Exact Multiplicity of Positive Solutions for a p-Laplacian Equation with Positive Convex Nonlinearity*

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Abstract

A p-Laplacian nonlinear elliptic equation with positive and p-superlinear nonlinearity and Dirichlet boundary condition is considered. We first prove the existence of two positive solutions when the spatial domain is symmetric or strictly convex by using a priori estimates and topological degree theory. For the ball domain in $\mathbb{R}^N$ with $N \geq 4$ and the case that $1 < p < 2$, we prove that the equation has exactly two positive solutions when a parameter is less than a critical value. Bifurcation theory and linearization techniques are used in the proof of the second result.

Keywords: p-Laplacian; positive solution; existence; exact multiplicity; topological degree; bifurcation.

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1 Introduction

The p-Laplacian is a second order quasilinear differential operator which arises from the studies of nonlinear phenomena in non-Newtonian fluids, reaction-diffusion prob-

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lems, non-linear elasticity, torsional creep problem, glacelogy, radiation of heat, etc. [37, 46, 51].

In this paper, we consider the positive solutions of Dirichlet boundary value problem

$$\begin{cases} \Delta_p u + \lambda f(x,u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 1$ is the p-Laplacian operator, Ω is a bounded smooth domain in $\mathbb{R}^N$ with $N > p$, λ is a nonnegative parameter, and the function f is at least continuous.

The existence and multiplicity of positive solutions to (P_λ) have been investigated extensively in recent years by using various methods. Existence of (possibly multiple) positive solutions of (P_λ) for various nonlinear functions f have been proved in, for example, [5, 7, 10, 12, 18, 23, 39, 40, 48, 58] for a higher dimensional bounded domain, [14, 20, 28, 29, 34, 38] for radially symmetric solutions on a ball or an interval. Usually the existence of positive solutions can be proved via variational methods, topological degree theory and other methods.

The uniqueness and exact multiplicity of the positive solution of (P_λ) is a much harder question. For a p-sublinear problem, the uniqueness of the positive solution of (P_λ) was proved in [22] (see also [33]) for a general bounded domain. For radially symmetric solutions, the uniqueness of radially symmetric positive solution on a finite ball, an annulus, or entire space was studied in [11, 25, 27, 41, 56, 59] by using a monotone separation theorem and related Pohozaev type equalities. An approach based on maximum principle, linearized equations and implicit function theorem was used in [1, 2] to prove the nondegeneracy and uniqueness of positive solution of (P_λ) on a ball. For the one-dimensional problem, the equation (P_λ) can be integrated via a quadrature method and the uniqueness or exact multiplicity of positive solutions of (P_λ) can be proved by analyzing the associated time-map, see [13, 21, 35, 36, 62]. Very recently, the exact multiplicity of positive solutions of (P_λ) on an annulus was showed in [57] by using the Kolodner-Coffman method. Note that except the one-dimensional case, all these previous results are about the uniqueness of positive solutions, and we do not know any exact multiplicity result with more than one solutions for the positive solutions of (P_λ), even in the radially symmetric case.

For the case $p = 2$ (Laplacian) and ball domains, exact multiplicity of positive solutions of (P_λ) has been considered under various hypotheses on f, see [42, 43, 44, 49, 50] and references therein. One of new ingredients in these works is a saddle-node bifurcation theorem for a mapping defined in a Banach space [15], and the turning direction of a solution curve at a bifurcation point can be determined from the properties of the nonlinearity f.

Central to the implicit function theorem and bifurcation method for the solvability of partial differential equations is the differentiability of the nonlinear operator generated by the equation. When $p = 2$, the p-Laplacian operator becomes the classical Laplacian operator, which is linear and clearly differentiable. When $p \neq 2$, the differentiability of the nonlinear p-Laplacian operator is an obstacle for applying the implicit function theorem and a bifurcation method. In [8], the differentiability of the inverse of the p-Laplacian operator was proved for $p \neq 2$, and this result was used in [30, 54] to obtain the uniqueness and a global bifurcation diagram of the positive solutions of (P_λ) for p-sublinear $f(x, s)$. That is, f satisfies $(p-1)f(x, s)-f_s(x, s)s \geq 0$ for $x \in \Omega$ and $s \geq 0$. Note that the uniqueness of a positive solution for a p-superlinear case, that is, $(p-1)f(x, s) - f_s(x, s)s < 0$, has been proved in [1, 2, 41], and it is also shown that the positive solution has Morse index 1 in that case, while the solution in the p-sublinear case is stable (Morse index 0). These two results indicate that when $(p-1)f(x, s) - f_s(x, s)s$ does not change sign, then any positive solution of (P_λ) is non-degenerate and the implicit function theorem can be used for a continuation of the solution curve. Hence, all positive solutions of (P_λ) lie on a continuous curve with no turning point, from which the uniqueness of positive solutions can be obtained.

Our main result in this paper is that when $f(x, s) \equiv f(s)$ is a positive, monotone increasing and convex nonlinearity, under a suitable subcritical growth condition on f (see (3.35)) and if Ω is a unit ball in $\mathbb{R}^N$, $(N \geq 4)$, then there exists $\lambda^* > 0$, such that for $1 < p < 2$, (P_λ) has no positive solution for $\lambda > \lambda^*$, has exactly one positive solution for $\lambda = \lambda^*$, and has exactly two positive solutions for $0 < \lambda < \lambda^*$. Moreover all positive solutions of (P_λ) lie on a single smooth solution curve Σ, which starts from $(\lambda, u) = (0, 0)$, continues as the minimal solution to some saddle-node bifurcation point (λ^*, u^*), then turns back and continues as the large solution to $(\lambda, u) = (0, \infty)$. Thus Σ is a $\supset$-shaped curve.

This type of multiplicity result for positive and convex f was first proved in [15] for an ordinary differential equation in form of $u'' + \lambda f(u) = 0$ for $x \in (0, \pi)$ with Dirichlet boundary condition, and it was generalized in [16] to a semilinear elliptic partial differential equation in form of $\Delta u + \lambda f(u) = 0$ for $x \in \Omega \subset \mathbb{R}^N$ with Dirichlet boundary condition. In these works, the existence of two positive solutions was proved under a subcritical growth condition on f. For the ball in $\mathbb{R}^N$, the global bifurcation diagram for (P_λ) with $p = 2$ and $f(x, u) = f(u)$ is exactly $\supset$-shaped, see [50]. Hence our main result in this paper is a generalization of this well-known exact multiplicity result for the semilinear case.

To establish the exact multiplicity of positive solution of (P_λ) with $1 < p < 2$ on a ball, we need to resolve several difficulties: (i) the C^2 differentiability of the nonlinear mapping defined by the equation; (ii) non-oscillatory property of the solution of the linearized equation. We adopt the bifurcation approach in [15, 16, 50] to a nonlinear mapping involving the inverse of the p-Laplacian operator as in [30, 54], in the functional setting of [2]. We further extend the C^1 differentiability of the inverse of the p-Laplacian operator in [8] to C^2 differentiability, so the turning direction at a turning point of bifurcation curve can be calculated. We also prove the positivity of solution of the linearized equation of (P_λ) by an approach of [50]. By overcoming these difficulties caused by the nonlinearity of the p-Laplacian operator, we obtain a complete characterization of the set of positive solutions of (P_λ) for $1 < p < 2$ and Ω being the unit ball in $\mathbb{R}^N$ with $N \geq 4$. As far as we know, it is the first exact multiplicity result for radially symmetric solutions of (P_λ) when $N \geq 2$.

The outline of the paper is as follows. In Section 2, we first present some preliminary lemmas. Then we prove the existence of two positive solutions of (P_λ) when λ is in a certain range (Theorem 2.5). The exact multiplicity of solutions to (P_λ) is analysed in Section 3. The main result of our paper is Theorem 3.9.

2 Preliminaries

2.1 Solutions of p-Laplace equations

Throughout this paper we assume $f \in C(\overline{\Omega} \times [0,\infty), (0,\infty))$ and satisfies the growth condition:

$$0 < f(x,s) \leq C_0(1 + s^{q-1}) \quad \text{for all } x \in \overline{\Omega} \text{ and } s \in [0,\infty), \tag{2.1}$$

where $C_0 > 0$ and $q > 1$ are constants.

A function $u \in W_0^{1,p}(\Omega)$ is said to be a weak solution to problem (P_λ) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx = \lambda \int_\Omega f(x,u) v dx \quad \text{for all } v \in C_c^\infty(\Omega).$$

If $q \leq Np/(N-p)$ in (2.1), then every solution u to problem (P_λ) belongs to $L^\infty(\Omega)$. Indeed, for subcritical case $q < Np/(N-p)$, this is a consequence of the result in Serrin [55] on the L^∞-estimates, and for critical case $q = Np/(N-p)$, a proof of this fact could be found in Peral [52, Appendix E]. Consequently $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ due to interior regularity in Tolksdorf [60, Theorem 1] and regularity near the boundary in Lieberman [45, Theorem 1]. Moreover it follows from the strong maximum principle in Vázquez [61] that $u > 0$ in Ω and $\partial u/\partial \nu < 0$ on $\partial\Omega$, since 0 is not a solution to problem (P_λ).

Let $\mathcal{K}_1 := \{u \in C_0^1(\overline{\Omega}) : u \geq 0\}$ be the ordered cone in $C_0^1(\overline{\Omega})$. Define a mapping $T : [0,\infty) \times \mathcal{K}_1 \to \mathcal{K}_1$ by

$$T(\lambda, u) := (-\Delta_p)^{-1}(\lambda N_f)u \;\; \text{for } (\lambda, u) \in [0,\infty) \times \mathcal{K}_1.$$

Here $(-\Delta_p)^{-1} : L^\infty(\Omega) \to C_0^1(\overline{\Omega})$ is the inverse p-Laplacian operator for a bounded domain Ω with $\partial\Omega \in C^{1,\beta}$ for some $\beta \in (0,1)$. It is completely continuous and order-preserving (see, e.g., [7, Lemma 1.1]). The Nemitskii operator $N_f : \mathcal{K}_1 \to L^\infty(\Omega)$ is defined as

$$N_f(u)(x) = f(x, u(x)) \quad \text{for } u \in \mathcal{K}_1 \text{ and } x \in \Omega,$$

and it is continuous and maps bounded sets in $\mathcal{K}_1$ into bounded sets in $L^\infty(\Omega)$. Hence T is well-defined, and it is completely continuous on $[0,\infty)\times\mathcal{K}_1$. Furthermore, problem (P_λ) has a positive solution u if and only if $T(\lambda,\cdot)$ has a fixed point u in $\mathcal{K}_1$ for $\lambda > 0$.

Next we recall a well-known fixed point index theorem which will be used to prove the existence of positive solutions of (P_λ).

Theorem 2.1. ([32]) *Let X be a Banach space, $\mathcal{K}$ a cone in X and $\mathcal{O}$ bounded and open in X. Let $0 \in \mathcal{O}$ and $A : \mathcal{K}\cap\overline{\mathcal{O}} \to \mathcal{K}$ be completely continuous. Suppose that $Ax \neq \nu x$ for all $x \in \mathcal{K}\cap\partial\mathcal{O}$ and all $\nu \geq 1$. Then $i(A, \mathcal{K}\cap\mathcal{O}, \mathcal{K}) = 1$.*

2.2 Existence of two positive solutions

In this subsection we prove the existence of two positive solutions of (P_λ) when Ω and f satisfy certain conditions. For the nonlinear eigenvalue problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

it is known that the principal eigenvalue λ_1^* for problem (2.2) exists, and $\lambda_1^* \in (0,\infty)$ is a simple isolated eigenvalue for problem (2.2) with the associated eigenfunction $w^* > 0$ in Ω (see, *e.g.*, [6]).

