ON COMPACTNESS AND RELATED CONCEPTS

A Thesis

Presented to

The Faculty of the Department of Mathematics

The College of William and Mary in Virginia

In Partial Fulfillment

Of the Requirements for the Degree of

Master of Arts

By

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May 1963

APPROVAL SHEET

This thesis is submitted in partial fulfillment of

the requirements for the degree of

Master of Arts

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ACKNOWLEDGMENTS

The writer wishes to express his gratitude to Professor F. W. Weiler under whose guidance this investigation was conducted. The author is indebted to Professor J. R. Lee and Professor S. H. Lawrence for their careful reading and criticism of the manuscript. He wishes also to acknowledge the use of the facilities at the Langley Research Center where part of the research was undertaken.

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ABSTRACT

In this paper we consider the concept of compactness and the related concepts of countable compactness, sequential compactness, metacompactness, and (briefly) paracompactness and conditions under which these concepts are equivalent. We assume the axiom of choice.

A subset E of a topological space is termed compact if and only if every open cover of E contains a finite subcover. In Part I we prove this concept is equivalent to the following statements:

(1) Every nest or direction in E of sets closed in E has a nonempty intersection.

(2) Every filter, net, or directed function in E has a cluster point in E.

(3) Every filter, net, or directed function in E contains respectively an ultrafilter, subnet, or subdirected function convergent to a point of E.

In Part II we consider countable compactness, sequential compactness, metacompactness, and paracompactness. A subset E of a topological space is countably compact if and only if every countable cover of E contains a finite subcover. Using well-known theorems for the real line as definitions in an abstract space, we define an intersection property, a Cantor property, a limit superior property, and a Bolzano-Weierstrass property. We prove these properties and countable compactness are equivalent. (In the case of the Bolzano-Weierstrass property we assume the space is T_1.) The set E is sequentially compact if and only if every sequence of points in E contains a subsequence convergent to a point of E. In a T_1 second axiom space compactness, countable compactness and sequential compactness are equivalent.

A topological space is metacompact if and only if every open cover has an open point-finite refinement. A T_1 space is compact if and only if it is both metacompact and countably compact. A space is paracompact if and only if it is Hausdorff and every open cover contains an open locally finite refinement. Every paracompact space is metacompact, and in a Hausdorff space compactness implies paracompactness. In a countably compact Hausdorff space compactness, paracompactness, and metacompactness are equivalent.

In the appendix we prove the Hausdorff maximal principle and justify maximal arguments used in the proofs of the theorems. We assume here Zorn's lemma.

ON COMPACTNESS AND RELATED CONCEPTS

INTRODUCTION

The concept of compactness is of fundamental importance in mathematics. A subset E of a topological space is compact (or bicompact) provided every open cover of E contains a finite subcover. If every countable open cover of E has a finite subcover, then E is countably compact. The set E is sequentially compact if every sequence of points in E contains a subsequence convergent to a point of E. In the literature the term compact may refer to bicompactness, countable compactness, or sequential compactness.

In this paper we are concerned with equivalent formulations of these concepts and conditions under which the concepts themselves are equivalent. Compactness can be defined in terms of convergence instead of open covers. In Part I we consider three types of convergence based respectively on filters, nets, and directed functions. The concept of a filter was formulated by H. Cartan. The theory of convergence based on filters is developed fully in N. Bourbaki (2). (Numbers in parentheses refer to the bibliography.) The concept of a net is due to E. H. Moore and H. L. Smith (8). The theory of Moore-Smith convergence can be found in their paper or in Kelley (6). The concept of a directed function is the basis of the theory of convergence developed by E. McShane and T. A. Botts (7). There are grounds for the belief that these three theories of convergence are essentially the same. We have attempted to state and prove the theorems in such a way as to bring out this interrelationship. In particular, we have indicated how a directed function can be associated with every net and a net with any directed function. We assume here the axiom of choice.

In Part II we consider countable compactness. The well-known theorems of Cantor and Bolzano-Weierstrass for the real line become definitions of Cantor and Bolzano-Weierstrass properties for a set E in an abstract topological space. These properties and countable compactness are equivalent. (In the case of the Bolzano-Weierstrass property we assume the T_1 axiom.) We consider conditions under which these and other properties and compactness and sequential compactness are equivalent. The concept of paracompactness is due to J. Dieudonné (3). R. Arens and J. Dugundji (1) have introduced the concept of metacompactness. A compact space or a paracompact space is always metacompact, and in a Hausdorff space compactness implies paracompactness. In a countably compact Hausdorff space the concepts of compactness, paracompactness, and metacompactness are equivalent.

In the appendix we justify certain maximal arguments used in the proofs of the theorems. We assume here Zorn's lemma.

The expression "if and only if" is denoted throughout by "iff" and "with respect to" by "wrt".

We use the following notation: The symbol $\mathbb{X}$ represents a topological space, and E is a subset of $\mathbb{X}$. The letters O and G represent sets open in $\mathbb{X}$, while Q represents a set open in E. The letter F usually denotes a closed set. A German letter represents a family of sets. Let S be a subset of $\mathbb{X}$. Then

$\mathcal{C}S$ is the complement of S in $\mathbb{X}$

$\mathcal{K}S$ is the closure of S in $\mathbb{X}$

$\mathcal{J}S$ is the interior of S in $\mathbb{X}$

$\mathcal{D}S$ is the derived set of S in $\mathbb{X}$

If $S \subseteq E$ and the closure operator is applied relative to E we write $\mathcal{K}_E S$. This remark applies to all operators.

PART I

Let $\mathbb{X}$ be a topological space and E a subset of $\mathbb{X}$.

DEFINITION. A family $\mathcal{G}$ is an open cover of E iff each G in $\mathcal{G}$ is an open set and

$$E \subseteq \bigcup \{G \mid G \in \mathcal{G}\}.$$

DEFINITION. The set E is said to possess the <u>strong Heine-Borel</u> property iff every open cover of E has a finite subcover.

The open cover of E may be uncountable.

Clearly E possesses the strong Heine-Borel property iff any family of open sets containing no finite subfamily covering E itself fails to cover E.

If $\mathcal{B}$ is a base for the topology $\mathcal{D}$, then the set E possesses the strong Heine-Borel property iff every subfamily $\mathcal{H}$ of $\mathcal{B}$ that covers E contains a finite subcover. For if $\mathcal{H}$ is a subfamily of $\mathcal{B}$ that covers E, then $\mathcal{H}$ is an open cover of E and so possesses a finite subcover. Conversely, let $\mathcal{G}$ be an open cover of E. Then since $\mathcal{B}$ is a base for $\mathcal{D}$, there exists a subfamily $\mathcal{B}_{\mathcal{G}}$ of $\mathcal{B}$ such that $\mathcal{B}_{\mathcal{G}}$ is a base for $\mathcal{G}$. Moreover, $\mathcal{B}_{\mathcal{G}}$ covers E. Hence, by assumption there exists a finite subcover $\mathcal{B}^*$ of $\mathcal{B}_{\mathcal{G}}$. Now for every $B \in \mathcal{B}^*$ there exists a $G_B \in \mathcal{G}$ such that $B \subseteq G_B$. Let

$$\mathcal{G}^* = \left\{ G_B \middle| B \in \mathcal{B}^* \right\}.$$

Then $\mathcal{G}^*$ is a finite cover of E.

DEFINITION. A family $\mathcal{F}$ of sets possesses the finite intersection property wrt E iff for every finite subfamily $\mathcal{F}^*$ of $\mathcal{F}$ the intersection

$$E \cap \bigcap \left\{ F \middle| F \in \mathcal{F}^* \right\}$$

is nonempty.

Every family of sets closed in E and possessing the finite intersection property wrt E is contained in a maximal family of sets closed in E and possessing the finite intersection property wrt E. (See the appendix for the proof.)

DEFINITION. A family $\mathcal{F}$ of sets possesses the intersection property wrt E iff

$$E \cap \bigcap \left\{ F \middle| F \in \mathcal{F} \right\}$$

is nonempty.

DEFINITION. The set E possesses the strong intersection property iff every family $\mathcal{F}$ of closed sets which has the finite intersection property wrt E also has the intersection property wrt E.

Clearly E possesses the strong intersection property iff every family $\mathcal{F}$ of closed sets which does not possess the intersection property wrt E contains a finite subfamily $\mathcal{F}^*$ such that

$$E \cap \bigcap \left\{ F \middle| F \in \mathcal{F}^* \right\}$$

is empty.

THEOREM. The set E possesses the strong Heine-Borel property iff E possesses the strong intersection property.

