P_k \equiv P(z(t_o^k),z(t_1^k),\ldots,z(t_{n_k}^k),z(t_o^k))$ (see D2.1), where the points $z(t_o^k),z(t_1^k),\ldots,z(t_{n_k}^k)$ are not necessarily distinct. Then, keeping in mind that $[z(t_j^k)-z(t_{j-1}^k)]\equiv\int_{L_j^k} dz(t)$, i.e., since $z(t_j^k)$ and $z(t_{j-1}^k)$ are the endpoints of the continuous rectifiable curve L_j^k and also, that $f(z(t_j^k))\int_{L_j^k} dz(t)$ $= \int_{L_j^k} f(z(t_j^k))dz(t)$, we have that

$$\left|\int_{\Gamma(P_k)} f(z)dz - \overline{S}_k\right| = \left|\sum_{j=1}^{n_k}\left(\int_{L_j^k} f(z(t))dz(t) - f(z(t_j^k))[z(t_j^k)-z(t_{j-1}^k)]\right)\right|$$

$$= \left| \sum_{j=1}^{n_k} \int_{L_j^k} [f(z(t))-f(z(t_j^k))]dz(t) \right|$$

$$\leq \sum_{j=1}^{n_k} \left| \int_{L_j^k} [f(z(t))-f(z(t_j^k))]dz(t) \right|$$

$$\leq \sum_{j=1}^{n_k} \max_{z\in L_j^k} \left| f(z(t))-f(z(t_j^k)) \right| \cdot l(L_j^k)$$

$$\leq \sum_{j=1}^{n_k} \frac{\varepsilon}{2l(C)} \cdot l(L_j^k) \quad \text{(since } L_j^k \subseteq K \text{ and } l(L_j^k)=|z(t_j^k)-z(t_{j-1}^k)|<\delta)$$

$$= \frac{\varepsilon}{2l(C)} \cdot \sum_{j=1}^{n_k} l(L_j^k)$$

$$\leq \frac{\varepsilon}{2l(C)} \cdot l(C) = \frac{\varepsilon}{2}, \quad \text{provided } k>k_1.$$

. Now, let m be a number such that $m>\max(k_1,k_o)$ and set $P(\varepsilon)\equiv P_m$. Now, $\left|\int_C f(z)dz - \int_{\Gamma(P(\varepsilon))} f(z)dz \right|$

$$= \left|[\int_C f(z)dz - \overline{S}_m] - [\int_{\Gamma(P(\varepsilon))} f(z)dz - \overline{S}_m] \right|$$

$$\leq \left| \int_C f(z)dz - \overline{S}_m \right| + \left| \int_{\Gamma(P(\varepsilon))} f(z)dz - \overline{S}_m \right|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ since } m>k_o \Longrightarrow \left|\int_C f(z)dz-\overline{S}_m\right| < \frac{\varepsilon}{2} \text{ and}$$

$$m>k_1 \Longrightarrow \left|\int_{\Gamma(P_m)} f(z)dz - \overline{S}_m \right| < \frac{\varepsilon}{2}, \quad (P_m\equiv P(\varepsilon)).$$

Hence, $\left| \int_C f(z)dz - \int_{\Gamma(P(\varepsilon))} f(z)dz \right| < \varepsilon \Longrightarrow \int_C f(z)dz=0$ since by T5.2 we have already shown that the integral over the boundary of a closed polygon is zero, and thus $\int_{\Gamma(P(\varepsilon))} f(z)dz=0$, and the theorem is proved.

CHAPTER VI

PROOF BY APPROXIMATION BY POLYNOMIALS

In this chapter, Cauchy's Theorem will be proved by a method not usually found in the regular textbooks on complex variables. This method uses an approximation by polynomials, i.e., we will work our way up to the point where an analytic function defined on a bounded simply connected domain is shown to be the limit of an almost uniformly convergent sequence of polynomials. Since every polynomial has a primitive function, Cauchy's Theorem follows almost immediately. However, it takes some very detailed and tedious steps to arrive at this level, and we shall have to fill in some background material and insert a few basic theorems at the start of the chapter.

As was mentioned in the introduction, the lemmas and theorems proved in this chapter follow a very intensive investigation of those used by Saks and Zygmund [5], but they have been prepared for use with polynomials rather than with rational functions entirely as they do in their text.

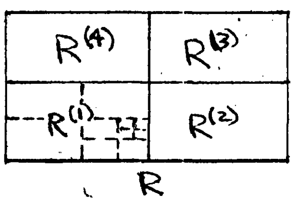
<u>T</u>6.1 (Cauchy's Theorem for a Rectangle)

<u>H</u> Consider an oriented closed rectangle R and an open set $O \subseteq Z \ni R \subseteq O$. Given also the function $f:O \to Z \ni f(z)$ is

analytic on R.

C $\ni \int_{\Gamma(R)} f(z)dz=0.$

P Since f(z) is analytic on R, f(z) is continuous on R, and hence is continuous on $\Gamma(R)$ which is a rectifiable curve, and thus $\exists \int_{\Gamma(R)} f(z)dz$. (By this argument, all the integrals that occur in this proof exist). Let $\varepsilon>0$ be given. We prove $|\int_{\Gamma(R)} f(z)dz|<\varepsilon$. For brevity, we define $I(R^*)\equiv\int_{\Gamma(R^*)} f(z)dz$ where R* is any oriented closed rectangle $R^* \subseteq R$. We divide R into four congruent closed rectangles $R^{(1)},R^{(2)},R^{(3)}$, and $R^{(4)}$, similar to R. Then,



(figure 6.1)

$I(R)=I(R^{(1)})+I(R^{(2)})+I(R^{(3)})+I(R^{(4)})$ by T3.10. Now, at least one of the four rectangles $R^{(k)}$, (k=1,2,3,4) must satisfy $\frac{1}{4}|I(R)|\leq|I(R^{(k)})|$ (same reasoning as in T5.1). Choose one such rectangle and denote it by R_1. Now we divide R_1 into four congruent closed rectangles similar to R_1, and for one of these, which we denote by R_2, we have that $\frac{1}{4^2}|I(R)|\leq|I(R_2)|$. Proceeding in this manner, we obtain a sequence of nested rectangles $R\equiv R_0\supset R_1\supset R_2\supset\ldots\supset R_n\supset\ldots$ satisfying $\frac{1}{4^n}|I(R)|\leq|I(R_n)|$. As was done similarly in T5.1, we can show that there exists a point $z_0\in R \ni (z_0)=\bigcap_{n=0}^{\infty} R_n$.

By assumption, $f(z)$ has a derivative at z_o. Put $L\equiv l(\Gamma(R))$, and let d denote the diameter of R. Then, $\exists\ \delta>0 \ni z\in N_\delta(z_o)$ and $z\neq z_o$ together $\Rightarrow \exists\ f(z)$ and $\left|\dfrac{f(z)-f(z_o)}{z-z_o} - f'(z_o)\right| < \dfrac{\varepsilon}{dL}$.

Note that $z\in N_\delta(z_o) \Rightarrow |f(z)-f(z_o)-(z-z_o)f'(z_o)| < \dfrac{\varepsilon}{dL}\cdot|z-z_o|$.

Let d_n denote the diameter of R_n. Then $d_n = \dfrac{d}{2^n}$, so that

$\exists \lim\limits_{n\to\infty} d_n = 0$, and hence $\exists\ N \ni d_N<\delta$. Here, $z,z_o\in R_N \Rightarrow |z-z_o|$

$\leq d_N<\delta \Rightarrow z\in N_\delta(z_o)$, i.e., $R_N \subseteq N_\delta(z_o)$. Next, we note that

$z\in\Gamma(R_N) \subseteq R_N \subseteq N_\delta(z_o) \Rightarrow |f(z)-f(z_o)-(z-z_o)f'(z_o)| \leq \dfrac{\varepsilon}{dL}|z-z_o|$

$\leq \dfrac{\varepsilon\cdot d_N}{dL}$. We now write $I(R_N)\equiv\displaystyle\int_{\Gamma(R_N)} f(z)dz = \int_{\Gamma(R_N)} f(z)dz - f(z_o)\int_{\Gamma(R_N)} dz$

$-f'(z_o)\displaystyle\int_{\Gamma(R_N)} zdz + z_of'(z_o)\int_{\Gamma(R_N)} dz$ (the last three $\int$'s equal 0)

$= \displaystyle\int_{\Gamma(R_N)} [f(z) - f(z_o) - (z-z_o)f'(z_o)]dz$. It follows that

$|I(R_N)| \leq \max\limits_{z\in\Gamma(R_N)} |f(z)-f(z_o)-(z-z_o)f'(z_o)|\cdot l(\Gamma(R_N))$

$\leq \dfrac{\varepsilon\cdot d_N}{dL}\cdot l(\Gamma(R_N))$. Here, $d_N = \dfrac{d}{2^N}$ and $l(\Gamma(R_N)) = \dfrac{L}{2^N}$, and

so $|I(R)| \leq 4^N|I(R_N)| \leq 4^N\cdot\dfrac{d_N}{dL}\cdot l(\Gamma(R_N)) = 4^N\cdot\dfrac{\varepsilon}{dL}\cdot\dfrac{d}{2^N}\cdot\dfrac{L}{2^N} = \varepsilon$.

Hence, $\left|\displaystyle\int_{\Gamma(R)} f(z)dz\right| < \varepsilon \Rightarrow \int_{\Gamma(R)} f(z)dz = 0$, and the theorem is proved.

The following is a familiar theorem and is stated without proof:

<u>T</u>6.2 <u>H</u> Given an oriented closed rectangle R and an open set

$0 \subseteq Z \ni R \subseteq 0$. Consider also the function $f:0 \rightarrow Z \ni f(z)$ is analytic on R.

