FOURIER TRANSFORMS IN EUCLIDEAN K SPACE

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ABSTRACT

The purpose of this thesis is to study the Fourier transform in Euclidean K Space (E_k).

Chapter One develops the properties of the Fourier transform of a Lebesque integrable function (L_1) in E_k. A theorem is proved that the Fourier transform of an L_1 function f is Gauss summable to the function f almost everywhere. Under certain conditions on the transform, an integral formula can be obtained for the inversion of the Fourier transform.

In Chapter Two, the Plancherel theorem is proved by three different methods. The one-dimensional proofs due to Bochner, Riesz, and Wiener are extended to E_k. The Parseval relation between two L_2 functions and their transforms is also proved.

Chapter Three shows the equivalence of L_1 positive definite functions, autocorrelation functions of an L_2 function, and bounded, continuous L_1 functions with a non-negative transform.

FOURIER TRANSFORMS IN EUCLIDEAN K SPACE

INTRODUCTION

The purpose of this thesis is to study the Fourier transform in Euclidean K Space. The thesis does not attempt to cover all that is known about Fourier transforms in E_k. However, it considers the problems of extending many of the basic results of Fourier transforms of functions of one variable.

The reader is assumed to be familiar with Lebesgue integration, in particular the convergence theorems due to Lebesgue and Fubinis theorem. The reader is referred to McShane and Botts [7]. Also the properties of the Fourier transform and the Hermite functions of a single variable, are assumed known to the reader. Here the reader is referred to Titchmarsh [12].

The properties of the Fourier transform of a Lebesgue integrable function in E_k are developed in Chapter One. Also the composition and convolution theorems are proved. The Gauss summability of the L_1 Fourier transform of f, to the function f, almost everywhere, is also proved. From this result, two theorems giving integral formulae for transform inversion, are proved. This development parallels that outlined by Bochner and Chanderskaran [1].

In Chapter Two, the one dimensional proofs due to

Bochner, Riesz, and Wiener of the Plancherel theorem are extended to E_k. From the Plancherel theorem, the Parseval relation between two L_2 functions and their transforms is also proved.

Chapter Three shows the equivalence of L_1 positive definite functions, autocorrelation functions of an L_2 function, and bounded continuous L_1 functions with non-negative transforms. The equivalence of the L_1 positive definite functions and continuous, bounded L_1 functions is due to Bochner [2]. The relation between continuous, bounded, L_1 functions with positive transform and autocorrelation functions of an L_2 function is suggested by Lee and Paupoli. However their work is in the one variable case, with some obscurity in the hypothesis which is less troublesome in engineering than mathematics.

CHAPTER I

FOURIER TRANSFORMS IN $L_1(E_k)$

Before we examine the properties of Fourier transforms, we will define some of the terms and concepts used throughout the paper.

Definition 1: E_k will denote Euclidean K Space and X a point in E_k, i.e., $X = (x_1, x_2, \ldots, x_k)$. Also we will denote $\sqrt{x_1^2 + x_2^2 + \ldots + x_k^2}$ by $|X|$, and $x_1y_1 + x_2y_2 + \ldots + x_ky_k$ by $(X \cdot Y)$.

The characteristic function of a set A is

$$K_A(X) = \begin{cases} 1 \text{ if } X \in A \\ 0 \text{ if } X \notin A \end{cases}$$

A step function in E_k is defined by $s(X) = \sum_{j=1}^{n} a_j K_{M_i}(X)$ where the a's are real or complex numbers and the M's are disjoint intervals in E_k.

Definition 2: A Lebesgue measurable function f is said to be a member of $L_p(E_k)$ if $||f||_p = [\int_{E_k} |f(X)|^p dV_x]^{1/p} < \infty$ where $dV_x = dx_1 dx_2 \ldots dx_k$. $||f||_p$ is called the L_p norm of f. The distance between f and g, two points in L_p, is $||f-g||_p$, where $||f-g||_p$ is equal to zero if and only if f = g almost everywhere (a.e.), that is, f = g everywhere except on a set of measure zero. By considering the class of functions equal a.e. as an equivalence class, we find that this distance function is a suitable metric.

Note 1: Throughout the paper several well-known integrals appear and are used in many proofs. The following two L_1, continuous and bounded functions

$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-m}{\sigma})^2}$ and $e^{-ium - \frac{1}{2}u^2\sigma^2}$ given in Parzen [9] page 218-19 give rise to these well-known integrals

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad \text{and}$$

$$\int_{-\infty}^{\infty} e^{-x^2 + ixy} dx = \sqrt{\pi}\, e^{\frac{-y^2}{4}} \quad \text{and finally}$$

$$\lim_{u \to \infty} \int_0^u \frac{\sin Rt}{t} = \frac{\pi}{2}.$$

See Weiner [13] pages 47, 48, 49.

$$\text{Also} \quad \frac{R^k 2^{-k+1}}{\Gamma(\frac{k}{2})} \int_0^\infty e^{-R^2t^2} t^{k-1} dt = \frac{2^{-k+1}}{\Gamma(\frac{k}{2})} \int_0^\infty e^{-\frac{y^2}{4}} y^{k-1} dy$$

$$\text{letting } R^2t^2 = y^2, \qquad = \frac{1}{\Gamma(\frac{k}{2})} \int_0^\infty e^{-t} t^{\frac{k}{2} - 1} dt$$

$$\text{letting } \frac{y^2}{4} = t, \qquad = \frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{k}{2})} = 1.$$

We will now define the Fourier transform of an L_1 function, say f, by

$$T(f) = \Phi(Y) = \int_{E_k} f(X)\, e^{i(X \cdot Y)} dV_x.$$

We shall denote the function and its transform by by f(X) and $\Phi(Y)$ respectively or $\Phi_f(Y)$ where confusion may otherwise arise. We shall now extend to k-dimensions

(E_k) many of the well-known results of Fourier transforms of one real variable.

Property 1: If $f(X) \in L_1(E_k)$, then $\Phi(Y)$ exists and is bounded by $\|f\|_1$ for all $Y \in E_k$.

Proof: Trivial.

Property 2: If $f(X) \in L_1(E_k)$, then $\Phi(Y)$ is uniformly continuous.

Proof: If $T = (t_1, t_2, \ldots, t_k)$, then consider

$$(1.1) \quad |\Phi(Y+T) - \Phi(Y)| = \left|\int_{E_k} f(X)e^{i(X\cdot Y)}(e^{i(X\cdot T)}-1)dV_x\right|$$
$$\leq \int_{E_k} |f(X)|\ |e^{i(X\cdot T)}-1|\ dV_x$$

now
$$|e^{i(X\cdot T)}-1| = [\sin^2(X\cdot T) + (\cos(X\cdot T)-1)^2]^{\frac{1}{2}}$$
$$= [2 - 2\cos(X\cdot T)]^{\frac{1}{2}}$$
$$= 2\left|\sin\frac{(X\cdot T)}{2}\right|$$

Hence (1.1) becomes

$$|\Phi(Y+T) - \Phi(Y)| \leq 2\int |f(X)|\ |\sin[(X\cdot T)^{\frac{1}{2}}]|dV_x$$
$$\leq 2\int_{E_k-R_j} |f(X)|dV_x + J\cdot \sum_{j=1}^{k} |t_j|\int_{R_J} |f(X)|dV_x$$

where $J > 0$ and $R_J = \{X \mid |x_j| \leq J \text{ for } j = 1, 2, \ldots k\}$. Given $\epsilon > 0$, we can make first integral $< \epsilon/2$. Then with J fixed we let $|T| \to 0$ so the second integral is $< \epsilon/2$. Hence

$$|\Phi(Y+T) - \Phi(Y)| < \epsilon \text{ for } |T| \text{ small.}$$

Property 3: If $f_1, f_2 \in L_1(E_k)$ and c_1 and c_2 are scalars, the $T(c_1f_1 + c_2f_2) = c_1T(f_1) + c_2T(f_2)$.

Proof: Immediate since integration is linear.

Property 4: If R is a real number, then

$$T[f(RX)](Y) = R^{-k}T[f(X)](R^{-1}Y)$$

and

$T[\overline{f}(X)](Y) = \overline{T[f(X)](-Y)}$ where $\overline{}$ denotes the complex conjugate.

Proof: $T[f(RX)](Y) = \int_{E_k} f(RX)e^{i(X\cdot Y)}dV_x$

let $RX = \tilde{X}$ $\qquad = R^{-k}\int_{E_k} f(\tilde{X})e^{i(R^{-1}(Y\cdot\tilde{X}))}dV_{\tilde{X}}$

$$= R^{-k}T[f(X)](R^{-1}Y)$$

and

$$T[\overline{f}(X)](Y) = \int_{E_k} \overline{f}(X)\ e^{i(X\cdot Y)}dV_x$$

$$= \overline{\int_{E_k} f(X)e^{i(X\cdot(-Y))}dV_x}$$

$$= \overline{T[f(X)](-Y)}$$

Property 5: If $f_n \to f$ in L_1 norm then $\Phi_n(Y) \to \Phi(Y)$ uniformly for all $Y \in E_k$.

Proof: By Hypothesis if $\epsilon > 0$, then there exists an integer N such that for all $n \geq N$

$$\int_{E_K} |f_n(X) - f(X)|dV_x < \epsilon.$$

Now let $Y \in E_k$ and we have

$$|\Phi_n(Y) - \Phi(Y)| = |\int_{E_k} e^{i(X\cdot Y)}[f_n(X) - f(X)]dV_x|$$

$$\leq \int_{E_k} |f_n(X) - f(X)| dV_x$$

$$< \epsilon \text{ for all } n \geq N \text{ and } Y \in E_k.$$

Now we are in a position to state and prove a useful theorem.

Theorem 1: (The composition Theorem) If $f, g \in L_1$ and $T(f) \equiv \Phi$ and $T(g) \equiv \psi$ then

$$\int_{E_k} \Phi(Y)\, g(Y) dV_y = \int_{E_k} f(Y) \psi(Y) dV_y.$$

Proof: $\int_{E_k} \int_{E_k} |e^{i(X \cdot Y)} f(X) g(Y)| dV_x dV_y$

$$= \int_{E_k} \int_{E_k} |f(X)| \cdot |g(Y)| dV_x dV_y = \|f\|_1 \cdot \|g\|_1 < \infty$$

Hence we can use the Fubini Theorem

$$\therefore \int_{E_k} \Phi(Y) g(Y) dVy = \int_{E_k} \int_{E_k} e^{i(X \cdot Y)} f(X) g(Y) dV_x dV_y$$

$$= \int_{E_k} \int_{E_k} e^{i(X \cdot Y)} f(X) \cdot g(Y) dV_y dV_x$$

$$= \int_{E_k} \psi(X) \cdot f(X) dV_x$$

At this point it would be advantageous to define a special step function which will be used in many proofs throughout the paper. Let

$$g_j(x_j) = \begin{cases} 1 \text{ if } a_j \le x_j \le b_j \\ 0 \text{ if } x_j \notin [a_j, b_j] \end{cases}$$

Now let

$$(1.A) \quad G(X) = \prod_{j=1}^{k} g_j(y_j)$$

Theorem 2: (Riemann Lebesgue Lemma) If $f(X) \in L_1(E_k)$, then $\lim_{|Y|\to\infty} \Phi_f(Y) = 0$.

Proof: We shall prove the theorem holds for step functions, then extend the result to all L_1 functions. Let $G(X)$ be as (1.A) and denote $T[G(X)]$ by $\psi(Y)$.

$$\psi(Y) = \int_{E_k} G(X)e^{i(X\cdot Y)}dV_x$$

$$= \prod_{j=1}^{k} \int_{a_j}^{b_j} e^{ix_jy_j}\, dx_j$$

Now each of these k individual integrals goes to zero as the y_j involved $\to \infty$ by the Riemann-Lebesgue lemma for one variable. $\therefore \psi(Y)\to 0$ as $|Y|\to\infty$. Hence by definition of step functions the conclusion is true for them also.

A well-known theorem which states that for each L_p function f there exists a sequence of step functions converging in L_p norm to f. Thus if $\epsilon > 0$ and $f \in L_1$ there exists a step function f such that $\|f - f\| < \epsilon$. Hence by Property 3

$$\Phi_f(Y) = \Phi_{f-f_\epsilon}(Y) + \Phi_{f_\epsilon}(Y)$$

and by Property 1

$$|\Phi_f(Y)| \leq ||f - f_\epsilon||_1 + |\Phi_{f_\epsilon}(Y)| < \epsilon + |\Phi_{f_\epsilon}(Y)|$$

Therefore $\overline{\lim}_{|Y|\to\infty} |\Phi_f(Y) < \epsilon + \overline{\lim}_{|Y|\to\infty} |\Phi_{f_\epsilon}(Y)|$

and since $f_\epsilon(X)$ is a step function

$\overline{\lim}_{|Y|\to\infty} |\Phi_{f_\epsilon}(Y)| = 0$ it follows that

$\overline{\lim}_{|Y|\to\infty} |\Phi_f(Y)| < \epsilon$ for arbitrary ϵ.

Hence we have proved the theorem.

The next theorem is important not only as a building block for future results, but it is used extensively in many areas of applied mathematics.

Theorem 3: (Convultion Theorem) If $f, g \in L_1(E_k)$ then their convultion (resultant or faltung) $h(X)$ is defined as

$$h(X) = \int_{E_k} f(X-Y)g(Y)dV_y = \int_{E_k} f(Y)g(X-Y)dV_y$$

which exists for almost all X, belongs to L_1 and $T[h(X)] = T[f(X)] \cdot T[g(X)]$.

