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A matricial proof of the symmetric exchange axiom for eigenvalues of principal submatrices of a complex Hermitian matrix

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ABSTRACT

In [C.R. Johnson, B. Kroschel, M. Omladič, Eigenvalue multiplicities in principal submatrices, Linear Algebra Appl. 390 (2004) 111–120] a result constraining the eigenvalues of principal submatrices of complex Hermitian matrices, based upon matroid theory, was given. Here we give a matricial proof of this result which also enables us to find a generalization of the original result.

1. Introduction

For $\alpha \subseteq N = \{1, 2, \ldots, n\}$ and $A \in M_n(\mathbb{C})$ let $A[\alpha]$ denote the principal submatrix of A lying in the rows and columns indexed by α and let $A(\alpha)$ denote the complementary principal submatrix, i.e., the submatrix resulting from the deletion of the rows and the columns indexed by α. Similarly, let $v[\alpha]$ denote the subvector of $v \in \mathbb{C}^n$ containing the components of v indexed by α and let $v(\alpha)$ denote the complementary subvector. Let $\sigma(A)$ denote the set of all eigenvalues of A, and for $\lambda \in \sigma(A)$ denote the geometric multiplicity of λ in A by $g_\lambda(A)$.

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Given a vector $u \in \mathbb{R}^{2^n}$, the principal minor assignment problem asks when is there an $n \times n$ matrix having its 2^n principal minors given by u. This problem was introduced in [4] and treated in several papers since [3,5,8]. Inverse principal rank characteristic problem, defined and treated in [2], is a subproblem of the principal minor assignment problem. It explores possible arrangements of the presence or absence of a nonzero principal minor of each possible size. More precisely, given a sequence $r_0, r_1, \ldots, r_n$ of 0s and 1s, the problem is to determine if there exist an $n \times n$ real symmetric matrix that has a principal submatrix of rank k if and only if $r_k = 1$, for all $0 \leqslant k \leqslant n$.

In [6] the very general related question of the possible arrangement of multiplicities, of a given eigenvalue, among principal submatrices of a complex Hermitian (real symmetric) matrix is raised. One restriction on this hierarchy of multiplicities is claimed in [1] (and reported in [6]), based upon matroid theory, the so-called symmetric exchange axiom (SEA). The result uses the symmetric difference, $\alpha \Delta \beta$, which is the set of elements in either of the sets α or β, but not in both.

Theorem 1 (SEA). *Suppose that $A \in M_n(\mathbb{C})$ is a Hermitian matrix and let $\alpha_1, \alpha_2 \subseteq \{1, 2, 3, \ldots, n\}$ be such that $g_\lambda(A[\alpha_i]) = 0$ for $i = 1, 2$. Then for every $j \in \alpha_1 \Delta \alpha_2$, there is a $k \in \alpha_1 \Delta \alpha_2$ such that*

$$g_\lambda(A[\alpha_1 \Delta \{j, k\}]) = 0.$$

In this note we address a question that was raised in [6] of finding a matrix theoretical proof of this result. Here we give a generalization of the SEA and a brief matricial proof that is valid for principal submatrices of a complex Hermitian matrix.

2. Background

In [7] dimensions of special subspaces of the eigenspaces associated with λ of a general matrix A, were considered. Here we will use a result for complex Hermitian matrices. As in [7] we define these special spaces as follows:

$$RE_\alpha^\lambda(A) = \{x \in \mathbb{C}^n; Ax = \lambda x,\ x(\alpha) = 0\}.$$

When $\lambda = 0$, we denote $RN_\alpha(A) = RE_\alpha^0(A)$.

In [7] the following result is given.

Theorem 2 ([7], Corollary 2). *Let $A \in M_n(\mathbb{C})$ be Hermitian. For $\alpha \subseteq N$ with $|\alpha| = n - k$*

$$\dim(RE_\alpha^\lambda(A)) \geqslant \frac{g_\lambda(A) + g_\lambda(A[\alpha]) - k}{2}.$$

From Theorem 2 we can deduce the following two lemmas.

Lemma 3. *Let A in $M_n(\mathbb{C})$ be a Hermitian matrix with $g_\lambda(A) = t$, and let $v_1, v_2, \ldots, v_t$ be linearly independent eigenvectors of A associated with the eigenvalue λ. Let $\nu = \{k; g_\lambda(A(k)) \geqslant t\}$. Then*

$$v_i[\nu] = 0 \text{ for } i = 1, 2, \ldots, t,$$

and $t \leqslant n - |\nu|$.

Proof. Using translation we can assume that $\lambda = 0$. Take $k \in \nu$. By Theorem 2

$$\dim(RN_{N\setminus\{k\}}(A)) \geqslant \frac{g_0(A) + g_0(A(k)) - 1}{2} \geqslant t - \frac{1}{2}.$$

It follows that $\dim(RN_{N\setminus\{k\}}(A)) \geqslant t$, and since $g_0(A) = t$ this implies that $v_i \in RN_{N\setminus\{k\}}(A)$ for $i = 1, \ldots, t$. This proves that $v_i[k] = 0$ for $i = 1, 2, \ldots, t$. We repeat this argument for every $k \in \nu$ to show that $v_i[\nu] = 0$ for $i = 1, 2, \ldots, t$. $\square$

Lemma 4. *Let $A[\alpha]$ be a submatrix of a Hermitian matrix A in $M_n(\mathbb{C})$ with $g_\lambda(A[\alpha]) = t$ and let $v_1, v_2, \ldots, v_t$ be linearly independent eigenvectors of $A[\alpha]$ associated with λ. Let*

$$\mu = \{k \in N \setminus \alpha; g_\lambda(A[\alpha \cup \{k\}]) \geqslant t\}.$$

Define vectors w_i in the following way: $w_i[\alpha] = v_i$ and $w_i[\mu] = 0$, for $i = 1, 2, \ldots, t$. Then w_i are the eigenvectors of $A[\alpha \cup \mu]$ corresponding to λ.