Proof. Suppose E possesses the strong Heine-Borel property but does not possess the strong intersection property. Then there exists a family $\mathfrak{F}$ of closed sets with the finite intersection property wrt E such that

$$E \cap \bigcap \{F \mid F \in \mathfrak{F}\}$$

is empty. Then

$$E \subseteq C \bigcap \{F \mid F \in \mathfrak{F}\} = \bigcup \{CF \mid F \in \mathfrak{F}\}$$

so that

$$\{CF \mid F \in \mathfrak{F}\}$$

is an open cover of E. By assumption there exists a finite subfamily $\mathfrak{F}^*$ of $\mathfrak{F}$ such that

$$\{CF \mid F \in \mathfrak{F}^*\}$$

covers E. Then

$$E \subseteq \bigcup \{CF \mid F \in \mathfrak{F}^*\} = C \bigcap \{F \mid F \in \mathfrak{F}^*\}$$

which implies that

$$E \cap \bigcap \{F \mid F \in \mathfrak{F}^*\}$$

is empty, a contradiction.

Now assume E possesses the strong intersection property but does not possess the strong Heine-Borel property. Let $\mathfrak{G}$ be an open cover of E with no finite subcover. Then for all finite subfamilies $\mathfrak{G}^*$ of $\mathfrak{G}$,

$$E \cap C \bigcup \{G \mid G \in \mathfrak{G}^*\} = E \cap \bigcap \{CG \mid G \in \mathfrak{G}^*\}$$

is nonempty. Hence the family

$$\{CG \mid G \in \mathfrak{G}\}$$

possesses the finite intersection property wrt E, so that

$$E \cap \bigcap \{ C\, G \mid G \in \mathcal{G} \}$$

is nonempty. This implies

$$E \cap C \bigcup \{ G \mid G \in \mathcal{G} \}$$

is nonempty, so that $\mathcal{G}$ does not cover E.

DEFINITION. The set E is said to be compact (or bicompact) iff E possesses the strong Heine-Borel property or equivalently E possesses the strong intersection property.

It is easily verified that E is compact in $\mathfrak{X}$ iff E is compact in E.

We note that every finite set is compact.

DEFINITION. A nonempty family $\mathfrak{A}$ of nonempty subsets of a set E is said to be directed downward by inclusion provided that for every $A_1 \in \mathfrak{A}$, $A_2 \in \mathfrak{A}$ there exists a set $A \in \mathfrak{A}$ such that $A \subseteq A_1 \cap A_2$. The family $\mathfrak{A}$ is then called a direction in E. ((7), p. 33.)

It is clear that a direction $\mathfrak{A}$ in E possesses the finite intersection property wrt E.

DEFINITION. A nonempty family $\mathfrak{N}$ of nonempty subsets of E is a nest (or tower or chain) in E iff whenever N_1 and N_2 are members of $\mathfrak{N}$, then either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$.

It is immediate that every nest in E is also a direction in E. In the appendix we prove the following statement:

HAUSDORFF MAXIMAL PRINCIPLE. If $\mathcal{U}$ is a family of subsets of E and $\mathcal{N}$ is a nest in E contained in $\mathcal{U}$, then there is a maximal nest $\mathcal{M}$ in E contained in $\mathcal{U}$ which contains $\mathcal{N}$. ((6), p. 32.)

A nest $\mathcal{M}$ in E contained in $\mathcal{U}$ is maximal in $\mathcal{U}$ iff no nest in E contained in $\mathcal{U}$ contains $\mathcal{M}$ properly.

DEFINITION. A nonempty family $\mathfrak{S}$ of nonempty subsets of E is termed a <u>filter on E</u> provided:

(i) If T is a subset of E which contains a subset S in $\mathfrak{S}$, then T is in $\mathfrak{S}$.

(ii) If $S_1 \in \mathfrak{S}$ and $S_2 \in \mathfrak{S}$, then $S_1 \cap S_2 \in \mathfrak{S}$.

Note that if $\mathfrak{S}$ is a filter on E, then $E \in \mathfrak{S}$. Clearly every filter on E is a direction in E.

DEFINITION. A filter $\mathfrak{U}$ on E is termed an <u>ultrafilter on E</u> provided no filter on E contains $\mathfrak{U}$ properly.

Thus if $\mathfrak{U}$ is an ultrafilter on E and $\mathfrak{S}$ is a filter on E such that $\mathfrak{U} \subseteq \mathfrak{S}$, then $\mathfrak{U} = \mathfrak{S}$. Every filter $\mathfrak{S}$ on E is contained in some ultrafilter $\mathfrak{U}$ on E. (See the appendix for the proof.)

DEFINITION. Let $\mathfrak{S}$ be a filter on E and x a point of E. Then x <u>adheres to $\mathfrak{S}$</u> or x is a <u>point of adherence of $\mathfrak{S}$</u> iff

$$x \in \bigcap \{ K_E S \mid S \in \mathfrak{S} \}.$$

The set of all adherence points of $\mathfrak{S}$ is called the <u>adherence set of $\mathfrak{S}$</u>.

A set V_x such that $x \in \mathcal{J}_E V_x$ is called a neighborhood of x.

DEFINITION. A filter $\mathfrak{S}$ on E is said to converge to a point x in E iff every neighborhood V_x of x in E contains an S in $\mathfrak{S}$. The point x is then termed a limit of $\mathfrak{S}$.

It can be shown that every convergent filter on E has a unique limit in E iff E is a Hausdorff space.

A filter $\mathfrak{S}$ on E converges to a point x in E iff every neighborhood V_x of x in E is in $\mathfrak{S}$. For assume $\mathfrak{S}$ converges to x. If V_x is a neighborhood of x in E, then there exists a set S in $\mathfrak{S}$ such that $S \subseteq V_x$. Then by definition of a filter V_x is in S. Conversely, assume every neighborhood V_x of x is in $\mathfrak{S}$. Then every neighborhood V_x of x contains an S in $\mathfrak{S}$ (namely V_x itself), so that $\mathfrak{S}$ converges to x.

DEFINITION. For any $x \in E$ we define

$$\mathcal{B}_x = \{ V_x | V_x \text{ is a neighborhood of } x \text{ in } E \}.$$

It is easily verified that $\mathcal{B}_x$ is a filter on E. We call $\mathcal{B}_x$ the filter of neighborhoods of x. By the above remark we have that a filter $\mathfrak{S}$ on E converges to a point x of E iff $\mathcal{B}_x \subseteq \mathfrak{S}$.

If a filter $\mathfrak{S}$ on E converges to $x \in E$, then x adheres to $\mathfrak{S}$. For let $S \in \mathfrak{S}$ and let Q_x be a set containing x which is open in E. Then Q_x is a neighborhood of x in E, so that $Q_x \in \mathfrak{S}$. Hence $Q_x \cap S \in \mathfrak{S}$, so that $Q_x \cap S$ is nonempty. Thus $x \in \mathcal{K}_E S$, and hence x adheres to $\mathfrak{S}$.

THEOREM. For any set E the following statements are equivalent:

(a) E is compact.

(b) If $\mathcal{U}$ is a direction in E and each A in $\mathcal{U}$ is closed in E, then $\mathcal{U}$ possesses the intersection property wrt E. ((7), p. 50.)

(c) If $\mathcal{N}$ is a nest in E and each N in $\mathcal{N}$ is closed in E, then $\mathcal{N}$ possesses the intersection property wrt E.

(d) Every filter on E possesses at least one adherence point. ((2), p. 91.)

(e) Every ultrafilter on E converges to a point of E. ((2), p. 91.)

Proof. We show first that (a), (b), and (c) are equivalent.

To show that (a) implies (b) we recall that every direction $\mathcal{U}$ in E possesses the finite intersection property wrt E. Also $\mathcal{U}$ is a family of sets closed in E. By assumption E is compact and so possesses the strong intersection property. Hence $\mathcal{U}$ possesses the intersection property wrt E.

Let $\mathcal{N}$ be a nest in E such that each N in $\mathcal{N}$ is closed in E. Then $\mathcal{N}$ is a direction in E and so by assumption $\mathcal{N}$ possesses the intersection property wrt E. Hence (b) implies (c).