The following lemma gives an extent of the parameter λ for which (P_λ) has at least one positive solution. Assume that $f(x,s)$ satisfies:

$(H0)$ There exist positive constants C_1, C_2 and q such that $q \in \left(p, \dfrac{Np}{N-p}\right]$ and

$$C_1 s^{p-1} < f(x,s) < C_2(1+s^{q-1}) \quad \text{for all } x \in \overline{\Omega} \text{ and } s \in [0,\infty).$$

Lemma 2.2. *Suppose f satisfies $(H0)$. Then*

1. *(P_λ) has no positive solution for $\lambda \geq \lambda_1^* C_1^{-1}$.*
2. *there exists $\lambda^* > 0$ such that (P_λ) has no positive solution for $\lambda > \lambda^*$, and (P_λ) has a minimal positive solution for $0 < \lambda < \lambda^*$.*

Proof. 1. We first provide an upper bound of the parameter λ for which (P_λ) has a positive solution. Assume on the contrary that there exists a positive solution u_λ of problem (P_λ) with $\lambda \geq \lambda_1^* C_1^{-1}$. Let $u = w^*$, $v = u_\lambda$, $A = B = 1$, $a(x) = \lambda_1^*$, $b(x) = \lambda f(x,u_\lambda)/u_\lambda^{p-1}$ in [3, Theorem 1]. Then $a(x) \leq b(x)$ in Ω, and

$$\int_\Omega L(w^*, u_\lambda)dx \leq 0,$$

since $u_\lambda > 0$ in Ω. Here

$$L(w^*, u_\lambda) = |w^*|^p - p\left(\frac{w^*}{u_\lambda}\right)^{p-1} |u_\lambda|^{p-2}\nabla u_\lambda \nabla w^* + (p-1)\left(\frac{w^*}{u_\lambda}\right)^p |\nabla u_\lambda|^p. \tag{2.3}$$

On the other hand, $L(u_\lambda, w^*) \geq 0$ by Picone's identity (see, for example, [4, Theorem 1.1]). Thus $L(u_\lambda, w^*) = 0$, for almost everywhere in Ω, which implies $u_\lambda = kw^*$ for some constant k, and one can easily reach a contradiction.

2. Let $\bar{w}$ be the unique positive solution of

$$\begin{cases} \operatorname{div}(|\nabla \bar{w}|^{p-2}\nabla \bar{w}) + 1 = 0, & x \in \Omega, \\ \bar{w} = 0, & x \in \partial\Omega. \end{cases}$$

Then $\bar{w} \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$ by well-known regularity results (*e.g.*, [60, Theorem 1] and [45, Theorem 1]). Moreover $\bar{w} > 0$ in Ω and $\partial\bar{w}/\partial\nu < 0$ on $\partial\Omega$ by [61, Theorem 5]. One can see that $\bar{w}$ is a super-solution of problem (P_λ) for sufficiently small $\lambda > 0$. Since 0 is a sub-solution to problem (P_λ) for each $\lambda > 0$, then there exists a positive solution $u(\lambda)$ to problem (P_λ) for small $\lambda > 0$ such that $0 < u(\lambda) \leq \bar{w}$ in Ω. Setting

$$\lambda^* := \sup\{\lambda : (P_\lambda) \text{ has a positive solution}\}, \tag{2.4}$$

then it follows from the result of part 1 that $\lambda^* \in (0, \infty)$ is well defined. For small $\delta > 0$, there exists $\bar{\lambda} \in (\lambda^* - \delta, \lambda^*)$ such that problem $(P_{\bar{\lambda}})$ has a positive solution $\bar{u}$. Since the solution $\bar{u}$ is a super-solution of (P_λ) for all $0 < \lambda < \bar{\lambda}$ and 0 is a sub-solution of (P_λ) for all $\lambda > 0$, problem (P_λ) has at least one positive solution for all $0 < \lambda \leq \bar{\lambda}$, and thus problem (P_λ) has a minimal positive solution $u_m(\lambda)$ for all $\lambda \in (0, \lambda^*)$ such that $u_m(\lambda)$ is nondecreasing with respect to λ in view of [24, Theorem 4.11]. $\square$

To obtain the existence of a second positive solution of (P_λ), we give an additional assumption on $f(x,s)$:

$(H1)$ There exist positive constants C_3, C_4 and q such that $q \in \left(p, \frac{(N-1)p}{N-p}\right]$ and

$$C_3(1 + s^{q-1}) \leq f(x,s) \leq C_4(1 + s^{q-1}) \quad \text{for all } x \in \Omega \text{ and } s \in [0, \infty).$$

Definition 2.3. *Assume $\Omega \subseteq \mathbb{R}^N$ is open, bounded and $0 \in \Omega$. Let e_i be the i-th unit vector in $\mathbb{R}^N$, i.e., $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$. We say that Ω is Steiner symmetric w.r.t. the coordinate hyperplane $H_i = \{x : x \cdot e_i = 0\}$ if the intersection of Ω with lines $\{P + te_i : t \in \mathbb{R}\}$, $P \in H_i$ is either empty or of the form $\{P + te_i : t \in (-t_0(P), t_0(P))\}$. We say that Ω is n-fold Steiner symmetric if Ω is Steiner symmetric w.r.t. the n coordinate hyperplanes $H_1, \cdots, H_n$.*

The following lemma can be proved by using the scaling argument of C. Azizieh and P. Clément [7]. For the sake of completeness, we give the proof of it.

Lemma 2.4. *Assume that $(H1)$ is satisfied and one of the following is also satisfied:*

1. *$f(x,s)$ is strictly decreasing in x_i for all $i = 1, 2, \cdots, N$, when Ω is a bounded n-fold Steiner symmetric domain in $\mathbb{R}^N$;*
2. *$f(x,s) = f(|x|,s)$ is weakly decreasing in the first variable when $\Omega = B_1(0)$ is a unit ball;*
3. *$f(x,s) = f(s)$ is locally Lipschitz continuous on $(0,\infty)$ when Ω is a strictly convex domain in $\mathbb{R}^N$ with C^2 boundary and $1 < p \le 2$.*

Then for given $\epsilon > 0$ and $M > 0$, there exists a constant $C(\epsilon, M)$ such that

$$\|u_\lambda\|_{C^1} \le C(\epsilon, M),$$

for any positive solution u_λ of problem (P_λ) with $\lambda \in [\epsilon, M]$.

Proof. Assume on the contrary that there is a sequence $\{(\lambda_n, u_n)\}_{n=1}^\infty$ of solutions to problem (P_{λ_n}) such that $\lambda_n \in [\epsilon, M]$ and $\|u\|_\infty \to \infty$ as $n \to \infty$. In view of the symmetry results for positive solutions due to Brock [9], u_n attains its maximum only at $x = 0$ for all $n \in \mathbb{N}$ (see, e.g., [26, Section 2]). Therefore the function $v_n(x) := c_n^{-1} u_n(c_n^{-\gamma} x)$ with $\gamma = (q-p)/p$ and $c_n = \|u_n\|_\infty = u_n(0)$ is defined in $B_n := B_{c_n^\gamma}(0)$ and satisfies

$$\begin{cases} \int_{B_n} |\nabla v_n|^{p-2} \nabla v_n \cdot \nabla \psi dx = \lambda_n c_n^{1-q} \int_{B_n} f(c_n^{-\gamma} x, c_n v_n(x)) \psi dx, \\ v_n \in C_0^1(\overline{B}_n), \ v_n \ge 0 \text{ on } B_n, \ \|v_n\|_\infty = 1 \end{cases} \tag{2.5}$$

for all $\psi \in C_c^\infty(B_n)$. By $(H1)$,

$$\epsilon C_3 (c_n^{1-q} + v_n^{q-1}(x)) \le \lambda_n c_n^{1-q} f(c_n^{-\gamma} x, c_n v_n(x)) \le M C_4 (c_n^{1-q} + v_n^{q-1}(x)) \quad \text{in } B_n.$$

Fixing $n_0 > 0$ and denoting $B = B_{n_0}$, it follows from Lieberman [45, Theorem 1] that there exist $K > 0$ and $\alpha \in (0,1)$ depending only on N, p, B such that

$$v_n \in C^{1,\alpha}(\overline{B}) \text{ and } \|v_n\|_{C^{1,\alpha}(\overline{B})} \le K, \quad \text{for all } n \ge n_0.$$

Thus there exist a function v_B and a convergent subsequence $v_{n'} \to v_B$ in $C^1(\overline{B})$ in view of Arzelà-Ascoli theorem. Taking test functions in $C_c^\infty(B)$ in (2.5) and passing to the limit, we have

$$\begin{cases} \int_B |\nabla v_B|^{p-2} \nabla v_B \cdot \nabla \psi dx \ge \epsilon C_3 \int_B v_B^{q-1} \psi dx \quad \text{for all } \psi \in C_c^\infty(B), \\ v_B \in C^1(\overline{B}), \ v_B \ge 0 \text{ in } B, \ \|v_B\|_\infty = 1, \end{cases}$$

Moreover, by the strong maximum principle due to Vázquez [61], $v_B(x) > 0$ for all $x \in B$ since $v_B \not\equiv 0$. Taking balls bigger and bigger we obtain the existence of a function $v \in C^1(\mathbb{R}^N)$ satisfying

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \cdot \nabla\psi dx \geq \epsilon C_3 \int_{\mathbb{R}^N} v^{q-1}\psi dx, & \text{for all } \psi \in C_c^\infty(\mathbb{R}^N), \\ v > 0 \text{ in } \mathbb{R}^N, \ \|v\|_\infty = 1, \end{cases}$$

which contradicts a nonexistence result for differential inequalities by Mitidieri and Pohozaev [47].

For part 3, by [7, Proposition 4.1], there exists $\delta > 0$ such that $d(x_n, \partial\Omega) \geq \delta > 0$ for all $n \in \mathbb{N}$ where $u_n(x_n) = \|u_n\|_\infty$. The function $v_n(x) := c_n^{-1} u_n(c_n^{-\gamma} x + x_n)$ with $\gamma = (q-p)/p$ is defined on $B_n := B_{c_n^\gamma \delta}(0)$. By similar argument as the above, we can obtain the same result. □

The main existence result of this subsection is the following theorem.

Theorem 2.5. *Assume that the hypotheses in Lemma 2.4 are satisfied. Assume in addition that $f(x,s)$ is nondecreasing in s. Then problem (P_λ) has at least two positive solutions for $\lambda \in (0, \lambda^*)$, one positive solution for $\lambda = \lambda^*$ and no positive solution for $\lambda > \lambda^*$.*

Proof. By definition of λ^* and an easy compactness argument, problem (P_λ) has a positive solution for $\lambda \in (0, \lambda^*]$ and no solutions for $\lambda > \lambda^*$. We will show that (P_λ) has at least two positive solutions for all $\lambda \in (0, \lambda^*)$.

Let λ be fixed with $0 < \lambda < \lambda^*$, and let u_* be a positive solution of (P_{λ^*}). Consider the modified problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda \overline{f}(x,u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \qquad (\overline{P}_\lambda)$$

where $\overline{f}(x,s) : \Omega \times \mathbb{R} \to [0,\infty)$ is defined as

$$\overline{f}(x,s) = \begin{cases} f(x, u_*(x)), & \text{if } s > u_*(x), \\ f(x,s), & \text{if } 0 \leq s \leq u_*(x), \\ f(x,0), & \text{if } u < 0. \end{cases}$$

Note that $\overline{f}(x,s) \leq f(x, u_*(x))$ for all $x \in \Omega$ and $s \in \mathbb{R}$ since $f(x,s)$ is nondecreasing in s.

