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Smith, Charles William, III

THE STRUCTURE OF AXISYMMETRIC TURBULENCE

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THE STRUCTURE OF AXISYMMETRIC TURBULENCE

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In Partial Fulfillment

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Doctor of Philosophy

by

Charles William Smith

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APPROVAL SHEET

This dissertation is submitted in partial fulfillment of the requirements for the degree of

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Charles William Smith

Charles William Smith

Approved, July 10, 1981

David Montgomery

David C. Montgomery, Chairman

Frederic R. Crownfield

Frederic R. Crownfield

George M. Vahala

Hans C. von Baeyer

Hans C. von Baeyer

Thomas A. Zang

Thomas A. Zang
Department of Math

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Dedication

I am very grateful to David Montgomery for his patience, instruction, and hours of discussion and for suggesting this topic. For much the same reasons, I am grateful to George Vahala, Fred Crownfield, and Bill Matthaeus. I am indebted to all those who have instructed me through the years, most particularly the faculty at William and Mary. And I am indebted to my friends who have made my stay here a good one. This work, meager as it is, is dedicated to the members of my family, all of whom have shown me more patience than I deserve.

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ABSTRACT

A wide range of laboratory and naturally occuring plasmas are frequently attributed a fluid description and, as such, demonstrate turbulent flows. We will investigate a variety of forms which may be taken by the correlation functions of these turbulent flows. The most commonly discussed isotropic symmetry is not generally anisotropic. This dissertation will develop an axisymmetric description from which the magnetic helicity may be extracted together with its spectrum. This description will be compared to the form taken by axisymmetric, helical Navier Stokes turbulence which will also be derived here. The microscales for this geometry will be tabulated and for completeness, the von Karman-Howarth equations will be derived.

THE STRUCTURE OF AXISYMMETRIC TURBULENCE

I. INTRODUCTION

This dissertation is concerned with the turbulent flow of a large class of fluids. We will concentrate primarily on magnetohydrodynamic fluids although Navier-Stokes fluids will occasionally be discussed in contrast with the MHD results. Flows which display a fundamental axisymmetry in their statistical description will be our primary concern although isotropic systems will be reviewed and one result independent of rotational symmetry will be given which allows for the measurement of the magnetic helicity. It will be argued in Chapter I that the axisymmetric description is more appropriate to a wide variety of plasma systems than is the more familiar and simpler isotropic description.

Also in Chapter I it will be argued that turbulence represents a random outcome of the flow. For this reason, some basic results of random function theory will be discussed in Chapter II. From these basic ideas, a reasonable definition for the ensemble average will be given on which to base arguments of continuity and differentiability and the Schwartz inequality will be used to restrict the correlation functions.

In Chapter III, we will look at the correlation functions which describe the ensemble average of the mean value of the product of two fields at two different points in space and time. We will concentrate on systems which demonstrate axisymmetry in this correlation and will construct the most general form for such a correlation. This form will be examined as one passes to the limit of the more familiar isotropic symmetry. By comparing this real space description with the similarly constructed Fourier space description, we will arrive at an exact method for measuring the turbulent magnetic helicity. The von Karman-Howarth equations for this system will be derived and a minimum scalar representation will be given. The slab limit which has been employed in solar wind discussions and the two-dimensional limit which is relevant to discussions of many confinement devices will also be discussed. First, a general introduction to magnetohydrodynamics is in order.

One effort at a statistical treatment of the bulk motion of a highly collisional collection of particles is to construct a fluid theory. A fluid theory, by definition and construction, assumes that volume elements of many particles may be formed and that these volumes maintain a degree of integrity and persist through time. Such a theory cannot deal with structure that is not large compared to the inter-particle spacing and it cannot deal with time scales which are not large compared with the collision times. It is expected that such a description, while representing a great simplification of the particle description, should still contain a great deal of structure and should still be applicable to a wide variety of phenomena.

The magnetohydrodynamic approximation is one such fluid theory.[1-5] It is constructed to be a fluid theory describing the large scale motion of a highly collisional collection of charged particles. The MHD equations for an incompressible fluid are

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla P + \frac{1}{\rho c}\mathbf{j}\otimes\mathbf{B} + \nu\nabla^2\mathbf{v} \,, \tag{1a}$$

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla\otimes(\mathbf{v}\otimes\mathbf{B}) + \frac{c^2}{4\pi\sigma}\nabla^2\mathbf{B} \,, \tag{1b}$$

$$\nabla\otimes\mathbf{B} = \frac{4\pi}{c}\mathbf{j} \,, \tag{1c}$$

$$\nabla\cdot\mathbf{B} = 0 \,, \tag{1d}$$

and

$$\nabla\cdot\mathbf{v} = 0 \,. \tag{1e}$$

where $\mathbf{v}$ is the fluid velocity, $\mathbf{B}$ is the magnetic field, ρ the mass density, P the pressure, $\mathbf{j}$ the current density, σ the conductivity, c the speed of light in vacuum, ν the kinematic viscosity, and $\frac{c^2}{4\pi\sigma} \equiv \lambda$ the magnetic diffusivity.

These equations are non-linear and this fact presents the principal difficulty in their solution. Let us first consider the case where the magnetic field is everywhere zero for all time. This yields the Navier-Stokes equation[6-9]

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{v} \,. \tag{2}$$

The last term in the above equation, the dissipative term, acts to damp away perturbations in the flow. It has its greatest influence on the small structure of the flow, but also effects large scale features of laminar flow by imposing no-slip boundary conditions. It is, quite generally, a term which acts to eliminate the smallest features of the flow and leaves behind the large scale features which are more easily discussed.

The term $\underset{\sim}{v}\cdot\nabla\underset{\sim}{v}$ in the convective derivative acts to create the small scale structure and increase the complexity of the flow. This makes for an interesting opposition and the degree of complexity which the flow may attain is determined by the relative importance of these two terms.[10-14] In order to estimate this, we can define V_0 to be a "typical" velocity of the flow and L_0 a "typical" length over which the velocity varies. We can then define a Reynold's number, R, by:

$$R = \frac{\underset{\sim}{v}\cdot\nabla\underset{\sim}{v}}{\nu\nabla^2\underset{\sim}{v}} \sim \frac{V_0 L_0}{\nu} . \tag{3}$$

This number can be inserted into the Navier-Stokes equation by scaling the equation using V_0, L_0 and the characteristic time scale, L_0/V_0, to give

$$\frac{\partial}{\partial t}\underset{\sim}{v}' + \underset{\sim}{v}'\cdot\nabla'\underset{\sim}{v}' = -\frac{1}{\rho'}\nabla' P' + \frac{1}{R}\nabla'^2\underset{\sim}{v}' . \tag{4}$$

We have used the definitions:

$$\underset{\sim}{v} = V_0\underset{\sim}{v}', \; \underset{\sim}{x} = L_0\underset{\sim}{x}', \; t = \frac{L_0}{V_0}t' ,$$

$$\nabla' \equiv \frac{\partial}{\partial \underset{\sim}{x}'} \text{ and } R \equiv V_0 L_0/\nu .$$

It is apparent from the definition of the Reynold's number, Eq. (3), and the manner in which it appears in the Navier-Stokes equation, Eq. (4), that as the Reynold's number becomes larger, the convective term grows in importance, adding complexity to the flow. Experimental and theoretical studies indicate that a fluid system becomes turbulent when the Reynold's number reaches 600 to a few thousand.[15-20] Several standard texts exist on the subject of fluid turbulence and a comparison of laminar and turbulent flow is in order.

Consider the properties that characterize laminar, or stable, fluid flow. For given initial conditions the fluid velocity at any one point and time may be predicted with reasonable certainty. If a perturbation is introduced, its effect is minimal and the fluid responds to damp the disturbance until it disappears leaving the fluid once again in the same relatively quiescent state. This implies that the fluid properties at point $\underset{\sim}{x}_1$ and time t_1 are well correlated with the properties at point $\underset{\sim}{x}_2$ and t_2 for all separations in space and time. In addition, since the flow is quiescent, the principal mechanism for diffusion of fluid inhomogeneities is molecular diffusion.

Turbulent flow is strikingly different. D. J. Tritton[21] defines turbulence as "a state of continuous instability." This allows the fluid flow to display unpredictable, chaotic behavior. The fluid properties at any one point cannot be predicted to any degree of accuracy, but vary randomly over a wide range. If a perturbation is introduced, it quickly loses its single identity by interacting with the flow, further distorting the flow so as not to allow it to return to its initial state. For this reason, correlation lengths in a turbulent flow are finite and typically less than the dimensions of the flow.[22] In addition, the variation in the flow acts to enhance diffusion, allowing mixing to occur on a time scale which is orders of magnitude faster than molecular diffusion.[23]

Turbulent fluid flow is normally considered to be dissipative. Non-dissipative flow is sometimes discussed in an effort to simplify discussions of the non-linear terms, but it is seldom treated as a physically realizable state. This means that the flow must constantly be forced so that energy may be supplied to the system. There are several ways to accomplish this. One is by the coupling of the turbulent fluctuations to the mean flow.[24] A second is by externally supplied random fluctuations. This second source exists in many forms. We must assume that our ability to construct a flow is limited. The tolerance of the machining of walls, operation of driving mechanisms and our ability to isolate the system must be limited and all of these limitations will act to perturb the fluid. If such a random disturbance remains small, it may be appropriate to ignore it all together. But, if it gains energy and begins to interact with other disturbances, the system will enter into "a state of continuous instability." Systems with these characteristics require a statistical treatment so that some degree of reproducibility may be established. This reproducibility is established theoretically by taking an ensemble average over a collection of similarly prepared flows. In practice, such a collection may be prohibitively expensive or technologically impossible and so time averages, spatial averages and wave-number averages are typically substituted based on assumptions of statistical near-stationarity, near-homogeneity, etc. This averaging process removes the intractable details of the flow and concentrates on the larger scale statistical properties of the random fluctuations which are to some degree controllable. The branch of mathematics which deals with such chaotic, irreproducible vector and scalar fields is known as random function theory[25] and some results are presented in Chapter II of this dissertation.

To this point, we have limited our discussions to neutral fluid flows. Using the MHD equations to include the magnetic field adds a great deal of complexity. To begin with, there are now two Reynold's numbers, the second being the magnetic Reynold's number, R_M, defined to be $V_0 L_0/\lambda$. The importance of this quantity may be seen by comparing the convective term in Eq. (1b) to the dissipative term in the same equation

$$R_M = \frac{\nabla \otimes (\underset{\sim}{v} \otimes \underset{\sim}{B})}{\lambda \nabla^2 \underset{\sim}{B}} \sim V_0 L_0/\lambda \ . \tag{5}$$

This shows than in analogy with the velocity field, large values of the magnetic Reynold's number should indicate a non-negligible contribution from the convective term which adds complexity and small scale structure to the magnetic field. Unlike Navier-Stokes flow where some simple flows have been examined to find the value of the Reynold's number for which turbulence sets in, it is not clear at what values of the magnetic and mechanical Reynold's numbers MHD turbulence will develop or what role the different Reynold's numbers will play in predicting the onset of MHD turbulence. We will postulate here that due to the similarity between the Navier-Stokes and MHD equations, the Reynold's numbers for transition to turbulence should be comparable.

All of this would be academic if no systems existed which seemed good candidates for turbulent MHD flow. In fact, many systems exist which are regularly described by the MHD approximation and which have been seen to display turbulent flows. We can use the Reynold's numbers of these systems as a first indication of what values of the

Reynold's numbers lead to turbulent flow. To do this, a brief description of these systems would seem to be in order. Therefore, the following describes some of the most widely studied plasmas such as the solar wind, and such laboratory plasma confinement devices as pinches and tokamaks.

The solar wind is composed of solar matter which is expelled from our sun. The ion component is 96% H^+ and 4% He^{++} by number density and the solar wind is charge neutral with an average density of 6 protons/cm^3 in the ecliptic plane at the Earth's orbit. At this radius, there is a mean flow speed of 4×10^7cm/sec which is radially directed and which forms a 45^o angle with the mean magnetic field of 5×10^{-5} Gauss. The radial streaming velocity is naturally a function of distance from the sun. At larger distances where the expansion velocity is less, there is less stretching of the mean magnetic field. This causes the mean magnetic field to take on an azimuthal direction further from the sun. Little is known of the solar wind outside the ecliptic plane. Within this plane, the typical proton temperature is 10^5 ^oK with the electron temperature running a factor of two higher. This implies an ion sound speed of $[\frac{5}{3} K(T_e + T_p)/m_p]^{1/2} = 5 \times 10^6$cm/sec and an Alfven speed of $\frac{B_0}{\sqrt{4\pi\rho}} =$ 4.4×10^6cm/sec. Since the streaming velocity is an order of magnitude larger than both the ion sound speed and the Alfven speed, sound waves and Alfven waves are convected away from the sun regardless of the propagation direction in the rest frame of the fluid. Due to this high streaming velocity, shocks are not uncommon and bow shocks surround the planets. Shocks are an effective source of turbulence. A second region where stirring may occur is the surface of the sun where the solar wind is initially expelled. This is not a well probed region, but observations of the sun suggest that it is highly turbulent and that it has regions of large shear in the magnetic field. One example of a high magnetic shear region is the collection of boundaries which are not confined to any single region of the sun, but cover the surface. It is thought that at some depth below the surface, the magnetic field returns to a more homogeneous form. This would seem to imply that most of the escaping material penetrates the surface at these regions of high magnetic flux density and large shear, to be returned to a homogeneous density as the mean magnetic field does likewise. A third source of turbulent mixing would be the sector boundaries. The solar wind may be divided azimuthally into four sectors of alternating direction of the mean magnetic field.[26] This implies the existence of a current sheet at the boundary. Such a structure is a strong source of turbulent flow. Sources aside, turbulent magnetic field fluctuations have been observed in the solar wind.[27,28]

The early z-pinch devices were linear systems with electrodes at either end. A current is passed through the plasma from one electrode to another causing the plasma column to collapse radially through the interaction of the current elements with the self consistent magnetic field.[29] Ohmic heating heats the plasma, but a rapid increase in the current produces rapid collapse of the column and the resulting shocks act to further heat the center of the column. Aside from its instability, the principal difficulty with this as a fusion device is the inability of the end electrodes to survive thermonuclear temperatures. This prompted the construction of toroidal devices which eliminated the need for magnetic mirrors to achieve confinement. The current is then driven by a segmented liner which induces a current to flow in the plasma.

Axial magnetic fields can be introduced by constructing solenoidal windings which carry azimuthal currents. This is the theta-pinch. Early theta-pinches were more effective in separating the plasma from the walls of the containment vessel. This led to the diffuse pinch where the axial magnetic field is comparable in magnitude to the a imuthal field. Zeta is a toroidal machine of this design.

Zeta was also observed to be extremely turbulent. The Zeta references given in Table I indicate that the turbulence within the Zeta machine could be reasonably well described by the axisymmetric description to be put forth in this dissertation. Although the analysis was not in the manner that might be characterized as standard within turbulence literature, involving frequency decomposition, it still indicates that the isotropic description is entirely inadequate. In addition, it was quite definitely observed to be turbulent and provides a reference point for the Reynold's number arguments given earlier.

The tokamak is a toroidal confinement device initially described by Artsimovich in the USSR. There are many excellent review articles which discuss the operation of the various tokamaks, since the tokamak represents the most extensively investigated fusion confinement device. The parameters of the various machines vary considerably, but the general design is common to all.[30,31] A strong toroidal (long way around the torus) magnetic field is provided by external poloidal (short way around) windings and this is the main source of confinement. A smaller poloidal field is provided by driving a toroidal current in the plasma. This current is produced by allowing the plasma to be the passive element of a large transformer through which a capacitor bank is discharged, making this a pulsed device. The principal means of heating the plasma is ohmic resistance which may be supplemented by microwave or neutral injection. This system is of particular interest to this dissertation due to the large current which is forced to flow along the magnetic field lines. This current is a source of magnetic helicity and as such, is discussed in Appendix IV. Tokamaks, like all confinement devices, are inherently inhomogeneous devices, but we may hope that average values of density, temperature, etc. might be appropriate in the description of global properties and the properties of regions far from the walls.

Tokamak turbulence has been discussed in the context of the Macrotor tokamak. Again, the analysis is not standard for a turbulence discussion but it indicates a nearly axisymmetric description. In addition, it provides a second reference point for the Reynold's number arguments.

In an attempt to suggest that turbulent flow should be widespread throughout many MHD flows and a variety of confinement devices, we have calculated the Reynold's numbers for a variety of systems and compare them to the well-documented turbulent flows of the solar wind, Macrotor tokamak and Culham Zeta pinch. There are three distinct characteristic velocities that may be used in constructing these numbers. In analogy with Poiseuille flow, we may choose a streaming fluid velocity.

For a toroidal device such as a tokamak, this may be a toroidal velocity or a poloidal rotation of the plasma column. Fluid velocity is due to the mean motion of the ions. Motion of the electrons relative to the ions constitutes current which induces a magnetic field which defines a self-consistent Alfven speed. This is also a velocity which might be used to generate the Reynold's numbers.

To accomplish this, a few quantities must be defined.[32] Let

$e \equiv 4.8 \times 10^{-10}$ esu,

$I_p \equiv$ plasma current,

$T_e \equiv$ temperature of the electrons in units of eV.,

$T_i \equiv$ temperature of the ions in units of eV.,

$\Lambda \equiv$ Coulomb logarithm,

$m_e \equiv 9.1 \times 10^{-28}$ grams,

$m_p \equiv 1.7 \times 10^{-24}$ grams,

$N \equiv$ mean electron density,

$Z \equiv$ mean proton number of the ions,

$K \equiv 1.4 \times 10^{-16}$ ergs/$^{\circ}$K,

$V_0 \equiv$ streaming velocity or toroidal velocity,

$V_r \equiv$ rotation velocity,

$V_A \equiv$ self-consistent Alfven speed,

$\tau_e \equiv$ electron-ion collision time in sec. $= \dfrac{(3.5 \times 10^4) T_e^{3/2}}{\Lambda Z N}$,

$\tau_i \equiv$ ion-ion collision time in sec. $= \dfrac{(3.0 \times 10^6) T_i^{3/2}}{\sqrt{2}\ \Lambda Z^3 N}$,

$\sigma_e \equiv$ electrical conductivity $= \dfrac{Ne^2}{m_e} \quad e$,

$\lambda \equiv$ magnetic diffusivity $= \dfrac{c^2}{4\pi\sigma_e}$,

$$[V_{rms}^{ions}]^2 = \frac{3k\mathcal{T}_i(11,600)}{m_p} ,$$

$$\nu \equiv [V_{rms}^{ion}]^2 \, \mathcal{T}_i/3 ,$$

and $P_m \equiv$ magnetic Prandtl number $= \nu/\lambda$.