Proof: Since $f(X)$ is measureable in X then $f(X)$ is measureable in X, Y. [We know we have sequence of Baire functions converging almost everywhere to $f(X)$ and X-Y is measureable in $E_k \times E_k$ and a Baire function is measureable, McShane and Botts.] To show that $h(X)$

exists almost everywhere, let us consider

$$\int_{E_k} |f(X-Y)|\ |g(y)|dV_x = |g(Y)| \int_{E_k} |f(X)|dV_x$$

$$= |g(Y)|\ \ \|f\|_1 \text{ , an } L_1$$

function. Then it follows that $\int_{E_k}\int_{E_k} |f(X-Y)|\ |g(Y)|dV_x dV_y$

exists, so by the Fubini theorem we have the preceding integral equals

$$\int_{E_k}\int_{E_k} |f(X-Y)|\cdot|g(Y)|dV_y dV_x \text{ which is}$$

$\int|h(X)|dV_x$. Therefore $h(X) \in L_1(E_k)$ and $h(X)$ exixts for almost all X in E_k.

$$T[h(X)](Z) = \int_{E_k} e^{i(X\cdot Z)} h(X)dV_x$$

$$= \int_{E_k}\int_{E_k} e^{i(X\cdot Z)} f(X-Y)g(Y)dV_y dV_x$$

$$= \int_{E_k}\int_{E_k} f(X-Y)e^{i[Z(X-Y)]} g(Y)e^{i(Z\cdot Y)} dV_y dV_x$$

$$= \int_{E_k} g(Y)e^{i(Z\cdot Y)} \int_{E_k} f(X-Y)e^{i[Z(X-Y)]} dV_x dV_y$$

let X-Y = T and we obtain

$$\int_{E_k} g(Y)e^{i(Z\cdot Y)} \int_{E_k} f(T)e^{i(Z\cdot T)} dV_T dV_y$$

$$= T[g(Y)](Z)\cdot T[f(T)](Z).$$

The justification for changing the order of integration again follows from Fubini's theorem with $f(X-Y)e^{i[Z\cdot(X-Y)]}g(Y)e^{i(Z\cdot Y)}$ replacing $f(X-Y)g(Y)$ in the earlier argument.

Note 2: If $f(X) \in L_1(E_k)$ is such that

$$f(X) = \prod_{j=1}^{k} f_j(x_j) \text{ and } f_j(x_j) \in L_1(-\infty, \infty)$$

for $j = 1, 2, \ldots k$, then $\Phi_f(Y) = \prod_{j=1}^{k} \Phi_{f_j}(y_j)$

where $\Phi_j(y_j) = \int_{-\infty}^{\infty} f_j(x_j)e^{ix_jy_j}\,dx_j$

Also clearly if $f(X)$ has transform $\Phi(Y)$ then $f(X)e^{i(X\cdot H)}$ has transform $\Phi(Y+H)$ and $f(X+H)$ has transform $\Phi(Y)e^{-i(Y\cdot H)}$.

An important consideration of many mathematical operations is the question of uniqueness. The following theorem shows that any two L_1 functions in E_k having the same Fourier transform are equal almost everywhere.

<u>Theorem 4</u>: If $f(X) \in L_1(E_k)$ and $\Phi(Y) = 0$, then $f(X) = 0$ almost everywhere.

Proof: Bochner [1] page 59.

We shall now investigate what extensions the usual inversion techniques yield in E_k. That is, under what conditions is

$$f(X) = (2\pi)^{-k}\int_{E_k} e^{-i(X\cdot Y)}\Phi_f(Y)dV_y$$

at a point X in E_k.

A result which is very useful in one dimension is known as the Riemann Localization theorem which states that for $f \in L_1(-\infty,\infty)$ the inversion of Φ_f at a point x depends only on the behavior of f in some neighborhood of x. Bochner [1] states and proves a theorem that if $f \in L_1(-\infty,\infty)$, $\delta > 0$ is some real number and

$$\int_0^\delta \left|\frac{f(x+t) + f(x-t)}{2} - f(x)\right| dt < \infty$$

then $f(x) = \lim_{R\to\infty} (2\pi)^{-1}\int_{-R}^{R} e^{-iyx}\Phi_f(y)dy.$

We shall now state and prove a theorem which extends the one dimensional case. Following the proof we shall examine the problems in getting a more general result without a stronger hypothesis or using a Kernel function [summability argument].

<u>Theorem 5</u>: If $f(X) \in L_1(E_k)$ can be written as

$f(X) = \prod_{j=1}^{k} f_j(x_j)$ where $f_j(x_j) \in L_1(-\infty,\infty)$ and if

there exists $\delta > 0$ such that

$$(1.C) \quad \int_0^\delta \left|\frac{\frac{f(x_j+t_j) + f(x_j-t_j)}{2} - f(x_j)}{t_j}\right| dt_j < \infty$$

for $j = 1,2,\ldots k$,

then

$$f(X) = \lim_{R\to\infty} \frac{1}{(2\pi)^k}\int_{J_r} e^{-i(X\cdot Y)}\Phi_f(Y)dV_y$$

where $J_r = \{x \mid |x_j| \leq R\}$.

Proof: $\lim_{R\to\infty} (2\pi)^{-k}\int_{J_R} e^{-i(X\cdot Y)}\Phi_f(Y)dV_y$

$$= \lim_{R\to\infty} \prod_{j=1}^{k} \int_{-R}^{R} e^{-ix_jy_j}\, \Phi_{f_j}(y_j)dy_j \text{ by Note 2}$$

$$= \prod_{j=1}^{k} \lim_{R\to\infty}\int_{-R}^{R} e^{-ix_jy_j}\, \Phi_{f_j}(y_j)dy_j$$

$$= f_1(x_1)\cdot f_2(x_2)\dots f_k(x_k) = f(X) \text{ by a}$$

theorem for functions of one variable from Bochner [1] page 9.

The condition 1.C is fulfilled if each f_j is normalized by $[f_j(x_j+0) + f_j(x_j-0)]$ that is the average of f_j's right and left limits at x_j.

We shall now generalize the hypothesis of the theorem to the two variable case, in order to show the difficulties which arise in extending the theorem. Let $f \in L_1(E_2)$ with transform Φ and let

$$S_R(x_1,x_2) = (2\pi)^{-2}\int_{-R}^{R}\int_{-R}^{R} e^{-i(y_1x_1+y_2x_2)}\Phi(y_1,y_2)dy_1dy_2$$

$$= (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{[-R,R]x[-R,R]}(y_1,y_2)e^{-i(y_1x_1+y_2x_2)}\Phi(y_1,y_2)dy_1dy_2$$

which by the composition theorem and Note 1

$$= (2\pi)^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T[K]f(x_1+y_1,x_2+y_2)dy_1dy_2.$$

Now consider

$$T[K_{[-R,R]x[-R,R]}(y_1,y_2)] = \int_{-R}^{R}\int_{-R}^{R} e^{ix_1y_1+ix_2y_2)}dx_1dx_2$$

$$= \frac{2\sin Ry_1}{y_1} \cdot \frac{2\sin Ry_2}{y_2}$$

Hence we have

$$S_R(x_1,x_2) = \pi^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin Ry_1}{y_1} \cdot \frac{\sin Ry_2}{y_2} f(y_1+x_1,y_2+x_2)dy_1dy_2$$

$$= \pi^{-2}[\int_0^{\infty}\int_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}] \frac{\sin Ry_1 \sin Ry_2}{y_1 \cdot y_2} f(y_1+x_1,y_2+x_2)dy_1dy_2$$

letting $y_2 = -y_2$ in the second integral and adding the resulting integrals we get

$$= 2\pi^{-2}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{\sin Ry_1}{y_1} \cdot \frac{\sin Ry_2}{y_2}$$

$$[\frac{f(x_1+y_1,x_2+y_2) + f(x_1+y_1,x_2-y_2)}{2}]dy_1dy_2$$

and repeating the above process on the inner integral we have

$$(1.D)\quad S_R(x_1,x_2) = 2^2\pi^{-2}\int_0^{\infty}\int_0^{\infty}\frac{\sin Ry_1 \sin Ry_2}{y_1y_2} \times$$

$$\times [\frac{f(x_1+y_1,x_2+y_2)+f(x_1+y_1,x_2-y_2)+f(x_1-y_1,x_2+y_2)}{4}$$

$$\frac{f(x_1-y_1,x_2-y_2)}{}]dy_1dy_2$$

Now setting

$$g_x(y_1,y_2) = [(f(x_1+y_1,x_2+y_2)+f(x_1-y_1,x_2+y_2)+f(x_1+y_1,x_2-y_2)$$

$$+f(x_1-y_1,x_2-y_2))/4 - f(x_1,x_2)]$$

The generalization of the one variable case would be: If $f(x_1,x_2) \in L_1(E_2)$ and for some $\delta > 0$

$$\text{(1.E)} \quad \int_0^{\delta}\int_0^{\delta} \left|\frac{g_x(y_1,y_2)}{y_1 y_2}\right| dy_1 dy_2 < \infty \text{ then}$$

$S_R(x_1,x_2) \to f(x_1,x_2)$ as $R \to \infty$.

Essentially there are two problems. First the product of sine functions in (1.D) rules out the use of the Riemann Lebesgue Lemma, as in the proof of the one variable case since the real part of $e^{i(x_1y_1+x_2y_2)}$ is $(\sin x_1y_1 \cdot \sin x_2y_2 - \cos y_1x_1 \cdot \cos y_2x_2)$. Also notice the denominator of (1.E) is the product $y_1 \cdot y_2$. This makes the integrand undefined for values of y_1 and y_2 where $y_1 \cdot y_2 = 0$; that is when either $y_1 = 0$ or $y_2 = 0$, even though the distance from $(0,0)$ to (y_1,y_2) may be large.

Because of these difficulties, we shall use a different approach to reach the desired results, namely inversion criteria.

Note 3: Let us define a special function $g(X) =$

$e^{-\varepsilon^2|X|^2} = \prod_{j=1}^{k} e^{-\varepsilon^2 x_j^2}$ where ε is real and $g(X) \in L_1(E_k)$

since $\int_{E_k} |e^{-\varepsilon^2|X|^2}| dV_x = \varepsilon^{-2}\ \pi^{\frac{1}{2}k}$. See Note 1.

Now $T[g(X)] \equiv \psi(Y)$ which by Note 2 $\psi(Y) = \prod_{j=1}^{k} \psi_j(y_j)$

with $\psi_j(y_j) = \int_{-\infty}^{\infty} e^{-\varepsilon^2 x_j^2} e^{iy_j x_j} dx_j = \frac{1}{\varepsilon}\ \pi\ e^{-4^{-1}\varepsilon^{-2} y_j^2}$.

[Note 1]

Hence $\psi(Y) = \dfrac{\pi^{\frac{k}{2}} e^{-4^{-1}\varepsilon^{-2}|Y|^2}}{\varepsilon^k}$

and letting $R = \frac{1}{\varepsilon}$ we have

$$T[e^{-\frac{|X|^2}{R^2}}] = R^k \pi^{\frac{k}{2}}\, e^{\frac{-R^2|Y|^2}{4}} .$$

Note 4: If $f \in L_1(E_k)$ and $Tf = \Phi$, we will define

$$(4.a) \quad S_R(X) = (2\pi)^{-k} \int_{E_k} \Phi(Y) e^{-i(X\cdot Y)} e^{-R^{-2}|Y|^2} dV_y .$$

Notice $S_R(X)$ exists since $|\Phi(Y)| \le \|f\|_1$ and $e^{-R^{-2}|Y|^2}$ is L_1.

Now we will show that $S_R(X) - f(X)$ can be expressed in terms of an integral of a single variable. By the composition theorem, Notes 2 and 3, considering $\Phi(Y)e^{-i(X\cdot Y)}$ as the transform and $e^{-R^{-2}|Y|^2}$ as the function we get

$$(4.b) \quad S_R(X) = \pi^{\frac{k}{2}} \cdot (2\pi)^{-k} R^k \int_{E_k} f(X+Z) e^{\frac{-R^2|Z|^2}{4}} dV_z$$

Now the family of K-dimensional surfaces $z_1^2 + z_2^2 + \dots + z_k^2 =$ constant covers E_k in such a way that for every point $Z \in E_k$ one and only one surface passes thru Z. By making the change of variables $Z_j = t \cdot u_j$ for $j = 1,2,\ldots,k-1$ and $Z_k = t(1 - \sum_{j=1}^{k-1} u_j^2)$ with Jacobian

$$\frac{\partial(z_1,z_2,\ldots,z_k)}{\partial(t,u_1,\ldots,u_{k-1})} = \frac{t^{k-1}}{\sqrt{1-\sum_{j=1}^{k-1} u_j^2}}$$

and then changing the order of integration and integrating over the surface of the k dimensional sphere $\sigma: u_1^2+u_2^2+\ldots+u_k^2 = 1$ instead of the volume of the sphere. We get

$$S_R(X) = R^k 2^{-k} \pi^{-k} \int_0^\infty e^{-4^{-1}R^2t^2} t^{k-1} \int_\sigma f(X+tU)\,d\sigma_u\,dt$$

where $d\sigma$ is the k-1 dimension volume element and σ is as above. The above follows directly from Courant [4] Volume II, page 300-303. Also Courant shows that the k-1 dimensional volume of $\sigma: w_{k-1} = \frac{2\pi}{\Gamma(\frac{k}{2})}$ where $\Gamma(\frac{k}{2})$ is the gamma function.