Define $M : \mathcal{K}_1 \to \mathcal{K}_1$ as

$$Mu := (-\Delta_p)^{-1}(\lambda N_{\overline{f}})u \text{ for } u \in \mathcal{K}_1.$$

Then M is completely continuous on $\mathcal{K}_1$, and u is a positive solution of $(\overline{P}_\lambda)$ if and only if $u = Mu$ on $\mathcal{K}_1$. By the property of $(-\Delta_p)^{-1}$ and boundedness of $\overline{f}$, there exists $R_1 > 0$ such that $\|Mu\|_{C^1} < R_1$ for all $u \in \mathcal{K}_1$. Applying Theorem 2.1 with $O = B_{R_1}$, we get

$$i(M, B_{R_1} \cap \mathcal{K}_1, \mathcal{K}_1) = 1. \tag{2.6}$$

Putting

$$\Sigma = \left\{ u \in C_0^1(\overline{\Omega}) : |u| < u_* \text{ in } \Omega,\ \|u\|_{C^1} < R_1 \text{ and } \left|\frac{\partial u}{\partial \nu}\right| < \left|\frac{\partial u_*}{\partial \nu}\right| \text{ on } \partial\Omega \right\},$$

then Σ is bounded and open in $C_0^1(\overline{\Omega})$. For any positive solution $\bar{u}$ of $(\overline{P}_\lambda)$, it follows from the strong comparison principle due to Cuesta and Takáč [17, Theorem 2.1] that $\bar{u} \in \Sigma \cap \mathcal{K}_1$. By (2.6) and excision property, we get

$$i(M, \Sigma \cap \mathcal{K}_1, \mathcal{K}_1) = i(M, B_{R_1} \cap \mathcal{K}_1, \mathcal{K}_1) = 1. \tag{2.7}$$

Since problem (P_λ) is equivalent to problem $(\overline{P}_\lambda)$ on $\Sigma \cap \mathcal{K}_1$, we conclude that problem (P_λ) has a positive solution in $\Sigma \cap \mathcal{K}_1$. We may assume $T(\lambda, \cdot)$ has no fixed point in $\partial\Sigma \cap \mathcal{K}_1$, otherwise the proof is done. Then $i(T(\lambda, \cdot), \Sigma \cap \mathcal{K}_1, \mathcal{K}_1)$ is well defined and by (2.7), we have

$$i(T(\lambda, \cdot), \Sigma \cap \mathcal{K}_1, \mathcal{K}_1) = 1. \tag{2.8}$$

It follows from Lemma 2.2 that problem (P_{λ_0}) has no solution in $\mathcal{K}_1$ for $\lambda_0 > \lambda_1^* C_1^{-1}$. By Lemma 2.4, there exists R_2 $(> R_1)$ such that for all possible positive solutions u of problem (P_μ) with $\mu \in [\lambda, \lambda_0]$, we have

$$\|u\|_{C^1} < R_2.$$

Define $h : [0,1] \times (\overline{B}_{R_2} \cap \mathcal{K}_1) \to \mathcal{K}_1$ by

$$h(\tau, u) = T(\tau\lambda_0 + (1-\tau)\lambda, u).$$

Then h is completely continuous on $[0,1] \times \mathcal{K}_1$ and $h(\tau, u) \neq u$ for all $(\tau, u) \in [0,1] \times (\partial B_{R_2} \cap \mathcal{K}_1)$. By homotopy invariance and the solution properties, we obtain

$$i(T(\lambda, \cdot), B_{R_2} \cap \mathcal{K}_1, \mathcal{K}_1) = i(T(\lambda_0, \cdot), B_{R_2} \cap \mathcal{K}_1, \mathcal{K}_1) = 0.$$

Thus by the additivity property and (2.8), we get

$$i(T(\lambda, \cdot), (B_{R_2} \setminus \overline{\Sigma}) \cap \mathcal{K}_1, \mathcal{K}_1) = -1.$$

This implies that problem (P_λ) has another positive solution in $(B_{R_2} \setminus \overline{\Sigma}) \cap \mathcal{K}_1$, and consequently the proof is complete. □

2.3 Bifurcation theory

For discussing the exactly multiplicity of solutions of (P_λ), we recall a bifurcation theorem by Crandall and Rabinowitz [15]:

Theorem 2.6. *Let X and Y be Banach spaces, let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F : U \to Y$ be a continuously differentiable mapping. Assume that $F(\lambda_0, u_0) = 0$. At (λ_0, u_0), F satisfies*

(1) *$dim N(F_u(\lambda_0, u_0)) = codim R(F_u(\lambda_0, u_0)) = 1$, and $N(F_u(\lambda_0, u_0)) = span\{w\}$,*

(2) *$F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.*

Let Z be any complement of $span\{w\}$ in X. Then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) form a curve $\{(\lambda(s), u(s)) : s \in I = (-\epsilon, \epsilon)\}$, where $\lambda : I \to \mathbb{R}$, $z : I \to Z$ are C^1 functions such that $u(s) = u_0 + sw + sz(s)$, $\lambda(0) = \lambda_0$, $\lambda'(0) = 0$, $z(0) = z'(0) = 0$. Moreover if F is C^2 near (λ_0, u_0), then $(\lambda(s), u(s))$ is also in C^2 class, and

$$\lambda''(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w, w]\rangle}{\langle l, F_\lambda(\lambda_0, u_0)\rangle},$$

where $l \in Y^*$ satisfying $N(l) = R(F_u(\lambda_0, u_0))$.

3 Exact multiplicity

In this section we assume that $\Omega = B$, the unit ball in $\mathbb{R}^N$, and $f(x, s) = f(s)$ for $s \in [0, \infty)$. If $1 < p \leq 2$, then in view of Gidas, Ni and Nirenberg [31] and Damascelli and Pacella [19], positive solutions to problem (P_λ) are radially symmetric, and hence problem (P_λ) can be equivalently written as

$$\begin{cases} (r^{N-1}|u'|^{p-2}u')' + \lambda r^{N-1} f(u) = 0, & r \in (0, 1), \\ u'(0) = u(1) = 0. \end{cases} \tag{3.1}$$

From now on, we suppose f satisfies $(H1)$ and

$(H2)$ $f \in C^2([0, \infty), (0, \infty))$;

$(H3)$ $f'(s) > 0$ and $f''(s) > 0$ for all $s \in [0, \infty)$.

For convenience, we use some notations given by Genoud [30]. For $p > 1$, set

$$p^* := \frac{1}{p-1} \text{ and } p' := p^* + 1 = \frac{p}{p-1}.$$

Define $\phi_p(\xi) := |\xi|^{p-2}\xi$, for $\xi \in \mathbb{R}$. It is well known that $\phi_p : \mathbb{R} \to \mathbb{R}$ is continuous and its inverse function $\phi_{p'}$ is p^*-homogeneous, that is, $\phi_{p'}(t\xi) = t^{p^*}\phi_{p'}(\xi)$, for $t \in [0, \infty)$.

Let (λ, u) be a positive solution of (3.1). Then

$$u'(r) = -\lambda^{p^*}\left(\frac{1}{r^{N-1}}\int_0^r \tau^{N-1}f(u(\tau))d\tau\right)^{p^*}, \quad r \in (0,1].$$

Clearly $u'(r) < 0$ for $r \in (0, 1]$. Moveover, one can derive a precise behavior of u near the origin using the L'Hospital's rule, see Aftalion and Pacella [2]. Indeed,

$$\lim_{r\to 0^+} r^{-p^*}u'(r) = -\left(\frac{\lambda}{N}f(u(0))\right)^{p^*}, \tag{3.2}$$

and

$$\lim_{r\to 0^+} r^{-(p^*-1)}u''(r) = -p^*\left(\frac{\lambda}{N}f(u(0))\right)^{p^*}. \tag{3.3}$$

We will work in various function spaces. For $n = 0, 1, 2$, $C^n[0, 1]$ denotes the space of n times continuously differentiable functions defined in $[0, 1]$, with the usual Sup norm $||\cdot||_{C^n[0,1]}$, and $W^{1,1}(0, 1)$ is the Sobolev space. Following [2], define

$$\begin{aligned} X := &\{v \in C^2(0,1] \cap C^1[0,1] : v'(0) = v(1) = 0,\\ &\text{with } |v'(x)| \le cx^{p^*},\ |v''(x)| \le cx^{p^*-1},\quad x \in (0,1]\}, \end{aligned} \tag{3.4}$$

where $c > 0$, then X is a Banach space with the norm (see [2])

$$||v||_X = ||v||_{C^2[1/2,1]} + \sup_{0<s<1/2}\frac{|v'(s)|}{s^{p^*}} + \sup_{0<s<1/2}\frac{|v''(s)|}{s^{p^*-1}}. \tag{3.5}$$

If (λ, u) is a positive solution of (3.1), then $u \in X$ by (3.2) and (3.3).

For any $h \in C^0[0, 1]$, the problem

$$\begin{cases} (r^{N-1}\phi_p(u'))' + r^{N-1}h(r) = 0, & r \in (0,1),\\ u'(0) = u(1) = 0, \end{cases} \tag{3.6}$$

has a unique solution $u(h) \in C^1[0, 1]$ given by

$$u(h)(r) = \int_r^1 \phi_{p'}\left[\int_0^s (\frac{t}{s})^{N-1}h(t)dt\right]ds. \tag{3.7}$$

From (3.7), we define the following operators:

$$\begin{aligned} &S_p:\ C^0[0,1] \to C^1[0,1],\quad S_p(h)(r) := u(h)(r);\\ &J:\ C^0[0,1] \to C^1[0,1],\quad J(h)(s) := \int_0^s (\frac{t}{s})^{N-1}h(t)dt;\\ &\Phi_q:\ C^0[0,1] \to C^0[0,1],\quad \Phi_q(g)(s) := \phi_q(g(s)),\ \text{for any}\ \ q > 1;\\ &I:\ \ C^0[0,1] \to C^1[0,1],\quad I(k)(r) := \int_r^1 k(s)ds. \end{aligned} \tag{3.8}$$

Obviously we have $S_p = I \circ \Phi_{p'} \circ J$. The continuity and differentiability of the operator S_p has been proved in, for example, Theorem 3.3 and Theorem 3.5 of [30], which we recall here:

Lemma 3.1. *Suppose that $p > 1$ and $N \geq 1$. Let $S_p : C^0[0,1] \to C^1[0,1]$ be defined as in (3.7) and (3.8).*

1. *S_p is continuous, bounded and compact.*
2. *If in addition $1 < p < 2$, then S_p is continuously differentiable at any $h \in C^0[0,1]$, and for any $\bar{h}_1 \in C^0[0,1]$, we have*

$$DS_p(h)[\bar{h}_1] = p^* I(|J(h)|^{p^*-1} J(\bar{h}_1)). \tag{3.9}$$

Furthermore, $v = DS_p(h)[\bar{h}_1]$ is equivalent to $v \in C^1[0,1]$ and v satisfies

$$\begin{cases} (r^{N-1}|u(h)'(r)|^{p-2}v'(r))' + p^* r^{N-1}\bar{h}_1(r) = 0, & r \in (0,1), \\ v'(0) = v(1) = 0, \end{cases} \tag{3.10}$$

where $u(h) = S_p(h)$.

Note that by differentiating (3.7) one gets

$$|u(h)'|^{2-p} = |J(h)|^{p^*-1}. \tag{3.11}$$

So (3.10) comes from (3.9) and (3.11).

To prove the exact multiplicity of positive solutions, we also need the information of the twice differentiability of the operator S_p. To prove that, we state a lemma which is similar to Lemma 2.1 in [8], and we omit its proof.

Lemma 3.2. *Suppose that $\alpha > -2$ and $v_0 \in C^1[0,1]$ has only simple zeros. Then there exists a neighborhood V_0 of v_0 in $C^1[0,1]$ such that if $v \in V_0$ then $|v|^\alpha v \in L^1(0,1)$ and the mapping $v \to |v|^\alpha v$ is continuous from $V_0 \to L^1(0,1)$.*

We can now prove the twice differentiability of S_p as follows.

Lemma 3.3. *Suppose that $1 < p < 2$, $h_0 \in C^0[0,1]$ satisfying $h_0(r) > 0$ for $r \in [0,1]$, and let $u(h_0)$ be defined as in (3.7). Then there exists a neighborhood V_1 of h_0 in $C^0[0,1]$ such that the mapping $h \mapsto |J(h)|^{p^*-3}J(h) : V_1 \to L^1(0,1)$ is continuous, $S_p : V_1 \to W^{1,1}(0,1)$ is C^2, and for all $h \in V_1$, $\bar{h}_1$, $\bar{h}_2 \in C^0[0,1]$,*

$$\begin{aligned} D^2S_p(h)[\bar{h}_1, \bar{h}_2] =& p^*(p^*-1) I(|J(h)|^{p^*-3} J(h) J(\bar{h}_1) J(\bar{h}_2)) \\ =& p^*(p^*-1) I(|u(h)'|^{4-3p} J(h) J(\bar{h}_1) J(\bar{h}_2)), \end{aligned} \tag{3.12}$$

where $u(h) = S_p(h)$.