Notice that as the temperature of the plasma increases, keeping all other parameters the same, the viscosity increases and the diffusivity decreases. This means that the magnetic Prandtl number increases with temperature suggesting that as confinement machines are pushed into the fusion regime, the ratio of the turbulent energy of the magnetic field to the turbulent energy of the velocity field should increase.

Table I

	$\frac{V_0 L_0}{\nu}$	$\frac{V_0 L_0}{\lambda}$	$\frac{V_0 L_0}{\nu}$	$\frac{V_r L_0}{\lambda}$	$\frac{V_A L_0}{\nu}$	$\frac{V_A L_0}{\lambda}$	P_m
(33) Solar Wind at 1AU	48	4×10^{35}					8.8×10^{12}
(34) Macrotor			1.1×10^{-2}	5.6×10^{3}	1.6×10^{-1}	8.4×10^{4}	5.0×10^{5}
(35) Zeta			.15	36.	2.6	640.	240.
(36) Terp					48.	3.8×10^{-2}	7.9×10^{-4}
(37) TFR (z=1)					3.8×10^{-3}	1.1×10^{5}	2.9×10^{8}
(37) TFR (z=10)					3.8	1.1×10^{4}	2.9×10^{3}
(37) TFR					2.2×10^{-5}	2.2×10^{7}	10^{12}
(38) TFR 600					4.3×10^{-3}	3.2×10^{3}	7.5×10^{5}
(39) Synchromak					.29	1.0×10^{4}	3.5×10^{4}
(40) Tosca						1.7×10^{5}	
(41) LT-3	1.1×10^{-5}	1.6×10^{2}	3.5×10^{-6}	52.	1.7×10^{-4}	2.5×10^{3}	1.5×10^{7}
(42) ACT-1							
(43) Alcator-A						1.2×10^{5}	
(44) FM-1 Spherator	5.0	3.5×10^{-2}					7.1×10^{-3}

As can be seen from Table I, the magnetic Reynold's number as calculated for Macrotor is typical if not small when compared to the magnetic Reynold's numbers of the other systems with the exception of the LT-3 tokamak, the FM-1 spherator and Terp. The magnetic Reynold's number for Zeta is exceptionally small when compared to the rest; and yet, it displays a highly-turbulent flow. The exception to this is again the Terp theta-pinch-like device. This would seem to imply that most of the systems tabulated should display turbulent flow, suggesting that the phenomenon is not uncommon.

As a last example in an effort to justify an investigation of turbulent MHD flow, the parameters for the first D-T tokamak test reactor, INTOR, have been suggested. This study suggests a minor radius of $a = 1.3$ meters, plasma current of $I_p = 6$ MA, toroidal magnetic field of $B_t = 55$ KG, mean number density of $n = 1.3 \times 10^{14} \mathrm{cm}^{-3}$, mean temperature of 10 keV and an energy confinement time of 1.3 seconds. These parameters imply a magnetic Reynold's number of $\frac{V_A a}{\lambda} = 2.8 \times 10^9$ and a mechanical Reynold's number of $\frac{V_A a}{\nu} = 3.0 \times 10^{-3}$. As was said earlier, it is not clear what role the different Reynold's numbers will play in predicting the onset of turbulence. But, comparison of this magnetic Reynold's number with the value given for the Macrotor tokamak, a known turbulent device of similar design, leads one to suspect that a fusion device such as INTOR would have to display a turbulent magnetic field. A turbulent magnetic field would be expected to stir the velocity field, although the small mechanical Reynold's number would lead to rapid dissipation of these fluctuations. The implications of this stirring followed by rapid dissipation are not entirely clear. It is doubtful that this could contribute significantly to the thermal energy of the fluid.

At this point, it would appear that turbulent MHD flow is a phenomena of widespread interest and applicability. We would like to discuss the axisymmetric turbulent description. To do this, it will be necessary to build on a few ideas basic to random function theory. These points will now be discussed.

II. RANDOM FUNCTION THEORY AND ISOTROPIC SYMMETRY

A. Implications of Random Function Theory

Probability theory as it is first introduced to students is based upon the outcome of a collection of observations of a random variable. As a typical example of a discrete variable, consider the roll of a die. The outcome of a single roll is considered to be a completely random event which results in an unpredictable element of the set of possible outcomes. Since the roll is in fact a causal process, the assumption of randomness represents ignorance concerning the preparation and execution. The probability distribution governing the random outcome may be approximated by a collection of similarly prepared rolls called an ensemble, the approximation improving as the size of the ensemble increases. We denote the probability distribution by $f^{(n)}$ where n is the number of events in the ensemble.

Probability theory for a continuous variable develops in an analogous manner. By the phrase "continuous variable," I mean a variable which may take on any value within a continuous range of values. Examples of such a variable arise often in the field of quality control where sacks or boxes must be filled to within a prescribed tolerance with cereal or grain, etc. The net weight of the container is a continuous variable, but the outcome of the packaging process is a single measurement: the weight of that one box or sack. If the continuous variable is represented by the symbol η and the ensemble generated probability distribution is represented by $f^{(n)}(\eta)$, then the outcome must be viewed by asking for the probability, P, that the result falls within a given range with

$$P(\eta_1 \leqslant \eta \leqslant \eta_2) = \int_{\eta_1}^{\eta_2} f^{(n)}(\eta)\, d\eta \tag{6}$$

Within the discussion of turbulent flows, an example of a continuous variable might be the flow velocity at a prechosen point in space and time. In this way, we would not be discussing the evolution of the flow, but simply a single value which the flow attains at some instant at some point.

Random function theory is the next natural generalization of the idea of a random variable.[25] In this case, the random outcome is not a single value, but a sequence of values. We may choose, for instance, to discuss the density as a function of time at some chosen point or we may choose a component of the magnetic field or the velocity field at that point. For turbulent flows, these would be random functions of time. The entire turbulent flow as a function of space and time is a random function of space and time. It is a random function in the sense that from one outcome to the next, from one execution of the experiment to the next, the flows are independent. The flow of a single execution at time t_1 is not independent from time t_2 and this allows us to correlate the flow over temporal separations. The same is true for spatial separations. This is the result of an underlying probability distribution governing the evolution of the system. Let us examine this point further.

For simplicity, we first consider that this random sequence is a function simply of time. In this way $\eta(t)$ represents a single event, the time dependent outcome of a measurement, and the ensemble is represented $\eta^{(1)}(t)$, $\eta^{(2)}(t),\ldots,$ $\eta^{(n)}(t)$. The individual outcomes, $\eta^{(i)}(t)$, are unlike each other in detail, but share common features reflecting the "similarly prepared" assumption. Causality of the system in question is represented by the time evolution of each $\eta^{(i)}(t)$. This causality which is assumed to be present may not allow extrapolation of $\eta^{(i)}(t)$ from times t_1 to t_2 since $\eta^{(i)}(t)$ may represent only a limited knowledge of the system. For instance, it may be a component of the turbulent field measured for all time but at only one position. It is therefore meaningful to discuss multiple time probability distributions such as $f_m = f_m(\eta_1,t_1;\eta_2,t_2;\ldots;\eta_m,t_m)$. For systems which are not causal, this reduces to $f_m = f_1(\eta_1,t_1)f_1(\eta_2,t_2)$ $\ldots f_1(\eta_m,t_m)$. In general, we are interested in these systems for turbulent purposes, but restrict ourselves to systems where the multiple time distributions become uncorrelated only for large enough temporal separations. The parameterization of the random variable, η, may be extended to include functions of position as well. This is labeled by $\eta(x,t)$ and forms the random field that is the basic entity of random function theory as it applies to turbulence.

Without assigning any particular choice of identity to the field η, it is possible to prove several general results. To do this, we must first note that it is the expectation value of a function which we will seek and not the probability distribution which governs its appearance.

Then, rather than discuss the probability governing the value of η at a point $\underset{\sim}{x}$ or the simultaneous value of η at points $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$, as before, it is more convenient to discuss the probability, $P[\eta(\underset{\sim}{x},t)]$ that $\eta(\underset{\sim}{x},t)$ take on a particular functional form. This is analogous to discussing not just the probability of the outcome of the roll of a die, but the probability of the sequence of values an observer may see as the die rolls to a stop.

Let us signify the ensemble average by brackets, $\langle\rangle$, and from this point on we will restrict our discussion to random variables for which the ensemble average is zero. Clearly, any random variable may be brought into this form by subtracting its average (which may be time dependent in the case of non-stationary processes or position dependent in the case of inhomogeneous processes).

Assuming that $\eta(\underset{\sim}{x},t)$ now satisfies this requirement, we may proceed to define the auto-correlation function by means of an integral over the functional space of $\eta(\underset{\sim}{x},t)$ in a manner that is analogous to Riemann integration over the real numbers.[45,46] Functional integration proceeds as follows. Let the space $(\underset{\sim}{x},t)$ be subdivided into volume elements $(\delta\underset{\sim}{x}^i, \delta t^i) \equiv \Delta\underset{\sim}{x}^i$ and let $\underset{\sim}{x}^i$ be a point within $\Delta\underset{\sim}{x}^i$. Furthermore, let $f_n(\eta(\underset{\sim}{x}^1), \eta(\underset{\sim}{x}^2), \ldots \eta(\underset{\sim}{x}^n))$ be a function which is constant in each $\Delta\underset{\sim}{x}^i$ and which approximates the function $f(\eta(\underset{\sim}{x},t))$ in each interval. Then, by integrating over incremental values of η within each interval $\Delta\underset{\sim}{x}^i$, we may write

$$I_n \equiv \int_{A_n}^{B_n}\cdots\int_{A_1}^{B_1} f_n(\eta(\underset{\sim}{x}^1), \eta(\underset{\sim}{x}^2), \ldots \eta(\underset{\sim}{x}^n))\, d\eta(x^1)\, d\eta(x^2)\, d\eta(x^3)\ldots d\eta(x^n)$$

and define

$$\int\cdots\int_D f[\eta(\underset{\sim}{x},t)]d[\eta(\underset{\sim}{x},t)] \equiv \lim_{n\to\infty} I_n$$

where D is a prescribed domain in functional space. In this manner, we may define the auto-correlation function to be

$$\langle\eta(\underset{\sim}{x}_1,t_1)\eta(\underset{\sim}{x}_2,t_2)\rangle \equiv \int d[\eta(\underset{\sim}{x},t)]\eta(\underset{\sim}{x}_1,t_1)\eta(\underset{\sim}{x}_2,t_2)P[\eta(\underset{\sim}{x},t)] \quad (7)$$

where $P[\eta(\underset{\sim}{x},t)]$ is the probability distribution governing the form of $\eta(\underset{\sim}{x},t)$. The region D is specified according to the properties of the system and the physical constraints which restrict the form which may be assumed by $\eta(\underset{\sim}{x},t)$. For our purposes, it will not be necessary to specify D.

This definition allows us to show that the ensemble average is a linear operator

$$\begin{aligned}&\langle\lambda_1 f_1[\eta(\underset{\sim}{x},t)] + \lambda_2 f_2[\eta(\underset{\sim}{x},t)]\rangle \\ &= \int d[\eta(\underset{\sim}{x},t)]\left\{\lambda_1 f_1[\eta(\underset{\sim}{x},t)] + \lambda_2 f_2[\eta(\underset{\sim}{x},t)]\right\} P[\eta(\underset{\sim}{x},t)] \\ &= \lambda_1\langle f_1[\eta(\underset{\sim}{x},t)]\rangle + \lambda_2\langle f_2[\eta(\underset{\sim}{x},t)]\rangle .\end{aligned} \quad (8)$$

We may then expand $\left\langle\left\{\eta(\underset{\sim}{x}_1,t) \pm \eta(\underset{\sim}{x}_2,t_2)\right\}^2\right\rangle \geq 0$ to write

$$\begin{aligned}&\langle\eta(\underset{\sim}{x}_1,t_1)\eta(\underset{\sim}{x}_1,t_1)\rangle + \langle\eta(\underset{\sim}{x}_2,t_2)\eta(\underset{\sim}{x}_2,t_2)\rangle \\ &\pm 2\langle\eta(\underset{\sim}{x}_1,t_1)\eta(\underset{\sim}{x}_2,t_2)\rangle \geq 0.\end{aligned} \quad (9)$$

Therefore, if $\eta(\underset{\sim}{x},t)$ is a homogeneous, stationary random function, $\langle\eta(\underset{\sim}{x}_1,t_1)\eta(\underset{\sim}{x}_2,t_2)\rangle$ attains its maximum value for zero spatial and temporal separation. For a realistic system that is neither homogeneous nor stationary, the peak is close to the origin on the length or time scale defining the departure from homogeneity and stationarity.

It is also necessary to consider continuity and differentiability of ensemble averaged quantities. Assume that the fluid variables defined over the region of interest and continuous and differentiable, there are two implications. First, the region of interest must be far from large-scale singularities (shocks, current sheets, etc). Second, it must be possible for the small scale turbulent structures to be described by a fluid approximation. Since the fluid is assumed to be viscous, there must be a minimum length scale associated with the flow where the viscous terms dominate to drive the energy in that length scale to zero. Differentiability and continuity are by definition properties of the very smallest scales and we would expect that these assumptions should allow a reasonable approximation to the large scale motions. Having now assumed spatial differentiability, Eq. (1) implies temporal differentiability.

If the realized flows are differentiable, then ensemble averages of products of properties of the flow must also be differentiable. This is so because the ensemble average is simply an integral over the set of realizations, all possessing those same traits. The probability function is not a function of space or time. We can then write that

$$\begin{aligned} &\left\langle \eta(\underset{\sim}{x}_1,t_1) \frac{\partial}{\partial \underset{\sim}{x}_2} \eta(\underset{\sim}{x}_2,t_2) \right\rangle \\ &\equiv \int d[\eta(\underset{\sim}{x},t)]\, \eta(\underset{\sim}{x}_1,t_1) \frac{\partial}{\partial \underset{\sim}{x}_2} \eta(\underset{\sim}{x}_2,t_2) P[\eta(\underset{\sim}{x},t)] \\ &= \frac{\partial}{\partial \underset{\sim}{x}_2} \left\langle \eta(\underset{\sim}{x}_1,t_1)\, \eta(\underset{\sim}{x}_2,t_2) \right\rangle \end{aligned} \tag{10}$$

and $\left\langle \eta(\underset{\sim}{x}_1,t_1) \frac{\partial}{\partial t_2} \eta(\underset{\sim}{x}_2,t_2)\right\rangle = \frac{\partial}{\partial t_2}\left\langle \eta(\underset{\sim}{x}_1,t_1)\eta(\underset{\sim}{x}_2,t_2)\right\rangle$.

We have already shown, Eq. (8), that the ensemble average is a linear operator. This, together with the trivial results

$$\left\langle f_1[\eta]f_2[\eta]\right\rangle = \left\langle f_2[\eta]f_1[\eta]\right\rangle, \tag{11a}$$

$$\left\langle f[\eta]f[\eta]\right\rangle \geq 0 \text{ and} \tag{11b}$$

$$\left\langle f[\eta]f[\eta]\right\rangle = 0 \Rightarrow f \equiv 0 \tag{11c}$$

allows us to apply the Schwartz inequality.[42]

$$\left\langle f_1[\eta]f_2[\eta]\right\rangle \leq \|f_1\|\|f_2\| \tag{12}$$

where $\|f\| \equiv \langle ff\rangle^{1/2}$. We may immediately use Eq. (12) to discuss cross-correlations of several functions.

Let $\beta(\underset{\sim}{x},t)$ also be a random scalar field of the same physical system which possesses $\eta(\underset{\sim}{x},t)$. We may then ask: what is $\left\langle \eta(\underset{\sim}{x}_1,t_1)\beta(\underset{\sim}{x}_2,t_2)\right\rangle$? Defining this to be

$$\left\langle \eta(\underset{\sim}{x}_1,t_1)\beta(\underset{\sim}{x}_2,t_2)\right\rangle \equiv \int d[\eta(\underset{\sim}{x},t)]d[\beta(\underset{\sim}{x},t)]\eta(\underset{\sim}{x}_1,t_1)\beta(\underset{\sim}{x}_2,t_2)P[\eta(\underset{\sim}{x},t)\beta(\underset{\sim}{x},t)],$$

we may apply the Schwartz inequality to assert that

$$\left\langle \eta(\underset{\sim}{x}_1,t_1)\beta(\underset{\sim}{x}_2,t_2)\right\rangle \leq \sqrt{\left\langle \eta(\underset{\sim}{x}_1,t_1)\eta(\underset{\sim}{x}_1,t_1)\right\rangle\left\langle \beta(\underset{\sim}{x}_2,t_2)\beta(\underset{\sim}{x}_2,t_2)\right\rangle} \tag{13}$$

This places an upper limit on such cross-correlations functions and might be paraphrased by the statement that "no two entities may be better correlated one-to-another than each is correlated to itself.