C For all $z \in \mathcal{J}R \Rightarrow f(z) = \frac{1}{2\pi i} \int_{\Gamma(R)} \frac{f(\zeta)d}{\zeta - z}$.

(Cauchy's Integral Formula for a Rectangle).

D6.1 We shall call a *square net of order n*, and denote it by $\mathfrak{Q}^{(n)}$, the countable family of squares $Q_{h,k}^{(n)} = [\frac{h}{2^n}, \frac{(h+1)}{2^n}; \frac{k}{2^n}, \frac{(k+1)}{2^n}]$, where h and k range independently through I.

R6.1 This net covers the entire open plane.

T6.3 The relation, between pairs of points in a set A, of being "connected in A" is an equivalence relationship.

D6.2 Every one of the subsets into which a set A is decomposed by the equivalence relationship of being "connected in A" is called a *component* of A.

R6.2 A component is hence a connected subset which is not contained in any larger connected subset.

R6.3 Every two components of a set are then either identical or disjoint.

D6.3 A closed and connected set containing more than one element is called a *continuum*.

R6.4 Every component of a closed set is also a closed set, and hence is either a continuum or reduces to a point.

D6.4 A *rational function* (of the variable z) is the quotient of two polynomials (of the variable z).

D6.5 The poles of a rational function are the zeros of the polynomial in the denominator.

D6.6 Given the set $A \ni \emptyset \neq A \subseteq Z$ and the sequence of functions $\{f_k(z)\}_{k=1}^{\infty} \ni k \Rightarrow f_k: A \to Z$ and consider the function $f(z) \ni f: A \to Z$. Then, $f_k(z) \Longrightarrow f(z)$ on A means the sequence $\{f_k(z)\}_{k=1}^{\infty}$ converges uniformly to f(z) on the set A, i.e., for every $\varepsilon > 0 \Rightarrow [\exists\, K = K(\varepsilon) \ni \forall\ k > K$ and $\forall\ z \in A \Rightarrow |f_k(z) - f(z)| < \varepsilon]$.

D6.7 Given an open set $G \subseteq Z$ and the sequence of functions $\{f_k(z)\}_{k=1}^{\infty} \ni k \Rightarrow f_k: G \to Z$ and consider the function $f(z) \ni f: G \to Z$. Then, $f_k(z) \overset{\simeq}{\Longrightarrow} f(z)$ on G means the sequence $\{f_k(z)\}_{k=1}^{\infty}$ converges almost uniformly to f(z) on G, i.e., for every compact set $E \ni \emptyset \neq E \subseteq G \Rightarrow [\ f_k(z) \Longrightarrow f(z)$ on E $]$.

T6.4 H Given the open set $G \subseteq Z$ and the sequence of functions $\{f_k(z)\}_{k=1}^{\infty} \ni k \Rightarrow f_k: G \to Z$. Consider also the function $f(z) \ni f: G \to Z$, and let the following conditions be satisfied:

1) $\forall\, \zeta \in G \Rightarrow \exists\, r = r(\zeta) > 0 \ni N_r(\zeta) \subseteq G$, and
2) $f_k(z) \Longrightarrow f(z)$ on $N_r(\zeta)$.

C $f_k(z) \overset{\simeq}{\Longrightarrow} f(z)$ on G.

L6.1 H Given a set G such that G is a bounded simply connected domain. Also, given $\varepsilon > 0$ and let $\exists\, b, c \in Z \ni b \neq c$ and $b, c \in \mathfrak{S}G$.

C $\exists$ a sequence of points $b = p_1, p_2, \ldots, p_n = c$ of $\mathfrak{S}G$ such that $d(p_k, p_{k+1}) < \varepsilon$, $(1 \le k \le n-1)$. (Condition of Cantor).

P Since G is a bounded simply connected domain, $\mathfrak{S}G$ is is closed and connected and hence is a continuum by D6.3, since it contains more than one element, b and c.

Let $A \equiv \{z \mid z \in Z \ni \exists$ a chain in $\mathcal{C}G$ from b to $z\}$, and let $B \equiv \{\mathcal{C}A \cap \mathcal{C}G\}$. Assume the lemma is false, i.e., the above-mentioned sequence does not exist. Then $B \neq \emptyset$. Since $b \in A$, $A \neq \emptyset$. Clearly $A \cap B = \emptyset$ and $A \cup B = \mathcal{C}G$. We need only to show that the juncture of the sets A and B is empty, i.e., $\mathfrak{J}(A,B)=\emptyset$, and we will thus have that $\mathcal{C}G=A/B$, i.e., $\mathcal{C}G$ is not connected, which is a contradiction. We therefore must show that A and B are closed sets.

Let $a \in \mathcal{K}A$. Pick $a_1 \in N_\varepsilon(a)$ so that $d(a,a_1) < \varepsilon$. Let $b=p_1,p_2,\dots,p_n=a_1$ be a sequence of points of the set $A \ni d(p_k,p_{k+1}) < \varepsilon$, $(1 \leq k \leq n-1)$. Thus the sequence $b=p_1,p_2,\dots$ $\dots,p_n=a,p_{n+1}=a_1$ is therefore a sequence $\ni d(p_k,p_{k+1}) < \varepsilon$ for $1 \leq k \leq n$, and is thus a chain joining a with b. Hence $a \in A$, and thus $\mathcal{K}A \subseteq A \Rightarrow \mathcal{K}A=A \Rightarrow A$ is closed.

Now, let $c \in \mathcal{K}B$. Pick $c_1 \in N_\varepsilon(c)$ so that $d(c,c_1) < \varepsilon$. Assume $c \notin B$. Thus $c \in A$ and hence can be joined with b by a chain as above. But this means that c_1 can also be joined with b by a chain, contradicting $c_1 \in B$. Thus, $c \in B$ and thus $\mathcal{K}B \subseteq B \Rightarrow \mathcal{K}B=B \Rightarrow B$ is closed. Thus A is closed and B is closed, and $A \cap B=\emptyset$, so that $\mathfrak{J}(A,B)=\emptyset$, and hence $\mathcal{C}G=A/B$, a contradiction since $\mathcal{C}G$ is a continuum $\Rightarrow \mathcal{C}G$ is connected. Thus the lemma must be true.

D6.8 Consider an open set $O \subseteq Z$ and the two functions $f:O \to Z$ and $F:O \to Z$. Then, $F(z)$ is a <u>primitive function</u> (or <u>anti-</u>

derivative) of $f(z)$ iff for every $z \in O \Rightarrow \exists F'(z)=f(z)$.

R6.5 Clearly, by the Fundamental Theorem, if $F(z)$ is an antiderivative of $f(z)$, then $\int_\alpha^\beta f(z)dz = F(\beta) - F(\alpha)$. Hence, if C is a piecewise differentiable closed curve, and $F(z)$ is an antiderivative of $f(z)$, then $\int_C f(z)dz=0$ since $F(\beta) - F(\alpha)=0$ since $\alpha=\beta$ (C being closed $\Rightarrow$ the initial point=the terminal point).

The following is another familiar theorem and will be stated without proof:

T6.5 H Given the continuous rectifiable curve C and the sequence of functions $\{f_k(z)\}_{k=1}^{\infty} \ni k \Rightarrow f_k: C \to Z$, and each is continuous on C. Consider also the function $f(z)$ such that $f: C \to Z$.

C $\exists \lim_{k\to\infty} \int_C f_k(z)dz = \int_C f(z)dz$.

L6.2 H Consider the open set $G \subseteq Z \ni G$ is bounded, and also a non-empty compact set $F \subseteq G$.

C $\exists \{L_j\}_{j=1}^{n} \ni 1 \leq j \leq n \Rightarrow L_j$ is an oriented line segment contained in $G \cap \complement F$ and $\ni$ for every analytic function $f: G \to Z$ and for every $\zeta \in F \Rightarrow f(\zeta) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{L_j} \frac{f(z)dz}{z-\zeta}$.

P G being bounded and contained in $Z \Rightarrow \complement G \neq \emptyset$, and $F \neq \emptyset$ by hypothesis. Since F is compact and G is open $\Rightarrow \complement G$ is closed, we have that $d(F, \complement G) > 0$ by a previous lemma. Now, choose $N \ni \frac{1}{2^{N-1}} < d(F, \complement G)$. Let $\Omega^{(N)}$ denote the net of squares of order N, see D6.1. If $Q \in \Omega^{(N)}$, then the diameter of Q, $\delta Q = \sqrt{(\frac{1}{2^N})^2+(\frac{1}{2^N})^2} = \sqrt{\frac{2}{2^{2N}}} = \frac{\sqrt{2}}{2^N} \cdot < \frac{2}{2^N} = \frac{1}{2^{N-1}}$ and $\frac{1}{2^{N-1}} < d(F, \complement G)$. Thus, we must have that for every $Q \in \mathfrak{Q}^{(N)} \ni Q \cap F \neq \emptyset \Longrightarrow Q \subseteq G$. Assume the contrary and let $z \in Q \cap \complement G$ and $\zeta \in Q \cap F$. Then, $z, \zeta \in Q \Rightarrow d(z,\zeta) \leq \delta Q < d(F, \complement G)$ and $z \in \complement G$ and $\zeta \in F \Longrightarrow d(z,\zeta) \geq d(F, \complement G)$, a contradiction.