Proof. Define $v_0 := J(h_0) \in C^1[0,1]$. Then by (3.6) and (3.7),

$$u(h_0)'(r) = -\phi_{p'}(J(h_0)), \quad \text{or} \quad -\phi_p(u(h_0)'(r)) = J(h_0),$$

and

$$(r^{N-1}\phi_p(u(h_0)'(r)))' + r^{N-1}h_0(r) = 0, \quad r \in [0,1].$$

We claim that v_0 has a unique simple zero $r = 0$. From the condition $h_0(r) > 0$ as $r \in [0,1]$, we obtain $r = 0$ is the unique zero point of v_0. On the other hand, for all $r \in (0,1]$, from (3.8),

$$\begin{aligned} v_0'(r) = J(h_0)'(r) =&(1-N)r^{-N}\int_0^r r^{N-1}h_0(t)dt + h_0(r) \\ =&(1-N)r^{-1}J(h_0)(r) + h_0(r) = (1-N)r^{-1}v_0(r) + h_0(r). \end{aligned} \tag{3.13}$$

Since $v_0(0) = 0$, by (3.13),

$$v_0'(0) = (1-N)\lim_{r\to 0}\frac{v_0(r)}{r} + h_0(0) = (1-N)v_0'(0) + h_0(0).$$

It implies that $v_0'(0) = h_0(0)/N \neq 0$. Therefore, v_0 has only a simple zero $r = 0$ in $[0,1]$.

Now for $1 < p < 2$, from Lemma 3.2 and the continuity of $J(h)$, we can find a neighborhood V_1 of h_0 in $C^0[0,1]$ such that

$$\text{the mapping } h \mapsto |J(h)|^{p^*-3}J(h) : V_1 \to L^1(0,1) \text{ is continuous.} \tag{3.14}$$

For any $h \in V_1$, $\bar{h}_1$, $\bar{h}_2 \in C^0[0,1]$ and $\delta \in \mathbb{R}$,

$$\begin{aligned} &DS_p(h+\delta\bar{h}_2)[\bar{h}_1] - DS_p(h)[\bar{h}_1] \\ =&p^*\int_r^1 \left|\int_0^s (\frac{t}{s})^{N-1}(h(t)+\delta\bar{h}_2(t))dt\right|^{p^*-1} J(\bar{h}_1)ds - \int_r^1 \left|\int_0^s (\frac{t}{s})^{N-1}h(t)dt\right|^{p^*-1} J(\bar{h}_1)ds \\ =&p^*\int_r^1 \left[\left|\int_0^s (\frac{t}{s})^{N-1}(h(t)+\delta\bar{h}_2(t))dt\right|^{p^*-1} - \left|\int_0^s (\frac{t}{s})^{N-1}h(t)dt\right|^{p^*-1}\right] J(\bar{h}_1)ds \\ =&p^*(p^*-1)\int_r^1 |J(h+\theta\delta\bar{h}_2)|^{p^*-3}J(h+\theta\delta\bar{h}_2)\delta J(\bar{h}_2)J(\bar{h}_1)ds \end{aligned}$$

with some $\theta \in (0,1)$. By using (3.11), we have

$$\begin{aligned} &\lim_{\delta\to 0}\frac{DS_p(h+\delta\bar{h}_2)[\bar{h}_1] - DS_p(h)[\bar{h}_1]}{\delta} \\ =&p^*(p^*-1)\int_r^1 |J(h)|^{p^*-3}J(h)J(\bar{h}_2)J(\bar{h}_1)ds \\ =&p^*(p^*-1)\int_r^1 |u(h)'|^{4-3p}J(h)J(\bar{h}_2)J(\bar{h}_1)ds. \end{aligned} \tag{3.15}$$

Since $h_0(r) > 0$ for all $r \in [0,1]$, we can choose the neighborhood V_1 of h_0 such that $h(r) > 0$ for all $r \in [0,1]$ and any $h \in V_1$. From $u(h)'(r) = -\phi_{p'}(\int_0^r (t/r)^{N-1}h(t)dt)$, there exists two constant numbers

$$C_1(h) = \left(\frac{1}{N}\min_{r\in[0,1]} h(r)\right)^{p^*} > 0, \quad C_2(h) = \left(\frac{1}{N}\max_{r\in[0,1]} h(r)\right)^{p^*} > 0$$

such that

$$C_1(h)r^{p*} < |u(h)'(r)| \leq C_2(h)r^{p*}, \qquad r \in (0,1].$$

It follows that, for any $r \in [0,1]$ and $h \in V_1$, $\bar{h}_1$, $\bar{h}_2 \in C^0[0,1]$

$$\left|\int_r^1 |u(h)'|^{4-3p} J(h)J(\bar{h}_2)J(\bar{h}_1)\ ds\right| \leq \int_r^1 |u(h)'|^{3-2p} \cdot |J(\bar{h}_2)| \cdot |J(\bar{h}_1)| ds < \infty \quad (3.16)$$

provided $1 < p < 2$. By (3.14), (3.15) and (3.16), we conclude that $S_p : V_1 \to W^{1,1}(0,1)$ is C^2 if $1 < p < 2$ and the equation (3.12) holds. $\square$

Remark 3.4. *We note that the operators S_p, J, Φ_q and I defined in* (3.8) *are all positive operators in the sense that for any $h \in C^0[0,1]$ satisfying $h(r) \geq 0$ for $r \in [0,1]$, then $K(h)(r) \geq 0$ for $r \in [0,1]$ with $K = S_p$, J, Φ_q or I. Similarly from* (3.9) *and* (3.12)*, for any $h \in C^0[0,1]$ satisfying $h(r) \geq 0$ for $r \in [0,1]$, the operators $DS_p(h) : C^0[0,1] \to C^1[0,1]$ and $D^2S_p(h) : C^0[0,1] \times C^0[0,1] \to C^1[0,1]$ (if exists) are also positive if $1 < p < 2$.*

We use a framework of nonlinear mappings to analyze the solutions of (3.1). Define $F : [0,\infty) \times C^0[0,1] \to C^0[0,1]$ by

$$F(\lambda, u) := u - S_p(\lambda f(u)) = u - \lambda^{p^*} S_p(f(u)). \quad (3.17)$$

Then (λ_0, u_0) is a positive solution of (3.1) if and only if $F(\lambda_0, u_0) = 0$. By Lemma 3.1, $F : [0,\infty) \times C^0[0,1] \to C^0[0,1]$ is a nonlinear C^1 mapping. Furthermore, suppose that (λ_0, u_0) is a positive solution of (3.1). Then $\lambda_0 > 0$, $u_0(r) > 0$ as $r \in [0,1)$. Put $h_0(r) = \lambda_0 f(u_0(r))$. Then $h_0(r) > 0$ as $r \in [0,1]$ by the assumption $(H2)$. Hence S_p is C^2 in a neighborhood of $h_0 = \lambda_0 f(u_0)$ in $C^0[0,1]$ by Lemma 3.3, which implies that F is C^2 in a neighborhood of (λ_0, u_0) in $(0,\infty) \times C^0[0,1]$. By standard calculation, at (λ_0, u_0), for $\psi_1, \psi_2 \in C^0[0,1]$, we have

$$\begin{aligned} F_u(\lambda_0, u_0)[\psi_1] =& \psi_1 - \lambda_0^{p^*} DS_p(f(u_0))[f'(u_0)\psi_1], \\ F_{uu}(\lambda_0, u_0)[\psi_1, \psi_2] =& -\lambda_0^{p^*}\{D^2S_p(f(u_0))[f'(u_0)\psi_1, f'(u_0)\psi_2] \\ & + DS_p(f(u_0))[f''(u_0)\psi_1\psi_2]\}, \\ F_\lambda(\lambda_0, u_0) =& -p^*\lambda_0^{p^*-1} S_p(f(u_0)). \end{aligned} \quad (3.18)$$

Since F is smooth, the set of solutions of (3.1) is locally a curve in $(0,\infty) \times C^0[0,1]$ near any nondegenerate positive solution. Next we explore the structure of the solution set of (3.1) near a degenerate positive solution (λ_0, u_0). In that case, the null space $N(F_u(\lambda_0, u_0))$ is nonempty. Let $w \in N(F_u(\lambda_0, u_0))$ and $w \not\equiv 0$. Then, by (3.10), $w(r)$ satisfies

$$\begin{cases} (r^{N-1}|u_0'(r)|^{p-2}w'(r))' + \lambda_0 p^* r^{N-1} f'(u_0(r))w(r) = 0, & r \in (0,1), \\ w'(0) = w(1) = 0. \end{cases} \tag{3.19}$$

About the operators F and $F_u(\lambda_0, u_0)$, we have the following properties.

Lemma 3.5. *Let $F : [0,\infty) \times C^0[0,1] \to C^0[0,1]$ be defined by (3.17) and let $F_u(\lambda_0, u_0) : C^0[0,1] \to C^0[0,1]$ be defined by (3.18), where (λ_0, u_0) is a positive solution of $F(\lambda, u) = 0$. Recall X is as defined in (3.4).*

(i) If $\lambda \in [0,\infty)$ and $u \in X$, then $F(\lambda, u) \in X$.

(ii) If $w \in C^0[0,1]$ such that $F_u(\lambda_0, u_0)[w] = 0$, then $w \in X$.

(iii) If $h \in X$ and if there exists $\phi \in C^0[0,1]$ such that $F_u(\lambda_0, u_0)[\phi] = h$, then $\phi \in X$.

Proof. (i) Suppose that $u \in X$, and $F(\lambda, u) = g$. By (3.17), g satisfies

$$u - S_p(\lambda f(u)) = g. \tag{3.20}$$

From (3.7) we have

$$u(r) - g(r) = S_p(\lambda f(u))(r) = \int_r^1 \phi_{p'}\left[\int_0^s (\frac{t}{s})^{N-1}\lambda f(u(t))dt\right] ds, \tag{3.21}$$

and

$$u'(r) - g'(r) = -\phi_{p'}\left[\int_0^r (\frac{t}{r})^{N-1}\lambda f(u(t))dt\right]. \tag{3.22}$$

Clearly, $g(1) = 0$ and $g'(0) = 0$, since $u \in X$ and

$$\lim_{r\to 0^+} \int_0^r (\frac{t}{r})^{N-1}\lambda f(u(t))dt = 0.$$

Moreover, by using L'Hospital's rule, we have

$$\begin{aligned} &\lim_{r\to 0^+} \frac{u'(r) - g'(r)}{r^{p^*}} = \lim_{r\to 0^+} \frac{u''(r) - g''(r)}{p^* r^{p^*-1}} \\ &= -\lim_{r\to 0^+} \frac{\left[r^{1-N}\int_0^r t^{N-1}\lambda f(u(t))dt\right]^{p^*-1}}{r^{p^*-1}} \left[\lambda f(u(r)) + \frac{1-N}{r^N}\int_0^r t^{N-1}\lambda f(u(t))dt\right] \\ &= -\left(\frac{\lambda f(u(0))}{N}\right)^{p^*}. \end{aligned} \tag{3.23}$$

From (3.23), we have

$$\lim_{r\to 0^+} \frac{u''(r)-g''(r)}{r^{p^*-1}} = -p^* \left(\frac{\lambda f(u(0))}{N}\right)^{p^*}. \tag{3.24}$$

Combining (3.23), (3.24) with $u \in X$, we obtain that $g \in X$. Hence, the mapping F maps $[0,\infty)\times X$ into the space X.

(ii) If $w \in C^0[0,1]$ such that $F_u(\lambda_0, u_0)[w] = 0$, then

$$\lambda_0^{p^*} DS_p(f(u_0))[f'(u_0)w] = w. \tag{3.25}$$

By the definition of DS_p, it is easy to see that $w \in C^1[0,1] \cap C^2(0,1]$ and $w(1) = 0$.