The last generalization of the notation of a random function is that of a random vector field. The random vector field $\underset{\sim}{\eta}(\underset{\sim}{x},t)$ may be thought of as a triplet of random scalar fields: $\eta^{(1)}(\underset{\sim}{x},t)$, $\eta^{(2)}(\underset{\sim}{x},t)$

and $\eta^{(3)}(\underset{\sim}{x},t)$ which may be constrained by such equations as are given by solenoidal requirements: $\nabla\cdot\underset{\sim}{\eta} = 0$ for all $\underset{\sim}{x}$ and t. If we discuss correlations of the random vector field, $\langle\underset{\sim}{\eta}(\underset{\sim}{x}_1,t_1)\,\underset{\sim}{\eta}(\underset{\sim}{x}_2,t_2)\rangle$, in terms of its components, $\langle\eta_i(\underset{\sim}{x}_1,t_1)\,\eta_j(\underset{\sim}{x}_2,t_2)\rangle$, we may apply Schwartz's inequality and assert that

$$\langle\eta_i(\underset{\sim}{x}_1,t_1)\,\eta_j(\underset{\sim}{x}_2,t_2)\rangle \le [\langle\eta_i(\underset{\sim}{x}_1,t_1)\,\eta_i(\underset{\sim}{x}_1,t_1)\rangle\,\langle\eta_j(\underset{\sim}{x}_2,t_2)\,\eta_j(\underset{\sim}{x}_2,t_2)\rangle]^{1/2} \tag{14}$$

This inequality places an upper bound on the cross-correlation of different components of the field as well as spatially and temporally separated values of the same component and sets for the upper bound the square-root of the product of the auto-correlations taken at zero space-time separations.

B. Review of Isotropic Symmetry

Further discussion of random fields as they pertain to turbulent systems requires that some assumption be made of the symmetry. The most often discussed condition is isotropic symmetry.[10] In this case, the correlation $\langle\hat{\alpha}\cdot\underset{\sim}{\eta}(\underset{\sim}{x}_1,t_1)\,\hat{\beta}\cdot\underset{\sim}{\eta}(\underset{\sim}{x}_2,t_2)\rangle$ is invariant under any rigid rotation of the vectors $\hat{\alpha}$, $\hat{\beta}$ and $\underset{\sim}{x}_2-\underset{\sim}{x}_1$ which keeps the relative orientation of the arbitrary unit vectors $\hat{\alpha}$ and $\hat{\beta}$ and the vector $\underset{\sim}{x}_2-\underset{\sim}{x}_1$ fixed. It has also most often been assumed that the above correlation is invariant under reflection in an arbitrary plane. We would like to suspend this assumption of reflection invariance.

From this point on we will make the standard assumption of tensor notation that like alphabetic indexes are summed unless otherwise stated. This assumption has not been made prior to this point.

Let us assume that the members of the ensemble in question display isotropic symmetry. Then the general form for $\langle \eta_i(\underset{\sim}{x}_1,t)\, \eta_j(\underset{\sim}{x}_2,t)\rangle$ may be given. If the random variable $\underset{\sim}{\eta}(\underset{\sim}{x},t)$ is associated with the turbulent velocity field or the turbulent magnetic field, then $\langle \underset{\sim}{\eta}(\underset{\sim}{x}_1,t)\cdot \underset{\sim}{\eta}(\underset{\sim}{x}_1,t)\rangle$ is proportional to the mean energy density of that field at the space-time point $(\underset{\sim}{x}_1,t)$. If the turbulence is homogeneous then this term is proportional to the total turbulent energy of that field. In view of the earlier discussion concerning the differentiability of the correlation functions, we will assume that we may Taylor expand the elements of the matrix $\langle \eta_i(\underset{\sim}{x},t)\; \eta_j(\underset{\sim}{x}+\underset{\sim}{r},t)\rangle$ about $\underset{\sim}{r}=0$ and we will see that the first non-trivial coefficient of this expansion is proportional to the rate at which the turbulent energy is dissipated.

To do this, we construct the isotropic form of $\langle \eta_i(\underset{\sim}{x},t)\, \eta_j(\underset{\sim}{x}+\underset{\sim}{r},t)\rangle$. The corresponding technique for the axisymmetric case is discussed at length in the first section of Chapter III. Briefly, $\langle \hat{\alpha}\cdot\underset{\sim}{\eta}(\underset{\sim}{x},t)\,\hat{\beta}\cdot\underset{\sim}{\eta}(\underset{\sim}{x}+\underset{\sim}{r},t)\rangle$ is constructed to be the most general isotropic, homogeneous, scalar function of the variables $\underset{\sim}{r}$, $\hat{\alpha}$, and $\hat{\beta}$. From this, properties of $\langle \eta_i(\underset{\sim}{x},t)\; \eta_j(\underset{\sim}{x}+\underset{\sim}{r},t)\rangle$ may be deduced. It may be written as

$$\langle \eta_i(\underset{\sim}{x},t)\; \eta_j(\underset{\sim}{x}+\underset{\sim}{r},t)\rangle = \eta^2_{rms}\, R_{ij}(\underset{\sim}{r}) \qquad (15)$$

where η^2_{rms} is the mean square value of a single component of the field $\underset{\sim}{\eta}(\underset{\sim}{x},t)$ and

$$R_{ij}(r) = \frac{f(r) - g(r)}{r^2}\; r_i r_j + g(r)\,\delta_{ij} + h(r)\,\epsilon_{ijk} r_k \qquad (16)$$

where $f(r)$, $g(r)$ and $h(r)$ are unknown scalar functions of $r \equiv (\underset{\sim}{r}\cdot\underset{\sim}{r})^{1/2}$. By the definition of η^2_{rms} we can see that $f(0) = g(0) = 1$ while $h(0)$ remains unspecified. The trace of the correlation matrix, where the correlation matrix is given by Eq. (15), evaluated at $r = 0$ is $3\eta^2_{rms}$ and this is proportional to the energy associated with the field $\underset{\sim}{\eta}(\underset{\sim}{x},t)$ if $\underset{\sim}{\eta}(\underset{\sim}{x},t)$ represents the turbulent velocity or the magnetic field.

A Taylor expansion of $f(r)$ and $g(r)$ produces

$$f(r) = 1 - \frac{1}{2}\lambda_f^{-2}\, r^2 + \mathcal{O}(r^4)$$

and (17)

$$g(r) = 1 - \frac{1}{2}\lambda_g^{-2}\, r^2 + \mathcal{O}(r^4).$$

Only even powers of r are present since we are assuming that $R_{ij}(\underset{\sim}{r})$ is an analytic function and the Schwartz inequality, Eq. (14), requires that the first non-trivial terms in the expansion be negative. We may relate the two coefficients λ_f and λ_g by noting that $\nabla\cdot\underset{\sim}{B} = 0$ or by assuming incompressible flow, $\nabla\cdot\underset{\sim}{v} = 0$. This implies the result $\frac{\partial}{\partial r_j} R_{ij}(\underset{\sim}{r}) = 0$ and requires $\lambda_g = \frac{1}{\sqrt{2}}\lambda_f$. The coefficient λ_f is called the Taylor micro-scale and it is not only a measure of the separation at which the turbulent field becomes uncorrelated, but it can also be related to the rate at which the energy is dissipated. To see this, we need to write the energy equation.

From Eqs. (1a) and (1b) we find

$$\frac{\partial}{\partial t}\mathcal{E} = -\nabla\cdot\left[\frac{v^2}{2}\underset{\sim}{v} + \frac{P}{\rho}\underset{\sim}{v} + \frac{B^2}{4\pi\rho}\underset{\sim}{v} + \frac{\underset{\sim}{v}\cdot\underset{\sim}{B}}{4\pi\rho}\underset{\sim}{B}\right] + \nu\,\underset{\sim}{v}\cdot\nabla^2\underset{\sim}{v} + \frac{\lambda}{4\pi\rho}\underset{\sim}{B}\cdot\nabla^2\underset{\sim}{B} \tag{18}$$

where $\mathcal{E} = \frac{v^2}{2} + \frac{B^2}{8\pi\rho}$ is the energy density of the incompressible, uniform density flow. The total energy is found by integrating over the volume of the fluid. The first term on the right may then be converted from a volume integral to a surface integral by the divergence theorem; for a variety of boundary conditions this term will vanish. If the system has periodic boundary conditions, or if the velocity goes to zero at the wall, or if the flow is localized so that the terms of interest go to zero faster than r^{-2}, allowing the boundary to be pushed to infinity, or if any appropriate combination of these boundary conditions is used, we may write:

$$\frac{\partial}{\partial t} \int_V \mathcal{E}\, d^3x = \int_V d^3\underset{\sim}{x}[\nu\, \underset{\sim}{v} \cdot \nabla^2 \underset{\sim}{v} + \frac{\lambda}{4\pi\rho} \underset{\sim}{B} \cdot \nabla^2 \underset{\sim}{B}] \tag{19}$$

For homogeneous turbulence, $\langle \mathcal{E} \rangle$ should be a good indication of the value of $\frac{1}{|V|}\int_V \mathcal{E}\, d^3x$ and so $\langle \nu\, \underset{\sim}{v} \cdot \nabla^2 \underset{\sim}{v} + \frac{\lambda}{4\pi\rho} \underset{\sim}{B} \cdot \nabla^2 \underset{\sim}{B} \rangle$ is a measure of the rate at which the mean energy density is dissipated.

If λ_v is the Taylor microscale for the velocity field (in analogy with λ_f) and if λ_B is the Taylor microscale for the magnetic field then

$$\langle \nu\, \underset{\sim}{v} \cdot \nabla^2 \underset{\sim}{v} + \frac{\lambda}{4\pi\rho} \underset{\sim}{B} \cdot \nabla^2 \underset{\sim}{B} \rangle = 15[\, \frac{\nu V^2_{rms}}{\lambda^2_v} + \frac{\lambda B^2_{rms}}{4\pi\rho\lambda^2_B}\,] \tag{20}$$

where V^2_{rms} and B^2_{rms} are the mean square values of a single component of the velocity and magnetic fields respectively. Thus the Taylor microscales not only give a correlation length for the turbulence, but they also describe the rate at which the energy dissipates.

As a third property, the Taylor microscales describe the size of the small energy containing eddies. This would be expected since it is a property of the Fourier transform that small scale behavior is represented by large $\underset{\sim}{k}$ (small wavelength) and large scale behavior is represented by the small values of $\underset{\sim}{k}$. A quick calculation makes this plausible. Suppose as a first approximation, we assume that f(r) and g(r) are Gaussian so that $f(r) \sim e^{-r^2/2\lambda^2}$ and $g(r) \sim e^{-r^2/2\lambda^2}$. We may apply the Fourier transform, defining

$$S_{ii}(\underset{\sim}{k}) = \frac{1}{8\pi^3} \int d\underset{\sim}{r}\, R_{ii}(\underset{\sim}{r}) e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \tag{21}$$

so that

$$\mathcal{E} = \int d\underset{\sim}{k}\, S_{ii}(\underset{\sim}{k}) \tag{22}$$

which in the isotropic case allows us to define an energy density in Fourier space, $\mathcal{E}_F$:

$$\mathcal{E}_F \equiv 4\pi k^2 S_{ii}(k) . \tag{23}$$

Taking f(r) and g(r) to have the previous forms implies that

$$\mathcal{E}_F \sim k \exp(-\tfrac{1}{4} k^2 \lambda^2) \tag{24}$$

to leading order. This function peaks at $k_p = \frac{\sqrt{2}}{\lambda}$, so λ represents the size of the small energy containing eddies.

The fact that the microscales should be related to the size of the small eddies is only reasonable. As λ is decreased, f(r) and g(r) become more sharply peaked. This means that the correlation functions become more sharply peaked and the field becomes uncorrelated at relatively smaller values of the separation. But what allows two points to become uncorrelated? It is the small energy containing eddies. When the two points are sufficiently separated, one may be influenced by a given eddy structure while another is not. The more separated they become, the greater the percentage of eddies which are available for the decorrelation process. The length which marks the beginning of this process is the size of the smallest eddies. And so, the microscales must be related to the size of the smallest eddies.

There is a third term in Eq. (16) which has not yet been discussed. It is the term $h(r)\,\epsilon_{ijh} r_h$, which does not contribute to the energy. For basically this reason, it has typically been ignored in discussions of isotropic turbulence with one notable exception.[47] It is just this term which is of interest to us now.

For simplicity, let us consider the case when the magnetic field is zero so that we are examining a traditional Navier-Stokes fluid. The helicity, $\int_V d^3\underset{\sim}{x}\ \underset{\sim}{v}\cdot\underset{\sim}{\omega}$ where $\underset{\sim}{\omega} = \nabla\otimes\underset{\sim}{v}$, is a conserved quantity when viscosity vanishes.[48] When viscosity is non-zero, the helicity is postulated to affect the dynamics of the turbulent flow. A Taylor expansion of $h(r)$ gives

$$h(r) = h_0[1 + \frac{1}{2}\frac{r^2}{\lambda_h{}^2} + \mathcal{O}(r^4)]. \tag{25}$$

In analogy with the energy discussion, we may write

$$\frac{\partial}{\partial t}\int_V d^3\underset{\sim}{x}\ \underset{\sim}{v}\cdot\underset{\sim}{\omega} = \int_V d^3\underset{\sim}{x}\ (\underset{\sim}{v}\cdot\nabla^2\underset{\sim}{\omega} + \underset{\sim}{\omega}\cdot\nabla^2\underset{\sim}{v}) \tag{26}$$

and so $(\underset{\sim}{v}\cdot\nabla^2\underset{\sim}{\omega} + \underset{\sim}{\omega}\cdot\nabla^2\underset{\sim}{v})$ relates to the rate at which the mean helicity density, $-6v_{rms}^2\ h_0$, is dissipated. In terms of the Taylor microscale, λ_h, this is

$$\nu\langle\underset{\sim}{v}\cdot\nabla^2\underset{\sim}{\omega} + \underset{\sim}{\omega}\cdot\nabla^2\underset{\sim}{v}\rangle = v_{rms}^2\ h_0\ \frac{60}{\lambda_h^2} \tag{27}$$

The Taylor microscale, λ_h, may also be related to the length scale of a typical helicity containing eddy. By analogy with the energy discussion, λ_h is the length scale of a typical helicity containing eddy.

Reinstating the magnetic field adds a considerable degree of difficulty. The helicity is no longer a conserved quantity for non-viscous flow, but rather the magnetic helicity

$$\int_V d^3\underset{\sim}{x}\ \underset{\sim}{A}\cdot\underset{\sim}{B}$$

is conserved.[49] We will postpone discussion of this until the next chapter. The principal difficulty lies in the fact that it is difficult to measure the magnetic vector potential, $\underset{\sim}{A}(\underset{\sim}{x},t)$. However, we will be able to formulate a discussion analogous to that just presented, expressing all of the needed quantities in terms of the measurable fields.

III. AXISYMMETRIC TURBULENCE

In this chapter, we will develop a description of the correlations in axisymmetric turbulence. We will follow the technique of Robertson, Chandrasekhar and others[47,50,56] as briefly discussed in the previous chapter, but it will also be necessary to introduce several new ideas at a very fundamental level in order to avoid the description of Moffatt[57] which contains too many independent tensor forms. We will not assume reflection invariance of the description because, as we shall demonstrate, this results in all helicity vanishing.

We will assume that the fluid is incompressible and the reader should be reminded that the magnetic field is solenoidal. Furthermore, we are free to choose a gauge in which to define the vector potential. The most convenient choice is the Coulomb gauge where the vector potential is also solenoidal. We assume that the fluid is homogeneous so that while the correlations of the fields are dependent upon the relative locations of the points in question they are independent under any translation which keeps the relative orientation of these points the same. We denote the ensemble average of a quantity by brackets, $\langle\rangle$.

A. Construction of R_{ij}

We will represent the fields of interest in the usual way, representing the cross product by the symbol $\otimes$. Then we define the fields by stating velocity = $\underset{\sim}{v}(\underset{\sim}{x},t)$ with $\langle \underset{\sim}{v} \rangle = 0$ and $\nabla \cdot \underset{\sim}{v} = 0$,

vorticity = $\underset{\sim}{\omega}(\underset{\sim}{x},t) = \nabla \otimes \underset{\sim}{v}$,

magnetic field = $\underset{\sim}{\mathcal{B}} = \underset{\sim}{B}_0 + \underset{\sim}{B}(\underset{\sim}{x},t)$ with $\langle \underset{\sim}{B} \rangle = 0$,

current = $\underset{\sim}{j} = \frac{c}{4\pi} \nabla \otimes \underset{\sim}{B}$,

and the magnetic vector potential $\underset{\sim}{\mathcal{A}} = \underset{\sim}{A}_0 + \underset{\sim}{A}(\underset{\sim}{x},t)$ such that

$\underset{\sim}{B}_0 = \nabla \otimes \underset{\sim}{A}_0$, $\underset{\sim}{B} = \nabla \otimes \underset{\sim}{A}$ and $\nabla \cdot \underset{\sim}{A} = 0$.

The mean magnetic field, B_0, is independent of space and time. This is a much simplified model of the more complex systems mentioned earlier. Clearly $\nabla \cdot \underset{\sim}{\omega} = 0$ and $\nabla \cdot \underset{\sim}{j} = 0$ and so we will treat $\underset{\sim}{v}, \underset{\sim}{\omega}, \underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{j}$ as random solenoidal vector fields of zero mean which are in general non-stationary. We will assume they are continuous and differentiable in space and time as are their correlations and we will denote them in general by the vector variable $\underset{\sim}{\eta}$. As is appropriate, we will choose to associate $\underset{\sim}{\eta}$ with one field or another.

It will also be assumed that the spatial and temporal development of the fields may be described by the MHD equations, Eqs. (1a-e). We will assume that there is no mean fluid velocity. Elsasser[58] has shown that a uniform mean fluid velocity can be transformed away by a Galilean transformation which leaves the magnetic field unchanged within the MHD approximation. Thus, any anisotropy introduced by a uniform mean fluid velocity can be transformed away.

We will occasionally discuss systems where the axis of symmetry is defined by something other than a mean magnetic field, although the latter will be our principal system of interest.