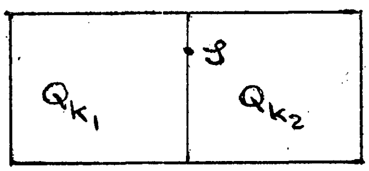
Now, define $\mathcal{G} \equiv \{Q \mid Q \in \mathfrak{Q}^{(N)}$ and $Q \cap F \neq \emptyset\}$. Since F is compact, F must be bounded and thus $\mathcal{G}$ is finite. Let $\mathcal{G} = \{Q_k\}_{k=1}^{m}$ for some $m \in I^+$. Define the set $\mathcal{L} \equiv \{L \mid \exists$ exactly one $Q \in \mathcal{G} \ni L$ is a side of the square $Q\}$. Then $\mathcal{G}$ being finite $\Longrightarrow \mathcal{L}$ is finite, so that $\mathcal{L} = \{L_j\}_{j=1}^{n}$ for some $n \in I^+$. Since $Q \in \mathcal{G} \Rightarrow Q \subseteq G$, then $L \in \mathcal{L} \Rightarrow L \subseteq G$. Furthermore, $L \in \mathcal{L} \Rightarrow L \cap F = \emptyset$, for otherwise, L would occur as a side of at least two squares in $\mathcal{G}$, contrary to the definition of $\mathcal{L}$. We then have that $1 \leq j \leq n \Longrightarrow L_j \subseteq G \cap \complement F$. We note at this point that if $L_j \in \mathcal{L}$ and L_j is a side of $Q_k \in \mathcal{G}$, then L_j is oriented in agreement with the boundary $\Gamma(Q_k)$, which itself is assumed to be oriented in the usual way.

Now, let $f: G \to Z$ such that f is analytic, and let $\zeta \in F$ be given. First we will assume $\exists\, Q_{k_o} \in \mathcal{G} \ni \zeta \in \partial Q_{k_o}$. Then, $k = k_o \Longrightarrow \frac{f(z)}{z-\zeta}$ is analytic on Q_k, and by Cauchy's Theorem for a Rectangle, $k \neq k_o \Longrightarrow \int_{\Gamma(Q_k)} \frac{f(z)dz}{z-\zeta} = 0$. Also, by Cauchy's Integral Formula for a Rectangle, $f(\zeta) = \frac{1}{2\pi i} \int_{\Gamma(Q_{k_o})} \frac{f(z)dz}{z-\zeta}$ so that we could write $f(\zeta)=\frac{1}{2\pi i}\sum_{k=1}^{m}\int_{\Gamma(Q_k)}\frac{f(z)dz}{z-\zeta}$, and if we let L_k^m denote the oriented sides of Q_k, then $f(\zeta)=\frac{1}{2\pi i}\sum_{k=1}^{m}\cdot\sum_{m=1}^{4}\int_{L_k^m}\frac{f(z)dz}{z-\zeta}$.

But if $L_k^m \notin \mathcal{L}$, then $-L_k^m$ also occurs in the above sum, and $\int_{L_k^m}\frac{f(z)dz}{z-\zeta} + \int_{-L_k^m}\frac{f(z)dz}{z-\zeta}=0$, so that $f(\zeta)=\frac{1}{2\pi i}\sum_{k=1}^{m}\cdot\sum_{m=1}^{4}\int_{L_k^m}\frac{f(z)dz}{z-\zeta}$

$$=\frac{1}{2\pi i}\sum_{L}\int_{L}\frac{f(z)dz}{z-\zeta} = \frac{1}{2\pi i}\sum_{j=1}^{n}\int_{L_j}\frac{f(z)dz}{z-\zeta}.$$

Second, we assume $\exists\, Q_{k_1}\in\mathcal{G} \ni \zeta\in\Gamma(Q_{k_1})$. First we will assume that ζ is not a vertex of Q_{k_1}. Thus, $\exists\, Q_{k_2}\in\mathcal{G}\ni \zeta\in\Gamma(Q_{k_2})$ also, where Q_{k_1} and Q_{k_2} are adjacent squares and ζ is not a vertex of Q_{k_2} either.



(figure 6.2)

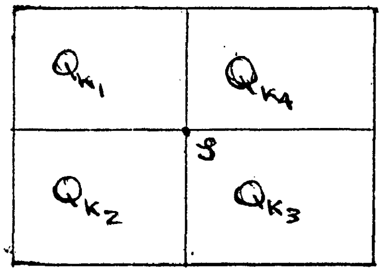
Let R denote the rectangle defined by $R=Q_{k_1}\cup Q_{k_2}$. Then, $\zeta\in\partial R$ and $k\neq k_1$ and $k\neq k_2 \Rightarrow \frac{f(z)}{z-\zeta}$ is analytic on Q_k and again by Cauchy's Theorem for a Rectangle, $\int_{\Gamma(Q_k)}\frac{f(z)dz}{z-\zeta}=0$.

Also, $f(\zeta)=\frac{1}{2\pi i}\int_{\Gamma(R)}\frac{f(z)dz}{z-\zeta}$, and hence

$$f(\zeta)=\frac{1}{2\pi i}\sum_{k=1}^{m}\int_{\Gamma(Q_k)}\frac{f(z)dz}{z-\zeta} + \frac{1}{2\pi i}\int_{\Gamma(R)}\frac{f(z)dz}{z-\zeta},\quad (k\neq k_1,k_2).$$

We now note that the side common to Q_{k_1} and Q_{k_2} does not belong to $\mathcal{L}$ by definition, and hence, by the same argument used earlier, $f(\zeta)=\frac{1}{2\pi i}\sum_{j=1}^{n}\int_{L_j}\frac{f(z)dz}{z-\zeta}$.

Finally, we assume that ζ is a vertex of $Q_{k_1} \in \mathcal{G}$. Thus $\exists$



(figure 6.3)

$Q_{k_2}, Q_{k_3}, Q_{k_4} \in \mathcal{G} \ni \zeta$ is a vertex of each of these also. Let Q denote the square defined by $Q = \bigcup_{i=1}^{4} Q_{k_i}$. Then, $\zeta \in \partial Q$ and thus $k \neq k_1, k_2, k_3, k_4 \Longrightarrow \frac{f(z)}{z-\zeta}$ is analytic on Q_k, and again by Cauchy's Theorem for a Rectangle, $\int_{\Gamma(Q_k)} \frac{f(z)dz}{z-\zeta} = 0$. Also, $f(\zeta) = \frac{1}{2\pi i} \int_{\Gamma(Q)} \frac{f(z)dz}{z-\zeta}$. Hence,

$$f(\zeta) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\Gamma(Q_k)} \frac{f(z)dz}{z-\zeta} + \frac{1}{2\pi i} \int_{\Gamma(Q)} \frac{f(z)dz}{z-\zeta}, \quad (k \neq k_i,\ 1 \leq i \leq 4).$$

Again we note that none of the four sides common to two of the squares $\{Q_{k_i}\}_{i=1}^{4}$ belongs to $\mathcal{L}$, by definition, and hence, by the same argument used earlier,

$$f(\zeta) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{L_j} \frac{f(z)dz}{z-\zeta}.$$

This exhausts the possibilities to be considered, and thus the lemma is proved.

The following two lemmas are offered to cover the necessary details needed to prove the validity of the properties of continuity possessed by a function of the variables z and t, where z ranges over the compact set F contained in Z, and t ranges over the closed interval [a,b].

<u>L</u>6.3 <u>H</u> Given a non-empty compact set $F \subseteq Z$ and the interval $I=[a,b]$. Consider the continuous function $\varphi(z,t): F \times I \to Z$ (meaning: for every $(z_o, t_o) \in F \times I$ and $\varepsilon > 0 \Longrightarrow [\exists\ \delta = \delta(z_o,$

$t_o,\varepsilon)>0 \ni \forall\ z\in F, t\in I \ni |z-z_o|<\delta$ and $|t-t_o|<\delta \Longrightarrow$ $|\varphi(z,t)-\delta(z_o,t_o)|<\varepsilon])$.

C For every $\varepsilon>0 \Longrightarrow [\exists\ \delta=\delta(\varepsilon)\ni\forall\ t_1,t_2\in I \ni |t_1-t_2|<\delta$ and $\forall\ z\in F \Longrightarrow |\varphi(z,t_1)-\varphi(z,t_2)|<\varepsilon]$.

P Assume the contrary. Then, for some $\varepsilon>0$, there does not exist $\delta(\varepsilon)$ satisfying the above conclusion. This means $\exists\ \varepsilon_o>0 \ni [\forall\ n \Longrightarrow \exists t_1^n, t_2^n \in I$ and $\exists\ z_n\in F \ni |t_1^n-t_2^n|<\frac{1}{n}$ and $|\varphi(z_n,t_1^n)-\varphi(z_n,t_2^n)|\leq \varepsilon_o]$. Since $\{t_1^n\}_{n=1}^{\infty} \ni n \Longrightarrow t_1^n\in I$ and I being compact (which implies that I is sequentially compact) $\Longrightarrow$ by definition of sequentially compactness that $\exists$ $\{n_j\}_{j=1}^{\infty}$ and $\exists\ t_o\in I \ni \exists \lim_{j\to\infty} t_1^{n_j}=t_o$. We next show that $\exists$ $\lim_{j\to\infty} t_2^{n_j}=t_o$. Let $\eta>0$ be given. Then $\exists\ J \ni [\ \frac{1}{n_J}<\frac{\eta}{2}$ and for every $j>J \Longrightarrow |t_1^{n_j}-t_o|<\frac{\eta}{2}\ ]$. Hence, we have that for all $j>J \Longrightarrow |t_2^{n_j}-t_o|\leq|t_2^{n_j}-t_1^{n_j}| + |t_1^{n_j}-t_o|<\frac{1}{n_j}+\frac{\eta}{2}<\frac{1}{n_J}+\frac{\eta}{2}$ $<\frac{\eta}{2}+\frac{\eta}{2}<\eta$, so that $\exists \lim_{j\to\infty} t_2^{n_j}=t_o$ also. Now, associated with the sets $\{t_1^{n_j}\}_{j=1}^{\infty}$ and $\{t_2^{n_j}\}_{j=1}^{\infty}$ is the set $\{z_{n_j}\}_{j=1}^{\infty}$ where $j \Longrightarrow z_{n_j}\in F$. But Fis compact and thus F is sequentially compact $\Longrightarrow \exists\ \{j_k\}_{k=1}^{\infty}$ and $\exists\ z_o\in F \ni \exists \lim_{k\to\infty} z_{n_{j_k}}=z_o$. We show next that $\exists \lim_{k\to\infty}\varphi(z_{n_{j_k}},t_1^{n_{j_k}})=\varphi(z_o,t_o)$. We will use $\mathfrak{z}_k\equiv z_{n_{j_k}}$ and $\tau_k\equiv t_1^{n_{j_k}}$ to simplify notation. Let $\lambda>0$ be given. Then $\varphi(z,t)$ being continuous at $(z_o,t_o) \Longrightarrow [\exists\ \delta=\delta(z_o,t_o,\lambda)>0 \ni \forall$ $z\in F, t\in I \ni |z-z_o|<\delta$ and $|t-t_o|<\delta \Longrightarrow |\varphi(z,t)-\varphi(z_o,t_o)|<\lambda]$.