We discuss the behavior of w' near $r = 0$ by calculating

$$\begin{aligned}
\lim_{r\to 0^+} \frac{w'(r)}{u_0'(r)} &= \lim_{r\to 0^+} \frac{r^{N-1}|u_0'(r)|^{p-2}w'(r)}{r^{N-1}|u_0'(r)|^{p-2}u_0'(r)} = \lim_{r\to 0^+} \frac{(r^{N-1}|u_0'(r)|^{p-2}w'(r))'}{(r^{N-1}|u_0'(r)|^{p-2}u_0'(r))'}\\
&= \lim_{r\to 0^+} \frac{p^* f'(u_0(r))w(r)}{f(u_0(r))} = \frac{p^* f'(u_0(0))w(0)}{f(u_0(0))} < \infty.
\end{aligned}$$

Combining this with (3.2) yields

$$\lim_{r\to 0^+} r^{-p^*} w'(r) = -\left(\frac{\lambda_0}{N} f(u_0(0))\right)^{p^*} \frac{p^* f'(u_0(0))w(0)}{f(u_0(0))}. \tag{3.26}$$

It implies that $|w'(r)| \le cr^{p^*}$ near the origin and $w'(0) = 0$.

Similarly, one can derive the behavior of w'' near $r = 0$ from the equation (3.19). Indeed from (3.19), we have

$$w'(r) = -\lambda_0 p^* |u_0'(r)|^{2-p} \int_0^r (\frac{t}{r})^{N-1} f'(u_0(t))w(t)dt, \quad r \in (0,1]. \tag{3.27}$$

It follows that

$$\begin{aligned}
w''(r) =& -\lambda_0 p^* \{(2-p)|u_0'(r)|^{-p} u_0'(r) u_0''(r) \int_0^r (\frac{t}{r})^{N-1} f'(u_0(t))w(t)dt\\
&+ |u_0'(r)|^{2-p} f'(u_0(r))w(r) + \frac{1-N}{r^N}|u_0'(r)|^{2-p} \int_0^r t^{N-1} f'(u_0(t))w(t)dt\}\\
=& -\lambda_0 p^* \left[(2-p)I_1(r) + I_2(r) + I_3(r)\right],
\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}
I_1(r) =& |u_0'(r)|^{-p} u_0'(r) u_0''(r) \int_0^r (\frac{t}{r})^{N-1} f'(u_0(t))w(t)dt,\\
I_2(r) =& |u_0'(r)|^{2-p} f'(u_0(r))w(r),\\
I_3(r) =& \frac{1-N}{r^N}|u_0'(r)|^{2-p} \int_0^r t^{N-1} f'(u_0(t))w(t)dt
\end{aligned} \tag{3.29}$$

for $r \in (0,1]$. Then,

$$\lim_{r\to 0^+} \frac{w''(r)}{r^{p^*-1}} = -\lambda_0 p^* \lim_{r\to 0^+} \left[(2-p)\frac{I_1(r)}{r^{p^*-1}} + \frac{I_2(r)}{r^{p^*-1}} + \frac{I_3(r)}{r^{p^*-1}}\right]. \tag{3.30}$$

Note that $u_0'(r) < 0$ in $(0,1]$ and $p^*-1 = p^*(2-p)$. By (3.2) and (3.3), we have

$$\begin{aligned}
\lim_{r\to 0^+} \frac{I_1(r)}{r^{p^*-1}} &= -\lim_{r\to 0^+} \frac{u_0''(r)\int_0^r (\frac{t}{r})^{N-1} f'(u_0(t))w(t)dt}{r^{p^*-1}|u_0'(r)|^{p-1}} \\
&= p^* \left[\frac{\lambda_0}{N} f(u_0(0))\right]^{p^*} \lim_{r\to 0^+} \frac{\int_0^r (\frac{t}{r})^{N-1} f'(u_0(t))w(t)dt}{r|r^{-p^*}u_0'(r)|^{p-1}} \\
&= \frac{p^*}{N}\left[\frac{\lambda_0}{N} f(u_0(0))\right]^{p^*} f'(u_0(0))w(0)\left[\frac{\lambda_0}{N} f(u_0(0))\right]^{-1} \\
&= \frac{p^*}{N}\left[\frac{\lambda_0}{N} f(u_0(0))\right]^{p^*(2-p)} f'(u_0(0))w(0),
\end{aligned}$$

and

$$\begin{aligned}
\lim_{r\to 0^+} \frac{I_2(r)}{r^{p^*-1}} &= \lim_{r\to 0^+} \frac{|u_0'(r)|^{2-p} f'(u_0(r))w(r)}{r^{p^*-1}} = f'(u_0(0))w(0)\lim_{r\to 0^+}\left|\frac{u_0'(r)}{r^{p^*}}\right|^{2-p} \\
&= \left[\frac{\lambda_0}{N} f(u_0(0))\right]^{p^*(2-p)} f'(u(0))w(0),
\end{aligned}$$

and

$$\lim_{r\to 0^+} \frac{I_3(r)}{r^{p^*-1}} = \frac{1-N}{N}\left[\frac{\lambda_0}{N} f(u_0(0))\right]^{p^*(2-p)} f'(u_0(0))w(0).$$

By (3.30),

$$\lim_{r\to 0^+} \frac{w''(r)}{r^{p^*-1}} = -\lambda_0 \frac{p^{*2}}{N}\left[\frac{\lambda_0}{N} f(u_0(0))\right]^{p^*(2-p)} f'(u_0(0))w(0).$$

Therefore w lies in the space X defined in (3.4).

(iii) Let $h \in X$. If there exists $\phi \in C[0,1]$ such that $F_u(\lambda_0, u_0)[\phi] = h$, then

$$\lambda_0^{p^*} DS_p(f(u_0))[f'(u_0)\phi] = \phi - h. \tag{3.31}$$

By the definition of DS_p, it is easy to see that $\phi \in C^1[0,1] \cap C^2(0,1]$ and $\phi(1) = 0$ since $h \in X$. From (3.10), we rewrite (3.31) as

$$(r^{N-1}|u_0'(r)|^{p-2}(\phi(r) - h(r))')' + \lambda_0 p^* r^{N-1} f'(u_0(r))\phi(r) = 0,\ r \in (0,1). \tag{3.32}$$

Using L'Hospital's rule, we have

$$\begin{aligned}\lim_{r\to0^+}\frac{\phi'(r)-h'(r)}{u_0'(r)} &= \lim_{r\to0+}\frac{r^{N-1}|u_0'(r)|^{p-2}(\phi'(r)-h'(r))}{r^{N-1}|u_0'(r)|^{p-2}u_0'(r)}\\ &= \lim_{r\to0+}\frac{(r^{N-1}|u_0'(r)|^{p-2}(\phi'(r)-h'(r)))'}{(r^{N-1}|u_0'(r)|^{p-2}u_0'(r))'}\\ &= \lim_{r\to0+}\frac{p^*f'(u_0(r))\phi(r)}{f(u_0(r))}=\frac{p^*f'(u_0(0))\phi(0)}{f(u_0(0))}<\infty.\end{aligned}$$

Combining this with (3.2) yields

$$\lim_{r\to0^+}r^{-p^*}(\phi'(r)-h'(r))=-\left(\frac{\lambda_0}{N}f(u_0(0))\right)^{p^*}\frac{p^*f'(u_0(0))\phi(0)}{f(u_0(0))}. \tag{3.33}$$

It implies that $|\phi'(r)|\le cr^{p^*}$ near the origin and $\phi'(0)=0$, since $h\in X$. On the other hand, from (3.8), (3.9) and (3.31), we obtain that

$$\begin{aligned}\lim_{r\to0^+}\frac{\phi''(r)-h''(r)}{r^{p^*-1}} &= -p^*\lambda_0^{p^*}\lim_{r\to0+}\frac{\left(|\int_0^r(\frac{t}{r})^{N-1}f(u_0(t))dt|^{p^*-1}\int_0^r(\frac{t}{r})^{N-1}f'(u_0(t))\phi(t)dt\right)'}{r^{p^*-1}}\\ &= -p^*\lambda_0^{p^*}\lim_{r\to0+}\frac{(1-N)p^*Z_1(r)+(p^*-1)Z_2(r)+Z_3(r)}{r^{p^*-1}},\end{aligned}$$

where

$$\begin{aligned}Z_1(r)=&r^{(1-N)p^*-1}\left(\int_0^r t^{N-1}f(u_0(t))dt\right)^{p^*-1}\int_0^r t^{N-1}f'(u_0(t))\phi(t)dt,\\ Z_2(r)=&r^{(1-N)p^*}\left(\int_0^r t^{N-1}f(u_0(t))dt\right)^{p^*-2}r^{N-1}f(u_0(r))\int_0^r t^{N-1}f'(u_0(t))\phi(t)dt,\\ Z_3(r)=&r^{(1-N)p^*}\left(\int_0^r t^{N-1}f(u_0(t))dt\right)^{p^*-1}r^{N-1}f'(u_0(r))\phi(r)\end{aligned} \tag{3.34}$$

for $r\in(0,1]$. Also by using L'Hospital's rule, we have

$$\begin{aligned}\lim_{r\to0^+}\frac{Z_1(r)}{r^{p^*-1}} &= \lim_{r\to0^+}\frac{\left(\int_0^r t^{N-1}f(u_0(t))dt\right)^{p^*-1}\int_0^r t^{N-1}f'(u_0(t))\phi(t)dt}{r^{Np^*}}\\ &= \lim_{r\to0^+}\left(\frac{\int_0^r t^{N-1}f(u_0(t))dt}{r^N}\right)^{p^*-1}\cdot\lim_{r\to0^+}\frac{\int_0^r t^{N-1}f'(u_0(t))\phi(t)dt}{r^N}\\ &= \frac{f(u_0(0))^{p^*-1}f'(u_0(0))\phi(0)}{N^{p^*}}.\end{aligned}$$

Similarly,

$$\lim_{r\to0^+}\frac{Z_2(r)}{r^{p^*-1}}=\lim_{r\to0^+}\frac{Z_3(r)}{r^{p^*-1}}=\frac{f(u_0(0))^{p^*-1}f'(u_0(0))\phi(0)}{N^{p^*-1}}.$$

It follows that $|\phi''(r)|\le cr^{p^*-1}$ near the origin since $h\in X$. Hence, $\phi\in X$. □

Because of the results in Lemma 3.5, from now on, we restrict the domain of the mapping F to $[0,\infty)\times X$. Then Lemma 3.5 implies that $F:[0,\infty)\times X\to X$, and consequently the Fréchet derivatives also satisfy $F_u(\lambda_0,u_0):X\to X$ and $F_{uu}(\lambda_0,u_0):X\times X\to X$.

The following non-oscillatory property of a nontrivial solution of the linearized equation plays a vital role in proving the exact multiplicity of positive solutions.

Lemma 3.6. *Assume that $1<p<2$, $N\geq 4$, $f(s)$ satisfies $(H1)$-$(H3)$ and*

$(H4)$

$$\frac{-(N-4)(p-1)}{N-p}\leq\frac{sf'(s)}{f(s)}\leq\frac{N(p-1)}{N-p},\quad \textit{for}\ \ s\in[0,\infty). \tag{3.35}$$

Let (λ_0,u_0) be a positive solution of (3.1), and let w be a nontrivial solution of (3.19). Then w can be chosen as positive in $[0,1)$ and $w'(r)<0$ for $r\in(0,1]$.