Next we prove that (c) implies (a). Let $\mathcal{F}$ be an arbitrary family of sets closed in E possessing the finite intersection property wrt E. We prove

$$\bigcap \{F \mid F \in \mathcal{F}\}$$

is nonempty, whence E is compact. Let $\mathcal{B}$ be a maximal family of sets closed in E which contains $\mathfrak{F}$ and possesses the finite intersection property wrt E. By the Hausdorff maximal principle there exists a maximal nest $\mathcal{N}$ in E contained in $\mathcal{B}$. By hypothesis

$$\bigcap \{N \mid N \in \mathcal{N}\}$$

is nonempty. We prove

$$\bigcap \{N \mid N \in \mathcal{N}\} = \bigcap \{B \mid B \in \mathcal{B}\}$$

whence

$$\bigcap \{B \mid B \in \mathcal{B}\} \subseteq \bigcap \{F \mid F \in \mathfrak{F}\}$$

implies

$$\bigcap \{F \mid F \in \mathfrak{F}\} .$$

is nonempty. Let B be any set in $\mathcal{B}$. We assert (1)

$$B \cap \bigcap \{N \mid N \in \mathcal{N}\}$$

is nonempty, and (2)

$$B \cap \bigcap \{N \mid N \in \mathcal{N}\}$$

is in $\mathcal{B}$. Here

$$\{B \cap N \mid N \in \mathcal{N}\}$$

is a nest of sets closed in E, so that by assumption

$$\bigcap \{B \cap N \mid N \in \mathcal{N}\}$$

is nonempty. Hence (1) is true. Consider a finite sequence

$$\{B_j \mid 1 \leqq j \leqq k\}$$

of sets in $\mathcal{B}$. The family defined by

$$\mathcal{B}' = \mathcal{B} \cup \{\bigcap \{B_j \mid 1 \leqq j \leqq k\}\}$$

consists of sets closed in E, possess the finite intersection property wrt E and contains $\mathfrak{F}$. Hence $\mathcal{B}' = \mathcal{B}$ so that

$$\cap \{B_j \mid 1 \leq j \leq k\}$$

is in $\mathcal{B}$. Also, the family

$$\mathcal{B}'' = \mathcal{B} \cup \{B \cap \cap \{N \mid N \in \mathfrak{N}\}\}$$

consists of sets closed in E and contains $\mathfrak{F}$. Moreover $\mathcal{B}''$ possesses the finite intersection property wrt E, for if every B_j is in $\mathcal{B}$, then

$$\cap \{B_j \mid \leq j \leq k\} \cap B \cap \cap \{N \mid N \in \mathfrak{N}\} = B' \cap \cap \{N \mid N \in \mathfrak{N}\}$$

is nonempty. Hence $\mathcal{B}'' = \mathcal{B}$, so that

$$B \cap \cap \{N \mid N \in \mathfrak{N}\}$$

is in $\mathcal{B}$. This proves (2). Finally, we assert that if B is any set in $\mathcal{B}$, then

$$\cap \{N \mid N \in \mathfrak{N}\} \subseteq B$$

whence

$$\cap \{N \mid N \in \mathfrak{N}\} = \cap \{B \mid B \in \mathcal{B}\}.$$

Suppose there exists a set $B_0 \in \mathcal{B}$ such that

$$\cap \{N \mid N \in \mathfrak{N}\} \not\subseteq B_0 .$$

Then

$$B_0 \cap \cap \{N \mid N \in \mathfrak{N}\}$$

is properly contained in

$$\cap \{N \mid N \in \mathfrak{N}\}.$$

Let

$$\mathfrak{N}' = \mathfrak{N} \cup \{B_0 \cap \cap \{N \mid N \in \mathfrak{N}\}\}.$$

Clearly $\mathfrak{N}^*$ is a nest in E contained in $\mathcal{B}$ and $\mathfrak{N} \leq \mathfrak{N}^*$ so that $\mathfrak{N} = \mathfrak{N}^*$. But then

$$B_0 \cap \bigcap \{N \mid N \in \mathfrak{N}\}$$

is in $\mathfrak{N}$ and so

$$\bigcap \{N \mid N \in \mathfrak{N}\} \leq B_0 \cap \bigcap \{N \mid N \in \mathfrak{N}\}$$

a contradiction.

We prove next the equivalence of (c), (d), and (e). Let $\mathfrak{G}$ be a filter on E. If S_1 and S_2 are in $\mathfrak{G}$, then $S_1 \cap S_2$ is in $\mathfrak{G}$. Also

$$\mathcal{K}_E(S_1 \cap S_2) \leq \mathcal{K}_E S_1 \cap \mathcal{K}_E S_2 .$$

Hence

$$\{\mathcal{K}_E S \mid S \in \mathfrak{G}\}$$

is a direction in E of sets closed in E. By assumption E is compact, so that

$$\bigcap \{\mathcal{K}_E S \mid S \in \mathfrak{G}\}$$

is nonempty. Hence (c) implies (d).

Next we show that (d) implies (e). Let $\mathfrak{U}$ be an ultrafilter on E, and let x be an adherence point of $\mathfrak{U}$. We prove that $\mathfrak{U}$ converges to x. Let V_x be a neighborhood of x in E; we show that V_x is in $\mathfrak{U}$. Let S_0 be any set in $\mathfrak{U}$. Then $S_0 \cap V_x$ is nonempty, since x is in $\mathcal{K}_E S_0$. We may assume $V_x \neq E$ and $S_0 \nsubseteq V_x$, for otherwise V_x is in $\mathfrak{U}$. Define

$$W_1 = S_0 \cap V_x$$

and

$$W_2 = S_0 \cap \mathcal{C}_E V_x$$

Here

$$(x) \cap \mathcal{K}_E W_2 \subseteq \mathcal{J}_E V_x \cap \mathcal{K}_E S_0 \cap \mathcal{K}_E \mathcal{C}_E V_x = \emptyset$$

so that x is not in $\mathcal{K}_E W_2$, and hence W_2 is not in $\mathcal{U}$. Now for every S in $\mathcal{U}$ the set $S \cap W_1$ is not empty. Otherwise

$$\begin{aligned} S \cap S_0 &= S \cap (W_1 \cup W_2) \\ &= S \cap W_2 \subseteq W_2 \end{aligned}$$

implies W_2 is in $\mathcal{U}$. Consider the family

$$\mathfrak{F} = \{F \mid F \subseteq E \text{ and there exists an } S \in \mathcal{U} \text{ such that } S \cap W_1 \subseteq F\}$$

Clearly $\mathfrak{F}$ is a filter on E containing the ultrafilter $\mathcal{U}$, and so $\mathfrak{F} = \mathcal{U}$. Since W_1 is in $\mathfrak{F}$, it follows that W_1 is in $\mathcal{U}$. Hence V_x is in $\mathcal{U}$. This proves $\mathcal{B} \subseteq \mathcal{U}$, so that $\mathcal{U}$ converges to x.

Now assume (e) is true, and let $\mathfrak{S}$ be a filter on E. Let $\mathcal{U}$ be an ultrafilter on E such that $\mathfrak{S} \subseteq \mathcal{U}$. Then $\mathcal{U}$ converges to a point x of E which is an adherence point of $\mathfrak{S}$. Hence (d) is true.

Finally, we show that (d) implies (a). Let $\mathcal{H}$ be a direction in E such that each A in $\mathcal{H}$ is closed in E. Define

$$\mathfrak{S} = \{S \mid S \subseteq E \text{ and there exists an } A \in \mathcal{H} \text{ such that } A \subseteq S\}.$$

It is easily verified that $\mathfrak{S}$ is a filter on E containing $\mathcal{H}$, so that

$$\begin{aligned} \emptyset &\neq \bigcap \{\mathcal{K}_E S \mid S \in \mathfrak{S}\} \\ &\subseteq \bigcap \{A \mid A \in \mathcal{H}\} \end{aligned}$$

Hence E is compact.

DEFINITION. A cover $\mathcal{G}$ of the set E is said to be a *minimal* (or *irreducible*) *cover* of E iff $\mathcal{G}$ contains no proper subcover of E.

THEOREM. A set E is compact iff every open cover of E contains a minimal subcover.

Proof. An open cover of a compact set E has a finite and hence a minimal subcover. We prove the converse. Assume every open cover of E contains a minimal subcover. It is clear then that every cover of E by sets open in E has a minimal subcover. Suppose E is not compact. Then there exists a nest $\mathcal{N}$ in E such that each $N \in \mathcal{N}$ is closed in E and

$$\cap\{N \mid N \in \mathcal{N}\}$$

is empty. Consider the family

$$\{C_E N \mid N \in \mathcal{N}\}$$

Clearly

$$\{C_E N \mid N \in \mathcal{N}\}$$

is a cover of E by sets open in E. Hence by assumption there exists a subfamily $\mathcal{N}^*$ of $\mathcal{N}$ such that

$$\{C_E N \mid N \in \mathcal{N}^*\}$$

is a minimal cover of E. Choose $N_0 \in \mathcal{N}^*$. Now not every $N \in \mathcal{N}^*$ contains N_0, for then

$$\{C_E N \mid N \in \mathcal{N}^*\}$$

would fail to cover the points of E in N_0. Hence there exists a set $N_1 \in \eta^*$ such that N_1 is a proper subset of N_0. But then $\mathcal{C}_E N_0$ is a proper subset of $\mathcal{C}_E N_1$, so that

$$\{\mathcal{C}_E N \mid N \in \eta^*, N \neq N_0\}$$

is a proper subcover of

$$\{\mathcal{C}_E N \mid N \in \eta^*\}$$

a contradiction. Hence E is compact.

DEFINITION. Let U be an arbitrary nonempty set and $\geqq$ a binary relation in U. Then $\geqq$ directs U provided

(i) If u, v, and w are elements of U such that $u \geqq v$ and $v \geqq w$, then $u \geqq w$.

(ii) If $u \in U$, then $u \geqq u$.

(iii) If u and v are elements of U, then there exists a $w \in U$ such that $w \geqq u$ and $w \geqq v$.

$\{U, \geqq\}$ is termed a <u>directed set</u> iff $\geqq$ directs U.

Let $\{U, \geqq\}$ be a directed set, and let f be a function mapping U into a nonempty subset E of $\mathbb{X}$. Then $\{f, U, \geqq\}$ is called a net in E.

