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PION-DEUTERON ELASTIC SCATTERING
AND THE PION-PION SCATTERING LENGTHS.

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DEEP-INELASTIC ELECTROPRODUCTION OF PIONS IN A QUARK-PARTON MODEL

PION-DEUTERON ELASTIC SCATTERING AND THE PION-PION SCATTERING LENGTHS

A Dissertation

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In Partial Fulfillment

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Doctor of Philosophy

by

Mary Elizabeth Vislay

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APPROVAL SHEET

This dissertation is submitted in partial fulfillment of

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ABSTRACT

In part one of this thesis invariant cross sections for the processes $e + p \rightarrow e + \pi^{\pm}$ + anything are calculated in a quark-parton model of hadrons. A statistical model proposed by Kuti and Weisskopf was used to calculate the distributions of the quarks inside a proton, and the invariant cross sections were written as sums of these distributions. We were able to demonstrate that the spectrum of final state mesons going backward in the photon-proton center of momentum frame is the same as the spectrum of large momentum fraction quarks immediately after the interaction. The agreement between experimental data and our theory is good.

In part two of this thesis the pion-pion scattering term of the multi-scattering series for pion-deuteron elastic scattering is calculated. Using Weinberg's off-shell pion-pion matrix element to describe the scattering of real and virtual pions, we found that his value pion-pion scattering lengths are consistent with the pion-deuteron experiments. The pion-deuteron cross section was also written as a function of the pion-pion scattering length a_0, and a value of $a_0 = 0.15\ m_\pi^{-1}$ was extracted from the data. This value is comparable to values of a_0 obtained from other experiments.

PART ONE

DEEP-INELASTIC ELECTROPROTUCTION OF PIONS IN A QUARK-PARTON MODEL

I. INTRODUCTION

Invariant cross sections for the single particle inclusive process $e + p \to e + \pi^{\pm}$ + anything have been calculated[1] in a quark-parton model of hadrons. Before presenting a detailed description of the calculation, a brief history of the theories and experiments leading up to this work will be given.

In the late 1960's Feynman[2] speculated that nucleons are bound states of some basic constituent particles which he named partons. Since then many people, both theorists and experimentalists, have tried to determine the nature of these partons. The photon has been used to probe the nucleon structure, via electron scattering, because the theory of the photon's interaction with electrons, QED, is experimentally known to be highly accurate. Proton structure functions, whether elastic or inelastic, can be extracted from data on electron-proton scattering. For the case of elastic scattering these structure functions are a function of a single variable, Q^2, the four-momentum transfer squared. But for inelastic scattering, a second variable, the energy loss of the electron, ν, is needed to describe the scattering completely. The variable ν can be written in invariant form as $\nu = q \cdot P/M$, where q and P are the momenta of the photon and proton respectively, and M is the mass of the proton. The variable Q^2 is $Q^2 = -q^2$. Experimentally, both Q^2 and ν can be varied independently, providing a wealth of data.

Much has been observed about electron-proton scattering, one of the more interesting phenomena being that of scaling. For inelastic electron-proton scattering at large Q^2 and ν it was found that the structure function depends, not on the individual variables Q^2 and ν, but rather on a single variable. This is surprising because, as indicated above, this dependence on a single variable is a property of elastic scattering, while the structure function was measured for inelastic scattering. This is significant for future work because it implies that in the right kinematic region the virtual photon of electron-proton scattering interacts with a single one of the partons, rather than with the proton as a whole, and that this photon-parton scattering is elastic.

The proton is a bound state of partons, and in the study of any bound state system where the details of the interactions among the constituents is not known, the application of an impulse approximation is the first line of theoretical investigation. To use an impulse approximation, it must be possible to treat the partons as free particles during the interaction between electron and proton, and the virtual photon must interact with only one of the partons, the others being spectators. These notions will be put on firm quantitative ground in Sec. II, but for now let us suppose that there is a kinematic region where these conditions are satisfied. From scaling we learned that the photon-parton scattering is elastic, so that the cross section for virtual photon-proton scattering is simply a sum of these individual photon-parton cross sections weighted by the probability of the proton being a particular configuration of partons.

From the beginning[3] it was tempting to identify partons with the quarks suggested by Gell-Mann and Zweig[4] because of the approximate SU(3) symmetry displayed by nature. In the quark model hadrons are composed of quarks, and the quantum numbers of a hadron are determined by the quantum numbers of the individual quarks of which it is built. In the simplest quark model, which will be used here, there are three quarks labeled in standard notation up (u), down (d), and strange (s). The baryons are composed of a set of three of these quarks, while the mesons are composed of a quark and an anti-quark. This simple model has had to be modified somewhat in light of our present understanding of the strong quark-quark interaction. Because the interaction is strong, effects like vacuum polarization, in which particle anti-particle pairs are created from the vacuum, occur copiously. That is, a hadron is not composed simply of the few quarks which carry its quantum numbers, but rather of these quarks plus a large number of virtual quark anti-quark ($q\bar{q}$) pairs. From experiments which measure the average momentum of the charged constituents and the average charge of the constituents of the proton, it has been further determined that the proton also contains many neutral bosons, called gluons, as well. The picture that emerges is one in which the proton is composed of three "valence" quarks (u, u, d) which carry the proton quantum numbers, and of a "core" (or "sea") of $q\bar{q}$ pairs and neutral gluons which has the quantum numbers of the vacuum, and which carries a goodly fraction of the proton's momentum.

In order to calculate the structure functions for the inelastic process $e + p \to e +$ anything, it is necessary to have a model for the distribution in momentum space of the quarks and gluons inside a proton. Several authors[5-7] have devised models for these quark momentum distributions. The models of Ref. 6 are based on sound field theoretic arguments, and are known to give excellent results for the calculation of the structure functions of inclusive electroproduction. However, it is difficult to extend these models to a calculation of one particle inclusive processes because they do not lend themselves to a derivation of the necessary correlated momentum distributions of two quarks in a proton. Therefore, we choose to use a phenomenological model proposed by Kuti and Weisskopf.[5] In their model the momentum distribution of quarks and gluons in the proton core is proportional to the relativistic phase space of these particles, and the momentum distribution of the valence quarks is based on Regge considerations. From these distributions the probability of finding a specified type of quark with a given fraction of the proton's momentum was calculated. Then a cross section was computed, and the constants of proportionality were determined by fitting the calculated cross section to the data for inclusive electron-proton inelastic scattering over the entire kinematic region.

In this work cross sections for $e + p \to e + \pi^{\pm}$ + anything for pions goind backward in the virtual photon (γ_v) - proton (p) center of momentum (c.m.) frame are calculated giving tolerable agreement with the data. The calculation of the cross section is described in the next section (Sec. II), beginning with a more detailed explanation of the quark-parton model. Then both qualitative and quantitative arguments are given to show that the spectrum of final state mesons going backward in the γ_v - p c.m. frame is the same as the spectrum of quarks which have an appreciable fraction of the proton's momentum after the interaction. Finally, the cross section is written in terms of the joint probabilities of finding two quarks with given momentum fractions inside the proton. The details of the Kuti and Weisskopf model are presented in Sec. III, including a calculation of the necessary joint probabilities. A comparison of these results with the data along with some final comments are given in Sec. IV.

II. CROSS SECTIONS FOR $e + p \to e + \pi^{\pm}$ + ANYTHING

A. The Impulse Approximation

In order to apply the impulse approximation to electron-proton scattering, the interaction time of the virtual photon must be much less than the lifetimes of the virtual parton states. That is, using the uncertainty principle,

$$\frac{1}{q_0} \ll \frac{1}{E_b} \tag{1}$$

where q_0 is the energy of the virtual photon, and E_b is the binding energy of the quarks. If, in addition, $Q^2 \gg E_b^2$, where Q^2 is the four-momentum transfer, then the scattering from the individual partons will be incoherent, and the electron-proton cross section will be a sum of the cross sections for scattering from individual partons.

The above conditions are realized in the infinite momentum frame of the proton. This frame is a specialization of the electron-proton center of momentum frame. The momenta of the virtual photon and the proton in the electron-proton c.m. are:

$$q = (q_0, \vec{q}) \quad ,$$

$$P = (\sqrt{P^2+M^2}, 0, 0, P) \; .$$

To obtain the infinite momentum frame, a boost is made so $P \to \infty$ for the proton's momentum, and the three-momentum of the virtual photon is perpendicular to P (see Fig. 1).

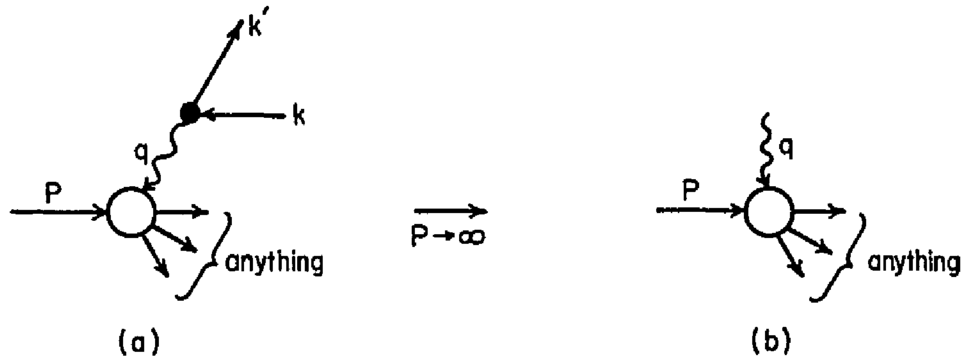

Fig. 1. Kinematics for electron-proton scattering in a) the electron-proton c.m. frame, and b) the infinite momentum frame.

Then q and P in the infinite momentum frame are

$$q = (q_0, \vec{q}_\perp, 0) ,$$

and

$$P = \left(P + \frac{M^2}{2P}, 0, 0, P \right) .$$

Employing the invariant product $q \cdot P = M\nu$, q_0 can be written in terms of lab (proton rest frame) variables as

$$q_0 = \frac{M\nu}{P}$$

In the infinite momentum frame the proton can be visualized as a stream of partons whose interactions with each other are slowed down by time dilation. For large momentum transfer the virtual photon interacts with a single one of the partons (see Fig. 2).

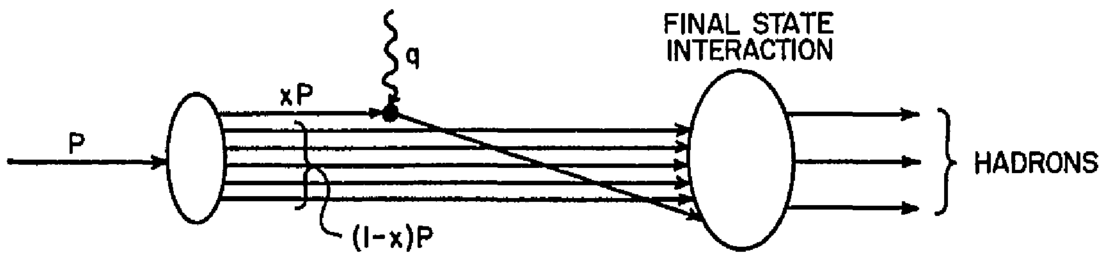

Fig. 2. Kinematics for virtual photon-proton scattering in the proton's infinite momentum frame.

If this parton has a fraction x of the proton's momentum, then its momentum is

$$P(x) = \left(xP + \frac{\mu^2 + P_\perp^2}{2xP} \, , \, \vec{P}_\perp \, , \, xP \right)$$

where μ is the parton mass. If we consider a bound state of just two partons, the binding energy of the parton with momentum fraction x is,

$$E_b = E(x) + E(1-x) - E_p$$

$$= \frac{\mu^2 + P_\perp^2}{2xP} + \frac{\mu^2 + P_\perp^2}{2(1-x)P} - \frac{M^2}{2P}$$

where the sum of the momentum fractions is one. In order to satisfy the first condition of the impulse approximation, Eq. (1), which can be rewritten as

$$M\nu \gg \frac{\mu^2 + P_\perp^2}{2x} + \frac{\mu^2 + P_\perp^2}{2(1-x)} - \frac{M^2}{2}$$

it is sufficient to have $\nu \gg M$ and $x \neq 0$ or 1, if $P_\perp$ is limited.