Proof. We define two test functions to be $v_1(r)=ru_0'(r)+p^*(N-p)u_0(r)$ and $v_2(r)=r^{2-N}v_1(r)$ and a linear differential operator

$$L_{u_0}[v](r):=(r^{N-1}|u_0'(r)|^{p-2}v'(r))'+\lambda_0p^*r^{N-1}f'(u_0(r))v(r). \tag{3.36}$$

First, we calculate

$$\begin{aligned}
L_{u_0}[ru_0'] =&(r^{N-1}|u_0'|^{p-2}(ru_0')')'+\lambda_0p^*r^{N-1}f'(u_0)(ru_0')\\
=&(r^{N-1}|u_0'|^{p-2}u_0'+r\cdot r^{N-1}|u_0'|^{p-2}u_0'')'+\lambda_0p^*r^Nf'(u_0)u_0'\\
=&(r^{N-1}|u_0'|^{p-2}u_0')'+(r\cdot r^{N-1}|u_0'|^{p-2}u_0'')'+\lambda_0p^*r^Nf'(u_0)u_0'\\
=&-\lambda_0r^{N-1}f(u_0)+r^{N-1}|u_0'|^{p-2}u_0''+r(r^{N-1}|u_0'|^{p-2}u_0'')'\\
&+\lambda_0p^*r^Nf'(u_0(r))u_0'.
\end{aligned} \tag{3.37}$$

On the other hand, differentiating (3.1) with respect to r, we obtain

$$\begin{aligned}
&(p-1)(r^{N-1}|u_0'|^{p-2}u_0'')'\\
=&-\lambda_0(N-1)r^{N-2}f(u_0)-\lambda_0r^{N-1}f'(u_0)u_0'-(N-1)(r^{-1}r^{N-1}|u_0'|^{p-2}u_0')'\\
=&-\lambda_0(N-1)r^{N-2}f(u_0)-\lambda_0r^{N-1}f'(u_0)u_0'\\
&+(N-1)r^{-2}(r^{N-1}|u_0'|^{p-2}u_0')-(N-1)r^{-1}(r^{N-1}|u_0'|^{p-2}u_0')'\\
=&-\lambda_0r^{N-1}f'(u_0)u_0'+(N-1)r^{N-3}|u_0'|^{p-2}u_0'.
\end{aligned}$$

Thus,

$$(r^{N-1}|u_0'|^{p-2}u_0'')'=-\lambda_0p^*r^{N-1}f'(u_0)u_0'+(N-1)p^*r^{N-3}|u_0'|^{p-2}u_0' \tag{3.38}$$

Substituting (3.38) into (3.37), we get

$$L_{u_0}[ru_0']=-\lambda_0p^*pr^{N-1}f(u_0). \tag{3.39}$$

Similarly we can calculate $L_{u_0}[p^*(N-p)u_0]$. Then for $r \in (0,1)$, we obtain

$$L_{u_0}[v_1] = L_{u_0}[ru_0' + p^*(N-p)u_0] = \lambda_0 p^* r^{N-1}[p^*(N-p)f'(u_0)u_0 - Nf(u_0))]. \quad (3.40)$$

Since $1 < p < 2$ and $f(u_0) > 0$,

$$v_1'(r) = ru_0'' + \frac{N-1}{p-1}u_0' = p^*[(p-1)ru_0'' + (N-1)u_0'] = -\lambda_0 p^* \frac{rf(u_0)}{|u_0'|^{p-2}} < 0$$

for $r \in (0,1)$. Note that

$$v_1(0) = p^*(N-p)u_0(0) > 0, \quad v_1(1) = u_0'(1) < 0.$$

Then there exists a unique zero $r_0 \in (0,1)$ such that $v_1(r_0) = 0$, and

$$v_1(r) > 0 \text{ for } r \in (0, r_0), \qquad v_1(r) < 0 \text{ for } r \in (r_0, 1). \quad (3.41)$$

Next we calculate

$$\begin{aligned}
L_{u_0}[v_2] =& (r^{N-1}|u_0'|^{p-2}(r^{2-N}v_1)')' + \lambda_0 p^* r^{N-1} f'(u_0)(r^{2-N}v_1) \\
=& (2-N)(|u_0'|^{p-2}v_1)' + [r^{2-N}(r^{N-1}|u_0'|^{p-2}v_1')]' + \lambda_0 p^* r f'(u_0)v_1 \\
=& (2-N)(p-2)|u_0'|^{p-4}u_0''v_1 + 2(2-N)|u_0'|^{p-2}v_1' \\
& + r^{2-N}[\lambda_0 p^* r^{N-1}(p^*(N-p)f'(u_0)u_0 - Nf(u_0))] \\
=& (2-N)(p-2)|u_0'|^{p-4}u_0''v_1 - 2(2-N)\lambda_0 p^* r f(u_0) \\
& + \lambda_0 p^* r[p^*(N-p)f'(u_0)u_0 - Nf(u_0)] \\
=& (2-N)(p-2)|u_0'|^{p-4}u_0''v_1 \\
& + \lambda_0 p^* r[p^*(N-p)f'(u_0)u_0 + (N-4)f(u_0)].
\end{aligned} \quad (3.42)$$

Combining (3.40)-(3.42) and the assumption $(H4)$, we have

$$L_{u_0}[v_1(r)] = g_1(r) \le 0, \;\; v_1(r) > 0 \;\text{ for } r \in (0, r_0), \quad (3.43)$$

and

$$L_{u_0}[v_2(r)] = g_2(r) \ge 0, \;\; v_2(r) < 0 \;\text{ for } r \in (r_0, 1). \quad (3.44)$$

Now we prove that w does not have any zeros in $[0, r_0)$. Without loss of generality we assume that $w(0) > 0$. On the contrary, suppose $r_1 \in (0, r_0)$ is the smallest positive zero of w in $(0, r_0)$. Then $w(r) > 0$ in $(0, r_1)$ and $w(r_1) = 0$. Multiplying the equation in (3.43) by w, multiplying (3.19) by v_1, subtracting and integrating on the interval $(0, r_1)$, we obtain

$$\int_0^{r_1} (L_{u_0}[v_1]w - L_{u_0}[w]v_1)\,dr = \int_0^{r_1} g_1 w dr \le 0. \quad (3.45)$$

But on the other hand, (3.45) is equal to

$$\begin{aligned}\int_0^{r_1} (L_{u_0}[v_1]w - L_{u_0}[w]v_1)\,dr =& r^{N-1}|u_0'|^{p-2}(wv_1' - v_1w')\Big|_0^{r_1}\\ =& -r_1^{N-1}|u_0'(r_1)|^{p-2}v_1(r_1)w'(r_1) > 0,\end{aligned} \tag{3.46}$$

which is a contradiction. Next we show w does not have any zeros in $[r_0, 1)$ by comparing w with v_2. On the contrary, suppose $r_2 \in [r_0, 1)$ is the largest positive zero of w in $[r_0, 1)$. Without loss of generality, we assume that $w(r) > 0$ in $[r_2, 1)$ and $w(r_2) = w(1) = 0$. Multiplying the equation in (3.44) by w, multiplying (3.19) by v_2, subtracting the two equations, and integrating over the interval $[r_2, 1)$, we have

$$\int_{r_2}^{1} (L_{u_0}[v_2]w - L_{u_0}[w]v_2)\,dr = \int_{r_2}^{1} g_2 w dr \geq 0. \tag{3.47}$$

However (3.47) is also equal to

$$\begin{aligned}&\int_{r_2}^{1} (L_{u_0}[v_2]w - L_{u_0}[w]v_2)\,dr\\ =& r^{N-1}|u_0'|^{p-2}(wv_2' - v_2w')\Big|_{r_2}^{1}\\ =& -\left[v_2(1)|u_0'(1)|^{p-2}w'(1) - v_2(r_2)r_2^{N-1}|u_0'(r_2)|^{p-2}w'(r_2)\right]\\ <&0,\end{aligned} \tag{3.48}$$

which is a contradiction. Hence, w can be chosen as positive in $[0, 1)$. Moreover, from (3.19) and condition $(H3)$, $w' < 0$ for $r \in (0, 1]$. $\square$

Now we are in a position to apply the saddle-node bifurcation theorem (Theorem 2.6) near a degenerate positive solution of (3.1).

Lemma 3.7. *Assume that $1 < p < 2$, $N \geq 4$, and $f(s)$ satisfies $(H1)$-$(H4)$. Let (λ_0, u_0) be a positive solution of* (3.1) *such that* (3.19) *has a nontrivial solution w which is chosen as positive as in Lemma 3.6. Then the solutions of* (3.1) *near (λ_0, u_0) form a curve $\{(\lambda(s), u(s)) : s \in (-\epsilon, +\epsilon)\}$, where $\lambda : (-\epsilon, +\epsilon) \to \mathbb{R}$ is a C^2 function such that $\lambda(0) = \lambda_0$, $\lambda'(0) = 0$, and $\lambda''(0) < 0$.*

Proof. Let X be the space defined in (3.4), and define $F : [0, \infty) \times X \to X$ as in (3.17) (the validity of $F(\lambda, \cdot) : X \to X$ is shown in Lemma 3.5). Suppose that (λ_0, u_0) is a positive solution of (3.1) such that (3.19) has a nontrivial solution w, then $w \in X$ from Lemma 3.5 (ii) and w can be chosen as positive from Lemma 3.6. From the uniqueness of solution to the initial value problem (3.19), we know that

$$N(F_u(\lambda_0, u_0)) = \{tw : t \in \mathbb{R}\} = \text{span}\{w\}. \tag{3.49}$$

For a bifurcation analysis, we need an integral characterization of the range space $R(F_u(\lambda_0, u_0))$. Let X^* be the dual space of X, and define $l \in X^*$ by

$$\langle l, h\rangle = \int_0^1 r^{N-1} f'(u_0(r))w(r)h(r)dr, \quad \text{for } h \in X. \tag{3.50}$$

We will prove that

$$R(F_u(\lambda_0, u_0)) = \{h \in X : \langle l, h\rangle = 0\}.$$

Suppose that $h \in R(F_u(\lambda_0, u_0))$. Since $h \in X$, then $h \in C^1[0,1] \cap C^2(0,1]$ satisfies

$$h'(0) = h(1) = 0, \text{ and } |h'(r)| \le cr^{p^*}, \; |h''(r)| \le \bar{c}r^{p^*-1}, \; r \in (0,1]. \tag{3.51}$$

Moreover there exists a $\psi \in X$ such that $h = F_u(\lambda_0, u_0)[\psi]$, *i.e.*

$$\lambda_0^{p^*} DS_p(f(u_0))[f'(u_0)\psi] = \psi - h. \tag{3.52}$$

From (3.10), we obtain that

$$(r^{N-1}|u_0'(r)|^{p-2}(\psi(r) - h(r))')' + \lambda_0 p^* r^{N-1} f'(u_0(r))\psi(r) = 0, \; r \in (0,1). \tag{3.53}$$

Note that equation (3.53) can be written as

$$L_{u_0}[\psi](r) = (r^{N-1}|u_0'(r)|^{p-2}h'(r))', \quad r \in (0,1), \tag{3.54}$$

where L_{u_0} is defined in (3.36). Multiplying (3.54) by w, (3.19) by ψ, subtracting and integrating on $(0,1)$, one has

$$\int_0^1 (wL_{u_0}[\psi] - \psi L_{u_0}[w])dr = \int_0^1 (r^{N-1}|u_0'(r)|^{p-2}h'(r))'w(r)dr. \tag{3.55}$$

By using integral by parts on the right hand side of (3.55), we get

$$\begin{aligned} \int_0^1 (r^{N-1}|u_0'|^{p-2}h')'wdr =& r^{N-1}|u_0'|^{p-2}h'w\Big|_0^1 - \int_0^1 r^{N-1}|u_0'|^{p-2}h'w'dr \\ =& -\int_0^1 r^{N-1}|u_0'|^{p-2}w'dh \\ =& -r^{N-1}|u_0'|^{p-2}w'h\Big|_0^1 + \int_0^1 (r^{N-1}|u_0'|^{p-2}w')'hdr \\ =& -\lambda^{p^*}\int_0^1 r^{N-1}f'(u_0(r))w(r)h(r)dr \end{aligned} \tag{3.56}$$

since $h(1) = 0$ and $|h'(r)| \le cr^{p^*}$ near the origin. For the left hand side of (3.55), we have

$$\begin{aligned} &\int_0^1 (wL_{u_0}[\psi] - \psi L_{u_0}[w])dr = r^{N-1}|u_0'(r)|^{p-2}[\psi'(r)w(r) - w'(r)\psi(r)]\Big|_0^1 \\ =& -\lim_{r\to 0^+} r^{N-1}|u_0'(r)|^{p-2}\psi'(r)w(r) + \lim_{r\to 0^+} r^{N-1}|u_0'(r)|^{p-2}w'(r)\psi(r) = 0. \end{aligned} \tag{3.57}$$

The last equality is obtained by using the precise behavior of $u_0(r)$, $w(r)$ and $\psi(r)$ near the origin, *i.e.*, (3.2) and (3.26). Combining (3.55), (3.56) and (3.57), we obtain

$$\int_0^1 r^{N-1} f'(u_0(r))w(r)h(r)dr = 0. \tag{3.58}$$

Hence $R(F_u(\lambda_0, u_0)) \subseteq \{h \in X : \langle l, h\rangle = 0\}$, where l is defined in (3.50).