But $\exists \lim_{k\to\infty} \zeta_k = z_o$ and $\exists \lim_{k\to\infty} \tau_k = t_o$ implies $\exists\, K=K(\delta)>0 \ni \forall$ $k>K \Longrightarrow |\zeta_k - z_o|<\delta$ and $|\tau_k - t_o|<\delta$ which in turn gives $|\varphi(\zeta_k,\tau_k)-\varphi(z_o,t_o)|<\lambda$. Hence, $\exists \lim_{k\to\infty}\varphi(\zeta_k,\tau_k)=\varphi(z_o,t_o)$ which is the same as saying $\exists \lim_{k\to\infty}\varphi(z_{n_{j_k}}, t_1^{n_{j_k}})=\varphi(z_o,t_o)$.

In a similar manner, we can also show that $\exists \lim_{k\to\infty}\varphi(z_{n_{j_k}}, t_2^{n_{j_k}})$ $=\varphi(z_o,t_o)$. Choose $k'>0$ this time so that $|\varphi(z_{n_{j_{k'}}}, t_1^{n_{j_{k'}}})$ $-\varphi(z_o,t_o)|<\frac{\varepsilon_o}{2}$ and $|\varphi(z_{n_{j_{k'}}}, t_2^{n_{j_{k'}}})-\varphi(z_o,t_o)|<\frac{\varepsilon_o}{2}$. Then,

$$|\varphi(z_{n_{j_{k'}}}, t_1^{n_{j_{k'}}})-\varphi(z_{n_{j_{k'}}}, t_2^{n_{j_{k'}}})| \le |\varphi(z_{n_{j_{k'}}}, t_1^{n_{j_{k'}}})-\varphi(z_o,t_o)|$$

$$+ |\varphi(z_{n_{j_{k'}}}, t_2^{n_{j_{k'}}})-\varphi(z_o,t_o)| < \frac{\varepsilon_o}{2}+\frac{\varepsilon_o}{2}=\varepsilon_o,$$ but this is a contradiction, for $z_{n_{j_{k'}}}$, $t_1^{n_{j_{k'}}}$, and $t_2^{n_{j_{k'}}}$ by their original choice gave $|\varphi(z_{n_{j_{k'}}}, t_1^{n_{j_{k'}}})-\varphi(z_{n_{j_{k'}}}, t_2^{n_{j_{k'}}})| \ge \varepsilon_o$.

<u>L</u>6.4 <u>H</u> Consider the continuous curve $C: \zeta=\zeta(t)$, $t\in I\equiv[a,b]$ and the continuous function $f: C\to Z$. Given also the non-empty compact set $F\subseteq Z \ni F\cap C=\emptyset$. Let $\varphi(z,t)\equiv\frac{f(\zeta(t))}{\zeta(t)-z}$. Note that $\forall\,(z,t)\in F\times I \Longrightarrow \varphi(z,t): F\times I\to Z$ and this implies that $[z\in F$ and $\zeta(t)\in C$ and $F\cap C=\emptyset] \Longrightarrow \zeta(t)\neq z$. Also, since $f: C\to Z$ and $\zeta: I\to C$ are both continuous, $f(\zeta(t))$ is also continuous on I.

<u>C</u> $\varphi(z,t)$ is continuous on $F\times I$.

P Let $(z_o,t_o)\in F\times I$. We want to prove that $\exists \lim\limits_{\substack{z\to z_o\\ t\to t_o\\ (z,t)\in F\times I}} \varphi(z,t) = \varphi(z_o,t_o)$.

We note first that C is compact and F is compact and $F\cap C=\emptyset \Longrightarrow d(F,C)>0$ by T4.2. Thus, $\forall\ (z,t)\in F\times I \Longrightarrow [z\in F \text{ and } \gamma(t)\in C] \Longrightarrow |\gamma(t)-z|\geq d(F,C)>0$ so that $\frac{1}{|\gamma(t)-z|}\leq\frac{1}{d(C,F)}<\infty$.

Now let $\epsilon>0$ be given. First, $f(\gamma(t))$ is continuous on I and $\gamma(t)$ is continuous on $I \Longrightarrow \exists\ \delta=\delta(\epsilon)>0 \ni |t-t_o|<\delta$ $\Longrightarrow |f(\gamma(t))-f(\gamma(t_o))|<\min\left(\frac{[d(C,F)]^2}{4|\gamma(t_o)|},\ \frac{[d(C,F)]^2}{4|z_o|}\right)$ and that $|\gamma(t)-\gamma(t_o)|<\frac{[d(C,F)]^2}{4|f(\gamma(t_o))|}$. It can also be assumed that $\delta<\frac{[d(C,F)]^2}{4|f(\gamma(t_o))|}$. We shall use here that $\frac{[d(C,F)]^2}{0}\equiv\infty$ and that $0\cdot\infty\equiv 0$.

Then, $(z,t)\in F\times I \ni |z-z_o|<\delta$ and $|t-t_o|<\delta$ implies that

$$|\varphi(z,t)-\varphi(z_o,t_o)| = \left|\frac{f(\gamma(t))}{\gamma(t)-z}-\frac{f(\gamma(t_o))}{\gamma(t_o)-z_o}\right|$$

$$= \left|\frac{f(\gamma(t))\cdot[\gamma(t_o)-z_o]-f(\gamma(t_o))\cdot[\gamma(t)-z]}{[\gamma(t)-z]\cdot[\gamma(t_o)-z_o]}\right|$$

$$= \frac{|f(\gamma(t))\cdot\gamma(t_o)-f(\gamma(t))\cdot z_o-f(\gamma(t_o))\cdot\gamma(t)+f(\gamma(t_o))\cdot z|}{|\gamma(t)-z|\cdot|\gamma(t_o)-z_o|}$$

$$\leq\frac{1}{[d(C,F)]^2}\cdot|f(\gamma(t))\cdot\gamma(t_o)-f(\gamma(t_o))\cdot\gamma(t)$$
$$+f(\gamma(t_o))\cdot z-f(\gamma(t)\cdot z_o|$$

$$\leq\frac{1}{[d(C,F)]^2}\cdot\big[|f(\gamma(t))\cdot\gamma(t_o)-f(\gamma(t_o))\cdot\gamma(t_o)|$$
$$+|f(\gamma(t_o))\cdot\gamma(t_o)-f(\gamma(t_o))\cdot\gamma(t)|+|f(\gamma(t_o))\cdot z$$

$$-f(\zeta(t_0))z_0| + |f(\zeta(t_0))z_0 - f(\zeta(t))z_0|]$$

$$= \frac{1}{[d(C,F)]^2}\cdot[\,|\zeta(t_0)|\cdot|f(\zeta(t))-f(\zeta(t_0))| + |f(\zeta(t_0))|\cdot|\zeta(t_0)$$

$$-\zeta(t)| + |f(\zeta(t_0))|\cdot|z-z_0| + |z_0|\cdot|f(\zeta(t_0))-f(\zeta(t))|\,]$$

$$\leq \frac{1}{[d(C,F)]^2}\cdot[\,|\zeta(t_0)|\cdot\frac{[d(C,F)]^2}{4|\zeta(t_0)|} + |f(\zeta(t_0))|\cdot\frac{[d(C,F)]^2}{4|f(\zeta(t_0))|}$$

$$+ |f(\zeta(t_0))|\cdot\frac{[d(C,F)]^2}{4|f(\zeta(t_0))|} + |z_0|\cdot\frac{[d(C,F)]^2}{4|z_0|}\,]$$

$$< [\,\frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}\,] = \varepsilon \,.$$

The next lemma we prove will allow us to approximate the integral of certain functions over certain curves by a rational function all of whose poles lie on the given curve.

<u>L</u>6.5 <u>H</u> Given the non-empty compact set $F \subseteq Z$ and let $C: \zeta = \zeta(t)$, $t \in I \equiv [a,b]$ be a piecewise differentiable curve. Also, consider the continuous function $f: C \to Z$, let $\varepsilon > 0$ be given, and let the condition that $F \cap C = \emptyset$ be satisfied. Note that $\forall\, z \in F \Longrightarrow \frac{f(\zeta)}{\zeta - z}$ is continuous on C and C is a piecewise differentiable curve so that $\exists \int_C \frac{f(\zeta)d\zeta}{\zeta - z}$.