Now $S_R(X) - f(X) =$

$$R^k 2^{-k} \pi^{-k} \int_0^\infty e^{\frac{-R^2t^2}{4}} t^{k-1} \int_\sigma f(X+tU)\,d\sigma_u\,dt - f(X)$$

$$= \frac{R^k 2^{-k+1}}{\Gamma(\frac{k}{2})} \int_0^\infty e^{-4^{-1}R^2t^2} t^{k-1} \left[\frac{1}{w_{k-1}} \int_\sigma f(X+tU)\,d\sigma_u \right] dt - f(X)$$

$$= \frac{R^k}{2^{k-1}\Gamma(\frac{k}{2})} \int_0^\infty e^{-4^{-1}R^2t^2} t^{k-1} \left[\frac{1}{w_{k-1}} \int_\sigma f(X+tU)\,d\sigma_u \right] dt -$$

$$- f(X) \frac{R^k}{2^{k-1}\Gamma(\frac{k}{2})} \int_0^\infty e^{-4^{-1}R^2t^2} t^{k-1} dt$$ See Note 1.

$$= \frac{R^k}{2^{k-1}\Gamma(\frac{k}{2})} \int_0^\infty e^{-4^{-1}R^2t^2} t^{k-1}[\frac{1}{w_{k-1}} \int_\sigma f(X+tU)d\sigma_u - f(X)]dt$$

and letting

$$(1.F) \quad g_x(t) = t^{k-1}[\frac{1}{w_{k-1}} \int_\sigma f(X+tU)d\sigma_u - f(X)]$$

we have

$$(1.H) \quad S_R(X) - f(X) = \frac{R^k}{2^{k-1}\Gamma(\frac{k}{2})} \int_0^\infty e^{-4^{-1}R^2t^2} g_x(t)\, dt$$

<u>Theorem 6</u>: Let $f(X) \in L_1(E_k)$ and $g_x(t)$ be defined by (1.F). If $H(t)$ for $0 \leq t < \infty$ is bounded and $H(t) \downarrow 0$ as $t \to \infty$ and if $t^k H(t)$ is continuous, bounded, and decreases monotonely to zero for all $t \geq$ some t_0, and $H(t) = o(\frac{1}{t^k})$, and $H(t) = O(\frac{1}{t^{k+\varepsilon}})$ as $t \to \infty$ for some $\varepsilon > 0$, then for a fixed X, the condition

$$G_x(t) = \int_0^t g_x(y)dy = o(t^k) \text{ as } t \to 0$$

implies

$$R^k \int_0^\infty g_x(t)H(Rt)dt = o(1) \text{ as } R \to \infty.$$

Proof:

$$R^k \int_0^\infty g_x(t)H(Rt)dt = R^k \int_0^u g_x(t)H(Rt)dt + R^k \int_u^\infty g_x(t)H(Rt)dt$$

$$= I_1 + I_2$$

$$I_1 = R^k\int_0^u g_x(t)H(Rt)dt = R^k \frac{1}{R} \int_0^{uR} g_x(\tfrac{s}{R})H(s)ds$$

and by integrating by parts (See Halmos [5] page 269) we get

$$I_1 = R^k G_x(\tfrac{S}{R})\, H(s)\Big|_0^{uR} - R^k\int_0^{uR} G_x(\tfrac{S}{R})d_sH(s)$$

$$= (uR)^k \frac{G_x(u)}{u^k} H(Ru) - R^k\int_0^{uR} G_x(\tfrac{S}{R})d_sH(s)$$

the first part goes to zero as $R \to \infty$ since this takes $(uR)^k \to \infty$ for all $u > 0$. Hence by hypothesis $(uR)^k H(Ru) \to 0$ so we have

$$(1.\text{I}) \quad I_1 = o(1) - \int_0^{uR} G_x(\tfrac{S}{R})R^k d_sH(s).$$

Now since $G_x(t) = o(t^k)$ as $t \to 0$ and by definition of $G_x(t)$ we have $v(t) = \dfrac{G_x(t)}{t^k}$ is a continuous function with $\lim\limits_{t\to 0} v(t) = 0$, hence define $v(0) = 0$. Now let $\delta > 0$, then for all $t \in [\overline{N_\delta(0)} - \{t \mid t < 0\}]$ $v(t)$ is bounded since v is continuous. Call that bound B_1. Now for for all $t > \delta$ let us consider

$$|v(t)| = \left|\frac{G_x(t)}{t^k}\right| = \left|\frac{1}{t^k}\int_0^t g_x(s)ds\right| =$$

$$= \left|\frac{1}{t^k}\int_0^t s^{k-1}f_x(s)ds - \frac{1}{t^k}\int_0^t s^{k-1}f(X)ds\right.$$

$$\le \frac{1}{\delta^k}\left|\int_0^t s^{k-1}f_X(s)ds\right| + \frac{f(X)}{k}$$

$$\leq \frac{1}{\delta^k} \left| \int_0^{\infty} s^{k-1} f_X(s)ds \right| + \frac{|f(X)|}{k}$$

$$\leq \frac{1}{w_{k-1}\delta^k} \| f \|_1 + \frac{|f(X)|}{k} = B_2 \text{ say}$$

and B_2 is finite for all X such that $|f(X)| < \infty$ and this holds almost everywhere. Now we can say $|N(t)| \leq B = \max \{B_1, B_2\}$ hence $v(t)$ is bounded for all $t > 0$. Hence

$G_X(\frac{S}{R}) = v(\frac{S}{R})(\frac{S}{R})^k$ now returning to (1.I)

$$I_1 = o(1) - \int_0^{uR} G_X(\frac{S}{R})R^k d_s H(s)$$

$$= o(1) - \int_0^{uR} (\frac{S}{R})^k \cdot v(\frac{S}{R})R^k d_s H(s)$$

$$= o(1) - \int_0^{uR} s^k v(\frac{S}{R}) d_s H(s)$$

$$= o(1) - \int_0^{s} s^k v(\frac{S}{R}) d_s H(s) - \int_s^{uR} s^k v(\frac{S}{R}) d_s H(s).$$

Now let us consider the first integral in the above expression. Where S is some arbitrary constant to be fixed later.

For finite S, s^k is bounded and $v(\frac{S}{R})$ is bounded everywhere, hence by dominated convergence we can pass the limit under the integral and $\lim_{R\to\infty} \frac{S}{R} = 0$ for all finite $\mathfrak{S}$ and $v(0)$ is zero, hence the first integral goes to zero as $R \to \infty$. Now the second integral

$$(1.J)\quad \left|\int_S^{Ru} s^k v(\tfrac{S}{R})d_sH(s)\right| \leq \left|B\int_S^{Ru} s^k d_sH(s)\right| \text{ and}$$

integrating by parts we get the integral (1.J)

$$= |B(Ru)^k \cdot H(Ru) - B\cdot S^k H(S) - Bk\int_S^{Ru} s^{k-1}H(s)ds|$$

and since $H(t) = O(\frac{1}{t^{k+\epsilon}})$ for some $\epsilon > 0$, $|H(t)| \leq \frac{C}{t^{k+\epsilon}}$

for all $t \geq t_o$ and S will be $> t_o$, hence we can say

(1.J)

$$\leq |B(Ru)^k\cdot H(Ru) - B\cdot S^k H(S) - kC\int_S^{Ru} s^{k-1}s^{-(k+\epsilon)}ds|$$

$$= |B(Ru)^k\cdot H(Ru) - B\cdot S^k H(S) - \frac{kC}{-\epsilon}(\frac{1}{(Ru)^\epsilon} - \frac{1}{S^\epsilon})|$$

which can be made as small as we wish by choosing S large since $\frac{1}{S^\epsilon}$ is monotone and $S^kH(S)$ also can be made as small as necessary for S large. Hence we can say

$$I_1 = o(1) \text{ as } R \to \infty.$$

Now we shall examine

$$I_2 = \int_u^\infty g_x(t)R^kH(Rt)dt$$

$$= \int_u^\infty t^{k-1}[f_x(t) - f(X)]\, R^kH(Rt)dt$$

$$= \int_u^\infty t^{k-1}R^kH(Rt)f_x(t)dt - \int_u^\infty t^{k-1}f(X)R^kH(Rt)dt$$

$$= I_{2,1} - I_{2,2} \text{ say and}$$

$$|I_{2,1}| = |\int_u^\infty t^{k-1} \frac{R^k H(Rt)}{w_{k-1}} \int_\sigma f(X+tZ)d\sigma_z dt|$$

and by the monotoneity of $H(t)$ we have

$$|I_{2,1}| \leq \left|\frac{(uR)^k H(Ru)}{w_{k-1}(u)^k} \int_u^\infty t^{k-1}\int_\sigma f(X+tZ)d\sigma_z dt\right|$$

but that integral is the integral $\int f(X+Y)dV_y$ over $E_k - J$ where J is the k-dimensional sphere of radius u and center at the origin hence we have

$$|I_{2,1}| \leq \frac{(uR)^k H(Ru)}{w_{k-1}(u)^k} \|f\|_1$$ and letting $R \to \infty$ by the monotoneity of $t^k H(t)$ for $t > t_o$ $|I_{2,1}| = o(1)$ as $R \to \infty$.

And for

$$|I_{2,2}| = |\int_u^\infty t^{k-1} R^k H(Rt) f(X)dt|$$

$$= |f(X)| \left|\int_u^\infty t^{k-1} R^k H(Rt)dt\right|$$

and by letting $Rt = s$, we have

$$|I_{2,2}| = |f(X)| \cdot \left|\int_{uR}^\infty s^{k-1} H(s)ds\right|.$$

And by hypothesis $H(s) = O(\frac{1}{s^{k+\varepsilon}})$ for $\varepsilon > 0$ which implies there exists C, a constant, such that $|H(s)| \leq \frac{C}{s^{k+\varepsilon}}$.

Hence

$$|I_{2,2}| = |f(X)|\int_{uR}^{\infty} s^{k-1} \frac{C}{s^{k+\epsilon}}\,ds = |f(X)|\int_{uR}^{\infty} \frac{1}{s^{1+\epsilon}}\,ds$$

$$= o(1) \text{ as } R \to \infty.$$

Hence $R^k\int_0^{\infty} g_x(t)H(Rt)dt = o(1)$ as $R \to \infty$.

Now using Theorem 6 and formula (1.H) in Note 4 we shall prove that the Fourier transform of every L_1 function f is Gauss-Summable to f almost everywhere. In other words, we have

Theorem 7: If $f \in L_1(E_k)$ and has transform $\Phi(Y)$, then

$$\lim_{R\to\infty} (2\pi)^{-k}\int_{E_k} \Phi(Y)e^{-i(X\cdot Y)}e^{-R^{-2}|Y|^2}dV_y = F(X)$$

almost everywhere.

Proof: $H(t) = \frac{1}{2^{k-1}\Gamma(\frac{k}{2})} e^{-\frac{t^2}{4}}$ satisfies all the hypothesis on H in Theorem 6. $|H(t)| \leq \frac{1}{2^{k-1}\Gamma(\frac{k}{2})}$;

H(t) decreases montonicly to zero, for all $t > \sqrt{2k}$ t^k

H(t) goes to zero,

$$H(t) = 2^{-k+1}(\Gamma(\tfrac{k}{2}))^{-1}e^{-\frac{t^2}{4}} = o(t^k) \text{ as } t \to \infty$$

and $H(t) = O(\frac{1}{t^{k+\epsilon}})$ for any finite $\epsilon > 0$ as $t \to \infty$. Now in order to apply Theorem 6 we must show that for all $f(X) \in L_1(E_k)$; $G_x(t) = \int_0^t g_x(y)dy = o(t^k)$ as $t \to \infty$ a.e.

where $g_X(y)$ is as in (1.F).

Consider

$$t^{-k}G_X(t) = t^{-k}\int_0^t g_X(y)dy$$

$$= t^{-k}\int_0^t y^{k-1}[\frac{1}{w_{k-1}} \int_\sigma f(X+yZ)d\sigma_Z - f(X)]dy$$

$$= \frac{1}{t^k w_{k-1}} \int_0^t y^{k-1}\int_\sigma [f(X+yZ) - F(X)]d\sigma_Z dy$$

$$= \frac{1}{k} \cdot \frac{1}{V} \int_0^t y^{k-1}\int_\sigma [f(X+yZ)\ f(X)]d\sigma_Z dy$$

where V is the k-dimensional volume of a sphere with radius t. Now if $t_n \to 0$ monotonely and V_n is the volume of the k-dimensional sphere with radius t_n, centered at X, then

$$\lim_{t_n\to 0} t_n^{-k}G_X(t_n) = \frac{1}{k} \lim_{t_n\to 0} \frac{1}{V_n} \int_0^t g^{k-1}\int_\sigma [f(X+yZ) - f(X)]d\sigma_Z dy$$

$$= \frac{1}{k} \lim_{n\to\infty} \int_{V_n} [f(X+Y) - f(X)]dV_y$$

$$= \frac{1}{k} \cdot 0 = 0 \text{ a.e.}$$

Thus $\lim_{t\to 0} t^{-k}G_X(t) = 0$ a.e.

The limit in the above is zero a.e. by an adaptation from k-dimensional cubes to k-dimensional spheres of a Theorem in McShane and Botts [7] page 189.

Now we are ready to prove two theorems concerning inversion of a Fourier transform for $L_1(E_k)$.