Note that $w \in X$ satisfies

$$\langle l, w\rangle = \int_0^1 r^{N-1} f'(u_0(r))w^2(r)dr > 0. \tag{3.59}$$

Thus,

$$w \notin R(F_u(\lambda_0, u_0)). \tag{3.60}$$

From the definition of $F_u(\lambda_0, u_0)$ and Lemma 3.1, as a mapping from $C^0[0,1]$ to $C^0[0,1]$, $F_u(\lambda_0, u_0)$ is a compact perturbation of the identity map on $C^0[0,1]$. Therefore

$$\dim N(F_u(\lambda_0, u_0)) = \operatorname{codim} R(F_u(\lambda_0, u_0)) = 1 \tag{3.61}$$

holds when $F_u(\lambda_0, u_0)$ is considered as a mapping $C^0[0,1] \to C^0[0,1]$.

We claim that (3.61) also holds for $F_u(\lambda_0, u_0) : X \to X$. For that purpose, we clarify the definitions of F_u on different spaces to be

$$L_X = F_u(\lambda_0, u_0)|_X, \qquad L_C = F_u(\lambda_0, u_0)|_{C^0[0,1]}. \tag{3.62}$$

Then (3.61) holds for $C^0[0,1]$ implies that $\dim N(L_C) = \operatorname{codim} R(L_C) = 1$. From Lemma 3.5 (ii), we have $N(L_C) = N(L_X) = \mathit{span}\{w\}$. So it remains to prove that $\operatorname{codim} R(L_X) = 1$. From (3.60), we know that $\operatorname{codim} R(L_X) \geq 1$. So we need to show that $\operatorname{codim} R(L_X) \leq 1$. On the contrary, suppose $\operatorname{codim} R(L_X) \geq 2$. Since $\operatorname{codim} R(L_C) = 1$, there exists some $q \in X$ such that $q \in R(L_C)$ but $q \notin R(L_X)$. Hence there exists $\vartheta \in C^0[0,1] \setminus X$ such that $F_u(\lambda_0, u_0)[\vartheta] = q$, which contradicts Lemma 3.5 (iii). Therefore $\operatorname{codim} R(L_X) = \operatorname{codim} R(F_u(\lambda_0, u_0)) = 1$, and consequently $R(F_u(\lambda_0, u_0)) = \{h \in X : \langle l, h\rangle = 0\}$. This shows that the condition (1) in Theorem 2.6 is satisfied.

The assumptions $(H2)$ and $(H3)$ imply that $f(u_0(r)) > 0$, $f'(u_0(r)) > 0$ and $f''(u_0(r)) > 0$ pointwisely for $r \in [0,1]$, and from Remark 3.4, the operator S_p maps positive functions to positive functions, then from (3.18), we obtain that

$$\langle l, F_\lambda(\lambda_0, u_0)\rangle = -p^* \lambda_0^{p^*-1} \int_0^1 r^{N-1} f'(u_0) w S_p(f(u_0))dr < 0, \tag{3.63}$$

which implies that $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$, thus the condition (2) in Theorem 2.6 is also satisfied. Therefore, from Theorem 2.6, the solutions of (3.1) near (λ_0, u_0)

form a curve $\{(\lambda(s), u(s)) : s \in (-\epsilon, +\epsilon)\}$, where $\lambda : (-\epsilon, +\epsilon) \to \mathbb{R}$ is a C^1 function such that $\lambda(0) = \lambda_0$, $\lambda'(0) = 0$, Moreover since f is C^2, then from (3.18), we have

$$\begin{aligned}
\langle l, F_{uu}(\lambda_0, u_0)[w, w]\rangle =& \int_0^1 r^{N-1} f'(u_0) w F_{uu}(\lambda_0, u_0)[w, w]) dr \\
=& -\lambda_0^{p^*}(p-1)\left\{ \int_0^1 r^{N-1} f'(u_0) w D^2 S_p(f(u_0))[f'(u_0)w, f'(u_0)w] dr \right. \\
& \left. + \int_0^1 r^{N-1} f'(u_0) w D S_p(f(u_0))[f''(u_0)w^2] dr \right\} \\
<& 0,
\end{aligned}$$

as $f'(u_0) > 0$, $f''(u_0) > 0$, $w > 0$, and from Remark 3.4, DS_p and $D^2 S_p$ are positive operators when $1 < p < 2$. From Theorem 2.6, we obtain that

$$\lambda''(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w, w]\rangle}{\langle l, F_\lambda(\lambda_0, u_0)\rangle} < 0. \tag{3.64}$$

□

Lemma 3.7 shows that if (λ_0, u_0) is a degenerate positive solution of (3.1), then all positive solutions of (3.1) near (λ_0, u_0) form a parabola-like curve, and the turning direction of the curve is to the left.

To complete the proof of global bifurcation diagram, we prove that the set of all positive solutions of (3.1) can be parameterized by the initial value $u(0)$ by using a well-known scaling argument.

Lemma 3.8. *Assume that $p > 1$, $f(s) > 0$ and $f(s)$ is locally Lipschitz continuous for $s \in [0, \infty)$. Then for each $\alpha > 0$, there exists at most one positive solution (λ, u) of problem (3.1) such that $u(0) = \alpha$.*

Proof. Assume on the contrary that there are two positive solutions (λ_1, u_1) and (λ_2, u_2) to problem (3.1) such that $u_1(0) = u_2(0) = \alpha$. Clearly, $\lambda_1 \neq \lambda_2$ by uniqueness for the initial value problem (see, e.g., [53, Theorem 4]). Then $u_1(\lambda_1^{-1/p} x)$ and $u_2(\lambda_2^{-1/p} x)$ are both solutions of the same initial value problem

$$(x^{N-1}\varphi_p(v'(x)))' + x^{N-1} f(v(x)) = 0,\ v(0) = \alpha,\ v'(0) = 0,$$

and hence $u_1(\lambda^{-1/p} x) = u_2(\lambda^{-1/p} x)$. This is impossible, since the first function vanishes at $x = \lambda_1^{1/p}$, while the second one vanishes at $x = \lambda_2^{1/p}$. □

Now we are ready to prove our main result on the exact multiplicity of positive solutions of (3.1).

Theorem 3.9. *Assume that* $1 < p < 2$, $N \geq 4$, *and* $f(s)$ *satisfies* $(H1)$-$(H4)$. *Then*

1. *There exists* $\lambda^* > 0$ *such that* (3.1) *has no positive solution for* $\lambda > \lambda^*$, *has exactly one positive solution for* $\lambda = \lambda^*$, *and has exactly two positive solutions for* $0 < \lambda < \lambda^*$.

2. *All positive solutions of* (3.1) *lie on a single smooth solution curve* Σ *which has two branches denoted by* $\Sigma^+ = \{(\lambda, u_+(\lambda)) : 0 < \lambda < \lambda^*\}$ *(the upper branch) and* $\Sigma^- = \{(\lambda, u_-(\lambda)) : 0 < \lambda \leq \lambda^*\}$ *(the lower branch). Moreover, for* $0 < \lambda < \lambda^*$, $u_-(\lambda)(r) < u_+(\lambda)(r)$ *for* $r \in [0, 1)$,

$$\lim_{\lambda \to 0^+} u_-(\lambda) = 0, \quad \lim_{\lambda \to 0^+} ||u_+(\lambda)||_\infty = \infty, \quad \lim_{\lambda \to (\lambda^*)^-} u_+(\lambda) = u_-(\lambda^*), \tag{3.65}$$

and there is a unique turning point $(\lambda^*, u_-(\lambda^*))$ *on the curve* Σ *where the curve bends to the left.*

Proof. When $\lambda = 0$, $u \equiv 0$ is the unique solution of (3.1), and $F_u(0, 0) : X \to X$ is the identity map. Hence by the implicit function theorem, there exists $\epsilon > 0$ and a C^1 function $u_- : (0, \epsilon) \to X$, such that $F(\lambda, u_-(\lambda)) = 0$ for $\lambda \in (0, \epsilon)$. That is, there exists a solution curve bifurcating from $(0, 0)$ and the curve continues to the right. We denote this curve by Σ^- and extend Σ^- to the right as far as possible. Define

$$\lambda^* = \sup\{\lambda > 0 : F(s, u_-(s)) = 0, F_u(s, u_-(s)) \text{ is nondegenerate for } s \in (0, \lambda)\}.$$

We have shown that $\lambda^* > 0$. From Lemma 2.2, problem (3.1) has no positive solutions for large $\lambda > 0$. Hence $\lambda^* < \infty$. From Theorem 2.5 and Lemma 2.4, $u_-(\lambda)$ is bounded in $C^1(\overline{\Omega})$ for $\lambda \in [\varepsilon, \lambda^*)$. Hence we have, subject to a subsequence, $u_-(\lambda) \to u_*$ in $C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$. From elliptic regularity theory, $u_* \in C^{1,\beta}(\overline{\Omega})$, and it is a positive solution of (3.1) with $\lambda = \lambda^*$. We shall still label u_* by $u_-(\lambda^*)$, by continuity. From the definition of λ^*, the solution $u_-(\lambda^*)$ is necessarily degenerate. From Lemma 3.7, the solutions of (3.1) near $(\lambda^*, u_-(\lambda^*))$ form a curve $\{(\lambda(s), u(s)) : s \in (-\epsilon, \epsilon)\}$ such that $\lambda(0) = \lambda^*$, $\lambda'(0) = 0$ and $\lambda''(0) < 0$. Thus $(\lambda^*, u_-(\lambda^*))$ is a turning point of the solution curve where the curve bends to the left. So for $\lambda \in (\lambda^* - \delta, \lambda^*)$, (3.1) has exactly two solutions near the turning point $(\lambda^*, u_-(\lambda^*))$. We name the solution other than $u_-(\lambda)$ to be $u_+(\lambda)$. Then $\Sigma^+ = \{(\lambda, u_+(\lambda)) : \lambda \in (\lambda^* - \delta, \lambda^*)\}$ is also a smooth curve. We extend this curve further left. Then there is no other degenerate solution on that curve since at any degenerate solution $(\tilde{\lambda}, \tilde{u})$, one can apply Lemma 3.7 to conclude that the solution set near $(\tilde{\lambda}, \tilde{u})$ is a curve $\{(\tilde{\lambda}(s), \tilde{u}(s)) : s \in (-\epsilon, \epsilon)\}$ such that $\tilde{\lambda}(0) = \tilde{\lambda}$, $\tilde{\lambda}'(0) = 0$ and $\tilde{\lambda}''(0) < 0$, which contradicts with the assumption that Σ^+ approaches $(\tilde{\lambda}, \tilde{u})$ from the right. Moreover Lemma 2.4 implies that the solution $u_+(\lambda)$ is bounded for all $\lambda \in [\varepsilon, \lambda^*)$ where $\varepsilon > 0$ is an arbitrary small positive number. Hence Σ^+ cannot blow up at a positive $\hat{\lambda} > 0$ so that $||u_+(\lambda)|| \to \infty$ as $\lambda \to \hat{\lambda}^+$. Therefore Σ^+ can be extended to the value $\lambda = 0$. Suppose that $\lim\limits_{\lambda \to 0^+} ||u_+(\lambda)||_\infty < \infty$.