Experimentally, lepton scattering reactions show that $P_\perp \lesssim 500$ MeV. Thus the requirements for using the impulse approximation in the infinite momentum frame of the proton are

$$\nu \gg M$$

$$Q^2 \gg M^2$$

This use of the impulse approximation to study high energy electron-proton scattering, together with the picture of the proton as a set of point constituents, is called the "parton model", and it is in this sense that the term will be used in this paper.

B. The Virtual Photon-Proton Center of Momentum Frame

To get some qualitative idea of what electron-proton scattering looks like at high energies, one can examine a reference frame in which the virtual photon is pure space-like,

$$q = (0, 0, 0, Q)$$

and the proton's momentum is

$$P = (E, 0, 0, -P)$$

For P and Q large, the parton picture is valid, and the virtual photon interacts with a single one of the partons.

When the photon interacts with any one of the partons, it disturbs the parton configuration, which then no longer forms a proton. The proton "fragments" and the partons, which cannot themselves appear in the final state, recombine into observable physical particles. The details of the recombination are unknown because the details of the strong parton-parton interactions, or the nature of the possible parton fragmentation into hadrons are not known. One may despair of these problems and believe that they limit the parton model to totally inclusive lepton induced reactions. But any model is likely to be plagued with similar problems, so let us be positive and ask where (i.e., some special kinematic region) a simple picture could give correct quantitative results. This is most easily done by using Feynman's picture representation[8] of the proton wave function in the above reference frame. Suppose that in the initial state one of the partons has a large fraction, y, of the proton's momentum, and that all of the other partons have very small momentum fractions. These partons with very small momentum fractions will be called "wee" partons. Feynman would draw the wave function for this case thus:

Fig. 3

If the virtual photon interacts with one of the wee partons, the final state wave function is:

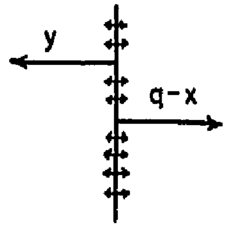

Fig. 4

After the interaction, the parton with momentum fraction y, together with some other appropriate wee partons look very much like a final state hadron going backward (with respect to the direction of the virtual photon) in this frame. Thus, one can hypothesize that backward going final state hadrons (e.g. pions) have the same momentum distribution as the large momentum fraction partons. Note that the requirement of a large momentum fraction is necessary to ensure that the final state hadron will have the same momentum as the final state parton.

The above reference frame can be compared with the virtual photon-proton c.m. system where

$$q = (q_0, 0, 0, P)$$

and

$$P = (E, 0, 0, -P)$$

The only change is that q is no longer purely space-like, and that P does not have to be large. However, the qualitative arguments we have made can still be expected to hold. That is, the spectrum of large momentum hadrons going backward in the γ_v - p c.m. system should be about the same as the spectrum of large momentum fraction partons in the infinite momentum frame.

C. The Proton's Infinite Momentum Frame

One can also examine electron-proton scattering in a frame of reference in which the incoming proton is moving infinitely fast (take

$P \to \infty$ below) in a direction perpendicular to the photon's three-momentum. Explicitly,

$$q = \left(\frac{M\nu}{P}\,,\ \vec{q}_\perp\,,\ 0\right)$$

$$P = \left(P + \frac{M^2}{2P}\,, 0, 0,\ P\right)$$

where M is the proton's mass, and $q \cdot P = M\nu$. The proton is replaced by a beam of non-interacting partons and the virtual photon is viewed as interacting with a single free parton, the remaining partons being spectators. In terms of the fraction, x_i, of the proton's momentum carried by the parton, the momentum of any given parton will be

$$P_i = \left(x_i P + \frac{\mu^2}{2x_i P}\ ,\ \vec{P}_{i\perp}\ ,\ x_i P\right)$$

provided $x_i \gg M/P$ and where μ is the parton mass. We expect $|\vec{P}_{i\perp}| < 500$ MeV, so that for x_i not very small and for $\nu \gg M$, we have $x_i = q \cdot P_i / M\nu$.

In this frame it can be shown that large momentum fraction partons do not interact strongly with the wee partons. Let us first note that there are few large momentum fraction partons: obviously there is not more than one parton with $x_i > 1/2$, not more than two partons (if that many) with $x_i > 1/3$, etc., and most partons will be in the wee region. For a parton with large momentum fraction, y, and a wee parton with momentum fraction, x,

$$s_{12} \equiv (P_1 + P_2)^2 \cong \mu^2\left(\frac{y}{x} + \frac{x}{y}\right)$$

where $P_\perp$ has been neglected. The rapidity difference, $\log s_{12}/\mu^2$, determines the decoupling of the two partons by well known arguments,[9] with more than one or two units difference meaning negligible interaction. For y and x very unequal, s_{12} can be a large factor times μ^2. For example, for $y = .5$ and $x = .05$, the rapidity difference becomes log 10.1, which is sufficient to have the large momentum fraction partons decoupled from the wee partons. Of course, one does not know the parton mass. Instead of taking it to be some typical hadron mass, one may take $\mu = 0$ for east of calculation,[10] argue that $\mu \approx 300$ MeV, or even that it is infinite.[11] If μ is zero, it may be that $\log s_{12}/\mu^2$ is undefined, but recent theories[12] indicate that the partons are bound in an approximate linear potential, and that their mass is about 300 MeV. It now seems reasonable to assume that a parton with a large fraction of the proton's momentum does not interact strongly with the wee partons.

Suppose that the virtual photon does not interact directly with the large momentum fraction parton discussed above, but that it does carry enough energy ($\nu >> M$) to make the parton picture valid. In this picture the function of the incoming photon is only to fragment the proton freeing its constituents (freeing them to become other things, that is). What happens to the parton with a large fraction of the proton's momentum? It is certainly true that any parton leaving the interaction zone must pick up at least one other parton to make a hadron. Since the interaction is weak, it is likely that only one extra parton will be picked up,[13] and since this extra parton will probably be wee, the momentum spectrum of the fast emerging mesons will be the same as that of the large momentum fraction partons.

D. Partons as Quarks

Let us suppose that the partons have the quantum numbers of quarks. The standard[4] quark notation will be used, namely, there are three quarks and three anti-quarks; the up quarks labeled u ($\bar{u}$) with charges +2/3 (-2/3), the down quarks labeled d ($\bar{d}$) with charges -1/3 (+1/3), and the strange quarks labeled s ($\bar{s}$) with charges -1/3 (+1/3). The proton consists of three valence quarks (u, u, d) which carry the quantum numbers of the proton, and of a core of $q\bar{q}$ pairs and neutral gluons. The probability of finding a u-quark in the proton with momentum fraction between x and x + dx is given in standard notation by u(x) dx. The probabilities $\bar{u}(x)$ dx, d(x) dx, etc. are similarly defined. When sums over the probabilities are being performed, it is more convenient to use a slightly different notation. To this end, define $P^u(x)\,dx = u(x)\,dx$. Then sums over $P^a(x)$ can easily be considered, where a takes the values u, $\bar{u}$, etc.

These single probabilities can be derived from the joint probability of finding n quarks in the proton with momentum fractions between x_1 and $x_1 + dx$, . . . x_n and $x_n + dx_n$. We write this differential joint probability as

$$dP_n(x_1 \cdots x_n) \equiv J_n(x_1 \cdots x_n)\ \delta(1 - \sum_{j=1}^{n} x_j)\ dx_1 \cdots dx_n \quad .$$

The δ-function arises because the sum of the individual quark momenta must equal the proton's momentum. The single probabilities are obtained from the joint probability by integrating over all of the momentum fractions save one, and by summing over all of the possible quark distributions, n.

$$P(x)\,dx \equiv \left\{ \sum_{n=0}^{\infty} \int J_n(x, x_2, \cdots x_n)\ \delta(1 - x - \sum_{j=2}^{n} x_j)\ dx_2 \cdots dx_n \right\}\ dx .$$

In this work we choose to normalize the single probabilities to the number of quarks in the proton. For the valence quarks in a proton,

$$\int_0^1 u_v(x)\ dx = 2$$

and

$$\int_0^1 d_v(x)\ dx = 1$$

For the quarks in the core,

$$\int_0^1 P_c^a(x)\ dx \rightarrow \infty$$

that is, there is an uncountable number of quarks in the core.

The joint probability of finding a quark of type a with momentum fraction between x and x + dx, and a quark of type b with momentum fraction between y and y + dy will be given by $J^{ab}(x, y)$. This is obtained from the differential joint probability as follows,

$$J(x,y)\,dx\,dy \equiv \left\{ \sum_{n=0}^{\infty} \int J_n(x, y, x_3 \cdots x_n)\ \delta(1 - x - y - \sum_{j=3}^{n} x_j)\ dx_3 \cdots dx_n \right\}\ dx\ dy .$$

Note that the joint probability is not a product of two independent probabilities, ($J^{ab}(x,y) \neq P^{a}(x)\ P^{b}(y)$), but rather, that there is a correlation between finding a parton with momentum fraction x and another parton with momentum fraction y in the proton. This correlation is a result of requiring that the sum of the momentum fractions be one.

E. The Cross Section

In general, the inelastic scattering of an electron by a proton can be written in terms of two structure functions, usually named W_1 and W_2. These structure functions depend on the independent variables of the scattering. In the lab (rest frame of the proton) these variables are,

$$\nu = \frac{q \cdot P}{M} = E' - E$$

and

$$Q^2 = -q^2 = 4E'E \sin^2(\tfrac{\theta}{2})$$

where the various momenta are defined in Fig. 5.

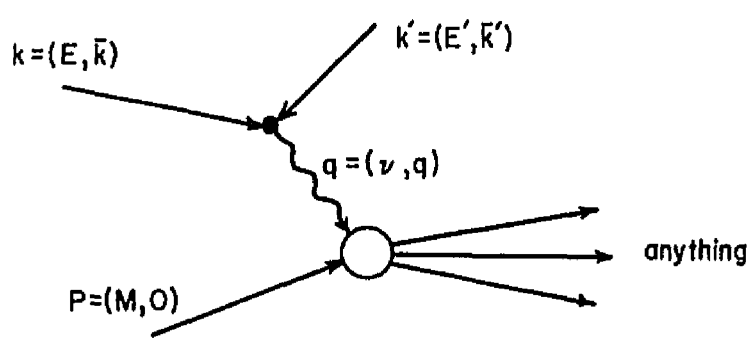

Fig. 5. Kinematics for electron-proton scattering in the lab.

The cross section for scattering an electron through an angle θ in the lab is

$$\frac{d\sigma}{d\Omega\, dE'} = \frac{4\alpha^2 E'^2}{Q^4} \left[W_2(Q^2,\nu) \cos^2\left(\frac{\theta}{2}\right) + 2W_1(Q^2,\nu) \sin^2\left(\frac{\theta}{2}\right) \right]$$

In the infinite momentum frame, the electron is scattered through an angle $\theta \approx Q/P$, which is quite small. Setting $\theta = 0$ in Eq. (2), we find that the cross section depends only on the structure function W_2. In what follows we will work close to the forward direction.