We begin the discussion of axisymmetric turbulence by constructing the correlation matrix. Choosing two position vectors $\underset{\sim}{x}$ and $\underset{\sim}{x} + \underset{\sim}{r}$ and two unit vectors $\hat{a}$ and $\hat{c}$, we may ask what is the same-time autocorrelation function of the component $\eta_a = \underset{\sim}{\eta} \cdot \hat{a}$ at the point $\underset{\sim}{x}$ and of the component $\eta_c = \underset{\sim}{\eta} \cdot \hat{c}$ at the point $\underset{\sim}{x} + \underset{\sim}{r}$. We will define this function to be

$$R(\underset{\sim}{\eta};\hat{a},\underset{\sim}{x};\hat{c},\underset{\sim}{x} + \underset{\sim}{r};t) \equiv \left\langle \hat{a}\cdot \underset{\sim}{\eta}(\underset{\sim}{x})\hat{c}\cdot \underset{\sim}{\eta}(\underset{\sim}{x} + \underset{\sim}{r})\right\rangle \tag{28}$$

The system of interest contain, by assumption, only one global vector field whose mean is non-zero and this is the mean magnetic field. It is unreasonable to assume that this global property does not introduce a fundamental anisotropy. Numerous measurements of the solar wind,[33] the Macrotor tokamak[34] and the Culham Zeta pinch[35] have demonstrated this. In discussing systems such as these not only is $R(\underset{\sim}{\eta};\hat{a},\underset{\sim}{x};\hat{c},\underset{\sim}{x} + \underset{\sim}{r};t)$ invariant under arbitrary translation ($\underset{\sim}{x} \to \underset{\sim}{x}' = \underset{\sim}{x} + \underset{\sim}{\xi}$ where $\underset{\sim}{\xi}$ is an arbitrary vector), but it is also invariant under a rigid rotation of the entire $\underset{\sim}{r}$,$\hat{a}$, and $\hat{c}$ configuration about the mean field $\underset{\sim}{B}_0$.

Let us define $\hat{b} \equiv \underset{\sim}{B}_0/|B_0|$ and construct the most general form that $R(\underset{\sim}{\eta};\ldots)$ may have and still be homogeneous and invariant under rigid rotation about $\underset{\sim}{B}_0$. The independent scalars under such rotations constructed from the vectors available from the specification of $R(\underset{\sim}{\eta};\ldots)$ are: $r \equiv (\underset{\sim}{r}\cdot\underset{\sim}{r})^{1/2}$;$\underset{\sim}{r}\cdot\hat{a}$, $\underset{\sim}{r}\cdot\hat{c}$, $\underset{\sim}{r}\cdot\hat{b} \equiv z$, $\hat{b}\cdot\hat{a}$, $\hat{b}\cdot\hat{c}$, $\hat{a}\cdot\underset{\sim}{r} \otimes \hat{b}$, $\hat{c}\cdot\underset{\sim}{r} \otimes \hat{b}$, $\hat{a}\cdot\hat{c}$, $\underset{\sim}{r}\cdot\hat{a} \otimes \hat{c}$, $\hat{b}\cdot\hat{a} \otimes \hat{c}$, $|\underset{\sim}{B}_0|$ and t. The most general form that $R(\underset{\sim}{\eta};\ldots)$ may have that is a bilinear function of $\hat{a}$ and $\hat{c}$ is

$$\begin{aligned}
&R(\underset{\sim}{\eta};\hat{a};\underset{\sim}{x};\hat{c};\underset{\sim}{x}+\underset{\sim}{r},\hat{b},t)\\
&=[R^{(1)}(\underset{\sim}{r}\cdot\hat{a})(\underset{\sim}{r}\cdot\hat{c})+R^{(2)}(\hat{a}\cdot\hat{c})+R^{(3)}(\hat{a}\cdot\hat{b})(\hat{c}\cdot\hat{b})\\
&+R^{(4)}(\underset{\sim}{r}\cdot\hat{a})(\hat{b}\cdot\hat{c})+R^{(5)}(\underset{\sim}{r}\cdot\hat{a})(\hat{c}\cdot\hat{b}\otimes\underset{\sim}{r})\\
&+R^{(6)}(\hat{b}\cdot\hat{a})(\hat{c}\cdot\hat{b}\otimes\underset{\sim}{r})+R^{(7)}(\underset{\sim}{r}\cdot\hat{a}\otimes\hat{c})+R^{(8)}(\hat{b}\cdot\hat{a}\otimes\hat{c})\\
&+R^{(9)}(\underset{\sim}{r}\cdot\hat{c})(\hat{b}\cdot\hat{a})+R^{(10)}(\underset{\sim}{r}\cdot\hat{c})(\hat{a}\cdot\hat{b}\otimes\underset{\sim}{r})\\
&+R^{(11)}(\hat{b}\cdot\hat{c})(\hat{a}\cdot\hat{b}\otimes\underset{\sim}{r})]\ \eta^2_{rms}
\end{aligned}\tag{29}$$

where the functions $R^{(i)}$ are functions of the variables r, z, $|\underset{\sim}{B}_0|$ and t and η^2_{rms} is chosen so that $R^2(r = 0) = 1$.

We can then define the correlation matrix $R_{ij}(\underset{\sim}{r})$ to be

$$\eta^2_{rms}\,R_{ij}(\underset{\sim}{r}) \equiv \left\langle \eta_i(\underset{\sim}{x})\,\eta_j(\underset{\sim}{x}+\underset{\sim}{r})\right\rangle \tag{30}$$

so that $R(\underset{\sim}{\eta};\ldots) = \eta^2_{rms}\,a_iR_{ij}(\underset{\sim}{r})c_j$. The matrix R_{ij} can then be shown from the previous form to be

$$\begin{aligned}
R_{ij}(\underset{\sim}{r}) \equiv\ & R^{(1)}r_ir_j + R^{(2)}\delta_{ij} + R^{(3)}b_ib_j,\\
&+ R^{(4)}r_ib_j + R^{(5)}r_i\epsilon_{jhl}b_hr_l + R^{(6)}b_i\epsilon_{jhl}b_hr_l\\
&+ R^{(7)}\epsilon_{ijh}r_h + R^{(8)}\epsilon_{ijh}b_h + R^{(9)}b_ir_j\\
&+ R^{(10)}r_j\epsilon_{ihl}b_hr_l + R^{(11)}b_j\epsilon_{ihl}b_hr_l.
\end{aligned}\tag{31}$$

But, R_{ij} is a matrix and can be written as the sum of a symmetric part and an antisymmetric part:

$$R_{ij} = \frac{1}{2}(R_{ij}+R_{ji}) + \frac{1}{2}(R_{ij}-R_{ji})\ . \tag{32}$$

Although we have decided not to assume that R_{ij} is invariant under coordinate inversion, we may ask how the symmetric and antisymmetric parts of R_{ij} behave under coordinate inversion, Inv.*

*See Appendix I.

To do this, we must first show that the symmetric part of R_{ij}, $R^s_{ij} \equiv \langle \eta_i(\underset{\sim}{x})\, \eta_j(\underset{\sim}{x} + \underset{\sim}{r}) \rangle^s$, and the antisymmetric part of R_{ij}, $R^a_{ij} \equiv \langle \eta_i(\underset{\sim}{x})\, \eta_j(\underset{\sim}{x} + \underset{\sim}{r}) \rangle^a$, obey the homogeneity assumption independently. This is quickly seen from Eq. (32) to follow directly from the assumption that R_{ij} in homogeneous. With this point established, we may ask for the coordinate inversion of R^s_{ij}

$$\begin{aligned}
\mathrm{Inv}\, R^s_{ij} &= \mathrm{Inv} \langle \eta_i(\underset{\sim}{x})\, \eta_j(\underset{\sim}{x} + \underset{\sim}{r}) \rangle^s \\
&= \langle \eta_i(-\underset{\sim}{x})\, \eta_j(-\underset{\sim}{x} - \underset{\sim}{r}) \rangle^s \\
&= \langle \eta_i(\underset{\sim}{x} + \underset{\sim}{r})\, \eta_j(\underset{\sim}{x}) \rangle^s \qquad \text{(homogeneity)} \qquad\qquad (33) \\
&= \langle \eta_j(\underset{\sim}{x})\, \eta_i(\underset{\sim}{x} + \underset{\sim}{r}) \rangle^s \\
&= \langle \eta_i(\underset{\sim}{x})\, \eta_j(\underset{\sim}{x} + \underset{\sim}{r}) \rangle^s \qquad \text{(symmetry)} \\
&= R^s_{ij}.
\end{aligned}$$

In the same manner, we write

$$\begin{aligned}
\mathrm{Inv}\, R^a_{ij} &= \langle \eta_i(-\underset{\sim}{x})\, \eta_j(-\underset{\sim}{x} - \underset{\sim}{r}) \rangle^a \\
&= \langle \eta_i(\underset{\sim}{x} + \underset{\sim}{r})\, \eta_j(\underset{\sim}{x}) \rangle^a \qquad \text{(homogeneity)} \qquad\qquad (34) \\
&= - \langle \eta_i(\underset{\sim}{x})\, \eta_j(\underset{\sim}{x} + \underset{\sim}{r}) \rangle^a \quad \text{(antisymmetry)} \\
&= - R^a_{ij}.
\end{aligned}$$

So, R^s_{ij} is invariant under inversion and is therefore a symmetric, proper tensor labeled R^{st}_{ij} while R^a_{ij} changes sign under inversion and is therefore an antisymmetric, pseudo-tensor labeled R^{ap}_{ij}. There is an important point since the functional dependence of R_{ij} upon z is unknown and since z is a pseudo-scalar, we will be able to restrict the form of R_{ij} in order that it behave properly under inversion.

We can further reduce the number of independent forms by applying the solenoidal constraint. Recall that we are concerned with turbulent fields, $\underset{\sim}{\eta}(\underset{\sim}{x},t)$, for which $\nabla \cdot \underset{\sim}{\eta} = 0$. It then follows that

$$
\begin{aligned}
0 &= \left\langle \eta_i(\underset{\sim}{x}) \nabla \cdot \underset{\sim}{\eta}(\underset{\sim}{x} + \underset{\sim}{r}) \right\rangle \\
&= \left\langle \eta_i(\underset{\sim}{x}) \frac{\partial}{\partial x_j} \eta_j(\underset{\sim}{x} + \underset{\sim}{r}) \right\rangle \\
&= \left\langle \eta_i(\underset{\sim}{x}) \frac{\partial}{\partial r_j} \eta_j(\underset{\sim}{x} + \underset{\sim}{r}) \right\rangle \\
&= \frac{\partial}{\partial r_j} \left\langle \eta_i(\underset{\sim}{x}) \eta_j(\underset{\sim}{x} + \underset{\sim}{r}) \right\rangle \\
&= \frac{\partial}{\partial r_j} R_{ij} \; .
\end{aligned} \tag{35}
$$

The derivative and ensemble average are assumed to commute since the ensemble average is an integration over a functional space. But if $\frac{\partial}{\partial r_j} R_{ij} = 0$, we may also write

$$
\frac{\partial}{\partial r_j} (\text{Inv } R_{ij}) = - \text{Inv}\left(\frac{\partial}{\partial r_j} R_{ij}\right) = 0 \tag{36}
$$

showing that the coordinate inversion operation leaves R_{ij} solenoidal. Then using Eq. (32) we may write that

$$
R^{st}_{ij} = \frac{1}{2}[R_{ij} + \text{Inv } R_{ij}]
$$

and (37)

$$
R^{ap}_{ij} = \frac{1}{2}[R_{ij} - \text{Inv } R_{ij}]
$$

which together with Eq. (36) allows us to assert that

$$
\frac{\partial}{\partial r_j} R^{st}_{ij} = 0
$$

and (38)

$$
\frac{\partial}{\partial r_j} R^{ap}_{ij} = 0 \; .
$$

After constructing R_{ij}^{st} and R_{ij}^{ap}, we can write with a trivial re-indexing of the scalar functions

$$\begin{aligned} R_{ij}^{st} = R^{(1)} r_i r_j + R^{(2)} \delta_{ij} + R^{(3)} b_i b_j + R^{(4)} (r_i b_j + r_j b_i) \\ + R^{(5)} (r_i \epsilon_{jhl} b_h r_l + r_j \epsilon_{ihl} b_h r_l) \\ + R^{(6)} (b_i \epsilon_{jhl} b_h r_l + b_j \epsilon_{ihl} b_h r_l) \end{aligned} \tag{39}$$

and

$$\begin{aligned} R_{ij}^{ap} = R^{(7)} \epsilon_{ijh} r_h + R^{(8)} \epsilon_{ijh} b_h \\ + R^{(9)} (r_i b_j - r_j b_i) \\ + R^{(10)} (r_i \epsilon_{jhl} b_h r_l - r_j \epsilon_{ihl} b_h r_l) \\ + R^{(11)} (b_i \epsilon_{jhl} b_h r_l - b_j \epsilon_{ihl} b_h r_l) \ . \end{aligned} \tag{40}$$

The identities

$$r_i \epsilon_{jhl} b_h r_l - r_j \epsilon_{ihl} b_h r_l = r^2 \epsilon_{ijh} b_h - z \epsilon_{ijh} r_h \tag{41a}$$

and

$$b_i \epsilon_{jhl} b_h r_l - b_j \epsilon_{ihl} b_h r_l = z \epsilon_{ijh} b_h - \epsilon_{ijh} r_h \tag{41b}$$

enable us to eliminate $R^{(10)}$ and $R^{(11)}$ as non-independent forms which can be absorbed into the still unspecified functions $R^{(7)}$ and $R^{(8)}$.

To apply the solenoidal constraints, we must be able to differentiate the scalar functions present in the description. This is possible, if the correlations are differentiable. The conditions implied by this assumption were discussed in Chapter II. Differentiation of the scalar functions is carried out by applying the chain rule

$$\begin{aligned} \frac{\partial}{\partial r_j} R^{(i)} &= \left(\frac{\partial r}{\partial r_j}\right) \frac{\partial R^{(i)}}{\partial r} + \left(\frac{\partial z}{\partial r_j}\right) \frac{\partial R^{(i)}}{\partial z} \\ &= \frac{r_j}{r} \frac{\partial}{\partial r} R^{(i)} + b_j \frac{\partial}{\partial z} R^{(i)} . \end{aligned} \tag{42}$$

Application of the solenoidal constraint, Eq. (38) to Eq. (39) and (40) then yields two vector equations. These equations must hold component by component, or equivalently, when the dot product is taken with any arbitrary vector. Judicious choices for the dot product vector quickly result in the following six constraint equations for the scalar functions

$$r\,\frac{\partial R^{(1)}}{\partial r} + z\,\frac{\partial R^{(1)}}{\partial z} + 4R^{(1)} + r^{-1}\,\frac{\partial R^{(2)}}{\partial r} + zr^{-1}\,\frac{\partial R^{(4)}}{\partial r} + \frac{\partial R^{(4)}}{\partial z} = 0 \tag{43}$$

$$\frac{\partial R^{(2)}}{\partial z} + zr^{-1}\,\frac{\partial R^{(3)}}{\partial r} + \frac{\partial R^{(3)}}{\partial z} + r\,\frac{\partial R^{(4)}}{\partial r} + z\,\frac{\partial R^{(4)}}{\partial z} + 4R^{(4)} = 0 \tag{44}$$

$$r\,\frac{\partial R^{(5)}}{\partial r} + z\,\frac{\partial R^{(5)}}{\partial z} + 5R^{(5)} + zr^{-1}\,\frac{\partial R^{(6)}}{\partial r} + \frac{\partial R^{(6)}}{\partial z} = 0 \tag{45}$$

$$\frac{\partial R^{(7)}}{\partial z} - r^{-1}\,\frac{\partial R^{(8)}}{\partial r} = 0 \tag{46}$$

$$-2R^{(9)} - r\,\frac{\partial R^{(9)}}{\partial r} - z\,\frac{\partial R^{(9)}}{\partial z} = 0 \tag{47}$$

and

$$zr^{-1}\,\frac{\partial R^{(9)}}{\partial r} + \frac{\partial R^{(9)}}{\partial z} = 0. \tag{48}$$

The last two constraint equations readily show that $R^{(9)} \equiv 0$*. This leaves

$$R^{ap}_{ij} = (R^{(7)} r_h + R^{(8)} b_h)\,\epsilon_{ijh} \tag{49}$$

as the only remaining terms in the pseudo-tensor part. The symmetric part remains unchanged except for the implicit restrictions of the constraint equations. Further discussion of the constraint equations will be reserved for later.

Having constructed the matrix R_{ij} to be the sum of a symmetric proper tensor and an antisymmetric pseudo-tensor, we now know whether the scalar functions contained in that description are proper scalars or pseudo-scalars. Since they are functions of a proper scalar, r, and

*See Appendix II.

a pseudo-scalar, z, we know immediately whether each function is an even or an odd function in z. Namely; $R^{(1)}$, $R^{(2)}$, $R^{(3)}$, $R^{(5)}$ and $R^{(7)}$ are all even functions of z while $R^{(4)}$, $R^{(6)}$ and $R^{(8)}$ are all odd functions of z. And since all of the functions are assumed to be continuous, we know that $R^{(4)}(z = 0) = R^{(6)}(z = 0) = R^{(8)}(z = 0)$ for all values of r. Further, since $R^{(2)}(r = 0)$ represents one third of the isotropic contribution to the mean square fluctuation strength, we know that it is necessarily greater than or equal to zero. In fact, it would be equal to zero only if the fluctuating vector were always parallel to the mean field. In like manner, the parallel direction cannot have a negative contribution and so it must be true that $R^{(2)}(r = 0) + R^{(3)}(r = 0) \geq 0$. The scalar function $R^{(3)}$ can in fact have a negative value at $\underset{\sim}{r} = 0$ and the minimum value it can attain, $- R^{(2)}(r = 0)$, corresponds to a restriction of the fluctuations in the perpendicular plane.

There is a particularly interesting point which should be made at this time. We have been concerned throughout with symmetry about a pseudo-vector, $\underset{\sim}{B}_0$. If we were to apply this line of thought to axisymmetry about a proper vector, such as a streaming velocity in a Navier-Stokes fluid, or had we simply chosen the direction of the mean field, $\underset{\sim}{B}_0$, in some preferred coordinate system such as a right-handed system, we would have a very different correlation fluctuation. Suppose this proper vector field were given by $\hat{\lambda}$ and we define $w \equiv \underset{\sim}{r} \cdot \hat{\lambda}$. The variable w is then a proper scalar as is r and now all of the scalar functions of R_{ij} would be proper scalars, being functions only of r and w. This forces $R^{(5)}$ and $R^{(6)}$ to be identically zero. This is seen by noting that

$$r_i \epsilon_{jhl} \lambda_h r_l + r_j \epsilon_{ihl} \lambda_h r_l \text{ and } \lambda_i \epsilon_{jhl} \lambda_h r_l + \lambda_j \epsilon_{ihl} \lambda_h r_l$$

are both pseudo-tensors forcing $R^{(5)}$ and $R^{(6)}$ to be pseudo-scalars. But, no pseudo-scalar function can be constructed of the proper scalars r and w except the function zero. This changes the nature of the correlations in a fundamental way, reducing the number of independent scalars needed and uncorrelating elements of the field which were correlated in the pseudo-vector representation.