Then $\lim_{\lambda\to 0^+} u_+(\lambda)$ exists and it is a solution of (3.1) with $\lambda = 0$. But the unique solution of (3.1) when $\lambda = 0$ is $u \equiv 0$, and from the implicit function theorem, the only solution near $u \equiv 0$ when λ is near 0 is $u_-(\lambda)$, which is a contradiction. Thus we must have $\lim\limits_{\lambda\to 0^+} ||u_+(\lambda)||_\infty = \infty$. Finally Lemma 3.8 ensures that all positive solutions of (3.1) lie on $\Sigma = \Sigma^+ \cup \Sigma^-$, since (3.65) has been proved. □

Theorem 3.9 is a natural extension of Theorem 6.21 in [50], where a similar result was obtained for $p = 2$. To conclude the paper, we point out that the following functions satisfy the conditions of Theorem 3.9: (i) $f(u) = c+u^q$, (ii) $f(u) = (c+u)^q$, where $1 < p < 2$, $N \geq 4$, $c > 0$ and $1 < q \leq \dfrac{N(p-1)}{N-p}$.

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References

[1] A. Aftalion and F. Pacella. Uniqueness and nondegeneracy for some nonlinear elliptic problems in a ball. *J. Differential Equations*, 195(2):380–397, 2003.

[2] A. Aftalion and F. Pacella. Morse index and uniqueness for positive solutions of radial p-Laplace equations. *Trans. Amer. Math. Soc.*, 356(11):4255–4272, 2004.

[3] W. Allegretto. Sturm theorems for degenerate elliptic equations. *Proc. Amer. Math. Soc.*, 129(10):3031–3035, 2001.

[4] W. Allegretto and Y.-X. Huang. A Picone's identity for the p-Laplacian and applications. *Nonlinear Anal.*, 32(7):819–830, 1998.

[5] A. Ambrosetti, J. Garcia Azorero, and I. Peral. Multiplicity results for some nonlinear elliptic equations. *J. Funct. Anal.*, 137(1):219–242, 1996.

[6] A. Anane. Simplicité et isolation de la première valeur propre du p-laplacien avec poids. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(16):725–728, 1987.

[7] C. Azizieh and P. Clément. A priori estimates and continuation methods for positive solutions of p-Laplace equations. *J. Differential Equations*, 179(1):213–245, 2002.

[8] P. A. Binding and B. P. Rynne. The spectrum of the periodic p-Laplacian. *J. Differential Equations*, 235(1):199–218, 2007.

[9] F. Brock. Continuous rearrangement and symmetry of solutions of elliptic problems. *Proc. Indian Acad. Sci. Math. Sci.*, 110(2):157–204, 2000.

[10] F. Brock, L. Iturriaga, and P. Ubilla. A multiplicity result for the p-Laplacian involving a parameter. *Ann. Henri Poincaré*, 9(7):1371–1386, 2008.

[11] D. Castorina. Uniqueness of positive radial solutions for quasilinear elliptic equations in an annulus. *Nonlinear Anal.*, 72(5):2195–2203, 2010.

[12] D. Castorina, P. Esposito, and B. Sciunzi. Degenerate elliptic equations with singular nonlinearities. *Calc. Var. Partial Differential Equations*, 34(3):279–306, 2009.

[13] J.-G. Cheng. Uniqueness results for the one-dimensional p-Laplacian. *J. Math. Anal. Appl.*, 311(2):381–388, 2005.

[14] J. Cossio, S. Herrón, and C. Vélez. Infinitely many radial solutions for a p-Laplacian problem p-superlinear at the origin. *J. Math. Anal. Appl.*, 376(2):741–749, 2011.

[15] M. G. Crandall and P. H. Rabinowitz. Bifurcation, perturbation of simple eigenvalues and linearized stability. *Arch. Rational Mech. Anal.*, 52:161–180, 1973.

[16] M. G. Crandall and P. H. Rabinowitz. Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. *Arch. Rational Mech. Anal.*, 58(3):207–218, 1975.

[17] M. Cuesta and P. Takáč. A strong comparison principle for positive solutions of degenerate elliptic equations. *Differential Integral Equations*, 13(4-6):721–746, 2000.

[18] A. Ćwiszewski and M. Maciejewski. Positive stationary solutions for p-Laplacian problems with nonpositive perturbation. *J. Differential Equations*, 254(3):1120–1136, 2013.

[19] L. Damascelli and F. Pacella. Monotonicity and symmetry of solutions of p-Laplace equations, $1 < p < 2$, via the moving plane method. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(4):689–707, 1998.

[20] M. del Pino, P. Drábek, and R. Manásevich. The Fredholm alternative at the first eigenvalue for the one-dimensional p-Laplacian. *J. Differential Equations*, 151(2):386–419, 1999.

[21] J. I. Díaz, J. Hernández, and F. J. Mancebo. Branches of positive and free boundary solutions for some singular quasilinear elliptic problems. *J. Math. Anal. Appl.*, 352(1):449–474, 2009.

[22] P. Drábek and J. Hernández. Existence and uniqueness of positive solutions for some quasilinear elliptic problems. *Nonlinear Anal.*, 44(2):189–204, 2001.

[23] P. Drábek and S. B. Robinson. Resonance problems for the p-Laplacian. *J. Funct. Anal.*, 169(1):189–200, 1999.

[24] Y.-H. Du. *Order structure and topological methods in nonlinear partial differential equations. Vol. 1. Maximum principles and applications.* Series in Partial Differential Equations and Applications. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[25] L. Erbe and M.-X. Tang. Uniqueness theorems for positive radial solutions of quasilinear elliptic equations in a ball. *J. Differential Equations*, 138(2):351–379, 1997.

[26] J. Fleckinger and W. Reichel. Global solution branches for p-Laplacian boundary value problems. *Nonlinear Anal.*, 62(1):53–70, 2005.

[27] B. Franchi, E. Lanconelli, and J. Serrin. Existence and uniqueness of nonnegative solutions of quasilinear equations in $\mathbf{R}^n$. *Adv. Math.*, 118(2):177–243, 1996.

[28] M. García-Huidobro, R. Manásevich, and J. R. Ward. Positive solutions for equations and systems with p-Laplace-like operators. *Adv. Differential Equations*, 14(5-6):401–432, 2009.

[29] J. García-Melián and J. Sabina de Lis. A local bifurcation theorem for degenerate elliptic equations with radial symmetry. *J. Differential Equations*, 179(1):27–43, 2002.

[30] F. Genoud. Bifurcation along curves for the p-Laplacian with radial symmetry. *Electron. J. Differential Equations*, pages No. 124, 17, 2012.

[31] B. Gidas, W.-M. Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68(3):209–243, 1979.

[32] D.-J. Guo and V. Lakshmikantham. *Nonlinear problems in abstract cones*, volume 5 of *Notes and Reports in Mathematics in Science and Engineering*. Academic Press Inc., Boston, MA, 1988.

[33] D. D. Hai. Uniqueness of positive solutions for a class of quasilinear problems. *Nonlinear Anal.*, 69(8):2720–2732, 2008.

[34] D. D. Hai, K. Schmitt, and R. Shivaji. Positive solutions of quasilinear boundary value problems. *J. Math. Anal. Appl.*, 217(2):672–686, 1998.

[35] K.-C. Hung and S.-H. Wang. A complete classification of bifurcation diagrams of classes of multiparameter p-Laplacian boundary value problems. *J. Differential Equations*, 246(4):1568–1599, 2009.

[36] K.-C. Hung and S.-H. Wang. Classification and evolution of bifurcation curves for a multiparameter p-Laplacian Dirichlet problem. *Nonlinear Anal.*, 74(11):3589–3598, 2011.

[37] B. Kawohl. On a family of torsional creep problems. *J. Reine Angew. Math.*, 410:1–22, 1990.

[38] C.-G. Kim, E. K. Lee, and Y.-H. Lee. Existence of the second positive radial solution for a p-Laplacian problem. *J. Comput. Appl. Math.*, 235(13):3743–3750, 2011.

[39] C.-G. Kim and J.-P. Shi. Existence and multiplicity of positive solutions to a quasilinear elliptic equation with strong Allee effect growth rate. *Results Math.*, 64(1-2):165–173, 2013.

[40] C.-G. Kim and J.-P. Shi. Multiple positive solutions for p-Laplacian equation with weak Allee effect growth rate. *Differential Integral Equations*, 26(7-8):707–720, 2013.

[41] P. Korman. Existence and uniqueness of solutions for a class of p-Laplace equations on a ball. *Adv. Nonlinear Stud.*, 11(4):875–888, 2011.

[42] P. Korman. *Global solution curves for semilinear elliptic equations.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[43] P. Korman, Y. Li, and T.-C. Ouyang. Exact multiplicity results for boundary value problems with nonlinearities generalising cubic. *Proc. Roy. Soc. Edinburgh Sect. A*, 126(3):599–616, 1996.

[44] P. Korman, Y. Li, and T.-C. Ouyang. An exact multiplicity result for a class of semilinear equations. *Comm. Partial Differential Equations*, 22(3-4):661–684, 1997.

[45] G. M. Lieberman. Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal.*, 12(11):1203–1219, 1988.

[46] P. Lindqvist. *Notes on the p-Laplace equation*, volume 102 of *Report. University of Jyväskylä Department of Mathematics and Statistics.* University of Jyväskylä, Jyväskylä, 2006.

[47] È. Mitidieri and S. I. Pokhozhaev. Absence of global positive solutions of quasilinear elliptic inequalities. *Dokl. Akad. Nauk*, 359(4):456–460, 1998.

[48] D. Motreanu, V. V. Motreanu, and N. S. Papageorgiou. A multiplicity theorem for problems with the p-Laplacian. *Nonlinear Anal.*, 68(4):1016–1027, 2008.

[49] T.-C. Ouyang and J.-P. Shi. Exact multiplicity of positive solutions for a class of semilinear problems. *J. Differential Equations*, 146(1):121–156, 1998.

[50] T.-C. Ouyang and J.-P. Shi. Exact multiplicity of positive solutions for a class of semilinear problem. II. *J. Differential Equations*, 158(1):94–151, 1999.

[51] M.-C. Pélissier and L. Reynaud. Étude d'un modèle mathématique d'écoulement de glacier. *C. R. Acad. Sci. Paris Sér. A*, 279:531–534, 1974.

[52] I. Peral. Some results on quasilinear elliptic equations: growth versus shape. In *Nonlinear functional analysis and applications to differential equations (Trieste, 1997)*, pages 153–202. World Sci. Publ., River Edge, NJ, 1998.

[53] W. Reichel and W. Walter. Radial solutions of equations and inequalities involving the p-Laplacian. *J. Inequal. Appl.*, 1(1):47–71, 1997.

[54] B. P. Rynne. A global curve of stable, positive solutions for a p-Laplacian problem. *Electron. J. Differential Equations*, pages No. 58, 12, 2010.

[55] J. Serrin. Local behavior of solutions of quasi-linear equations. *Acta Math.*, 111:247–302, 1964.

[56] J. Serrin and M.-X. Tang. Uniqueness of ground states for quasilinear elliptic equations. *Indiana Univ. Math. J.*, 49(3):897–923, 2000.

[57] I. Sim and S. Tanaka. The exact multiplicity of positive solutions for a class of two-point boundary-value problems with the one-dimensional p-Laplacian. *Proc. Roy. Soc. Edinburgh Sect. A*, 144(1):187–203, 2014.

[58] S. Takeuchi. Multiplicity result for a degenerate elliptic equation with logistic reaction. *J. Differential Equations*, 173(1):138–144, 2001.

[59] M.-X. Tang. Uniqueness and global structure of positive radial solutions for quasilinear elliptic equations. *Comm. Partial Differential Equations*, 26(5-6):909–938, 2001.

[60] P. Tolksdorf. Regularity for a more general class of quasilinear elliptic equations. *J. Differential Equations*, 51(1):126–150, 1984.

[61] J. L. Vázquez. A strong maximum principle for some quasilinear elliptic equations. *Appl. Math. Optim.*, 12(3):191–202, 1984.

[62] S.-H. Wang and T.-S. Yeh. Exact multiplicity and ordering properties of positive solutions of a p-Laplacian Dirichlet problem and their applications. *J. Math. Anal. Appl.*, 287(2):380–398, 2003.