Now consider the elastic scattering of an electron from a point spin-1/2 particle of momentum xP and a charge e_a (in units of e). The structure function W_2 for this process can be derived from the one calculated in the Appendix by making the replacements $p \to xP$ and $m \to xM$,

$$W_2^{\mathrm{POINT}}(Q^2,\nu) = e_a^2 \ \delta\left(\frac{Q^2}{2Mx} - \frac{q\cdot P}{M} \right) .$$

Making the replacement $q\cdot P = M\nu$, the cross section in the forward direction is

$$\frac{d\sigma^{\mathrm{POINT}}}{d\Omega\, dE'} = \frac{4\alpha^2 E'^2}{Q^4} \quad e_a^2 \ \frac{x}{\nu} \ \delta\left(x - \frac{Q^2}{2M\nu} \right) \qquad (3)$$

This is the cross section for scattering from a single free quark-parton. Note that the δ-function constrains the virtual photon to interact only with a parton whose momentum fraction is $Q^2/2M\nu$. Under the assumptions of the parton model the inelastic cross section for electron-proton scattering is the superposition of the point cross sections, Eq. (3), weighted by the probability distributions of quarks inside a proton. Therefore, the totally inclusive cross section is

$$\frac{d\sigma}{d\Omega\, dE'} = \frac{4\alpha^2 E'^2}{Q^4} \sum_a e_a^2 \; \frac{1}{\nu} \sum_{n=0}^{\infty} \int J_n(x_1 \cdots x_n)\, \delta\!\left(1-\sum_{j=1}^{n} x_j\right)\, x_1\, \delta\!\left(x_1 - \frac{Q^2}{2M\nu}\right) dx_1 \cdots dx_n \tag{4}$$

$$= \frac{4\alpha^2 E'^2}{Q^4}\; \frac{x}{\nu} \sum_a e_a^2\, P^a(x) \;,$$

where $x \equiv Q^2/2M\nu$ and $J(x_1 \ldots x_n)dx_1 \ldots dx_n$ is the differential probability of finding n quarks with momentum fractions $x_1 \ldots x_n$ in a proton. The index $\underline{a}$ runs only over the charged quarks, which can interact with the photon, and not over the index of the neutral gluons.

Comparing Eq. (4) with Eq. (2) in the forward direction, the scaling result is obtained,

$$\nu W_2(x) = x \sum_a e_a^2\, P^a(x) \;, \tag{5}$$

that is, νW_2 is a function of the single variable $x \equiv Q^2/2M\nu$ only.

The spectrum of final state partons with momentum fraction y is obtained from Eq. (4) by not doing the integral over y,

$$\frac{d\sigma}{d\Omega\, dE'\, dy} = \frac{4\alpha^2 E'^2}{Q^4}\; \frac{x}{\nu} \sum_a e_a^2\, J^{ab}(x,y) \;,$$

or, a more convenient form for later use,

$$y\, \frac{d\sigma}{d\Omega\, dE'\, dy} = \frac{4\alpha^2 E'^2}{Q^4}\; \frac{xy}{\nu} \sum_a e_a^2\, J^{ab}(x,y) \;.$$

III. THE KUTI-WEISSKOPF MODEL

The proton is composed of three valence quarks, and a core of $q\bar{q}$ pairs and neutral gluons. In the Kuti-Weisskopf[5] model the quarks in the core are distributed statistically,

$$dP_c(x) \sim g \frac{dx}{(x^2 + \mu^2/P^2)^{1/2}} \tag{6}$$

where μ is the effective mass of the quarks and P is the proton's momentum in the infinite momentum frame ($P \to \infty$). The momentum distribution of the valence quarks is based on Regge considerations,

$$dP_v(x) \sim \frac{x^{1-\alpha(0)}\, dx}{(x^2 + \mu^2/P^2)^{1/2}}$$

where $\alpha(0) = 1/2$ is the intercept of the nondiffractive trajectories. Kuti and Weisskopf assume a statistical distribution for the gluons, the constant g in Eq. (6) being replaced by g' to get the gluon probability distribution.

In this model the proton is an n-quark state composed of three valence quarks (u, u, d), and a core of k_1 $u\bar{u}$ pairs, k_2 $d\bar{d}$ pairs, k_3 $s\bar{s}$ pairs, and ℓ neutral gluons, where $n = 3 + k_1 + k_2 + k_3 + \ell$. Then the probability distribution of such an n-quark state is

$$dP_n(x_1 \cdots x_n) = Z \prod_{i=1}^{n} \frac{dx_i}{(x_i^2 + \mu^2/P^2)^{1/2}} \delta\left(1 - \sum_{j=1}^{n} x_j\right) x_1^{1-\alpha(0)} x_2^{1-\alpha(0)} x_3^{1-\alpha(0)}$$

$$\cdot \frac{(\frac{1}{3} g)^{k_1} (\frac{1}{3} g)^{k_2} (\frac{1}{3} g)^{k_3} g'^{\ell}}{k_1! \; k_2! \; k_3! \; \ell!} ,$$

where Z is a normalization constant obtained from

$$\sum_{n=0}^{\infty} \int dP_n(x_1 \cdots x_n) = 1 ,$$

and $(1/3\ g)^{k_i} / k_i\,!$ is the statistical weight for a symmetric k_i-particle state. The indices k_i can only take on the values 0, 2, 4, . . ., because the distribution has been written to include both a quark and an anti-quark in the factor $(1/3\ g)^{k_i} / k_i\,!$. Instead of following Kuti and Weisskopf on this, we could have used the statistical weight $(1/6\ g)^{k_i} / k_i\,!$, $k_i = 0, 1, 2, \ldots$ for the individual quarks in the core, or used the statistical weight $g^k / k!$, $k = 0, 6, \ldots$, lumping all three types of quarks together. It makes no difference which form we choose since all three forms will give the same result, providing we keep the counting straight.

A. Single Probabilities For a Proton

As discussed in Ref. 5, the probability of finding a u-quark in the proton valence with momentum fraction between x and x + dx is,

$$u_v(x) = a \cdot Z \frac{x^{1-\alpha(0)}}{(x^2 + \mu^2/P^2)^{1/2}} \sum_{n=0}^{\infty} \int \prod_{i=1}^{n-1} \frac{dx_i}{(x_i^2 + \mu^2/P^2)^{1/2}} \delta\left(1 - x - \sum_{j=1}^{n-1} x_j\right)$$

$$\cdot x_1^{1-\alpha(0)} x_2^{1-\alpha(0)} \frac{(\frac{1}{3} g)^{k_1} (\frac{1}{3} g)^{k_2} (\frac{1}{3} g)^{k_3} (g')^{\ell}}{k_1! \; k_2! \; k_3! \; \ell!} .$$

For the other quarks

$$d_v(x) = 1/2\, u_v(x)$$

$$s_v(x) = 0$$

Because electron-proton scattering is an electromagnetic process, it will not be necessary to calculate the probability distribution of the neutral gluons. It is only necessary to include the statistical weight for the gluons in $dP_n(x_1 \ldots x_n)$. For the $q\bar{q}$ pairs in the core,

$$u_c(x) = d_c(x) = s_c(x)$$

where

$$u_c(x) = Z \frac{1}{(x^2+\mu^2/P^2)^{1/2}} \sum_{n=0}^{\infty} \int \prod_{i=1}^{n-1} \frac{dx_i}{(x_i^2+\mu^2/P^2)^{1/2}} \delta\left(1-x-\sum_{j=1}^{n-1} x_j\right)$$

$$\cdot\; x_1^{1-\alpha(0)}\, x_2^{1-\alpha(0)}\, x_3^{1-\alpha(0)} \quad k_1 \cdot \frac{(\frac{1}{3}g)^{k_1} (\frac{1}{3}g)^{k_2} (\frac{1}{3}g)^{k_3}\, g'^{\ell}}{k_1!\; k_2!\; k_3!\; \ell!}$$

The results of the integrations in the limit $P \to \infty$ are,

$$u_v(x) = 2 \cdot d_v(x) = 2 \cdot B^{-1}[1-\alpha(0)\, ,\ \gamma + 2(1-\alpha(0))]\; x^{-\alpha(0)}\, (1-x)^{-1+\gamma+2(1-\alpha(0))},$$

and

$$u_c(x) = d_c(x) = s_c(x) = \frac{1}{3}\, g\, x^{-1}\, (1-x)^{-1+\gamma+3(1-\alpha(0))}\, ,$$

where $\gamma \equiv g + g'$ and $B(\ldots)$ is a Beta function. To keep the counting straight, recall that $u_c(x)$ actually is $u_c(x) \equiv u_c(x) + \bar{u}_c(x)$. These momentum distributions are used in Eq. (5) to yield $\nu W_2(x)$ for totally inclusive electron-proton scattering. Drell and Yan[14] speculated that near x = 1,

$$\nu W_2(x) \sim (1 - x)^3.$$

This implies that $\gamma = 3$ for the Kuti-Weisskopf model. The remaining free parameter was determined from fits to experimental data for totally inclusive electron-proton scattering which yielded g = 1.

B. Joint Probabilities for a Proton

The probability of finding a u-quark with momentum fraction between x and x + dx, and a d-quark with momentum fraction between y and y + dy in the proton valence is

$$J_{vv}^{ud}(x,y) = 2 \cdot Z \frac{x^{1-\alpha(0)}}{(x^2+\mu^2/P^2)^{1/2}} \cdot \frac{y^{1-\alpha(0)}}{(y^2+\mu^2/P^2)^{1/2}} \sum_{n=0}^{\infty} \int \prod_{i=1}^{n-2} \frac{dx_i}{(x_i^2+\mu^2/P^2)^{1/2}} \tag{7}$$

$$\cdot\, \delta\Big(1-x-y-\sum_{j=1}^{n-2} x_j\Big) \cdot x_1^{1-\alpha(0)} \frac{(\frac{1}{3}g)^{k_1}(\frac{1}{3}g)^{k_2}(\frac{1}{3}g)^{k_3} g'^{\ell}}{k_1!\; k_2!\; k_3!\; \ell!}$$

Elementary calculation also yields

$$J_{vv}^{ud}(x,y) = J_{vv}^{du}(x,y) = J_{vv}^{uu}(x,y).$$

Note that $J_{vv}^{dd}(x,y) = 0$ because there is only one d-quark in the proton valence. In addition to the above probability distribution, Eq. (7), there are only three other independent joint probability functions. The results of the integrations (limit $P \to \infty$) to obtain these functions are:

$$J_{vv}^{ud}(x,y) = J_{vv}^{du}(x,y) = J_{vv}^{uu}(x,y)$$

$$= 2 \cdot x^{-\alpha(0)} \, y^{-\alpha(0)} \, (1-x-y)^{-1+\gamma+(1-\alpha(0))} \, \frac{\Gamma[\gamma+3(1-\alpha(0))]}{\Gamma^2(1-\alpha(0))\,\Gamma[\gamma+(1-\alpha(0))]}$$

$$J_{vc}^{uu}(x,y) = 2\, J_{vc}^{du}(x,y) = J_{vc}^{ud}(x,y) = 2\, J_{vc}^{dd}(x,y) = J_{vc}^{us}(x,y) = 2\, J_{vc}^{ds}(x,y)$$

$$= 2 \cdot \tfrac{1}{3} g \; x^{-\alpha(0)} \, y^{-1} \, (1-x-y)^{-1+\gamma+2(1-\alpha(0))} \, \frac{\Gamma[\gamma+3(1-\alpha(0))]}{\Gamma(1-\alpha(0))\,\Gamma[\gamma+2(1-\alpha(0))]}$$

$$J_{cv}^{uu}(x,y) = 2\, J_{cv}^{ud}(x,y) = J_{cv}^{du}(x,y) = 2\, J_{cv}^{dd}(x,y) = J_{cv}^{su}(x,y) = 2\, J_{cv}^{sd}(x,y)$$

$$= 2 \cdot \tfrac{1}{3} g \; x^{-1} \, y^{-\alpha(0)} \, (1-x-y)^{-1+\gamma+2(1-\alpha(0))} \, \frac{\Gamma[\gamma+3(1-\alpha(0))]}{\Gamma(1-\alpha(0))\,\Gamma[\gamma+2(1-\alpha(0))]}$$

$J_{cc}^{ab}(x,y)$ $\quad$ a, b = u, d, s are all equal

$$= \tfrac{1}{9} \, g^2 \, x^{-1} \, y^{-1} \, (1-x-y)^{-1+\gamma+3(1-\alpha(0))} \quad .$$

For convenience later on the four joint probability distributions will be renamed $J_{vv}(x,y)$, $J_{vc}(x,y)$, $J_{cv}(x,y)$, and $J_{cc}(x,y)$ in an obvious notation.