This implies that there is a fundamental difference between the two turbulent flows. We choose here to assume that the turbulent flow is axisymmetric about the mean magnetic field pseudo-vector, rather than some similarly oriented proper vector, because it is the magnetic field pseudo-vector which appears in the MHD equations. In addition, the MHD equations, which we have assumed to be the dynamical equations governing the evolution of the turbulent flow, are invariant under coordinate inversion. It is only reasonable to expect that this invariance would be similarly reflected in the interaction between the fluctuating, turbulent field and that physical entity which defines the symmetry axis. The mean magnetic field pseudo-vector accomplishes this.

B. Fourier Space Correlation Representation

The entire constructive process that has been discussed so far may also be paralleled in Fourier space. There are advantages to this representation. One is that the solenoidal constraint equations are algebraic rather than differential equations. A second advantage is that it will allow the magnetic field correlation matrix to be uncurled to obtain the magnetic helicity.

To begin, we may apply the developments of the previous sections to write

$$S_{ij}(\underset{\sim}{k}) = \widetilde{R}_{ij} \equiv \frac{1}{(2\pi)^3}\int d\underset{\sim}{r}\ e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} R_{ij}(\underset{\sim}{r}) \ , \tag{50}$$

and

$$S_{ij}(\underset{\sim}{k}) = S^{st}_{ij}(\underset{\sim}{k}) + S^{ap}_{ij}(\underset{\sim}{k}) \ , \tag{51}$$

where

$$\begin{aligned} S^{st}_{ij}(\underset{\sim}{k}) = {} & S^{(1)}k_ik_j + S^{(2)}\delta_{ij} + S^{(3)}b_ib_j + S^{(4)}(k_ib_j + k_jb_i) \\ & + S^{(5)}(k_i\epsilon_{jhl}b_hk_l + k_j\epsilon_{ihl}b_hk_l) \\ & + S^{(6)}(b_i\epsilon_{jhl}b_hk_l + b_j\epsilon_{ihl}b_hk_l) \ . \end{aligned} \tag{52}$$

and

$$S^{ap}_{ij}(\underset{\sim}{k}) = S^{(7)}\epsilon_{ijh}k_h + S^{(8)}\epsilon_{ijh}b_h + S^{(9)}(k_ib_j - k_jb_i), \tag{53}$$

Here, the scalar functions $S^{(i)}$ are functions of $k = |\underset{\sim}{k}|$ and $\underset{\sim}{k}\cdot\hat{b}$ and by the earlier developments, the functions $S^{(1)}$, $S^{(2)}$, $S^{(3)}$, $S^{(5)}$ and $S^{(9)}$ are even functions of $\underset{\sim}{k}\cdot\hat{b}$ while $S^{(4)}$, $S^{(6)}$ and $S^{(8)}$ are odd functions of $\underset{\sim}{k}\cdot\hat{b}$.

The solenoidal constraint is now $k_jS_{ij} = 0$ and as before, $k_jS^{st}_{ij} = 0$ while $k_jS^{ap}_{ij} = 0$. This implies that

$$S^{(1)}k^2 + S^{(2)} + (\underset{\sim}{k}\cdot\hat{b})S^{(4)} = 0 \ , \tag{54}$$

$$S^{(3)}(\underset{\sim}{k}\cdot\hat{b}) + k^2S^{(4)} = 0 \ , \tag{55}$$

$$k^2S^{(5)} + (\underset{\sim}{k}\cdot\hat{b})S^{(6)} = 0 \ , \tag{56}$$

$$S^{(8)} = S^{(9)} = 0 \ , \tag{57}$$

and $S^{(7)}$ is unconstrained. This leaves S^{ap} in the particularly simple form of $S^{ap}_{ij} = S^{(7)}\epsilon_{ijh}k_h$. It is particularly interesting to note that in the case of turbulence without any degree of symmetry, it can be shown that this is still the complete form of S^{ap}_{ij} except that no symmetry is assumed in the scalar function $S^{(7)}(k)$.*

The Fourier space representation is one means by which the configuration space constraint equations may be satisfied. Application of the solenoidal constraint in Fourier space is algebraic and when S_{ij} is written with a minimum number of scalar functions, it can be inverse transformed to the configuration space representation which satisfies the constraint equations and contains only a minimum number of independent scalar functions. This will be done in a later section.

In writing Eqs. (51, (52) and (53) we used the arguments of the previous section to construct the Fourier space representation. We could also have chosen to apply Eq. (50) in a straightforward manner[59] and it is gratifying to see that both approaches give the same result. In writing Eq. (50), we defined the Fourier transform of R_{ij} to be $R_{ij} = S_{ij}$. Let us keep to this notation and define the Fourier transform of the scalar function $R^{(i)}$ to be $\tilde{R}^{(i)}$. Carrying out the transformation of R_{ij}, we find that:

$$S^{(1)} = - k^2 \frac{\partial^2}{\partial k^2} \tilde{R}^{(1)} + k^{-3} \frac{\partial}{\partial k} \tilde{R}^{(1)} \ , \tag{58a}$$

$$S^{(2)} = - k^{-1} \frac{\partial}{\partial k} \tilde{R}^{(1)} + \tilde{R}^{(2)} \ , \tag{58b}$$

$$S^{(3)} = - \frac{\partial^2}{\partial(\underset{\sim}{k}\cdot\hat{b})^2} \tilde{R}^{(1)} + \tilde{R}^{(3)} + 2i \frac{\partial}{\partial(\underset{\sim}{k}\cdot\hat{b})} \tilde{R}^{(4)} , \tag{58c}$$

*See Appendix III.

$$S^{(4)} = -k^{-1}\frac{\partial}{\partial k}\frac{\partial}{\partial(\underset{\sim}{k}\cdot\hat{b})}\tilde{R}^{(1)} + ik^{-1}\frac{\partial}{\partial k}\tilde{R}^{(4)} , \tag{58d}$$

$$S^{(5)} = -k^{-2}\frac{\partial^2}{\partial k^2}\tilde{R}^{(5)} + k^{-3}\frac{\partial}{\partial k}\tilde{R}^{(5)}, \tag{58e}$$

$$S^{(6)} = -k^{-1}\frac{\partial}{\partial k}\frac{\partial}{\partial(\underset{\sim}{k}\cdot\hat{b})}\tilde{R}^{(5)} + ik^{-1}\frac{\partial}{\partial k}\tilde{R}^{(6)}, \tag{58f}$$

$$S^{(7)} = ik^{-1}\frac{\partial}{\partial k}\tilde{R}^{(7)} , \tag{58g}$$

$$S^{(8)} = S^{(9)} = 0 . \tag{58h}$$

C. Energy Spectrum

We may associate $\eta^2_{rms}R_{ij}(r) \equiv \langle \eta_i(\underset{\sim}{x})\,\eta_j(\underset{\sim}{x} + \underset{\sim}{r})\rangle$ with the correlation of the velocity field or the magnetic field and to within a constant of $\frac{1}{2}$ or $\frac{1}{8\pi\rho}$ respectively, we may write that the mean energy density contained in the field is $\eta^2_{rms}R_{ii}(0) = \langle \eta_i(\underset{\sim}{x})\,\eta_i(\underset{\sim}{x})\rangle$ where the sum convention is in effect. This gives, to within a factor of η^2_{rms}, $E = 3R^{(2)}(0) + R^{(3)}(0)$. The ratio of $R^{(2)}(0)$ to $R^{(3)}(0)$ is an indication of the degree of anisotropy in the field. This quantity can be measured with relative ease compared to the energy spectrum. However, most of the well developed theory on the evolution of energy in a turbulent field is discussed within the context of the energy spectrum.

In order to relate the energy spectrum to the previous formalism, we note that the energy spectrum is simply the trace of the Fourier space correlation matrix. This is clearly seen by the definition

$$R_{ij}(\underset{\sim}{r}) = \int d\underset{\sim}{k}\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}S_{ij}(\underset{\sim}{k})$$

and so, (59)

$$\text{Energy density} \sim R_{ii}(0) = \int d\underset{\sim}{k}\; S_{ii}(\underset{\sim}{k}) .$$

The energy spectrum is then seen to be

$$E(\underset{\sim}{k}) = S_{ii}(\underset{\sim}{k}) = S^{(1)}k^2 + 3S^{(2)} + 2(\underset{\sim}{k}\cdot\hat{b})S^{(4)} \tag{60}$$

together with the constraint equations

$$S^{(1)}k^2 + S^{(2)} + (\underset{\sim}{k}\cdot\hat{b})S^{(4)} = 0$$

and

$$(\underset{\sim}{k}\cdot\hat{b})S^{(3)} + k^2S^{(4)} = 0.$$

This allows the energy spectrum to be written as:

$$E(k) = 2S^{(2)} + (1 - \cos^2\theta\ )S^{(3)} \tag{61}$$

where $\underset{\sim}{k}\cdot\hat{b} \equiv k \cos\theta$ and the functions $S^{(2)}(k,\underset{\sim}{k}\cdot\hat{b})$ and $S^{(3)}(k,\underset{\sim}{k}\cdot\hat{b})$ are unconstrained, independent scalar functions, even in $(\underset{\sim}{k}\cdot\hat{b})$, but remaining to be measured.

To obtain $S^{(2)}(k,\underset{\sim}{k}\cdot\hat{b})$ and $S^{(3)}(k,\underset{\sim}{k}\cdot\hat{b})$ requires correlation measurements to be made over a range of r from r = 0 to several correlation lengths and over a range of $\underset{\sim}{r}\cdot\hat{b}$. Therefore, $\underset{\sim}{r}$ must range over the half plane rather than simply in any one direction as was true in the isotropic analysis. Such a complete set of measurements is not available. It is therefore necessary to discuss the reduced correlation functions where correlations are known only for values of $\underset{\sim}{r}$ along a line given by $\hat{1}$. This is given by

$$S_{ij}(k_1) \equiv \int dk_2 dk_3 S_{ij}(\underset{\sim}{k}) \tag{62}$$

where k_2 and k_3 are the two components of $\underset{\sim}{k}$ in mutually orthogonal directions perpendicular to $\hat{1}$. The correlation $S_{ij}(k_1)$ is found from the correlations measured in configuration space by

$$S_{ij}(k_1) = \frac{1}{(2\pi)} \int dr_1 \, e^{-ik_1 r_1} R_{ij}(r_1 \hat{1}). \tag{63}$$

D. Magnetic Helicity

The magnetic helicity has been a topic of theoretical discussions for some time. In the case of infinite conductivity, the magnetic helicity is a constant of the motion for the system* and has been postulated to participate in selective decay of incompressible, 3-D MHD turbulence.[60-63]

Until now, there has been no clearly acceptable means of measuring this quantity in a real system due to inherent difficulties in measuring the vector potential. In this section, a means will be presented which relates measurements of the magnetic field to correlations between the magnetic field and the magnetic vector potential.

We previously defined the magnetic field $\underset{\sim}{\mathcal{B}} = \underset{\sim}{B}_0 + \underset{\sim}{B}(\underset{\sim}{x},t)$ and the magnetic vector potential $\underset{\sim}{\mathcal{A}} = \underset{\sim}{A}_0 + \underset{\sim}{A}(\underset{\sim}{x},t)$ so that $\langle \underset{\sim}{\mathcal{B}} \rangle = \underset{\sim}{B}_0$ and $\langle \underset{\sim}{\mathcal{A}} \rangle = A_0$. This allows us to write that the mean magnetic helicity density is given by $\langle \underset{\sim}{\mathcal{A}} \cdot \underset{\sim}{\mathcal{B}} \rangle = \langle \underset{\sim}{A}_0 \cdot \underset{\sim}{B}_0 \rangle + \langle \underset{\sim}{A} \cdot \underset{\sim}{B} \rangle$. The term $\langle \underset{\sim}{A}_0 \cdot \underset{\sim}{B}_0 \rangle$ is the magnetic helicity density of the mean field while $\langle \underset{\sim}{A} \cdot \underset{\sim}{B} \rangle$ is the contribution from the fluctuating field. There is no contribution from the interaction of the fluctuating field with the mean field, since $\langle \underset{\sim}{A} \cdot \underset{\sim}{B}_0 \rangle = \langle \underset{\sim}{A} \rangle \cdot \langle \underset{\sim}{B}_0 \rangle = 0 = \langle \underset{\sim}{A}_0 \cdot \underset{\sim}{B} \rangle$.

For this section, we will identify the random vector field $\underset{\sim}{\eta}(\underset{\sim}{x})$ with $\underset{\sim}{B}(\underset{\sim}{x},t)$ and write $B_{rms}^2 R_{ij}(\underset{\sim}{r}) = \langle B_i(\underset{\sim}{x}) B_j(\underset{\sim}{x} + \underset{\sim}{r}) \rangle$. If we define

*See Appendix IV.

$$B_{rms}^2 A_{ij}(\underset{\sim}{r}) \equiv \left\langle B_i(\underset{\sim}{x}) A_j(\underset{\sim}{x} + \underset{\sim}{r}) \right\rangle \tag{64}$$

and

$$\widetilde{A}_{ij}(\underset{\sim}{k}) \equiv \frac{1}{(2\pi)^3} \int d\underset{\sim}{r}\; e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} A_{ij}(\underset{\sim}{r}) \tag{65}$$

then

$$S_{ij}(\underset{\sim}{k}) = i\,\epsilon_{jhl} k_h \widetilde{A}_{il}\ . \tag{66}$$

Taking the vector product of $\underset{\sim}{k}$ with the second index in (66) yields:

$$\begin{aligned} \epsilon_{mnj} k_n S_{ij} &= i\,\epsilon_{mnj}\,\epsilon_{jhl} k_h k_n \widetilde{A}_{il} \\ &= i[k_m k_l - \delta_{ml} k^2]\widetilde{A}_{il} \\ &= -i\,k^2\,A_{im}\ , \end{aligned} \tag{67}$$

if the Coulomb gauge is assumed. Taking the trace and recalling the form of $S_{ij}(\underset{\sim}{k})$ gives for the magnetic helicity density spectrum:

$$A_{ii} = 2iS^{(7)}(k, \underset{\sim}{k}\cdot\hat{b})\ . \tag{68}$$

Recall that

$$\begin{aligned} R_{ij}^{ap} &= \int d\underset{\sim}{k}\;{}^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} S_{ij}^{ap} \\ &= \epsilon_{ijh}(-i)\frac{\partial}{\partial r_h}\int d\underset{\sim}{k} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} S^{(7)} \\ &= \epsilon_{ijh}\frac{\partial}{\partial r_h}\Phi(r, \underset{\sim}{r}\cdot\hat{b}) \end{aligned} \tag{69}$$

where $\Phi \equiv -i\int d\underset{\sim}{k}\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} S^{(7)}$. This is one solution to the constraint Eq. (46). But, the mean magnetic helicity density of the fluctuating field, $\langle \underset{\sim}{A}\cdot\underset{\sim}{B}\rangle$, is simply $\int d\underset{\sim}{k}\;\widetilde{A}_{ii}$ which implies that

$$\langle \underset{\sim}{A}\cdot\underset{\sim}{B}\rangle \equiv -2B_{rms}^2\Phi(r = 0)\ . \tag{70}$$

But, when $R_{ij}(\underset{\sim}{r})$ is the measured quantity, $\Phi(0)$ is still an integration constant which must be obtained by integration of the correlation matrix. This is done in Appendix III for the general case of a non-symmetrical turbulent field which includes the case of an axisymmetric system.

In the case of an axisymmetric system, we may choose to integrate along a straight line path in the plane perpendicular to the mean field. If we choose two orthogonal directions $\hat{\alpha}$ and $\hat{\beta}$ so that $\hat{\alpha} \otimes \hat{\beta} = \hat{r}$, we can write that

$$H_m = \langle \underset{\sim}{B} \cdot \underset{\sim}{A} \rangle = -2 \int_0^{\infty} dr \langle \hat{\alpha} \cdot \underset{\sim}{B}(x)\, \hat{\beta} \cdot \underset{\sim}{B}(\underset{\sim}{x} + \underset{\sim}{r}_1) \rangle \,. \tag{71}$$

By this equation, the mean magnetic helicity density is related to measurements of the magnetic field which may be taken at points sufficiently separated as to render the prescription feasible. This approach should be compared to the direct technique of trying to calculate the vector potential based upon measurements of the magnetic field or current density taken at points more closely spaced than the smallest length scale of the turbulent spectrum. This would not normally be practical.

A few more words should probably be said about the significance of this quantity. If a wildly varying magnetic field is viewed as a turbulent interaction of Alfven waves, the magnetic helicity will give the relative amount of right circularly polarized waves versus left circularly polarized waves. In this way, it describes a handedness of the flow. If this handedness becomes concentrated at the small wave numbers, as is postulated by inverse cascade arguments, it will produce a large scale helical structure which might be viewed as a distortion of the mean field from uniformity.

E. Reduction of R_{ij} to Minimum Scalars

As mentioned previously, there are several approaches which will allow one to apply the constraint equations which result from $\frac{\partial}{\partial r_j} R^{st}_{ij} = 0 = \frac{\partial}{\partial r_j} R^{ap}_{ij}$. All share the characteristic that the constraint equations as they are given by Eqs. (43) to (49) are avoided. Rather, in the first case, $S_{ij}(\underset{\sim}{k})$ is considered with the constraints $k_j S^{st}_{ij} = 0 = k_j S^{ap}_{ij}$ which is then transformed to configuration space thereby satisfying Eqs. (43) to (48).

The second approach is to note that the divergence of the curl of any quantity is zero. This allows us to define a sufficiently general generating tensor which when curled will produce R_{ij} in a form which satisfies the constraints by construction.