<u>C</u> $\exists\; Q(z) = \sum_{j=0}^{n-1} \frac{a_j}{z-b_j} \ni 0 \leq j \leq n-1 \Longrightarrow b_j \in C$, $(a_j \in Z)$ and $\ni$

$$\forall\, z \in F \Longrightarrow \left| \int_C \frac{f(\zeta)d\zeta}{\zeta - z} - Q(z) \right| \leq \varepsilon .$$

(Thus $Q(z)$ is a rational function whose poles all lie on the curve C and Q of course depends on ε).

<u>P</u> We will first prove that the lemma is true when C is differentiable and then use this to prove it true for C being a piecewise differentiable curve. Thus, C being

differentiable $\Longrightarrow \gamma'(t)$ is continuous on I where I is compact. Since a function that is continuous on a compact set is bounded, there exists $M>0 \ni$ for every $t \in I$ $\Longrightarrow |\gamma'(t)| \leq M$. Now, by L6.4, $\frac{f(\gamma(t))}{\gamma(t)-z}$ is continuous on $F \times I$, where F and I are both compact. Thus, applying L6.3, $\exists\, \delta=\delta(\varepsilon)>0 \ni [\forall\, z \in F$ and $\forall\, t,t' \in I \ni |t-t'|<\delta] \Longrightarrow$

$$\left| \frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(t'))}{\gamma(t')-z} \right| < \frac{\varepsilon}{M(b-a)} .$$

Choose $\Delta:\{t_j\}_{j=1}^{n}$ to be a partition of $I \ni \|\Delta\|<\delta$. Then, define $\forall\, z \in F \Longrightarrow Q(z) \equiv \sum_{j=0}^{n-1} \frac{f(\gamma(t_j))}{z-\gamma(t_j)}[\gamma(t_j)-\gamma(t_{j+1})]$. (Note that $0 \leq j \leq n-1 \Longrightarrow \gamma(t_j) \in C$). Then, $\forall\, z \in F \Longrightarrow$

$$\left|\int_C \frac{f(\gamma)d\gamma}{\gamma-z} - Q(z)\right| = \left| \int_a^b \frac{f(\gamma(t))}{\gamma(t)-z}\cdot\gamma'(t)dt \right.$$

$$\left. - \sum_{j=0}^{n-1} \frac{f(\gamma(t_j))}{z-\gamma(t_j)}[\gamma(t_j)-\gamma(t_{j+1})]\right|$$

$$= \left|\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \frac{f(\gamma(t))}{\gamma(t)-z} - \sum_{j=0}^{n-1} \frac{f(\gamma(t_j))}{\gamma(t_j)-z}[\gamma(t_{j+1})-\gamma(t_j)]\right|$$

$$= \left|\sum_{j=0}^{n-1} \left[\int_{t_j}^{t_{j+1}} \frac{f(\gamma(t))}{\gamma(t)-z}\cdot\gamma'(t)dt - \frac{f(\gamma(t_j))}{\gamma(t_j)-z} \int_{t_j}^{t_{j+1}} \gamma'(t)dt\right]\right|$$

$$\left(\text{since } \int_{t_j}^{t_{j+1}} d\gamma(t) = [\gamma(t_{j+1})-\gamma(t_j)]\right),$$

$$= \left|\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(t_j))}{\gamma(t_j)-z} \right]\cdot\gamma'(t)dt \right|$$

$$\leq \sum_{j=0}^{n-1} \left| \int_{t_j}^{t_{j+1}} \left[\frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(t_j))}{\gamma(t_j)-z} \right]\cdot\gamma'(t)dt \right|$$

$$\leq \sum_{j=0}^{n-1} \frac{\varepsilon}{M(b-a)} \cdot M|t_{j+1} - t_j| \qquad \text{(using } \underline{T}3.7)$$

$$= \frac{\varepsilon}{b-a} \cdot \sum_{j=0}^{n-1} |t_{j+1} - t_j| = \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon \, .$$

We next assume that C is piecewise differentiable. Thus, $\exists \Delta : \{t_k\}_{k=0}^{m}$, a partition of $I \ni 1 \leq k \leq m \Longrightarrow$ C is differentiable on $I_k \equiv [t_{k-1}, t_k]$. Let $1 \leq k \leq m \Longrightarrow C_k : \zeta = \zeta(t),\ t \in I_k$. We now have that $1 \leq k \leq m \Longrightarrow \exists\ Q_k(z) = \sum_{j=0}^{n_k - 1} \frac{a_j^k}{z - b_j^k} \ni 0 \leq j \leq n_k \Longrightarrow$

$b_j^k \in C_k \subseteq C$ and $\ni \forall z \in F \Longrightarrow \left| \int_{C_k} \frac{f(\zeta)d\zeta}{\zeta - z} - Q_k(z) \right| \leq \frac{\varepsilon}{m}$, which is

proved for each C_k as we did above for C. Then, if we define $\forall\ z \in F \Longrightarrow Q(z) \equiv \sum_{k=1}^{m} Q_k(z)$, we have that $\forall\ z \in F \Longrightarrow$

$\left| \int_C \right.$ $\forall\ z \in F \Longrightarrow Q(z) \equiv \sum_{k=1}^{m} Q_k(z)$, we have that $\forall\ z \in F \Longrightarrow$ C_k

$$= \left| \sum_{k=1}^{m} \left[\int_{C_k} \frac{f(\zeta)d\zeta}{\zeta - z} - Q_k(z) \right] \right| \leq \sum_{k=1}^{m} \left| \int_{C_k} \frac{f(\zeta)d\zeta}{\zeta - z} - Q_k(z) \right|$$

$\leq \sum_{k=1}^{m} \frac{\varepsilon}{m} = m \cdot \frac{\varepsilon}{m} = \varepsilon$, and clearly all of the poles of $Q(z)$ are on C.

The following two lemmas deal with "translation of poles". The first of these allows us to approximate a rational function having a single complex number as its only pole, by another rational function having some other complex number as its only pole, if the two poles are "close enough" to each other. The second lemma allows us to approximate a rational function having a single pole by a polynomial, if this pole is "far enough out."

The above "intuitive" introduction of the following two lemmas will become less vague as they are stated and proved in detail.

L6.6 H Given the non-empty compact set $F \subseteq Z$ and consider the two points $z_1, z_2 \in \mathscr{C}F \ni |z_1 - z_2| \leq \frac{1}{2}d(z_2,F)$, (since $z_2 \in \mathscr{C}F$ and F is closed and (z_2) is compact, $d(z_2,F)>0$ by T4.2). Consider also the rational function $R(z) \equiv \sum_{j=0}^{n} \frac{a_j}{(z-z_1)^j}$

$$= a_o + \frac{a_1}{(z-z_1)} + \frac{a_2}{(z-z_2)^2} + \cdots + \frac{a_n}{(z-z_1)^n}, \ (z \neq z_1), \text{ and let}$$

$\varepsilon > 0$ be given. Note that $R(z)$ is a rational function whose only pole is z_1.

C $\exists$ $Q(z) = \sum_{k=0}^{m} \frac{b_k}{(z-z_2)^k} = b_o + \frac{b_1}{(z-z_2)} + \frac{b_2}{(z-z_2)^2} + \cdots + \frac{b_n}{(z-z_2)^n}$,

$(z \neq z_2)$, $\ni \forall z \in F \implies |R(z) - Q(z)| \leq \varepsilon$.

(Here the approximating function $Q(z)$ is a rational function whose only pole is z_2. Of course, $Q(z)$ depends on ε).

P Without loss of generality, we may assume that the coefficients $a_j \neq 0$ for $0 \leq j \leq n$. Choose $j \ni 1 \leq j \leq n$. Anytime $\zeta \in Z \ni |\zeta| < 1$ we can write $\frac{1}{(1-\zeta)^j} = \sum_{k=0}^{\infty} \binom{k+j-1}{k} \zeta^k$, where

$\binom{k+j-1}{k} \equiv \frac{(k+j-1)!}{k!(j-1)!}$. We have that $\forall z \in F \implies |z-z_2| \geq d(z_2,F)$

since $z_2 \in \mathscr{C}F$, and thus $\frac{1}{|z-z_2|} \leq \frac{1}{d(z_2,F)}$ so that

$\left| \frac{z_1 - z_2}{z - z_2} \right| = \frac{|z_1 - z_2|}{|z - z_2|} \leq \frac{|z_1 - z_2|}{d(z_2,F)} \leq \frac{1}{2} < 1$, since by hypothesis $|z_1 - z_2| \leq \frac{1}{2}d(z_2,F)$.