IV. RESULTS AND COMMENTS

The cross section for finding a large momentum fraction quark in the final state is

$$y \frac{d\sigma}{d\Omega\, dE' dy} = \frac{4\alpha^2 E'^2}{Q^4} \frac{xy}{\nu} \sum_{a,b,i,j} e_a^2 \, J^{ab}_{ij}(x,y) \qquad (8)$$

where the indices a, b run over u, d, s and the indices i, j run over v, c. It should be emphasized that $x \equiv Q^2/2M\nu$ is the scaling variable. This cross section must be multiplied by the probability that a given quark will contribute to a specified final state hadron. By taking the extreme position that the large momentum fraction quark goes into the valence of the final state particle, and that only one additional quark is picked up (i.e. the final state hadron is a meson) an estimate of these probabilities can be made. For example, only joint probabilities of the types $J^{au}_{ij}(x,y)$ and $J^{ad}_{ic}(x,y)$ can contribute to the π^+ valence $(u\bar{d})$, while the joint probabilities $J^{ad}_{ij}(x,y)$ and $J^{au}_{ic}(x,y)$ will contribute to the π^- valence $(\bar{u}d)$. In particular, for final state π^+ the sum in Eq. (8) is

$$\sum_{a,i} e_a^2 \left[J^{au}_{iv}(x,y) + \tfrac{1}{2} J^{au}_{ic}(x,y) + \tfrac{1}{2} J^{ad}_{ic}(x,y) \right] .$$

The factor of 1/2 is necessary because the joint probability $J^{ab}_{ic}(x,y)$

gives the probability of finding a $b\bar{b}$ pair in the proton core while only one-half of that pair contributed to the π^+ valence. Likewise, for a final state π^- the sum in Eq. (8) is

$$\sum_{a,i} e_a^2 \left[J_{i\ v}^{a\ b}(x,y) + \frac{1}{2} J_{i\ c}^{a\ d}(x,y) + \frac{1}{2} J_{i\ c}^{a\ u}(x,y) \right] .$$

There are, of course, other possibilities, such as the fast parton picking up more than one additional parton or the fast parton not going into the valence, that can affect the pion cross section. However, we shall assume that the probability of these occurences is small, and compare the data to the simplest assumption described above. Therefore, using the compact notation introduced at the end of Sec. III C, the cross section for $e + p \to e + \pi^+ +$ anything is

$$y \frac{d\sigma}{d\Omega\, dE'\, dy} = \frac{4\alpha^2 E'^2}{Q^4} \frac{xy}{\nu} \left[\frac{5}{9} J_{vv}(x,y) + \frac{1}{3} J_{vc}(x,y) + \frac{2}{3} J_{cv}(x,y) + \frac{2}{3} J_{cc}(x,y) \right]$$

The cross section for $e + p \to e + \pi^- +$ anything is

$$y \frac{d\sigma}{d\Omega\, dE'\, dy} = \frac{4\alpha^2 E'^2}{Q^4} \frac{xy}{\nu} \left[\frac{4}{9} J_{vv}(x,y) + \frac{3}{4} J_{vc}(x,y) + \frac{1}{3} J_{cv}(x,y) + \frac{2}{3} J_{cc}(x,y) \right]$$

To be able to compare these cross sections with the experiments, they must be rewritten in terms of the invariant cross section $E(d^3\sigma/dp^3)$ for the process $\gamma_v + p \to \pi +$ anything. In the infinite momentum frame of the proton the invariant cross section is

$$E \frac{d^3\sigma}{dp^3} = \frac{y}{\pi} \frac{d\sigma}{dy\, dP_\perp^2} \tag{9}$$

where, as before, y is the momentum fraction of the outgoing parton. In terms of the cross sections calculated above,

$$\frac{d\sigma}{d\Omega dE' dy} = \Gamma \frac{d\sigma}{dy} \tag{10}$$

where Γ is the flux of virtual photons. Combining Eqs. (9) and (10), integrating over $P_\perp^2$, and normalizing to the total cross section yields the function

$$F(y) = \frac{1}{\sigma_{tot}\,\pi\Gamma} \cdot y \cdot \frac{d\sigma}{d\Omega\, dE'\, dy}$$

where σ_{tot} is the total electroproduction cross section. The total cross section can be written in terms of $W_2(x)$ as

$$\sigma_{tot} = \frac{4\pi^2\alpha}{K} \cdot \frac{Q^2+\nu^2}{Q^2} \cdot W_2(x)$$

where

$$K \equiv \nu - Q^2/2M \ .$$

The Kuti-Weisskopf form for $\nu W_2(x)$ will be used to calculate σ_{tot}.

The function F(y) as defined above is an invariant function. That is, the function F can be obtained in another reference frame by replacing y in F(y) by its transformation in the other frame. In the γ_v - p center of momentum frame, define the variable $z = P_\parallel / P_{max}$ for the pion's momentum. Then in the γ_v - p c.m. F(z) = F(y) where y = y(z). It is the function F(z) that is given by the experiments, and to which this theory will be compared.

As discussed in Ref. 5 and above, the choices $\alpha(0) = 1/2$, $\gamma = 3$, and $g = 1$ gives the best agreement of Kuti-Weisskopf's $\nu W_2(x)$ with the totally inclusive electroproduction data for $x > .2$. These values of the parameters also give a nice fit for $F(z)$.

The experimental data[15,16] for $\gamma_v + p \to \pi^- +$ anything and the results of this calculation are plotted in Fig. 6. For the backward goind pions ($z < 0$) a reasonable fit to the data is obtained. The data for $z > 0$ has also been plotted although the model described here cannot be used to make predictions in that region.

The function $F(z)$ for the process $\gamma_v + p \to \pi^+ +$ anything is compared to the data[15] in Fig. 7. Again, a good fit in the backward direction is obtained. Note that for large z there is contamination from elastic electron-proton scattering in the experimental data.

Some quark-parton model considerations of the ratio $\sigma(\pi^+)/\sigma(\pi^-)$ have been made by Dakin and Feldman.[17] Experimental determinations of this ratio have been made for a number of W and Q^2 values, but only in the forward direction in the γ_v - p c.m. system. That is, the experimental points lie in the region where this model cannot be used to make accurate predictions. The experiments show the ratio increasing as γ_v becomes more virtual, and ranging between one and two. A calculation of the ratio for $z < 0$ has yielded $\sigma(\pi^+)/\sigma(\pi^-) = 1.3$, with little variation as a function of Q^2.

There is another possibility which should be discussed here. After the interaction with the virtual photon, the collection of partons undergoes some final state interactions to produce physical hadrons. During this final state interaction, a parton which is isolated in rapidity

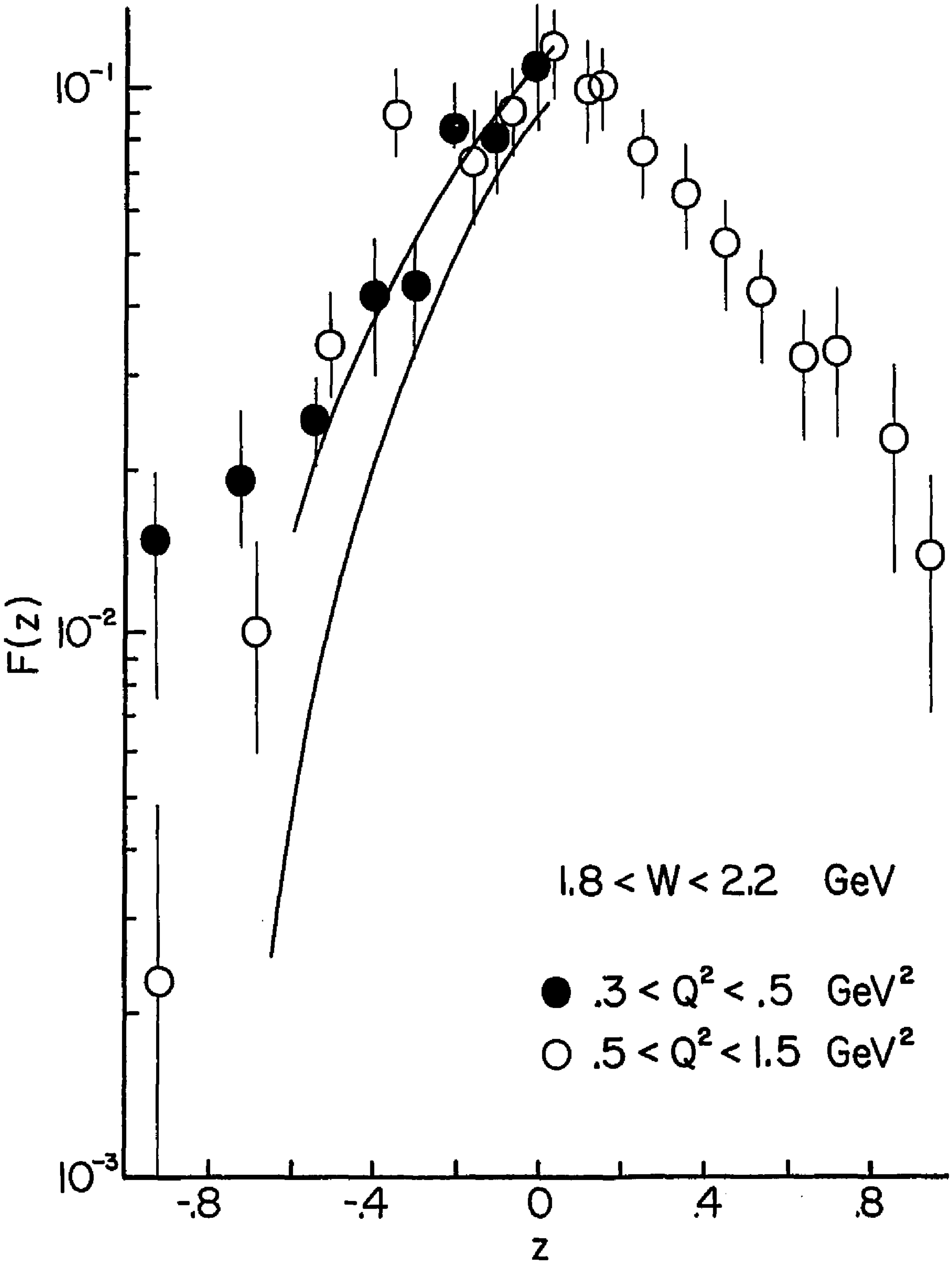

Fig. 6. The invariant function F(z) for inclusive electroproduction of π^-. The top curve for W = 2.2 GeV, Q^2 = .3 GeV^2, and the bottom curve is for W = 2.2 GeV, Q^2 = 1.5 GeV^2. Both curves have g = 1.