DEFINITION. Let $\{f, U, \geqq\}$ be a net in E, and let x be a point of E. Then x is a <u>cluster point</u> of $\{f, U, \geqq\}$ iff for every open set O_x containing x it is true that for every $u \in U$ there is an element v (depending on O_x and u) in U such that $v \geqq u$ and $f(v) \in O_x$.

A net in E may have one, none, or many cluster points in E. ((6), p. 71.)

DEFINITION. Let $\{f, U, \geqq\}$ be a net in E and $x \in E$. Then $\{f, U, \geqq\}$ <u>converges to</u> <u>x</u> iff for every open set O_x containing x there is an element u (depending on O_x) in U such that if $v \in U$ and $v \geqq u$, then $f(v) \in O_x$.

A net $\{f, U, \geqq\}$ in E may converge to a unique limit in E in the sense that if $\{f, U, \geqq\}$ converges to $x \in E$ and also to $y \in E$, then $x = y$. It can be shown that every convergent net in E has a unique limit in E iff E is a Hausdorff space. ((6), p. 67.)

Clearly if a net in E converges to a point x in E, then x is a cluster point of the net. The converse is false, even if the net has a unique cluster point. ((6), p. 71.)

DEFINITION. Let $\{f, U, \geqq\}$ and $\{g, Y, \geqq\}$ be nets in E. Then $\{g, Y, \geqq\}$ is called a <u>subnet</u> of $\{f, U, \geqq\}$ iff there exists a function v mapping Y into U such that

(i) If $y \in Y$, then $g(y) = f(v(y))$.

(ii) For each $u \in U$ there is a $y \in Y$ such that if $z \in Y$ and $z \geqq y$, then $v(z) \geqq u$.

DEFINITION. Let U be a nonempty arbitrary set, $\mathcal{U}$ a direction in U and f a mapping of U into a nonempty subset E of $\mathbb{X}$. Then $\{f, \mathcal{U}\}$ is termed a <u>directed function in E</u>. ((7), p. 35.)

DEFINITION. Let $\{f, \mathcal{U}\}$ be a directed function in E and x a point of E. Then x is a <u>cluster point</u> of $\{f, \mathcal{U}\}$ iff for every open set O_x containing x and every $A \in \mathcal{U}$ there exists a u (depending on O_x and A) in A such that $f(u) \in O_x$.

A directed function in E may have one, none, or many cluster points.

DEFINITION. Let $\{f, \mathcal{U}\}$ be a directed function in E and $x \in E$. Then $\{f, \mathcal{U}\}$ <u>converges to</u> x iff for every open set O_x containing x there is an A (depending on O_x) in $\mathcal{U}$ such that if $u \in A$, then $f(u) \in O_x$.

A directed function $\{f, \mathcal{U}\}$ in E may converge to a unique limit in E in the sense that if $\{f, \mathcal{U}\}$ converges to $x \in E$ and also to $y \in E$, then $x = y$. It can be proved that every convergent directed function in E has a unique limit point in E iff E is a Hausdorff space. ((7), p. 41.)

Clearly if a directed function in E converges to a point $x \in E$, then x is a cluster point of the directed function.

DEFINITION. Let $\{f, \mathcal{U}\}$ be a directed function in E and $\mathcal{B}$ a direction in U. Then $\mathcal{B}$ is a <u>subdirection</u> of $\mathcal{U}$ iff for every $A \in \mathcal{U}$ there is a $B \in \mathcal{B}$ such that $B \subseteq A$. The directed function $\{f, \mathcal{B}\}$ is termed a <u>subdirected function</u> of $\{f, \mathcal{U}\}$ iff $\mathcal{B}$ is a subdirection of $\mathcal{U}$.

THEOREM. Let $\{f, U, \geq\}$ be a net in E. For $u \in U$ define

$$A_u = \{ v \mid v \in U \text{ and } v \geq u \}$$

Let

$$\mathcal{U} = \{A_u \mid u \in U\}.$$

Then $\{f, \mathcal{U}\}$ is a directed function in E.

Proof. By the definition of a net f is a mapping from U into E. We assert that $\mathcal{U}$ is a direction in U. Since U is nonempty and $u \geqq u$ for every $u \in U$, $\mathcal{U}$ is a nonempty family of nonempty sets. Let A_{u_1} and A_{u_2} be sets in $\mathcal{U}$. There is an element $u_3 \in U$ such that $u_3 \geqq u_1$ and $u_3 \geqq u_2$. If $u \geqq u_3$, then $u \geqq u_1$ and $u \geqq u_2$, and hence $A_{u_3} \subseteq A_{u_1} \cap A_{u_2}$.

THEOREM. Let $\{f, \mathcal{U}\}$ be a directed function in E. For A_1 and A_2 in $\mathcal{U}$ let $A_1 \geqq A_2$ iff $A_1 \subseteq A_2$. By the axiom of choice there exists a mapping c of $\mathcal{U}$ into U such that $c(A) \in A$ for every $A \in \mathcal{U}$. Let μ be the mapping of $\mathcal{U}$ into E defined by $\mu(A) = f(c(A))$. Then $\{\mu, \mathcal{U}, \geqq\}$ is a net in E.

Proof. Clearly $A \geqq A$ for every A in $\mathcal{U}$, and if $A_1 \geqq A_2$ and $A_2 \geqq A_3$, then $A_1 \geqq A_3$. Let A_1 and A_2 be sets in $\mathcal{U}$. Since $\mathcal{U}$ is a direction, there exists a set A_3 in $\mathcal{U}$ such that $A_3 \subseteq A_1 \cap A_2$. Then $A_3 \geqq A_1$ and $A_3 \geqq A_2$. Hence $\{\mathcal{U}, \geqq\}$ is a directed set, and so $\{\mu, \mathcal{U}, \geqq\}$ is a net in E.

THEOREM. The following statements are equivalent.

(a) E is compact.

(b) Every net in E has a cluster point in E.

(c) Every directed function in E has a cluster point in E.

(d) Every directed function in E has a subdirected function in E convergent to a point of E.

(e) Every net in E has a subnet in E convergent to a point of E.

Proof. Assume E is compact. Let $\{f, U, \geqq\}$ be a net in E. For each $u \in U$ define

$$F_u = \mathcal{K}_E \{f(v) \mid v \in U \text{ and } v \geqq u\}$$

Define

$$\mathfrak{F} = \{F_u \mid u \in U\}$$

Consider a finite subfamily

$$\{F_{u_j} \mid 1 \leqq j \leqq k\} \quad \text{of } \mathfrak{F}$$

Since $\geqq$ directs U, there is a $v \in U$ such that $v \geqq u_j$ for $1 \leqq j \leqq k$. Then $f(v) \in F_{u_j}$ for $1 \leqq j \leqq k$, so that

$$f(v) \in \cap \{F_{u_j} \mid 1 \leqq j \leqq k\}$$

Hence $\mathfrak{F}$ is a family of sets closed in E possessing the finite intersection property, so that by assumption

$$\cap \{F_u \mid u \in U\}$$

is nonempty. Let

$$x \in \cap \{F_u \mid u \in U\}$$

Let Q_x be an open set in E containing x, and let $u \in U$. Then $x \in \mathcal{K}_E \{f(v) \mid v \in U \text{ and } v \geqq u\}$, so that $Q_x \cap \cap \{f(v) \mid v \in U \text{ and } v \geqq u\}$ is nonempty. Hence there is a $v \in U$ such that $v \geqq u$ and $f(v) \in Q_x$. It follows that x is a cluster point of $\{f, U, \geqq\}$. Hence (a) implies (b).

Next we show that (b) implies (c). Let $\{f, \mathcal{U}\}$ be a directed function in E, and let $\{\mu, \mathcal{U}, \geqq\}$ be the net in E as defined above. By assumption there is an element $x \in E$ such that x is a cluster point of $\{\mu, \mathcal{U}, \geqq\}$. Choose an open set Q_x containing x and a set $A \in \mathcal{U}$. Then there is a set $A' \in \mathcal{U}$ such that $A' \geqq A$ and $\mu(A') \in Q_x$. Let $u = c(A')$, where c is the choice function which occurs in the definition of the net $\{\mu, \mathcal{U}, \geqq\}$. Then $u \in A' \subseteq A$ and $f(u) = f(c(A')) = \mu(A') \in Q_x$. Hence x is a cluster point of $\{f, \mathcal{U}\}$.