Now, define $\forall\ z \in F \Longrightarrow \mathfrak{z} = \varphi(z) \equiv \dfrac{z_1 - z_2}{z - z_2}$. Thus, $\forall\ z \in F$ $\Longrightarrow |\mathfrak{z}| \leq \frac{1}{2} < 1$ and hence $\forall\ z \in F \Longrightarrow$

$$\frac{1}{\left(1 - \frac{z_1 - z_2}{z - z_2}\right)^j} = \frac{1}{(1 - \mathfrak{z})^j} = \sum_{k=0}^{\infty} \binom{k+j-1}{k} \mathfrak{z}^k = \sum_{k=0}^{\infty} \binom{k+j-1}{k} \left(\frac{z_1 - z_2}{z - z_2}\right)^k .$$

Hence, as a result of the following manipulations, $\forall\ z \in F \Longrightarrow$

$$\frac{1}{(z-z_1)^j} = \frac{1}{(z-z_2)^j} \cdot \frac{(z-z_2)^j}{(z-z_1)^j} = \frac{1}{(z-z_2)^j} \cdot \frac{1}{\frac{(z-z_1)^j}{(z-z_2)^j}}$$

$$= \frac{1}{(z-z_2)^j} \cdot \frac{1}{\left(\frac{z - z_2 - (z_1 - z_2)}{z - z_2}\right)^j} = \frac{1}{(z-z_2)^j} \cdot \frac{1}{\left(1 - \frac{z_1 - z_2}{z - z_2}\right)^j}$$

$$= \frac{1}{(z-z_2)^j} \cdot \sum_{k=0}^{\infty} \binom{k+j-1}{k} \left(\frac{z_1 - z_2}{z - z_2}\right)^k .$$

Now, $\sum_{k=0}^{\infty} \binom{k+j-1}{k} \mathfrak{z}^k \overset{\sim}{\Longrightarrow}$ on the open unit disk, $N_1(0)$, and the function $\varphi : F \to N_1(0)$ is continuous and F being compact gives us that $\varphi(F)$ is compact (by the fact that the continuous image of a compact set is compact) and also, $\varphi(F) \subseteq N_1(0)$ so that by definition of almost uniform convergence, $\sum_{k=0}^{\infty} \binom{k+j-1}{k} \mathfrak{z}^k \Longrightarrow$ on $\varphi(F)$. From this we have that $\varepsilon > 0 \Longrightarrow \exists\ L = L(\varepsilon) \ni \forall\ \mathfrak{z} \in \varphi(F) \Longrightarrow$

$$\left| \sum_{k=0}^{\infty} \binom{k+j-1}{k} \mathfrak{z}^k - \sum_{k=0}^{L} \binom{k+j-1}{k} \mathfrak{z}^k \right| \leq \frac{[d(z_2, F)]^j}{n |a_j|} .$$

We now define $Q_j(z) \equiv \dfrac{1}{(z-z_2)^j} \cdot \sum_{k=0}^{L} \binom{k+j-1}{k} \dfrac{(z_1 - z_2)^k}{(z-z_2)^k}$, so that

$$\forall\ z\in F \Longrightarrow \left| \frac{1}{(z-z_1)^j} - Q_j(z) \right|$$

$$= \left| \frac{1}{(z-z_2)^j}\cdot \sum_{k=0}^{\infty}\binom{k+j-1}{k}\left(\frac{z_1-z_2}{z-z_2}\right)^k - \frac{1}{(z-z_2)^j}\cdot \sum_{k=0}^{\infty}\binom{k+j-1}{k}\left(\frac{z_1-z_2}{z-z_2}\right)^k \right|$$

$$= \left| \frac{1}{(z-z_2)^j} \right|\cdot\left| \sum_{k=0}^{\infty}\binom{k+j-1}{k}\zeta^k - \sum_{k=0}^{L}\binom{k+j-1}{k}\zeta^k \right|$$

$$\leq \left| \frac{1}{(z-z_2)^j} \right|\cdot\frac{\varepsilon[d(z_2,F)]^j}{n|a_j|} \leq \frac{1}{[d(z_2,F)]^j}\cdot\frac{\varepsilon[d(z_2,F)]^j}{n|a_j|}$$

$$= \frac{\varepsilon}{n|a_j|},\quad \left(\text{since } \frac{1}{|z-z_2|} \leq \frac{1}{d(z_2,F)}\right).$$

We now define $Q(z)\equiv a_o+a_1Q_1(z)+a_2Q_2(z)+\ldots+a_nQ_n(z)$.

Thus, $\forall\ z\in F \Longrightarrow |R(z)-Q(z)|$

$$= \left| \sum_{j=0}^{n}\frac{a_j}{(z-z_1)^j} - \left(a_o+\sum_{j=1}^{n}a_jQ_j(z)\right) \right|$$

$$= \left| a_o+\sum_{j=1}^{n}\frac{a_j}{(z-z_1)^j} - a_o-\sum_{j=1}^{n}a_jQ_j(z) \right|$$

$$= \left| \sum_{j=1}^{n}\left[\frac{a_j}{(z-z_1)^j} - a_jQ_j(z)\right] \right|$$

$$\leq \sum_{j=1}^{n}|a_j|\cdot\left| \frac{1}{(z-z_1)^j} - Q_j(z) \right|$$

$$\leq \sum_{j=1}^{n}|a_j|\cdot\frac{\varepsilon}{n|a_j|} = \sum_{j=1}^{n}\frac{\varepsilon}{n} = n\cdot\frac{\varepsilon}{n} = \varepsilon .$$

<u>L</u>6.7 <u>H</u> Given the non-empty compact set $F\subseteq Z$ and consider the point $z_1\in\mathcal{C}F \ni \forall\ z\in F \Longrightarrow |z|\leq \frac{1}{2}|z_1|$, $z_1\neq 0$. Also, consider the rational function $R(z)\equiv \sum_{j=0}^{n}\frac{a_j}{(z-z_1)^j} = a_o+\frac{a_1}{(z-z_1)}$ $+\frac{a_2}{(z-z_1)^2}+\ldots\ldots+\frac{a_n}{(z-z_1)^n}$, $(z\neq z_1)$, and let $\varepsilon>0$ be given.

Note that $R(z)$ is a rational function whose only pole is z_1.

<u>C</u> $\exists$ a polynomial $P(z)\ni\forall\ z\in F \Longrightarrow |R(z)-P(z)|\leq\varepsilon$.

P This proof is similar to that of the last lemma. We first assume $0 \leq j \leq n \Longrightarrow a_j \neq 0$, and then pick $j \ni 1 \leq j \leq n$. Again, $\forall\, \zeta \in F \ni |\zeta| \leq 1 \Longrightarrow \frac{1}{(1-\zeta)^j} = \sum_{k=0}^{\infty} \binom{k+j-1}{k} \zeta^k$, where $\binom{k+j-1}{k} \equiv \frac{(k+j-1)!}{k!(j-1)!}$. Here, we have that $\forall z \in F \Longrightarrow |z| \leq \frac{1}{2}|z_1|$ by hypothesis, and thus if $z_1 \neq 0$, $|\frac{z}{z_1}| = \frac{|z|}{|z_1|} \leq \frac{1}{2} < 1$. Now, define $\forall\, z \in F \Longrightarrow \zeta = \varphi(z) \equiv \frac{z}{z_1}$. Thus, $\forall\, z \in F \Longrightarrow |\zeta| \leq \frac{1}{2} < 1$ so that $\forall\, z \in F \Longrightarrow \frac{1}{(1-\frac{z}{z_1})^j} = \frac{1}{(1-\zeta)^j} = \sum_{k=0}^{\infty} \binom{k+j-1}{k} \zeta^k$

$= \sum_{k=0}^{\infty} \binom{k+j-1}{k} (\frac{z}{z_1})^k$. Next we show that $\forall\, z \in F \Longrightarrow$

$$\frac{1}{(z-z_1)^j} = \frac{1}{(-1)^j (z_1-z)^j} = \frac{(-1)^j}{(z_1-z)^j} = \frac{(-1)^j}{z_1{}^j (1-\frac{z}{z_1})^j}$$

$= \frac{(-1)^j}{z_1{}^j} \sum_{k=0}^{\infty} \binom{k+j-1}{k} (\frac{z}{z_1})^k$. As in the last proof, since $\sum_{k=0}^{\infty} \binom{k+j-1}{k} \zeta^k$ is almost uniformly convergent on $N_1(0)$ and φ is continuous on a compact set F and hence $\varphi(F)$ is compact, we have convergence on $\varphi(F)$ as before, so that again, for $\varepsilon > 0 \Longrightarrow \exists\, L = L(\varepsilon) \ni \forall \zeta \in \varphi(F) \Longrightarrow$

$|\sum_{k=0}^{\infty} \binom{k+j-1}{k} \zeta^k - \sum_{k=0}^{L} \binom{k+j-1}{k} \zeta^k| \leq \frac{|z_1|^j}{n|a_j|}$. We now let $P_j(z) \equiv \frac{(-1)^j}{(z_1)^j} \cdot \sum_{k=0}^{L} \binom{k+j-1}{k} (\frac{z}{z_1})^k$. Then, $\forall\, z \in F \Longrightarrow$

$$\left| \frac{1}{(z-z_1)^j} - P_j(z) \right| = \left| \frac{(-1)^j}{z_1{}^j} \cdot \sum_{k=0}^{\infty} \binom{k+j-1}{k} (\frac{z}{z_1})^k - \frac{(-1)^j}{(z_1)^j} \cdot \sum_{k=0}^{L} \binom{k+j-1}{k} (\frac{z}{z_1})^k \right|$$

$$= \left|\frac{(-1)^j}{(z_1)^j}\right|\cdot\left|\sum_{k=0}^{\infty}\binom{k+j-1}{k}\mathfrak{z}^k - \sum_{k=0}^{L}\binom{k+j-1}{k}\mathfrak{z}^k\right|$$

$$\leq \frac{1}{|z_1|^j}\cdot\frac{|z_1|^j}{n|a_j|} = \frac{\varepsilon}{n|a_j|}\,.$$

We now define $P(z)\equiv a_0+a_1P_1(z)+\ldots+a_nP_n(z)$. Thus, $P(z)$ is a polynomial, and $\forall\, z\in F \Longrightarrow |R(z)-P(z)|$

$$= \left|\sum_{j=0}^{n}\frac{a_j}{(z-z_1)^j} - \left(a_0+\sum_{j=1}^{n}a_jP_j(z)\right)\right|$$

$$= \left|\sum_{j=1}^{n}\left[\frac{a_j}{(z-z_1)^j} - a_jP_j(z)\right]\right|$$

$$\leq \sum_{j=1}^{n}|a_j|\cdot\left|\frac{1}{(z-z_1)^j} - P_j(z)\right|$$

$$\leq \sum_{j=1}^{n}|a_j|\cdot\frac{\varepsilon}{n|a_j|} = \sum_{j=1}^{n}\frac{\varepsilon}{n} = n\cdot\frac{\varepsilon}{n} = \varepsilon\,.$$

We are now ready to make use of these last few lemmas in proving the following theorem which will allow us to approximate, by polynomials, functions which are analytic on a bounded simply connected domain, under certain conditions.