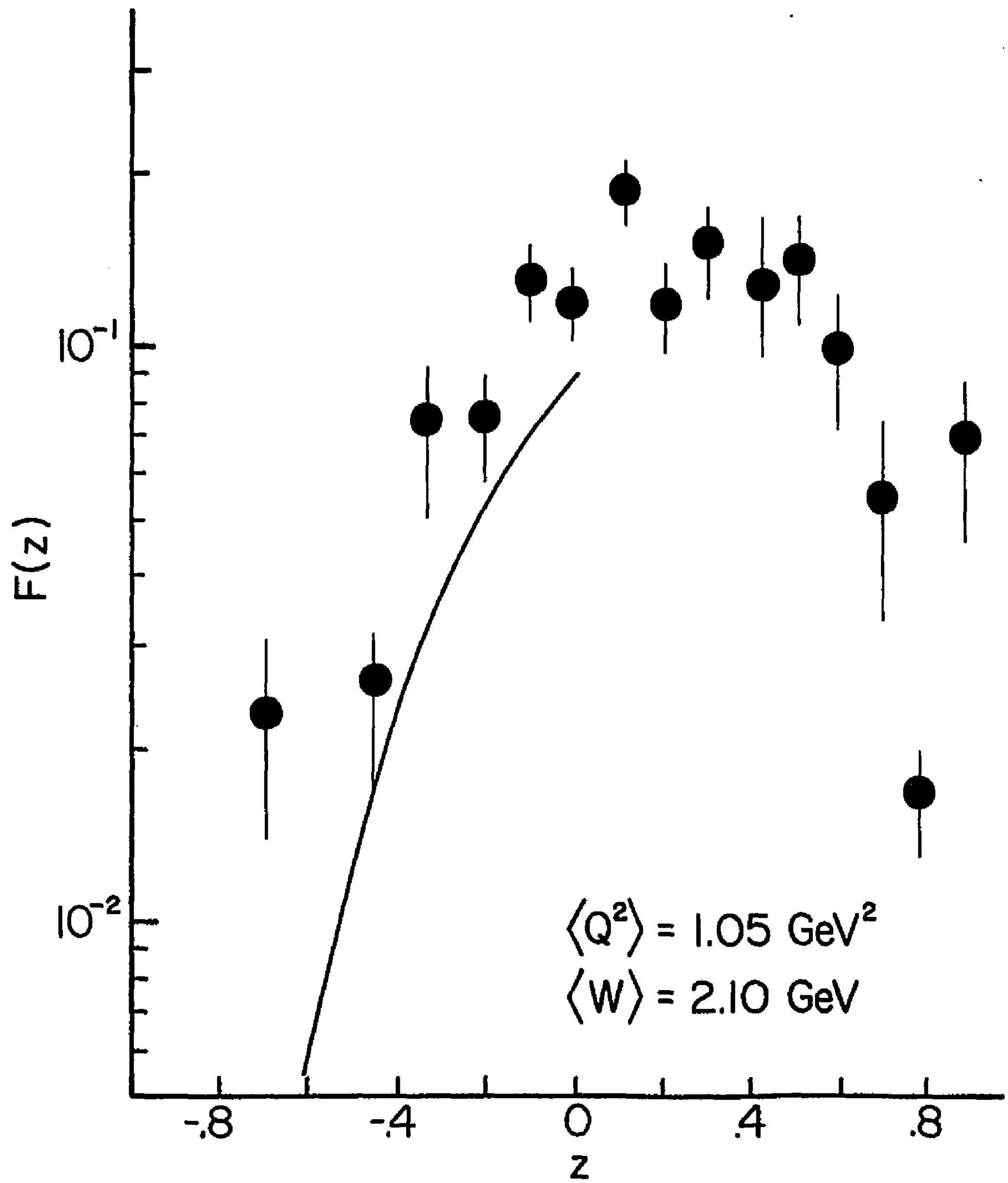

Fig. 7. The invariant function F(z) for inclusive electroproduction of π^+ with W = 2.1 GeV, and Q^2 = 1.05 GeV^2.

space might itself fragment into several hadrons. The nature of this fragmentation has been studied by several authors[18] for the case of partons which have been isolated by virtue of their large transverse momenta. The spectrum of the resulting hadrons is independent of the momentum of the parent parton. The probability of producing a given hadron depends only on the fraction of the parton's momentum that it carries. It has been speculated[18,19] that the distribution of hadrons resulting from the fragmentation of a parton has the same functional form as the distribution of partons inside a hadron, namely $(1 - \chi)^n$, where χ is the hadron's momentum fraction. The effect of including the possibility of parton fragmentation in the above model is to reduce the cross section for z near -1, and to enhance it by about the same amount near $z = 0$. In addition, because the parton can fragment into a variety of hadrons other than pions, the over-all normalization of the cross section would be less than that given above. As can be seen from Figs. 6 and 7, any scheme which predicts smaller cross sections than the ones calculated here would not be compatible with the data.

It can be concluded that the $z < 0$ region at present experimental energies represents partons in an intermediate kinematic region where "grabbing" a single additional parton is what they are most likely to do. For $z = 0$ there will be multi-parton interactions, while at higher energies the parton itself may show scaling behavior as it fragments. If this is true, it will be interesting to observe relatively fewer $z < 0$ pions produced in higher energy experiments.

Thus, a quark-parton model has been used to calculate cross sections for $e + p \to e + \pi^{\pm}$ + anything. The values of the parameters that were used were taken from the work of Kuti and Weisskopf[5] where they were determined by fitting to the totally inclusive experiment $e + N \to e$ + anything. No new parameters are needed for the extension to the one particle inclusive reactions $\gamma_v + p \to \pi^{\pm}$ + anything, and this calculation of the longitudinal momentum spectrum of the pions has given reasonable agreement with the data in the kinematic region where agreement was expected.

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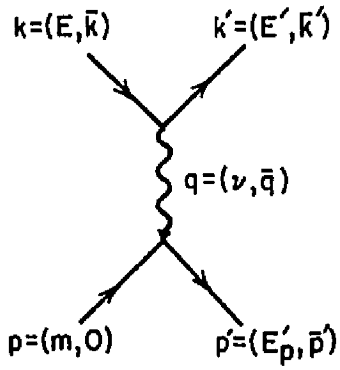
APPENDIX

Structure Function for Point Scattering

In this Appendix the cross section and the structure function $\nu W_2(Q^2,\nu)$ are calculated for the elastic scattering of an electron from a point spin-1/2 particle of charge e_a. The mass of the electron will be taken to be small in comparison to other relevant energies, and so will be neglected ($m_e \simeq 0$). The calculation will be done in the rest frame of the target particle.



$$\nu = E - E' = \frac{q \cdot p}{m}$$

$$Q^2 = -(k-k')^2 = -q^2$$

Fig. 8. Kinematics in the target rest frame (lab).

The cross section in the lab is

$$\sigma = \frac{1}{4mE} \int \frac{d^3k'}{(2\pi)^3 2E'} \frac{d^3p'}{(2\pi)^3 2E_p'} (2\pi)^4 \delta^4(p'+k'-p-k) \frac{e^4 e_a^2}{Q^4}$$

$$\cdot \frac{1}{4} \sum_{\text{SPINS}} \left| \bar{u}(k') \gamma_\mu u(k) \, \bar{u}(p') \gamma^\mu u(p) \right|^2 .$$

It is straightforward to show that

$$\frac{1}{4} \sum_{\text{SPINS}} |\cdots|^2 = 16\, mE'E \left[m \cos^2(\tfrac{\theta}{2}) + \nu \sin^2(\tfrac{\theta}{2}) \right] .$$

Rewriting the integral in terms of the energy E' and doing the d^3p' integral, the differential cross section is,

$$\frac{d\sigma}{d\Omega\, dE'} = \frac{4\alpha^2 E'^2}{Q^4} \quad e_a^2 \quad \frac{m}{E_p'} \quad \delta(E_p' + E' - m - E) \left[\cos^2(\tfrac{\theta}{2}) + \frac{\nu}{m} \sin^2(\tfrac{\theta}{2}) \right] .$$

The energy δ-function can be rewritten as follows:

$$\begin{aligned} \delta(E_p' + E' - m - E) &= \delta(E_p' - m - \nu) \\ &= 2E_p' \quad \delta(E_p'^2 - (m+\nu)^2) \\ &= 2E_p' \quad \delta(p'^2 - (p+q)^2) \\ &= 2E_p' \quad \delta(-2p\cdot q - q^2) \\ &= (E_p'/m) \cdot \quad \delta\left(\frac{Q^2}{2m} - \frac{p\cdot q}{m}\right) \end{aligned}$$

where $Q^2 = -q^2$. Thus,

$$\frac{d\sigma}{d\Omega\, dE'} = \frac{4\alpha^2 E'^2}{Q^4} \quad e_a^2 \; \delta\left(\frac{Q^2}{2m} - \frac{p\cdot q}{m}\right) \left[\cos^2(\tfrac{\theta}{2}) + \frac{\nu}{m} \sin^2(\tfrac{\theta}{2}) \right]$$

Comparing this with the cross section, Eq. (2), for scattering from an extended particle, we find that the structure functions for the elastic scattering of an electron from a point particle are

$$W_2(Q^2, \nu) = e_a^2 \ \delta\left(\frac{Q^2}{2m} - \frac{q \cdot p}{m}\right)$$

and

$$W_1(Q^2, \nu) = e_a^2 \ \frac{\nu}{2m} \ \delta\left(\frac{Q^2}{2m} - \frac{q \cdot p}{m}\right) .$$

PART TWO

PION-DEUTERON ELASTIC SCATTERING AND THE PION-PION SCATTERING LENGTHS

I. INTRODUCTION

In this research a calculation of the π-π scattering term of the multi-scattering series for pion-deuteron elastic scattering has been made. This term is interesting for two distinct and complementary reasons. First, since present calculations, which include single and double scattering contributions, do not fit the data at intermediate pion energies (150-300 MeV) for backward scattering of the pion, a calculation of the π-π term would help to bring the theory into closer agreement with the data. Second, gaining an understanding of the interactions of the hadrons is of fundamental interest in physics. Since the π-π term depends directly on the π-π scattering lengths, it would be useful if this calculation could be used to extract π-π scattering lengths from experimental pion-deuteron elastic scattering data.

Attempts to explain pion-deuteron elastic scattering in the backward direction for incident pion energies up to 300 MeV have met with a degree of success only in this decade.[1] This success is a result of using improved deuteron wave functions to calculate the single and double scattering contributions (Figs. 1a and 1b). However, there is still a discrepancy between the theory and the experimental data. We have examined the contribution of the π-π term (Fig. 1c) in an attempt to close this gap. Our motivation for doing this was a calculation by Vickson[2] in which he had estimated that the π-π term for zero-energy forward scattering and on-shell pions is about the same size as the single

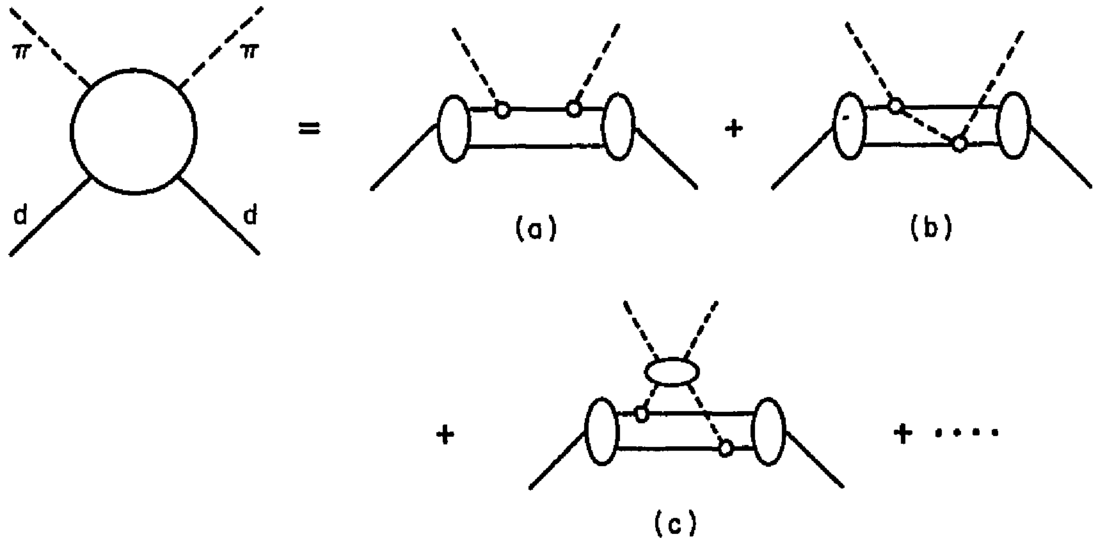


Fig. 1. The a) single scattering, b) double scattering, and c) π-π scattering terms of the multi-scattering series for pion-deuteron elastic scattering.