Proof. Again we can assume that $\lambda = 0$ using translation. Take $k \in \mu$. Then

$$\dim(RN_\alpha(A[\alpha \cup \{k\}])) \geqslant \frac{g_0(A[\alpha \cup \{k\}]) + g_0(A[\alpha]) - 1}{2} \geqslant t - \frac{1}{2}$$

and

$$\dim(RN_\alpha(A[\alpha \cup \{k\}])) \geqslant t.$$

This proves that $A[\alpha \cup \{k\}]$ has eigenvectors w_i^k with $w_i^k[k] = 0$ for $i = 1, 2, \ldots, t$. But then $w_i^k[\alpha]$ have to be the eigenvectors of $A[\alpha]$ corresponding to 0 and $w_i^k[\alpha] = v_i$. Repeating this argument for every $k \in \mu$ proves the vectors w_i, defined by $w_i[\alpha] = v_i$ and $w_i[\mu] = 0$, for $i = 1, 2, \ldots, t$, are the eigenvectors of $A[\alpha \cup \mu]$ corresponding to λ. $\square$

3. Main result

Now we give our main result which, in the special case when $s = 0$, reduces to the SEA.

Theorem 5. *Let A in $M_n(\mathbb{C})$ be Hermitian, $\alpha_1, \alpha_2 \subseteq N$ and $0 \leqslant s \leqslant n$ an integer. Assume that $g_\lambda(A[\alpha_i]) \leqslant s$ for $i = 1, 2$.*

Then for every $j \in \alpha_1 \Delta \alpha_2$ there exists a $k \in \alpha_1 \Delta \alpha_2$ such that

$$g_\lambda(A[\alpha_1 \Delta \{j, k\}]) \leqslant s$$

and there is a $k' \in \alpha_1 \Delta \alpha_2$ such that

$$g_\lambda(A[\alpha_2 \Delta \{j, k'\}]) \leqslant s.$$

Proof. It suffices to prove the conclusion for j and k. Using translation we can assume that $\lambda = 0$. Choose $j \in \alpha_1 \Delta \alpha_2$ and assume that $g_0(A[\alpha_1 \Delta \{j, k\}]) \geqslant s + 1$ for every $k \in \alpha_1 \Delta \alpha_2$. In particular, suppose, without loss of generality, that $g_0(A[\alpha_1 \Delta \{j\}]) = s + 1$. Let $v_1, v_2, \ldots, v_{s+1}$ be linearly independent eigenvectors of $A[\alpha_1 \Delta \{j\}]$ associated with 0.

We define the following sets:

$$\gamma_1 = (\alpha_1 \setminus \alpha_2) \Delta \{j\}$$

and

$$\gamma_2 = (\alpha_2 \setminus \alpha_1) \Delta \{j\}.$$

Choose $k \in \gamma_1$, $k \neq j$. Then $\alpha_1 \Delta \{j, k\} = (\alpha_1 \Delta \{j\}) \setminus \{k\}$ and $g_0(A[\alpha_1 \Delta \{j, k\}]) \geqslant s + 1$ by our assumption. Now Lemma 3 tells us that $v_i[\gamma_1] = 0$ and

$$s + 1 \leqslant |\alpha_1 \Delta \{j\}| - |\gamma_1| = |\alpha_1 \cap \alpha_2|.$$

Choose $k \in \gamma_2$. Then $\alpha_1 \Delta \{j, k\} = (\alpha_1 \Delta \{j\}) \cup \{k\}$ and $g_0(A[\alpha_1 \Delta \{j, k\}]) \geqslant s + 1$ by our assumption. Now $(\alpha_1 \Delta \{j\}) \cup \gamma_2 = \alpha_1 \cup \alpha_2$ and Lemma 4 tells us that the matrix $A[\alpha_1 \cup \alpha_2]$ has eigenvectors $w_1, w_2, \ldots, w_{s+1}$ of the form $w_i[\alpha_1 \Delta \{j\}] = v_i$ and $w_i[\gamma_2] = 0$.

Since $w_i[\alpha_1 \setminus \alpha_2] = 0$, we conclude that w_i, $i = 1, \ldots, s+1$, are the eigenvectors of the matrix $A[\alpha_2]$ associated with zero. This gives us a contradiction to the assumption that $g_0(A[\alpha_2]) \leqslant s$. $\square$

If we take $s = 0$ in the previous theorem we get the SEA. Taking $\lambda = 0$ gives us the following corollary, which is a special case of the SEA, and it is given in [1] and in [6].

Corollary 6. *Let A in $M_n(\mathbb{C})$ be Hermitian. For any two nonsingular matrices $A[\alpha_1]$ and $A[\alpha_2]$ and for each $j \in \alpha_1 \Delta \alpha_2$ there is a $k \in \alpha_1 \Delta \alpha_2$ such that the matrix $A[\alpha_1 \Delta \{j, k\}]$ is also nonsingular.*

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