To prove that (c) implies (a) we suppose that E is not compact. Let $\mathcal{G}$ be an open cover of E such that no finite subfamily of $\mathcal{G}$ covers E. Define $\mathcal{U} = \{A \mid \exists$ a finite subfamily $\mathcal{G}^*$ of $\mathcal{G}$ such that $A = E \cap C \cup \{G \mid G \in \mathcal{G}^*\}\}$. Here $\mathcal{U}$ is a nonempty family of nonempty sets. Let $A_1 = E \cap C \cup \{G \mid G \in \mathcal{G}_1^*\}$ and $A_2 = E \cap C \cup \{G \mid G \in \mathcal{G}_2^*\}$ be sets in A. Define $A = E \cap C \cup \{G \mid G \in \mathcal{G}_1^* \cup \mathcal{G}_2^*\}$. Then $A \in \mathcal{U}$ and $A \subseteq A_1 \cap A_2$. Hence $\mathcal{U}$ is a direction in E. Let $f : E \to E$ be the identity mapping of E onto E. By definition $\{f, \mathcal{U}\}$ is a directed function in E. But for every $x \in E$ there is a $G \in \mathcal{G}$ containing x, and then $E \cap C\, G$ is a set in $\mathcal{U}$ such that $f(u) = u$ belongs to $C\, G$ for all $u \in E \cap C\, G$. Hence $\{f, \mathcal{U}\}$ has no cluster point in E.

Hence (a), (b), and (c) are equivalent.

We show next that (c) implies (d). Let $\{f, \mathcal{U}\}$ be a directed function in E, and let $x \in E$ be a cluster point of $\{f, \mathcal{U}\}$. Let $\mathcal{B}$ be the family of all sets of the type $A \cap f^{-1}(Q_x)$, where $A \in \mathcal{U}$ and Q_x is open in E and contains x. We assert that $\mathcal{B}$ is a direction in U. Since x is a cluster point of $\{f, \mathcal{U}\}$, for every Q_x and $A \in \mathcal{U}$ there is an element $u \in A$ such that $f(u) \in Q_x$ and therefore

$u\epsilon A \cap f^{-1}(Q_x)$. Hence $\mathcal{B}$ is a nonempty family of nonempty subsets of U. Let B_1 and B_2 be sets in $\mathcal{B}$. Then there exists sets A_1 and A_2 in $\mathcal{U}$ and Q_x^1 and Q_x^2 such that $B_1 = A_1 \cap f^{-1}(Q_x^1)$ and $B_2 = A_2 \cap f^{-1}(Q_x^2)$. Since $\mathcal{U}$ is a direction in U, there is an $A\epsilon\mathcal{U}$ such that $A \subseteq A_1 \cap A_2$. Put $Q_x = Q_x^1 \cap Q_x^2$, and let $B = A \cap f^{-1}(Q_x)$. Then B is in $\mathcal{B}$ and $B \subseteq B_1 \cap B_2$. Hence $\mathcal{B}$ is a direction in U. By definition it is clear that $\mathcal{B}$ is an subdirection of $\mathcal{U}$, so that $\{f, \mathcal{B}\}$ is a subdirected function of $\{f, \mathcal{U}\}$. Let $A\epsilon\mathcal{U}$ be arbitrary. Then for every Q_x there is a set $B = A \cap f^{-1}(Q_x)$ in $\mathcal{B}$ such that if $u\epsilon B$, then $f(u)\epsilon Q_x$. Hence $\{f, \mathcal{B}\}$ converges to x.

We prove now (d) implies (e). Let $\{f, U, \geqq\}$ be a net in E, and let $\{f, \mathcal{U}\}$ be the directed function in E as defined above. By assumption there is a subdirected function $\{f, \mathcal{B}\}$ of $\{f, \mathcal{U}\}$ and a point x in E such that $\{f, \mathcal{B}\}$ converges to x. Let $\{\mu, \mathcal{B}, \geqq\}$ be the net in E as defined above. We assert that $\{\mu, \mathcal{B}, \geqq\}$ is a subnet of $\{f, U, \geqq\}$. Define $v = c$ where c is the choice function mapping $\mathcal{B}$ into U which occurs in the definition of $\{\mu, \mathcal{B}, \geqq\}$. Then $B\epsilon\mathcal{B}$ implies that

$$\begin{aligned}\mu(B) &= f(c(B)) \\ &= f(v(B)).\end{aligned}$$

Let $u\epsilon U$ be arbitrary. Then the set

$$A_u = \{v \mid v\epsilon U \text{ and } v \geqq u\}$$

is in $\mathcal{U}$ by definition, so that, since $\mathcal{B}$ is a subdirection of $\mathcal{U}$, there is a $B \in \mathcal{B}$ such that $B \subseteq A_u$. Now if $B' \in \mathcal{B}$ and $B' \geqq B$, then $B' \subseteq B$ and therefore $\nu(B') = c(B') \in B' \subseteq A_u$, so that $\nu(B') \geqq u$. Hence $\{\mu, \mathcal{B}, \geqq\}$ is a subnet of $\{f, U, \geqq\}$. Now let Q_x be an open set in E which contains x. Since $\{f, \mathcal{B}\}$ converges to x, there is a $B \in \mathcal{B}$ such that if $u \in B$, then $f(u) \in Q_x$. If $B' \in \mathcal{B}$ and $B' \geqq B$, then $c(B') \in B' \subseteq B$ and therefore $\mu(B') = f(c(B')) \in Q_x$. Hence $\{\mu, \mathcal{B}, \geqq\}$ converges to x.

Finally we show that (c) implies (b). Let $\{f, U, \geqq\}$ be a net in E. By assumption there is a subnet $\{g, Y, \geqq\}$ of $\{f, U, \geqq\}$ and a point x in E such that $\{g, Y, \geqq\}$ converges to x. We assert that x is a cluster point of $\{f, U, \geqq\}$. Let Q_x be an open set in E which contains x. Then there is an element $y_1 \in Y$ such that if $s \in Y$ and $s \geqq y_1$, then $g(s) \in Q_x$. Let $u \in U$. Since $\{g, Y, \geqq\}$ is a subnet, there is an element $y_2 \in Y$ such that if $s \in Y$ and $s \geqq y_2$, then $\nu(s) \geqq u$, where ν is the function which occurs in the definition of a subnet. Here $\{Y, \geqq\}$ is a directed set, so that there is an element $y \in Y$ such that $y \geqq y_1$ and $y \geqq y_2$. Define $v = \nu(y)$. Note $v \in U$. Then $y \geqq y_2$ implies that $v = \nu(y) \geqq u$ and $y \geqq y_1$ that $f(v) = f(\nu(y)) = g(y) \in Q_x$. Hence x is a cluster point of $\{f, U, \geqq\}$.

PART II

Let E be a subset of a topological space $\mathfrak{X}$.

DEFINITION. We define the following properties:

(a) E possesses the <u>weak Heine-Borel</u> property iff every countable open cover of E contains a finite subcover.

(b) E possesses the <u>weak intersection</u> property iff every countable family of closed sets which has the finite intersection property wrt E has the intersection property wrt E.

(c) E possesses the <u>Cantor</u> property iff every countable family

$$\{F_n | n \geq 1\}$$

of closed sets such that $F_{n+1} \subseteq F_n$ and $E \cap F_n \neq \emptyset$ for all n has the intersection property wrt E.

(d) E possesses the <u>limit superior</u> property iff for every infinite sequence

$$\{E_n | n \geq 1\}$$

of nonempty subsets of E, it is true that $E \cap l_s E_n$ is nonempty.

THEOREM. For any set E the following properties are equivalent:

(a) Weak Heine-Borel.

(b) Weak intersection.

(c) Cantor.

(d) Limit superior.

Proof. We prove first that (a) implies (b). Let

$$\{F_n \mid n \geq 1\}$$

be a family of closed sets which has the finite intersection property wrt E. Suppose

$$E \cap \bigcap \{F_n \mid n \geq 1\}$$

is empty. Then

$$E \subseteq \mathcal{C} \bigcap \{F_n \mid n \geq 1\}$$
$$= \bigcup \{\mathcal{C} F_n \mid n \geq 1\}$$

so that

$$\{\mathcal{C} F_n \mid n \geq 1\}$$

is an open cover of E. Hence by assumption

$$E \subseteq \bigcup \{\mathcal{C} F_n \mid 1 \leq n \leq N\}$$
$$= \mathcal{C} \bigcap \{F_n \mid 1 \leq n \leq N\}$$

for some N. This implies that

$$E \cap \bigcap \{F_n \mid 1 \leq n \leq N\}$$

is empty, a contradiction.

Next we prove that (b) implies (c). Let

$$\{F_n \mid n \geq 1\}$$

be a countable family of closed sets such that $F_{n+1} \subseteq F_n$ and $E \cap F_n$ is nonempty for every n. Here

$$F_n = \bigcap \{F_j \mid 1 \leqq j \leqq n\}$$

so that

$$E \cap \bigcap \{F_j \mid 1 \leqq j \leqq n\}$$

is nonempty. Hence by assumption

$$E \cap \bigcap \{F_j \mid j \geqq 1\}$$

is nonempty. Thus E has the Cantor property.