<u>Theorem 8:</u> If $f(X) \in L_1(E_k)$ and $T(f) = \Phi(Y) \in L_1(E_k)$, then

$$f(X) = (2\pi)^{-k}\int_{E_k} \Phi(T)e^{-i(X\cdot Y)}dV_y \text{ a.e.}$$

Proof: $$(2\pi)^{-k}\int_{E_k} \sup_R|\Phi(Y)e^{-i(X\cdot Y)}e^{-R^{-2}|Y|^2}dV_y$$

$$= (2\pi)^{-k}\int_{E_k} |\Phi(Y)|dV_y < \infty \text{ since } \Phi \in L_1$$

$\therefore$ By the dominated convergence Theorem and Theorem 7

$$f(X) = \lim_{R\to\infty} (2\pi)^{-k}\int_{E_k} \Phi(Y)e^{-i(X\cdot Y)}e^{-|Y|^2R^{-2}}dV_y$$

$$= (2\pi)^{-k}\int_{E_k} \lim_{R\to\infty} \Phi(Y)e^{-i(X\cdot Y)}e^{-|Y|^2R^{-2}}dV_y$$

$$= (2\pi)^{-k}\int_{E_k} \Phi(Y)e^{-(X\cdot Y)}dV_y \text{ a.e.}$$

<u>Theorem 9</u>: If $f(X) \in L_1(E_k)$, $|f(X)| \leq M$ and the transform of $f(X)$, $\Phi(Y) \geq 0$, then $\Phi(Y) \in L_1(E_k)$ and

$$f(X) = (2\pi)^{-k}\int_{E_k} \Phi(Y)e^{-i(X\cdot Y)}dV_y \text{ a.e.}$$

Proof: In Note 4, we have the forms 4.a and 4.b for $(2\pi)^{-k}\int_{E_k} \Phi(Y)e^{-i(X\cdot Y)}e^{-|Y|^2R^{-2}}dV_y = S_R(X)$

using 4.b we have

$$|S_R(X)| = |(\pi)^{\frac{k}{2}} \cdot (2\pi)^{-\frac{1}{2}k} R^k \int_{E_k} f(X+Z) e^{\frac{-R^2|Z|^2}{4}} dV_z|$$

which letting $RZ = Y$ and taking the absolute values under the integral yields

$$|S_R(X)| \leq 2^{-k} \pi^{-\frac{1}{2}k} \int_{E_k} |f(X+\tfrac{1}{R}Y)| e^{\frac{-|Y|^2}{4}} dV_y$$

$$\leq M 2^{-k} \pi^{-\frac{1}{2}k} \int_{E_k} e^{\frac{-|Y|^2}{4}} dV_y$$

$$= M \text{ by Note 1.}$$

Hence we have

$|S_R(0)| \leq M$ but from form (4.a) $S_R(0) =$

$$(2\pi)^{-k} \int_{E_k} \Phi(Y) e^{\frac{-|Y|^2}{R^2}} dV_y \leq M$$

and by Property 1 $\Phi(Y)$ is bounded and we have $e^{\frac{-|Y|^2}{R^2}}$ integrable by dominated convergence

$$M \geq \lim_{R\to\infty} (2\pi)^{-k} \int_{E_k} \Phi(Y) e^{\frac{-|Y|^2}{R^2}} dV_y$$

$$= (2\pi)^{-k} \int_{E_k} \Phi(Y) dV_y \text{ but } \Phi(Y) \geq 0$$

hence $|\Phi(Y)| = \Phi(Y)$. This means $\Phi(Y) \in L_1(E_k)$. Now by applying Theorem 8, the inversion holds a.e.

<u>Corollary</u>: If $\Phi_f(Y)$ is a transform as in the hypothesis of Theorem 8 or 9, then if $f(X) \in L_1 (E_k)$ is

continuous the inversion holds everywhere. Moreover, if f is continuous at some point X, in E_k, then the inversion holds at X_1.

Proof: Consider

$$f_0(X) = (2\pi)^{-k}\int_{E_k} \Phi_f(Y)e^{-i(X\cdot Y)}dV_y$$

where $f \in L_1$ is as in Theorem 8 or 9. Then we have $f_0(X) = f(X)$ a.e. By Property 2 $f_0(X)$ is uniformly continuous. Therefore if $f(X)$ is continuous at X_1, then for $\epsilon > 0$, there exists $\delta_1 > 0$ such that $|f(X) - f(X_1)| < \frac{\epsilon}{2}$ whenever $|X - X_1| < \delta_1$. And by the continuity of $f_0(X)$ we know that there exists $\delta_2 > 0$ such that $|f_0(X) - f_0(X_1)| < \frac{\epsilon}{2}$ for $|X - X_1| < \delta_2$. If we choose $\delta = \min\{\delta_1, \delta_2\}$

$$|f(X_1) - f_0(X_1)| \leq |f(X_1) - f(X')| + |f(X') - f_0(X')| + |f_0(X') - f_0(X_1)|$$ where

X' is a point in the δ neighborhood of X_1 such that $f(X') = f_0(X')$. We know such a point exists since $f(X) = f_0(X)$ a.e. and the measure of any $\delta > 0$ neighborhood is not zero. Hence

$$|f(X_1) - f_0(X_1)| \leq |f(X_1) - f(X')| + |f(X') - f_0(X_1)|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and since $\epsilon > 0$ is arbitrary $f(X_1) = f_0(X_1)$.

Note 5: Denote $f(x_1+h_1, x_2+h_2, \ldots, x_k+h_k)$ by $f(X+H)$ or f_h and let $f(X) \in L_1(E_k)$. That is

$$\| f \|_1 = [\int_{E_k} |f(X)|\, dV_x] = A < \infty$$

hence $\| f_h \|_1 = [\int_{E_k} |f(X+H)|\, dV_x] = A < \infty$.

We shall let $w(H) = \| f(X)-f(X+H) \|_1 = \int_{E_k} |f(X)-f(X+H)|\, dV_x$.

Now consider $w(-H) = \int_{E_k} |f(X) - f(X-H)|\, dV_x$ and letting $X = Y+H$ we have

$$w(-H) = \int_{E_k} |-f(Y) + f(Y+H)|\, dV_y = w(H)$$

hence $w(H)$ is even and clearly $w(H) \geq 0$.

Therefore

$$0 \leq w(H) = \| f - f_h \|_1 \leq \| f \|_1 + \| f_h \|_1 = 2A < \infty$$

by Minkowski inequality hence $0 \leq w(H) \leq 2A$. Now, consider

$$w(H_1+H_2) = \int_{E_k} |f(X) - f(X+H_1+H_2)|\, dV_x$$

and letting $X = Y - H_1$

$$w(H_1+H_2) = \int_{E_k} |f(Y-H_1) - f(Y) + f(Y) - f(Y+H_2)|\, dV_y$$

$$\leq \int_{E_k} |f(Y) - f(Y-H_1)|dV_y + \int_{E_k} |f(Y) - f(Y+H_2)|dV_y$$

$$= w(-H_1) + w(H_2) = w(H_1) + w(H_2)$$

and clearly $w(0) = 0$ since $\int_{E_k} |0|dV_x = 0$.

<u>Theorem 10</u>: $\lim\limits_{|H|\to 0} w(H) = 0$ where $w(H)$ is as in Note 5, and $|H|$ is given in Definition 1.

Proof: Let $G(X)$ be the function defined by (1.A). Let $A = [a_1,b_1] \times [a_2,b_2] \times \ldots \times [a_k,b_k]$ and

$\Delta = [a_1+h_1, b_1+h_1] \times [\ldots] \times [a_k-h_k, b_k+h_k]$ then

$$w_G(H) = \int_{E_k} |G(X+H) - G(X)|dV_x = 2\int_{E_k} |G(X)|dV_x - 2\int_{A\cap\Delta} dV_x$$

$$= 2\|G\|_1 - 2 \prod_{j=1}^{k} (b_j-a_j-h_j)$$

$$= 2[\prod_{j=L}^{k} (b_j-a_j) - \prod_{j=1}^{k} (b_j-a_j-h_j)] \to 0$$

as $|H| \to 0$. That is, if $\epsilon > 0$, then there exists $\delta_2 > 0$ such that for all $|H| < \delta_1$ $w_G(H) < \epsilon$. Hence the result holds for characteristic functions. And since

$$w_{f_1+f_2}(H) = \int_{E_k} |f_1(X) + f_2(X) - f_1(X+H) - f_2(X+H)|dV_x$$

$$\leq \int_{E_k} |f_1(X) - f_1(x+H)|dV_x + \int_{E_k} |f_2(X) - f_2(X+H)|dV_x$$

$= w_{f_1}(H) + w_{f_2}(H)$, the result holds for step functions, since step functions are linear combinations of characteristic functions. Now let $f(X) \in L_1(E_k)$ and let $\{f_n\}$ be a sequence of step functions such that $f_n \xrightarrow[L_1]{} f$ as $n \to \infty$. Then

$$w_f(H) = \int_{E_k} |f(X) - f(X+H)|\,dV_x$$

$$= \int_{E_k} |f(X) - f_n(X) - f(X+H) + f_n(X+H) + f_n(X) - $$

$$-f_n(X+H)|\,dV_x$$

$$= \int_{E_k} |f(X) - f_n(X)|\,dV_x + \int_{E_k} |f_n(X) - f_n(X+H)|\,dV_x +$$

$$+ \int_{E_k} |f(X+H) - f_n(X+H)|\,dV_x$$

$= 2\,\|f - f_n\|_1 + w_{f_n}(H)$ and since $f_n \xrightarrow[L_1]{} f$ if $\epsilon > 0$, for sufficiently large n $2\,\|f - f_n\|_1 < \frac{\epsilon}{2}$ and from there exists $\delta > 0$ such that for all $|H| < \delta$ $w_{f_n}(H) < \frac{\epsilon}{2}$.

Hence $w_f(H) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So $\lim_{|H| \to 0} w_f(H) = 0$ for all $f(X) \in L_1(E_k)$.

<u>Theorem 11</u>: If $f(X)$, $g(X)$ are bounded L_1 functions then

$$h(X) = \int_{E_k} f(X+Y)\overline{g(Y)}\,dV_y = \int_{E_k} f(X-Y)\overline{g(-Y)}\,dV_y$$

exists for all $X \in E_k$ and $h(X)$ is bounded, uniformly continuous and $L_1(E_k)$.

Proof: Since g is bounded $|g(X)| \leq M$ for some real M. Clearly then $h(X)$ exists for all X. By letting $Y = -Y$ the second integral follows from the first. And

$$|h(X)| = \left|\int_{E_k} f(X+Y)\overline{g(Y)}dV_y\right| \leq \int_{E_k} |f(X+Y)|\ |\overline{g(Y)}|dV_y$$

$$\leq M\int_{E_k} |f(X+Y)|dV_y = M\cdot\|f\|_1 < \infty, \text{ so } h \text{ is bounded.}$$

Now

$$|h(X) - h(X+T)| \leq \int_{E_k} |f(X+Y) - f(X+Y+T)||\overline{g(Y)}|dV_y$$

$$\leq M\int_{E_k} |f(X+Y) - f(X+Y+T)|dV_y$$

and letting $X+Y = \tilde{Y}$ we have $|h(X) - h(X+T)| \leq w_f(T)\cdot M$ and by the previous theorem $w_f(T) \to 0$ as $|T| \to 0$. Therefore $h(X)$ is uniformly continuous since is $\epsilon > 0$ there exist $\delta > 0$ such that $w_f(T) < \frac{\epsilon}{M}$. And the δ depends on f not the point X. So $|h(X) - h(X+T)| < \epsilon$ for all $X \in E_k$ when $|T| < \delta$.

Consider

$$\int_{E_k} |h(X)|dV_x = \int_{E_k} \left|\int_{E_k} f(X+Y)\overline{g(Y)}dV_y\right| dV_x$$

$$\leq \int_{E_k}\int_{E_k} |f(X+Y)||\overline{g(Y)}|dV_y dV_x \text{ and by the}$$

Fubini Theorem

$$= \int_{E_k}\int_{E_k} |f(X+Y)|\,|\overline{g(Y)}|\,dV_x dV_y$$

$$= \|g\|_1 \cdot \|f\|_1 < \infty \text{ hence } h(X) \in L_1(E_k).$$

The following two theorems give us a kind of Parseval relation for bounded L_1 functions which will be used in Chapter 2 since bounded L_1 functions are L_2.

<u>Theorem 12</u>: If $f(X)$ a bounded L_1 function, then

$$\int_{E_k} |f(X)|^2 dV_x = (2\pi)^{-k}\int_{E_k} |\Phi(Y)|^2 dV_y < \infty.$$

Proof: Since f is bounded, there exists a real number $M \geq 0$ with $|f(X)| \leq M$ for all $X \in E_k$. Hence

$$\int_{E_k} |f(X)|^2 dV_x \leq M\int_{E_k} |f(X)|\,dV_x = M\cdot\|f\|_1 < \infty$$

that is $f \in L_2(E_k)$ also. Now consider

$$h(X) = \int_{E_k} f(X+Y)f(Y)dV_y = \int_{E_k} f(X-Y)\overline{f(-Y)}dV_y$$

now let $g(Y) = \overline{f(-Y)}$ and by the convolution Theorem and Property 4

$$T[h(X)](Y) = \Phi_f(Y)\cdot T[g][T] = \Phi_f(Y)\cdot\overline{\Phi_f(Y)} = |\Phi(Y)|^2$$

$$\geq 0$$

and by the previous theorem $h(X)$ is a continuous L_1 function. Applying the corollary to Theorem 9, we have

$$h(X) = \int_{E_k} f(X+Y)\overline{f(Y)}dV_y = (2\pi)^{-k}\int_{E_k} |\Phi_f(Y)|^2 e^{-i(X\cdot Y)}dV_y$$

everywhere. Now let $X = (0,0,0,\ldots 0)$ and we have

$$\int_{E_k} |f(Y)|^2 dV_y = (2\pi)^{-k}\int_{E_k} |\Phi_f(Y)|^2 dV_y.$$

Theorem 13: If f,g are bounded L_1 functions, then

$$\int_{E_k} f(X)\overline{g(X)}dV_x = (2\pi)^{-k}\int_{E_k} \Phi(Y)\overline{\Psi(Y)}dV_y$$

where $T(f) \equiv \Phi$ and $T(g) \equiv \Psi$.