T6.6 Given the set $G \ni G$ is a bounded simply connected domain $\ni G \subseteq Z$ and consider the analytic function $f:G\to Z$. Let F be a non-empty compact set $\ni F\subseteq G$ and let $\varepsilon>0$ be given.

C $\exists$ a polynomial $P(z)\ni\forall\, z\in F \Longrightarrow |f(z)-P(z)|<\varepsilon$.

P We first prove the following needed lemma:

L H Define $H=\{z\mid d(z,\complement G)\geq \frac{1}{2}d(z,F)$ or $d(z,\complement G)\geq\frac{1}{2}d(F,\complement G)\}$.

N F is compact and $\complement G$ is closed and $F\cap\complement G=\emptyset$ so that $d(F,\complement G)>0$.

C $\exists$ a rational function $Q(z)$ all of whose poles lie in

$G \cap \mathcal{C}H$ and which satisfies the condition $|f(z)-Q(z)|<\frac{\epsilon}{2}$ for all $z \in H$.

P The second part of the definition of H above gives us that $F \subseteq H$. We can also show that H is closed: Clearly H can be written as $H=H_1 \cup H_2$, where

$H_1 \equiv \{z | d(z, \mathcal{C}G) \geq \frac{1}{2}d(z,F)\}$ and $H_2 \equiv \{z | d(z, \mathcal{C}G) \geq \frac{1}{2}d(F, \mathcal{C}G)\}$.

Let $z_o \in \mathcal{K}H_1$. Thus every nhood that contains z_o contains points in H_1. Thus, $\forall\ n \in I^+ \Longrightarrow \exists\ z_n \in H_1 \ni z_o, z_n \in N_{\frac{1}{n}}(z_o)$, and $d(z_n, \mathcal{C}G) \geq \frac{1}{2}d(z_n,F)$ and $|z_o-z_n|<\frac{1}{n}$. Thus, a sequence $\{z_n\}_{n=1}^{\infty}$ of points in H_1 is formed where $\lim_{n\to\infty} z_n=z_o$. Now, the function $\varphi(z)\equiv\varphi(z,E)$, where $\emptyset \neq E \subseteq Z$ is a continuous function on Z, so that $\exists \lim_{z_n\to z_o} d(z_n, \mathcal{C}G)=d(z_o, \mathcal{C}G)$, and also, $\exists \lim_{z_n\to z_o} \frac{1}{2}d(z_n,F)=\frac{1}{2}d(z_o,F)$. Hence,

$$\exists \lim_{z_n\to z_o} d(z_n, \mathcal{C}G) \geq \lim_{z_n\to z_o} \frac{1}{2}d(z_n,F),\ \forall\ z_n \in H_1, \text{ and thus}$$

$d(z_o, \mathcal{C}G) \geq \frac{1}{2}d(z_o,F) \Longrightarrow z_o \in H_1$. Therefore, $\mathcal{K}H_1 \subseteq H_1$ and thus H_1 is closed.

Similarly, if $z_o \in \mathcal{K}H_2$, then as before, there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of points in $H_2 \ni \lim_{n\to\infty} z_n=z_o$. Again, we have that $\exists \lim_{z_n\to z_o} d(z_n, \mathcal{C}G)=d(z_o, \mathcal{C}G)$. Hence,

$$\exists \lim_{z_n\to z_o} d(z_n, \mathcal{C}G) \geq \lim_{z_n\to z_o} \frac{1}{2}d(F, \mathcal{C}G),\ \forall\ z_n \in H_2, \text{ and thus}$$

$d(z_o, \mathcal{C}G) \geq \frac{1}{2}d(F, \mathcal{C}G) \Longrightarrow z_o \in H_2$. Therefore $\mathcal{K}H_2 \subseteq H_2$

and hence H_2 is closed. $H=H_1 \cup H_2$ implies that H is closed therefore.

Now, by L6.2, $\exists$ segments $L_1, L_2, \ldots, L_n$ in $G \cap \complement H \ni$ $f(z)=\frac{1}{2\pi i}\sum_{j=1}^{n}\int_{L_j}\frac{f(\zeta)d\zeta}{\zeta-z}$, $\forall\, z\in H$. Let $f_i(z)\equiv\frac{1}{2\pi i}\int_{L_i}\frac{f(\zeta)d\zeta}{\zeta-z}$, where $1\leq i\leq n$. Then $f(z)=f_1(z)+f_2(z)+\ldots.+f_n(z)$. Consider $f_1(z)=\frac{1}{2\pi i}\int_{L_1}\frac{f(\zeta)d\zeta}{\zeta-z}$. By L6.5, there exists a rational function $Q_1(z)$ having poles exclusively on L_1 $\ni$ $|2\pi i\cdot f_1(z)-Q_1(z)|<\frac{\varepsilon\pi}{n}$, $\forall\, z\in H$. But,

$$|2\pi i\cdot f_1(z)-Q_1(z)|=|2\pi i[f_1(z)-\frac{Q_1(z)}{2\pi i}]|=2\pi|f_1(z)-\frac{Q_1(z)}{2\pi i}|<\frac{\varepsilon\pi}{n},$$

so that $|f_1(z)-\frac{Q_1(z)}{2\pi i}|<\frac{\varepsilon}{2n}$. Thus, for $f_i(z)$, $1\leq i\leq n$, $\exists$ a rational function $Q_i(z)$ having poles exclusively on L_i $\ni$ $|f_i(z)-\frac{Q_i(z)}{2\pi i}|<\frac{\varepsilon}{2n}$, $\forall\, z\in H$. Let $Q(z)\equiv\frac{Q_1(z)}{2\pi i}+\frac{Q_2(z)}{2\pi i}+\ldots.+\frac{Q_n(z)}{2\pi i}$. Then, $|f(z)-Q(z)|=|[f_1(z)+\ldots.+f_n(z)]+[\frac{Q_1(z)}{2\pi i}+\ldots+\frac{Q_n(z)}{2\pi i}]|=|[f_1(z)-\frac{Q_1(z)}{2\pi i}]+\ldots+[f_n(z)-\frac{Q_n(z)}{2\pi i}]|$ $\leq|f_1(z)-\frac{Q_1(z)}{2\pi i}|+\ldots.+|f_n(z)-\frac{Q_n(z)}{2\pi i}|<\frac{\varepsilon}{2n}+\ldots+\frac{\varepsilon}{2n}=n\cdot\frac{\varepsilon}{2n}=\frac{\varepsilon}{2}$, and clearly the rational function $Q(z)$ has poles exclusively in $G\cap\complement H$, completing the proof of the lemma.

Here, $Q(z)$ found above in our lemma can be rewritten in the form $Q(z)\equiv Q_1(z)+Q_2(z)+\ldots+Q_m(z)$ where each of the functions $Q_k(z)$, $1\leq k\leq m$, is rational and has only one pole.

Consider $Q_1(z)$. Let us denote its only pole by a. By T4.2, since (a) is compact and $\complement G$ is closed and $a\in G \Longrightarrow$ $(a)\cap\complement G=\emptyset$ so that $d(a,\complement G)>0$. Hence $\exists$ a point $b\in\complement G$ $\ni$

$d(a,b)=d(a,\complement G)$.

Since $a \in G \cap \complement H \subseteq \complement H$, the conditions for membership in H are not satisfied when $z=a$. Hence, since $b \in \complement G$,

$$d(a,b)=d(a,\complement G)< \tfrac{1}{2}d(F,\complement G)\leq \tfrac{1}{2}d(F,b) \qquad (1).$$

Now, let c be a point satisfying the hypothesis for L6.7, i.e., $c \in \complement G$ and $c \neq 0 \ni |z| \leq \frac{1}{2}|c|, \forall z \in F$. Now, G is a bounded simply connected domain so that $\complement G$ is a component (see D6.2) and since $b,c \in \complement G$, we can use the "Condition of Cantor" (see previous theorem), i.e., there exists a sequence of points $b=p_1,p_2,p_3,\ldots,p_{n-1},p_n=c$ all in $\complement G \ni$ $d(p_k,p_{k+1})< \frac{1}{2}d(F,\complement G)$ for $1 \leq k \leq n-1$. (Note that we are considering our "ϵ" here to be $\frac{1}{2}d(F,\complement G)$).

Let $p_0 \equiv a$. Then for $k \geq 1$, we have that $d(p_k,p_{k+1}) < d(p_{k+1},F)$ by (1) above since all the p_k's belong to $\complement G$. As for $k=0$, $p_0 \equiv a$ and we have that $d(p_0,p_1)< \frac{1}{2}d(F,\complement G) \leq \frac{1}{2}d(F,p_1)$ since $p_1 \in \complement G$ and from Cantor's Condition above.

Thus the hypothesis for L6.6 is now satisfied, and we can now apply this lemma successively to every pair of points p_k,p_{k+1}, getting a finite sequence of rational functions $R_0(z),R_1(z),\ldots,R_n(z)$ (where $R_0(z) \equiv Q_1(z)$) satisfying the following conditions: $|R_{k+1}(z)-R_k(z)|< \frac{\epsilon}{2m(n+1)}$, $\forall z \in F$, $0 \leq k \leq n-1$, and the only pole of the function $R_k(z)$ is p_k. (Note that the only pole of $R_n(z)$ is c).