We prove now (c) implies (d). Let

$$\{E_n \mid n \geqq 1\}$$

be an infinite sequence of nonempty subsets of E. Define

$$F_n = \mathcal{K} \cup \{E_j \mid j \geqq n\}$$

Then

$$\begin{aligned} F_{n+1} &= \mathcal{K} \cup \{E_j \mid j \geqq n+1\} \\ &\subseteq \mathcal{K} \cup \{E_j \mid j \geqq n\} \\ &= F_n \end{aligned}$$

and

$$\begin{aligned} E \cap F_n &= E \cap \mathcal{K} \cup \{E_j \mid j \geqq n\} \\ &\supseteq E \cap E_n \\ &= E_n \\ &\neq \emptyset \end{aligned}$$

for every n. By assumption

$$\emptyset \neq E \cap \bigcap \{F_n | n \geqq 1\}$$
$$= E \cap \bigcap \{\mathcal{K} \cup \{E_j | j \geqq n\} \mid n \geqq 1\}$$
$$= E \cap l_s E_n$$

so that E possesses the limit superior property.

Finally, we show that (d) implies (a). Let

$$\{G_n | n \geqq 1\}$$

be a countably infinite open cover of E such that

$$E \cap \complement \cup \{G_n | 1 \leqq n \leqq N\} = E \cap \bigcap \{\complement G_n | 1 \leqq n \leqq N\}$$

is nonempty for every N. Define

$$E_N = E \cap \bigcap \{\complement G_n | 1 \leqq n \leqq N\}$$

By assumption $E \cap l_s E_n$ is nonempty. Let $x \varepsilon E \cap l_s E_n$. Since

$$E \subseteq \cup \{G_n | n \geqq 1\}$$

there exists an n_o such that $x \varepsilon G_{n_o}$. Now $N > n_o$ implies

$$G_{n_o} \cap E_N = G_{n_o} \cap E \cap \bigcap \{\complement G_n | 1 \leqq n \leqq N\}$$
$$\subseteq G_{n_o} \cap \complement G_{n_o}$$

is empty, so that $x \notin l_s E_n$.

We recall that x_o is a limit point of an infinite sequence

$$\{x_n | n \geqq 1\}$$

iff for every open set O containing x_0, there exists a subsequence

$$\{x_{n_j} | j \geqq 1\}$$

such that x_{n_j} is contained in O for every j. Hence x_0 is a limit point of

$$\{x_n | n \geqq 1\}$$

iff x_0 belongs to $l_S(x_n)$. It follows that (d) is equivalent to the statement (d'): Every sequence of points in E has a limit point in E.

DEFINITION. The set E is said to be <u>countably compact</u> iff E possesses the weak Heine-Borel property.

Clearly a compact set is countably compact but the converse is not true. ((4), p. 290 and (5), pp. 55-56.)

DEFINITION. The set E possesses the <u>Bolzano-Weierstrass</u> property iff for every infinite subset S of E, the set $E \cap \mathcal{D}S$ is nonempty.

Every finite set vacuously possesses the Bolzano-Weierstrass property.

THEOREM. Every countably compact set E possesses the Bolzano-Weierstrass property.

Proof. Let S be an infinite subset of E, and let

$$\{x_n | n \geqq 1\}$$

be an infinite sequence of points in S, where $x_i \neq x_j$ for $i \neq j$.

By assumption E possesses the limit superior property, so that $E \cap \ell_S(x_n)$ is nonempty. Let $x \in E \cap \ell_S(x_n)$. If O is any open set containing x, there exists a subsequence

$$\{x_{n_j} \mid j \geqq 1\}$$

such that x_{n_j} is in O and $x_{n_j} \neq x$ for every j. Therefore

$$\emptyset \neq x_{n_j} \cap O \cap \mathcal{C}(x)$$
$$\subseteq S \cap O \cap \mathcal{C}(x)$$

Hence $x \in \mathcal{D}S$, so that $E \cap \mathcal{D}S$ is nonempty.

THEOREM. In a T_1 space every set E possessing the Bolzano-Weierstrass property is countably compact.

Proof. Suppose E is not countably compact. Then there exists a countably infinite open cover

$$\{O_n \mid n \geqq 1\}$$

of E which contains no finite subcover. Then

$$E \cap \mathcal{C} \cup \{O_n \mid 1 \leqq n \leqq k\}$$

is nonempty for every k. We assert in fact that

$$E \cap \mathcal{C} \cup \{O_n \mid 1 \leqq n \leqq k\}$$

is infinite for each k. Otherwise there exists a k_0 such that

$$E \cap \mathcal{C} \cup \{O_n \mid 1 \leqq n \leqq k_0\} = \{y_1, y_2, \ldots, y_{j_0}\}$$

Let $y_j \in O_{n_j}$ for $1 \leq j \leq j_0$. Then

$$E = (E \cap \cup\{O_n | 1 \leq n \leq k_0\}) \cup (E \cap \complement \cup \{O_n | 1 \leq n \leq k_0\})$$

$$\subseteq \cup\{O_n | 1 \leq n \leq k_0\} \cup \cup\{(y_j) | 1 \leq j \leq j_0\}$$

$$\subseteq \cup\{O_n | 1 \leq n \leq k_0\} \cup \cup\{O_{n_j} | 1 \leq j \leq j_0\}$$

so that

$$\{O_n | n \geq 1\}$$

contains a finite subcover. Now for each k let

$$x_k \in E \cap \complement \cup \{O_n | 1 \leq n \leq k\}$$

where $x_r \neq x_s$ for $r \neq s$. Define

$$S = \cup\{(x_k) | k \geq 1\}$$

By assumption there exists a point $x_0 \in E \cap \mathcal{D}S$. Let $x_0 \in O_{n_0}$. Since the space is T_1 and $x \in \mathcal{D}S$ it follows that the intersection $S \cap O_{n_0}$ is infinite. But $k > k_0$ implies that $x_k \in \complement O_{n_0}$. This is a contradiction.

COROLLARY. For any set E in a T_1 space the following properties are equivalent:

(a) Weak Heine-Borel.

(b) Weak intersection.

(c) Cantor.

(d) Limit superior.

(e) Bolzano-Weierstrass.

THEOREM. In a T_1 space every infinite open cover of a countably compact set E has a proper subcover.

Proof. Let $\mathcal{G}$ be an infinite open cover of E such that no proper subfamily of $\mathcal{G}$ covers E. Then for every G in $\mathcal{G}$ there is a point $x_G \in E$ such that $x_G \in G$ but

$$x_G \notin \cup \{ G' \mid G' \in \mathcal{G}, G' \neq G \}.$$

Define

$$S = \{ x_G \mid G \in \mathcal{G} \}.$$

Suppose $G_1 \neq G_2$. Then $x_{G_1} \in G_1$ and $x_{G_1} \notin G_2$ so that $x_{G_1} \neq x_{G_2}$. Hence S is infinite, so that by assumption $E \cap \mathcal{D}S$ is nonempty. Let $x_0 \in E \cap \mathcal{D}S$. Since $\mathcal{G}$ covers E, there is a $G_0 \in \mathcal{G}$ such that $x_0 \in G_0$. Since $x_0 \in \mathcal{D}S$ and the space is T_1, the intersection $G_0 \cap S$ is infinite. Hence G_0 contains two distinct points of S, a contradiction.

The assumption that the cover be infinite is necessary, as may be seen by considering a finite set.

THEOREM. A T_1 space is countably compact iff every infinite open cover has a proper subcover.

Proof. Suppose the T_1 space $\mathbb{X}$ is not countably compact. Let S be an infinite subset of $\mathbb{X}$ such that $\mathcal{D}S$ is empty. Then S is closed and hence isolated, so that every point x of S is contained in an open set G_x which contains no other points of S.

Define

$$\mathcal{G} = \{ G_x | x \in S \} .$$

If $\mathcal{G}$ covers $\mathbb{X}$, then $\mathcal{G}$ is an infinite open cover which has no proper subcover. If $\mathcal{G}$ does not cover $\mathbb{X}$, then

$$\mathcal{H} = \mathcal{G} \cup \{ CS \}$$

is an infinite open cover with no proper subcover.

THEOREM. A countable set E is compact iff it is countably compact.

Proof. Let

$$E = \{ x_n | n \geqq 1 \} .$$

Assume E is countably compact, and let $\mathcal{G}$ be an open cover of E. For each n let G_n be a set of the cover $\mathcal{G}$ containing x_n. Then

$$\{ G_n | n \geqq 1 \}$$

is a countable open cover of E and hence contains a finite subcover. Therefore $\mathcal{G}$ contains a finite subcover, and hence E is compact.

DEFINITION. A topological space is termed a <u>Lindelöf</u> space iff every family of open sets $\mathcal{G}$ contains a countable subfamily $\mathcal{G}^*$ such that

$$\cup \{ G | G \in \mathcal{G} \} = \cup \{ G | G \in \mathcal{G}^* \} .$$

THEOREM. In a Lindelöf space a set E is compact iff it is countably compact.

Proof. Assume E is countably compact. Then every open cover of E contains a countable subcover and hence a finite subcover.

DEFINITION. A topological space satisfies the <u>second axiom of countability</u> iff there exists a countable base for its topology. The space is the termed a <u>second axiom space</u>.

Every second axiom space is a Lindelöf space.

THEOREM. In a second axiom space a set E is compact iff it is countably compact.