Proof: By Theorem 11 $\int_{E_k} |\Phi(Y)|^2 dV_y < \infty$ and

$\int_{E_k} |\Psi(Y)|^2 dV_y < \infty$, hence by the Schwarz inequality

$\int_{E_k} |\Phi(Y)\overline{\Psi(Y)}|dV_y$ exists and by Theorem 11

$h(X) = \int_{E_k} f(X+Y)\overline{g(Y)}dV_y$ is continuous, bounded and L_1.

So by Theorem 8

$$h(X) = \int_{E_k} f(X+Y)\overline{g(Y)}dV_y = (2\pi)^{-k}\int_{E_k} \Phi(Y)\overline{\Psi(Y)}e^{-i(X\cdot Y)}dV_y$$

Now let $X = (0,0,\ldots 0)$ and we have

$$\int_{E_k} f(Y)\overline{g(Y)}dV_y = (2\pi)^{-k}\int_{E_k} \Phi(Y)\overline{\Psi(Y)}dV_y$$

Theorem 14: If $f \in L_1(E_k)$ and $ix_j f(X) \in L_1$ then

$\frac{\partial\Phi_f(Y)}{\partial y_j}$ exists and $\Phi_{ix_jf}(Y) = \frac{\partial\Phi_f(Y)}{\partial y_j}$.

Proof: $\frac{\Phi(Y+H_j) - \Phi(Y)}{h_j} = T[f(X)(\frac{e^{i(X\cdot H_j)}-1}{h_j})]$ (1.K)

where $H_j = \{0,0,\ldots 0,h_j,0\ldots 0\}$. (1.K) holds by Note 2.

Denote by $T[f_{h_j}(X)]$. Now $f_{h_j}(X) \xrightarrow[L_1]{} ix_jf(X)$ for all x_j

since $\lim_{h_j\to 0} f(X)(\frac{e^{i(X\cdot H_j)}-1}{h_j}) = f(X) \lim_{h_j\to 0} (\frac{e^{ix_jh_j}-1}{h_j})$

$= ix_jf(X)$ and

$|f(X)(\frac{e^{i(X\cdot H_j)}-1}{h_j})| \leq |f(X)| \left|\frac{e^{i(x_jh_j)}-1}{h_j}\right| \leq |x_j||f(X)|$

which is $L_1(E_k)$ by hypothesis. Hence by dominated convergence

$$\lim_{h_j\to 0} \int_{E_k} |f(X)[\frac{e^{i(X\cdot H_j)}-1}{h_j} - ix_j]|dV_x$$

$$= \int_{E_k} \lim_{h_j\to 0} |f(X)|\left|\frac{e^{i(X\cdot H_j)}-1}{h_j} - ix_j\right|dV_x$$

$= 0$ hence $f_{h_j} \to ix_jf(X)$ in L_1 norm. Hence by Property 5 $T[f_{h_j}(X)] \to T[ix_jf(X)]$ uniformly as $h_j\to 0$.

Then since the transform exists for all y_j

$$\frac{\partial\Phi_f(Y)}{\partial y_j} = \Phi_{ix_jf}(Y)$$

by (1.K) and the definition of a partial derivative.

Bochner, in his book on Fourier Integrals [2] proves another type of inversion theorem for $L_1(E_k)$ which we shall state without proof.

Theorem A: If $f(X) \in L_1(E_k)$ is continuous, $f(X) \to 0$ as $|X| \to \infty$, and

$\frac{\partial^k f(X)}{\partial x_1 \partial x_2 \cdots \partial x_k}$ exists, is $L_1(E_k)$ and continuous, then

$$f(X) = (2\pi)^{-k} \int_{E_k} \Phi_f(Y) e^{-i(X \cdot Y)} dV_Y$$ where the integral

is to be taken as a Cauchy principal value. That is let $N = \{X \mid |x_j| \leq n_j$ where the n_j's are natural numbers$\}$ and we have

$$f(X) = \lim_{n_1 n_2 \cdots n_k \to \infty} (2\pi)^{-k} \int_N e^{-i(X \cdot Y)} \Phi_f(Y) dV_y$$

Proof: Bochner [2] pages 249-255.

This theorem gives the type of result which is analogous to the usual inversion theorem for functions of one variable.

CHAPTER II

L_2 AND THE PLANCHEREL THEOREM

When considering Fourier transforms in L_2, we find that for an arbitrary L_2 function f the transform as defined in Chapter 1 may not even exist. The reason for this statement is the fact that although $f \in L_2$, f may not be a member of L_1, hence the transform would not exist. However, the existence of an inner product in L_2, along with the properties of the distance [norm] function for L_2 [Definition 2] makes L_2 a Hilbert Space. This inner product of two functions $f,g \in L_2(E_k)$ is defined by

(a) $\langle f,g\rangle = \int_{E_k} f(X)\cdot\overline{g(X)}dV_x$ which exists and is finite

by the Schwarz inequality. For all $f,g \in L_2(E_k)$, the inner product also has the following properties

(b) $\langle g,f\rangle = \overline{\langle f,g\rangle}$

(c) $\langle f,f\rangle = \|f\|_2^{\,2}$

(d) $\langle f,ag\rangle = \bar{a}\langle f,g\rangle$ where a is a scalar

(e) $\langle f,g_1+g_2\rangle = \langle f,g_1\rangle + \langle f,g_2\rangle$

(f) $|\langle f,g\rangle| \leq \|f\|_2\cdot\|g\|_2$

(g) $\langle f,g\rangle$ is a continuous function of $L_1 x L_2$

Proofs: See Stone [11] pages 3-5, and pages 23-32.

Definition 3: If S_1 and S_2 are two subsets of $L_2(E_k)$, a <u>linear transformation</u> (operator) $\mathcal{T}$ is a function with domain S_1 and range S_2 such that if $f, g \in S_1$ and α, β are complex numbers then $\mathcal{T}(\alpha f+\beta g) = \alpha\mathcal{T}(f) + \beta\mathcal{T}(g)$. $\mathcal{T}$ is called <u>isometric</u> if for all $f, g \in S_1$, $\langle \mathcal{T}f, \mathcal{T}g\rangle = \langle f, g\rangle$. $\mathcal{T}$ is called unitary if the domain and range are the same and $\langle \mathcal{T}f, \mathcal{T}g\rangle = \langle f, g\rangle$.

Two standard theorems on linear operators on the Hilbert Space L_2 are stated without proof. The proofs follow from two theorems in Stone [11] page 57 and 76, which are more general in scope. However for our purposes these will be sufficient.

<u>Theorem B</u>: Let $\mathcal{T}_0$ be a linear operator defined on a linearly dense set S in L_2 (linear combinations of the elements of S are dense in L_2) with $\langle \mathcal{T}_0 f, \mathcal{T}_0 f\rangle = \langle f, f\rangle$ for all f in S. Then $\mathcal{T}_0$ has a unique extension to an isometric linear operator $\mathcal{T}$ with domain L_2 and on S $\mathcal{T}_0 = \mathcal{T}$.

<u>Theorem C</u>: If $\mathcal{T}$ is an isometric linear operator with domain L_2, then $\mathcal{T}$ is a 1 to 1 function and its range is closed. If the range of $\mathcal{T}$ is dense in L_2, then the range is L_2 and $\mathcal{T}$ is unitary and has an inverse $\mathcal{T}^{-1}$ which is a unitary operator.

The theorem of major significance in the study of the Fourier transform in $L_2(E_k)$ is that of Plancherel. We shall prove the theorem by three methods.

Theorem 15: (Plancherel) The set $S \subseteq L_2$ which are Fourier transforms of functions in $L_1 \cap L_2$ is dense in L_2 and the operator $\mathcal{J}_0$ defined on $L_1 \cap L_2$ with range S by

$$\mathcal{J}_0(f)(Y) = (2\pi)^{-\frac{k}{2}} \int_{E_k} f(X) e^{i(X \cdot Y)} dV_y$$

has a unique extension to a unitary operator $\mathcal{J}$ with domain and range L_2.

Proof 1: This proof is outlined in Bochner and Chandrasekharan [1]. However we shall reproduce it here with some addition in order to give a clear picture of some of the methods of proof of the important theorem.

Let $\varepsilon > 0$ and

$$\Phi^{\varepsilon}_{a_j b_j}(y_j) = \begin{cases} \text{if } 0, \text{ then } y_j < a_j - \varepsilon \\ \text{if } 1, \ a_j \leq y_j \leq b_j \\ \text{if } \dfrac{y_j - a_j + \varepsilon}{\varepsilon}, \ a_j - \varepsilon \leq y_j \leq a_j \\ \text{if } \dfrac{b_j + \varepsilon - y_j}{\varepsilon}, \ b_j \leq y_j \leq b_j + \varepsilon \\ \text{if } 0, \ y_j > b_j + \varepsilon \end{cases}$$

and $b_j > a_j$. Now let

$$\Phi^{\varepsilon}_{a,b}(Y) = \prod_{j=1}^{k} \Phi^{\varepsilon}_{a_j, b_j}(y_j).$$

Clearly $\Phi^{\varepsilon}_{a,b}$ is a bounded continuous function in

$L_1 \cap L_2$ and the class of R determined by linear combinations of functions of the form $\Phi^{\varepsilon}_{a,b}$ is dense in $L_2(E_k)$, since its closure contains the step functions which are dense in $L_2(E_k)$.

We now <u>define</u>

$$f^{\varepsilon}_{a,b}(X) = (2\pi)^{-\frac{k}{2}}\int_{E_k} \Phi^{\varepsilon}_{a,b}(Y)e^{-i(X\cdot Y)}dV_y$$

then $f^{\varepsilon}_{a,b}$ is again a bounded continuous function belonging to L_1 hence L_2. To show $f^{\varepsilon}_{a,b}$ is L_1 from Note 2 it is sufficient to show that

$$f^{\varepsilon}_j(x_j) = \int_{-\infty}^{\infty} \Phi^{\varepsilon}_{a_j b_j}(y_j)x^{-ix_j y_j}dy_j \text{ is } L_1 \ (-\infty,\infty)$$

integrating by parts we have

$$f^{\varepsilon}_j(x_j) = \Phi^{\varepsilon}_{a_j b_j}(y_j)\,\frac{e^{-ix_j y_j}}{-ix_j}\Bigg|_{a_j-\varepsilon}^{b_j+\varepsilon} - \int_{a_j-\varepsilon}^{b_j+\varepsilon} \Phi'^{\varepsilon}_{a_j b_j}(y_j)\frac{e^{-ix_j y_j}}{-ix_j}dy_j$$

$$= 0 + \frac{1}{\varepsilon}\int_{a_j-\varepsilon}^{a_j} \frac{e^{-ix_j y_j}}{ix_j}\,dy_j - \frac{1}{\varepsilon}\int_{b_j}^{b_j+\varepsilon}\frac{e^{-ix_j y_j}}{ix_j}\,dy_j$$

$$= \frac{1}{\varepsilon x_j^2}\left[e^{-ia_j x_j} - e^{-ia_j x_j}e^{i\varepsilon y_j} - e^{-ib_j x_j}e^{-i\varepsilon x_j} + e^{-ib_j x_j}\right]$$

and since $f^{\varepsilon}_j(x_j)$ is a transform it is bounded, so on any finite interval $[-N_j, N_j]$ $f_j(x_j)$ is integrable and outside that interval from above we have

$$|f_j(x_j)| \le \frac{A}{\varepsilon x_j^2} \quad \text{which is integrable hence}$$

hence $f_j(x_j) \in L_1(-\infty,\infty)$.

And by Theorem 8, and the corollary

$$\Phi^{\varepsilon}_{a,b}(Y) = (2\pi)^{-\frac{k}{2}}\int_{E_k} f^{\varepsilon}_{a,b}(X)e^{i(X\cdot Y)}dV_x \text{ everywhere.}$$

Now let S be the set of bounded functions in $L_1 \cap L_2$ and define the linear operator $\mathcal{J}_0$ on S by

$$\mathcal{J}_0(f)(Y) = (2\pi)^{-\frac{1}{2}k}\int_{E_k} f(X)e^{i(X\cdot Y)}dV_x$$

S is dense in L_2 since S contains the step functions. And by Theorem 12 for all $f \in S$

$$\int_{E_k} |f(X)|^2 dV_x = \int_{E_k} |\mathcal{J}_0(f)(Y)|^2 dV_y$$

That is $\mathcal{J}_0$ is isometric on a linearly dense subset of L_2. Hence by Theorem B $\mathcal{J}_0$ has a unique extension to an isometric linear operator $\mathcal{J}$ with domain L_2. And therefore by Theorem C $\mathcal{J}$ is a one to one function and its range is closed. On S, $\mathcal{J}=\mathcal{J}_0$, then $\mathcal{J}$ has R as a subset of its range and R is dense in L_2. Hence by Theorem C again $\mathcal{J}$ is unitary with domain and range L_2. And there exists an inverse $\mathcal{J}^{-1}$ which is also unitary.