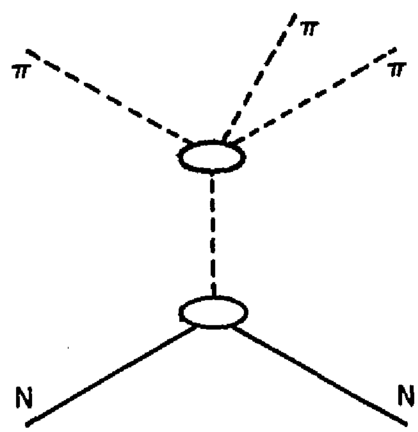


Fig. 2. Diagram for the Chew-Low reaction $\pi + N \rightarrow 2\pi + N$.

scattering term. Unfortunately this was a gross overestimate of the π-π term, mainly because of Vickson's poor choice of a deuteron wave function.[3] In this work we found that the π-π term does not contribute significantly to forward scattering, but that it does contribute to backward scattering. For the larger incident pion energies and backward scattering, the single scattering term is quite small because it depends on the deuteron overlap integral (break-up function) which falls off rapidly with increasing momentum transfer. Qualitatively, we can view the situation as follows: in single scattering some additional momentum is imparted to only one of the nucleons, and the greater the momentum difference between the nucleons, the less likely they are to remain bound together as a deuteron. The momentum transfered to the nucleon increases with increasing incident pion energy, and with larger scattering angles, suppressing the single scattering.

In double scattering, the momentum transfer is shared between the two nucleons, so the break-up function does not fall so rapidly with increasing pion energy. However, the interaction is mostly p-wave, so that the double scattering amplitude is largest for scattering angles near 90°. This means that the single and double scattering are both suppressed in the backward direction for larger incident pion energies, and that the π-π term will be most likely to give an appreciable contribution in this region. Indeed, we have found that the π-π term gives the greatest contribution for the backward scattering of pions with incident energies > 200 MeV.

There are no direct means of measuring the π-π scattering lengths, because there are no pion targets or existing pion colliding beams. Therefore, it is necessary to rely on indirect experiments to measure the π-π scattering lengths. Chew and Low[4] proposed that the π-π scattering lengths could be determined from $\pi + N \rightarrow 2\pi + N$ cross sections (Fig. 2) suitably extrapolated from the physical region to the π exchange pole ($t = m_\pi^2$). There are problems with such an extrapolation because it is not clear to what extent other exchange mechanisms (besides the π exchange) contribute to the scattering. At present numerous authors[5] have attempted this extrapolation, and have determined values for π-π phase shifts and scattering lengths.

The π-π scattering lengths have also been extracted from an analysis of the decay $K^\pm \rightarrow \pi^+ + \pi^- + e^\pm + \nu$ (K_{e4}).[6] The decay amplitude is expanded in terms of the π-π phase shifts, the phase shift difference $\delta_s - \delta_p$ appearing in the s-wave, p-wave interference terms. In a region where δ_p can be neglected, the scattering length a_0 has also been determined.

In this work we have explored the possibility of using pion-deuteron elastic scattering data to obtain a value of the π-π scattering length a_0. To do this it was necessary to have a form for the π-π interaction. Following Weinberg[7] the amplitude was expanded to second order in the momenta. This resulted in three expansion coefficients which had to be determined. Weinberg used current algebra arguments to impose three conditions on the π-π amplitude, thus determining both a_0 and a_2.

Ideally, the most general form of the amplitude should have been used in the calculation of pion-deuteron elastic scattering. But this would have required a three parameter fit to the data to obtain values for a_0 and a_2, an impracticality for us because of the disparity between the theory and the data. Therefore, two of Weinberg's conditions were imposed on the π-π amplitude, allowing a one parameter fit to the data to be made. In this manner a value of the scattering length a_0 is tentatively extracted from the data.

In Sec. II an outline of the calculation of the π-π scattering term is presented. The result of this calculation is compared to the data in Sec. III, and our value of a_0 is compared with Weinberg's result, and with the values found from the other experiments. Some estimates of other processes which might contribute to pion-deuteron scattering are also given.

II. CALCULATION OF THE π-π SCATTERING TERM

In this section a detailed calculation of the π-π current term for pion-deuteron elastic scattering will be given. Figure 3 is a diagram of the π-π term showing the kinematics in the initial deuteron rest frame, which we will call the lab frame.

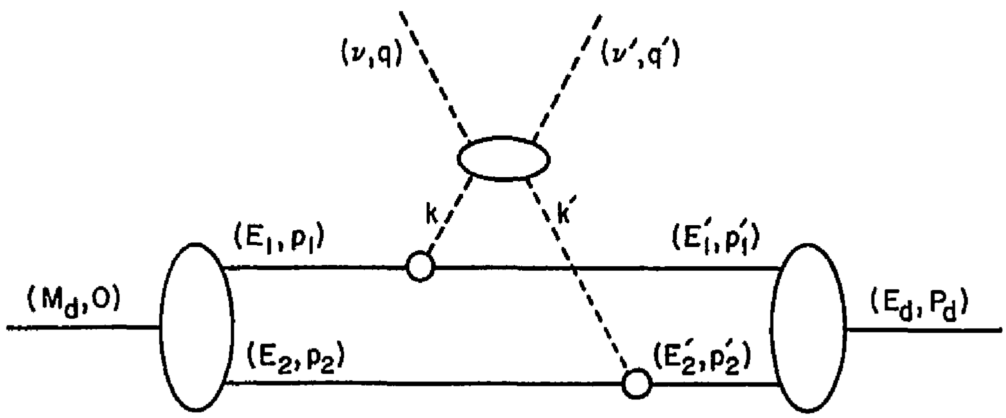


Fig. 3. Kinematics for the π-π term in the lab.

In order to evaluate this diagram we must be able to write expressions for the "bubbles" in Fig. 3. Before doing that, we make several preliminary observations. First, note that we do not consider the graph in which both virtual pion legs couple to the same nucleon. That graph is already included in the single scattering term via measured π-N phase shifts. Also, to facilitate the calculation, we will be using a form of the impulse approximation. The nucleons will be treated as free particles during the scattering, and the amplitude will be written in the limit of non-relativistic nucleons. These approximations are in keeping with the approximations made by Carlson[1] in calculating the single and double scattering amplitudes. The "bubbles" will be discussed in the following order. First, we will introduce a form for the pion-nucleon coupling. This coupling did not appear in the calculation of the single and double scattering because it was possible to use measured π-N phase shifts there. That cannot be done here because no two of the four pion legs couples to the same nucleon. Next, the π-π scattering "bubble" is discussed, and it is shown that the internal pion legs must be treated as off-shell. Finally, we indicate the form of the deuteron wave function.

A pseudoscalar coupling will be used for the πNN interaction, given by the following Hamiltonian density,

$$H_{\pi NN} = i g_0 \; \bar{\psi}_N \, \gamma_5 \, \boldsymbol{\tau} \cdot \boldsymbol{\varphi} \; \psi_N$$

where ψ_N is the nucleon field, and $\boldsymbol{\varphi}$ is the field of the pion. It should be noted that a pseudovector coupling could also have been used here, because in the non-relativistic limit the pseudovector coupling gives the same result as the pseudoscalar coupling. Of course, the actual π-N interaction is not described by such an ideal point coupling, and the Hamiltonian density $H_{\pi NN}$ should be multiplied by some form factor. In analogy to electromagnetism, the form factor will be taken to be $(1 - k^2/M_f^2)^{-2}$ where k^2 is the four-momentum squared of the pion, and M_f is the form factor mass. The pion-nucleon interaction is not mediated by the exchange of a single intermediate particle, but rather, there are numerous particles which can contribute to the form factor. However, these exchanges can be represented by the exchange of a single phenomenological particle, as in Fig. 4. The form factor mass has been estimated to be about five pion masses.[7]

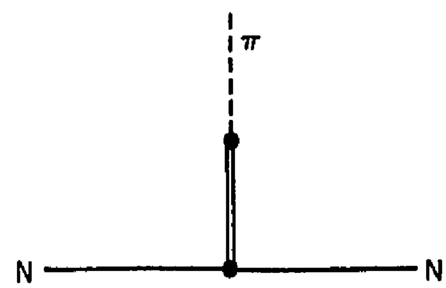


Fig. 4. The pion-nucleon form factor represented by the exchange of a single phenomenological particle.

This calculation is not sensitive to the exact value of M_f, as long as it is on the order of GeV or so.

Next, a choice must be made for the π-π scattering amplitude. It is instructive to first estimate how far the two virtual pion legs are off shell. From Fig. 3 it is seen that

$$q' - q = k - k'$$

For backward scattering and 300 MeV incident pions a rough estimate gives $k' - k \simeq 400$ MeV. If this is divided between the two pion legs, the virtual pions are a few pion masses off-shell. Thus, it is unrealistic to make any on-shell approximations for the virtual pions. Instead, Weinberg's form of the off-shell π-π scattering amplitude will be used to describe the scattering of real and virtual pions.

Invoking crossing symmetry, isospin conservation, and Bose statistics, Weinberg[8] found that the expansion for the π-π scattering amplitude to second order in the momenta is

$$M(\pi\pi \to \pi\pi) = \delta_{ab}\,\delta_{cd}\,[A + B(s+u) + C\,t + \cdots]$$
$$+ \delta_{ad}\,\delta_{cb}\,[A + B(s+t) + C\,u + \cdots] \qquad (1)$$
$$+ \delta_{ac}\,\delta_{bd}\,[A + B(u+t) + C\,s + \cdots]$$

where s, t, u are Mandelstan variables for π-π scattering, and A, B, C are some constant coefficients which are related to the scattering lengths as follows;

$$a_0 \cong -\frac{1}{32\pi m_\pi}\left[5A + 8m_\pi^2 B + 12 m_\pi^2 C\right]$$

$$a_2 \cong -\frac{1}{32\pi m_\pi}\left[2A + 8m_\pi^2 B\right].$$

The subscripts a, b, c, d are the pion isospin indices indicated in Fig. 5.

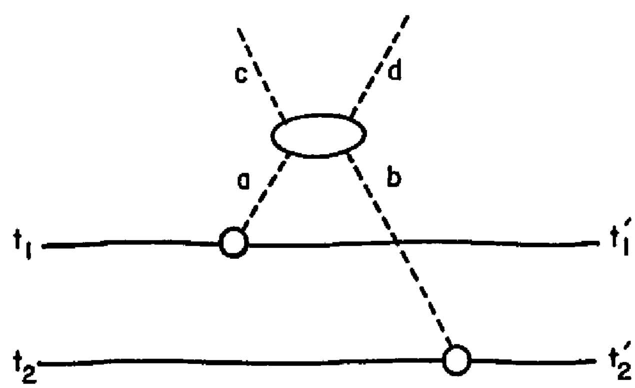


Fig. 5. Isospin indices for the π-π term.