DEFINITION. The set E is said to be <u>sequentially compact</u> iff every sequence of points in E contains a subsequence converging to a point in E.

THEOREM. Every sequentially compact set E is countably compact.

Proof. We prove that E possesses the limit superior property. Let

$$\{E_n | n \geq 1\}$$

be an infinite sequence of sets such that E_n is a nonempty subset of E for each n. Choose a sequence

$$\{x_n | n \geq 1\}$$

such that $x_n \in E_n$. By assumption there exists a point x_0 of E and a subsequence

$$\{x_{n_j} | j \geq 1\}$$

of

$$\{x_n | n \geqq 1\}$$

such that $\exists \lim x_{n_j} = x_o$. Then

$$(x_o) \subseteq E \cap li(x_{n_j})$$
$$\subseteq E \cap lsE_n$$

so that E possesses the limit superior property.

DEFINITION. A topological space satisfies the <u>first axiom of countability</u> iff there exists a countable base at each point of the space. The space is then termed a <u>first axiom space</u>.

THEOREM. In a first axiom space a set E is countably compact iff it is sequentially compact.

Proof. Assume E is countably compact. Let

$$\{x_n | n \geqq 1\}$$

be an infinite sequence of points in E. By assumption E possesses the limit superior property, so that $E \cap ls(x_n)$ is nonempty. Let x_o be a point of $E \cap ls(x_n)$. Since the space is first axiom, there exists a subsequence

$$\{x_{n_j} | j \geqq 1\}$$

such that $\exists \lim x_{n_j} = x_o$. Hence E is sequentially compact.

Combining results we obtain the following:

THEOREM. In a first axiom T_1 space the following statements are equivalent:

(a) E possesses the Bolzano-Weierstrass property.

(b) E is countably compact.

(c) E is sequentially compact.

Every second axiom space is a first axiom space. Hence the following:

THEOREM. In a second axiom space the following statements are equivalent:

(a) E is compact.

(b) E is countably compact.

(c) E is sequentially compact.

If in addition the space is T_1, then (a), (b), and (c) are equivalent to the statement

(d) E possesses the Bolzano-Weierstrass property.

In particular the Euclidean spaces R_n are second axiom T_1 spaces.

DEFINITION. Let $\mathfrak{U}$ and $\mathfrak{B}$ be two open covers of the topological space $\mathbb{X}$. Then $\mathfrak{U}$ is called a refinement of $\mathfrak{B}$ iff for each $A \in \mathfrak{U}$ there is a $B \in \mathfrak{B}$ containing A.

DEFINITION. The cover $\mathfrak{U}$ is point-finite iff each point of $\mathbb{X}$ is contained in only finitely many members of $\mathfrak{U}$.

DEFINITION. A topological space $\mathfrak{X}$ is <u>metacompact</u> iff every open cover of $\mathfrak{X}$ has an open point-finite refinement. ((1), p. 142.)

Every compact space is metacompact, since any open cover of the space has a finite subcover which is clearly an open point-finite refinement of the given cover. On the other hand, a metacompact space need not be compact. For example, any infinite discrete space is metacompact but not compact.

Clearly every discrete space is metacompact.

Every point-finite open cover of a space $\mathfrak{X}$ has a minimal subcover. (See the appendix for the proof.)

THEOREM. A T_1 space $\mathfrak{X}$ is compact iff it is both metacompact and countably compact. ((1), p. 142.)

Proof. Assume $\mathfrak{X}$ is countably compact and metacompact. Let $\mathcal{G}$ be an open cover of $\mathfrak{X}$. By assumption there exists an open point-finite refinement $\mathcal{H}$ of $\mathcal{G}$. Let $\mathcal{H}^*$ be a minimal subcover of $\mathcal{H}$. Then $\mathcal{H}^*$ must be finite, for an infinite open cover of a countably compact $\mathfrak{X}$ has a proper subcover. Each H in $\mathcal{H}^*$ is contained in some G in $\mathcal{G}$, so that $\mathcal{G}$ contains a finite subcover. Hence the space is compact.

This theorem can be restated in the following way:

THEOREM. Let $\mathfrak{X}$ be a countably compact T_1 space. Then $\mathfrak{X}$ is compact iff it is metacompact. ((1), p. 142.)

DEFINITION. A cover $\mathcal{G}$ of a topological space $\mathbb{X}$ is termed <u>locally finite</u> iff for each point x of $\mathbb{X}$ there is an open set containing x which intersects only a finite number of sets of $\mathcal{G}$.

DEFINITION. A space is <u>paracompact</u> iff it is Hausdorff and each open cover has an open locally finite refinement. (3)

We note that some authors (e.g., (6)) specify "regular" rather than "Hausdorff". It can be shown that every paracompact space is regular and normal.

Every paracompact space is evidently metacompact. Hence, we have the following:

THEOREM. Let $\mathbb{X}$ be a paracompact space. Then $\mathbb{X}$ is compact iff $\mathbb{X}$ is countably compact. ((1), p. 143.)

Clearly every compact Hausdorff space is paracompact. Hence the preceding theorem can be stated in the following way:

THEOREM. Let $\mathbb{X}$ be a countably compact Hausdorff space. Then $\mathbb{X}$ is compact iff $\mathbb{X}$ is paracompact.

Combining the above theorems we have the following:

THEOREM. Let $\mathbb{X}$ be a countably compact Hausdorff space. Then the following statements are equivalent:

(a) $\mathbb{X}$ is compact.

(b) $\mathbb{X}$ is paracompact.

(c) $\mathbb{X}$ is metacompact.

DEFINITION. Let $\mathcal{U}$ be a family of subsets of a space $\mathbb{X}$, and let $x \in \mathbb{X}$. Then the <u>star S_x at x of $\mathcal{U}$</u> is defined to be the union of all the sets of $\mathcal{U}$ containing x.

A cover $\mathcal{U}$ of $\mathbb{X}$ is a <u>star-refinement</u> of a cover $\mathcal{B}$ of $\mathbb{X}$ iff the family of stars of $\mathcal{U}$ at points of $\mathbb{X}$ is a refinement of $\mathcal{B}$.

A topological space $\mathbb{X}$ is termed <u>fully normal</u> iff for every open cover $\mathcal{G}$ of $\mathbb{X}$ there is an open cover $\mathcal{H}$ of $\mathbb{X}$ which is a star-refinement of $\mathcal{G}$. (11)

Every fully normal space is normal.

The following theorem was proved by A. H. Stone ((10), pp. 977-980):

THEOREM. A T_1 space $\mathbb{X}$ is fully normal iff it is paracompact.

For additional equivalent formulations of paracompactness see (6), p. 156.

APPENDIX

In this section we assume Zorn's lemma and prove the Hausdorff maximal principle and justify the maximal arguments used in the proofs of the theorems.

DEFINITION. Let S be a nonempty set and $\leqq$ an ordering relation. Then $\{S, \leqq\}$ is termed a <u>partially ordered set</u> iff

(i) If $a \leqq b$ and $b \leqq a$, then $a = b$.

(ii) If $a \leqq b$ and $b \leqq c$, then $a \leqq c$.

A partially ordered set is termed a <u>simply ordered set</u> iff

(iii) For every a, b in S either $a \leqq b$ or $b \leqq a$.

An <u>upper bound</u> b of a subset A of S is an element of S such that $a \leqq b$ for all $a \in A$.

An element u of S is termed a <u>maximal element</u> of S iff u has the property that if $s \in S$ and $u \leqq s$, then $u = s$.

We accept the following statement as an axiom.

ZORN'S LEMMA. If every simply ordered subset of a partially ordered set S has an upper bound, then S has a maximal element. ((6), p. 33.)

We prove the following:

HAUSDORFF MAXIMAL PRINCIPLE. If $\mathcal{U}$ is a nonempty family of nonempty sets and $\mathcal{N}$ is a nest contained in $\mathcal{U}$, then there is a maximal nest $\mathcal{M}$ contained in $\mathcal{U}$ which contains $\mathcal{N}$.