Proof 2: This proof is a k-dimensional adaptation of the one variable proof due to F. Riesz given in Titchmarsh [12]. We define the operator $\mathcal{J}_0$ on $L_1 \cap L_2$, a dense subset of L_2, by the equation

$$\mathcal{J}_0(f)(Y) = F(Y) = (2\pi)^{-\frac{1}{2}k}\int_{E_k} f(X)e^{i(X\cdot Y)}dV_x$$

For any real $\delta \neq 0$ we have

$$\text{(2-A)}\quad \int_{E_k} e^{\frac{-\delta^2|Y|^2}{2}}|F(Y)|^2 dV_y =$$

$$= (2\pi)^{-k}\int_{E_k} e^{\frac{-\delta^2|Y|^2}{2}}\int_{E_k} f(U)e^{i(Y\cdot U)}\int_{E_k}\overline{f(Z)}e^{-i(Y\cdot Z)}\cdot$$

$$\cdot dV_z dV_u dV_y$$

$$= (2\pi)^{-k}\int_{E_k} f(U)\int_{E_k}\overline{f(Z)}\int_{E_k} e^{\frac{-\delta^2|Y|^2}{2} + i[Y\cdot(U-Z)]}\cdot$$

$$\cdot dV_y dV_z dV_u$$

by the Fubini Theorem since $f \in L_1$ and $e^{\frac{-\delta^2|Y|^2}{2}}$ is integrable. Now integrating the inner integral we get

$$\text{(2-B)}\quad \int_{E_k} e^{-\frac{1}{2}\delta^2|Y|^2}|F(Y)|^2 dV_y =$$

$$= \delta^{-k}(2\pi)^{-\frac{1}{2}k}\int_{E_k} f(U)\int_{E_k}\overline{f(Z)}e^{-\delta^{-2}2^{-1}|U-Z|^2}dV_z dV_u$$

and using the Cauchy-Schwarz inequality on the integral on the right in (2-B) we have letting $\Delta = -\frac{|U-Z|^2}{4\delta^2}$

$$\left|\int_{E_k}\int_{E_k} f(U)\cdot\overline{f}(Z)e^{2\Delta}dV_z dV_u\right| = \left|\int_{E_k}\int_{E_k} f(U)e^{\Delta}\cdot\overline{f(Z)}e^{\Delta}dV_z dV_u\right|$$

$$\leq \left[\int_{E_k}\int_{E_k}|f(U)|^2 e^{2\Delta}dV_z dV_u\right]^{\frac{1}{2}}\left[\int_{E_k}\int_{E_k}|f(Z)|^2 e^{2\Delta}dV_z dV_u\right]^{\frac{1}{2}}$$

$= \int_{E_k}\int_{E_k} |f(U)|^2 e^{2\Delta} dV_z dV_u$ and letting $T = U - Z$, we have

$$\left|\int_{E_k}\int_{E_k} f(U)\cdot\overline{f(Z)}e^{2\Delta}dV_z dV_u\right| \leq \int_{E_k} |f(U)|^2 \int_{E_k} e^{\frac{-|T|^2}{2\delta^2}} dV_t dV_u$$

$$= \delta^k (2\pi)^{\frac{k}{2}} \|f\|_2^2$$

hence substituting this result in 2-B we have

$$\int_{E_k} e^{-\frac{1}{2}\delta^2|Y|^2} |F(Y)|^2 dV_y \leq \|f\|_2^2.$$

And by Property 1, Chapter I $|F(Y)|^2 \leq (2\pi)^{-k}\|f\|_1^2$.

Also notice that if $\delta_o > \delta$,

$$e^{-\frac{1}{2}\delta^2|Y|^2}|F(Y)|^2 > e^{-\frac{1}{2}\delta_o^2|Y|^2}|F(Y)|^2$$

hence if we pick any sequence of δ's $\{\delta_1 \delta_2 \dots\}$ going to zero by the monotone convergence theorem we have

$$\lim_{n\to\infty} \int_{E_k} e^{-\frac{1}{2}\delta^2|Y|^2}|F(Y)|^2 dVy = \int_{E_k} |F(Y)|^2 dV_y$$

moreover since this holds for <u>every</u> sequence $\{\delta_n\}\to 0$ we have

$$\text{(2-C)} \quad \lim_{\delta\to 0} \int_{E_k} e^{-\frac{1}{2}\delta^2|Y|^2}|F(Y)|^2 dV_y = \int_{E_k} |F(Y)|^2 dV_y.$$

Hence $F(Y) \in L_2$. Now from (2-B) we have

$$\int_{E_k} |F(Y)|^2 e^{-\frac{1}{2}\delta^2|Y|^2} dV_y = \delta^{-k}(2\pi)^{-\frac{1}{2}k} \int_{E_k} f(U) \int_{E_k} \overline{f(Z)} e^{2\Delta} dV_z dV_u$$

$$= \delta^{-k}(2\pi)^{-\frac{1}{2}k}\int_{E_k} f(U)\int_{E_k} \overline{f}(U+T)e^{\frac{-|T|^2}{2\delta^2}}dV_t dV_u$$

by letting T = Z-U. We again apply the Fubini theorem and we have

$$(2\text{-D}) \quad \int_{E_k} |F(Y)|^2 e^{-\frac{1}{2}\delta^2|Y|^2} dV_y = \delta^{-k}(2\pi)^{-\frac{1}{2}k}\int_{E_k} e^{\frac{-|T|^2}{2\delta^2}} G(T)dV_t$$

where

$$G(T) = \int_{E_k} f(U)\overline{f}(U+T)dV_u;$$ G(T) is bounded by the Cauchy-Schwarz inequality since $f \in L_2$. And G(T) is continuous since

$$|G(T+H)-G(T)| = |\int_{E_k} f(U)[\overline{f}(U+T+H)-\overline{f}(U+T)]dV_u|$$

$$\leq \|f\|_2 \|\overline{f}(U+T+H)-\overline{f}(U+T)\|_2 < \epsilon$$

since by a well-known theorem $\|\overline{f}(U+T+H)-\overline{f}(U+T)\|_2$ goes to zero as $|H|\to 0$.

Now using (2-C) and (2-D) we have

$$\int_{E_k} |F(Y)|^2 dV_y = \lim_{\delta\to 0}\int_{E_k} e^{-\frac{1}{2}\delta^2|Y|^2}|F(Y)|^2 dV_y$$

$$= \lim_{\delta\to 0}\int_{E_k} d^{-k}(2\pi)^{-\frac{1}{2}k}e^{\frac{-|T|^2}{2\delta^2}} G(T)dV_t$$

Now let $\frac{1}{\delta}T = Y$, then

$$\int_{E_k} |F(Y)|^2 dV_y = \lim_{\delta\to 0}\int_{E_k} e^{-\frac{1}{2}|Y|^2}(2\pi)^{-\frac{1}{2}k}G(\delta T)dV_y$$

The integrand on the right is dominated by $g(Y) = e^{-\frac{1}{2}|Y|^2}M$, where M is the bound of G, g(Y) is integrable hence by dominated convergence

$$\int_{E_k} |F(X)|^2 dV_x = \int_{E_k} \lim_{\delta\to 0} e^{-\frac{1}{2}|Y|^2}(2\pi)^{-\frac{1}{2}k}G(\delta Y)dV_y$$

$$= (2\pi)^{-\frac{1}{2}k}G(0)\int_{E_k} e^{-\frac{1}{2}|Y|^2}dV_y$$

$$= G(0) = \int_{E_k} |f(X)|^2 dV_x.$$

Therefore $\mathcal{J}_0$ is isometric on $L_1 \cap L_2$. Hence by Theorem B $\mathcal{J}_0$ has a unique extension to an isometric operator $\mathcal{J}$ with domain L_2 and range a closed subset of L_2. Now we will show that the range of $\mathcal{J}$ is dense.

So, let us define the operator $\mathcal{J}_1'$ on $L_1 \cap L_2$ by

$$\mathcal{J}_1'(f)(Y) = (2\pi)^{-\frac{1}{2}k}\int_{E_k} f(X)e^{-i(X\cdot Y)}dV_y.$$

By an argument similiar to that we used on $\mathcal{J}_0$, we know that $\mathcal{J}_1'$ has a unique extension to an isometric linear operator $\mathcal{J}_1$ with domain L_2 and range a closed subset of L_2. If $N = \{X \mid |x_j| \leq n \text{ for } J = 1,2\ldots k\}$ and $I = \{X \mid |x_j| \leq 1 \text{ for } j = 1,2\ldots k\}$, then

$$\mathcal{J}[K_I(X)](Y) = (2\pi)^{-\frac{1}{2}k} \prod_{j=1}^{k} \frac{e^{iy_j} - e^{-iy_j}}{iy_j}$$

$$= \prod_{j=1}^{k} h_j(x_j).$$

Now let $g_N(Y) = \left(\prod_{j=1}^{k} h_j(x_j)\right)K_N(Y)$

then $\lim_{n\to\infty} g_N(Y) = [K_I(X)](Y)$ in the L_2 sense and

$g_N(Y) \in L_1 \cap L_2$. Since $\mathcal{J}_1$ is a unique extension

$$\mathcal{J}_1(\mathcal{J}(K_I(X)))(Z) = \lim_{n\to\infty} \mathcal{J}_1'(g_N(Y))(Z)$$

Now if $\lim_{n\to\infty} \mathcal{J}_1'(g_N(Y))(Z) = f(Z)$ a.e. then

$$f(Z) = \mathcal{J}_1[\mathcal{J}(K_I(X))(Y)](Z) \text{ a.e.}$$

$$\mathcal{J}_1'(g_N(Y))(Z) = \int_N \prod_{j=1}^{k} h_j(y_j)e^{-iy_jz_j}dV_y = \prod_{j=1}^{k}\int_{-n} h_j(y_j)e^{-y_jz_j}dy_j$$

$$= \prod_{j=1}^{k} R_{jN}$$

Now

(2-E) $$\lim_{n\to\infty} \mathcal{J}_1'(g_N(Y))(Z) = \lim_{n\to\infty} \prod_{j=1}^{k} R_{jN}(Z_j) = \prod_{j=1}^{k} \lim_{n\to\infty} R_{jN}.$$

Now let us evaluate one of these k products.

$$R_{jN} = (2\pi)^{-1}\int_{-n}^{n} \frac{(e^{iy_j}-e^{-iy_j})}{iy_j} e^{-iy_jz_j}\, dy_j$$

$$= (2\pi)^{-1}\int_{-n}^{n}\left[\frac{\cos[y_j(1-z_j)]+i\sin[y_j(1-z_j)]-}{iy_j}\right.$$

$$\left.\frac{-\cos[y_j(1+z_j)]+\sin[y_j(1+z_j)]}{}\right]dy_j$$

but the cosine terms can be dropped since $\frac{\cos y_j(1\pm z_j)}{y_j}$ is an odd function and since $\frac{\sin y_j(1\pm z_j)}{y_j}$ is even we have

$$R_{jN}(z_j) = (2\pi)^{-1}\int_{-n}^{n} \frac{\sin[y_j(1-z_j)] + \sin[y_j(1+z_j)]}{y_j}\, dy_j$$

and if $|z_j| > 1$, $R_{jN}(z_j)$ is zero since for these values the integrand is odd. So for $|z_j| \leq 1$ we have, since the integrand is even,

$$R_{jN}(z_j) = \frac{1}{\pi}\int_{0}^{n} \frac{\sin[y_j(1-z_j)]+\sin[y_j(1+z_j)]}{y_j}\, dy_j$$

Now if $|z_j| = 1$, one of the two sine functions is zero and using the well-known fact that

$$\lim_{n\to\infty}\int_0^n \frac{\sin Rt}{t}\,dt = \frac{\pi}{2}$$

we have $\lim\limits_{n\to\infty} R_{jN}(z_j) = \frac{1}{2}$ and if $|z_j| < 1$ we have the sum of two such integrals so for these z_j's, $\lim\limits_{n\to\infty} R_{jN}(z_j) = 1$.

Now using (2-E) and the above we have

$$\lim_{n\to\infty} \mathcal{J}_1{}'(g_N(Y))(Z) = \begin{cases} 0 \text{ if } Z \text{ is such that for one } j \\ \quad |z_j| > 1. \\ \frac{1}{2^k} \text{ if } Z \in \{Z \mid |z_j|=1,\ j=1,2\ldots k\} \\ 1 \text{ if } Z \in \{Z \mid |z_j|<1,\ j=1,2\ldots k\} \end{cases}$$

That is

$$\lim_{n\to\infty} \mathcal{J}_1{}'(g_N(Y))(Z) = K_I(Z) \text{ a.e. or}$$

$$\mathcal{F}_1[\mathcal{F}(K_I(X))(Y)](Z) = K_I(Z) \text{ a.e.}$$

By a similiar argument $\mathcal{F}\mathcal{F}_1[K_I] = K_I$ a.e.

Let $\mathcal{A}$ be the class of L_2 functions such that

$\mathcal{F}_1(\mathcal{F}f) = f$ a.e. by the above $K_I(Z) \in \mathcal{A}$. Now let

$f \in L_2(E_k)$ and $\{f_n\} \xrightarrow[L_2]{} f$ with $f_n \in L_1 \cap L_2$ for all n,

also let R be a real number, not zero.

Consider

$$\mathcal{F}(f(RX)(Y) = \lim_{n\to\infty} \mathcal{F}_0[f_n(RX)](Y)$$

$$= \underset{n\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{\frac{k}{2}}} \int_{E_k} \mathcal{F}f_n(RX)e^{i(X\cdot Y)}dV_x$$

and letting $RX = \tilde{X}$ we have the above equals

$$R^{-k} \underset{n\to\infty}{\text{l.i.m.}} (2\pi)^{-\frac{1}{2}k}\int_{E_k} f_n\tilde{X}e^{i(X\cdot\frac{Y}{R})}dV_x$$

$= R^{-k}\mathcal{F}[f(X)](\frac{Y}{R})$; the same relation clearly

holds for $\mathcal{F}_1$.