Weinberg now imposes several conditions on the matrix element. First, he is able to calculate the matrix element using PCAC and current algebra techniques in the limit of vanishing pion momenta. By comparing the matrix element thus calculated with Eq. (1), Weinberg obtains the condition on the expansion coefficients,

$$B - C = \left(\frac{g_v}{f_\pi}\right)^2 \tag{2}$$

where f_π [9] is the pion decay constant. Then Weinberg imposes the Adler self-consistency argument[10] which shows that the matrix element vanishes when any one of the four pion momenta vanish and the other three are on the mass shell. That is, M vanishes when $s = t = u = m_\pi^2$. This yields the relation,

$$A = - m_\pi^2 \left(2B + C \right) . \tag{3}$$

A final relation among the coefficients is obtained using the fact that commutators of pion fields with $\partial_\mu A^\mu$ are isoscalar. This result is true for both the σ-model and the free-quark model, and implies that the matrix element is proportional to a certain δ-function. From this the final relation is derived,

$$A = - m_\pi^2 \left(B + C \right) . \tag{4}$$

Thus, Weinberg has three relations for the three expansion coefficients A, B, and C, and he obtains the values for the scattering lengths:

$$a_0 = 0.20 \; m_\pi^{-1}$$

$$a_2 = - 0.06 \; m_\pi^{-1}$$

It would be best if A, B, and C could be individually fit to the experimental data, but this is probably asking too much for a first try. Instead we shall try to keep two of Weinberg's relations, and then write the π-π scattering as a function of one remaining parameter which we take to be a_0. Arguing that the first of the assumptions, Eq. (2), is the least reliable because of the mass extrapolation, we do not include it in our derivation of the matrix element. Imposing the two conditions, Eqs. (3) and (4), the low energy π-π matrix is

$$M(\pi\pi\to\pi\pi) = \left(\frac{32\pi}{7} a_0 m_\pi^{-1}\right)\left\{(m_\pi^2 - t)\,\delta_{ab}\,\delta_{cd} + (m_\pi^2 - u)\,\delta_{ad}\,\delta_{bc} + (m_\pi^2 - s)\,\delta_{ac}\,\delta_{bd}\right\} \tag{5}$$

Note that a_0 itself has units m_π^{-1} so that M is dimensionless. This differs from Weinberg's result by the replacement

$$\left(\frac{32\pi}{7} a_0 m_\pi^{-1}\right) \longleftrightarrow \left(\frac{g_v}{f_\pi}\right)^2$$

Finally, a choice must be made for the deuteron wave function. In coordinate space the deuteron wave function is

$$\Phi_m(\vec{x}) = N\left[\frac{u(r)}{r} Y_0^0(\Omega)\,\chi^m + \frac{w(r)}{r}\sum_n Y_2^{m-n}(\Omega)\,\chi^n\,C_{m-n,n}\right] \tag{6}$$

where χ^m is a combination of Pauli spinors with total spin 1 and projection m, and $C_{m-n,n}$ is the C-G coefficient

$$C_{m-n,n} \equiv \langle 2, m-n\,;\, 1, n \mid 2, 1\,;\, 1, m\rangle\ .$$

The normalization constant N is unity provided

$$\int_0^\infty [u^2(r) + w^2(r)]\, dr = 1 .$$

In this paper Moravcsik's[11] best approximation to the Gartenhause wave function will be used for u(r) and w(r). The momentum space wave function is the Fourier transform of Eq. (6),

$$\Phi_m(\vec{p}) = (2\pi)^{-3/2} \int d^3x \;\; \Phi_m(\vec{x})\, e^{-i\vec{p}\cdot\vec{x}} \quad .$$

Now, the M-matrix for the π-π current term of pion-deuteron elastic scattering is

$$M(\pi d \to \pi d) = -\int d^3k \;\; d^3p \;\; \Phi^*_{m'}[\vec{p} - \vec{k} + \tfrac{1}{2}(\vec{q}\,' - \vec{q})] \;\; \Phi_m(\vec{p})$$

$$\cdot \frac{\sqrt{4 M_d E_d}}{(2\pi)^3 \sqrt{16 E_{p_1} E_{p_2} E'_{p_1} E'_{p_2}}} \quad M(\pi NN \to \pi NN) \tag{7}$$

where the various momenta are defined above and in Fig. 3. It is straightforward to write M(πNN → πNN) from $H_{\pi NN}$ and obtain

$$M(\pi NN \to \pi NN) = g_0^2 \;\; \bar{u}_{t_2' s_2'}(\vec{p}_2\,')\, \gamma_5\, \tau_b \;\; u_{t_2 s_2}(\vec{p}_2)\, \frac{f(k'^2)}{k'^2 - m_\pi^2}\, \varphi^*_d \;\; M(\pi\pi \to \pi\pi)$$

$$\cdot\, \varphi_c \; \frac{f(k^2)}{k^2 - m_\pi^2} \;\; \bar{u}_{t_1' s_1'}(\vec{p}_1\,')\, \gamma_5\, \tau_a \;\; u_{t_1 s_1}(\vec{p}_1) \tag{8}$$

where $f(k^2) = (1 - k^2/M_f^2)^{-2}$ is the form factor discussed earlier, s_i and t_i are nucleon spin and isospin indices respectively, and the π-π matrix element M($\pi\pi \to \pi\pi$) is given by Eq. (5). For ease in writing the following, the isospin parts of the matrix element are combined as follows:

$$I_t \equiv u^{\dagger}_{t_2'} \tau_b u_{t_2} \varphi^*_d \varphi_c u^{\dagger}_{t_1'} \tau_a u_{t_1} \delta_{ab} \delta_{cd} \tag{9}$$

I_u and I_s are similarly defined. For the case of π^+-deuteron scattering these factors are $I_t = 3$, $I_u = 2$, and $I_s = 2$.

The approximation of non-relativistic nucleons requires that substitutions of the following type be made:

$$E_p \approx M$$

$$\bar{u}_{s'}(\vec{p}\,')\gamma_5 u_s(\vec{p}) \approx u^{\dagger}_{s'} \vec{\sigma}\cdot(\vec{p}\,'-\vec{p})\, u_s \quad \text{where } u_{1/2} = \binom{1}{0}, \text{ etc.} \tag{10}$$

$$k^2 - m_\pi^2 \approx -\vec{k}^{\,2} - m_\pi^2 = -\omega^2(k)$$

$$m_\pi^2 - t \approx m_\pi^2 + \vec{k}^{\,2} + \vec{k}'^{\,2} - 2\vec{k}\cdot\vec{k}'$$

Combining Eqs. (7) through (10), the following π-π contribution to the pion-deuteron matrix element is obtained,

$$M(\pi d \to \pi d) = g_0^2 \frac{\sqrt{4M_d E_d}}{(2\pi)^3\, 4M^2} \left(\frac{32\pi}{7} a_0 m_\pi^{-1}\right) \int d^3k\, F_{m'm}(\kappa) \frac{-(\vec{\sigma}\cdot\vec{k}')_{s_2's_2}(\vec{\sigma}\cdot\vec{k})_{s_1's_1}}{\omega^2(k')\,\omega^2(k)}$$

$$\cdot\left\{(m_\pi^2 + \vec{k}^2 + \vec{k}'^2 - 2\vec{k}\cdot\vec{k}')\, I_t + (-2\vec{q}\cdot\vec{k}' + \vec{k}'^2)\, I_u + (2\vec{q}\cdot\vec{k} + \vec{k}^2)\, I_s\right\}$$

where[12] $F_{m'm}(\kappa) \equiv \int d^3p\, \phi^*_{m'}(\vec{p} + \vec{\kappa})\, \phi_m(\vec{p})$ is the overlap between initial and final deuterons, and $\vec{\kappa} \equiv -\vec{k} + 1/2(\vec{q}\,' - \vec{q})$. The d^3k integration is most readily done by first making the change to the variable

$$\vec{\kappa} \equiv -\vec{k} + \vec{Q} \qquad \text{where} \qquad \vec{Q} = \tfrac{1}{2}(\vec{q}\,' - \vec{q})$$

and then quantizing the spin along the direction of $\vec{Q}$. Then $(\vec{\sigma}\cdot\vec{k})_{s_2' s_2}$ $(\vec{\sigma}\cdot\vec{k})_{s_1' s_1}$ is written as a function of κ, Q, and θ where θ is the angle between $\vec{\kappa}$ and $\vec{Q}$. The angular κ integration is then performed using standard techniques, the only non-zero contributions being:

$$\Omega_{-1-1}(\kappa) = \Omega_{+1+1}(\kappa) = -2\pi \int_{-1}^{+1} d\xi \quad \frac{\kappa^2\xi^2 - Q^2}{4\kappa^2Q^2\xi - (\kappa^2 + Q^2 + m_\pi^2)^2}$$

and

$$\Omega_{00}(\kappa) = 2\pi \int_{-1}^{+1} d\xi \quad \frac{2\kappa^2\xi^2 - (\kappa^2 + Q^2)}{4\kappa^2Q^2\xi^2 - (\kappa^2 + Q^2 + m_\pi^2)^2}$$

where $\xi \equiv \cos\theta$. This yields

$$M(\pi d \to \pi d) = g_0^2 \, \frac{\sqrt{4M_d E_d}}{(2\pi)^3\, 4M^2} \left(\frac{32\,\pi}{7}\, a_0 m_\pi^{-1}\right) \int_0^\infty \kappa^2 d\kappa \quad F_{m'm}(\kappa) \quad \Omega_{m'm}(\kappa)$$
$$\cdot \left\{ (2\,q\cdot\vec{Q} + \kappa^2 + Q^2)(I_s + I_u) + (m_\pi^2 + 4Q^2)\, I_t \right\} . \qquad (11)$$

This π-π contribution is added to the single and double scattering terms to give the total pion-deuteron matrix element,

$$M_{m'm} = M_{m'm}^{(\text{SINGLE})} + M_{m'm}^{(\text{DOUBLE})} + M_{m'm}^{(\pi\text{-}\pi)}$$

In terms of the M-matrix, the differential cross section for pion-deuteron elastic scattering is

$$\frac{d\sigma}{d\Omega} = R \cdot \frac{1}{3} \sum_{m',m} |M_{m'm}|^2$$

where the kinematic factors have been lumped into R.

$$R = \frac{1}{64\,\pi^2 M_d} \cdot \frac{q'}{q} \left[(\nu + M_d) - \frac{q\,\nu'}{q'} \cos\,\theta \right]^{-1}$$

III. RESULTS AND COMMENTS

The π-π current contribution to the differential cross section for the reaction $\pi^+ + d \rightarrow \pi^+ + d$ has been calculated. For incident pion energies up to about 150 MeV, $M^{(\pi\pi)}$ is small compared to the single scattering term. Therefore, cross sections for these energies are not displayed here because they are unchanged from ones given earlier.

For pion-deuteron scattering at 256 MeV, the theoretical cross section, including only the single and double scattering terms, is about twice as large as the experimental[13] cross section in the forward direction, as can be seen in Fig. 6. As mentioned in the introduction, the inclusion of $M^{(\pi\pi)}$ cannot improve the theory in the forward direction, so that the discrepancy there between theory and experiment is due to the inability to calculate the single and double scattering.