1. Fourier Space Approach

We have already argued that

$$S_{ij}(\underset{\sim}{k}) \equiv \frac{1}{(2\pi)^3} \int d\underset{\sim}{r}\; e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} R_{ij}(\underset{\sim}{r}) \tag{50}$$

may be constructed without consideration of the particular form of R_{ij}. It is given by Eqs. (51) to (53) together with the constraint Eqs. (54) to (57). One application of the constraint equations allows us to write

$$C(r,z) \equiv \int d\underset{\sim}{k}\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} k^{-2} S^{(2)}(k,\underset{\sim}{k}\cdot\hat{b}) \tag{72a}$$

$$D(r,z) \equiv \int d\underset{\sim}{k}\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} k^{-4} S^{(3)}(k,\underset{\sim}{k}\cdot\hat{b}) \tag{72b}$$

$$E(r,z) \equiv i\int d\underset{\sim}{k}\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} (\underset{\sim}{k}\cdot\hat{b})^{-1} S^{(5)}(k,\underset{\sim}{k}\cdot\hat{b}) \tag{72c}$$

and

$$F(r,z) \equiv -i\int d\underset{\sim}{k}\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} S^{(7)}(k,\underset{\sim}{k}\cdot\hat{b}) = \Phi(r,z). \tag{72d}$$

This allows us to write S_{ij} in terms of $S^{(2)}$, $S^{(3)}$, $S^{(5)}$ and $S^{(7)}$ only. Inverse transform to configuration space, and get

$$\begin{aligned} R^{st}_{ij}(r) = {} & \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j} C(r,z) + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\frac{\partial^2}{\partial r_b{}^2} D(r,z) - \delta_{ij}\nabla^2 C(r,z) \\ & + b_i b_j (\nabla^2)^2 D(r,z) - \left(b_j \frac{\partial}{\partial r_i} + b_i \frac{\partial}{\partial r_j}\right)\frac{\partial}{\partial r_b}\nabla^2 D(r,z) \\ & + \left(\frac{\partial}{\partial r_i}\epsilon_{jhl} b_h \frac{\partial}{\partial r_l} + \frac{\partial}{\partial r_j}\epsilon_{ihl} b_h \frac{\partial}{\partial r_l}\right)\frac{\partial}{\partial r_b} E(r,z) \\ & - \left(b_i \epsilon_{jhl} b_h \frac{\partial}{\partial r_l} + b_j \epsilon_{ihl} b_h \frac{\partial}{\partial r_l}\right)\nabla^2 E(r,z) \end{aligned} \tag{73}$$

and $R^{ap}_{ij}(r) = \epsilon_{ijh}\frac{\partial}{\partial r_h}\Phi(r,z)$ where $\frac{\partial}{\partial r_b} \equiv \hat{b}\cdot\nabla$ is the derivative with respect to the component of $\underset{\sim}{r}$ in the $\hat{b}$ direction.

Notice that this form contains fourth derivatives. This is characteristic of this approach. This problem can be avoided if the independent functions are chosen to be four functions in the Fourier representations rather than their appropriate transforms, but this is impractical since data is acquired in configuration space and would need to be transformed before the scalar functions could be obtained. There is a means by which the constraint equations may be applied which leaves a form for R_{ij} containing at most second derivatives.

2. Generating Tensor Approach

This is an extension of a technique set forth by Chandrasekhar[53] for the case of axisymmetric, inversion invariant, Navier-Stokes turbulence. The system discussed there corresponds in our notation to $R^{(5)} = R^{(6)} = R^{(7)} = R^{(8)} = 0$. For this case, since R^{st}_{ij} is already a symmetric matrix and even under $\underset{\sim}{r} \to -\underset{\sim}{r}$, it is trivial to show that $\frac{\partial}{\partial r_j} R^{st}_{ij} = 0$ implies $\frac{\partial}{\partial r_i} R^{st}_{ij} = 0$. We should then be able to define a matrix $Q_{ij}(\underset{\sim}{r})$ so that $R^{st}_{ij} \equiv \epsilon_{jlm} \frac{\partial}{\partial r_l} Q_{im}$ which implies by construction that $\frac{\partial}{\partial r_j} R^{st}_{ij} = 0$. This can be extended to include helical Navier Stokes (helical turbulence which is symmetric about a proper vector) by our now familiar result that

$$R^{ap}_{ij} = \epsilon_{ijh} \frac{\partial}{\partial r_h} \Phi = \epsilon_{jlm} \frac{\partial}{\partial r_l} \Phi\, \delta_{im}.$$

If this system only is to be discussed, then this description is a valid application of the constraint equation when

$$Q_{ij} \equiv Q^{(1)} \epsilon_{ijh} r_h + Q^{(2)} b_j \epsilon_{ilm} b_l r_m + \left(\frac{\partial Q^{(1)}}{\partial z}\right) r_j \epsilon_{ilm} b_l r_m. \tag{74}$$

This representation satisfies the solenoidal constraints by construction, possesses the proper symmetry of the indexes and produces a form for R_{ij} that contains at most second derivatives.

This is applicable to the more general form of R_{ij} present in the MHD description, but since we are considering symmetry about a pseudo-vector, we are allowed terms as given by $R^{(5)}$ and $R^{(6)}$, which are more difficult to arrive at by this approach. It is therefore useful to extend this approach by writing

$$R_{ij} = \epsilon_{ilm} \epsilon_{jhn} \frac{\partial}{\partial r_l} \frac{\partial}{\partial r_h} \mathbf{R}_{mn}\ . \tag{75}$$

This yields a form for R_{ij} which trivially satisfies the constraint equations and possesses the proper symmetry of the indexes. Since the operation of curling on both indexes preserves the symmetry, and maps proper tensor into proper tensor and pseudo-tensor into psueod-tensor, we can write

$$R_{ij}^{st} = \epsilon_{ilm} \epsilon_{jhn} \frac{\partial}{\partial r_l} \frac{\partial}{\partial r_h} \mathbf{R}_{mn}^{st} . \tag{76a}$$

and

$$R_{ij}^{ap} = \epsilon_{ilm} \epsilon_{jhn} \frac{\partial}{\partial r_l} \frac{\partial}{\partial r_h} \mathbf{R}_{mn}^{ap} . \tag{76b}$$

This allows $\mathbf{R}_{mn}^{st}$ and $\mathbf{R}_{mn}^{ap}$ to be constructed in the same manner as were R_{ij}^{st} and R_{ij}^{ap}.

The reader might find this last statement questionable, so the following line of thought is proposed. Suppose that R_{ij} is identified with the magnetic field correlation. Then, any combination of curls and time derivatives will produce a form which still displays axisymmetry. But, what of "uncurling"? Since R_{ij} is the measured quantity, is an axisymmetric form for $\mathbf{R}_{ij}$ completely general, or can other forms still produce an axisymmetric measurable? Since a curl on each index of $\mathbf{R}_{ij}$ produces R_{ij}, $\mathbf{R}_{ij}$ may at first be identified with the magnetic vector potential correlation. We may write

$$\underset{\sim}{A}(\underset{\sim}{x}) = \frac{1}{c} \int \frac{\underset{\sim}{j}(\underset{\sim}{x}')d^3\underset{\sim}{x}'}{|\underset{\sim}{x} - \underset{\sim}{x}'|} + \nabla \psi(\underset{\sim}{x}) \tag{77}$$

where $\underset{\sim}{j}(x)$ is the current density and $\psi(\underset{\sim}{x})$ is an arbitrary scalar function. Then,

$$\left\langle \underset{\sim}{A}(\underset{\sim}{x}_1)\underset{\sim}{A}(\underset{\sim}{x}_2)\right\rangle = \frac{1}{c^2}\int d^3\underset{\sim}{x}'d^3\underset{\sim}{x}'' \frac{1}{|\underset{\sim}{x}_1 - \underset{\sim}{x}'|}\frac{1}{|\underset{\sim}{x}_2 - \underset{\sim}{x}''|}\left\langle \underset{\sim}{j}(\underset{\sim}{x}')\underset{\sim}{j}(\underset{\sim}{x}'')\right\rangle$$
$$+ \frac{1}{c}\int d^3\underset{\sim}{x}' \frac{1}{|\underset{\sim}{x}_1 - \underset{\sim}{x}'|}\left\langle \underset{\sim}{j}(\underset{\sim}{x}')\nabla\psi(\underset{\sim}{x}_2)\right\rangle$$
$$+ \frac{1}{c}\int d^3\underset{\sim}{x}' \frac{1}{|\underset{\sim}{x}_2 - \underset{\sim}{x}'|}\left\langle \nabla\psi(\underset{\sim}{x}_1)\underset{\sim}{j}(\underset{\sim}{x}')\right\rangle \tag{78}$$
$$+ \left\langle \nabla\psi(\underset{\sim}{x}_1)\nabla\psi(\underset{\sim}{x}_2)\right\rangle$$

is the magnetic vector potential correlation.

It is apparent from the previous equation that non-axisymmetric magnetic vector potential correlations can yield axisymmetric magnetic field correlations since only the first term on the right is required to be axisymmetric. However, if we assume a Coulomb gauge, $\nabla\cdot\underset{\sim}{A} = 0$, and if we assume that the region of interest is infinite, we are then required to assert that $\nabla\psi = 0$. This yields an axisymmetric form for the magnetic vector potential correlation which is completely general within the assumption of the Coulomb gauge and this form may be constructed in analogy with the construction of the magnetic field correlation.

But, this implies that there are eight scalar functions present in $\left\langle A_i(\underset{\sim}{x})A_j(\underset{\sim}{x} + \underset{\sim}{r})\right\rangle$ and we know that only four are independent. If $\left\langle A_i(\underset{\sim}{x})A_j(\underset{\sim}{x} + \underset{\sim}{r})\right\rangle$ is to be used only to generate R_{ij}, that is if it is to be used to motivate $\mathbf{R}_{ij}$, we can show that only four of the tensor forms in $\left\langle A_i(\underset{\sim}{x})A_j(\underset{\sim}{x} + \underset{\sim}{r})\right\rangle$ contribute independently to $\left\langle B_i(\underset{\sim}{x})B_j(\underset{\sim}{x} + \underset{\sim}{r})\right\rangle$ and we can show that the other four forms may be discarded. Of course, while this entire discussion has been worded with the magnetic field correlation in mind, it applies in general to any case where R_{ij} is the measured quantity and where R_{ij} is solenoidal. We may now proceed to write $\mathbf{R}_{ij}$.

By the previous argument, we may write as a first effort

$$R_{ij}^{st} = R^{(1)}(r,z)b_i b_j + R^{(2)}(r,z)\,\delta_{ij} + R^{(3)}(r,z)(b_i \epsilon_{jhl} b_h r_l + b_j \epsilon_{ihl} b_h r_l)$$
$$+ R^{(5)}(r,z) r_i r_j + R^{(6)}(r,z)(r_i \epsilon_{jhl} b_h r_l + r_j \epsilon_{ihl} b_h r_l) \qquad (79)$$
$$+ R^{(7)}(r,z)(r_i b_j + r_j b_i)$$

and

$$R_{ij}^{ap} = R^{(4)}(r,z)\epsilon_{ijh} b_h + R^{(8)}(r,z)\epsilon_{ijh} r_h \ .$$

The solenoidal constraints are irrelevant for our purposes here, but it should be recalled that their existence allows us to write the above forms with the assurance that a completely general form for R_{ij} will be produced. In order to show that some of the forms represented in the previous equation may be eliminated without loss of generality to R_{ij}, we may consider the following equations where L, M and N are arbitrary scalar functions of r and z:

$$\frac{\partial}{\partial r_j} L r_i = L\,\delta_{ij} + \frac{r_i r_j}{r}\frac{\partial}{\partial r} L + r_i b_j \frac{\partial}{\partial z} L \qquad (80a)$$

$$\frac{\partial}{\partial r_i} L r_j = L\,\delta_{ij} + \frac{r_i r_j}{r}\frac{\partial}{\partial r} L + r_j b_i \frac{\partial}{\partial z} L \qquad (80b)$$

$$\frac{\partial}{\partial r_j} M b_i = \frac{r_j b_i}{r}\frac{\partial}{\partial r} M + b_i b_j \frac{\partial}{\partial z} M \qquad (80c)$$

$$\frac{\partial}{\partial r_i} M b_j = \frac{r_i b_j}{r}\frac{\partial}{\partial r} M + b_i b_j \frac{\partial}{\partial z} M \qquad (80d)$$

$$\frac{\partial}{\partial r_i} N \epsilon_{jhl} b_h r_l = \epsilon_{jhl} b_h r_l\left(\frac{r_i}{r}\frac{\partial}{\partial r} + b_i \frac{\partial}{\partial z}\right)N + N\epsilon_{ijh} b_h \qquad (80e)$$

$$\frac{\partial}{\partial r_j} N \epsilon_{ihl} b_h r_l = \epsilon_{ihl} b_h r_l\left(\frac{r_j}{r}\frac{\partial}{\partial r} + b_j \frac{\partial}{\partial z}\right)N - N\epsilon_{ijh} b_h \qquad (80f)$$

By adding Eqs. (80a) and (80b) and associating $\frac{\partial}{\partial r} L$ with $R^{(5)}$, we can see that $R^{(5)}$ may be dropped from Eq. (79) at the expense of adding two gradient terms. But since the curl of a gradient is zero, we may ignore these gradient terms since they do not contribute to R_{ij}. In the same manner, Eqs. (80c) and (80d) allow $R^{(7)}$ to be eliminated while Eqs. (80e) and (80f) allow us to drop $R^{(6)}$. Subtracting Eqs. (80f) and (80e) and recalling the Eqs. (41a) and (41b) allows us to eliminate $R^{(8)}$. This leaves us with a form for $\mathbf{R}_{ij}$ which generates, according to Eqs. (76a) and (76b), a completely general form for $\mathbf{R}_{ij}$ which possesses the proper symmetry and which is solenoidal and contains only four random functions explicitly represented. This form for $\mathbf{R}_{ij}$ is

$$\mathbf{R}_{ij}^{st} \equiv R^{(1)}(r,z) b_i b_j + R^{(2)}(r,z)\,\delta_{ij} + R^{(3)}(r,z)(b_i \epsilon_{jhl} b_h r_l + b_j \epsilon_{ihl} b_h r_l) \tag{81a}$$

and

$$\mathbf{R}_{ij}^{ap} \equiv \mathbf{R}^{(4)}(r,z)\,\epsilon_{ijh} b_h \tag{81b}$$

where $\mathbf{R}^{(1)}$ and $\mathbf{R}^{(2)}$ are even functions of z while $\mathbf{R}^{(3)}$ and $\mathbf{R}^{(4)}$ are odd.

Both of these approaches share a common complaint. That is that the independent scalar functions are several derivatives removed from the measured quantities. This is regrettable since it complicates any attempt to measure the correlation functions for a real system, but compared to the difficulties in measuring the correlation between any two directions as the separation vector varies over a range of values, this added difficulty is perhaps not so important. The key point is that for an axisymmetric system, many more measurements need to be made to measure the correlation function than were needed in the isotropic case. This is the principal reason why in any experimental analysis, we will need to confine our discussions to reduced correlation functions where correlations are only known for values of $\underset{\sim}{r}$ in a given direction.

F. von Karman-Howarth Equations

This section is presented primarily for completeness sake. The dynamical equations of the random scalar functions has been a topic of discussions for the simpler geometries. Characteristically, these equations do not form a closed set but rather form a hierarchy of equations which require some sort of closure scheme. A suitable closure has not yet been presented for any geometry. If we choose to present the dynamical equations for the scalar functions in the generating tensors presented in the previous sections we will again obtain a hierarchy of equations but the equations obtained will be of a particularly simple form. This simplicity is, however, misleading, and a closure scheme is not obvious.