And by a similiar argument

$$\mathcal{F}(f(X+H))(Y) = e^{-i(H\cdot Y)}\mathcal{F}(f(X))(Y)$$

and $\mathcal{F}_1(e^{-i(Y\cdot H)}f(Y))(X) = \mathcal{F}_1(f(Y))(X+H)$

Now let $f \in \mathcal{A}$, then

$$\mathcal{F}_1(\mathcal{F}(f(RX))(Y))(Z) = \mathcal{F}_1(R^{-k}\mathcal{F}f(X)(\frac{Y}{R}))(Z)$$

$$= R^{-k}R^{k}\mathcal{J}_1(\mathcal{J}(F(X))(Y))(RZ)$$

$$= f(RZ)$$

and

$$\mathcal{J}_1(\mathcal{J}(f(X+H))(Y))(Z) = \mathcal{J}_1(e^{-i(H\cdot Y)}\,(f(X))(Y))(Z)$$

$$= \mathcal{J}_1(\mathcal{J}(f(X))(Y))(Z+H) = f(Z+H)$$

Hence $f(RX) \to f(X+H) \in \mathcal{A}$. Hence this relation holds for all characteristic functions defined on k-dimensional cubes centered at the origin and translations of these cubes in E_k. By the linearity of $\mathcal{J}$ and $\mathcal{J}_1$ this relation also holds for all step functions defined on such cubes. These step functions are dense in L_2, hence the range of $\mathcal{J}$ and $\mathcal{J}_1$ is $L_2(E_k)$. And the Theorem is proved.

Proof 3: This proof is a generalization of a proof due to Weiner using the Hermite functions.

Definition 4: The single variable $n^{\underline{th}}$ degree Hermite polynomial is defined as

$$H_n(x) = (-1)^n\, e^{x^2}\, \frac{d^n e^{-x^2}}{dx^n}$$

And the $n^{\underline{th}}$ function for one variable

$$\psi_n{*}(x) = e^{\frac{-x^2}{2}}\, H_n(x)$$

And by normalizing this function we get the usual single variable $n^{\underline{th}}$ Hermite function

$$\varphi_n(x) = \frac{\psi_n{*}(x)}{\|\psi_n{*}(\bar{x})\|_2} \qquad \text{as shown in Titchmarsh [12]}$$

these functions are $L_1 \cap L_2$, or-thonormal, and the set of linear combinations of these functions is dense in L_2.

We shall now define the k-variable Hermite function,

$$\psi_{n_1 n_2 \ldots n_k}(X) = \prod_{j=1}^{k} \psi_{n_j}(x_j) \text{ where the } \psi_{n_j}\text{'s are}$$

single variable Hermite functions.

Obviously be the definition and the properties of single variable Hermite functions, these k-dimensional Hermite functions are $L_1 \cap L_2$ and orthonormal. Now we will show that the set of linear combinations of these k-dimensional Hermite functions is dense in $L_2(E_k)$.

Let I be an interval in E_k that is

$$I = [a_1,b_1] \times [a_2,b_2] \times \ldots \times [a_k,b_k] = I_1 \times I_2 \times \ldots \times I_k.$$

Let $\epsilon > 0$. Since linear combinations of the one dimensional Hermite functions are dense in $L_2(-\infty,\infty)$, Φ_{n_j} denote that linear combination of one dimensional Hermite functions such that in $L_2(-\infty,\infty)$

(2-F) $\quad \| K_{I_j}(x_j) - \Phi_{n_j}(x_j) \|_2^2 < \dfrac{\epsilon^2}{kM_j}$ where k is the dimension of the space and

$$M_1 = 2 \prod_{j=2}^{k} (b_j - a_j)$$

$$M_j = 2^j \prod_{h=j+1}^{k} (b_h - a_h) \prod_{h=1}^{k} \| \Phi_{n_h} \|_2^2; \text{ for } j = 2,3,\ldots,k-1$$

and

$$M_k = 2^{k-1} \prod_{j=1}^{k-1} \| \Phi_{n_j} \|_2^2$$

Now for any a,b,c, and d real or complex numbers

$$\begin{aligned} |ab-cd|^2 &= |ab-ad+ad-cd|^2 \\ &\leq (|a||b-d|+|d||a-c|)^2 \\ &\leq 2|a|^2|b-d|^2+2|d|^2|a-c|^2 \end{aligned}$$

Consider $K_I(X) = \prod_{l=1}^{k} K_{I_l}(x_l)$ and its associated $\Phi_{n_1 n_2 \ldots n_k}(X) = \prod_{j=1}^{k} \Phi_{n_j}(x_j)$ where the Φ_{n_j}'s are as in the preceeding paragraph and satisfy (2-F).

$$\| K_I(X)-\Phi_{n_1\ldots n_k}(X)\|_2^2 = \int_{E_k} |K_I(X)-\Phi_{n_1\ldots n_k}(X)|^2 dV_x$$

$$= \int_{E_k} \left| \prod_{j=1}^{k} K_{I_j}(x_j) - \prod_{j=1}^{k} \Phi_{n_j}(x_j)\right|^2 dV_x =$$

$$= \int_{E_k} \left| \prod_{j=2}^{k} K_{I_j}(x_j)\cdot K_{I_1}(x_1) - \prod_{j=2}^{k} \Phi_{n_j}(x_j)\cdot\Phi_{n_1}(x_1)\right|^2 dV_x$$

$$\leq 2\int_{E_k} \left| \prod_{j=2}^{k} K_{I_j}(x_j)\right|^2 |K_{I_1}(x_1)-\Phi_{n_1}(x_1)|^2 dx_1 dx_2 \ldots dx_k$$

$$+ 2\int |\Phi_{n_1}(x_1)|^2 \left| \prod_{j=2}^{k} K_{I_j}(x_j) - \prod_{j=2}^{k} \Phi_{n_j}(x_j)\right|^2 dx_1\ldots dx_k$$

$$= 2 \prod_{j=2}^{k} (b_j-a_j) \| K_{I_1}(x_1)-\Phi_{n_1}(x_1)\|_2^2 +$$

$$+ 2\| \Phi_{n_1}(x_1)\|_2^2 \int_{E_{k-1}} | \prod_{j=2}^{k} K_{I_j}(x_j) - \prod_{j=2}^{k} \Phi_{n_j}(x_j)|^2 dx_2 dx_3 \ldots dx_k$$

and by repeating the above process k-1 times we get

$$\| K_I(X) - \Phi_{n_1 \ldots n_k}(X)\|_2^2 \leq 2 \prod_{l=2}^{k} (b_l - a_l) \| K_{I_1}(x_1) - \Phi_{n_1}(x_1)\|_2^2 +$$

$$+ 2^2 \prod_{l=3}^{k} (b_l - a_l) \cdot \| \Phi_{n_1}(x_1)\|_2^2 \| K_{I_2}(x_2) - \Phi_{n_2}(x_2)\|_2^2 + \ldots +$$

$$+ 2^{k-1} \prod_{l=1}^{k-1} \| \Phi_{n_1}(x_1)\|_2^2 \| K_{I_k}(x_k) - \Phi_{n_k}(x_k)\|_2^2$$ and by

(2-F) we have then

$$\| K_I(X) - \Phi_{n_1 \ldots n_k}\|_2^2 < \frac{\epsilon^2}{k} + \frac{\epsilon^2}{k} + \ldots + \frac{\epsilon^2}{k} = \epsilon^2$$

Hence the linear combinations of the k-dimensional Hermite functions are dense in $L_2(E_k)$ since the step functions (dense in $L_2(E_k)$) are merely linear combinations of characteristic functions on intervals.

Let us now define $\mathcal{J}_0$ on $L_1 \cap L_2$ as in the other two proofs. Notice that

$$\mathcal{J}_0(\psi_{n_1, n_2, \ldots n_k}(X))(Y) = \prod_{j=1}^{k} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_{n_j}(x_j) e^{ix_j y_j} dx_j$$

$$= \prod_{j=1}^{k} (i)^{n_j} \psi_{n_j}(y_j)$$ from

a property of the single variable Hermite functions. Therefore

$$\mathcal{J}_0(\psi_{n_1 n_2 \ldots n_k}(X))(Y) = (i)^{\sum_{j=1}^{k} n_j} \psi_{n_1 \ldots n_k}(Y)$$

And since linear combinations of the φ's are dense in L_2 and $\mathcal{J}_0$ has a dense subset of L_2 as a subset of its range, we have by Theorems B and C, $\mathcal{J}_0$ has a unique extension to a unitary operator with domain and range L_2. Also, $\mathcal{J}$ has a unitary inverse $\mathcal{J}^{-1}$.

Let $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k)$ and $B = (b_1, b_2, \ldots, b_k)$ with $\mathcal{E}_j < b_j$ $j = 1,2,\ldots k$. Now let $A = \{X \mid \mathcal{E}_j < x_j < b_j\}$.

$$\int_A \mathcal{J}(F(X))(Y)dV_y = \langle \mathcal{J}f, \mathcal{J}K_A(X)\rangle = \langle f, \mathcal{J}^{-1}K_A\rangle$$

$$= \int_{E_k} f(X)(2\pi)^{-\frac{1}{2}k}\int_A e^{-i(X\cdot Y)}dV_y dV_x$$

$$= \int_A \frac{1}{(\sqrt{2\pi})^k}\int_{E_k} f(X)e^{i(X\cdot Y)}dV_x dV_y$$

by the Fubini Theorem since $f \in L_1 \cap L_2$. So

$$\int_A (2\pi)^{-\frac{1}{2}k}\int_{E_k} f(X)e^{-(X\cdot Y)}dV_x dV_y = \int_A \mathcal{J}_0(f(X))(Y)dV_y$$

and hence $\mathcal{J}_0(f) = \mathcal{J}(f)$ a.e. since A is any k-dimensional box. So we have for all $f \in L_1 \cap L_2$

$$\mathcal{J}_0(f) = \mathcal{J}(f) \text{ a.e.}$$

Hence we have

Corollary I: On $L_1 \cap L_2$, $\mathcal{J}_0(f) = \mathcal{J}(f)$ a.e. where $\mathcal{J}_0$ is defined in the usual manner and $\mathcal{J}$ is its extension.

Corollary II: $\mathcal{J}$ has a unitary inverse $\mathcal{J}^{-1}$.

Proof: Every proof of Theorem 15 fulfills all the hypothesis of Theorem C, hence we have its conclusion.

Corollary III: If $f,g \in L_2(E_k)$, then $\langle \mathfrak{J}f, \mathfrak{J}g \rangle = \langle f,g \rangle$, that is the Parseval relation holds.

Proof: $\mathfrak{J}$ is a unitary linear operator.

Hence $\langle \mathfrak{J}(f+g), \mathfrak{J}(f+g) \rangle = \langle f+g, f+g \rangle$ and

$$\begin{aligned} \langle \mathfrak{J}(f+g), \mathfrak{J}(f+g) \rangle &= \langle \mathfrak{J}f+\mathfrak{J}g, \mathfrak{J}f+\mathfrak{J}g \rangle \\ &= \langle \mathfrak{J}f, \mathfrak{J}f+\mathfrak{J}g \rangle + \langle \mathfrak{J}g, \mathfrak{J}f+\mathfrak{J}g \rangle \\ &= \langle \mathfrak{J}f, \mathfrak{J}f \rangle + \langle \mathfrak{J}f, \mathfrak{J}g \rangle + \langle \mathfrak{J}g, \mathfrak{J}f \rangle + \\ &\quad + \langle \mathfrak{J}g, \mathfrak{J}g \rangle \\ &= \langle f,f \rangle + 2\mathrm{Re}\langle \mathfrak{J}f, \mathfrak{J}g \rangle + \langle g,g \rangle \end{aligned}$$

and similarly $\langle f+g, f+g \rangle = \langle f,f \rangle + 2\mathrm{Re}\langle f,g \rangle + \langle g,g \rangle$

hence $\mathrm{Re}\,\langle \mathfrak{J}f, \mathfrak{J}g \rangle = \mathrm{Re}\,\langle f,g \rangle$

and if we replace g by $\bar{g}$ in the above argument we get

$$\mathrm{Im}\,\langle \mathfrak{J}f, \mathfrak{J}g \rangle = \mathrm{Im}\,\langle f,g \rangle$$

hence $$\langle \mathfrak{J}f, \mathfrak{J}g \rangle = \langle f,g \rangle$$

And since $\mathfrak{J}^{-1}$ is a unitary operator we have the same relation holding for $\mathfrak{J}^{-1}$.

Corollary IV: Let $\mathcal{E}$ be a point in E_k with $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2, \ldots \mathcal{E}_k)$ and let $J_{\mathcal{E}} = \{Y \mid 0 \le y_j \le \mathcal{E}_j;\ j = 1,2,\ldots k\}$ and if $f(X) \in L_2(E_k)$, then

$$\int_{J_{\mathcal{E}}} \mathfrak{J}^{-1}(f(X))(Y)dV_y = (2\pi)^{-\frac{1}{2}k} \int_{E_k} f(X) \prod_{j=1}^{k} \left(\frac{e^{-i\mathcal{E}_j x_j} - 1}{-ix_j}\right) dV_x$$

and $$\int_{J_{\mathcal{E}}} \mathfrak{J}(f(X))(Y)dV_y = (2\pi)^{-\frac{1}{2}k} \int_{E_k} f(X) \prod_{j=1}^{k} \left(\frac{e^{i\mathcal{E}_j x_j} - 1}{ix_j}\right) dV_x$$

Proof:

$$\int_{J_\varepsilon} \mathcal{F}^{-1}(f(X))(Y)dV_y = \langle \mathcal{F}^{-1}f, K_{J_\varepsilon} \rangle$$

$$= \langle f, \mathcal{F}(K_{J_\varepsilon}) \rangle = (2\pi)^{-\frac{1}{2}k} \int_{E_k} f(X) \prod_{j=1}^{k} \left(\frac{e^{-i\varepsilon_j x_j} - 1}{-ix_j}\right) dV_x .$$

Since $\mathcal{F}(K_{J_\varepsilon}) = (2\pi)^{-\frac{1}{2}k} \int_0^{\varepsilon_1} \int_0^{\varepsilon_2} \cdots \int_0^{\varepsilon_k} e^{-(X \cdot Y)} dV_y$

$$= (2\pi)^{-\frac{1}{2}k} \prod_{j=1}^{k} \left(\frac{e^{i\varepsilon_j x_j} - 1}{ix_j}\right)$$

A similar argument will give the second expression stated in the conclusion.