Now, applying L6.7, we have $c \in \complement F \ni [\forall z \in F \Rightarrow |z| \leq \frac{1}{2}|c|$, $c \neq 0]$ so we have that $\exists$ a polynomial $P_1(z)$ such that $|R_n(z)-P_1(z)|< \frac{\epsilon}{2m(n+1)}$. Then, $|Q_1(z)-P_1(z)|$

$$= |Q_1(z)-R_n(z)+R_n(z)-P_1(z)|$$

$$\leq |Q_1(z)+R_n(z)| + |R_n(z)-P_1(z)|$$

$$<|R_o(z)-R_n(z)| + \frac{\varepsilon}{2m(n+1)} \quad (\text{since } Q_1(z)\equiv R_o(z))$$

$$= |R_n(z)-R_o(z)| + \frac{\varepsilon}{2m(n+1)}$$

$$= |[R_n(z)-R_{n-1}(z)]+[R_{n-1}(z)-R_{n-2}(z)]+\ldots+[R_1(z)-R_o(z)]| + \frac{\varepsilon}{2m(n+1)}$$

$$\leq |R_n(z)-R_{n-1}(z)|+|R_{n-1}(z)-R_{n-2}(z)|+\ldots+|R_1(z)-R_o(z)| + \frac{\varepsilon}{2m(n+1)}$$

$$\leq \frac{\varepsilon}{2m(n+1)} + \frac{\varepsilon}{2m(n+1)} + \ldots\ldots + \frac{\varepsilon}{2m(n+1)} + \frac{\varepsilon}{2m(n+1)}$$

$$= (n+1)\cdot \frac{\varepsilon}{2m(n+1)} = \frac{\varepsilon}{2m}.$$

Similarly, for $Q_k(z)$, $\exists$ a polynomial $P_k(z)$ such that $|Q_k(z)-P_k(z)|< \frac{\varepsilon}{2m}$, for $k=2,3,\ldots,m$. Now, let $P(z)\equiv P_1(z)+P_2(z)+\ldots+P_m(z)$. (Note that $P(z)$ is a polynomial). Thus,

$$|Q(z)-P(z)|=|[Q_1(z)+\ldots+Q_m(z)]-[P_1(z)+\ldots+P_m(z)]|$$
$$=|[Q_1(z)-P_1(z)]+[Q_2(z)-P_2(z)]+\ldots+[Q_m(z)-P_m(z)]|$$
$$\leq \frac{\varepsilon}{2m} + \frac{\varepsilon}{2m} + \ldots + \frac{\varepsilon}{2m} = m\cdot \frac{\varepsilon}{2m} = \frac{\varepsilon}{2}.$$

Finally, $|f(z)-P(z)| = |f(z)-Q(z)+Q(z)-P(z)|$

$$\leq |f(z)-Q(z)| + |Q(z)-P(z)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It is in the following theorem that we finally show that an analytic function on a bounded simply connected domain can be written as the limit of a sequence of almost-uniformly convergent polynomials.

T6.7 H Given the set $G \ni G$ is a simply connected and bounded domain $\subseteq Z$ and consider the analytic function $f:G\to Z$.

C $\exists \{P_n(z)\}_{n=1}^{\infty} \ni n \Longrightarrow P_n(z)$ is a polynomial and

$$P_n(z) \overset{\sim}{\Longrightarrow} f(z) \text{ on } G.$$

P We make use of T6.4 which says that a sequence of polynomials which converges to an analytic function (defined on an open set in Z) on some open nhood of every point in that open set, will converge almost uniformly to that function on the set itself.

We first define the following open sets $\forall\, k \in I^+ \Longrightarrow$ $G_k \equiv \{z \mid z \in G \text{ and } d(z, \mathcal{C}G) > \frac{1}{k}\}$. Then since $\frac{1}{k} > \frac{1}{k+1}$, by the manner in which G_k is defined gives us that $G_k \subseteq G_{k+1}, \forall\, k$. We next prove that $G = \bigcup_{k=1}^{\infty} G_k$. First, $k \Longrightarrow G_k \subseteq G$ so that $\bigcup_{k=1}^{\infty} G_k \subseteq G$. Let $z \in G$. Then G being open $\Longrightarrow \mathcal{C}G$ is closed and (z) being compact $\Longrightarrow d(z, \mathcal{C}G) > 0$ by T4.2. Thus, $\exists\, k \in I^+$ $\ni d(z, \mathcal{C}G) > \frac{1}{k} \Longrightarrow z \in G_k$ (since $z \in G$) $\Longrightarrow z \in \bigcup_{k=1}^{\infty} G_k$ and thus $G \subseteq \bigcup_{k=1}^{\infty} G_k$ so that $G = \bigcup_{k=1}^{\infty} G_k$.

We next prove that $\forall\, k \Longrightarrow G_k \subseteq G$. Assume the contrary, i.e., suppose $\exists\, k \in I^+ \ni \mathcal{K}G_k \not\subseteq G$. Then $\exists\, \mathcal{S} \in \mathcal{K}G_k \cap \mathcal{C}G$. Hence, $\mathcal{S} \in \mathcal{K}G_k \Longrightarrow N_{\frac{1}{k}}(\mathcal{S}) \cap \mathcal{C}G \neq \emptyset$. Choose $z \ni z \in N_{\frac{1}{k}}(\mathcal{S}) \cap \mathcal{C}G$. Then $d(z, \mathcal{S}) < \frac{1}{k}$ but since $z \in G_k$ and $\mathcal{S} \in \mathcal{C}G$, we have that $d(z, \mathcal{S}) \geq d(z, \mathcal{C}G) > \frac{1}{k}$, a contradiction. Thus, $\forall\, k \Longrightarrow \mathcal{K}G_k \subseteq G$.

Since $\mathcal{K}G_k \subseteq G$ which is bounded $\Longrightarrow \mathcal{K}G_k$ is bounded and $\mathcal{K}G_k$ being closed $\Longrightarrow$ by Heine-Borel that $\mathcal{K}G_k$ is compact, $\forall\, k \in I^+$. We are thus ready to apply T6.6, which gives us that $\forall\, n \in I^+ \Longrightarrow [\exists$ a polynomial $P_n(z) \ni \forall\, z \in G_n \subseteq \mathcal{K}G_n$ which is compact $\Longrightarrow |f(z) - P_n(z)| < \frac{1}{n}$].

We now fix $k \in I^+$. Let $\varepsilon > 0$ be given. Choose $n_o > k \ni \frac{1}{n_o} < \varepsilon$. Then $\forall\, n \geq n_o$ and $\forall\, z \in G_k \subseteq G_{n_o} \subseteq G_n$ implies that $|f(z) - P_n(z)| < \frac{1}{n} \leq \frac{1}{n_o} < \varepsilon$. Thus, $P_n(z) \Longrightarrow f(z)$ on G_k. Since this can be done for $k \in I^+$, we have that the sequence of polynomials $\{P_n(z)\}$ converges to $f(z)$ in an open nhood of every point in G, i.e., $z \in G = \bigcup_{k=1}^{\infty} G_k \Longrightarrow \exists\, k \ni z \in G_k$ and $P_n(z) \Longrightarrow f(z)$ on $G_k \Rightarrow P_n(z) \overset{\sim}{\Longrightarrow} f(z)$ on G by T6.4. Thus f(z) is the limit of a sequence of almost uniformly convergent polynomials.

We are now ready to finally prove Cauchy's Theorem, which is rather easily handled now for a bounded simply connected domain in light of what has already been proved.

T6.8 H Consider the bounded simply connected domain $D \subseteq Z$ and the analytic function $f: D \to Z$. Also, let C be a piecewise differentiable closed curve $\ni C \subseteq D$.

C $\exists \int_C f(z)dz = 0$. (Cauchy's Theorem)

P By R6.5, since every polynomial has an antiderivative and since C is a piecewise differentiable closed curve, we have that $n \in I^+ \Longrightarrow \int_C P_n(z)dz = 0$, where $\{P_n(z)\}_{n=1}^{\infty}$ is the sequence from T6.7 $\ni P_n(z) \overset{\sim}{\Longrightarrow} f(z)$ on D. And, since $P_n(z) \overset{\sim}{\Longrightarrow} f(z)$ on D and C is compact $\subseteq D \Rightarrow P_n(z) \Longrightarrow f(z)$ on C, by D6.7.

Now, since C and $\{P_n(z)\}_{n=1}^{\infty}$ satisfy the hypothesis of T6.5, we must have that $\exists \lim_{n\to\infty} \int_C P_n(z)dz = \int_C f(z)dz$.

But, since $n \in I^+ \Longrightarrow \int_C P_n(z)dz=0$, our result is that $\int_C f(z)dz=0$, and Cauchy's Theorem is proved.

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VITA

Raoul Louis Weinstein

Born in Richmond, Virginia, June 10, 1940. Graduated from Newport News High School in June 1957. He received a Bachelor of Arts degree from the University of Richmond, June 1961. Entered employment with the Engineering Research Laboratory of the Newport News Shipbuilding and Dry Dock Company in June 1961. Entered the University of Rochester as a graduate computing center assistant in September 1962, and the College of William and Mary in September 1963 as a graduate assistant.

He joined the faculty of Christopher Newport College in September 1964 as an Assistant Instructor of Mathematics.

In August 1965, the author will have completed the requirements for and receive the Master of Arts degree from the College of William and Mary.