Recall that two of the MHD equations are the magnetic Navier-Stokes equation

$$\frac{\partial}{\partial t} \underset{\sim}{v} + \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} = -\nabla(P/\rho) + \frac{1}{4\pi\rho} (\nabla \otimes \underset{\sim}{B}) \otimes \underset{\sim}{B} + \nu \nabla^2 \underset{\sim}{v} \tag{82}$$

and the induction equation

$$\frac{\partial}{\partial t} \underset{\sim}{B} = \nabla \otimes (\underset{\sim}{v} \otimes \underset{\sim}{B}) + \lambda \nabla^2 \underset{\sim}{B} \tag{83}$$

where ν is viscosity, ρ the density, P the pressure and λ the magnetic diffusivity, and we will from here on assume constant density. From these equations we may write the two tensor equations:

$$\frac{\partial}{\partial t} R_{ij}^{st,ap} = \chi \nabla^2 R_{ij}^{st,ap} + T_{ij}^{st,ap}(\mathbf{r}) \tag{84}$$

where the identification of the coefficient χ and the tensors $T_{ij}^{st,ap}$ are dependent upon the choice of which vector field is to be represented by the random field $\boldsymbol{\eta}$. If we identify $\boldsymbol{\eta}$ with $\mathbf{v}$, then $\chi = 2\nu$ and

$$\begin{aligned} T_{ij} = {} & \frac{\partial}{\partial r_k} \left\langle v_j(\mathbf{x} + \mathbf{r}) v_i(\mathbf{x}) v_h(\mathbf{x}) \right\rangle - \frac{\partial}{\partial r_h} \left\langle v_i(\mathbf{x}) v_j(\mathbf{x} + \mathbf{r}) v_h(\mathbf{x} + \mathbf{r}) \right\rangle \\ & - \frac{1}{4\pi\rho} \frac{\partial}{\partial r_h} \left\langle v_j(\mathbf{x} + \mathbf{r}) B_i(\mathbf{x}) B_h(\mathbf{x}) \right\rangle + \frac{1}{4\pi\rho} \left\langle v_i(\mathbf{x}) B_j(\mathbf{x} + \mathbf{r}) B_h(\mathbf{x} + \mathbf{r}) \right\rangle \\ & + \frac{1}{\rho} \left[\frac{\partial}{\partial r_i} \left\langle v_j(\mathbf{x} + \mathbf{r}) P(\mathbf{x}) \right\rangle - \frac{\partial}{\partial r_j} \left\langle v_i(\mathbf{x}) P(\mathbf{x} + \mathbf{r}) \right\rangle \right] \\ & + \frac{1}{8\pi\rho} \left[\frac{\partial}{\partial r_j} \left\langle v_i(\mathbf{x}) \left| \mathbf{B}(\mathbf{x} + \mathbf{r}) \right|^2 \right\rangle - \frac{\partial}{\partial r_j} \left\langle v_i(\mathbf{x}) \left| \mathbf{B}(\mathbf{x} + \mathbf{r}) \right|^2 \right\rangle \right] . \end{aligned} \tag{85}$$

If we identify $\boldsymbol{\eta}$ with $\mathbf{B}$, then $\chi = 2\lambda$ and

$$\begin{aligned} T_{ij} = {} & \frac{\partial}{\partial r_h} \left\langle B_i(\mathbf{x}) v_j(\mathbf{x} + \mathbf{r}) B_h(\mathbf{x} + \mathbf{r}) \right\rangle - \frac{\partial}{\partial r_h} \left\langle v_i(\mathbf{x}) B_j(\mathbf{x} + \mathbf{r}) B_h(\mathbf{x}) \right\rangle \\ & + \frac{\partial}{\partial r_h} \left\langle B_i(\mathbf{x}) B_j(\mathbf{x} + \mathbf{r}) [v_h(\mathbf{x}) - v_h(\mathbf{x} + \mathbf{r})] \right\rangle . \end{aligned} \tag{86}$$

Using Eqs. (76a), (76b) and (84), we may write that

$$\frac{\partial}{\partial t} \mathcal{R}_{ij}^{st,ap} = \chi \nabla^2 \mathcal{R}_{ij}^{st,ap} + \mathcal{T}_{ij}^{st,ap} . \tag{87}$$

This is advantageous because we have now eliminated the dependent scalar functions and half of the tensor forms. Although we don't know the functional form of $\mathcal{T}_{ij}^{st}$, or $\mathcal{T}_{ij}^{ap}$, we do know that they can be written in the same form as R^{st}_{ij} and R^{ap}_{ij}. Therefore, we can write

$$T^{st}_{ij} = T^{(1)} b_i b_j + T^{(2)} \delta_{ij} + T^{(3)} (b_i \epsilon_{jhl} b_h r_l + b_j \epsilon_{ihl} b_h r_l), \tag{88a}$$

and

$$T^{ap}_{ij} = T^{(4)} \epsilon_{ijh} b_h , \tag{88b}$$

which implies

$$\frac{\partial}{\partial t} R^{(1)} = \chi \nabla^2 R^{(1)} + T^{(1)} , \tag{89a}$$

$$\frac{\partial}{\partial t} R^{(2)} = \chi \nabla^2 R^{(2)} + T^{(2)} , \tag{89b}$$

$$\frac{\partial}{\partial t} R^{(3)} = \chi (\nabla^2 + 2r^{-1} \frac{\partial}{\partial r}) R^{(3)} + T^{(3)} , \tag{89c}$$

and

$$\frac{\partial}{\partial t} R^{(4)} = \chi \nabla^2 R^{(4)} + T^{(4)} . \tag{89d}$$

These are the von Karman-Howarth equations for this application of the solenoidal constraints. Note that they form a heirarchy which needs to be closed by some scheme before the equations may be solved.

G. Cross-Field Correlation

If we choose a second random vector field, call it $\underset{\sim}{\beta}$, we can extend the previous discussion to include the cross correlation between two different vector fields. To begin, define

$$\langle \underset{\sim}{\eta} \cdot \underset{\sim}{\beta} \rangle C_{ij} \equiv \langle \eta_i(\underset{\sim}{x}) \beta_j(\underset{\sim}{x} + \underset{\sim}{r}) \rangle \tag{90}$$

and note that the usual homogeneity rule developed for R_{ij} no longer holds. Namely, it is not necessarily true that $C_{ij}(\underset{\sim}{r}) = C_{ji}(-\underset{\sim}{r})$. In order to apply the previous developments, it is necessary to render C_{ij} into a form appropriate to the homogeneity restriction. We can write

$$C_{ij}(\underset{\sim}{r}) = {}^{(+)}C_{ij}(\underset{\sim}{r}) + {}^{(-)}C_{ij}(\underset{\sim}{r}), \tag{91}$$

where

$${}^{(+)}C_{ij}(\underset{\sim}{r}) \equiv \frac{1}{2}\left\langle \eta_i(\underset{\sim}{x})\beta_j(\underset{\sim}{x}+\underset{\sim}{r}) + \eta_j(\underset{\sim}{x}+\underset{\sim}{r})\beta_i(\underset{\sim}{x})\right\rangle, \tag{92}$$

and

$${}^{(-)}C_{ij}(\underset{\sim}{r}) \equiv \frac{1}{2}\left\langle \eta_i(\underset{\sim}{x})\beta_j(\underset{\sim}{x}+\underset{\sim}{r}) - \eta_j(\underset{\sim}{x}+\underset{\sim}{r})\beta_i(\underset{\sim}{x})\right\rangle. \tag{93}$$

Homogeneity implies that ${}^{(+)}C_{ij}(\underset{\sim}{r}) = {}^{(+)}C_{ji}(-\underset{\sim}{r})$ and ${}^{(-)}C_{ij}(\underset{\sim}{r}) = -{}^{(-)}C_{ji}(-\underset{\sim}{r})$.

By applying the coordinate inversion operator, we find that

$${}^{(+)}C_{ij}(\underset{\sim}{r}) = {}^{(+)}C^{st}_{ij}(\underset{\sim}{r}) + {}^{(+)}C^{ap}_{ij}(\underset{\sim}{r}) \quad , \tag{94a}$$

and

$${}^{(-)}C_{ij}(\underset{\sim}{r}) = {}^{(-)}C^{sp}_{ij}(\underset{\sim}{r}) + {}^{(-)}C^{at}_{ij}(\underset{\sim}{r}). \tag{94b}$$

The term C_{ij} contains only one pseudo-scalar and it is $z = \underset{\sim}{r}\cdot\hat{b}$ even though $\langle \underset{\sim}{\eta}\cdot\underset{\sim}{\beta}\rangle$ may itself be a pseudo-scalar. We can now assert that any pseudo-scalar function of z is zero when z = 0 so that $\langle \underset{\sim}{\eta}\cdot\underset{\sim}{\beta}\rangle$ is defined by the trace of ${}^{(+)}C_{ij}(0)$. This is the correct framework in which to view the cross-helicity, $\langle \underset{\sim}{v}\cdot\underset{\sim}{B}\rangle$. By way of a purely pedagogical remark, the von Karman-Howarth equations for this system are of the same form as before and simply contain a different identification of S_{ij} which may be found directly.

As is evident from this section, measurements of cross-field correlations require twice as many operations as do measurements of single field correlations. This is a result of the inability to apply the homogeneity assumption in the more direct manner.

H. Taylor Microscales

As was discussed in the introduction to isotropic turbulence, the quantities which characterize the turbulent flow are the spectra of the conserved or nearly conserved quantities and the microscales. A prescription for measuring the energy spectrum has been available for some time. With the present work, a viable technique for measuring the magnetic helicity spectrum has also become available. The cross-helicity spectrum may also be measured in a manner analogous with the energy spectrum. That leaves the need for a framework in which to discuss the microscales.

It is for this purpose that we have paid close attention to the tensor nature of the individual terms in the correlations. We know that some of the scalar function contained in R_{ij} are even functions of z while the others are odd. Further, we know that due to the assumption of analyticity, only even powers of r may be included in the Taylor expansions of the scalar functions.

Some degree of notation is required at this point. We will denote the microscales of the scalar function $R^{(i)}$ by the superscript "i" and since axisymmetric flows have more microscales than do isotropic flows, we will use subscripts to designate the different terms in the Taylor expansion of each function. In this way, we can write:

$$R^{(1)}(r,z) = R^{(1)}(0) - \frac{r^2}{2}\frac{\partial^2}{\partial r^2}R^{(1)}(0) + \frac{z^2}{2}\frac{\partial^2}{\partial z^2}R^{(1)}(0) + \mathcal{O}(r^2z^2)$$

$$R^{(1)}(0) - \frac{r^2}{2}[\lambda^{(1)}{}_{rr}]^{-2} + \frac{z^2}{2}[\lambda^{(1)}{}_{zz}]^{-2} + + \mathcal{O}(r^2z^2) \qquad (95a)$$

$$R^{(2)}(r,z) = 1 - \frac{r^2}{2}[\lambda^{(2)}{}_{rr}]^{-2} + \frac{z^2}{2}[\lambda^{(2)}{}_{zz}]^{-2} + \mathcal{O}(r^2z^2) \qquad (95b)$$

$$R^{(3)}(r,z) = R^{(3)}(0) - \frac{r^2}{2}[\lambda^{(3)}{}_{rr}]^{-2} + \frac{z^2}{2}[\lambda^{(3)}{}_{zz}]^{-2} + \mathcal{O}(r^2z^2) \qquad (95c)$$

and

$$R^{(4)}(r,z) = z[\lambda^{(4)}{}_{z}]^{-2} - \frac{r^2}{2}[\lambda^{(4)}{}_{rr}]^{-2} + \mathcal{O}(r^2z) \; . \qquad (95d)$$

It should be recalled that the normalization decided upon at the beginning of Chapter III forces $R^{(2)}(0) \equiv 1$. Also, $R^{(4)}(r,z)$ must be an odd function of z and so $R^{(4)}(r = 0) = 0$.

The time rate of change of the mean energy density is, from Eq. (19),

$$\eta^2_{rms} \frac{\partial}{\partial t}[3R^{(2)}(0) + R^{(3)}(0)] = \langle \eta \cdot \nabla^2 \eta \rangle \qquad (96)$$

where χ is a constant dependent upon the identification given to .
From the Taylor expansions, we get

$$\eta^{-2}_{rms}\langle \eta \cdot \nabla^2 \eta \rangle = -3(3[\lambda^{(2)}{}_{rr}]^{-2} + [\lambda^{(3)}{}_{rr}]^{-2}) + (3[\lambda^{(2)}{}_{zz}]^{-2} + [\lambda^{(3)}{}_{zz}]^{-2}) + 4[\lambda^{(4)}{}_{z}]^{-2} + 6R^{(1)}(0) \; . \qquad (97)$$

This may be simplified by applying Eqs. (43) and (44) to the Taylor expanded forms, Eqs. (95a) to (95d). Equation (43) implies that

$$4R^{(1)}(0) - [\lambda^{(2)}{}_{rr}]^{-2} + [\lambda^{(4)}{}_{z}]^{-2} = 0 \qquad (98)$$

and Eq. (44) implies that

$$2[\lambda^{(2)}{}_{zz}]^{-2} - [\lambda^{(3)}{}_{rr}]^{-2} + [\lambda^{(3)}{}_{zz}]^{-2} + 5[\lambda^{(4)}{}_{z}]^{-2} = 0. \qquad (99)$$

Combining Eqs. (97), (98) and (99) gives

$$\eta^{-2}_{rms} \langle \underset{\sim}{\eta} \cdot \nabla^2 \underset{\sim}{\eta} \rangle = 10R^{(1)}(0) - 10[\lambda^{(2)}_{rr}]^{-2} - 2[\lambda^{(3)}_{rr}]^{-2} + + [\lambda^{(2)}_{zz}]^{-2} . \tag{100}$$

This result can quickly be seen to reduce to the isotropic form.

The microscales governing the decay of the magnetic helicity may be found in the same manner. As is shown in Appendix IV, the time rate of change of the mean magnetic helicity density is $\frac{-8\pi\lambda}{c} \langle \underset{\sim}{B} \cdot \underset{\sim}{j} \rangle$. We may express this in terms of the Taylor microscales of the magnetic field correlation by writing:

$$R^{(7)}(r,z) = R^{(7)}(0) - \frac{r^2}{2} [\lambda^{(7)}_{rr}]^{-2} + \frac{z^2}{2} [\lambda^{(7)}_{zz}]^{-2} + \mathcal{O}(r^2 z^2) \tag{101a}$$

and

$$R^{(8)}(r,z) = z[\lambda^{(8)}_{z}]^{-2} - \frac{r^2}{2} [\lambda^{(8)}_{rr}]^{-2} + \mathcal{O}(r^2 z) \tag{101b}$$

in which case

$$B^{-2}_{rms} \langle \underset{\sim}{B} \cdot \underset{\sim}{j} \rangle = 6R^{(7)}(0) + 2 \frac{\partial}{\partial z} R^{(8)}(0) = -6R^{(7)}(0) - 2[\lambda^{(8)}_{z}]^{-2} . \tag{102}$$

This reduces to the isotropic limit of $-6R^{(7)}(0)$. Using a microscale technique, it is possible to tell whether the magnetic helicity (measured by the integration procedure) is growing or being damped away.

A similar calculation may be performed to find the microscales governing the decay of $\langle \underset{\sim}{v} \cdot \underset{\sim}{\omega} \rangle$ which is of interest in Navier-Stokes turbulence. Referring to the microscales of the velocity correlation, we get

$$\nu\langle \underset{\sim}{v}\cdot\nabla^2\underset{\sim}{\omega} + \underset{\sim}{\omega}\cdot\nabla^2\underset{\sim}{v}\rangle$$
$$= -4\left\{+9[\lambda^{(7)}{}_{rr}]^{-2} + 6[\lambda^{(7)}{}_{rr}]^{-2} + 2[\lambda^{(7)}{}_{zz}]^{-2} + 4[\lambda^{(8)}{}_{rrz}]^{-2} + [\lambda^{(8)}{}_{zzz}]^{-2}\right\}. \tag{103}$$

The constraint Eq. (46) implies that

$$[\lambda^{(7)}{}_{zz}]^{-2} - [\lambda^{(8)}{}_{rrz}]^{-2} = 0 \tag{104}$$

which implies that Eq. (103) may be written as

$$2\nu\langle \underset{\sim}{v}\cdot\nabla^2\underset{\sim}{\omega}\rangle = -4\left\{9[\lambda^{(7)}{}_{rr}]^{-2} + 6[\lambda^{(7)}{}_{rr}]^{-2} + + 6[\lambda^{(7)}{}_{zz}]^{-2} + [\lambda^{(8)}{}_{zzz}]^{-2}\right\}. \tag{105}$$

I. Geometrical Limits

There are a variety of limits to the general axisymmetric geometry and for the sake of completeness, some of the more often discussed limits will be outlined in this section.

The most often discussed limit of isotropic turbulence was discussed in Chapter two to provide an introduction to the Taylor microscales. A comparison of Eqs. (16), (39) and (49) demonstrates that the isotropic forms are contained within the axisymmetric description and it is presumed that as $|\underset{\sim}{B}_0|$ is reduced, the correlations should approach the isotropic forms. Recall that the scalar functions are functions of $|\underset{\sim}{B}_0|$. An interesting result of this limit is that directions and separations which are correlated in the axisymmetric description become uncorrelated as the system becomes isotropic. Let $\hat{r}$ be the unit vector in the direction of the displacement vector $\underset{\sim}{r}$ and let $\hat{\theta}$ be perpendicular to both

$\hat{r}$ and $\hat{b}$. In the axisymmetric case, $\langle \hat{r} \cdot \underset{\sim}{\eta}(\underset{\sim}{x}) \hat{\theta} \cdot \underset{\sim}{\eta}(\underset{\sim}{x} + \underset{\sim}{r}) \rangle$ is a correlated quantity with contributions coming independently from both the tensor and pseudo-tensor halves of R_{ij}. However, in the isotropic case, these quantities become uncorrelated for all values of $\underset{\sim}{r}$. The pseudo-tensor contribution may be linked to the existence of a helical spectrum, but the proper tensor contribution involves the still not understood terms which are peculiar to MHD turbulence.

1. Slab Limit

As previously mentioned, the slab limit is the most often discussed geometrical limit for solar wind turbulence. Without discussing the validity of this geometry, it is a simple matter to see the form the correlations take as a system approaches such a geometry. The slab limit assumes that correlation lengths in the direction of the mean magnetic field are finite while infinite correlation lengths exist in the perpendicular direction. In addition, the fluctuations are restricted to lie in the perpendicular plane.

This limit is arrived at by assuming that all of the scalar functions are functions of the variable z alone in keeping with the microscales just mentioned. In this limit,

$$R_{ij} = (\delta_{ij} - b_i b_j) R^{(2)}(z) + R^{(8)}(z) \epsilon_{ijh} b_h \tag{106}$$

and the spectral representation is given by

$$S_{ij} = \delta(k_\perp)[(\delta_{ij} - b_i b_j) S^{(2)}(k_\parallel) + S^{(8)}(k_\parallel) \epsilon_{ijh} b_h]. \tag{107}$$

Notice that this limit allows for helical flow and that it is trivially solenoidal. Consistent with the microscale arguments, only wavenumber vectors, $\underset{\sim}{k}$, parallel to $\hat{b}$ are permitted. Unlike the isotropic limit or the limits about to be discussed, there is no motivation as to when a system should display a turbulent slab geometry.

2. Two Dimensional Geometry

An extensive review of turbulence in two dimensions, has recently been published by Kraichnan and Montgomery.[64] Another recent publication by Montgomery and Turner[65] argues that the 2-D geometry is a limit of the fully general axisymmetric form that is achieved when the mean magnetic field becomes large. Within this limit, fluctuations are once again restricted to the perpendicular plane, but correlation lengths are now infinite in the parallel direction while they are finite in the perpendicular plane. Only wavenumber vectors, $\underset{\sim}{k}$, perpendicular to the symmetry axis are allowed and so we can write for the correlation matrices:

$$R_{ij} = (\delta_{ij} - b_i b_j) R^{(2)}(r^2 - z^2) \tag{108}$$

and

$$S_{ij} = (\delta_{ij} - b_i b_j) S^{(2)}(k_\perp) \delta(k_\parallel). \tag{109}$$

Notice that this form does not allow for helical flow. The presence of helicity could therefore represent an obstacle to the collapse of the flow to a 2-D geometry.