CHAPTER III

POSITIVE DEFINITE AND AUTOCORRELATION FUNCTIONS

A question which naturally follows from Theorem 9 in Chapter I is what kind of bounded, continuous L_1 functions have a positive transform? In this chapter we shall give two characterizations of these functions and show their equivalence.

Definition 5: f(X) is positive difinite if it is

1) Bounded and continuous,

2) Hermitian, that is $f(X) = \overline{f(-X)}$,

3) And if it satisfies the following decisive condition. For any points $X_1, X_2, \ldots, X_m \in E_k$, $m = (2, 3, \ldots)$ and any numbers $a_1, a_2, \ldots, a_m$,

$$\sum_{j=1}^{m} \sum_{n=1}^{m} f(X_n - X_j) a_j \cdot \overline{a_n} \geq 0.$$

Theorem 16: If f is a continuous, bounded L_1 function with L_1 transform $\Phi(Y) \geq 0$, then f(X) is positive definite.

Proof: 1) By hypothesis f is bounded and continuous.

2) By Theorem 9 and the corollary, f equals the inverse transformation (L_1) of Φ everywhere. And since $\Phi(Y) \geq 0$, $\overline{\Phi(Y)} = \Phi(Y)$, then

$$f(X) = (2\pi)^{-k}\int_{E_k} e^{-i(X\cdot Y)}\Phi(Y)dV_y = (2\pi)^{-k}\int_{E_k} e^{-i(X\cdot Y)}\overline{\Phi(Y)}dV_y$$

$$= (2\pi)^{-k}\int_{E_k} \overline{e^{-i(X\cdot Y)}\Phi(Y)}dV_y = \overline{f(-X)}$$

So f is Hermitian.

3) If $X_1, X_2, \ldots, X_m$ are points in E_k and $a_1, a_2, \ldots, a_m$ are numbers where $m = (2, 3, \ldots)$, then since $\Phi(Y) \geq 0$,

$$0 \leq (2\pi)^{-k}\int_{E_k} \Big| \sum_{j=1}^{m} a_j\ e^{-i(Y\cdot X_j)}\Big|^2\Phi(Y)dV_y$$

$$= (2\pi)^{-k}\int_{E_k} \Big(\sum_{j=1}^{m} \bar{a}_j e^{i(Y\cdot X_j)} \cdot \sum_{n=1}^{m} a_n e^{-i(Y\cdot X_n)}\Big)\Phi(Y)dV_y$$

$$= (2\pi)^{-k}\sum_{j=1}^{m}\sum_{n=1}^{m} \bar{a}_j a_n \int_{E_k} e^{-i(Y(X_n - X_j))}\Phi(Y)dV_y$$

$$= \sum_{j=1}^{m}\sum_{n=1}^{m} \bar{a}_j a_n f(X_n - X_j)$$ by Theorem 9. Hence f is positive definite.

<u>Theorem 17</u>: (Bochner's Theorem) If $f(X) \in L_1(E_k)$ is positive definite, then $f(X)$ has a positive transform $\Phi_f \in L_1(E_k)$.

Proof: Let $F(X)$ be positive definite and L_1, now using that special class of bounded, continuous L_1 functions $f_{a,b}$ which we used in the first proof of the Plancherel Theorem, we can form the function

$$F(\mathcal{E}) = \int_{E_k}\int_{E_k} f(\mathcal{E}-X-Y)g(X)\overline{g(-Y)}dV_x dV_y$$

$$= \int_{E_k}\int_{E_k} f(\varepsilon+X-Y)g(-X)\overline{g(-Y)}dV_x dV_y$$

where $g(X)$ is in the special class of L_1 functions.

Let $|f(X)| \leq A$ and $|g(X)| \leq B$.

Let N and j be positive integers with $N > 1$. Also let m_i be an enumation of the points in $[-N^{j+1}, N^{j+1}]^k$ with integer co-ordinates and $x_\gamma{}^i = \frac{n_i}{N^j} \in [-N,N)^k = \mathcal{N}_N$, $X^i = (x_1{}^i, x_2{}^i, \ldots, x_k{}^i)$. Let $V_{ij} = [x_\gamma{}^i, x_\gamma{}^i + \frac{1}{N^j})^k$ and $Y^i = X^i$. Now we define

$$S_j(X,Y) = \sum_\alpha \sum_\beta f(X^\alpha - Y^\beta)g(-X^\alpha)\overline{g(-Y^\beta)}K_{V_{\alpha_j} \times V_{\beta_j}}(X,Y)$$

clearly $|S_j| \leq AB^2$.

Now

$$\int_{E_k}\int_{E_k} S_j(X,Y)dV_x dV_y = \frac{1}{N^{2jk}}\sum_\alpha\sum_\beta f(X^\alpha - Y^\beta)g(-X^\alpha)\overline{g(-Y^\beta)} \geq 0$$

since f is positive definite.

Now let $H_N(X,Y) = f(X-Y)g(-X)g(-Y)K_{\mathcal{N}_N \times \mathcal{N}_N}(X,Y) = \lim_{j\to\infty} S_j(X,Y)$ so by dominated convergence

$$\int_{E_k}\int_{E_k} H_N(X,Y)dV_x = \lim_{j\to\infty}\int_{E_k}\int_{E_k} S_j(X,Y)dV_x dV_y \geq 0$$

$|H_N| \leq A|g(-X)|\ |\overline{g(-Y)}|$ hence integrable over $E_k x E_k$.

So again applying dominated convergence

$$\int_{E_k}\int_{E_k} f(X-Y)g(-X)\overline{g(-Y)}dV_x dV_y = \lim_{N\to\infty}\int_{E_k}\int_{E_k} H_N(X,Y)dV_x dV_y \geq 0$$

that is $F(0) \geq 0$. And from Theorem 11 in Chapter I used twice $F(\mathcal{E})$ is continuous and $L_1(E_k)$. So if we now form the functions

$$\Phi(Y) = \int_{E_k} e^{i(X\cdot Y)}f(X)dV_x$$

and (B) $$\Gamma(Y) = \int_{E_k} e^{i(X\cdot Y)}g(X)dV_x$$

then $|\Gamma(Y)|^2\Phi(Y) =$

$$\int_{E_k} e^{i(X\cdot Y)}g(X)dV_x\int_{E_k} e^{i(Y\cdot U)}\overline{g(-U)}dV_u\int_{E_k} e^{i(Y\cdot J)}f(J)dV_j$$

$$= \int_{E_k}\int_{E_k}\int_{E_k} e^{i[Y\cdot(X+U+J)]}g(X)\overline{g(-U)}f(J)dV_x dV_u dV_j$$

and letting $\mathcal{E} = J+X+U$, that is changing J to $\mathcal{E}$ we get

$$|\Gamma(Y)|^2\Phi(Y) = \int_{E_k} e^{i(Y\cdot\mathcal{E})}F(\mathcal{E})dV_{\mathcal{E}}$$

and since $\Gamma(Y) \in L_1(E_k)$ then $|\Gamma(Y)|^2\Phi(Y)$ is also in $L_1(E_k)$ by Property 1 used on $\Gamma(Y)$ and $\Phi(Y)$ [transforms are bounded]. Hence by Theorem 8 and the corollary

$$F(\mathcal{E}) = (2\pi)^{-k}\int_{E_k} e^{-i(Y\cdot\mathcal{E})}|\Gamma(Y)|^2\Phi(Y)dV_y$$

and since $F(0) \geq 0$ we have

(C) $$(2\pi)^{-k}\int_{E_k}|\Gamma(Y)|^2\Phi(Y)dV_y \geq 0.$$

Since f is positive definite $\Phi(Y)$ is real,

$$\Phi(Y) = \int_{E_k} e^{i(X\cdot Y)} f(X)dV_x = \int_{E_k} e^{i(X\cdot Y)}\overline{f(-X)}dV_x =$$

$$= \overline{\int_{E_k} e^{i(X\cdot Y)} f(X)dV_x} = \overline{\Phi(Y)}.$$

Now let us assume that for some $Y \in E_k \Phi(Y) < 0$, since $\Phi(Y)$ is continuous there exists an interval A with measure > 0 for all $Y \in A$. And by the choice of the g's we can pick $\Gamma(Y)$ to be a continuous bounded L_1 function not zero on A, but vanishing on $E_k - A$. This implies

$$(2\pi)^{-k}\int_{E_k} |\Gamma(Y)|^2 \Phi(Y)dV_y \leq M(2\pi)^{-k}\int_A \Phi(Y)dV_y < 0 \text{ but}$$

this contradicts (C) which follows from the fact that f is positive definite. Hence $\Phi(Y) \geq 0$ and therefore by Theorem 9 $\Phi(Y) \in L_1(E_k)$.

<u>Definition 6</u>: The autocorrelation of a function $g(X) \in L_2(E_k)$ is defined to be the function

$$a(Y) = \int_{E_k} \overline{g(X)} g(X+Y)dV_x.$$

Notice that a(Y) is defined for all $Y \in E_k$ and by the Cauchy-Schwarz inequality, bounded, and continuous since $g \in L_2(E_k)$. a(Y) is also Hermitian since

$$\overline{a(-Y)} = \int_{E_k} \overline{\overline{g(X)} g(X-Y)}dV_x = \int_{E_k} g(X)\overline{g(X-Y)}dV_x$$

and letting $X-Y = \tilde{X}$ we get

$$\overline{a(-Y)} = \int_{E_k} g(\tilde{X}+Y)\overline{g(\tilde{X})}dV_x = a(Y).$$

<u>Theorem 18</u>: If $f(X) \in L_1(E_k)$ is bounded, continuous and has L_1 transform $\Phi_f \geq 0$, then f is an autocorrelation of some $g \in L_2$.

Proof: By Theorem 9 since $\Phi_f \geq 0$ and f is bounded we have $\Phi_f \in L_1$. Hence $\Phi(Y) \equiv \sqrt{\Phi_f(Y)} \in L_2$. By defining $\Phi(Y)$ in this way clearly $\Phi(Y)$ is real and even.

Now let $g(T) = (2\pi)^{-\frac{k}{2}}\mathcal{J}^{-1}(\Phi(Y))(T)$ where $\mathcal{J}^{-1}$ is the L_2 inverse transform. And from our L_2 theory we have

$$\mathcal{J}(g(T+X))(\varepsilon) = e^{-i(\varepsilon\cdot X)}\mathcal{J}(g(T))(\varepsilon)$$

hence by the Parseval relation for L_2 functions,

$$\int_{E_k}\overline{g(T)}g(T+X)dV_t = (2\pi)^{-\frac{k}{2}}(2\pi)^{-\frac{k}{2}}\int_{E_k}\overline{\Phi(\varepsilon)}\Phi(\varepsilon)e^{-i(\varepsilon\cdot X)}dV$$

$$= (2\pi)^{-k}\int_{E_k}\Phi_f(\varepsilon)e^{-i(\varepsilon\cdot X)} = f(X)$$

everywhere by Theorem 9. Hence $f(X)$ is the autocorrelation of $g(T)$ which by the Plancherel Theorem is in L_2.

<u>Theorem 19</u>: If $f(X) \in L_1(E_k)$ is the autocorrelation of some $g(T) \in L_2(E_k)$, then $f(X)$ has a non-negative L_1 transform Φ_f and $\Phi_f \in L_1(E_k)$.

Proof: If $g(T) \in L_2(E_k)$ is such that

$$(3.A)\quad f(X) = \int_{E_k} g(T+X)\overline{g(T)}dV_t = \int_{E_k} \Phi_g(Y)\overline{\Phi_g(Y)}e^{-i(X\cdot Y)}dV_y$$

where Φ_g is the L_2 transform of g and hence $\Phi_g \in L_2$. (3.A) followed from the Parseval relation for L_2 functions. Hence

$f(X) = \int_{E_k} |\Phi_g(Y)|^2 e^{-i(Y\cdot X)}dV_y$. Now $|\Phi_g(Y)|^2 \in L_1(E_k)$, hence by Theorem 9

$$|\Phi_g(Y)|^2 = (2\pi)^{-k}\int_{E_k} f(X)e^{i(X\cdot Y)}dV_x$$

that is $|\Phi_g(Y)|^2 = 2\pi\ \Phi_f(Y)$ a.e. and $|\Phi_g(Y)|^2 \geq 0$ by definition and since $\Phi_f(Y)$ is continuous $\Phi_f(Y) \geq 0$ for all $(Y) \in E_k$. Then by Theorem 9 $\Phi_f(Y) \in L_1(E_k)$.

Hence by Theorems 16, 17, 18, and 19 we can say these concepts are equivalent:

i) $f \in L_1$ is bounded and continuous, with $\Phi_f \geq 0$.

ii) $f \in L_1$ is positive definite.

iii) $f \in L_1$ is an autocorrelation of some L_2 function.

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