What about the backward direction where the π-π term can contribute? The theoretical single plus double scattering cross section is about eight times larger than the experiment. It is unrealistic to suppose that the addition of the π-π term could significantly improve the situation. We will simply try to minimize the backward cross section by adjusting the parameter a_0. It was found that for the best value of the scattering length, $a_0 = 0.15\ m_\pi^{-1}$, the inclusion of the π-π term decreases the cross section by only about 30%. In Fig. 6 the cross section for two values of a_0 is compared with the data. The solid line includes only

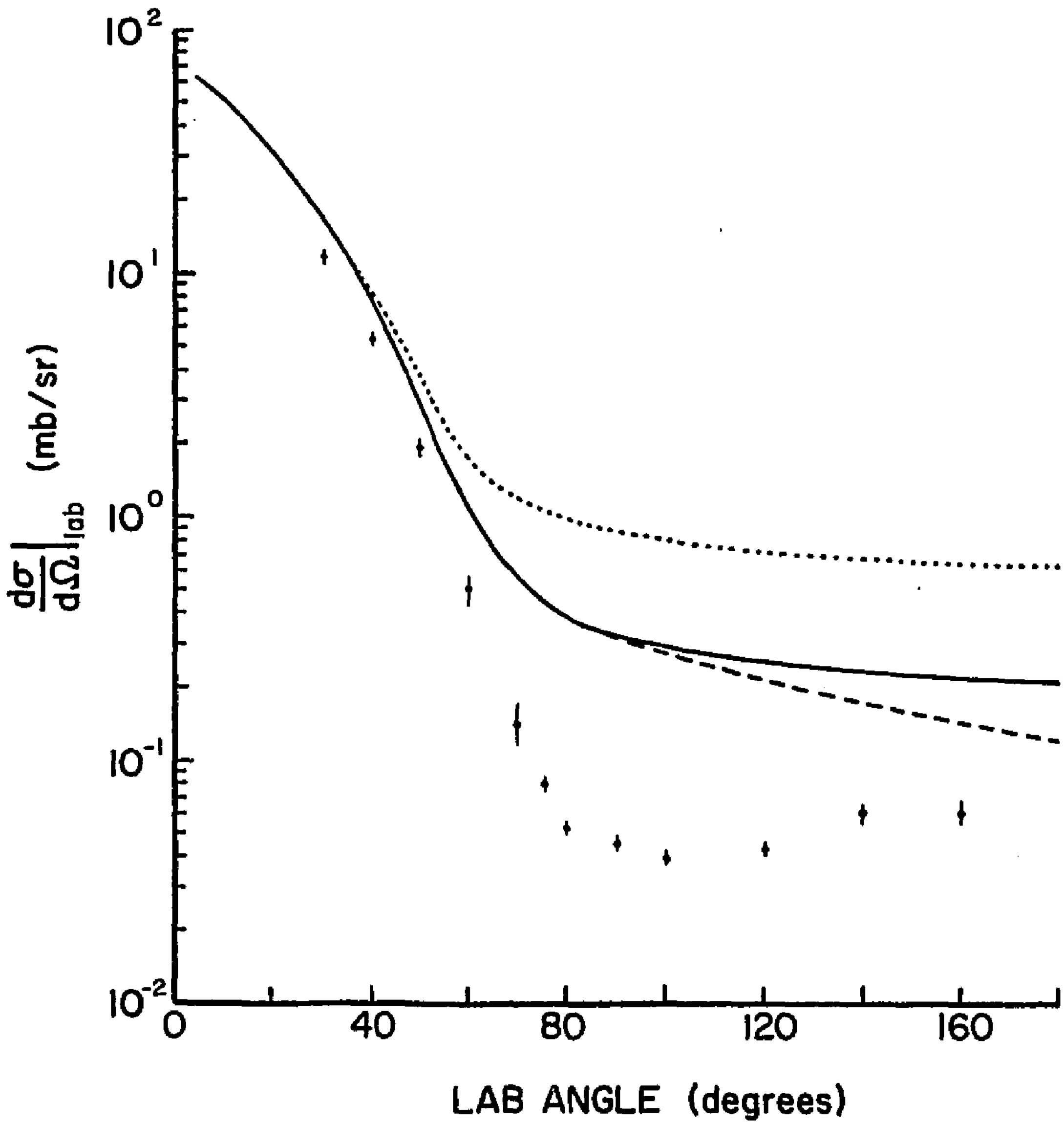


Fig. 6. π^+-deuteron elastic cross section for 256 MeV incident pions. The solid line includes only the single and double scattering, while the dashed line includes the π-π term with a value of $a_0 = 0.15\ m_\pi^{-1}$, and the dotted line includes the π-π term with a value of $a_0 = 0.6\ m_\pi^{-1}$.

the single and double scattering, while the dashed line includes the π-π term with a value of $a_0 = 0.15\ m_\pi^{-1}$, and the dotted line includes the π-π term with a value of $a_0 = 0.6\ m_\pi^{-1}$. Realizing the restrictions on our value of a_0, it can be compared the the value[6] $a_0 = 0.17 \pm 0.13\ m_\pi^{-1}$ from the K_{e4} decay and the value[5] $a_0 = 0.67$ from the π-N scattering experiment.

There may also be some question of whether contributions from an N^* might be larger than the contribution of the π-π current term. The presence of an N^* would alter the deuteron wave function,[14]

$$\psi_S + \psi_D \rightarrow \psi_S + \psi_D + \psi_{D^*}$$

where $|\psi_{D^*}|^2$ is perhaps as much as 1.5% of the total. This would alter the deuteron overlap function, but for low momentum transfers ($< 1\ \text{fm}^{-1}$) the change would not be significant. To get some idea of how small the change is, consider the effect of including the ordinary D-state in the overlap function. Using Moravscik's wave functions, the overlap for S-state along and S- plus D-states together are plotted vs. momentum transfer in Fig. 7. Adding the D-state does not change the overlap function by much, and adding the D^*-state would change it even less. Such a small change would only slightly alter the single scattering amplitude, the term most sensitive to the value of the overlap function. And in any case the π-π current term could still be added to the modified (including the N^*) single scattering term, lowering the backward cross section by about the same amount.

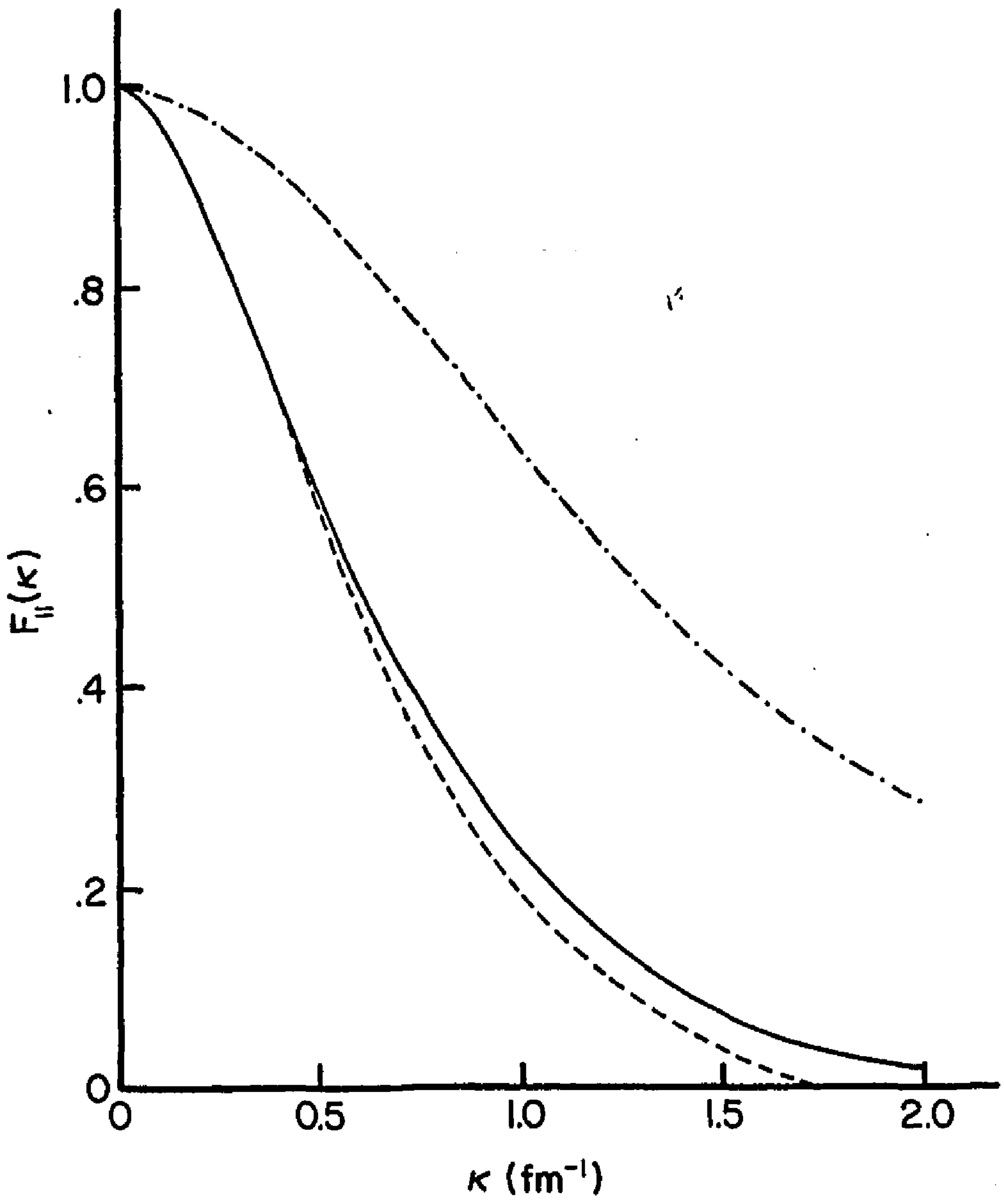


Fig. 7. The overlap F (κ) as a function of the momentum transfer κ for i) Moravcsik S-state wave function only (solid line), ii) Moravcsik S plus D-state wave function (dashed line), and iii) Ernst and Flugge S-state wave function (dash-dot line).

For comparison, the overlap function using the gaussian wave function of Ernst and Flugge[15] is also plotted in Fig. 7. From this it is obvious that the greatest improvement in the calculation of single scattering is due to using improved dueteron wave functions, not including factors like the N^*.

Contributions of other terms must be examined. The next lowest mass particle allowed is the ρ, as in Fig. 8.

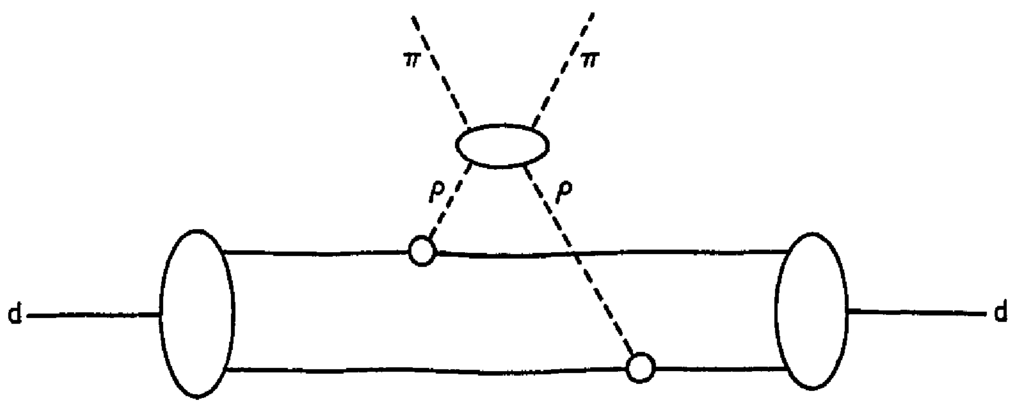


Fig. 8. π-ρ current term of pion-deuteron elastic scattering.

For an incident pion kinetic energy of 256 MeV, the maximum value of Q is 370 MeV. Also,at this energy the major contribution to the integral in Eq. (11) occurs at $\kappa \approx 3\ \mathrm{fm}^{-1} \approx 580$ MeV. Then the ratio of the π-ρ term to the π-π term is approximately

$$\frac{M^{(\pi-\rho)}}{M^{(\pi-\pi)}} \approx \frac{(k^2 + m_\pi^2)(k'^2 + m_\pi^2)}{(k^2 + m_\rho^2)(k'^2 + m_\rho^2)} \approx \frac{1}{18}$$

It is entirely within reason to neglect the π-ρ current term.

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$$\langle 0|\partial_\mu A^\mu(0)|\pi_{qb}\rangle = f_\pi\, m_\pi^2\, \delta_{ab}\, (2/q_0)^{1/2}(2\pi)^{-3/2}.$$

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