Recent work suggesting an alternative to the 2-D collapse may be the form recently developed by Montgomery and Turner.[66] In this limit, the microscales and wavenumber vectors are restricted in the same manner as in the 2-D limit. The difference is that fluctuations are allowed to leave the perpendicular plane allowing helical flows to be reinstated. The correlation representation for this geometry is:

$$R_{ij} = R^{(1)}(r^2 - z^2)r_i r_j + R^{(2)}(r^2 - z^2)\,\delta_{ij} + R^{(3)}(r^2 - z^2)b_i b_j + + R^{(7)}(r^2 - z^2)\,\epsilon_{ijh} r_h \tag{110}$$

and

$$S_{ij} = [S^{(1)}(k_\perp)k_i k_j + S^{(2)}(k_\perp)\,\delta_{ij} + S^{(3)}(k_\perp)b_i b_j + S^{(4)}(k_\perp)\,\epsilon_{ijh} k_h]\,\delta(k_\parallel). \tag{111}$$

Summary

In order to examine the turbulence of any given flow, it is necessary to have a framework in which to discuss it. This thesis has attempted to construct such a framework for a geometry that is believed to be appropriate to a large number of physical systems. The plasmas in some of these laboratory devices have been acknowledged to be turbulent; by comparing Reynold's numbers, we are led to suspect that many others should also be turbulent. Such turbulent flows seem to be high Prandtl number flows suggesting that most of the turbulent energy may lie in the magnetic field.

In developing the axisymmetric description we constructed the scalars under this set of rotations from the vectors available in the correlation function. Appropriate combinations were formed in accordance with a short theorem so that the various contributions to the correlation matrix behave properly under coordinate inversion. We saw that this constructive approach gave a form for the correlation matrix that contained four independent, unknown scalar functions in the case where the symmetry axis is the mean magnetic field. The von Karman-Howarth equations were derived for these scalar functions. The microscales for this geometry were outlined and they were related to the decay of the energy and the various helicities.

Perhaps the most important aspect of this dissertation is that it demonstrates a means for measuring the magnetic helicity together with its spectrum. I have not included an example of this technique in application because such efforts are at this stage still incomplete and very much ongoing. However, I should mention some results so that the reader may know that there are applications.

Recent application of this formalism to the solar wind has produced some remarkable results.[28] The solar wind is, as mentioned earlier, remarkably anisotropic. Fluctuations are well restricted to the perpendicular plane and the correlation lengths suggest a slab geometry. And yet, when the reduced correlation functions are Fourier transformed and the reduced energy spectrum, $\mathcal{E}(k_r)$, is calculated under the frozen flow hypothesis so that the total energy, E, is $\int \mathcal{E}(k)dk$, we find for large k that $\mathcal{E}(k) \sim k^{-5/3}$ which is the isotropic prediction of Kolmogorov. The magnetic helicity reduced spectrum is found to behave so that $H(k) \sim k^{-8/3}$ which again has the same wavenumber dependence as an isotropic result. No explanation of this can be given at this time, but it remains true. In addition, H(k), is seen to oscillate rapidly about zero. The net helicity exists in the lowest wavenumbers as theory predicts, but each wavenumber is seen to possess significant amounts of helicity indicating a polarization of the structures at that wavenumber.

As mentioned, these investigations are still young and it would not be appropriate to include them in detail in this dissertation. I mention them in closing simply to demonstrate that the formalism discussed here has applications.

APPENDIX I

It is important for the purposes of this dissertation to distinguish between proper tensors and pseudo-tensors. A tensor has two defining properties under coordinate transformation. They are:

1) its behavior under coordinate rotation and

2) its behavior under inversion of the coordinate system.

Under coordinate rotation (rigid rotation for simplicity) a vector $\underset{\sim}{r}$ is transformed into a vector $\underset{\sim}{r}'$ according to the rule $r'_\alpha = \Lambda_{\alpha\beta} r_\beta$ where $\Lambda_{\alpha\beta}$ has the property that $\Lambda_{\beta\alpha} \Lambda_{\beta\gamma} = \delta_{\alpha\gamma}$. The second rank tensor $\delta_{\alpha\beta}$ is defined in the usual manner $\delta_{\alpha\beta} = \begin{cases} 1 \text{ if } \alpha = \beta \\ 0 \text{ if } \alpha \neq \beta \end{cases}$. A second rank tensor transforms according to $T_{\alpha\beta} = \Lambda_{\alpha\gamma} \Lambda_{\beta\delta} T_{\gamma\delta}$ and so on. As an example, if r_i is a vector, one example for T_{ij} is $r_i r_j$. If b_i is another vector, then $r_i b_j$ also transforms under rotation as required. $r_i r_j r_h$ would be a third rank tensor, and so forth.

Inversion is a separate operation. A coordinate inversion cannot be accomplished by any combination of rigid rotations. Under inversion, a right-handed Cartesian system becomes a left-handed Cartesian system. A proper vector transforms under inversion (labeled Inv.) as would a position vector: it changes sign. If $\underset{\sim}{r}$ is such a proper vector, Inv $\underset{\sim}{r} = -\underset{\sim}{r}$. A pseudo-vector does not change sign: If $\underset{\sim}{b}$ is such a pseudo-vector, Inv $\underset{\sim}{b} = \underset{\sim}{b}$.

A second rank proper tensor does not change sign under inversion: Inv. $r_i r_j = r_i r_j$ and Inv. $b_i b_j = b_i b_j$. A second rank pseudo-tensor does change sign under inversion: Inv. $r_i b_j = -r_i b_j$. In general, an Nth rank proper tensor transforms under inversion according to: Inv. $T_{ij\ldots} = (-1)^N T_{ij\ldots}$ and an Nth rank pseudo-tensor transforms according to Inv. $T_{ij\ldots} = (-1)^{N+1} T_{ij\ldots}$.

A vector is, of course, only a 1st rank tensor and a scalar is a zeroeth rank tensor. Hence, proper scalars do not change under inversion and pseudo-scalars change sign. As examples, $r^2 \equiv \underset{\sim}{r}\cdot\underset{\sim}{r}$ is a proper scalar while $\underset{\sim}{r}\cdot\hat{b}$ is a pseudo-scalar. Therefore, $r^2 r_i r_j$ is still a proper tensor while $(\underset{\sim}{r}\cdot\hat{b})r_i r_j$ is a pseudo-tensor. Notice also that $\epsilon_{ijh} r_j b_h$ is a proper vector and $\epsilon_{ijh} \frac{\partial}{\partial r_j} b_h$ is also. Then, $r_i \epsilon_{jhl} r_h b_l$ is another example of a proper, second rank tensor while $b_i \epsilon_{jhl} r_h b_l$ is another pseudo-tensor of rank two.

So, we can change a proper tensor into a pseudo-tensor and vice-versa by multiplying it by a pseudo-scalar. Multiplication by a proper scalar leaves the "pseudo-ness" unchanged. Further, the sum of two proper tensors is a proper tensor and the sum of two psuedo-tensors is a pseudo-tensor, but the sum of a proper tensor and pseudo-tensor is not a tensor. It is simply a matrix, an object with indexes. Any function of a proper scalar is a proper scalar as is any even function of a pseudo-scalar. An odd function of a pseudo-scalar is a pseudo-scalar.

We are constructing a symmetry based upon a pseudo-vector $\hat{b}$. Contained within this description is the pseudo-scalar $\underset{\sim}{r}\cdot\hat{b}$. By determining that each contribution to R_{ij} is either proper tensor or pseudo-tensor, we may determine whether the scalar functions are even or odd functions of $\underset{\sim}{r}\cdot\hat{b}$. It also enables us to double the number of constraint equations imposed upon those functions. Also, by choosing a pseudo-vector to designate the symmetry axis we find that the correlations may include terms excluded by the proper vector case, indicating that it is a fundamentally different system.

APPENDIX II

The anti-symmetric, pseudo-tensor part of the correlation matrix $R_{ij}(\underset{\sim}{r})$ is written as $R^{ap}_{ij}(r) = R^{(7)}\epsilon_{ijh}r_h + R^{(8)}\epsilon_{ijh}b_h$. This is due to the fact that the candidate form $R^{(9)}(r_ib_j - r_jb_i)$ is forced to be zero by the constraint equations:

$$-2R^{(9)} - r\frac{\partial}{\partial r}R^{(9)} - z\frac{\partial}{\partial z}R^{(9)} = 0 \tag{47}$$

and

$$zr^{-1}\frac{\partial}{\partial r}rR^{(9)} + \frac{\partial}{\partial z}R^{(9)} = 0. \tag{48}$$

This can be seen by two different approaches which act to support the assertion that the constructive approach undertaken here to find $R_{ij}(\underset{\sim}{r})$ is equally applicable in Fourier space.

The first approach is to solve these equations directly. Equation (48) may be solved by separation of variables which gives $R^{(9)} = C\exp(r^2 - z^2)$ where C is any constant. This is not a solution of Eq. (47) and even if it were, we would be forced to assert C = 0 in order that correlations go to zero as $\underset{\sim}{r}$ goes to infinity for constant values of $r^2 - z^2$.

A second approach is worth noting since it relies upon our ability to apply the constructive approach to correlation in Fourier space. As can be seen in Appendix III, when the Fourier representation is transformed back to configuration space where the solenoidal constraints are trivially satisfied having been applied in Fourier space where they are simply algebraic, no term of the form $R^{(9)}(r_ib_j - r_jb_i)$ is obtained.

APPENDIX III

If there is no symmetry to the correlations, we may simply choose an arbitrary direction, $\underset{\sim}{\lambda}$, and write:

$$\underset{\sim}{\alpha} = \underset{\sim}{\lambda} \otimes \underset{\sim}{k}, \quad \underset{\sim}{\beta} = \underset{\sim}{k} \otimes (\lambda \otimes \underset{\sim}{k})$$

and

$$S_{ij}^{ap} = S^{(7)}(\underset{\sim}{k}) \epsilon_{ijh} k_h + S^{(8)}(\underset{\sim}{k}) \epsilon_{ijh} \alpha_h + S^{(9)} \epsilon_{ijh} \beta_h$$

$$+ S^{(10)}(\underset{\sim}{k})[k_i \alpha_h - k_j \alpha_i] + S^{(11)}(\underset{\sim}{k})[k_i \beta_j - k_j \beta_i]$$

$$+ S^{(12)}(\underset{\sim}{k})[\alpha_i \beta_j - \alpha_j \beta_i] .$$

The last three terms, although reasonable candidates for S_{ij}, may be seen to be not independent and the solenoidal constraint forces $S^{(8)} = 0$ and $S^{(9)} = 0$. This means that $S_{ij}^{ap}(\underset{\sim}{k}) = S^{(7)}(\underset{\sim}{k}) \epsilon_{ijh} k_h$ is the most general form for S_{ij}^{ap} and the assumption of a particular symmetry simply restricts the function $S^{(7)}(\underset{\sim}{k})$ to a simpler form. At this point, with no symmetry assumed, we know only that it is an even function of $\underset{\sim}{k}$.

An inverse transformation to real space gives

$$R_{ij}^{ap}(\underset{\sim}{r}) = \int d\underset{\sim}{k}\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} S_{ij}^{ap}(\underset{\sim}{k})$$

$$= \int d\underset{\sim}{k}\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} S^{(7)}(\underset{\sim}{k}) \epsilon_{ijh} k_h$$

$$= \epsilon_{ijh} \frac{\partial}{\partial r_h} (-i) \int d\underset{\sim}{k}\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} S^{(7)}(\underset{\sim}{k})$$

$$= \epsilon_{ijh} \frac{\partial}{\partial r_h} \Phi(\underset{\sim}{r})$$

where $\Phi(\underset{\sim}{r}) \equiv -i \int d\underset{\sim}{k}\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} S^{(7)}(\underset{\sim}{k})$.

This shows that even without any degree of symmetry in the problem, we can write that $R^{ap}_{ij}(\underset{\sim}{r}) = \epsilon_{ijh} \frac{\partial}{\partial r_h} \Phi(\underset{\sim}{r})$. Since $S^{ap}_{ij}(k)$ can still be written in the simple form just mentioned, it is clear that it will still uncurl in the same manner stating that $H_m = 2\Phi(0)$ in even the most general case. To relate this to the measured correlation matrix, we can choose two mutually perpendicular directions $\hat{\gamma}$ and $\hat{\delta}$ and a vector $\underset{\sim}{r}$ so that $\hat{\gamma} \times \hat{\delta} = \hat{r}$ and say:

$$\gamma_i R^{ap}_{ij} \delta_j = \epsilon_{ijh} \gamma_i \delta_j \frac{\partial}{\partial r_h} \Phi(\underset{\sim}{r}).$$

We can then parameterize a straight line curve $\underset{\sim}{r}(l)$ where l is the parameter such that $r(0) = 0$ and $r(\infty) = \infty$ and write:

$\int_{l=0}^{\infty} dr(l)\, \gamma_i R^{ap}_{ij} \delta_j = \Phi(\infty) - \Phi(0)$. But, R^{st}_{ij} is an even function of $\underset{\sim}{r}$ while R^{ap}_{ij} is an odd function of $\underset{\sim}{r}$ so we can write:

$$\begin{aligned}
&\int_{l=0}^{\infty} dr(l)\, \gamma_i [R^{ap}_{ij}(\underset{\sim}{r}) - R^{ap}_{ij}(-\underset{\sim}{r})]\, \delta_j \\
&= \int_{l=0}^{\infty} dr(l)\, \gamma_i [R_{ij}(\underset{\sim}{r}) - R_{ij}(-\underset{\sim}{r})]\, \delta_j \\
&= -2\Phi(0).
\end{aligned}$$

This gives the magnetic helicity to be:

$$H_m = 2\Phi(0) = -\int_{l=0}^{\infty} dr(l)\, \gamma_i [R_{ij}(\underset{\sim}{r}) - R_{ij}(-\underset{\sim}{r})]\, \delta_j .$$

APPENDIX IV

The fact that the magnetic helicity is a constant of the motion for non-resistive, three-dimensional MHD is worth reviewing here. It should be noted that boundary conditions must be assumed in this derivation and that these assumptions must be justified when the theorem is applied to a real physical system. Keeping this in mind, we review the result here.

If we define $H_M \equiv \int_V \underset{\sim}{A}\cdot\underset{\sim}{B}\, d^3\underset{\sim}{x}$ to be the magnetic helicity density integrated over some volume of the fluid, V, then we can write:

$$\frac{DH_M}{Dt} \equiv \frac{\partial}{\partial t} H_M + \underset{\sim}{v}\cdot\nabla H_M = \int_V \frac{D}{Dt}\left(\frac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{\rho}\right)\rho\, d^3\underset{\sim}{x} + \int_V \frac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{\rho}\frac{D}{Dt}\left(\rho\, d^3\underset{\sim}{x}\right) .$$

By conservation of mass, the second integral is zero. To evaluate the first integral, we can write

$$\frac{D}{Dt}\left(\frac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{\rho}\right) = \underset{\sim}{A}\cdot\frac{D}{Dt}\frac{\underset{\sim}{B}}{\rho} + \frac{\underset{\sim}{B}}{\rho}\cdot\frac{D}{Dt}\underset{\sim}{A}.$$

Since $\frac{\partial}{\partial t}\underset{\sim}{B} - \nabla\otimes(\underset{\sim}{v}\otimes\underset{\sim}{B}) = \frac{c}{4\pi\sigma}\nabla^2\underset{\sim}{B}$

and $\frac{\partial}{\partial t}\underset{\sim}{A} = \underset{\sim}{v}\otimes\underset{\sim}{B} + \frac{c}{4\pi\sigma}\nabla^2\underset{\sim}{A} - \nabla(\phi c)$

and $\frac{D\rho}{Dt} = -\rho\nabla\cdot\underset{\sim}{v}$, we can write

$$\frac{D}{Dt}\left(\frac{\underset{\sim}{A}\cdot\underset{\sim}{B}}{\rho}\right) = \underset{\sim}{B}\cdot\nabla(\underset{\sim}{A}\cdot v - \phi) + \lambda\,(\underset{\sim}{A}\cdot\nabla^2\underset{\sim}{B} + \underset{\sim}{B}\cdot\nabla^2\underset{\sim}{A})$$

where $\lambda \equiv \frac{c}{4\pi\sigma}$, is the conductivity and c is the speed of light, and ϕ is the electrostatic potential.

This allows us to write:

$$\frac{D}{Dt}\int_V d^3\underset{\sim}{x}\ \underset{\sim}{A}\cdot\underset{\sim}{B} = \int_S (\underset{\sim}{B}\cdot\hat{n})(\underset{\sim}{A}\cdot\underset{\sim}{v} - \phi)ds + \lambda\int_V d^3\underset{\sim}{x}(\underset{\sim}{A}\cdot\nabla^2\underset{\sim}{B} + \underset{\sim}{B}\cdot\nabla^2\underset{\sim}{A})$$

where $\hat{n}$ is the unit normal to the surface, S, of the volume, V.

In the case of infinite conductivity, $\lambda = 0$ and the magnetic field is convected by the fluid elements so that if V is initially chosen so that $\underset{\sim}{B}\cdot\hat{n} = 0$ everywhere on S, it will remain so for all time. If $\lambda \neq 0$, there are three choices for S which will eliminate the surface integral. They are:

1) The surface of a perfect conductor on which $\underset{\sim}{B}\cdot\hat{n}$ will be zero,

2) Periodic boundary conditions,

3) An infinite volume where the far field values go to zero sufficiently fast for the surface integral to go to zero. In either of these cases, or in any combination of these cases,

$$\frac{d}{dt}H_M = \lambda\int_V d^3\underset{\sim}{x}(\underset{\sim}{A}\cdot\nabla^2\underset{\sim}{B} + \underset{\sim}{B}\cdot\nabla^2\underset{\sim}{A}) = \frac{-8\pi}{c}\ \lambda\int_V d^3\underset{\sim}{x}\ \underset{\sim}{B}\cdot\underset{\sim}{j}\ .$$

These cases are not the same as the assumption we have made from the beginning; that the system is infinite and homogeneous. But if it is reasonable to believe that when the length scale on which the system is inhomogeneous is large compared to correlation lengths, then a homogeneous model should still be a good local approximation. This would seem to indicate that the mean magnetic helicity density calculated from a homogeneous model should still be pertinent to a description of the system.

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VITA

Charles W. Smith

Born in Baltimore, Maryland, November 18, 1955. Graduated from Mt. St. Joseph High School in that city, June 1973, from the University of Maryland with a B.S. in Physics and Mathematics, June 1977. M. S. degree received from William and Mary, August 1979.