Proof. Define $\bar{\bar{N}}$ to be the set of all nests $\mathcal{N}'$ contained in $\mathcal{U}$ such that $\mathcal{N} \subseteq \mathcal{N}'$. Clearly $\{\bar{\bar{N}}, \subseteq\}$ is a partially ordered set. Let $\bar{\bar{M}}$ be a simply ordered subset of $\bar{\bar{N}}$. We assert that there exists an $\mathcal{N}^{\#} \in \bar{\bar{N}}$ such that $\mathcal{N}^{\#}$ is an upper bound of $\bar{\bar{M}}$. Define

$$\mathcal{N}^{\#} = \{N \mid N \in \mathcal{U} \text{ and } \exists\, \mathcal{N}' \in \bar{\bar{M}} \ni N \in \mathcal{N}'\}.$$

We assert that $\mathcal{N}^{\#} \in \bar{\bar{N}}$. Clearly $\mathcal{N}^{\#}$ is a family of sets of $\mathcal{U}$ containing $\mathcal{N}$. We prove $\mathcal{N}^{\#}$ is a nest. Let N_1 and N_2 be two sets in $\mathcal{N}^{\#}$. Then there exist two nests $\mathcal{N}_1$ and $\mathcal{N}_2$ in $\bar{\bar{M}}$ such that $N_1 \in \mathcal{N}_1$ and $N_2 \in \mathcal{N}_2$. Since $\bar{\bar{M}}$ is simply ordered, either $\mathcal{N}_1 \subseteq \mathcal{N}_2$ or $\mathcal{N}_2 \subseteq \mathcal{N}_1$. Hence N_1 and N_2 are either both in $\mathcal{N}_1$ or both in $\mathcal{N}_2$, so that either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. It follows that $\mathcal{N}^{\#}$ is an upper bound of $\bar{\bar{M}}$ in $\bar{\bar{N}}$. Hence, by Zorn's lemma there exists a maximal nest $\mathcal{N}^*$ in $\bar{\bar{N}}$. Since any nest in $\mathcal{U}$ containing $\mathcal{N}^*$ is in $\bar{\bar{N}}$, it follows that $\mathcal{N}^*$ is maximal in $\mathcal{U}$.

THEOREM. Every filter $\mathcal{G}$ on a nonempty set E is contained in an ultrafilter on E.

Proof. Define $\bar{\bar{F}}$ to be family of all filters $\mathcal{G}'$ on E which contain $\mathcal{G}$. Clearly $\{\bar{\bar{F}}, \subseteq\}$ is a partially ordered set. Let $\bar{\bar{G}}$ be a simply ordered subset of $\bar{\bar{F}}$. We assert that there exists an upper bound $\mathcal{G}^{\#}$ of $\bar{\bar{G}}$ in $\bar{\bar{F}}$. Define

$$\mathcal{G}^{\#} = \{S \mid S \subseteq E \text{ and } \exists\, \mathcal{G}' \in \bar{\bar{G}} \ni S \in \mathcal{G}'\}$$

Clearly $\mathfrak{G}^{\#}$ is a nonempty family of nonempty subsets of E containing $\mathfrak{G}$. We prove that $\mathfrak{G}^{\#}$ is a filter on E. Let T be a subset of E and S a subset of T in $\mathfrak{G}^{\#}$. Then S and therefore T belongs to a filter $\mathfrak{G}'$ in $\overline{G}$, so that $T \in \mathfrak{G}^{\#}$. Now let S_1 and S_2 be two sets in $\mathfrak{G}^{\#}$. Then there exist filters $\mathfrak{G}_1$ and $\mathfrak{G}_2$ in $\overline{G}$ such that $S_1 \in \mathfrak{G}_1$ and $S_2 \in \mathfrak{G}_2$. Since $\overline{G}$ is simply ordered, either $\mathfrak{G}_1 \subseteq \mathfrak{G}_2$ or $\mathfrak{G}_2 \subseteq \mathfrak{G}_1$, so that S_1 and S_2 are both in $\mathfrak{G}_1$ or both in $\mathfrak{G}_2$. Hence either $S_1 \cap S_2 \in \mathfrak{G}_1$ or $S_1 \cap S_2 \in \mathfrak{G}_2$, so that $S_1 \cap S_2 \in \mathfrak{G}^{\#}$. Hence $\mathfrak{G}^{\#}$ is a filter on E. It follows that $\mathfrak{G}^{\#}$ is an upper bound of $\overline{G}$ in $\overline{F}$. Hence by Zorn's lemma there exists a maximal element $\mathfrak{U}$ in $\overline{F}$. Here $\mathfrak{U}$ is a filter on E containing $\mathfrak{G}$. Moreover every filter on E containing $\mathfrak{U}$ is in $\overline{F}$, so that $\mathfrak{U}$ is an ultrafilter on E.

THEOREM. Every family $\mathfrak{F}$ of sets closed in E and possessing the finite intersection property wrt E is contained in a maximal family of sets closed in E and possessing the finite intersection property wrt E.

Proof. Define $\overline{H}$ to be the set of all families $\mathfrak{F}'$ of sets closed in E possessing the finite intersection property wrt E and containing $\mathfrak{F}$. Clearly $\{\overline{H}, \subseteq\}$ is a partially ordered set. Let $\overline{I}$ be a simply ordered subset of $\overline{H}$. We assert that there exists an upper bound $\mathfrak{F}^{\#}$ of $\overline{I}$ in $\overline{H}$. Define

$$\mathfrak{F}^{\#} = \{ F \mid F \subseteq E \text{ and } \exists\, \mathfrak{F}' \in \overline{I} \ni F \in \mathfrak{F}' \}$$

Clearly $\mathfrak{F}^{\#}$ is a family of sets closed in E and contains $\mathfrak{F}$. We

prove that $\mathfrak{F}^{\#}$ possesses the finite intersection property wrt E.
Let

$$\{F_i \mid 1 \leqq i \leqq n\}$$

be a finite subfamily of $\mathfrak{F}^{\#}$. Then there exist families $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n$ in $\bar{\mathfrak{I}}$ such that $F_i \in \mathfrak{F}_i$ for $i = 1, 2, \ldots, n$. Since $\bar{\mathfrak{I}}$ is simply ordered one of these families say $\mathfrak{F}_K$ contains the others. Hence $F_i \in \mathfrak{F}_K$ for $i = 1, 2, \ldots, n$ so that

$$E \cap \bigcap \{F_i \mid 1 \leqq i \leqq n\}$$

is nonempty. Hence $\mathfrak{F}^{\#}$ possesses the finite intersection property wrt E. It follows that $\mathfrak{F}^{\#}$ is an upper bound of $\bar{\mathfrak{I}}$ in $\bar{H}$. Hence by Zorn's lemma there exists a maximal element $\mathfrak{M}$ in $\bar{H}$. Then $\mathfrak{M}$ is a family of sets closed in E possessing the finite intersection property wrt E. and containing $\mathfrak{F}$. Moreover, any family of sets closed in E possessing the finite intersection property wrt E and containing $\mathfrak{M}$ is in $\bar{H}$, so that $\mathfrak{M}$ is a maximal family.

THEOREM. Every point-finite open cover of a space $\mathbb{X}$ has a minimal subcover.

Proof. Let $\mathfrak{G}$ be a point-finite open cover of $\mathbb{X}$. We may assume $\mathfrak{G}$ is not minimal. Call a nonempty subfamily $\mathfrak{B}$ of $\mathfrak{G}$ redundant iff

$$\{G \mid G \in \mathfrak{G} \text{ and } G \notin \mathfrak{B}\}$$

covers $\mathbb{X}$. We assert that there exists a maximal redundant subfamily $\mathfrak{B}^*$ of $\mathfrak{G}$, whence

$$\{G \mid G \in \mathcal{G} \text{ and } G \notin \mathcal{B}^*\}$$

is a minimal subcover. Define $\bar{B}$ to be the set of all redundant subfamilies $\mathcal{B}$ of $\mathcal{G}$. Note $\bar{B}$ is not empty. Clearly $\{\bar{B}, \subseteq\}$ is a partially ordered set. Let $\bar{C}$ be a simply ordered subset of $\bar{B}$. We assert that there exists a $\mathcal{B}^{\#} \in \bar{B}$ such that $\mathcal{B}^{\#}$ is an upper bound of $\bar{C}$. Define

$$\mathcal{B}^{\#} = \{B \mid B \subset \mathbb{X} \text{ and } \exists\ \mathcal{B} \in \bar{C} \ni B \in \mathcal{B}\}$$

We assert that $\mathcal{B}^{\#}$ is redundant. Suppose

$$\{G \mid G \in \mathcal{G} \text{ and } G \notin \mathcal{B}^{\#}\}$$

is not a cover. Let x be a point such that no G in $\mathcal{G}$ not in $\mathcal{B}^{\#}$ contains x. Let $G_1, G_2, \ldots, G_n$ be all the sets of the point-finite cover $\mathcal{G}$ which contain x. Then $G_k \in \mathcal{B}^{\#}$ for $k = 1,2,\ldots,n$. Since $\bar{C}$ is simply ordered, it follows by the definition of $\mathcal{B}^{\#}$ that there exists a family $\mathcal{B}' \in \bar{C}$ such that $G_k \in \mathcal{B}'$ for $k = 1,2,\ldots,n$. Then no G in $\mathcal{G}$ not in $\mathcal{B}'$ contains x, that is, $\mathcal{B}'$ is not redundant. Hence $\mathcal{B}^{\#}$ is an upper bound of $\bar{C}$ in $\bar{B}$. It follows by Zorn's lemma that there exists a maximal redundant subfamily $\mathcal{B}^*$ of $\mathcal{G}$ in $\bar{B}$. The cover

$$\{G \mid G \in \mathcal{G} \text{ and } G \notin \mathcal{B}^*\}$$

is then minimal. Indeed if this family with a set G_0 deleted covers $\mathbb{X}$, then the family

$$\mathcal{B}^* \cup \{G_0\}$$

is redundant and contains $\mathcal{B}^*$ properly.

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